Euclidean Vector Spaces, Linear Dependence & Independence

Motivation: Most practical engineering problems involve large numbers of variables and we are thus forced to seek solutions in higher dimensional spaces. An elementary understanding of these spaces and how they operate is essential if we want to understand the underlying solution methods.

Understanding the idea of a basis is fundamental towards understanding the workings of many computational algorithms used in the numerical solution of large scale engineering problems. The concept of linear independence goes half way towards understanding the meaning of a basis.

Outcomes In today's lecture we will learn how to:

- Recognize the similarity between $\mathbb{R}^3$ and $\mathbb{R}^n$.
- Recognize subspaces of $\mathbb{R}^n$.
- Determine whether a given vector is a linear combination of others or not.
- Establish the linear dependence or independence of a given set of vectors.

Contents

- $\mathbb{R}^n$ as an extension of $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Length, angle and dot product in $\mathbb{R}^n$.
- Lines and Planes in $\mathbb{R}^n$.
- Vector Subspaces.
- The nullspace of a matrix.
- Linear combinations.
- Linear dependence and independence.
- Using determinants to establish dependence/independence.

Exercises

1. Find the distance between $P(0, -2, -1, 1)$ and $Q(-3, 2, 4, 4)$ in $\mathbb{R}^4$.
2. In each part below, determine whether the given vectors are orthogonal or not.

(a) $\boldsymbol{u} = [-4, 6, -10, 1]$, $\boldsymbol{v} = [2, 1, -2, 9]$ (b) $\boldsymbol{a} = [1, 2, 0, 1, -1]$ and $\boldsymbol{b} = [2, -1, 0, 1, 1]$.

3. Find the cosine of the angle between $\boldsymbol{x} = [0, 2, 3, 1, 0, -1]$ and $\boldsymbol{y} = [3, 0, -1, 0, 2, 1]$.

4. Determine whether or not the two lines

$$L_1 : \begin{cases} x_1 &= 3 + 4t \\ x_2 &= 2 + 6t \\ x_3 &= 3 + 4t \\ x_4 &= -1 - 2t \end{cases} \qquad L_2 : \begin{cases} x_1 &= \tau \\ x_2 &= 3 - 3\tau \\ x_3 &= 5 - 4\tau \\ x_4 &= 4 - 2\tau \end{cases}$$

intersect in $\mathbb{R}^4$.

5. Find the vector projection of $\boldsymbol{a} = [2, 1, 4, -1]$ on $\boldsymbol{b} = [-1, 1, 2, 3]$ in $\mathbb{R}^4$ (our usual formula applies in $\mathbb{R}^n$).

6. Determine which of the following are subspaces of $\mathbb{R}^3$.
   (a) All vectors of the form $[a, 0, 0]$, $a \in \mathbb{R}$.
   (b) All vectors of the form $[a, 1, 1]$, $a \in \mathbb{R}$.
   (c) All vectors of the form $[a, b, c]$, where $b = a + c$, $a, b, c \in \mathbb{R}$.
   (d) All vectors of the form $[a, b, c]$, where $b = a + c + 1$, $a, b, c \in \mathbb{R}$.
   (e) All vectors of the form $[a, b, c]$, where $a > 0$, $a, b, c \in \mathbb{R}$.

7. Show that $\boldsymbol{w} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ can be written as a l.c. of $\boldsymbol{v}_1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\boldsymbol{v}_3 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

8. Show that any vector in $\mathbb{R}^2$ can be written as a l.c. of $\boldsymbol{u}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

9. Show that the vectors

$$\boldsymbol{u}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \qquad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

are l.i. in $\mathbb{R}^4$.

10. For each of the following sets of vectors, decide whether they are l.i. or not.

(a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 10 \\ 0 \\ 1 \end{bmatrix} \right\}$ (d) $\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 3 \\ 0 \end{bmatrix} \right\}$

11. Think about each of the following statements:
    (a) Can three vectors in $\mathbb{R}^2$ be l.i. ?
    (b) Can two vectors in $\mathbb{R}^3$ be l.d. ?
    (c) Can two vectors in $\mathbb{R}^3$ be l.i. ?
    (d) Can five vectors in $\mathbb{R}^3$ be l.i. ?

