IPDA1005 Introduction to Probability and Data Analysis

Solution to Worksheet 4

1. During a power blackout, one hundred persons are arrested on suspicion of looting. Each is given a polygraph (lie-detector) test. From past experience it is known that the polygraph is 90% reliable when administered to a guilty suspect and 98% reliable when given to someone who is innocent. Suppose that of the one hundred persons taken into custody, only twelve were actually involved in any wrongdoing. What is the probability that a given suspect is innocent given that the polygraph says he is guilty?

   (**Hint**: Let B be the event "Polygraph says suspect is guilty", and let A_1 and $A_2(= A_1^c)$ be the events "Suspect is guilty" and "Suspect is not guilty", respectively.)

   Solution: First, it should be clear that we're being asked to calculate $P(A_1^c|B)$. Let's now see what information we're given: $P(B|A_1) = 0.90$; $P(B^c|A_1^c)$, the event that the polygraph says the suspect is not guilty when he's innocent, is 0.98; and $P(A_1) = 12/100$ and $P(A_1^c) = 88/100$. Using Bayes' theorem, we can write

   $$P(A_1^c|B) = \frac{P(B|A_1^c)P(A_1^c)}{P(B|A_1^c)P(A_1^c) + P(B|A_1)P(A_1)}$$

   We have all the quantities we require except $P(B|A_1^c)$ but it should be clear that $P(B|A_1^c) = 1 - P(B^c|A_1^c) = 1 - 0.98 = 0.02$. Substituting all of the require quantities into the expression above yields

   $$\begin{aligned} P(A_1^c|B) &= \frac{(0.02)(88/100)}{(0.02)(88/100) + (0.90)(12/100)} \\ &= 0.14 \end{aligned}$$

2. According to the manufacturer's specifications, your home burglar alarm has a 95% chance of going off if someone breaks into your house. During the two years you have lived there, the alarm has gone off on five different nights, each time for no apparent reason. Suppose the alarm goes off tomorrow night.

   (a) What is the probability that someone is trying to break into your house? (Note: Police statistics show that the chances of any particular house in your neighborhood being burglarized on any given night are two in ten thousand.)

   (b) If you now buy a much more expensive burglar alarm that has a 99.9% chance of going off if someone breaks into your house, what do you think will happen to the probability you were asked to calculate in part (a). Will it change a lot? Why or why not?

Solution: Let A be the event that "the alarm goes off"", and B and B^c be the events that "the house is being burgled" and that "it is not being burgled", respectively.

(a) The first line of part (a) is a continuation of the previous sentence, so what we're after here is $P(B|A)$, the probability that someone is trying to break into your house given that the alarm goes off. The information we're given is that $P(A|B) = 0.95$; $P(A|B^c) = 5/(2 \times 365)$; $P(B) = 2/10000$; and hence $P(B^c) = 9998/10000$. Intuitively, we might expect $P(B|A)$ to be close to one because the burglar alarm performs pretty well—$P(A|B)$ is close to one and $P(A|B^c)$ is close to zero. Nevertheless, using Bayes' theorem, we see that $P(B|A)$ is actually pretty small:

$$\begin{aligned} P(B|A) &= \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)} \\ &= \frac{(0.95)(2/10000)}{(0.95)(2/10000) + (5/730)(9998/10000)} \\ &= 0.027 \end{aligned}$$

(b) You should be able to see from the expression above that because $P(B^c)$ is so large (it's a very safe neighbourhood!), it dominates the expression in the denominator and effectively "washes out" the numerator. Hence, even if $P(A|B)$ increases from 0.95 to 0.999, the value of $P(B|A)$ will change very little.

3. The screens used for a certain type of cell phone are manufactured by 3 companies, A, B, and C. The proportions of screens supplied by A, B, and C are 0.5, 0.3, and 0.2, respectively, and their screens are defective with probabilities 0.01, 0.02, and 0.03, respectively. Given that the screen on such a phone is defective, what is the probability that Company A manufactured it?

