Vectors

Motivation: Vectors are an elementary tool in engineering design and analysis. In addition, most practical engineering problems involve functions depending on multiple variables and understanding their behaviour relies on a good knowledge of vectors.

Most engineering texts use the standard unit basis vector notation, so we will adopt it here as well. The dot product has got many uses, such as allowing us to work out the angle between vectors and finding projections.

Outcomes In today's lecture we will learn how to:

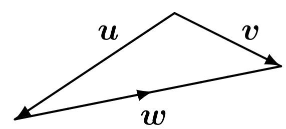
- Work with vectors in a geometric as well as in an algebraic setting.
- Understand the distinction between notation for a point and vector.
- Recognize that vectors are independent of location within 2 or 3 space.
- Work in standard unit basis vector notation.
- Understand and use of the two versions of the dot product.
- Determine scalar and vector projections.
- Understand the meaning of the direction cosines of a vector $\boldsymbol{a}$ and relation to the unit vector in the direction of $\boldsymbol{a}$.

Contents

- Geometric view of vectors and vector arithmetic in the plane.
- Coordinate vectors of points in the plane.
- Vector arithmetic in terms of coordinate vectors.
- The length of a vector.
- Extension of these concepts to vectors in 3 space.
- Standard unit basis vectors in 2 and 3 space and associated notation.
- The dot product - first form.
- The dot product - second form and geometrical interpretation.
- Scalar and vector projections.
- Work done by a force.
- Direction cosines.

Exercises

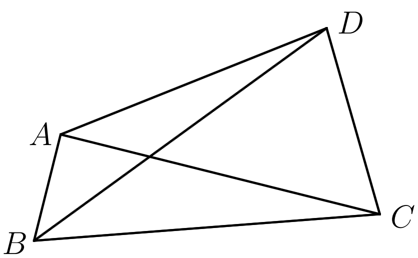
1. Exercise 12(a) (p. 643): 5, 6.
2. ABC is a triangle with D a point on BC such that BD is three-quaters of BC. If $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AC} = \boldsymbol{b}$ express $\overrightarrow{BC}$, $\overrightarrow{BD}$, $\overrightarrow{DC}$ and $\overrightarrow{AD}$ in terms of $\boldsymbol{a}$ and/or $\boldsymbol{b}$.
3. In the figure below, express $\boldsymbol{w}$ in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$.



4. In the figure below, write each combination of vectors as a single vector.

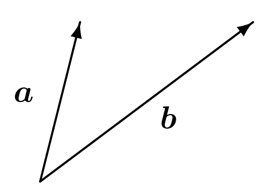
   (a) $\overrightarrow{AB} + \overrightarrow{BC}$ (b) $\overrightarrow{CD} + \overrightarrow{DA}$

   (c) $\overrightarrow{BC} - \overrightarrow{DC}$ (d) $\overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA}$



5. Copy the vectors in the figure below and use them to draw the following vectors.

   (a) $\boldsymbol{a} + \boldsymbol{b}$ (b) $\boldsymbol{a} - \boldsymbol{b}$ (c) $2\boldsymbol{a}$ (d) $-\frac{1}{2}\boldsymbol{b}$



6. In each of the following cases, find the vector given by $\boldsymbol{a} = \overrightarrow{AB}$. Draw $\overrightarrow{AB}$ and the equivalent representation starting at the origin.

   (a) $A(1, 3)$, $B(4, 4)$ (b) $A(-1, -1)$, $B(-3, 4)$

7. For each of the following, find (i) $||\boldsymbol{a}||$ (ii) $\boldsymbol{a}+\boldsymbol{b}$ (iii) $\boldsymbol{a}-\boldsymbol{b}$ (iv) $-2\boldsymbol{a}$ and (v) a unit vector in the direction of $\boldsymbol{a}$.

   (a) $\boldsymbol{a} = [-4, 3]$, $\boldsymbol{b} = [6, 2]$ (b) $\boldsymbol{a} = [6, 2, 3]$, $\boldsymbol{b} = [-1, 5, -2]$

8. Find a vector that has the same direction as $[-2, 4, 2]$ but has a length of 4.

9. If D, E and F are the vertices of a triangle, find $\overrightarrow{DE} + \overrightarrow{EF} + \overrightarrow{FD}$.

10. Exercise 12(b) (p. 649): 2.

11. Exercise 12(c) (p. 654): 1, 5, 7.

12. For each of the following, find (i) $\boldsymbol{a}+\boldsymbol{b}$ (ii) $4\boldsymbol{a}+3\boldsymbol{b}$ (iii) $||\boldsymbol{a}||$ (iv) and $||\boldsymbol{b}-\boldsymbol{a}||$.

    (a) $\boldsymbol{a} = 4\boldsymbol{i}+\boldsymbol{j}$, $\boldsymbol{b} = \boldsymbol{i}-2\boldsymbol{j}$ (b) $\boldsymbol{a} = 2\boldsymbol{i}-4\boldsymbol{j}+4\boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{j}-\boldsymbol{k}$

13. Given $\boldsymbol{a} = [-3, 2]$ and $\boldsymbol{b} = [4, 1]$, find $\boldsymbol{a}.\boldsymbol{b}$.

14. If $||\boldsymbol{a}|| = \sqrt{6}$, $||\boldsymbol{b}|| = 4$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $60°$, find $\boldsymbol{a}.\boldsymbol{b}$.

15. Determine the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

    (a) $\boldsymbol{a} = [5, 12]$, $\boldsymbol{b} = [-2, 5]$ (b) $\boldsymbol{a} = 4\boldsymbol{i}-3\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i}+2\boldsymbol{j}-2\boldsymbol{k}$

16. For the following pairs of vectors, determine if they are orthogonal, parallel or neither.

    (a) $\boldsymbol{a} = [4, -12, -8]$, $\boldsymbol{b} = [-3, 9, 6]$ (b) $\boldsymbol{a} = 2\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}$
    (c) $\boldsymbol{a} = [-d, e, 0]$, $\boldsymbol{b} = [e, d, f]$

17. Exercise 12(d) (p. 662): 1, 3, 6.

18. A rescue boat tows a life raft along the ocean. The rope makes an angle of $45°$ with the ocean and the tension in the rope is 800 N. How much work is done by the rescue boat in pulling the life raft 2km to shore?

19. For the vector $\boldsymbol{a} = [6, 3, -2]$ determine the direction cosines as well the direction angles.

20. Given that $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are vectors in 3 space, which of the following expressions are sensible and which are not?

