# More on Linear Systems & Inverses

**Motivation:** Homogeneous linear systems occur frequently in practical engineering problems. The Gauss Jordan procedure is used to avoid the need for back substitution and can also be used to determine the inverse of a matrix.

Solving systems of equations by inverting the coefficient matrix can be advantageous in some application problems as we will show.

**Outcomes** In today's lecture we will learn how to:

- Recognize the special nature of homogeneous systems and understand the associated terminology.
- Determine the inverse of a matrix by the Gauss Jordan method.
- Identify in which circumstances it is advantageous to solve $A\boldsymbol{x} = \boldsymbol{b}$ as $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.

## Contents

- The solution of homogeneous systems.
- The Gauss Jordan method to solve linear systems.
- Elementary matrices and their relation to e.r.o.'s.
- Using the Gauss Jordan method to determine inverses.
- The limited role of inverses in solving linear systems.

## Exercises

1. Solve the following homogeneous system.

$$\begin{aligned} Z_1 + Z_2 - 2Z_3 - Z_5 &= 0 \\ 2Z_1 + 2Z_2 - Z_3 + Z_5 &= 0 \\ -Z_1 - Z_2 + 2Z_3 - 3Z_4 + Z_5 &= 0 \\ Z_3 + Z_4 + Z_5 &= 0 \end{aligned}$$

2. For the following linear system, determine all values of $k$ for which the system has nontrivial solutions and describe these.

$$\begin{aligned} x_1 + 3x_2 + 6x_3 &= 0 \\ x_2 + 5x_3 &= 0 \\ 2x_1 + 3x_2 + kx_3 &= 0 \end{aligned}$$

3. For each of the following homogeneous systems, determine by inspection (*i.e.* you shouldn't have to write down any calculations) whether or not they have nontrivial solutions.

(a) $\begin{aligned} 3x_1+4x_2-7x_3 &= 0\\ 17x_2-5x_3 &= 0\\ 12x_3 &= 0\end{aligned}$ (b) $\begin{aligned} 5x_1+2x_2 &= 0\\ 15x_1+6x_2 &= 0\end{aligned}$

(c) $\begin{aligned} x_1+3x_2+2x_3+x_4 &= 0\\ 2x_1-x_2-x_3+x_4 &= 0\\ 14x_1+3x_2 &= 0\end{aligned}$ (d) $\begin{aligned} a_{11}x_1+a_{12}x_2+a_{13}x_3 &= 0\\ a_{21}x_1+a_{22}x_2+a_{23}x_3 &= 0\end{aligned}$

4. Solve Exercise 7 from Lecture 5 using the Gauss Jordan method.

5. Which of the following are elementary matrices?

(a) $\begin{bmatrix} 1 & 0\\ 3 & 1\end{bmatrix}$ (b) $\begin{bmatrix} -2 & 1\\ 1 & 0\end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0\\ 0 & 14\end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$

6. Find the inverses of each of the following matrices.

$$A=\begin{bmatrix} 1 & 4\\ 2 & 7\end{bmatrix}, B=\begin{bmatrix} 2 & 6 & 6\\ 2 & 7 & 6\\ 2 & 7 & 7\end{bmatrix}, C=\begin{bmatrix} -1 & 3 & -4\\ 2 & 4 & 1\\ -4 & 2 & -9\end{bmatrix}, D=\begin{bmatrix} a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{bmatrix}, E=\begin{bmatrix} \alpha & 0 & 0 & 0\\ 1 & \alpha & 0 & 0\\ 0 & 1 & \alpha & 0\\ 0 & 0 & 1 & \alpha\end{bmatrix}$$

7. Solve the following systems of linear equations by using the inverse of the coefficient matrix.

(a) $\begin{aligned} 3x_1+4x_2 &= 10\\ 7x_1+9x_2 &= 20\end{aligned}$ (b) $\begin{aligned} 5x_1+7x_2+4x_3 &= 1\\ 3x_1-x_2+3x_3 &= 1\\ 6x_1+7x_2+5x_3 &= 1\end{aligned}$

8. Exercise 11(c) (p. 597): 6.

These exercises should take around 80 minutes to complete.

