Determinants

Motivation: Determinants have a variety of uses such as deciding whether a matrix is invertible or not, calculating the inverse of a matrix and solving linear systems. They are also key to understanding the important concept of eigenvalues and eigenvectors used in many engineering applications.

Outcomes In today's lecture we will learn how to:

- Calculate determinants of square matrices of any size.
- Evaluate determinants of larger matrices by first applying appropriate elementary row or column operations.
- Identify the relationship between the determinant of a matrix and its invertibility.
- Calculate the inverse of a 2×2 matrix.
- Solve a system of equations using Cramer's rule.
- Calculate the cross product and scalar triple product using a determinant.

Contents

- Introduction to determinants.
- Submatrices, minors, cofactors and cofactor expansions to calculate determinants.
- Rules for calculating determinants and examples.
- Relation of determinants to invertibility.
- Cramer's rule for solving systems of equations.

Exercises

1. Calculate the determinants of the following matrices.

(a) $A = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$ (b) $B = \begin{bmatrix} -2 & 1 \\ 3 & -2 \end{bmatrix}$ (c) $C = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 5 \\ -6 & 3 & 1 \end{bmatrix}$

(d) $D = \begin{bmatrix} 2 & -10 & 11 \\ 5 & 3 & -4 \\ 7 & 9 & 12 \end{bmatrix}$ (e) $E = \begin{bmatrix} -12 & 9 & -5 \\ 3 & 9 & 1 \\ -7 & 2 & -2 \end{bmatrix}$ (f) $F = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 0 & 6 \\ 1 & 5 & 3 \end{bmatrix}$

(g) $G = \begin{bmatrix} 6 & 7 & 1 \\ 1 & 3 & 2 \\ 0 & 1 & 5 \end{bmatrix}$ (h) $H = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 3 & 7 \\ 0 & 2 & 9 \end{bmatrix}$ (i) $I = \begin{bmatrix} 9 & 5 & 1 \\ 13 & 0 & 2 \\ 11 & 0 & 3 \end{bmatrix}$

2. Exercise 11(b) (p. 584): 2, 3(a).

3. Exercise 11(c) (p. 597): 4.

4. In (a) and (b), use elementary row or column operations to help you calculate the given determinant.

$$\text{(a)} \begin{vmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{vmatrix} \qquad \text{(b)} \begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix}$$

5. Given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$, find

$$\text{(a)} \begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} \qquad \text{(b)} \begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix}$$

6. By inspection, explain why $\begin{vmatrix} -2 & 8 & 1 & 4 \\ 3 & 2 & 5 & 1 \\ 1 & 10 & 6 & 5 \\ 4 & -6 & 4 & -3 \end{vmatrix} = 0.$

7. Given that A is 3×3 and $\det(A) = -2$, find (a) $\det(3A)$ and (b) $\det(A^{-1})$.

8. Exercise 11(b) (p. 584): 4.

9. Find the inverse of the following matrices, if the inverse exists.

(a) $\begin{bmatrix} 3 & 4 \\ 7 & 9 \end{bmatrix}$ (b) $\begin{bmatrix} -7 & 4 \\ 8 & -5 \end{bmatrix}$ (c) $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ 5 & 4 \end{bmatrix}$

10. Use Cramer's rule to solve the following systems of linear equations.

(a) $\begin{aligned} 6x_1 + 12x_2 &= 33 \\ 4x_1 + 7x_2 &= 20 \end{aligned}$ (b) $\begin{aligned} \tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 &= 1 \\ \tfrac{1}{4}x_1 - \tfrac{1}{6}x_2 &= -\tfrac{3}{2} \end{aligned}$

(c) $\begin{aligned} 2x_1 - x_2 &= 5 \\ 5x_1 + 3x_3 &= 19 \\ 4x_2 + 7x_3 &= 17 \end{aligned}$

11. Use Cramer's rule to solve the following system for x_2 without solving for the remaining variables.

$$\begin{aligned} 4x_1 + x_2 + x_3 + x_4 &= 6 \\ 3x_1 + 7x_2 - x_3 + x_4 &= 1 \\ 7x_1 + 3x_2 - 5x_3 + 8x_4 &= -3 \\ x_1 + x_2 + x_3 + 2x_4 &= 3 \end{aligned}$$

12. For the following pairs of vectors, determine $\boldsymbol{a} \times \boldsymbol{b}$ by taking the determinant of an appropriate matrix.

