Curtin College

DIPLOMA OF ENGINEERING

LINEAR ALGEBRA & STATISTICS EMTH1019
VECTORS – WEEK 5

Curtin College

Vectors

- Vector Basics & Notation
- Dot Product
- Scalar & Vector Projections
- Direction Cosines

RESOURCES FOR LINEAR ALGEBRA

- Slides
- Lecture Notes
- Lecture problems on first few pages of notes
- Tutorial problems
- News Forum

RESOURCES & HOMEWORK

- The lecture notes are much more detailed than this PowerPoint
- You should be able to do all the exercises on page 2 and 3 of this weeks lecture notes
- The Khan Academy has some wonderful resources
  - Link to Khan Academy - Vectors

IMPORTANT

Everything we do with vectors relies on you learning these basic skills.

1. Make a vector from 2 points
2. Determine the magnitude (length) of a vector
3. Add and subtract vectors
4. 2 ways of multiplying vectors
   a. Scalar product – also known as Dot Product – produces a scalar
   b. Vector Product – also known as Cross Product – produces another vector

For instance:

- To find the angle between 2 vectors you need to be able to do 2 & 4a.
- To find how much of a vector lies in a different direction requires 2 & 4a.
- To find an R^3 vector orthogonal to 2 other vectors requires 4b & 2.

WHAT IS A VECTOR?

- Latin origin
  - From the Latin *vector,* "one who carries or conveys" or "one who rides"
- Physics
  - A quantity that has both direction and magnitude
- Mathematics
  - A matrix with one row or one column
- Biology
  - An organism, such as a mosquito, that transmits a disease of parasite from one living thing to another
- Aviation
  - A course taken by an aircraft

VECTORS & SCALARS

In Engineering

- Scalars only have magnitude (no direction)
    - 7 kg, 14 years, 21 kmh, 28 metres
- Vectors have both magnitude and direction
    - 11 km due west, 9.8 m/s^2 towards the earth, 12 miles north by north west

Curtin College

VECTOR REPRESENTATION

If we have 2 Points we can make a Vector.

- In R^3 (3 space) Point A (1, 3, 5) and Point B (4, -6, 7)
- A vector that goes from A to B can be written as
- $\overrightarrow{AB}$.

To calculate $\overrightarrow{AB}$

- Determine how far is it from the x coordinate of A to the x coordinate of B
- From 1 to 4 is a step of +3
  - This is the same as calculating (Point B)- (Point A)
  - Point A (**1**, 3, 5) and Point B (**4**, -6, 7)
- Do the same for the y and for the z
- The Vector $\overrightarrow{AB}$= $\langle 3, -9, 2 \rangle$
- I use <> for vectors but you also see [ ] but never ()

Calculate the vector $\overrightarrow{BA}$?

- Is it equal to $\overrightarrow{AB}$?

POINT AND VECTOR NOTATION

Points

- In R^3 (3 space) Point A (1, 3, 5) and Point B (4, -6, 7)
- A (1,3,5) and B(4,-6,7)
- You can describe the point B(4,-6,7) as a position vector $\overrightarrow{OB} = [4, -6, 7]$ = <4,-6,7>, but not round brackets.

Vectors

- A vector that goes from A to B can be written as $\overrightarrow{AB}$.
- A vector that goes from B to A can be written as $\overrightarrow{BA}$.
- You also see vectors written with one letter either a bold letter e.g. **d** or a letter with ~ (a tilde) underneath it.

Curtin College

MAGNITUDE OF A VECTOR

Consider a vector $\boldsymbol{a} =< 3,2 >$

- Draw it on a set of Cartesian axes starting at (0,0) and finishing at the point(3,2). Remember to add the arrow.
- Calculate the length of this vector.
  - Easy $||\boldsymbol{a}|| = \sqrt{3^2 + 2^2} = \sqrt{13}$

Calculate the length of the 3D vector $\boldsymbol{b} =< 3,2,1 >$

Calculate the length of the 4D vector $\boldsymbol{c} =< 1,2,3,4 >$

Thank goodness for Pythagoras

ARE THE VECTORS EQUAL?