12. Find all values of x for which the set

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ x \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$$

is l.i. in $\mathbb{R}^3$.

These exercises should take around 90 minutes to complete.

(Answers: 1. $\sqrt{59}$; 2. (a) No (b) Yes; 3. $\cos\theta = -\frac{4}{15}$; 4. No; 5. $\boldsymbol{p} = [-\frac{4}{15}, \frac{4}{15}, \frac{8}{15}, \frac{12}{15}]$; 6. (a) and (c) only; 10.(a) Yes (b) No (c) Yes (d) No; 11.(a) No (b) Yes (c) Yes (d) No; 12. Any $x \neq \frac{1}{2}$)

Euclidean Vector Spaces

Recall the notation of the real number line as $\mathbb{R}$. Following the same notation, we usually denote 2 space (*i.e.* the Cartesian plane) by $\mathbb{R}^2$ and 3 space by $\mathbb{R}^3$. Note that we can extend most of our work on vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ to vectors in general n space, *i.e.* $\mathbb{R}^n$. For $\boldsymbol{x} \in \mathbb{R}^n$, we write

$$\boldsymbol{x} = [x_1, x_2, \ldots, x_n]$$

and the definitions of addition, scalar multiplication, length, the dot product and orthogonality extend directly. In this context, $\mathbb{R}^n$, $n = 1, 2, 3, 4, \ldots$ are collectively known as the *Euclidean Vector Spaces*. This name is chosen to distinguish them from more abstract vector spaces which aren't studied until second year (such as vector spaces of matrices, vector spaces of functions *etc.*). In general, vector spaces are sets of objects which we can add and multiply by scalars and where these operations obey a certain set of rules.

Ex: Find the length of $\boldsymbol{a} = [1, 2, 5, -3, 1]$.

Soln: For $\boldsymbol{x} \in \mathbb{R}^n$, $||\boldsymbol{x}|| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$. Hence,

$$||\boldsymbol{a}|| = \sqrt{(1)^2 + (2)^2 + (5)^2 + (-3)^2 + (1)^2} = \sqrt{40}.$$

Ex: Find $\boldsymbol{a}.\boldsymbol{b}$, where $\boldsymbol{a} = [3, 2, 1, 1]$ and $\boldsymbol{b} = [-1, 1, -1, 1]$.

Soln: The dot product between two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^n$ is given by $\boldsymbol{x}.\boldsymbol{y} = x_1\,y_1 + x_2\,y_2 + \ldots + x_n\,y_n$. Hence,

$$\boldsymbol{a}.\boldsymbol{b} = (3)(-1) + (2)(1) + (1)(-1) + (1)(1) = -1$$

Note: Although the dot product can be naturally extended to $\mathbb{R}^n$, there is no version of the cross product in higher dimensional Euclidean space. Also, the dot product in $\mathbb{R}^n$ is often called the *Euclidean Inner Product* (inner products are effectively a generalization of the dot product to more abstract vector spaces).

Ex: Find the angle between $\boldsymbol{a} = [2, 3, 1, -1, 0]$ and $\boldsymbol{b} = [0, 1, -1, -1, 5]$ in $\mathbb{R}^5$.

Soln: Although we can't visualize these vectors in $\mathbb{R}^5$, the formula $\cos\theta = \frac{\boldsymbol{a}.\boldsymbol{b}}{||\boldsymbol{a}||\,||\boldsymbol{b}||}$ still holds. Thus

$$\cos\theta = \frac{\boldsymbol{a}.\boldsymbol{b}}{||\boldsymbol{a}||\,||\boldsymbol{b}||} = \frac{(2)(0) + (3)(1) + (1)(-1) + (-1)(-1) + (0)(5)}{\sqrt{(2)^2 + (3)^2 + (1)^2 + (-1)^2 + (0)^2}\sqrt{(0)^2 + (1)^2 + (-1)^2 + (-1)^2 + (5)^2}}$$

$$= \frac{3}{\sqrt{15}\sqrt{28}}, \text{ so } \theta = \cos^{-1}\left(\frac{3}{\sqrt{15}\sqrt{28}}\right) = 81.58^\circ \text{ (2 d.p.)}$$

Ex: Show that $\boldsymbol{u} = [2, 4, -1, 3]$ and $\boldsymbol{v} = [2, -2, -1, 1]$ are orthogonal in $\mathbb{R}^4$.