    (a) $(\boldsymbol{a}.\boldsymbol{b}).\boldsymbol{c}$ (b) $(\boldsymbol{a}.\boldsymbol{b})\boldsymbol{c}$ (c) $||\boldsymbol{a}||(\boldsymbol{b}.\boldsymbol{c})$ (d) $\boldsymbol{a}.(\boldsymbol{b}+\boldsymbol{c})$ (e) $\boldsymbol{a}.\boldsymbol{b}+\boldsymbol{c}$ (f) $||\boldsymbol{a}||.(\boldsymbol{b}+\boldsymbol{c})$

21. In each of the following cases, find $\boldsymbol{a}.\boldsymbol{b}$.

    (a) $\boldsymbol{a} = [2, 6, -3]$, $\boldsymbol{b} = [8, -2, -1]$

    (b) $||\boldsymbol{a}|| = 12$, $||\boldsymbol{b}|| = 15$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $60°$.

    (c) $\boldsymbol{a} = 4\boldsymbol{i} - 3\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} + 4\boldsymbol{j} + 6\boldsymbol{k}$.

22. Find, rounded to the nearest degree, the three angles in the triangle ABC, where $A(1, 2, 3)$, $B(6, 1, 5)$ and $C(-1, -2, 0)$.

23. Find the scaler and vector projections of $\boldsymbol{a}$ on $\boldsymbol{b}$ in each of the following cases.

    (a) $\boldsymbol{a} = [4, 1]$, $\boldsymbol{b} = [2, 3]$

    (b) $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j}$, $\boldsymbol{b} = \boldsymbol{i} + \boldsymbol{k}$

24. If $\boldsymbol{b} = [3, 0, -1]$, find a vector $\boldsymbol{a}$ such that the scalar projection of $\boldsymbol{a}$ on $\boldsymbol{b}$ is equal to 2.

These exercises should take around 2 hours to complete.

(Answers: 2. $\overrightarrow{BC} = \boldsymbol{b} - \boldsymbol{a}$, $\overrightarrow{BD} = \frac{3}{4}\boldsymbol{b} - \frac{3}{4}\boldsymbol{a}$, $\overrightarrow{DC} = \frac{1}{4}\boldsymbol{b} - \frac{1}{4}\boldsymbol{a}$, $\overrightarrow{AD} = \frac{1}{4}\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}$; 3. $\boldsymbol{v} - \boldsymbol{u}$; 4.(a) $\overrightarrow{AC}$ (b) $\overrightarrow{CA}$ (c) $\overrightarrow{BD}$ (d) $\overrightarrow{BA}$; 6.(a) $\boldsymbol{a} = [3, 1]$ (b) $\boldsymbol{a} = [-2, 5]$; 7.(a)(i) 5 (ii) $[2, 5]$ (iii) $[-10, 1]$ (iv) $[8, -6]$ (v) $[\frac{-4}{5}, \frac{3}{5}]$, (b) (i) 7 (ii) $[5, 7, 1]$ (iii) $[7, -3, 5]$ (iv) $[-12, -4, -6]$ (v) $[\frac{6}{7}, \frac{2}{7}, \frac{3}{7}]$; 8. $[-\frac{4}{\sqrt{6}}, \frac{8}{\sqrt{6}}, \frac{4}{\sqrt{6}}]$; 9. $\boldsymbol{0}$; 12.(a)(i) $\boldsymbol{a} + \boldsymbol{b} = 5\boldsymbol{i} - \boldsymbol{j}$ (ii) $4\boldsymbol{a} + 3\boldsymbol{b} = 19\boldsymbol{i} - 2\boldsymbol{j}$ (iii) $|a| = \sqrt{17}$ (iv) $|\boldsymbol{b} - \boldsymbol{a}| = 3\sqrt{2}$, (b)(i) $\boldsymbol{a} + \boldsymbol{b} = 2\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ (ii) $4\boldsymbol{a} + 3\boldsymbol{b} = 8\boldsymbol{i} - 10\boldsymbol{j} + 13\boldsymbol{k}$ (iii) $|a| = 6$ (iv) $|\boldsymbol{b} - \boldsymbol{a}| = \sqrt{65}$; 13. -10; 14. $2\sqrt{6}$; 15.(a) $\approx 44°$ (b) $\approx 48°$; 16.(a) parallel (b) neither (c) orthogonal; 18. ≈ 1131371 joules; 19. $[\cos\alpha, \cos\beta, \cos\gamma] = \left[\frac{6}{7}, \frac{3}{7}, -\frac{2}{7}\right]$, $\alpha \approx 31°$, $\beta \approx 65°$, $\gamma \approx 107°$; 20. Only (b), (c) and (d) make sense; 21.(a) 7 (b) 90 (c) -10; 22. $114°$, $33°$ and $33°$; 23.(a) $\frac{11}{\sqrt{13}}$, $[\frac{22}{13}, \frac{33}{13}]$ (b) $\frac{1}{\sqrt{2}}$, $\frac{1}{2}\boldsymbol{i} + \frac{1}{2}\boldsymbol{k}$; 24. $[0, 0, -2\sqrt{10}]$ is just one possibility)

Vectors

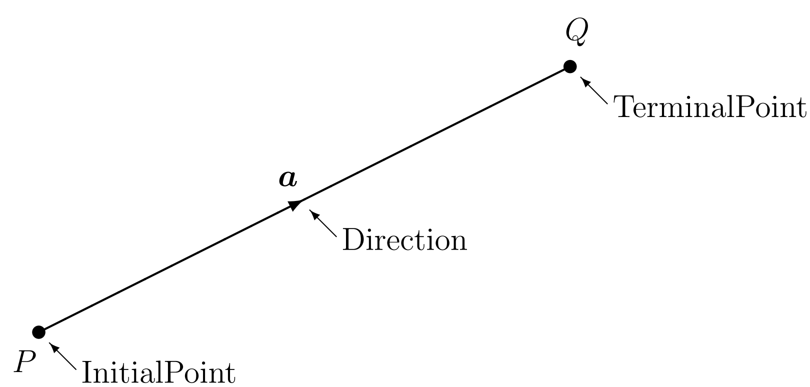
When talking about quantities in engineering, we generally distinguish between scalars and vectors.

A scalar quantity is characterized only by magnitude. Examples are mass, temperature, time, speed *etc.*

A vector quantity is characterized by both magnitude and direction. Examples are force, velocity *etc.*

Within a written mathematical statement, we use the following notation to distinguish between scalars and vectors. For scalar quantities, we use a plain (usually lower case) letter such as 'a'. For vector quantities, in typewritten text we use a boldface (and again usually lower case) letter such as '$\boldsymbol{a}$'. Since this is difficult to do in handwritten text, there we usually underline the letter with a tilde, *i.e.* $\underset{\sim}{a}$.

