Lecture 9

Determinants

Determinants

With each square (*i.e.* $n \times n$) matrix A, we can associate a number called the determinant, denoted by $\det(A)$ or $|A|$. Before showing how to calculate $\det(A)$, we need to grasp some other concepts.

The matrix we obtain after deleting one or more columns or rows from a matrix A is called a submatrix of A. For example, $\begin{bmatrix} 2 & 1 \\ 3 & 7 \end{bmatrix}$ is a submatrix of each of

$$\begin{bmatrix} 2 & 1 & 6 \\ 3 & 7 & 9 \\ -1 & 5 & 9 \end{bmatrix}, \begin{bmatrix} 2 & 5 & 1 \\ 3 & -1 & 7 \\ -1 & 5 & 9 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 0 & 1 \\ 8 & 5 & 1 \\ 3 & 7 & 7 \end{bmatrix}.$$

Consider an $n \times n$ matrix $A = [a_{ij}]$. The minor, M_{ij}, associated with the element a_{ij} is the determinant of the $(n-1) \times (n-1)$ submatrix obtained from A by deleting row i and column j.

$$e.g. \quad \text{if } A = \begin{bmatrix} 1 & 5 & 3 \\ 6 & 2 & 4 \\ 0 & 1 & 0 \end{bmatrix} \text{ then}$$

$$M_{21} = \det\left(\begin{bmatrix} 5 & 3 \\ 1 & 0 \end{bmatrix}\right) = \begin{vmatrix} 5 & 3 \\ 1 & 0 \end{vmatrix},$$

Finally, the cofactor associated with entry a_{ij} of A is given by

$$C_{ij} = (-1)^{i+j} M_{ij},$$

e.g. for A above, $C_{21} = (-1)^3 M_{21} = -\begin{vmatrix} 5 & 3 \\ 1 & 0 \end{vmatrix}$.

Consider the $n \times n$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

For $n = 1$, *i.e.* $A = [a_{11}]$, $\det(A) = a_{11}$.

For $n = 2$, *i.e.* $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$,

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

For $n > 2$,

$$\begin{aligned} \det(A) &= a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n} \\ &= \sum_{i=1}^{n} a_{1i}C_{1i} \end{aligned}.$$

This is called the cofactor expansion along the 1st row of A.

<u>Ex:</u> Find $\det(A)$ if $A = \begin{bmatrix} 3 & 5 \\ -2 & -4 \end{bmatrix}$.

<u>Ex:</u> Find $\det(B)$ if $B = \begin{bmatrix} 2 & -4 \\ 1 & 7 \end{bmatrix}$.

<u>Ex:</u> Find $|C|$ if $C = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix}$.

<u>Ex:</u> Find $|D|$ if $D = \begin{bmatrix} 1 & 3 & -4 \\ -2 & 1 & 2 \\ -9 & 15 & 0 \end{bmatrix}$.

We can take a cofactor expansion along any row or column:

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} \qquad i\text{–th row}$$

$$\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij} \qquad j\text{–th column}$$

<u>Ex:</u> Find $|E|$ if $E = \begin{bmatrix} 2 & 0 & 3 \\ 1 & 4 & -2 \\ 5 & 0 & -3 \end{bmatrix}$.

Rules for Calculating Determinants

Consider an $n \times n$ matrix A:

1. If A has a row or a column of zeros, $\det(A) = 0$. *e.g.*
$$\begin{vmatrix} 5 & 0 & 6 & -1 \\ 0 & 0 & 8 & -2 \\ 1 & 0 & -3 & 4 \\ 3 & 0 & 0 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 3 & -1 \end{vmatrix} = 0.$$

2. Multiply any one row of A by a scalar k to obtain A'. Then
$$\det\left(A'\right) = k \det(A)$$
(makes sense when you consider taking a cofactor expansion along that row) *e.g.* $\begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} = 5$ and $\begin{vmatrix} 3 & 9 \\ -1 & 2 \end{vmatrix} = 15$, as expected

3. Interchange any two rows in A to obtain A'. Then
$$\det\left(A'\right) = -\det(A)$$

e.g. $\begin{vmatrix} 13 & 1 \\ 2 & -1 \end{vmatrix} = -15$ and $\begin{vmatrix} 2 & -1 \\ 13 & 1 \end{vmatrix} =$ 15, as expected.