(Answers: 1. $Z_1=-s-t$, $Z_2=s$, $Z_3=-t$, $Z_4=0$, $Z_5=t$; 2. $k=-3$, $x_1=9t$, $x_2=-5t$, $x_3=t$; 3. (b), (c) and (d); 5. (a), (c) and (d); 6. $A^{-1}=\begin{bmatrix} -7 & 4\\ 2 & -1\end{bmatrix}$,

$$B^{-1}=\begin{bmatrix} \frac{7}{2} & 0 & -3\\ -1 & 1 & 0\\ 0 & -1 & 1\end{bmatrix}, C \text{ is not invertible}, D=\begin{bmatrix} \frac{1}{a} & 0 & 0\\ 0 & \frac{1}{b} & 0\\ 0 & 0 & \frac{1}{c}\end{bmatrix}, E=\begin{bmatrix} \frac{1}{\alpha} & 0 & 0 & 0\\ -\frac{1}{\alpha^2} & \frac{1}{\alpha} & 0 & 0\\ \frac{1}{\alpha^3} & -\frac{1}{\alpha^2} & \frac{1}{\alpha} & 0\\ -\frac{1}{\alpha^4} & \frac{1}{\alpha^3} & -\frac{1}{\alpha^2} & \frac{1}{\alpha}\end{bmatrix};$$

7.(a) $x_1=-10$, $x_2=10$ (b) $x_1=8$, $x_2=-1$, $x_3=-8$)

## Homogeneous Systems

Homogeneous systems are simply linear systems where the right hand side of each of the equations is equal to zero. The term *homogeneous* is used widely in engineering mathematics, usually referring to equations in the same way it does here. In general, we have

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= 0 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n &= 0 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn} &= 0
\end{aligned}$$

where the variables are $x_1$, $x_2$, $\ldots$, $x_n$ and the coefficients $a_{ij}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ are given. In the usual matrix form, we may write

$$A\boldsymbol{x} = \mathbf{0}$$

Note that $\boldsymbol{x} = \mathbf{0}$ is always a solution to these systems (we call this the trivial solution) and hence a homogeneous system is always consistent. Another way of looking at this is to view a homogeneous system as a special case of $A\boldsymbol{x} = \boldsymbol{b}$, with $\boldsymbol{b} = \mathbf{0}$. We have $\mathrm{r}([A|\boldsymbol{b}]) = \mathrm{r}([A|\mathbf{0}]) = \mathrm{r}(A)$, regardless of what the matrix $A$ is. Thus, the system will always be consistent and we only need to distinguish between two cases:

(i) If $\mathrm{r}(A) = n$, the trivial solution is the only solution.

(ii) If $\mathrm{r}(A) < n$, we get infinitely many solutions. Amongst these is the trivial (*i.e.* zero) solution as well as an infinite number of non trivial (*i.e.* nonzero) solutions.

In the case of infinitely many solutions, we again use parameters to describe these.

**<u>Ex:</u>** Solve the following homogeneous system:

$$\begin{array}{rrrcl}
4x_1 & +2x_2 & -x_3 & = & 0 \\
2x_1 & -3x_2 & +x_3 & = & 0 \\
6x_1 & -x_2 & & = & 0
\end{array}$$

**<u>Soln:</u>** $[A|\boldsymbol{b}] = [A|\mathbf{0}] = \left[\begin{array}{ccc|c} 4 & 2 & -1 & 0 \\ 2 & -3 & 1 & 0 \\ 6 & -1 & 0 & 0 \end{array}\right] \begin{array}{l} \mathrm{R}_1 \to \mathrm{R}_2 \\ \mathrm{R2} \to \mathrm{R}_1 \\ \phantom{x} \end{array} \sim \left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 4 & 2 & -1 & 0 \\ 6 & -1 & 0 & 0 \end{array}\right] \begin{array}{l} \phantom{x} \\ \mathrm{R}_2 \to \mathrm{R}_2 - 2\mathrm{R}_1 \\ \mathrm{R}_3 \to \mathrm{R}_3 - 3\mathrm{R}_1 \end{array}$