Geometrically, we represent vectors by a directed line segment as follows:

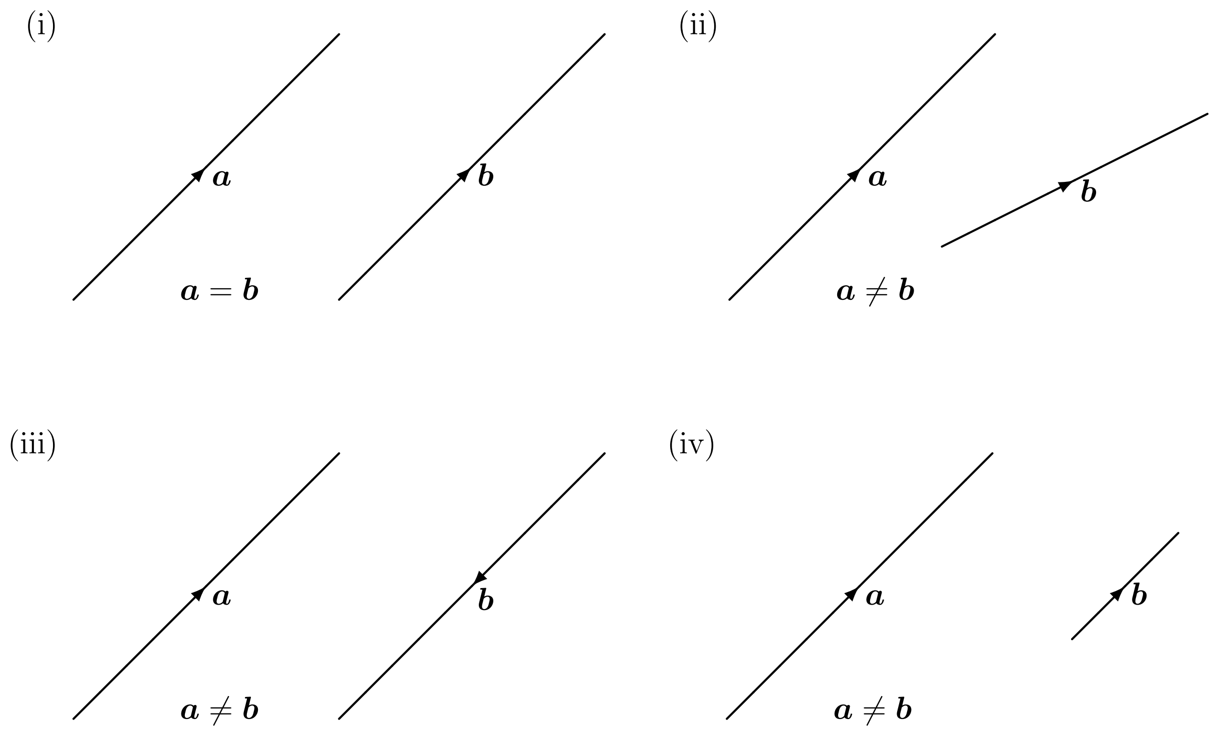


If the vector goes from a point P to a point Q as shown for $\boldsymbol{a}$ above, then we often use the notation $\overrightarrow{PQ}$ to denote the vector as well.

The magnitude or length of a vector $\boldsymbol{a}$ is denoted by $||\boldsymbol{a}||$ or $|\boldsymbol{a}|$. It is often also called the norm of $\boldsymbol{a}$.

We would like to do arithmetic with vectors in much the same way as we do with scalars. The first thing to consider is what we mean by the equality of two vectors.

We say that two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are equal if they have the same length and direction. For examples, in the cases below, $\boldsymbol{a} = \boldsymbol{b}$ only in case (i).



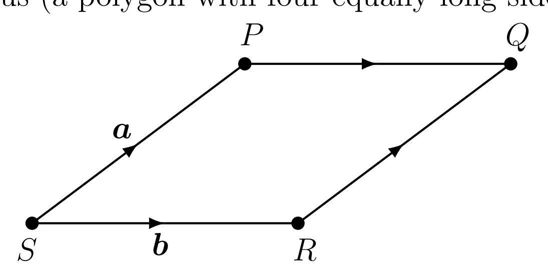
Notice that vectors are not tied to any particular location within the plane. As long as they share the same length and direction, they are considered to be equal, regardless of where in the plane we draw them.

Before talking about addition and subtraction of vectors, we need to define the vector equivalent of the scalar 0. The zero vector, denoted by $\mathbf{0}$, is simply defined to be a vector which has no particular direction and zero length.

Going back to the notion of length of a vector, note that

- $||\boldsymbol{u}|| \geq 0$ for any vector $\boldsymbol{u}$ and the only way in which $||\boldsymbol{u}|| = 0$ is if $\boldsymbol{u} = \mathbf{0}$.
- If $||\boldsymbol{u}|| = 1$ for a given vector $\boldsymbol{u}$, then we say that $\boldsymbol{u}$ is a unit vector.

<u>Ex:</u> Consider a rhombus (a polygon with four equally long sides) $PQRS$:



Clearly $\overrightarrow{PQ} = \overrightarrow{SR}$, $\overrightarrow{SP} = \overrightarrow{RQ}$, $||\boldsymbol{a}|| = ||\boldsymbol{b}||$, but $\boldsymbol{a} \neq \boldsymbol{b}$.

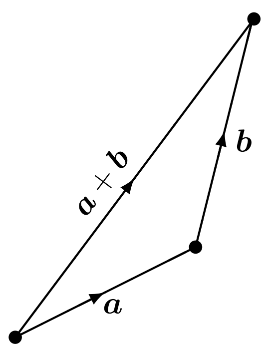
Vector Operations

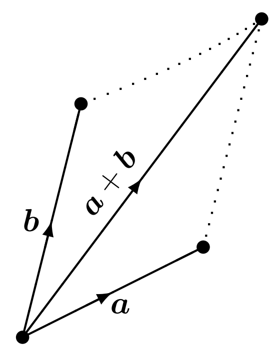
Ideally, we like to apply all of the usual arithmetic we use for scalars to vectors. For addition and subtraction, this is easily done. Multiplication is a little more difficult, while no equivalent of scalar division exists for vectors.

The sum of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is written as

$$\boldsymbol{a}+\boldsymbol{b}$$

and we can use either the triangle law or parallelogram law to work out the sum.





In the triangle law, to add $\boldsymbol{a}$ and $\boldsymbol{b}$, we simply place the vectors so that the initial point of $\boldsymbol{b}$ coincides with the terminal point of $\boldsymbol{a}$. $\boldsymbol{a}+\boldsymbol{b}$ is then simply the vector whose initial point is the initial point of $\boldsymbol{a}$ and whose terminal point is the terminal point of $\boldsymbol{b}$. In the parallelogram law, both vectors are drawn so that their initial points coincide and $\boldsymbol{a}+\boldsymbol{b}$ is then the diagonal of the parallelogram form by $\boldsymbol{a}$ and $\boldsymbol{b}$.

Note the special rule for addition with the zero vector. For any vector $\boldsymbol{u}$,

$$\boldsymbol{u}+\mathbf{0}=\mathbf{0}+\boldsymbol{u}=\boldsymbol{u}.$$

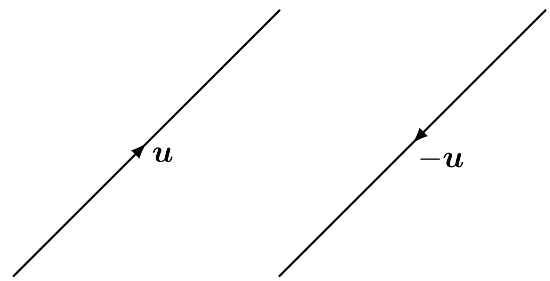
Also, it is clear from the workings of the parallelogram law that

$$\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}$$

for any pair of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

For a real (scalar) number, we know exactly what we mean by its negative. For vectors, we have the following idea of negatives.