If $\overrightarrow{\boldsymbol{AB}}$= $\overrightarrow{\boldsymbol{BA}}$ the both the magnitude and the direction must be equal

Magnitude (also called NORM) of the vector $\overrightarrow{AB}$ is written as $||\overrightarrow{AB}||$

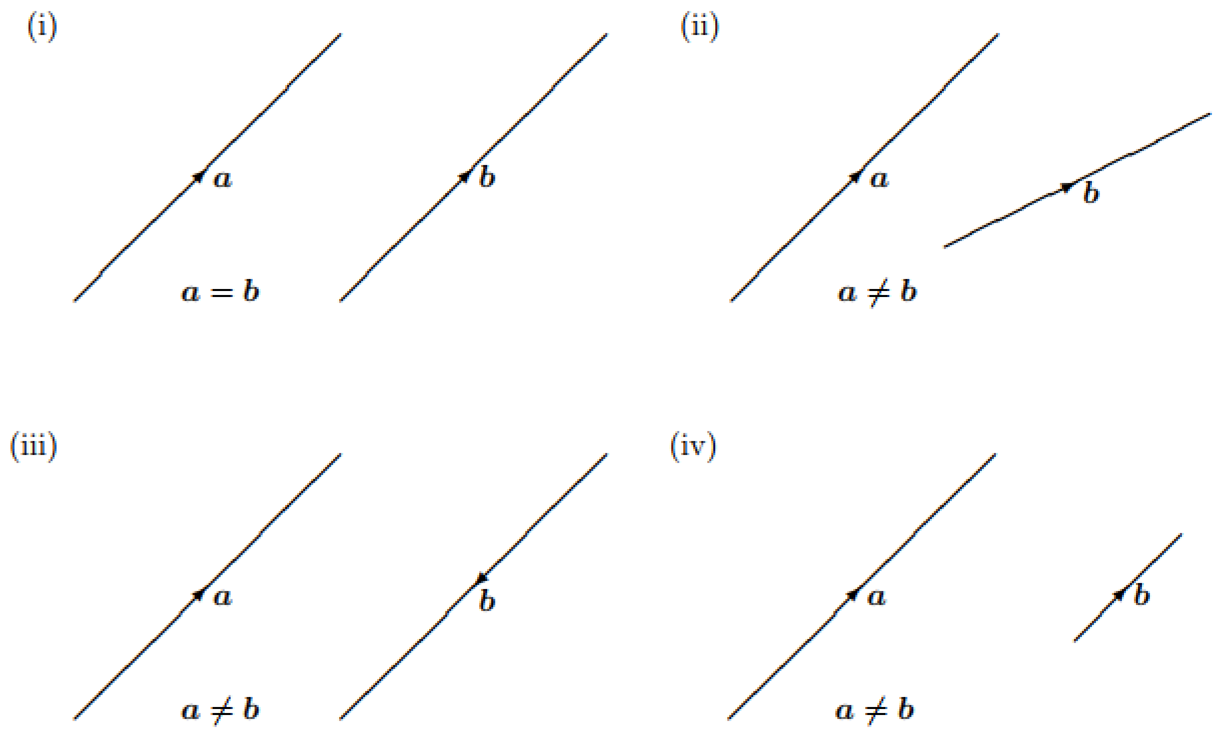
- $\sqrt{3^2 + (-9)^2 + 2^2} = \sqrt{9 + 81 + 4} = \sqrt{94}$
- What is $||\overrightarrow{BA}||$

To determine only the direction of a vector you divide a vector by its magnitude. This is called the unit vector and has a length of 1 unit.

- $\widehat{AB}$ is the unit vector. It has a little hat on top of it (to keep the rain off)
- $\widehat{AB} = \frac{<3,-9,2>}{\sqrt{94}} =< \frac{3}{\sqrt{94}}, \frac{-9}{\sqrt{94}}, \frac{2}{\sqrt{94}} >$

Calculate $\widehat{BA}$

- Are these 2 vectors equal?
    - Why?


(i)
a
b
a = b
(ii)
a
b
a ≠ b
(iii)
a
b
a ≠ b
(iv)
a
b
a ≠ b

ZERO & UNIT VECTORS

What length and direction does a **zero** vector have?

- $\boldsymbol{u} \;=< 0{,}0{,}0 > \quad or\; \boldsymbol{v} \;=< 0{,}0{,}0{,}0 >$
  - The length is 0 but what direction does a zero vector point?
  - Nowhere. It has no particular direction.

What length does a **unit** vector have?

- The length is always 1
- Is the vector ***z***=<*1,1,1*> a unit vector?