Soln: As usual, we simply need to show that their dot product is equal to zero, *i.e.*

$$\boldsymbol{u}.\boldsymbol{v} = (2)(2) + (4)(-2) + (-1)(-1) + (3)(1) = 0.$$

Hence $\boldsymbol{u} \perp \boldsymbol{v}$, as required.

Note that at most three vectors can be orthogonal to each other in $\mathbb{R}^3$, but $\mathbb{R}^4$ can have up to four vectors being orthogonal to one another. For example, check that $\boldsymbol{e}_1 = [1, 0, 0, 0]$, $\boldsymbol{e}_2 = [0, 1, 0, 0]$, $\boldsymbol{e}_3 = [0, 0, 1, 0]$ and $\boldsymbol{e}_4 = [0, 0, 0, 1]$ are orthogonal in $\mathbb{R}^4$.

The concepts of lines and planes can also be readily extended to $\mathbb{R}^n$. Note that planes in $\mathbb{R}^n$ for $n \geq 4$ are referred to as *hyperplanes*. They are widely used in the analysis of linear programming problems (which themselves are widely used in various industry and engineering disciplines for purposes such as production planning, optimal task scheduling and resource allocation).

Ex: Find the equation of the line passing through the points $P(0, 3, 4, 1)$ and $Q(7, -1, 0, 1)$ in $\mathbb{R}^4$.

Soln: The parametric equations are found in the same way as we do for lines in $\mathbb{R}^3$. A direction vector for the line is $\boldsymbol{a} = \overrightarrow{PQ} = [7, -4, -4, 0]$. Using it and the point P, the equations of the line are

$$\begin{aligned} x_1 &= 7t \\ x_2 &= 3 - 4t \\ x_3 &= 4 - 4t \\ x_4 &= 1 \end{aligned}$$

Ex: The general equation of a hyperplane in $\mathbb{R}^4$ takes the form $ax_1 + bx_2 + cx_3 + dx_4 = e$, where $\boldsymbol{n} = [a, b, c, d]$ is a normal vector to the plane (*i.e.* perpendicular to it). Find the equation of the plane with normal vector $\boldsymbol{n} = [-1, 2, -5, -3]$ and containing the point $P(2, -1, 7, -2)$.

Soln: $-x_1 + 2x_2 - 5x_3 - 3x_4 = e$ with $e = -(2) + 2(-1) - 5(7) - 3(-2) = -33$, *i.e.* the equation is $-x_1 + 2x_2 - 5x_3 - 3x_4 = -33$.

Vector Subspaces

Consider the Euclidean space $\mathbb{R}^n$ and let U be a subset of vectors in $\mathbb{R}^n$. If U is itself a vector space (*i.e.* it obeys all the necessary rules), we say that it is a *subspace* of $\mathbb{R}^n$. Subspaces are important in the understanding of solutions of systems of linear equations and in analyzing *linear transformations* (*i.e.* linear functions which take elements from one Euclidean space and map them to another Euclidean space).

A subset U of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if and only if

(a) for any $\boldsymbol{u}, \boldsymbol{v} \in U$, $\boldsymbol{u} + \boldsymbol{v} \in U$, and

(b) for any $\boldsymbol{u} \in U$ and any scalar s, $s\boldsymbol{u} \in U$.

We say U is a subspace of $\mathbb{R}^n$ if it is closed under addition and scalar multiplication.

Basically, a subspace must be such that we can not escape from it by adding vectors within it or by multiplying vectors within it by a scalar (where that scalar may be any real number). Once we can establish that a subset of vectors is a subspace, we can then associate a basis and a dimension with it (see later).

The first thing to note from the subspace requirements above is the following. Let U be a subspace of $\mathbb{R}^n$ containing a vector $\boldsymbol{u}$. Then, since U must be closed under scalar multiplication, $s\boldsymbol{u} \in U$ for any scalar s. Letting $s = 0$, we find that $(0)\boldsymbol{u} = \mathbf{0}$ must be in U, *i.e. any subspace of $\mathbb{R}^n$ must contain the zero vector!* Since it is often easy to look for the zero vector in a set, in practice, when checking for a subspace, we usually start by establishing that $\mathbf{0}$ is in the set.