    (a) $\boldsymbol{a} = [2, 1, 1]$, $\boldsymbol{b} = [2, -1, 2]$ (b) $\boldsymbol{a} = \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$

These exercises should take around 2 hours to complete.

(Answers: 1.(a) $\det(A) = 2$ (b) $\det(B) = 1$ (c) $\det(C) = -117$ (d) $\det(D) = 1288$ (e) $\det(E) = -114$ (f) $\det(F) = -42$ (g) $\det(G) = 44$ (h) $\det(H) = 13$ (i) $\det(I) = -85$; 4.(a) 39 (b) 6; 5.(a) -6 (b) -6; 7.(a) -54 (b) $-\frac{1}{2}$; 9.(a) $\begin{bmatrix} -9 & 4 \\ 7 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} -\frac{5}{3} & -\frac{4}{3} \\ -\frac{8}{3} & -\frac{7}{3} \end{bmatrix}$ (c) $\begin{bmatrix} 12 & -1 \\ -15 & \frac{3}{2} \end{bmatrix}$; 10.(a) $x_1 = \frac{3}{2}$, $x_2 = 2$ (b) $x_1 = -2$, $x_2 = 6$ (c) $x_1 = 2$, $x_2 = -1$, $x_3 = 3$; 11. $x_2 = 0$; 12.(a) [3,-2,-4] (b) [4,1,-1])

Determinants

With each square (*i.e.* $n \times n$) matrix A, we can associate a number called the determinant, denoted by $\det(A)$ or $|A|$. Before showing how to calculate $\det(A)$, we need to grasp some other concepts.

The matrix we obtain after deleting one or more columns or rows from a matrix A is called a submatrix of A. For example, $\begin{bmatrix} 2 & 1 \\ 3 & 7 \end{bmatrix}$ is a submatrix of each of the matrices

$$\begin{bmatrix} 2 & 1 & 6 \\ 3 & 7 & 9 \\ -1 & 5 & 9 \end{bmatrix}, \quad \begin{bmatrix} 2 & 5 & 1 \\ 3 & -1 & 7 \\ -1 & 5 & 9 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 & 1 \\ 8 & 5 & 1 \\ 3 & 7 & 7 \end{bmatrix}.$$

Consider an $n \times n$ matrix $A = [a_{ij}]$. The minor, M_{ij}, associated with the element a_{ij} is the determinant of the $(n-1) \times (n-1)$ submatrix obtained from A by deleting row i and column j.

$$e.g. \quad A = \begin{bmatrix} 1 & 5 & 3 \\ 6 & 2 & 4 \\ 0 & 1 & 0 \end{bmatrix} \quad \to \quad M_{21} = \det\left(\begin{bmatrix} 5 & 3 \\ 1 & 0 \end{bmatrix}\right) = \begin{vmatrix} 5 & 3 \\ 1 & 0 \end{vmatrix},$$

where we have yet to figure out how to calculate the determinant of this submatrix.

Finally, the cofactor associated with entry a_{ij} of A is given by

$$C_{ij} = (-1)^{i+j} M_{ij},$$

i.e. it's just the minor M_{ij} multiplied by either 1 or -1.

e.g. for the matrix A above, $C_{21} = (-1)^3 M_{21} = -\begin{vmatrix} 5 & 3 \\ 1 & 0 \end{vmatrix}$.

