Lecture 11

Euclidean Vector Spaces,

Linear Dependence

& Independence

Euclidean Vector Spaces

Recall the notation of the real number line as $\mathbb{R}$. Following the same notation, we usually denote 2 space (*i.e.* the Cartesian plane) by $\mathbb{R}^2$ and 3 space by $\mathbb{R}^3$. Note that we can extend most of our work on vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ to vectors in general n space, *i.e.* $\mathbb{R}^n$. For $\boldsymbol{x} \in \mathbb{R}^n$, we write

$$\boldsymbol{x} = [x_1, x_2, \ldots, x_n]$$

and the definitions of addition, scalar multiplication, length, the dot product and orthogonality extend directly. In this context, $\mathbb{R}^n$, $n = 1, 2, 3, 4, \ldots$ are collectively known as the *Euclidean Vector Spaces*.

Note: No version of the cross product in $\mathbb{R}^n$, $n \geq 4$. Also, the dot product in $\mathbb{R}^n$ is often called the *Euclidean Inner Product*.

Planes in $\mathbb{R}^n$ for $n \geq 4$ are referred to as *hyperplanes*.

Ex: Given $\boldsymbol{a} = [-2, 1, 0, 2, 3, -1]$ and $\boldsymbol{b} = [4, 2, -1, 2, 0, 1]$, find $3\boldsymbol{a} - \boldsymbol{b}$ and the scalar projection of $\boldsymbol{a}$ on $\boldsymbol{b}$.

Ex: Find the parametric equations of the line in $\mathbb{R}^5$ passing through the point $P(2, -4, 1, 0, -1)$ and is parallel to the line $\boldsymbol{r} = [5, 3, -2, 1, 1] + t[3, 1, 4, -2, 2]$.

Ex: Determine the equation of the plane passing through the point $P(5, 3, -1, 1, 2)$ and is parallel to the plane $4x_1 + x_2 - 2x_3 + 2x_4 - x_5 = -2$.

Vector Subspaces

Consider $I\!R^n$ and let $U \subset I\!R^n$. If U is itself a vector space, we say that it is a *subspace* of $I\!R^n$.

A subset U of $I\!R^n$ is a subspace of $I\!R^n$ if and only if

(a) for any $\boldsymbol{u}, \boldsymbol{v} \in U$, $\boldsymbol{u} + \boldsymbol{v} \in U$, and

(b) for any $\boldsymbol{u} \in U$ and any scalar s, $s\boldsymbol{u} \in U$.

We say U is a subspace of $I\!R^n$ if it is closed under addition and scalar multiplication.

Basically, a subspace must be such that we can not escape from it by adding vectors within it or by multiplying vectors within it by a scalar (where that scalar may be any real number).

Note that *any subspace of* $\mathbb{R}^n$ *must contain the zero vector!*

Ex: Show that the set of vectors in $\mathbb{R}^3$ where the second component is twice the first, and the third component is three times the first (*i.e.*, $[a, 2a, 3a]$) is a subspace of $\mathbb{R}^3$.

Ex: Let U denote all vectors in $\mathbb{R}^3$ of the form $[a, a^2, b]$. Show that U is not a subspace of $\mathbb{R}^3$.

Ex: Let W denote all vectors in $\mathbb{R}^3$ such that their first component is negative. Show that W is not a subspace of $\mathbb{R}^3$.

Solution Space of Homogeneous System

Consider a homogeneous system of m linear equations in n unknowns

$$A\boldsymbol{x} = \mathbf{0}$$

i.e. A is $m \times n$ and $\boldsymbol{x} \in \mathbb{R}^n$. Let

$$V = \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} = \mathbf{0}\}$$

V is a vector subspace of $\mathbb{R}^n$ and therefore a vector space. In this sense, we refer to V as the *null space of the matrix* A.

Linear Combinations

Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\} \subset I\!R^n$. If the vector $\boldsymbol{u}$ can be expressed in the form

$$\boldsymbol{u} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_m\boldsymbol{u}_m$$

for some scalars $c_1, c_2, \ldots, c_m \in I\!R$, we say that $\boldsymbol{u}$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$. Clearly, $\boldsymbol{u}$ is itself a vector in $I\!R^n$.

Ex: Let $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in I\!R^3$. Show that $\boldsymbol{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a linear combination of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$.

Linear Dependence and Independence

Consider $I\!R^n$. A set of vectors $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\}$ in $I\!R^n$ is said to be linearly dependent if there are scalars $c_1, c_2, \ldots, c_m$, not all zero, such that

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_m\boldsymbol{u}_m = \boldsymbol{0}.$$

On the other hand, if the only way this equation can hold is with $c_1 = c_2 = \ldots = c_m = 0$, then the set is called linearly independent.

Note the following:

(i) Notation. We usually write

- linearly dependent as l.d.

- linearly independent as l.i.

- linear combination as l.c.

(ii) Suppose $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$ are l.d. and $c_i \neq 0$ in the previous equation. Then

$$\boldsymbol{u}_i = -\frac{c_1}{c_i}\boldsymbol{u}_1 - \frac{c_2}{c_i}\boldsymbol{u}_2 - \ldots - \frac{c_{i-1}}{c_i}\boldsymbol{u}_{i-1} - \frac{c_{i+1}}{c_i}\boldsymbol{u}_{i+1} - \ldots - \frac{c_m}{c_i}\boldsymbol{u}_m,$$

i.e. $\boldsymbol{u}_i$ is a l.c. of the others, *i.e.* it 'depends' on them.

Testing for l.i. or l.d.

(i) If there only two vectors check to see if they're parallel, i.e. $v_1 = sv_2$. If they are parallel then they're l.d., else they're l.i. If there are more than two vectors go to (ii).

Ex: Decide whether the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is l.i. or l.d., where $\boldsymbol{v}_1 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} 6 \\ -12 \end{bmatrix}$.

(ii) Check to see if the number of vectors m is more than space n (i.e. $\mathbb{R}^n$), if $m > n$ then they're l.d., if not go to (iii).

Ex: Decide whether the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is l.i. or l.d., where $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$ and $\boldsymbol{v}_3 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

(iii) Are the number of vectors m the same as space n, i.e. $m = n$? If it is then set up matrix A (where the columns of A are the vectors) and calculate the determinant. If $\det(A) = 0$ then they're l.d., if $\det(A) \neq 0$ then they're l.i. If number of vectors isn't same as space, i.e. $m \neq n$, go to (iv).

Ex: Decide whether the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is l.i. or l.d., where $\boldsymbol{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 4 \\ -4 \end{bmatrix}$ and $\boldsymbol{v}_3 = \begin{bmatrix} -6 \\ 2 \\ 1 \end{bmatrix}$.

(iv) Set up augmented matrix $[A|0]$ then use E.R.O's to determine the rank of $r(A) = r(A|0)$. If $r(A) < m$ then they're l.d., if $r(A) = m$ then they're l.i.

Ex: Decide whether the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is l.i. or l.d., where $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 0 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 4 \\ 1 \\ -2 \end{bmatrix}$ and $\boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 3 \\ -1 \\ 2 \end{bmatrix}$.