Ex: Note that for any $\mathbb{R}^n$, $\mathbb{R}^n$ itself and $\{\mathbf{0}\}$ are subspaces. $\{\mathbf{0}\}$ is called the *zero vector space*, simply denoted as 0. The closure under addition and scalar multiplication for $\mathbb{R}^n$ is obvious (add any two vectors with n components and we get another vector with n components; multiply a vector with n components by a scalar and we again get a vector with n components). For the zero subspace, let $\boldsymbol{u}, \boldsymbol{v} \in 0$. Then $\boldsymbol{u} = \boldsymbol{v} = \mathbf{0}$ and $\boldsymbol{u} + \boldsymbol{v} = \mathbf{0} + \mathbf{0} = \mathbf{0} \in 0$. Similarly, $s\boldsymbol{u} = s\mathbf{0} = \mathbf{0} \in 0$ for any scalar s. Thus, 0 is closed under addition and scalar multiplication and it must be a subspace of $\mathbb{R}^n$.

Ex: Let U denote all vectors in $\mathbb{R}^3$ such that their second component is 1. Show that U is not a subspace of $\mathbb{R}^3$.

Soln: Note that $\mathbf{0} = [0, 0, 0]$ does not have its second component equal to 1, *i.e.* it is not in U. Thus, since U does not contain the zero vector, it cannot be a subspace of $\mathbb{R}^3$. Alternatively, $\boldsymbol{a} = [0, 1, 2]$ and $\boldsymbol{b} = [1, 1, 1]$ are both in U, but $\boldsymbol{a} + \boldsymbol{b} = [1, 2, 3] \notin U$, so the set is not closed under addition and it is thus not a subspace.

Ex: Show that the set of vectors in $\mathbb{R}^3$ where the third component equals the sum of the first two components is a subspace of $\mathbb{R}^3$.

Soln: The set can be written as

$$U = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_3 = x_1 + x_2 \right\}$$

To demonstrate closure under addition and multiplication, we need to show that these hold for general vectors within the set. Thus, let $\boldsymbol{u} = [u_1, u_2, u_3]$ and $\boldsymbol{v} = [v_1, v_2, v_3]$ be in U and let s be any scalar. Then $\boldsymbol{u}, \boldsymbol{v} \in U$ means $u_3 = u_1 + u_2$ and $v_3 = v_1 + v_2$.
Now consider $\boldsymbol{u}+\boldsymbol{v} = [u_1+v_1, u_2+v_2, u_3+v_3]$. Notice that $u_3+v_3 = (u_1+u_2)+(v_1+v_2) = (u_1+v_1)+(u_2+v_2)$ which shows that $\boldsymbol{u}+\boldsymbol{v} \in U$, *i.e.* U is closed under addition.
Next, consider $s\boldsymbol{u} = [su_1, su_2, su_3]$. Note that $su_3 = s(u_1+u_2) = su_1 + su_2$, *i.e.* $s\boldsymbol{u} \in U$ as well and U is therefore closed under scalar multiplication.
Combining these two properties, U must be a subspace of $\mathbb{R}^3$.
Still on this example, note that $\mathbf{0} \in U$, as required. Furthermore, the requirement $x_3 = x_1 + x_2$ can be easily rewritten as $x_1 + x_2 - x_3 = 0$, which simply describes a plane in $\mathbb{R}^3$. Note that this plane contains the origin. In the previous example, U was also just a plane in $\mathbb{R}^3$, namely $x_2 = 1$, but it did not contain the origin.
More generally, any plane or line in $\mathbb{R}^3$ containing the origin will form a subspace. Any line or plane not containing the origin will not form a subspace. Similarly, in $\mathbb{R}^2$, any line containing the origin will form a subspace (while those not containing the origin will not).

Ex: Show that the set defined by $\mathbb{R}^2_+ = \{[x_1, x_2] \in \mathbb{R}^2 \mid x_1 > 0 \text{ and } x_2 > 0\}$ is not a subspace of $\mathbb{R}^2$.

