EMTH1019 Linear Algebra & Statistics for Engineers

Tutorial 12 Plane Transformations & Least Squares

SOLUTIONS

1. (i) Here we have

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 9 & 14 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{(6)(14) - (9)(9)} \begin{bmatrix} 14 & -9 \\ -9 & 6 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 14 & -9 \\ -9 & 6 \end{bmatrix}$$

$$\operatorname{pinv}(A) = (A^T A)^{-1} A^T = \frac{1}{3} \begin{bmatrix} 14 & -9 \\ -9 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -4 & 5 & 1 \\ 3 & -3 & 0 \end{bmatrix}$$

The least squares solution is

$$\hat{\boldsymbol{x}} = \operatorname{pinv}(A)\boldsymbol{b} = \frac{1}{3} \begin{bmatrix} -4 & 5 & 1 \\ 3 & -3 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -4 \\ 6 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} \\ 2 \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{x}}_1 \\ \hat{\boldsymbol{x}}_2 \end{bmatrix}$$

Hence the best approximate solution is for $x_1 = -\frac{4}{3}$ and $x_2 = 2$,

(ii) Here we have

$$A = \begin{bmatrix} 2 & -2 \\ 1 & -2 \\ 1 & 1 \\ -2 & 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 2 & 1 & 1 & -2 \\ -2 & -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & -2 \\ 1 & 1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 10 & -9 \\ -9 & 13 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{(10)(13) - (-9)(-9)} \begin{bmatrix} 13 & -(-9) \\ -(-9) & 10 \end{bmatrix} = \frac{1}{49} \begin{bmatrix} 13 & 9 \\ 9 & 10 \end{bmatrix}$$

$$\operatorname{pinv}(A) = (A^T A)^{-1} A^T = \frac{1}{49} \begin{bmatrix} 13 & 9 \\ 9 & 10 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & -2 \\ -2 & -2 & 1 & 2 \end{bmatrix} = \frac{1}{49} \begin{bmatrix} 8 & -5 & 22 & -8 \\ -2 & -11 & 19 & 2 \end{bmatrix}$$

The least squares solution is

$$\hat{\boldsymbol{x}} = \operatorname{pinv}(A)\boldsymbol{b} = \frac{1}{49} \begin{bmatrix} 8 & -5 & 22 & -8 \\ -2 & -11 & 19 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{49} \begin{bmatrix} 14 \\ 21 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} \\ \frac{3}{7} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{x}}_1 \\ \hat{\boldsymbol{x}}_2 \end{bmatrix}$$

Hence the best approximate solution is for $x_1 = \frac{2}{7}$ and $x_2 = \frac{3}{7}$,

2. (i) Begin by forming A^TA and $A^T\boldsymbol{b}$ where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$$

Hence

$$A^TA = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}, \quad A^T\boldsymbol{b} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Now use Gaussian Elimination on the augmented matrix $[A^TA|A^T\boldsymbol{b}]$

$$\left[\begin{array}{cc|c} 6 & 5 & 1 \\ 5 & 6 & -1 \end{array}\right] R_2 = 6R_2 - 5R_1 \sim \left[\begin{array}{cc|c} 6 & 5 & 1 \\ 0 & 11 & -11 \end{array}\right]$$

Row 2: $11x_2 = -11 \Rightarrow x_2 = -1$

Row 1: $6x_1 + 5x_2 = 1 \Rightarrow 6x_1 + 5(-1) = 1 \Rightarrow 6x_1 = 6 \Rightarrow x_1 = 1$
Hence the best approximate solution is for $x_1 = 1$ and $x_2 = -1$.