Consider the vectors $\boldsymbol{t} =< 1{,}2 >, \boldsymbol{w} =< 2{,}4 >, \boldsymbol{m} =< 3{,}6 >$

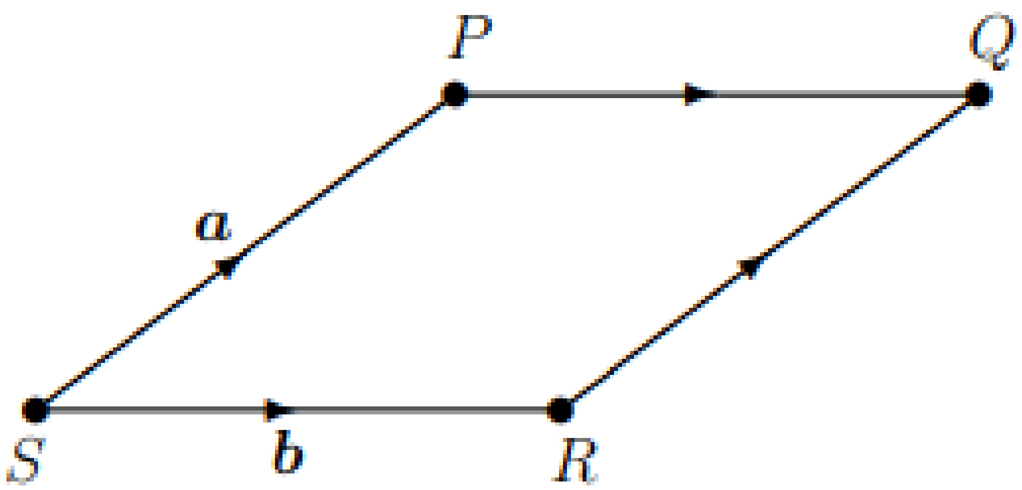
- They are not unit vectors as they have magnitudes of $\sqrt{5}, \sqrt{20}$ and $\sqrt{45}$, but which direction do they point in? Plot on a graph.
- How do you make $\boldsymbol{t}, \boldsymbol{w}$ and $\boldsymbol{m}$ into vectors with a magnitude of 1?
- Does $\hat{\boldsymbol{t}}$ have the same direction as $\boldsymbol{t}$?

QUESTION: CONSIDER A RHOMBUS PQRS

What is a rhombus?

If we let $\boldsymbol{a} = \overrightarrow{SP}\ and\ \boldsymbol{b} = \overrightarrow{SR}$,

- Does $\boldsymbol{a} = \boldsymbol{b}$**?**
- Does $||\boldsymbol{a}|| = ||\boldsymbol{b}||$?
- Express the following in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$
  - $\overrightarrow{PR}$
  - $\overrightarrow{QS}$



VECTOR OPERATIONS

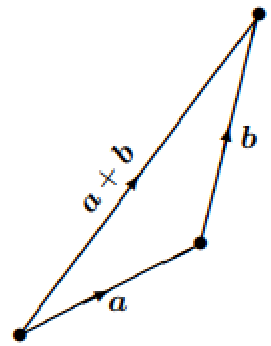
Sum of 2 vectors $\boldsymbol{a} + \boldsymbol{b} \;=\; \boldsymbol{b} + \boldsymbol{a}$

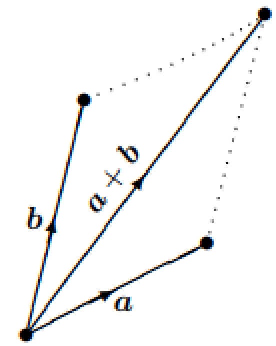
If $\boldsymbol{g} \;=\; \langle 3, -9, 2 \;\rangle$ and $\boldsymbol{h} \;=\; \langle 2, -10, -2 \;\rangle$

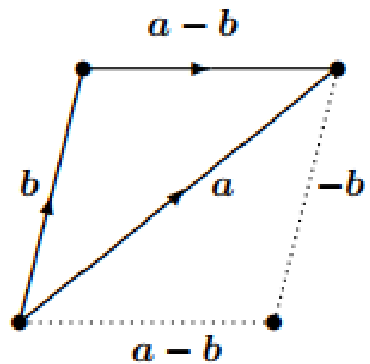
- $\boldsymbol{g} \;+\; \boldsymbol{h} \;=$
- $\boldsymbol{h} \;+\; \boldsymbol{g} \;=$

What happens if $\boldsymbol{z} = < 0{,}0{,}0 >$?