The negative of a vector $\boldsymbol{u}$ is denoted by $-\boldsymbol{u}$ and it is simply the vector which has the same length as $\boldsymbol{u}$ but is opposite in direction, *i.e.*

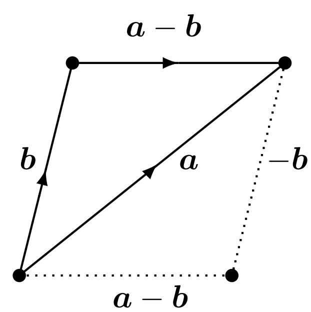


Clearly, $\boldsymbol{u} + (-\boldsymbol{u}) = \boldsymbol{0}$.

The notion of the negative of a vector now lets us find a way of subtracting vectors. Basically, the difference of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by

$$\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b}).$$

Note that we can determine $\boldsymbol{a} - \boldsymbol{b}$ easily from $\boldsymbol{a}$ and $\boldsymbol{b}$ by using a variation of the triangle law. Since $\boldsymbol{b} + (\boldsymbol{a} - \boldsymbol{b}) = \boldsymbol{a}$, we have



i.e. place $\boldsymbol{a}$ and $\boldsymbol{b}$ so that their initial points coincide. Then $\boldsymbol{a} - \boldsymbol{b}$ is the vector whose starting point coincides with the terminal point of $\boldsymbol{b}$ and its terminal point coincides with the terminal point of $\boldsymbol{a}$.

Now we come to the question of multiplication. Because vectors are more complicated objects than scalars, there are various ways in which we can multiply them. We'll meet actual multiplication of vectors in our next lecture. For now, we restrict ourselves to the idea of scalar multiplication.

Scalar Multiplication. Given a scalar c and a vector $\boldsymbol{a}$, the scalar multiple $c\boldsymbol{a}$ is a vector such that

(i) it has the same direction as $\boldsymbol{a}$ if $c > 0$;

(ii) its direction is opposite to that of $\boldsymbol{a}$ if $c < 0$.

In either case, the length of $c\boldsymbol{a}$ is $|c|$ times the length of $\boldsymbol{a}$, *i.e.* $||c\boldsymbol{a}|| = |c|\,||\boldsymbol{a}||$. *e.g.*

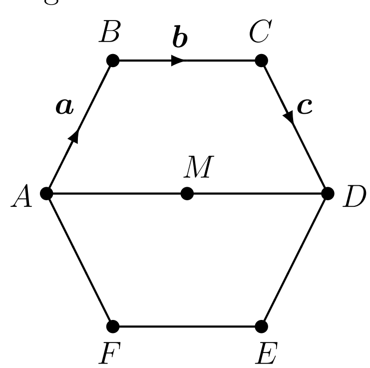
Note that multiplication by the scalar 0 always yields the zero vector, *i.e.* $(0)\boldsymbol{u} = \boldsymbol{0}$ for any vector $\boldsymbol{u}$.

Quite often in practice, it is useful to deal with vectors whose length is exactly equal to 1, known as unit vectors. These can be created easily as follows. For any vector $\boldsymbol{a} \neq \boldsymbol{0}$, the unit vector in the direction of $\boldsymbol{a}$ is denoted by $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{a}} = \dfrac{\boldsymbol{a}}{||\boldsymbol{a}||}$, *i.e.* it is simply $\boldsymbol{a}$ multiplied by the reciprocal of its own length.

Finally, notice that the only way in which two vectors can be parallel is if one can be expressed as a scalar multiple of the other, *i.e.* $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if $\boldsymbol{a} = c\boldsymbol{b}$ for some scalar c. Satisfy yourself that this requires the two vectors to have either the same or opposite directions.

Ex: Let $ABCDEF$ be a regular hexagon, M the midpoint of AD, $\overrightarrow{AB} = \boldsymbol{a}$, $\overrightarrow{BC} = \boldsymbol{b}$ and $\overrightarrow{CD} = \boldsymbol{c}$. Find $\overrightarrow{AC}$, $\overrightarrow{AD}$, $\overrightarrow{AM}$, $\overrightarrow{BE}$ and $\overrightarrow{FC}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

Soln: We begin by drawing a diagram.



Clearly, we have

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \boldsymbol{a} + \boldsymbol{b}$,

$\overrightarrow{AD} = \overrightarrow{AC} + \overrightarrow{CD} = \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$,

$\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AD} = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$,

$\overrightarrow{BE} = \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \boldsymbol{b} + \boldsymbol{c} + (-\boldsymbol{a}) = \boldsymbol{b} + \boldsymbol{c} - \boldsymbol{a}$,

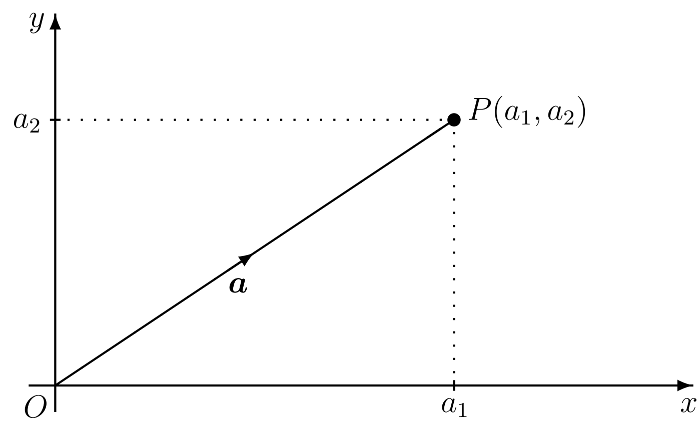
$\overrightarrow{FC} = \overrightarrow{FA} + \overrightarrow{AB} + \overrightarrow{BC} = (-\boldsymbol{c}) + \boldsymbol{a} + \boldsymbol{b} = \boldsymbol{a} + \boldsymbol{b} - \boldsymbol{c}$.