(ii) Begin by forming A^TA and $A^T\boldsymbol{b}$ where

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

Hence

$$A^TA = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 \\ 2 & 1 & 1 & 1 & 2 \\ 1 & -1 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 0 \\ 4 & 11 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A^T\boldsymbol{b} = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 \\ 2 & 1 & 1 & 1 & 2 \\ 1 & -1 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Now use Gaussian Elimination on the augmented matrix $[A^TA|A^T\boldsymbol{b}]$

$$\left[\begin{array}{ccc|c} 6 & 4 & 0 & 1 \\ 4 & 11 & 0 & 2 \\ 0 & 0 & 4 & 0 \end{array}\right] R_2 = 6R_2 - 4R_1 \sim \left[\begin{array}{ccc|c} 6 & 4 & 0 & 1 \\ 0 & 50 & 0 & 8 \\ 0 & 0 & 4 & 0 \end{array}\right]$$

Row 3: $4x_3 = 0 \Rightarrow x_3 = 0$

Row 2: $50x_2 = 8 \Rightarrow x_2 = \frac{4}{25}$

Row 1: $6x_1 + 4x_2 = 1 \Rightarrow 6x_1 + 4(\frac{4}{25}) = 1 \Rightarrow 6x_1 = \frac{9}{25} \Rightarrow x_1 = \frac{3}{50}$
Hence the best approximate solution is for $x_1 = \frac{3}{50}$, $x_2 = \frac{4}{25}$ and $x_3 = 0$.

3. (i) $[A|\boldsymbol{b}] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \\ 1 & 2 & -1 & 2 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -4 \\ 0 & 1 & -2 & -4 \end{array}\right] R_3 \to 2R_3 + R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -4 \\ 0 & 0 & -4 & -12 \end{array}\right].$$

Row 3: $-4x_3 = -12 \Rightarrow x_3 = 3$

Row 2: $-2x_2 = -4 \Rightarrow x_2 = 2$

Row 1: $x_1 + x_2 + x_3 = 6 \Rightarrow x_1 + 3 + 2 = 6 \Rightarrow x_1 = 1$

Hence the unique solution is for $x_1 = 1$, $x_2 = 2$ and $x_3 = 3$, *i.e.* $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(ii) Begin by forming $A^T A$ and $A^T \boldsymbol{b}$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix}$$

Hence

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 6 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

$$A^T \boldsymbol{b} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 6 \end{bmatrix}$$

Now use Gaussian Elimination on the augmented matrix $[A^T A | A^T \boldsymbol{b}]$

$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 10 \\ 2 & 6 & -2 & 8 \\ 1 & -2 & 3 & 6 \end{array}\right] \begin{array}{l} R_2 \to 3R_2 - 2R_1 \\ R_3 \to 3R_3 - R_1 \end{array} \sim \left[\begin{array}{ccc|c} 3 & 2 & 1 & 10 \\ 0 & 14 & -8 & 4 \\ 0 & -8 & 8 & 8 \end{array}\right] R_3 \to 14R_3 + 8R_2$

$$\sim \left[\begin{array}{ccc|c} 3 & 2 & 1 & 10 \\ 0 & 14 & -8 & 4 \\ 0 & 0 & 48 & 144 \end{array}\right].$$

Row 3: $48x_3 = 144 \Rightarrow x_3 = 3$

Row 2: $14x_2 - 8x_3 = 4 \Rightarrow 14x_2 - 8(3) = 4 \Rightarrow 14x_2 = 28 \Rightarrow x_2 = 2$

Row 1: $3x_1 + 2x_2 + x_3 = 10 \Rightarrow 3x_1 + 2(2) + 3 = 10 \Rightarrow 3x_1 = 3 \Rightarrow x_1 = 1$

Hence the least squares solution is for $x_1 = 1$, $x_2 = 2$ and $x_3 = 3$,

i.e. $\hat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(iii) The unique solution is the same as the lest squares solution, *i.e.* $\boldsymbol{x} = \hat{\boldsymbol{x}}$. This is to be expected when a system of linear equations is consistent and has a unique solution, that is the least squares solution will generate the unique solution.