Soln: This set is called the positive quadrant of $\mathbb{R}^2$. Note that $[1,1] \in \mathbb{R}^2_+$ but $(-3)[1,1] = [-3,-3] \notin \mathbb{R}^2_+$, so the set is clearly not closed under scalar multiplication and can't be a subspace of $\mathbb{R}^2$.

Solution Space of Homogeneous System

Consider a homogeneous system of m linear equations in n unknowns

$$A\boldsymbol{x} = \mathbf{0}$$

i.e. A is $m \times n$ and $\boldsymbol{x} \in \mathbb{R}^n$. Let

$$V = \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} = \mathbf{0}\}$$

i.e. the set of all possible solutions of the system.
Let $\boldsymbol{u}, \boldsymbol{v} \in V$, then $A\boldsymbol{u} = \mathbf{0}$ and $A\boldsymbol{v} = \mathbf{0}$. Furthermore, let s be any scalar.
Then $A(\boldsymbol{u}+\boldsymbol{v}) = A\boldsymbol{u} + A\boldsymbol{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$ which show that $\boldsymbol{u}+\boldsymbol{v} \in V$, *i.e.* V is closed under addition.
Similarly, $A(s\boldsymbol{u}) = sA\boldsymbol{u} = s\mathbf{0} = \mathbf{0}$ which shows that $s\boldsymbol{u} \in V$, *i.e.* V is closed under scalar multiplication.

Hence, V is a vector subspace of $\mathbb{R}^n$ and therefore a vector space. In this sense, we refer to V as the *null space of the matrix A*.

Linear Combinations

Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\} \subset I\!R^n$. If the vector $\boldsymbol{u}$ can be expressed in the form

$$\boldsymbol{u} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_m\boldsymbol{u}_m$$

for some scalars $c_1, c_2, \ldots, c_m \in I\!R$, we say that $\boldsymbol{u}$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$. Clearly, $\boldsymbol{u}$ is itself a vector in $I\!R^n$.

Ex: Let $\boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} \in I\!R^3$. Show that $\boldsymbol{w}_1 = \begin{bmatrix} 9 \\ 2 \\ 7 \end{bmatrix}$ is a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$, while $\boldsymbol{w}_2 = \begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix}$ is not.

Soln: In general, we need to solve a system of linear equations to address this type of problem. For the first vector,

$$c_1\boldsymbol{u} + c_2\boldsymbol{v} = \boldsymbol{w}_1 \text{ gives } c_1\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 7 \end{bmatrix} \quad i.e. \quad \begin{array}{rcl} c_1 + 6c_2 & = & 9 \\ 2c_1 + 4c_2 & = & 2 \\ -c_1 + 2c_2 & = & 7 \end{array}$$

The corresponding augmented matrix is
$$\left[\begin{array}{cc|c} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 8 & 16 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{array}\right].$$
Hence, $c_2 = 2$ and $c_1 = -3$, so $\boldsymbol{w}_1 = (-3)\boldsymbol{u} + (2)\boldsymbol{v}$, *i.e.* $\boldsymbol{w}_1$ is a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$.
For the second vector,

$$c_1\boldsymbol{u} + c_2\boldsymbol{v} = \boldsymbol{w}_2 \text{ gives } c_1\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix} \quad i.e. \quad \begin{array}{rcl} c_1 + 6c_2 & = & 4 \\ 2c_1 + 4c_2 & = & -1 \\ -c_1 + 2c_2 & = & 8 \end{array}$$

The corresponding augmented matrix is
$$\left[\begin{array}{cc|c} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 8 & 12 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & -8 & -9 \\ 0 & 0 & 3 \end{array}\right].$$
Hence, the system is inconsistent and has no solution, *i.e.* $\boldsymbol{w}_2$ can not be written as a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$.

Ex: Let $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ be the standard unit basis vectors in $I\!R^3$. Then, for any vector in $I\!R^3$, we may write

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k},$$

i.e. any vector in $I\!R^3$ is clearly a linear combination of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$.

Ex: Show that any vector in $I\!R^3$ can be written as a linear combination of the vectors $\boldsymbol{u}_1 = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$, $\boldsymbol{u}_2 = \begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix}$ and $\boldsymbol{u}_3 = \begin{bmatrix} -3 \\ 4 \\ 7 \end{bmatrix}$.