Solution: This problem is very similar to the one we did during Lecture 3. Let D be the event that a screen is defective, and let A, B, C be, respectively, the event that a screen is supplied by each of the corresponding manufacturers. Then, using Bayes' theorem,

$$\begin{aligned} P(A|D) &= \frac{P(D|A)P(A)}{P(D)} \\ &= \frac{P(D|A)P(A)}{P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C)} \\ &= \frac{(0.01)(0.5)}{(0.01)(0.5) + (0.02)(0.3) + (0.03)(0.2)} \\ &= 0.294 \end{aligned}$$

4. (**Defense attorney's fallacy**) A woman has been murdered, and her husband is put on trial for this crime. Evidence comes to light that the defendant had a history of abusing his wife. The defense attorney argues that the evidence of abuse should be excluded on grounds of irrelevance, since only 1 in 10,000 men who abuse their wives subsequently murder them. Should the judge grant the defense attorney's motion to bar this evidence from trial?

Suppose that the defense attorney's 1-in-10,000 figure is correct, and further assume the following facts: 1 in 10 men commit abuse against their wives, 1 in 5 married women who are murdered are murdered by their husbands, and 50% of husbands who murder their wives previously abused them.

Let A be the event that the husband commits abuse against his wife, and let G be the event that the husband is guilty. The defense's argument is that $P(G|A) = 1/10,000$, so guilt is still extremely unlikely conditional on a previous history of abuse. However, the defense attorney fails to condition on a crucial fact: in this case, we know that the wife was murdered. Therefore, the relevant probability is not $P(G|A)$, but $P(G|A, M)$, where M is the event that the wife was murdered.

(a) Ignoring the event M for the moment, use Bayes' theorem to write an expression for $P(G|A)$.

(b) Modify the expression in part (a) to add extra conditioning on the event M.

(c) Using the expression in part (b), calculate $P(G|A, M)$ and compare it with $P(G|M)$. What is the effect of considering the defendant's history of abuse?

Solution:

(a) Using Bayes' theorem, we can write $P(G|A)$, the probability that the husband is guilty given that abuse has occurred, as

$$\begin{aligned} P(G|A) &= \frac{P(A|G)P(G)}{P(A)} \\ &= \frac{P(A|G)P(G)}{P(A \cap G) + P(A \cap G^c)} \\ &= \frac{P(A|G)P(G)}{P(A|G)P(G) + P(A|G^c)P(G^c)} \end{aligned}$$

(b) Bayes' theorem with additional conditioning allows us to incorporate all the evidence: the fact that the husband has committed abuse and the fact that his wife has been murdered. It can be written as

$$P(G|A, M) = \frac{P(A|G, M)P(G|M)}{P(A|G, M)P(G|M) + P(A|G^c, M)P(G^c|M)}$$

(c) Finally, substituting the required quantities in the above expression yields

$$\begin{aligned} P(G|A, M) &= \frac{(0.5)(0.2)}{(0.5)(0.2) + (0.1)(0.8)} \\ &= \frac{5}{9} \end{aligned}$$

Clearly, including the evidence of abuse has increased the probability of guilt from 20% to over 50%, so the defendant's history of abuse provides important additional information about the chance of his guilt. Note that because *all* the evidence is used, $P(G|A) = 10^{-4}$ is irrelevant to the argument.

5. An oil exploration company currently has two active projects, one in Asia and the other in Europe. Let A be the event that the Asian project is successful and B be the event that the European project is successful. Suppose that A and B are independent events with $P(A) = 0.4$ and $P(B) = 0.7$.

(a) If the Asian project is not successful, what is the probability that the European project is also not successful? Explain your reasoning.

(b) What is the probability that at least one of the two projects will be successful?

(c) Given that at least one of the two projects is successful, what is the probability that only the Asian project is successful?

Solution: Recall that two events A and B are independent if $P(A|B) = P(A)$, in other words, knowing that B has occurred doesn't give us any additional information about A. Recall further from the definition of conditional independence that $P(A \cap B) = P(A|B)P(B)$ and hence independence implies that $P(A \cap B) = P(A)P(B)$.