$$\sim \left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 0 & 8 & -3 & 0 \\ 0 & 8 & -3 & 0 \end{array}\right] \begin{array}{l} \phantom{x} \\ \phantom{x} \\ \mathrm{R}_3 \to \mathrm{R}_3 - \mathrm{R}_2 \end{array} \sim \left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 0 & 8 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Note that it is not actually necessary to carry along the zero column for the augmented matrix here, but most people seem to do it naturally anyway. As $\mathrm{r}(A) = 2$ is less than the number of variables, $n = 3$, we get infinitely many solutions. The non-leading variable in this case is $x_3$, so we assign $x_3 = t$. Then $8x_2 - 3x_3 = 0$, *i.e.* $x_2 = \frac{3}{8}x_3 = \frac{3}{8}t$. Finally, $2x_1 - 3x_2 + x_3 = 0$, *i.e.* $x_1 = \frac{3}{2}x_2 - \frac{1}{2}x_3 = \frac{3}{2}(\frac{3}{8}t) - \frac{t}{2} = \frac{t}{16}$. In vector form, the solution is

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{t}{16} \\ \frac{3}{8}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{16} \\ \frac{3}{8} \\ 1 \end{bmatrix}.$$

This essentially describes a line passing through the origin in 3 space. Infinitely many solutions to homogeneous systems always generate lines or planes which pass through the origin and we'll consider this in more detail later. Finally, note that we end up solving homogeneous systems frequently when we look at eigenvectors towards the end of semester.

## Reduced Row Echelon Form and the Gauss Jordan Method

A variation of Gaussian elimination is the Gauss Jordan method, which involves manipulating the augmented matrix of a system into the reduced row echelon form. This simply means that all leading entries in the matrix are 1, any leading entry occurs to the right of the leading entry in the row above, all zero rows are at the bottom of the matrix and any column containing a leading entry has only zeros in the remaining entries. In practice, we first manipulate a given matrix into the row echelon form (working forward along the columns) and then use further e.r.o.'s (now working backwards along the columns) to obtain the reduced row echelon form. This is best illustrated by an example.

**Ex:** Solve
$$\begin{aligned} x_1 - x_2 + 2x_2 &= -1 \\ 2x_1 + x_2 - 2x_3 &= -2 \\ -x_1 + 2x_2 - 4x_3 &= 1 \end{aligned}$$

**Soln:** $[A|\boldsymbol{b}] = \left[\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 2 & 1 & -2 & -2 \\ -1 & 2 & -4 & 1 \end{array}\right] \begin{array}{l} \mathrm{R}_2 \to \mathrm{R}_2 - 2\mathrm{R}_1 \\ \mathrm{R}_3 \to \mathrm{R}_3 + \mathrm{R}_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 0 & 3 & -6 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \begin{array}{l} \mathrm{R}_2 \to \mathrm{R}_3 \\ \mathrm{R}_3 \to \mathrm{R}_2 \end{array}$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 0 & 1 & -2 & 0 \\ 0 & 3 & -6 & 0 \end{array}\right] \mathrm{R}_3 \to \mathrm{R}_3 - 3\mathrm{R}_2 \quad \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \mathrm{R}_1 \to \mathrm{R}_1 + \mathrm{R}_2$$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. We put $x_3 = t$, then $x_2 = 2t$ and $x_1 = -1$.

## Elementary Matrices and Calculating Inverses

An $n \times n$ matrix is called elementary if it can be obtained from the $n \times n$ identity matrix by a single elementary row operation. *e.g.* $E_1$, $E_2$ and $E_3$ below are all elementary matrices:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \mathrm{R}_1 \to \mathrm{R}_2 \\ \mathrm{R}_2 \to \mathrm{R}_1 \end{matrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathrm{R}_2 \to \mathrm{R}_2 - 3\mathrm{R}_1 \sim \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathrm{R}_3 \to 5\mathrm{R}_3 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} = E_3$$

To see the use of elementary matrices, consider the effect of pre-multiplying $A = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 1 & 3 & 7 \\ 4 & 2 & 2 & 1 \end{bmatrix}$ by each of these elementary matrices:

$$E_1 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 1 & 3 & 7 \\ 4 & 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 & 7 \\ 1 & 3 & 1 & 2 \\ 4 & 2 & 2 & 1 \end{bmatrix}$$

$$E_2 A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 1 & 3 & 7 \\ 4 & 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 2 \\ -1 & -8 & 0 & 1 \\ 4 & 2 & 2 & 1 \end{bmatrix}$$

$$E_3 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 1 & 3 & 7 \\ 4 & 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 1 & 3 & 7 \\ 20 & 10 & 10 & 5 \end{bmatrix}$$

