# EMTH1019 Linear Algebra & Statistics for Engineers

## Tutorial 7 Systems of Linear Equations & Gaussian Elimination

## SOLUTIONS

1. (i) $[A\,|\,b] = \left[\begin{array}{ccc|c} 2 & -1 & 0 & 3 \\ 4 & 3 & -1 & 1 \\ 0 & 3 & 1 & 0 \end{array}\right]$

   (ii) $[A\,|\,b] = \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & -2 \\ 0 & 1 & -3 & 0 & 1 & -1 \\ 1 & -2 & 4 & 0 & 3 & 0 \end{array}\right]$

2. (i) Yes it is in row-echelon form.

   (ii) No it is not in row-echelon form, since rows of zeros must be along the bottom row.

3. (i) $A = \begin{bmatrix} 2 & 1 \\ -4 & 0 \end{bmatrix} \; R_2 = R_2 + 2R_1 \quad \sim \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

   There are two non-zero rows, so rank $= 2$.

   (ii) $B = \begin{bmatrix} 2 & 3 & -1 \\ -8 & -7 & 6 \\ 6 & -1 & -7 \end{bmatrix} \; \begin{array}{l} R_2 = R_2 + 4R_1 \\ R_3 = R_3 - 3R_1 \end{array} \quad \sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & 5 & 2 \\ 0 & -10 & -4 \end{bmatrix} \; R_3 = R_3 + 2R_2$

   $$\sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

   There are two non-zero rows, so rank $= 2$.

   (iii) $C = \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \; \begin{array}{l} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 3R_1 \end{array} \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & -6 & 4 & 14 & -20 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \; \begin{array}{l} R_3 = R_3 - 2R_2 \\ R_4 = R_4 - 3R_2 \end{array}$

   $$\sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & -10 & 15 \end{bmatrix} \; R_4 = R_4 + \tfrac{5}{2}R_3 \quad \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   There are three non-zero rows, so rank $= 3$.

4. (i) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right] \; \begin{array}{l} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 3R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right] \; R_3 = -\tfrac{1}{5}R_3$

   $$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 1 & 2 & 4 \end{array}\right] \; R_3 = 7R_3 + R_2 \quad \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & 10 & 30 \end{array}\right]$$

$r(A) = r(A|b) = n = 3 \;\Rightarrow$ Unique solution.

Row 3: $10x_3 = 30 \;\Rightarrow\; \boxed{x_3 = 3}$

Row 2: $-7x_2 - 4x_3 = 2 \;\Rightarrow\; -7x_2 - 12 = 2 \;\Rightarrow\; -7x_2 = 14 \;\Rightarrow\; \boxed{x_2 = -2}$

Row 1: $x_1 + 2x_2 + 3x_3 = 6 \;\Rightarrow\; x_1 - 4 + 9 = 6 \;\Rightarrow\; x_1 = 6 - 5 \;\Rightarrow\; \boxed{x_1 = 1}$

(ii) $$\left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{array}\right] \begin{array}{l} \\ \\ R_3 = 2R_3 - 5R_1 \end{array} \sim \left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -1 & 4 & -3 \end{array}\right] \begin{array}{l} \\ \\ R_3 = R_3 + R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

$r(A) = 2 < r(A|b) = 3 \;\Rightarrow$ No solution.

(iii) $$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right] \begin{array}{l} R_2 = R_2 + R_1 \\ R_3 = R_3 - 2R_1 \end{array} \sim \left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{array}\right] \begin{array}{l} R_2 = \frac{1}{3}R_2 \\ R_3 = R_3 + 3R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$r(A) = 2 = r(A|b) < n = 3 \;\Rightarrow$ Infinitely many solutions.

Need $n - r = 3 - 1 = 1$ parameter.

Let $\boxed{x_3 = t}$, $t \in \mathbb{R}$

Row 2: $\boxed{x_2 = 2}$

Row 1: $3x_1 + 5x_2 - 4x_3 = 7 \;\Rightarrow\; 3x_1 + 10 - 4t = 7 \;\Rightarrow\; 3x_1 = 4t - 3 \;\Rightarrow$ $\boxed{x_1 = \frac{4}{3}t - 1}$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}t - 1 \\ 2 \\ t \end{bmatrix} = t \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}$$

5. Let $x_1$ be the number of days Factory A is scheduled, $x_2$ be the number of days Factory B is scheduled, and $x_3$ be the number of days Factory C is scheduled.

   The requirements for refrigerators, dishwashers and stoves can be written as:

$$\begin{aligned} 10x_1 + 20x_2 + 20x_3 &= 100 \\ 50x_1 + 40x_2 + 10x_3 &= 290 \\ 30x_1 + 10x_2 + 40x_3 &= 180 \end{aligned}$$

Now we solve the augmented matrix:

$$\left[\begin{array}{ccc|c} 10 & 20 & 20 & 100 \\ 50 & 40 & 10 & 290 \\ 30 & 10 & 40 & 180 \end{array}\right] \begin{array}{l} R_1 = \frac{1}{10}R_1 \\ R_2 = \frac{1}{10}R_2 \\ R_3 = \frac{1}{10}R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 10 \\ 5 & 4 & 1 & 29 \\ 3 & 1 & 4 & 18 \end{array}\right] \begin{array}{l} R_2 = R_2 - 5R_1 \\ R_3 = R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 10 \\ 0 & -6 & -9 & -21 \\ 0 & -5 & -2 & -12 \end{array}\right] \begin{array}{l} R_2 = -\frac{1}{3}R_2 \\ R_3 = 6R_3 - 5R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 10 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & 33 & 33 \end{array}\right]$$

$r(A) = r(A|b) = n = 3 \;\Rightarrow$ Unique solution.

Row 3: $33x_3 = 33 \;\Rightarrow\; \boxed{x_3 = 1}$

Row 2: $2x_2 + 3x_3 = 7 \;\Rightarrow\; 2x_2 + 3 = 7 \;\Rightarrow\; 2x_2 = 4 \;\Rightarrow\; \boxed{x_2 = 2}$

Row 1: $x_1 + 2x_2 + 2x_3 = 10 \;\Rightarrow\; x_1 + 4 + 2 = 10 \;\Rightarrow\; x_1 = 10 - 6 \;\Rightarrow\; \boxed{x_1 = 4}$

So Factory A should be scheduled for 4 days, Factory B for 2 days, and Factory C for 1 day.

6. Need to solve $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 = \boldsymbol{0}$

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

*i.e.*

$$\begin{array}{l} c_1 + 2c_2 - c_3 = 0 \\ c_2 + 4c_3 = 0 \\ 5c_1 + 8c_2 = 0 \end{array}$$

Now we solve the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 5 & 8 & 0 & 0 \end{array}\right] R_3 = R_3 - 5R_1 \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 5 & 0 \end{array}\right] R_3 = R_3 + 2R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{array}\right]$$

$r(A) = r(A|b) = n = 3 \;\Rightarrow$ Unique solution.

Row 3: $13c_3 = 0 \;\Rightarrow\; \boxed{c_3 = 0}$

Row 2: $c_2 + 4c_3 = 0 \;\Rightarrow\; c_2 + 4(0) = 0 \;\Rightarrow\; c_2 + 0 = 0 \;\Rightarrow\; \boxed{c_2 = 0}$

Row 1: $c_1 + 2c_2 - c_3 = 0 \;\Rightarrow\; c_1 + 2(0) - 0 = 0 \;\Rightarrow\; c_1 + 0 = 0 \;\Rightarrow\; \boxed{c_1 = 0}$

$\therefore$ Since $c_1 = c_2 = c_3 = 0$ they are linearly independent.