Soln: Let $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ be a general vector in $I\!R^3$, *i.e.* b_1, b_2 and b_3 can take on any values in $I\!R$. We want to show that no matter what the values of b_1, b_2 and b_3, it is always possible to find coefficients c_1, c_2 and c_3 so that

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + c_3\boldsymbol{u}_3 = \boldsymbol{b} \qquad i.e.\ \ c_1\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} + c_2\begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix} + c_3\begin{bmatrix} -3 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

In component form, we have the system of equations

$$\begin{aligned} 3c_1 + 4c_2 - 3c_3 &= b_1 \\ c_1 + 3c_2 + 4c_3 &= b_2 \\ -2c_1 + 6c_2 + 7c_3 &= b_3 \end{aligned}$$

The augmented matrix is

$$[A|\boldsymbol{b}] = \left[\begin{array}{ccc|c} 3 & 4 & -3 & b_1 \\ 1 & 3 & 4 & b_2 \\ -2 & 6 & 7 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_2 \\ 3 & 4 & -3 & b_1 \\ -2 & 6 & 7 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_2 \\ 0 & -5 & -15 & b_1 - 3b_2 \\ 0 & 12 & 15 & b_3 + 2b_2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_2 \\ 0 & 1 & 3 & -\frac{1}{5}b_1 + \frac{3}{5}b_2 \\ 0 & 12 & 15 & b_3 + 2b_2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_2 \\ 0 & 1 & 3 & -\frac{1}{5}b_1 + \frac{3}{5}b_2 \\ 0 & 0 & -21 & b_3 - \frac{26}{5}b_2 + \frac{12}{5}b_1 \end{array}\right]$$

From the final augmented matrix, $\mathrm{r}(A) = \mathrm{r}([A|\boldsymbol{b}]) = 3$, so the system clearly has a unique solution no matter what the values of b_1, b_2 or b_3. Thus, it is possible to write any vector in $I\!R^3$ as a l.c. of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$.

Alternative Solution: Since we are dealing with a system of 3 equations in three unknowns, $A\boldsymbol{c} = \boldsymbol{b}$, A is square and we can calculate its determinant. We have

$$\det(A) = \begin{vmatrix} 3 & 4 & -3 \\ 1 & 3 & 4 \\ -2 & 6 & 7 \end{vmatrix} = \begin{vmatrix} 0 & -5 & -15 \\ 1 & 3 & 4 \\ 0 & 12 & 15 \end{vmatrix} = -(1)\begin{vmatrix} -5 & -15 \\ 12 & 15 \end{vmatrix} = -(-75 + 180) = -105.$$

Since $\det(A) \neq 0$, A is invertible and therefore $\boldsymbol{c} = A^{-1}\boldsymbol{b}$ which shows that we can always find a set of coefficients $\boldsymbol{c}$ no matter what the vector $\boldsymbol{b} \in \mathbb{R}^3$ is. Thus, it is possible to write any vector in $\mathbb{R}^3$ as a l.c. of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$.

Linear Dependence and Independence

Consider $\mathbb{R}^n$. A set of vectors $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\}$ in $\mathbb{R}^n$ is said to be linearly dependent if there are scalars $c_1, c_2, \ldots, c_m$, not all zero, such that

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_m\boldsymbol{u}_m = \boldsymbol{0}.$$

On the other hand, if the only way this equation can hold is with $c_1 = c_2 = \ldots = c_m = 0$, then the set is called linearly independent.

Note the following:

(i) Notation. We usually write

- linearly dependent as l.d.
- linearly independent as l.i.
- linear combination as l.c.

(ii) Suppose $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ are l.d. and $c_i \neq 0$ in the previous equation. Then

$$\boldsymbol{u}_i = -\frac{c_1}{c_i}\boldsymbol{u}_1 - \frac{c_2}{c_i}\boldsymbol{u}_2 - \ldots - \frac{c_{i-1}}{c_i}\boldsymbol{u}_{i-1} - \frac{c_{i+1}}{c_i}\boldsymbol{u}_{i+1} - \ldots - \frac{c_n}{c_i}\boldsymbol{u}_n,$$

i.e. $\boldsymbol{u}_i$ is a l.c. of the others, *i.e.* it 'depends' on them.