Now notice what happens if we apply the same e.r.o.'s to $A$ directly

$$A = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 1 & 3 & 7 \\ 4 & 2 & 2 & 1 \end{bmatrix} \begin{matrix} \mathrm{R}_1 \to \mathrm{R}_2 \\ \mathrm{R}_2 \to \mathrm{R}_1 \end{matrix} \sim \begin{bmatrix} 2 & 1 & 3 & 7 \\ 1 & 3 & 1 & 2 \\ 4 & 2 & 2 & 1 \end{bmatrix} = E_1 A$$

$$A = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 1 & 3 & 7 \\ 4 & 2 & 2 & 1 \end{bmatrix} \mathrm{R}_2 \to \mathrm{R}_2 - 3\mathrm{R}_1 \sim \begin{bmatrix} 1 & 3 & 1 & 2 \\ -1 & -8 & 0 & 1 \\ 4 & 2 & 2 & 1 \end{bmatrix} = E_2 A$$

$$A = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 1 & 3 & 7 \\ 4 & 2 & 2 & 1 \end{bmatrix} \mathrm{R}_3 \to 5\mathrm{R}_3 \sim \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 1 & 3 & 7 \\ 20 & 10 & 10 & 5 \end{bmatrix} = E_3 A$$

**Observation:** Let $E$ be obtained from $I_m$ by a particular e.r.o. Let A be any $m \times n$ matrix. Then performing the same e.r.o. on $A$ will result in the matrix $EA$. (*i.e.* an e.r.o. is equivalent to a matrix multiplication by the corresponding elementary matrix).

Note that every elementary matrix is invertible and its inverse is also an elementary matrix (and simply corresponds to the e.r.o. which does the reverse of the original). For example, consider

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ \mathrm{R}_2 \to \mathrm{R}_2 + 3\mathrm{R}_1 \ \sim \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_4$$

Then, the operation used to create $E_4$ is just the reverse of the operation used to obtain $E_2$ and

$$E_2E_4 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

as expected, so $E_4 = E_2^{-1}$.

Note that we could reduce a matrix to row echelon form with a series of matrix multiplications by elementary matrices if we wanted to. For example, let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 3 & 4 & 5 \end{bmatrix}$ and

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}.$$

Then

$$E_4E_3E_2E_1A = E_4E_3E_2 \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 3 & 4 & 5 \end{bmatrix} = E_4E_3 \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & 2 \end{bmatrix} = E_4 \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

While this idea doesn't lend itself to hand computation, it is easy to incorporate into software. Our purpose for introducing elementary matrices is to justify the use of the Gauss Jordan technique for calculating the inverse of a matrix.

Let $A$ be an invertible $n \times n$ matrix (we also use the terminology that $A$ is non-singular). Then it turns out that we can always reduce $A$ to $I_n$ by a sequence of e.r.o.'s. Hence, there is a sequence of elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_kE_{k-1}\ldots E_2E_1A = I$$

Post multiplying each side by $A^{-1}$, we get

$$E_kE_{k-1}\ldots E_2E_1AA^{-1} = IA^{-1}$$

or

$$E_k E_{k-1} \dots E_2 E_1 I = A^{-1}$$

*i.e.* if we apply the same sequence of e.r.o.'s to $I$, we will end up with $A^{-1}$!

*Hence, the technique to obtain the inverse of $A$ is as follows.* Form the augmented matrix $[A|I]$ (*i.e.* append the identity matrix to the right hand side of $A$) and then apply e.r.o.'s to this augmented matrix until the left hand side turns into the identity. The right hand side will then automatically become $A^{-1}$.