4. (i) (a) We begin by setting up the matrix A and $\boldsymbol{b}$,

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix}$$

In computing pinv(A) we get

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{(3)(14) - (6)(6)} \begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix}$$

$$\text{pinv}(A) = (A^T A)^{-1} A^T = \frac{1}{6} \begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 8 & 2 & -4 \\ -3 & 0 & 3 \end{bmatrix}$$

The least squares solution is

$$\hat{\boldsymbol{x}} = \text{pinv}(A)\boldsymbol{b} = \frac{1}{6} \begin{bmatrix} 8 & 2 & -4 \\ -3 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -18 \\ 24 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

Hence the least squares line that best fits the data is $y = -3 + 4x$.

(b) Begin by forming $A^T A$ and $A^T \boldsymbol{b}$ where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix}$$

Hence

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}, \quad A^T \boldsymbol{b} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 15 \\ 38 \end{bmatrix}$$

Now use Gaussian Elimination on the augmented matrix $[A^T A | A^T \boldsymbol{b}]$

$$\left[\begin{array}{cc|c} 3 & 6 & 15 \\ 6 & 14 & 38 \end{array}\right] R_2 = R_2 - 2R_1 \quad \sim \left[\begin{array}{cc|c} 3 & 6 & 15 \\ 0 & 2 & 8 \end{array}\right]$$

Row 2: $2a_1 = 8 \Rightarrow a_1 = 4$

Row 1: $3a_0 + 6a_1 = 15 \Rightarrow 3a_0 + 6(4) = 15 \Rightarrow 3a_0 = -9 \Rightarrow a_0 = -3$
Hence the least squares line is $y = -3 + 4x$.

(ii) (a) We begin by setting up the matrix A and $\boldsymbol{b}$,

$$A = \begin{bmatrix} 1 & -3 \\ 1 & -1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 8 \\ 5 \\ 3 \\ 0 \end{bmatrix}$$

In computing pinv(A) we get

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 1 & -1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 20 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{(4)(20)-(0)(0)} \begin{bmatrix} 20 & 0 \\ 0 & 4 \end{bmatrix} = \frac{1}{80} \begin{bmatrix} 20 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\text{pinv}(A) = (A^TA)^{-1}A^T = \frac{1}{80} \begin{bmatrix} 20 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & -1 & 1 & 3 \end{bmatrix} = \frac{1}{80} \begin{bmatrix} 20 & 20 & 20 & 20 \\ -12 & -4 & 4 & 12 \end{bmatrix}$$

The least squares solution is

$$\hat{\boldsymbol{x}} = \text{pinv}(A)\boldsymbol{b} = \frac{1}{80} \begin{bmatrix} 20 & 20 & 20 & 20 \\ -12 & -4 & 4 & 12 \end{bmatrix} \begin{bmatrix} 8 \\ 5 \\ 3 \\ 0 \end{bmatrix} = \frac{1}{80} \begin{bmatrix} 320 \\ -104 \end{bmatrix} = \begin{bmatrix} 4 \\ -1.3 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

Hence the least squares line that best fits the data is $y = 4 - 1.3x$.

(b) Begin by forming A^TA and $A^T\boldsymbol{b}$ where

$$A = \begin{bmatrix} 1 & -3 \\ 1 & -1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 8 \\ 5 \\ 3 \\ 0 \end{bmatrix}$$

Hence

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 1 & -1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 20 \end{bmatrix}$$

$$A^T\boldsymbol{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 5 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 16 \\ -26 \end{bmatrix}$$

Now use Gaussian Elimination on the augmented matrix $[A^TA|A^T\boldsymbol{b}]$

$$\left[\begin{array}{cc|c} 4 & 0 & 16 \\ 0 & 20 & -26 \end{array}\right] \text{Matrix is already in row echelon form}$$

Row 2: $20a_1 = -26 \;\Rightarrow\; a_1 = -\frac{13}{10} = -1.3$

Row 1: $4a_0 = 16 \;\Rightarrow\; a_0 = 4$

Hence the least squares line is $y = 4 - 1.3x$.

