EMTH1019 Linear Algebra & Statistics for Engineers

Tutorial 12 Plane Transformations & Least Squares

During this workshop, students will work towards the following learning outcomes:

- determine least squares solutions to inconsistent systems of linear equations.
- calculate a least squares line for given data points.
- determine a best fit quadratic approximation for given data points.

Inconsistent systems

1. Use the pseudoinverse $\text{pinv}(A) = (A^TA)^{-1}A^T$ to find the least squares solution for the following inconsistent systems of linear equations.

(i) $$\begin{aligned} x_1 + 2x_2 &= 3 \\ x_1 + x_2 &= 1 \\ 2x_1 + 3x_2 &= 3 \end{aligned}$$

(ii) $$\begin{aligned} 2x_1 - 2x_2 &= -1 \\ x_1 - 2x_2 &= 0 \\ x_1 + x_2 &= 1 \\ -2x_1 + 2x_2 &= 0 \end{aligned}$$

2. For the following inconsistent linear systems, solve the normal system of equations $A^TA\boldsymbol{x} = A^T\boldsymbol{b}$ using Gaussian Elimination to determine the least squares solution.

(i) $$\begin{aligned} 2x_1 + x_2 &= 2 \\ x_1 + 2x_2 &= 0 \\ x_1 + x_2 &= -3 \end{aligned}$$

(ii) $$\begin{aligned} 2x_2 + x_3 &= 1 \\ x_1 + x_2 - x_3 &= 0 \\ 2x_1 + x_2 &= 1 \\ x_1 + x_2 + x_3 &= -1 \\ 2x_2 - x_3 &= 0 \end{aligned}$$

3. The following system of linear equations is consistent and has a unique solution.

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ x_1 - x_2 + x_3 &= 2 \\ x_1 + 2x_2 - x_3 &= 2 \end{aligned}$$

(i) Solve the system $A\boldsymbol{x} = \boldsymbol{b}$ using Gaussian Elimination to find this unique solution $\boldsymbol{x}$.

(ii) Determine the least square solution $\hat{\boldsymbol{x}}$ to the normal system of equations by using Gaussian Elimination.

(iii) What do you notice when you compare the unique solution $\boldsymbol{x}$ from (i) to the least squares solution $\hat{\boldsymbol{x}}$ from (ii).

Least squares lines

4. For each of the following given sets of data points, find the least squares line $y = a_0 + a_1 x$ by (a) using the pseudoinverse, and (b) solving the normal system using Gaussian Elimination.

   (i) $(1,1),(2,5),(3,9)$

   (ii) $(-3,8),(-1,5),(1,3),(3,0)$

   (iii) $(-2,3),(-1,1),(0,0),(1,-2),(2,-4)$

Quadratic approximations

5. Find a quadratic least squares approximating polynomial $y = a_0 + a_1 x + a_2 x^2$ for the data points $(-3,1)$, $(-2,0)$, $(0,1)$, $(2,3)$, $(3,5)$.