- $\boldsymbol{g} \;+\; \boldsymbol{z} \;=$
- $\boldsymbol{g} \;-\; \boldsymbol{z} \;=$
- $\boldsymbol{z}$ – $\boldsymbol{h} \;=$







Curtin College

NEGATIVE OF A VECTOR

The negative of the vector $\boldsymbol{g} = \langle 4, -6, 1.5 \rangle$ is $-\boldsymbol{g} = \langle -4, 6, -1.5 \rangle$

- These vectors have the same magnitude
- They are parallel
- But they point in opposite directions – it is as if one is pointing south and the other is pointing north.
- They are not equal
- They are 2 distinct vectors

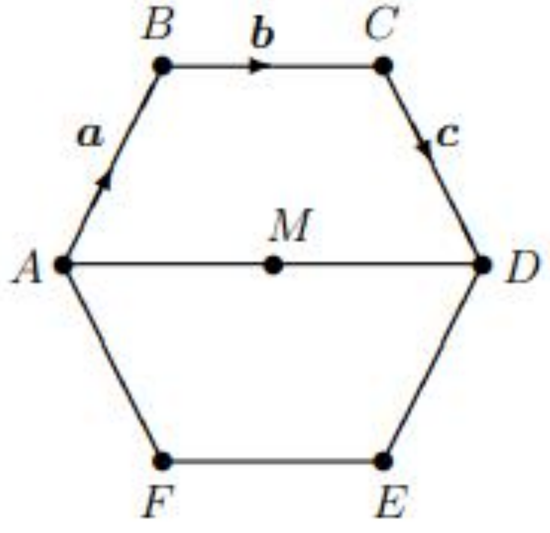
The negative of $\boldsymbol{a}$ is $-\boldsymbol{a}$

The negative of $\overrightarrow{AB}$ is $\overrightarrow{BA}$ or even $-\overrightarrow{AB}$

LET *ABCDEF* BE A REGULAR HEXAGON….

Ex: Let $ABCDEF$ be a regular hexagon, M the midpoint of AD, $\overrightarrow{AB} = \boldsymbol{a}$, $\overrightarrow{BC} = \boldsymbol{b}$ and $\overrightarrow{CD} = \boldsymbol{c}$. Find $\overrightarrow{AC}$, $\overrightarrow{AD}$, $\overrightarrow{AM}$, $\overrightarrow{BE}$ and $\overrightarrow{FC}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

Soln: We begin by drawing a diagram.



Curtin College

SCALAR MULTIPLICATION

What happens when you multiply a vector by a scalar?

If the vector $\boldsymbol{a} = < 4, -5, -1 >$ is multiplied by the scalar 7 you get…

If the vector $\boldsymbol{a}$ is multiplied by **<u>any</u>** scalar represented by $\boldsymbol{S}$ you get $\boldsymbol{Sa}$

- Is $\boldsymbol{Sa}$ a vector**?**

If $\boldsymbol{a} = < 4, -5, -1 >$ is multiplied by $\boldsymbol{S}$ you get …

- What is the magnitude of this new vector?
- Is $\boldsymbol{a}$ always equal to the vector $\boldsymbol{S} * \boldsymbol{a}$?
- Does $\boldsymbol{Sa}$ always have the same direction as $\boldsymbol{a}$?
- What is the magnitude of $\boldsymbol{Sa}$?
- What is the answer if $\boldsymbol{S} = 0$?

Curtin College

POSITION VECTORS AGAIN

Points have no direction.
A position vector tells you where a point is with reference to the origin

- Point P = (3, 6)
- The origin $\boldsymbol{O}$ = (0, 0)
- The Position Vector $\overrightarrow{OP}$ starts at the origin and ends at the point P
  - $\overrightarrow{OP} = < 3, 6 >$
  - Notice that the numbers are the same as the point P but I have used trianglet brackets to make it obvious that this is a position **VECTOR**
- What is $\overrightarrow{PO}$?
- If $\overrightarrow{OQ} = < 1, 2 >$ what does
  - $\overrightarrow{OP} + \overrightarrow{OQ} =$
  - $\overrightarrow{OP} - \overrightarrow{OQ} =$
  - $\overrightarrow{OQ} - \overrightarrow{OP} =$
  - $\overrightarrow{PQ} =$