(iii) As we've mentioned previously, any set of two vectors is l.d. if the vectors are parallel (*i.e.* a scalar multiple of one another). If the two vectors are not parallel, they are l.i.

Ex: Show that the set $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is l.d., where $\boldsymbol{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 3 \end{bmatrix}$, $\boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 5 \\ -1 \end{bmatrix}$, and $\boldsymbol{u}_3 = \begin{bmatrix} 7 \\ -1 \\ 5 \\ 8 \end{bmatrix}$.

Soln: We need to check for the possible solutions of

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + c_3\boldsymbol{u}_3 = \mathbf{0} \quad i.e. \quad c_1\begin{bmatrix} 2 \\ -1 \\ 0 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \\ 5 \\ -1 \end{bmatrix} + c_3\begin{bmatrix} 7 \\ -1 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{aligned} 2c_1 + c_2 + 7c_3 &= 0 \\ -c_1 + 2c_2 - c_3 &= 0 \\ 5c_2 + 5c_3 &= 0 \\ 3c_1 - c_2 + 8c_3 &= 0 \end{aligned} \qquad [A|\mathbf{0}] = \left[\begin{array}{ccc|c} 2 & 1 & 7 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 5 & 5 & 0 \\ 3 & -1 & 8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 2 & -1 & 0 \\ 2 & 1 & 7 & 0 \\ 0 & 5 & 5 & 0 \\ 3 & -1 & 8 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} -1 & 2 & -1 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 5 & 5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 2 & -1 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since $\mathrm{r}(A) = 2 < 3$, this homogeneous system has infinitely many solutions. In particular, it has nontrivial solutions (*e.g.* put $c_3 = 1$, then $c_2 = -1$ and $c_1 = -3$), so the set of vectors must be linearly dependent. Note that, for a set of l.d. vectors such as these, solving the associated equation will always lead to infinitely many solutions.

Ex: Show that $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is l.d., where $\boldsymbol{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\boldsymbol{u}_2 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ and $\boldsymbol{u}_3 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

Soln: As for the previous example, we check for the possible solutions of

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + c_3\boldsymbol{u}_3 = \mathbf{0} \quad i.e. \quad c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ 6 \end{bmatrix} + c_3\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $\begin{aligned} 2c_1 + 3c_2 + 4c_3 &= 0 \\ c_1 + 6c_2 + c_3 &= 0 \end{aligned}$ We have $[A|\mathbf{0}] = \left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 1 & 6 & 1 & 0 \end{array}\right]$, and we don't even need to row reduce this matrix to recognize that $\mathrm{r}(A) \leq 2 < 3$, *i.e.* this underdetermined homogeneous system must have infinitely many solutions (including nontrivial ones). Thus, the set of vectors must be l.d. **Note that exactly the same argument can be used to show that any set of more than n vectors in $\mathbb{R}^n$ must be l.d.**

Ex: Show that $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is l.i., where $\boldsymbol{u}_1 = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$, $\boldsymbol{u}_2 = \begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix}$ and $\boldsymbol{u}_3 = \begin{bmatrix} -3 \\ 4 \\ 7 \end{bmatrix}$.

Soln: $c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + c_3\boldsymbol{u}_3 = \mathbf{0}$, *i.e.* $c_1\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} + c_2\begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix} + c_3\begin{bmatrix} -3 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ or

$$\begin{aligned} 3c_1 + 4c_2 - 3c_3 &= 0 \\ c_1 + 3c_2 + 4c_3 &= 0 \\ -2c_1 + 6c_2 + 7c_3 &= 0 \end{aligned} \qquad \text{We have } [A|\mathbf{0}] = \left[\begin{array}{ccc|c} 3 & 4 & -3 & 0 \\ 1 & 3 & 4 & 0 \\ -2 & 6 & 7 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 3 & 4 & -3 & 0 \\ -2 & 6 & 7 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 0 & -5 & -15 & 0 \\ 0 & 12 & 15 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 12 & 15 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -21 & 0 \end{array}\right], \textit{ i.e. } \mathrm{r}(A) = 3$$

and this homogeneous system has only the trivial solution, $c_1 = c_2 = c_3 = 0$. Thus, the set of vectors must be l.i.