Analytic Representation of Vectors in 2 and 3 Space

Vectors in 2 space (*i.e.* the Cartesian plane) and 3 space can be readily described in terms of components. Consider the Cartesian plane with origin O. For any point $P(a_1, a_2)$ in the plane, we can write down its position vector as

$$\overrightarrow{OP} = [a_1, a_2] = \boldsymbol{a}.$$

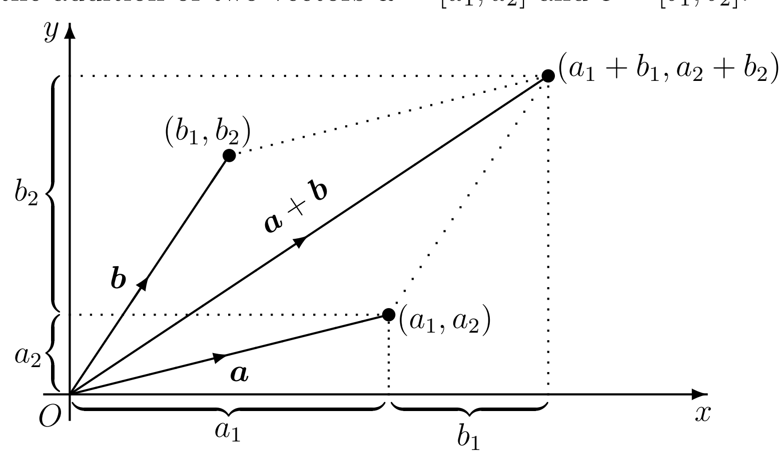
Note how we use square brackets when writing out the vector (as opposed to round brackets when we describe the coordinates of the point P).



Now let us revisit some of the earlier concepts in terms of this analytic vector notation. Suppose we have two vectors $\boldsymbol{a} = [a_1, a_2]$ and $\boldsymbol{b} = [b_1, b_2]$. Suppose that $\boldsymbol{a} = \boldsymbol{b}$. Locate both vectors in the plane such that the initial points coincide at the origin O. Then, clearly, their terminal points and their respective coordinates must also coincide which leads to $a_1 = b_1$ and $a_2 = b_2$. In other words, two vector being equal is equivalent to each of their components being equal.

Similarly, in 3 space, we can write a vector in component form as $\boldsymbol{a} = [a_1, a_2, a_3]$ and equality between vectors again means that their components must be equal.

Next we look at the addition of two vectors $\boldsymbol{a} = [a_1, a_2]$ and $\boldsymbol{b} = [b_1, b_2]$.



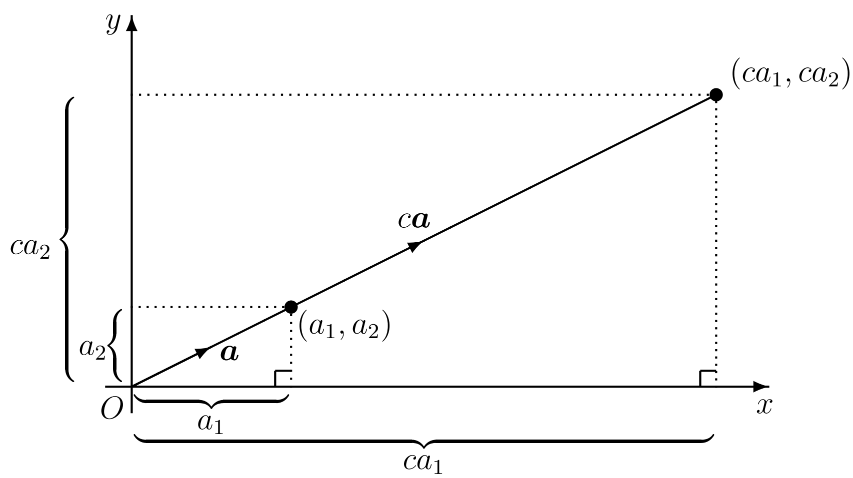
Clearly, from the diagram,

$$\boldsymbol{a} + \boldsymbol{b} = [a_1 + b_1, a_2 + b_2].$$

Similarly, for two vectors $\boldsymbol{a} = [a_1, a_2, a_3]$ and $\boldsymbol{b} = [b_1, b_2, b_3]$ in 3 space,

$$\boldsymbol{a} + \boldsymbol{b} = [a_1 + b_1, a_2 + b_2, a_3 + b_3].$$

The operation of scalar multiplication is also easily carried out in component form. Suppose we have $\boldsymbol{a} = [a_1, a_2]$ as shown below. What does $c\boldsymbol{a}$ look like?



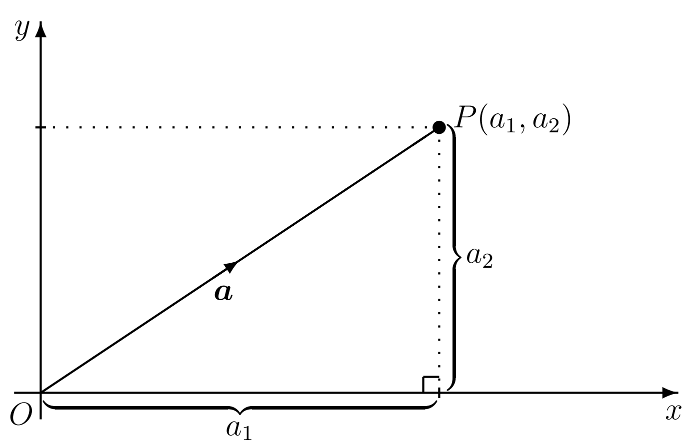
Using the fact that the ratios of the respective sides of the similar triangles in the diagram must be equal, we find that

$$c\boldsymbol{a} = [ca_1, ca_2].$$

Similarly, for $\boldsymbol{a} = [a_1, a_2, a_3]$ in 3 space,

$$c\boldsymbol{a} = [ca_1, ca_2, ca_3].$$

Finally, we look at how to determine the length of a vector in component form. Given $\boldsymbol{a} = [a_1, a_2]$, consider the diagram below.



Applying Pythagoras' Theorem to the triangle in the diagram, we have

$$||\boldsymbol{a}|| = ||\overrightarrow{OP}|| = \sqrt{a_1^2 + a_2^2}.$$

Similarly, for $\boldsymbol{a} = [a_1, a_2, a_3]$ in 3 space,

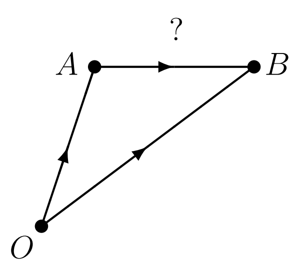
$$||\boldsymbol{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Note that these definitions satisfy the length properties we looked at before. Verify that $||c\boldsymbol{a}|| = |c|\,||\boldsymbol{a}||$ also holds in each case.

<u>Ex:</u> Find $\boldsymbol{a}+\boldsymbol{b}$, $k\boldsymbol{a}$, $\boldsymbol{a}-\boldsymbol{b}$ and $||\boldsymbol{a}||$ if $\boldsymbol{a} = [-2, 1, 0]$, $\boldsymbol{b} = [1, 3, 2]$ and $k = -3$.

<u>Soln:</u> We have
$\boldsymbol{a}+\boldsymbol{b} = [-2, 1, 0] + [1, 3, 2] = [-1, 4, 2]$,
$k\boldsymbol{a} = (-3)[-2, 1, 0] = [6, -3, 0]$,
$\boldsymbol{a}-\boldsymbol{b} = [-2, 1, 0] - [1, 3, 2] = [-3, -2, -2]$,
and $||\boldsymbol{a}|| = \sqrt{(-2)^2 + (1)^2 + (0)^2} = \sqrt{5}$.