(a) Because A and B are independent, the fact that A has not occurred is independent of B not occurring as well, and hence $P(B^c) = 1 - 0.7 = 0.3$.

(b) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)P(B)$
$= 0.4 + 0.7 - (0.4)(0.7) = 0.82$

(c) The probability we're after here is $P(A \cap B^c | A \cup B)$, and it's straightforwad to show using a Venn diagram or algebraically that

$$\begin{aligned} P(A \cap B^c | A \cup B) &= \frac{P(A \cap B^c)}{P(A \cup B)} \\ &= \frac{P(A) - P(A \cap B)}{P(A \cup B)} = \frac{0.4 - 0.28}{0.82} \\ &= 0.146 \end{aligned}$$

6. If A and B are independent events, show that A^c and B are also independent. (**Hint**: First establish a relationship among $P(A^c \cap B)$, $P(B)$, and $P(A \cap B)$.)

Solution: $P(B) = P(A^c \cap B) + P(A \cap B)$, so we can write

$$\begin{aligned} P(A^c \cap B) &= P(B) - P(A \cap B) \\ &= P(B) - P(A)P(B) \text{ (because of independence of } A \text{ and } B) \\ &= P(B)(1 - P(A)) \\ &= P(B)P(A^c) \end{aligned}$$

which implies independence.

7. If A and B are independent events, does it also follow that A^c and B^c are independent? (**Hint:** Come up with two different expressions for $P(A^c \cup B^c)$, equate them and then simplify.)

Solution: We need to show that if $P(A \cap B) = P(A)P(B)$, then $P(A^c \cap B^c) = P(A^c)P(B^c)$. To begin with, we can write $P(A^c \cup B^c)$ in two ways, as

$$P(A^c \cup B^c) = P(A^c) + P(B^c) - P(A^c \cap B^c)$$

and, using DeMorgan's laws, as

$$P(A^c \cup B^c) = P((A \cap B)^c) = 1 - P(A \cap B)$$

Combining these two equations yields

$$1 - P(A \cap B) = P(A^c) + P(B^c) - P(A^c \cap B^c)$$

Since A and B are independent, $P(A \cap B) = P(A)P(B)$, so

$$\begin{aligned} P(A^c \cap B^c) &= P(A^c) + P(B^c) - (1 - P(A)P(B)) \\ &= (1 - P(A))(1 - P(B)) \\ &= P(A^c)P(B^c) \end{aligned}$$

which implies that A^c and B^c are independent.

8. Suppose you've just purchased a string of 24 LED lights to decorate your flat. The LEDs are wired in series, which means that if at least one LED fails, the whole string fails. If each LED has a 99.9% chance of working the first time that current is applied, what is the probability that the string itself will *not* work?

   Solution: If at least one LED fails, the whole string fails. Let A_i be the event that the ith LED fails, for $i = 1, 2, \ldots, 24$. Then,

$$\begin{aligned} P(\text{string fails}) &= 1 - P(\text{string works}) \\ &= 1 - P(\text{all 24 LEDs work}) \\ &= 1 - P(A_1^c \cap A_2^c \cap \ldots \cap A_{24}^c) \end{aligned}$$

   If we assume that LED failures are independent events,

$$\begin{aligned} P(\text{string fails}) &= 1 - P(A_1^c)P(A_2^c) \cdots P(A_{24}^c) \\ &= 1 - (0.999)^{24} \\ &= 0.024 \end{aligned}$$

(Sources: Questions 1, 2, 7 & 8 are from Larsen and Marx (2012), Questions 3 & 4 are from Blitzstein and Hwang (2014), and Questions 5 & 6 are from Devore and Berk (2012).)

Bibliography

1. Blitzstein, J.K. and Hwang, J. (2014) *Introduction to Probability.* CRC Press/Taylor & Francis Group: Boca Raton, FL.

2. Devore, J.L. and Berk, K.N. (2012) *Modern Mathematical Statistics with Applications.* Springer: New York.

3. Larsen, R.J. and Marx, M.L. (2014) *An Introduction to Mathematical Statistics and Its Applications*, 5th ed. Prentice Hall: Boston.