Alternative Solution: Since we are dealing with a system of 3 equations in three unknowns, $A\boldsymbol{c} = \mathbf{0}$, A is square and we can calculate its determinant. We have

$$\det(A) = \begin{vmatrix} 3 & 4 & -3 \\ 1 & 3 & 4 \\ -2 & 6 & 7 \end{vmatrix} = \begin{vmatrix} 0 & -5 & -15 \\ 1 & 3 & 4 \\ 0 & 12 & 15 \end{vmatrix} = -(1)\begin{vmatrix} -5 & -15 \\ 12 & 15 \end{vmatrix} = -(-75+180) = -105.$$

Since $\det(A) \neq 0$, A is invertible and we thus get the unique trivial solution for this homogeneous system, *i.e.* $\boldsymbol{c} = A^{-1}\mathbf{0} = \mathbf{0}$, *i.e.* $c_1 = c_2 = c_3 = 0$ as before and the set of vectors must be l.i.

Note that a similar method can be used whenever we want to test whether a set of n vectors in $\mathbb{R}^n$ is l.i. or not. We simply put the vectors as columns into a matrix A and calculate $\det(A)$. If $\det(A) \neq 0$, the vectors will be l.i. as in this example. If $\det(A) = 0$, the homogeneous system $A\boldsymbol{c} = \mathbf{0}$ will have infinitely many (nontrivial) solutions, so the vectors must be l.d.

<u>**Ex:**</u> Show that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ is a l.i. set of vectors in $\mathbb{R}^4$, where

$$\boldsymbol{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 4 \\ 1 \\ 5 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_4 = \begin{bmatrix} 3 \\ 0 \\ 10 \\ 4 \end{bmatrix}.$$

<u>**Soln:**</u> Following the technique suggested in the previous example, we let $A = [\boldsymbol{v}_1 \,|\, \boldsymbol{v}_2 \,|\, \boldsymbol{v}_3 \,|\, \boldsymbol{v}_4]$. Then

$$\det(A) = \begin{vmatrix} 2 & 1 & 4 & 3 \\ 0 & 3 & 1 & 0 \\ 2 & 1 & 5 & 10 \\ 0 & 0 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 4 & 3 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 4 \end{vmatrix} = (2)(3)(1)(4) = 24 \neq 0,$$

so the set of vectors must be l.i.

<u>**Ex:**</u> Decide whether $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is l.i. or l.d., where

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \qquad \text{and} \quad \boldsymbol{v}_3 = \begin{bmatrix} -1 \\ 2 \\ -4 \end{bmatrix}.$$

Soln: Let $A = [\boldsymbol{v}_1 \,|\, \boldsymbol{v}_2 \,|\, \boldsymbol{v}_3]$. Then

$$\det(A) = \begin{vmatrix} 1 & 1 & -1 \\ 4 & 1 & 2 \\ 2 & 3 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 0 & -3 & 6 \\ 0 & 1 & -2 \end{vmatrix} = (1) \begin{vmatrix} -3 & 6 \\ 1 & -2 \end{vmatrix} = 6 - 6 = 0,$$

so the set must be l.d.

So in summary, when testing if a set of vectors is l.i. or l.d. we go through the following steps:

(i) If there only two vectors check to see if they're parallel, i.e. $v_1 = sv_2$. If they are parallel then they're l.d., else they're l.i. If there are more than two vectors go to (ii).

(ii) Check to see if the number of vectors m is more than space n (i.e. $\mathbb{R}^n$), if $m > n$ then they're l.d., if not go to (iii).

(iii) Are the number of vectors m the same as space n, i.e. $m = n$? If it is then set up matrix A (where the columns of A are the vectors) and calculate the determinant. If $\det(A) = 0$ then they're l.d., if $\det(A) \neq 0$ then they're l.i. If number of vectors isn't same as space, i.e. $m \neq n$, go to (iv).

(iv) Set up augmented matrix $[A|0]$ then use E.R.O's to determine the rank of $r(A) = r(A|0)$. If $r(A) < m$ then they're l.d., if $r(A) = m$ then they're l.i.