ADDING POSITION VECTORS $\overrightarrow{\boldsymbol{AB}} = \overrightarrow{\boldsymbol{OB}} - \overrightarrow{\boldsymbol{OA}}$

Remember that $\overrightarrow{\boldsymbol{AB}}$ is the vector that goes from point A to point B

If A(-1, 2, 3), B(3, -1, 0), find

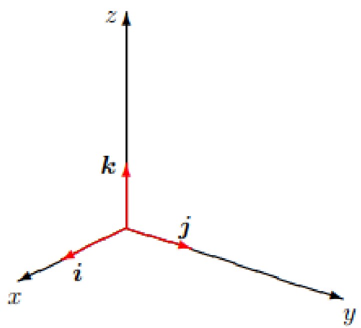
- $\overrightarrow{OA}$
- $\overrightarrow{OB}$
- $\overrightarrow{BA}$
- $\overrightarrow{AB}$
- What space are these vectors in?
- Does $\overrightarrow{AB} = \overrightarrow{BA}$?

STANDARD UNIT BASIS VECTORS

Big words for something simple

The i, j & k are the standard unit basis vectors and are unit vectors on the Cartesian axes.

- $\boldsymbol{i} = < 1, 0, 0 >$
- $\boldsymbol{j} = < 0, 1,\ 0 >$
- $\boldsymbol{k} = < 0, 0, 1 >$



We have been writing 3D vectors in the format <x, y, z>

- E.g. $a =< 3, 4, -5 >$

Another way of writing vectors is

- $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j} - 5\boldsymbol{k}$ or you will see me write $< 3\boldsymbol{i}, 4\boldsymbol{j}, -5\boldsymbol{k} >$
- They are interchangeable

PROBLEM

If $\boldsymbol{a} = \mathbf{3}\boldsymbol{i} - \boldsymbol{j} + \mathbf{2}\boldsymbol{k}$ and $\boldsymbol{b} =< \mathbf{5}, -\mathbf{2}, -\mathbf{1} > \ldots$

Calculate:

- $a + b$
- $b - a$
- $2a$
- $2a + b$
- $|a|$
- $||a + b||$

THE DOT PRODUCT (OR SCALAR PRODUCT)

There are 2 ways to multiply vectors the Dot Product and the Cross Product

- The Dot product produces a Scalar (just a number).
- The Cross Product produces another Vector and only exists in $R^{3.}$

The Dot Product of 2 vectors $\boldsymbol{a} =< a_1, a_2, a_3 > \, and \, \boldsymbol{b} =< b_1, b_2, b_3 >$ is

$$\boldsymbol{a}.\boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

For example

$$< 1,2,3 >.< -2,0,1 > = (1)(-2) + (2)(0) + (3)(1) = -2 + 0 + 3 = 1$$

- The answer is 1. This is a scalar quantity.

Also $\boldsymbol{a}.\boldsymbol{a} = a_1 a_1 + a_2 a_2 + a_3 a_3 = ||\boldsymbol{a}||^2$

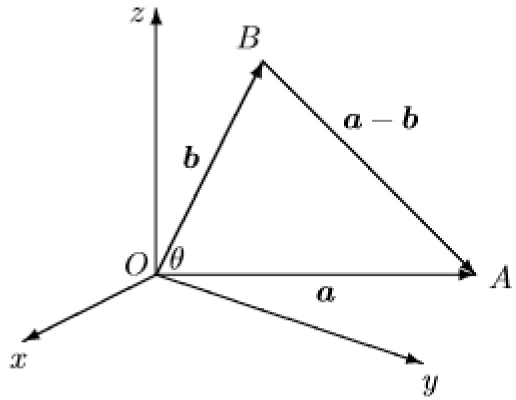
DOT PRODUCT RULE

The Dot Product Rule has a geometric meaning

Consider 2 vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ with an angle q between them.