4. Add a multiple of one row in A to another to obtain A'. Then

$$\det\left(A'\right) = \det(A)$$

e.g. let $A = \begin{bmatrix} 1 & 3 \\ -3 & 5 \end{bmatrix}$, then $\det(A) = 14$.

Consider $\begin{bmatrix} 1 & 3 \\ -3 & 5 \end{bmatrix}$ $R_2 \to R_2 + 3R_1$

$\sim \begin{bmatrix} 1 & 3 \\ 0 & 14 \end{bmatrix} = A'$ and $\det(A') = 14$, as expected.

5. If one row in A is a scalar multiple of another, then $\det(A) = 0$.

e.g. $\begin{vmatrix} 3 & -1 & 4 & 7 \\ 2 & 2 & 3 & -1 \\ -3 & 1 & -4 & -7 \\ 1 & 6 & 2 & 1 \end{vmatrix} = 0$,

since $R_3 = -R_1$.

6. $\det\left(A^{\top}\right) = \det(A)$. This rule basically allows us to apply Rules 2-5 to columns as well as rows.

7. $\det(k\,A) = k^n \det(A)$ (This is simply an expanded version of Rule 2.)

8. $\det(AB) = \det(A)\det(B)$ This is not an obvious rule, but a very important one which is used frequently in practice.

9. An upper triangular matrix is square with all entries below the main diagonal equal to zero. A lower triangular matrix is square with all entries above the main diagonal equal to zero. Finally, the determinant of a lower triangular or upper triangular matrix is the product of all the diagonal elements. For example,

   (i) $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 2 \end{vmatrix} = (1)(4)(2) = 8.$

(ii) $\begin{vmatrix} 2 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 3 & 0 & 5 & 0 \\ -1 & 1 & 2 & 4 \end{vmatrix} = (2)(1)(5)(4) = 40.$

In particular, note that the identity matrix is both upper and lower triangular, so $\det(I) = 1^n = 1$ for any order n.

10. A square matrix A is invertible if and only if $\det(A) \neq 0$.

From Rule 10, it follows that:

(i) $\det(A) = 0$ shows that A is singular, *i.e.* A^{-1} does not exist.

$\det(A) \neq 0$ shows that A is non-singular, *i.e.* A^{-1} does exist.

(ii) Note that if A is non-singular, then $AA^{-1} = I$. Hence,

$$\begin{array}{lrcl} \textit{i.e.} & \det\left(AA^{-1}\right) & = & \det(I) \\ \textit{i.e.} & \det(A)\det\left(A^{-1}\right) & = & 1 \\ \textit{i.e.} & \det\left(A^{-1}\right) & = & \dfrac{1}{\det(A)} \end{array}$$

Ex: Find $|F|$ if $F = \begin{bmatrix} 3 & 0 & -2 & 4 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 1 & -1 \\ 4 & 0 & 0 & -1 \end{bmatrix}$.

Calculating the Inverse of a 2×2 Matrix

Consider a general 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

To find the inverse A^{-1} of the matrix we can use the Gauss Jordan method.

Alternatively,

$$\boxed{A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

This is quite a useful formula to keep in mind.

Ex: Find the inverse of $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Cramer's Rule

Consider a system of n linear equations in n unknowns:

$$A\boldsymbol{x} = \boldsymbol{b},$$

where $\det(A) \neq 0$.
Let A_i be the matrix obtained from A by replacing the i-th column with $\boldsymbol{b}$. Then the solution of the system is given by

$$\boxed{x_i = \frac{\det(A_i)}{\det(A)}, \qquad i = 1, 2, \ldots, n}$$

Ex: Solve the following system of equations by Cramer's Rule:

$$\begin{aligned} x_1 &+ 2x_2 &= 4 \\ 3x_1 &+ 4x_2 &= 6 \end{aligned}$$

Ex: Use Cramer's rule to solve the following system for x_1 without solving for the remaining variables.

$$\begin{aligned} 2x_1 - x_2 + x_3 &= 3 \\ x_1 + x_2 - x_3 &= 0 \\ x_1 - x_2 + 2x_3 &= 5 \end{aligned}$$