What are the components of a vector joining two points $A(a_1, a_2)$ and $B(b_1, b_2)$ in the plane?



Letting O denote the origin, we have $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$. Re-arranging, we get $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$, *i.e.*

$$\overrightarrow{AB} = [b_1, b_2] - [a_1, a_2] = [b_1 - a_1, b_2 - a_2],$$

i.e. the components of a vector between two points are obtained by simply subtracting the coordinates of the initial point from the coordinates of the terminal point. In 3 space, the same applies: for points $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$,

$$\overrightarrow{AB} = [b_1 - a_1, b_2 - a_2, b_3 - a_3].$$

<u>Ex:</u> Given $A(-1, 2, 3)$ and $B(3, -1, 0)$, find $\overrightarrow{AB}$ and $\overrightarrow{BA}$.
<u>Soln:</u> $\overrightarrow{AB} = [3, -1, 0] - [-1, 2, 3] = [4, -3, -3]$. $\overrightarrow{BA} = [-1, 2, 3] - [3, -1, 0] = [-4, 3, 3]$.
(Note that $\overrightarrow{BA} = -\overrightarrow{AB}$, as we would expect. . .)

Standard Unit Basis Vectors

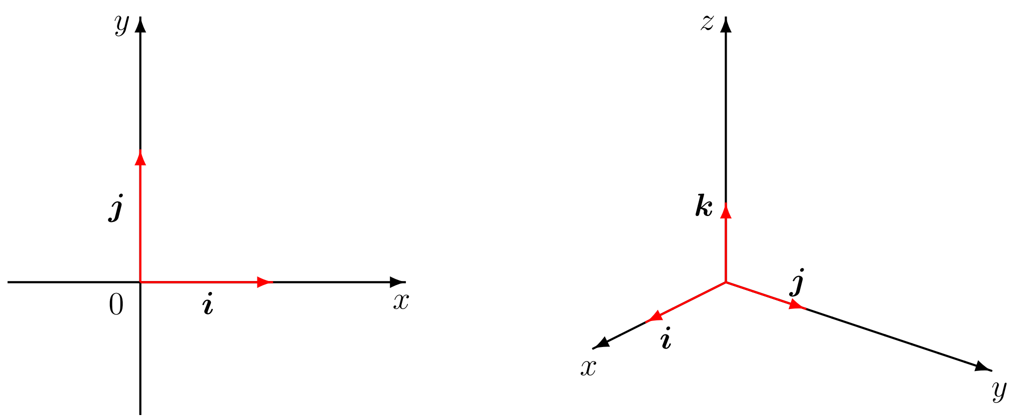
The vectors

$$\boldsymbol{i} = [1, 0] \qquad \text{and} \qquad \boldsymbol{j} = [0, 1]$$

in 2 space and

$$\boldsymbol{i} = [1, 0, 0], \qquad \boldsymbol{j} = [0, 1, 0] \qquad \text{and} \qquad \boldsymbol{k} = [0, 0, 1]$$

in 3 space are known as the standard unit basis vectors.



The reason for this terminology is as follows. Consider any $\boldsymbol{a} = [a_1, a_2, a_3]$ in 3 space. Then

$$\begin{aligned} \boldsymbol{a} &= [a_1, a_2, a_3] \\ &= [a_1, 0, 0] + [0, a_2, 0] + [0, 0, a_3] \\ &= a_1[1, 0, 0] + a_2[0, 1, 0] + a_3[0, 0, 1] \\ &= a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}, \end{aligned}$$

i.e. any vector in 3 space can be written as a corresponding combination of the standard unit basis vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$. *e.g.* $[3, 2, -4] = 3\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}$.

Similarly, for any $\boldsymbol{a} = [a_1, a_2]$ in 2 space, we may write

$$\boldsymbol{a} = [a_1, a_2] = a_1\boldsymbol{i} + a_2\boldsymbol{j}.$$

Ex: Given $\boldsymbol{a} = 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{b} = [5, -2, -1]$, find $\boldsymbol{a} + \boldsymbol{b}$, $2\boldsymbol{a}$ and $||\boldsymbol{a}||$.

Soln: $\boldsymbol{b} = 5\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$, so $\boldsymbol{a} + \boldsymbol{b} = (3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) + (5\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}) = 8\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k} = [8, -3, 1]$. $2\boldsymbol{b} = 2(5\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}) = 10\boldsymbol{i} - 4\boldsymbol{j} - 2\boldsymbol{k} = [10, -4, -2]$. Finally,

$$||\boldsymbol{a}|| = ||3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}|| = ||[3, -1, 2]|| = \sqrt{(3)^2 + (-1)^2 + (2)^2} = \sqrt{14}.$$

Note how the component notation and the standard unit basis vector notation is interchangeable.

The Dot Product

There are two commonly used ways to multiply two vectors. The first of these is the *dot product*. It is also often known as the *scalar product*, since the result of the multiplication is a scalar quantity.

The dot product (or scalar product) of $\boldsymbol{a} = [a_1, a_2, a_3]$ and $\boldsymbol{b} = [b_1, b_2, b_3]$ is

$$\boxed{\boldsymbol{a}.\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3}$$

For example, $[1, 2, 3].[-2, 0, 1] = (1)(-2) + (2)(0) + (3)(1) = 1$.

Note the following:

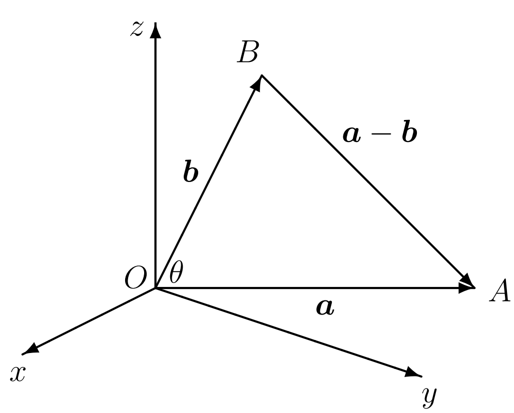
(i) $\boldsymbol{a}.\boldsymbol{b}$ is clearly a scalar quantity and $\boldsymbol{a}.\boldsymbol{b} = \boldsymbol{b}.\boldsymbol{a}$.

(ii) $\boldsymbol{a}.\boldsymbol{a} = a_1^2 + a_2^2 + a_3^2 = ||\boldsymbol{a}||^2$, *i.e.* $||\boldsymbol{a}|| = \sqrt{\boldsymbol{a}.\boldsymbol{a}}$.

(iii) For vectors in 2 space, if $\boldsymbol{a} = [a_1, a_2]$, $\boldsymbol{b} = [b_1, b_2]$, then $\boldsymbol{a}.\boldsymbol{b} = a_1b_1 + a_2b_2$.