Starting with the cosine rule

- $\left|\left|\overrightarrow{BA}\right|\right|^2 = \left|\left|\overrightarrow{OA}\right|\right|^2 + \left|\left|\overrightarrow{OB}\right|\right|^2 - 2\left|\left|\overrightarrow{OA}\right|\right|.\left|\left|\overrightarrow{OB}\right|\right|\cos(\theta)$
- $\overrightarrow{OA} = \boldsymbol{a}, \overrightarrow{OB} = \boldsymbol{b}, \overrightarrow{BA} = \boldsymbol{b} - \boldsymbol{a}$
- $||a - b||2 = ||a||2 + ||b||2 - 2||a|| * ||b||\cos(\theta)$



$$\boldsymbol{a.b = \left|\left|a\right|\right| * ||b||\cos(\theta)}$$

or

$$\boldsymbol{cos(\theta)} = \frac{\boldsymbol{a.b}}{\boldsymbol{\left|\left|a\right|\right| * ||b||}}$$

FIND THE ANGLE *BAC*

If the points are defined as A(1,0,2), B(-1,0,1) and C(1,-1,1)

- $\boldsymbol{cos(\theta)}$=
- $\boldsymbol{cos(\theta)}$=$\frac{\boldsymbol{<-2,0,-1>.<0,-1,-1>}}{\boldsymbol{\sqrt{5}*\sqrt{2}}} = \frac{\boldsymbol{0+0+1}}{\boldsymbol{\sqrt{10}}} = \frac{\boldsymbol{1}}{\boldsymbol{\sqrt{10}}}$ This is the cosine of the angle.
- Therefore $\boldsymbol{\theta} = ?$

IMPORTANT

- If two vectors have an angle of $\frac{\pi}{2}$ between them they are orthogonal
- If the angle between them is 0 or p then they are parallel

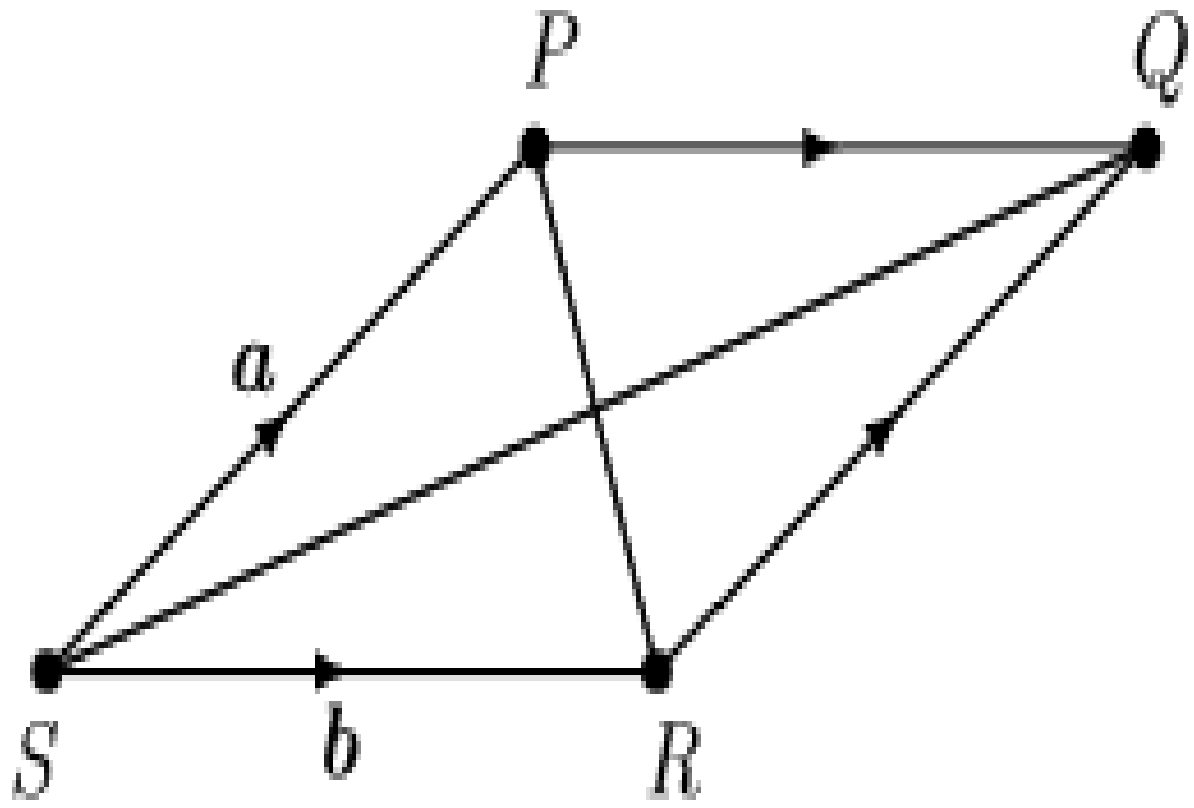
SHOW THAT THE DIAGONALS OF A RHOMBUS ARE PERPENDICULAR

What special properties does a rhombus have?