**Ex:** Find the inverse of $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$.

**Soln:** $[A|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 4 & -3 & 0 & 1 & 0 \\ 3 & 6 & -5 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \mathrm{R}_2 \to \mathrm{R}_2 - 2\mathrm{R}_1 \\ \mathrm{R}_3 \to \mathrm{R}_3 - 3\mathrm{R}_1 \end{array}$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -7 & -2 & 1 & 0 \\ 0 & 3 & -11 & -3 & 0 & 1 \end{array}\right] \mathrm{R}_2 \to \tfrac{1}{2}\mathrm{R}_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{2} & -1 & \frac{1}{2} & 0 \\ 0 & 3 & -11 & -3 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \mathrm{R}_3 \to \mathrm{R}_3 - 3\mathrm{R}_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{2} & -1 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & -\frac{3}{2} & 1 \end{array}\right] \begin{array}{l} \\ \\ \mathrm{R}_3 \to -2\mathrm{R}_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{2} & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 3 & -2 \end{array}\right] \begin{array}{l} \mathrm{R}_1 \to \mathrm{R}_1 - 2\mathrm{R}_3 \\ \mathrm{R}_2 \to \mathrm{R}_2 + \frac{7}{2}\mathrm{R}_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & -6 & 4 \\ 0 & 1 & 0 & -1 & 11 & -7 \\ 0 & 0 & 1 & 0 & 3 & -2 \end{array}\right] \begin{array}{l} \mathrm{R}_1 \to \mathrm{R}_1 - \mathrm{R}_2 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -17 & 11 \\ 0 & 1 & 0 & -1 & 11 & -7 \\ 0 & 0 & 1 & 0 & 3 & -2 \end{array}\right] = [I|A^{-1}],$$

*i.e.* $A^{-1} = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}$. Check for yourself that $AA^{-1} = I$.

What happens to this procedure if $A$ is not invertible?

**Ex:** Try to find the inverse of $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 1 & 3 & -5 \end{bmatrix}$.

**Soln:** $[A|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 4 & -3 & 0 & 1 & 0 \\ 1 & 3 & -5 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -7 & -2 & 1 & 0 \\ 0 & 2 & -7 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 - R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -7 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

*i.e.* this will show up as a row of zeros when trying to perform the Gauss Jordan procedure and we can't continue. A matrix which is not invertible is also called singular.

## Invertibility and Solutions of Systems

Consider a system of $n$ equations in $n$ unknowns, *i.e.*

$$A\boldsymbol{x} = \boldsymbol{b}$$

where $A$ is $n \times n$, $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.

Clearly, if $A$ is invertible with inverse $A^{-1}$, we have

$$\begin{aligned} A^{-1}A\boldsymbol{x} &= A^{-1}\boldsymbol{b} \\ \text{i.e. } I\boldsymbol{x} &= A^{-1}\boldsymbol{b} \\ \text{i.e. } \boldsymbol{x} &= A^{-1}\boldsymbol{b} \end{aligned}$$

**Ex:** Solve the system $A\boldsymbol{x} = \boldsymbol{b}$, where $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

**Soln:** From our previous lecture, we have $A^{-1} = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}$. Thus,

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 19 \\ -12 \\ -3 \end{bmatrix}.$$

This example raises the question as to why we should solve a system of $n$ equations in $n$ unknowns by Gaussian Elimination when we could just use the formula above? The answer is simply that the amount of computation required for Gaussian elimination is less than the amount of computation required in finding the inverse of the coefficient matrix. Particularly for large systems, it usually requires much less effort to employ Gaussian elimination.

However, in some practical problems, we end up having to solve $A\boldsymbol{x} = \boldsymbol{b}$ many times over for the same $A$ but for different vectors $\boldsymbol{b}$. Here, calculating $A^{-1}$ once is more efficient than solving several sets of equations by Gaussian elimination.

**Ex:** Solve the previous example again with $\boldsymbol{b}$ replaced by

$$\boldsymbol{b}' = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

**Soln:** $\boldsymbol{x} = A^{-1}\boldsymbol{b}' = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -21 \\ 14 \\ 4 \end{bmatrix}.$

Also note that $\boldsymbol{x} = A^{-1}\boldsymbol{b}$ does not work when $A$ is not invertible. Without any further work (*i.e.* Gaussian Elimination), we can only conclude that $A\boldsymbol{x} = \boldsymbol{b}$ is either inconsistent (no solution) or it has infinitely many solutions.

Finally, for the homogeneous system $A\boldsymbol{x} = \mathbf{0}$, if $A$ is invertible, then $\boldsymbol{x} = A^{-1}\mathbf{0} = \mathbf{0}$, *i.e.* the zero solution is the only solution. In this case, if $A$ is not invertible, we can conclude that $A\boldsymbol{x} = \mathbf{0}$ must have infinitely many solutions.

Clearly, then, we would like to be able to tell whether a square matrix is invertible or not, preferably without having to actually attempt to calculate the inverse. This is the goal of our next lecture.