There is another way to compute the dot product between vectors which gives a better insight into its geometrical meaning.

Consider two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ with an angle θ between them, where $0° \leq \theta \leq 180°$. Suppose we locate both vectors with their initial points at the origin, as shown below.



Let A be the terminal point of $\boldsymbol{a}$ and B the terminal point of $\boldsymbol{b}$. Then we can apply the cosine rule to $\triangle OAB$ to obtain

$$||\overrightarrow{BA}||^2 = ||\overrightarrow{OA}||^2 + ||\overrightarrow{OB}||^2 - 2||\overrightarrow{OA}||\,||\overrightarrow{OB}||\cos\theta$$

But $||\overrightarrow{OA}|| = ||\boldsymbol{a}||$, $||\overrightarrow{OB}|| = ||\boldsymbol{b}||$ and $||\overrightarrow{BA}|| = ||\boldsymbol{a}-\boldsymbol{b}||$. Substituting into the previous equation, we have

$$||\boldsymbol{a}-\boldsymbol{b}||^2 = ||\boldsymbol{a}||^2 + ||\boldsymbol{b}||^2 - 2||\boldsymbol{a}||\,||\boldsymbol{b}||\cos\theta$$

Note that $||\boldsymbol{a}-\boldsymbol{b}||^2 = (\boldsymbol{a}-\boldsymbol{b}).(\boldsymbol{a}-\boldsymbol{b}) = \boldsymbol{a}.\boldsymbol{a} - 2\boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{b} = ||\boldsymbol{a}||^2 - 2\boldsymbol{a}.\boldsymbol{b} + ||\boldsymbol{b}||^2$. Substituting into the previous equation, we get

$$||\boldsymbol{a}||^2 - 2\boldsymbol{a}.\boldsymbol{b} + ||\boldsymbol{b}||^2 = ||\boldsymbol{a}||^2 + ||\boldsymbol{b}||^2 - 2||\boldsymbol{a}||\,||\boldsymbol{b}||\cos\theta$$

i.e. $-2\boldsymbol{a}.\boldsymbol{b} = -2||\boldsymbol{a}||\,||\boldsymbol{b}||\cos\theta$, *i.e.*

$$\boxed{\boldsymbol{a}.\boldsymbol{b} = ||\boldsymbol{a}||\,||\boldsymbol{b}||\cos\theta}$$

i.e. the dot product between two vectors (be it in 2 space or in 3 space) is equal to the product of the lengths times the cosine of the angle between them.

It follows that if $\boldsymbol{a} \neq \boldsymbol{0}$ and $\boldsymbol{b} \neq \boldsymbol{0}$, then

$$\boxed{\cos\theta = \frac{\boldsymbol{a}.\boldsymbol{b}}{||\boldsymbol{a}||\,||\boldsymbol{b}||}}$$

<u>Ex:</u> Find the cosine of the angle BAC for $A(1,0,2)$, $B(-1,0,1)$ and $C(1,-1,1)$.

<u>Soln:</u> $\overrightarrow{AB} = [-2,0,-1]$, $\overrightarrow{AC} = [0,-1,-1]$.

$$\cos\theta = \frac{\overrightarrow{AB}.\overrightarrow{AC}}{||\overrightarrow{AB}||\,||\overrightarrow{AC}||} = \frac{[-2,0,-1].[0,-1,-1]}{\sqrt{(-2)^2+(0)^2+(-1)^2}\sqrt{(0)^2+(-1)^2+(-1)^2}} = \frac{1}{\sqrt{10}}$$

We say that $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal (or perpendicular) if the angle between them is 90°. In this case,

$$\frac{\boldsymbol{a}.\boldsymbol{b}}{||\boldsymbol{a}||\,||\boldsymbol{b}||} = \cos(90^\circ) = 0 \;\longrightarrow\; \boldsymbol{a}.\boldsymbol{b} = 0.$$

In fact, the only way in which to vectors can be orthogonal is if $\boldsymbol{a}.\boldsymbol{b} = 0$.

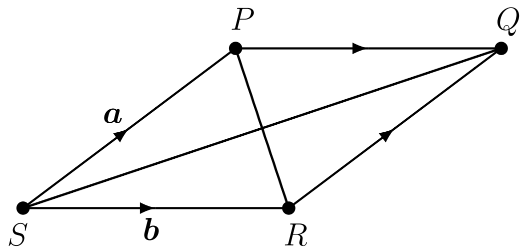
For example [1,-2,3].[2,1,0]=0, so vectors are orthogonal.

Note that we consider the zero vector to be perpendicular to any other vector, since $\boldsymbol{0}.\boldsymbol{u} = [0,0,0].[u_1,u_2,u_3] = 0$ for any $\boldsymbol{u}$. Furthermore, brackets involving dot products multiply out as for normal multiplication. For example, given $\boldsymbol{a} = [a_1,a_2]$, $\boldsymbol{b} = [b_1,b_2]$ and $\boldsymbol{c} = [c_1,c_2]$, we have

$$\boldsymbol{a}.(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}.([b_1,b_2]+[c_1,c_2]) = [a_1,a_2].[(b_1+c_1),(b_2+c_2)] = a_1b_1+a_1c_1+a_2b_2+a_2c_2 = \boldsymbol{a}.\boldsymbol{b}+\boldsymbol{a}.\boldsymbol{c}$$

<u>Ex:</u> Show that the diagonals of a rhombus are perpendicular.

<u>Soln:</u> Denote the rhombus as $PQRS$, *i.e.*



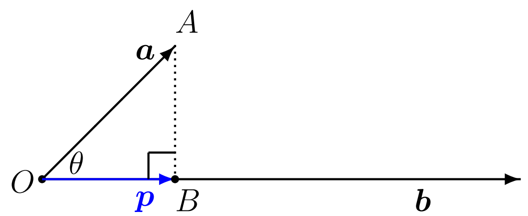
Let $\overrightarrow{SP} = \overrightarrow{RQ} = \boldsymbol{a}$ and $\overrightarrow{SR} = \overrightarrow{PQ} = \boldsymbol{b}$. Then $\overrightarrow{PR} = \overrightarrow{PS} + \overrightarrow{SR} = -\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} - \boldsymbol{a}$ and $\overrightarrow{SQ} = \overrightarrow{SP} + \overrightarrow{PQ} = \boldsymbol{a} + \boldsymbol{b}$. Hence,

$$\overrightarrow{PR}.\overrightarrow{SQ} = (\boldsymbol{a} - \boldsymbol{b}).(\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{a}.\boldsymbol{a} + \boldsymbol{a}.\boldsymbol{b} - \boldsymbol{b}.\boldsymbol{a} - \boldsymbol{b}.\boldsymbol{b} = ||\boldsymbol{a}||^2 - ||\boldsymbol{b}||^2 = 0,$$

i.e. the diagonals must be orthogonal.