What formulae do I need to use to see if 2 vectors are perpendicular?

What does it have to equal?

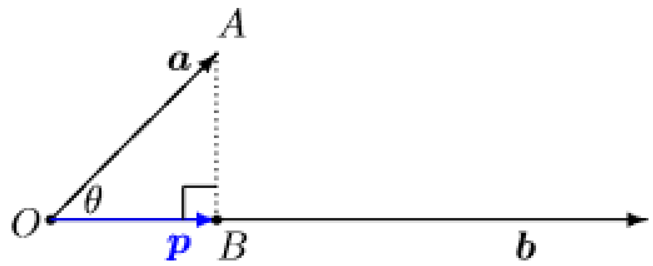
Draw and label a rhombus


P
Q
a
S
b
R

PROJECTION & COMPONENT OF A VECTOR

Consider 2 vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ that start at the origin.

- In this example $\boldsymbol{a}$ is pointing up and to the right and $\boldsymbol{b}$ lies on the x axis



How much of a projects on to the x axis?

- If we make $\boldsymbol{a}$ the hypotenuse of a right angled triangle you can see that the projection (or shadow) of $\boldsymbol{a}$ is $\boldsymbol{p}$, and if we know θ we could just use trigonometry and the Dot Product Rule

$$p = ||\overrightarrow{OB}|| = ||\overrightarrow{OA}|| \cos\theta = ||\boldsymbol{a}|| \frac{\boldsymbol{a}.\boldsymbol{b}}{||\boldsymbol{a}||\,||\boldsymbol{b}||} = \boldsymbol{a}.\left(\frac{\boldsymbol{b}}{||\boldsymbol{b}||}\right) = \boldsymbol{a}.\hat{\boldsymbol{b}},$$

SCALAR & VECTOR PROJECTION

Scalar Projection

- Scalar projection= $\boldsymbol{a}.\hat{\boldsymbol{b}}$
- sometimes you see Scalar projection written as p
- These 2 vectors are a dot product so p is a scalar (just a number)
- Scalar Projection only tells me the magnitude of the projection.

Vector Projection

- Vector Projection= $(Scalar\ Projection).\hat{\boldsymbol{b}}$
  - **Vector Projection = (Scalar Projection of a on b)* (Unit vector b)**
  - **Sometimes you see vector projection written as** $\boldsymbol{p}$
  - this is easy to confuse with the scalar projection p
- Vector projection tells us the magnitude of the projection and the direction of $\boldsymbol{b}$ **–** remember $\hat{\boldsymbol{b}}$ is the unit vector
- You must work out the Scalar Projection first

PROBLEM

Find the scalar and vector projection of a on b

$$a =< 2,1,-5 > \; and \; b =< 3,-4,0 >$$

Scalar projection of a on b means

- How much of the vector $\boldsymbol{a}$ lies in the $\boldsymbol{b}$ direction?

$$< \mathbf{2}, \mathbf{1}, -\mathbf{5} >. \frac{< \mathbf{3}, -\mathbf{4}, \mathbf{0} >}{\sqrt{\mathbf{9} + \mathbf{16} + \mathbf{0}}} = \frac{6 - 4 + 0}{\sqrt{25}} = \frac{2}{5}$$

The answer is a scalar

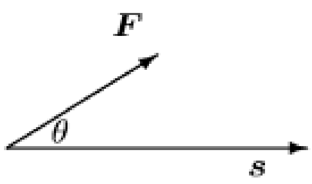
Vector Projection of a on b means

- (Scalar projection of a on b)* ($\boldsymbol{b}$ direction)
- $\frac{2}{5} * \frac{<\mathbf{3},-\mathbf{4},\mathbf{0}>}{\sqrt{\mathbf{9}+\mathbf{16}+\mathbf{0}}} = \frac{2}{25} *< 3,-4,0 >$
- The answer is a vector