(iii) (a) We begin by setting up the matrix A and $\boldsymbol{b}$,

$$A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ -2 \\ -4 \end{bmatrix}$$

In computing pinv(A) we get

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{(5)(10)-(0)(0)} \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix} = \frac{1}{50} \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\text{pinv}(A) = (A^TA)^{-1}A^T = \frac{1}{50} \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{bmatrix} = \frac{1}{50} \begin{bmatrix} 10 & 10 & 10 & 10 & 10 \\ -10 & -5 & 0 & 5 & 10 \end{bmatrix}$$

The least squares solution is

$$\hat{\boldsymbol{x}} = \text{pinv}(A)\boldsymbol{b} = \frac{1}{50} \begin{bmatrix} 10 & 10 & 10 & 10 & 10 \\ -10 & -5 & 0 & 5 & 10 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \\ -2 \\ -4 \end{bmatrix} = \frac{1}{50} \begin{bmatrix} -20 \\ -85 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} \\ -\frac{17}{10} \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

Hence the least squares line that best fits the data is $y = -\frac{2}{5} - \frac{17}{10}x$.

(b) Begin by forming A^TA and $A^T\boldsymbol{b}$ where

$$A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ -2 \\ -4 \end{bmatrix}$$

Hence

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$$

$$A^T\boldsymbol{b} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \\ -2 \\ -4 \end{bmatrix} = \begin{bmatrix} -2 \\ -17 \end{bmatrix}$$

Now use Gaussian Elimination on the augmented matrix $[A^TA|A^T\boldsymbol{b}]$

$$\left[\begin{array}{cc|c} 5 & 0 & -2 \\ 0 & 10 & -17 \end{array}\right] \text{Matrix is already in row echelon form}$$

Row 2: $10a_1 = -17 \ \Rightarrow \ a_1 = -\frac{17}{10}$

Row 1: $5a_0 = -2 \ \Rightarrow \ a_0 = -\frac{2}{5}$

Hence the least squares line is $y = -\frac{2}{5} - \frac{17}{10}x$.

5. Begin by forming A^TA and $A^T\boldsymbol{b}$ where

$$A = \begin{bmatrix} 1 & -3 & 9 \\ 1 & -2 & 4 \\ 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \\ 5 \end{bmatrix}$$

Hence

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -3 & -2 & 0 & 2 & 3 \\ 9 & 4 & 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & -3 & 9 \\ 1 & -2 & 4 \\ 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 26 \\ 0 & 26 & 0 \\ 26 & 0 & 194 \end{bmatrix}$$

$$A^T\boldsymbol{b} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -3 & -2 & 0 & 2 & 3 \\ 9 & 4 & 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 10 \\ 18 \\ 66 \end{bmatrix}$$

We now set up the augmented matrix $[A^TA|A^T\boldsymbol{b}]$ then reduce it into row echelon form

$$\left[\begin{array}{ccc|c} 5 & 0 & 26 & 10 \\ 0 & 26 & 0 & 18 \\ 26 & 0 & 194 & 66 \end{array}\right] \; R_3 = 5R_3 - 26R_1 \quad \sim \left[\begin{array}{ccc|c} 5 & 0 & 26 & 10 \\ 0 & 26 & 0 & 18 \\ 0 & 0 & 294 & 70 \end{array}\right]$$

Row 3: $294a_2 = 70 \;\Rightarrow\; a_2 = \frac{5}{21}$

Row 2: $26a_1 = 18 \;\Rightarrow\; a_1 = \frac{9}{13}$

Row 1: $5a_0 + 26a_2 = 10 \;\Rightarrow\; 5a_0 + 26(\frac{5}{21}) = 10 \;\Rightarrow\; 5a_0 = \frac{80}{21} \;\Rightarrow\; a_0 = \frac{16}{21}$

This means that the least squares approximating quadratic for the data points is $y = \frac{16}{21} + \frac{9}{13}x + \frac{5}{21}x^2$.