<u>Projection and Component of a Vector</u>

Consider two vector $\boldsymbol{a}$ and $\boldsymbol{b}$ drawn so that their initial points coincide at O. Let A be the terminal point of $\boldsymbol{a}$. We are interested in the *projection* of $\boldsymbol{a}$ onto $\boldsymbol{b}$. This can be thought of as the shadow of $\boldsymbol{a}$ on $\boldsymbol{b}$ as indicated by the blue vector in the diagram. Let B be the endpoint of this shadow, as indicated.



When talking about projections, we distinguish between *scalar* and *vector* projections, as described below.

The scalar projection of $\boldsymbol{a}$ on $\boldsymbol{b}$ is $p = ||\overrightarrow{OB}||$, *i.e.* it is simply the length of the shadow vector. If θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, note that

$$p = ||\overrightarrow{OB}|| = ||\overrightarrow{OA}|| \cos\theta = ||\boldsymbol{a}|| \frac{\boldsymbol{a}.\boldsymbol{b}}{||\boldsymbol{a}||\,||\boldsymbol{b}||} = \boldsymbol{a}.\left(\frac{\boldsymbol{b}}{||\boldsymbol{b}||}\right) = \boldsymbol{a}.\hat{\boldsymbol{b}},$$

where $\hat{\boldsymbol{b}} = \dfrac{\boldsymbol{b}}{||\boldsymbol{b}||}$ is the unit vector in the direction of $\boldsymbol{b}$, *i.e.*

$$\boxed{\text{scalar projection} = p = \boldsymbol{a}.\hat{\boldsymbol{b}}}$$

The vector projection of $\boldsymbol{a}$ on $\boldsymbol{b}$ is then simply the blue shadow vector itself, *i.e.*

$$\boxed{\text{vector projection} = \boldsymbol{p} = p\hat{\boldsymbol{b}},}$$

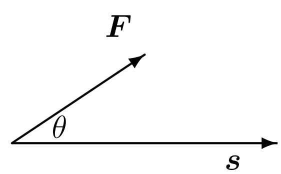
where p is the scalar projection defined above and $\hat{\boldsymbol{b}}$ is the unit vector in the direction of $\boldsymbol{b}$.

Ex: Find the vector projection of $[2, 1, -5]$ on $[3, -4, 0]$.

Soln: $||\boldsymbol{b}|| = \sqrt{(3)^2 + (-4)^2 + (0)^2} = \sqrt{25} = 5$, so $\hat{\boldsymbol{b}} = [\frac{3}{5}, -\frac{4}{5}, 0]$. Hence, scalar projection is $p = \boldsymbol{a}.\hat{\boldsymbol{b}} = [2, 1, -5].[\frac{3}{5}, -\frac{4}{5}, 0] = \frac{6}{5} - \frac{4}{5} = \frac{2}{5}$. The vector projection is therefore $\boldsymbol{p} = p\hat{\boldsymbol{b}} = \frac{2}{5}[\frac{3}{5}, -\frac{4}{5}, 0] = [\frac{6}{25}, -\frac{8}{25}, 0]$.

Work Done by a Force

Suppose a force $\boldsymbol{F}$ is applied to an object, resulting in a displacement $\boldsymbol{s}$.



$$\begin{aligned}
\text{Work} &= \text{force} \times \text{displacement} \\
&= \text{magnitude of } \boldsymbol{F} \text{ in direction of } \boldsymbol{s} \times ||\boldsymbol{s}|| \\
&= \text{scalar projection of } \boldsymbol{F} \text{ on } \boldsymbol{s} \times ||\boldsymbol{s}|| \\
&= ||\boldsymbol{F}|| \cos\theta ||\boldsymbol{s}|| \\
&= \boldsymbol{F}.\boldsymbol{s}
\end{aligned}$$

$$\boxed{\text{Work} = \boldsymbol{F}.\boldsymbol{s}}$$

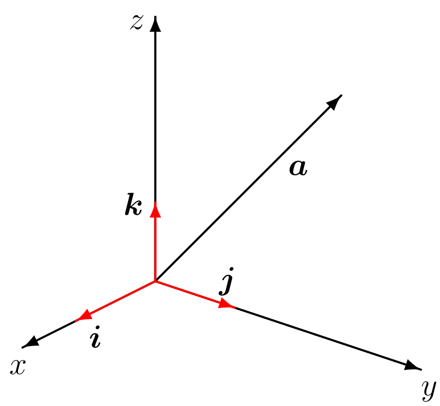
Ex: Find the work done by a force $\boldsymbol{F} = [2, -3, -1]$ in moving an object from $A(2, -1, 3)$ to $B(5, 3, -6)$.

Soln: Work $= \boldsymbol{F}.\boldsymbol{s} = \boldsymbol{F}.\overrightarrow{AB} = [2, -3, -1].[3, 4, -9] = 6 - 12 + 9 = 3$. If position was indicated in metres and the components of $\boldsymbol{F}$ measure in Newtons, the units for Work would, of course, be joules.

Direction Cosines

The direction cosines of a vector in 3 space are simply the cosines of the angles that the vector makes with respect to each of the standard unit basis vectors. They can be easily found from the vector itself and give us a simple way to visualize the orientation of the vector.

Consider a vector $\boldsymbol{a} = [a_1, a_2, a_3]$ in 3 space and let α, β and γ be the angles which $\boldsymbol{a}$ makes with $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, respectively. Then $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are the direction cosines of $\boldsymbol{a}$.



We have $\cos\alpha = \dfrac{\boldsymbol{a}.\boldsymbol{i}}{||\boldsymbol{a}||\,||\boldsymbol{i}||} = \dfrac{a_1}{||\boldsymbol{a}||}$, $\cos\beta = \dfrac{\boldsymbol{a}.\boldsymbol{j}}{||\boldsymbol{a}||\,||\boldsymbol{j}||} = \dfrac{a_2}{||\boldsymbol{a}||}$, $\cos\gamma = \dfrac{\boldsymbol{a}.\boldsymbol{k}}{||\boldsymbol{a}||\,||\boldsymbol{k}||} = \dfrac{a_3}{||\boldsymbol{a}||}$, so

$$[\cos\alpha, \cos\beta, \cos\gamma] = \frac{\boldsymbol{a}}{||\boldsymbol{a}||} = \hat{\boldsymbol{a}},$$

i.e. the direction cosines are simply given by the components of the unit vector in the direction of $\boldsymbol{a}$!

Ex: Find the direction cosines of $\boldsymbol{a} = [4, -5, 3]$.

Soln: $||\boldsymbol{a}|| = \sqrt{(4)^2 + (-5)^2 + (3)^2} = \sqrt{50}$. Thus

$$[\cos\alpha, \cos\beta, \cos\gamma] = \frac{\boldsymbol{a}}{||\boldsymbol{a}||} = \left[\frac{4}{\sqrt{50}}, -\frac{5}{\sqrt{50}}, \frac{3}{\sqrt{50}}\right].$$