Now find the vector projection of $\boldsymbol{b}$ on $\boldsymbol{a}$

WORK DONE BY A FORCE

A powerful application of Scalar Projection is when you ask how much work is done by a force $\boldsymbol{F}$ when it results in a displacement $\boldsymbol{s}$



$$
\begin{aligned}
\text{Work} &= \text{force} \times \text{displacement} \\
&= \text{magnitude of } \boldsymbol{F} \text{ in direction of } \boldsymbol{s} \times ||\boldsymbol{s}|| \\
&= \text{scalar projection of } \boldsymbol{F} \text{ on } \boldsymbol{s} \times ||\boldsymbol{s}|| \\
&= ||\boldsymbol{F}|| \cos\theta ||\boldsymbol{s}|| \\
&= \boldsymbol{F}.\boldsymbol{s}
\end{aligned}
$$

$$\boldsymbol{Work = F.s}$$

The Work Done is the **Dot Product** of the Force and the displacement

PROBLEM

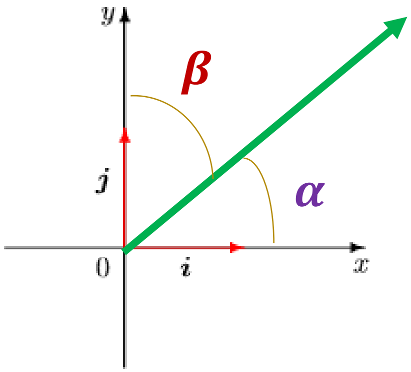
How much work is done by a force $\boldsymbol{F} = < 2, -3, -1 >$ in moving an object from $A(2,-1,3)$ to $B(5,3,-6)$?

DIRECTION COSINES IN 2 SPACE

The direction Cosines in 2 space are simply the cosines of the angles that the vector makes with respect to each of the standard unit basis vectors (or WRT to the axes)

Consider the vector $\boldsymbol{a} =< \boldsymbol{a_1}, \boldsymbol{a_2} >$

- $\boldsymbol{\alpha}\ (\boldsymbol{alpha})$and $\boldsymbol{\beta}(\boldsymbol{beta})$ are angles
- $||a|| = \sqrt{{a_1}^2 + {a_2}^2}$
- $\cos(\alpha) = \frac{a_1}{||a||}$
- $\cos(\beta) = \frac{a_2}{||a||}$

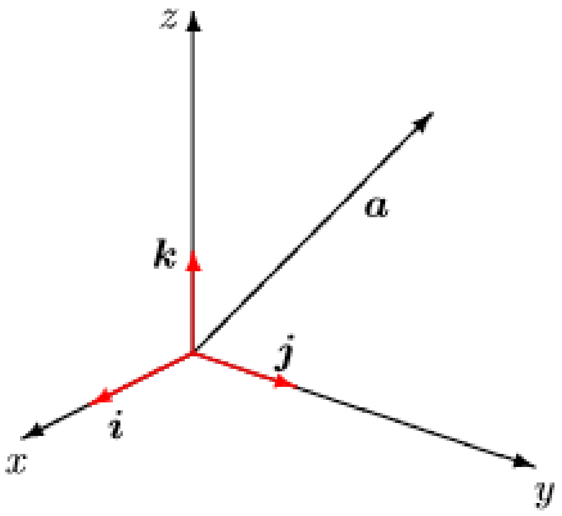


IMPORTANT

The direction cosines are the same as $\hat{a}$, which is the unit vector.

- $\hat{a} =< \cos(\alpha), \cos(\beta) >$

DIRECTION COSINES IN 3 SPACE



We have $\cos\alpha = \dfrac{\boldsymbol{a}.\boldsymbol{i}}{||\boldsymbol{a}||\,||\boldsymbol{i}||} = \dfrac{a_1}{||\boldsymbol{a}||}$, $\cos\beta = \dfrac{\boldsymbol{a}.\boldsymbol{j}}{||\boldsymbol{a}||\,||\boldsymbol{j}||} = \dfrac{a_2}{||\boldsymbol{a}||}$, $\cos\gamma = \dfrac{\boldsymbol{a}.\boldsymbol{k}}{||\boldsymbol{a}||\,||\boldsymbol{k}||} = \dfrac{a_3}{||\boldsymbol{a}||}$, so

$$[\cos\alpha, \cos\beta, \cos\gamma] = \frac{\boldsymbol{a}}{||\boldsymbol{a}||} = \hat{\boldsymbol{a}},$$

PROBLEM

Calculate the Direction Cosines for $\boldsymbol{c} = \langle 4, -5, 3 \rangle$