With these ideas mastered, we can now start to calculate determinants. Consider the $n \times n$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

For $n = 1$, *i.e.* $A = [a_{11}]$, $\det(A) = a_{11}$.

For $n = 2$, *i.e.* $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$.

For $n > 2$, $\det(A) = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n} = \sum_{i=1}^{n} a_{1i}C_{1i}$. This is called the cofactor expansion along the 1st row of A.

Ex: Find $\det(A)$ if $A = \begin{bmatrix} 1 & 2 \\ 7 & 5 \end{bmatrix}$.

Soln: $\begin{vmatrix} 1 & 2 \\ 7 & 5 \end{vmatrix} = (1)(5) - (2)(7) = -9$.

Ex: Find $|A|$ if $A = \begin{bmatrix} -1 & 4 & 5 \\ 3 & 6 & 2 \\ 4 & -3 & 0 \end{bmatrix}$.

Soln: First notice that it's a good idea to write the signs of the cofactors adjacent to the respective entries in the matrix, *i.e.* $\begin{bmatrix} -1^{+} & 4^{-} & 5^{+} \\ 3 & 6 & 2 \\ 4 & -3 & 0 \end{bmatrix}$. We have

$$\begin{aligned} \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= (-1)C_{11} + (4)C_{12} + (5)C_{13} \\ &= (-1)M_{11} + (4)(-1)M_{12} + (5)M_{13} \\ &= -\begin{vmatrix} 6 & 2 \\ -3 & 0 \end{vmatrix} - 4\begin{vmatrix} 3 & 2 \\ 4 & 0 \end{vmatrix} + 5\begin{vmatrix} 3 & 6 \\ 4 & -3 \end{vmatrix} \\ &= -((6)(0) - (2)(-3)) - 4((3)(0) - (4)(2)) + 5((3)(-3) - (6)(4)) \\ &= -6 + 32 - 165 \\ &= -139 \end{aligned}$$

Ex: Find $\det A$, if $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 2 & 4 & 5 \\ 3 & 5 & 6 & 2 \\ 4 & 8 & -3 & 0 \end{bmatrix}$.

Soln: $A = \begin{bmatrix} 0^{+} & 1^{-} & 0^{+} & 0^{-} \\ -1 & 2 & 4 & 5 \\ 3 & 5 & 6 & 2 \\ 4 & 8 & -3 & 0 \end{bmatrix}$

$$\det(A) = +(0)\begin{vmatrix} 2 & 4 & 5 \\ 5 & 6 & 2 \\ 8 & -3 & 0 \end{vmatrix} - (1)\begin{vmatrix} -1 & 4 & 5 \\ 3 & 6 & 2 \\ 4 & -3 & 0 \end{vmatrix} + (0)\begin{vmatrix} -1 & 2 & 5 \\ 3 & 5 & 2 \\ 4 & 8 & 0 \end{vmatrix} - (0)\begin{vmatrix} -1 & 2 & 4 \\ 3 & 5 & 6 \\ 4 & 8 & -3 \end{vmatrix} = -(-139) = 139,$$

using the result from the previous example.

As the above example show, it helps to have a lot of zeros in your matrix when calculating a determinant. They happened to occur in the first row this time, but what if they don't?

It turns out that we can calculate the determinant of a matrix A by a cofactor expansion along any row or column of A, *i.e.*

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} \qquad i\text{−th row}$$

$$\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij} \qquad j\text{−th column}$$

Ex: Evaluate $|A| = \begin{vmatrix} 1^+ & 0^- & 2^+ \\ 2^- & 1^+ & 1^- \\ 0^+ & 3^- & 0^+ \end{vmatrix}$.

Soln: Cofactor expansion along the first row:

$$\det(A) = (1)\begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} - (0)\begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix} + (2)\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = -3 - 0 + 2(6) = 9.$$

Cofactor expansion along the third column:

$$\det(A) = (2)\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} - (1)\begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} + (0)\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 2(6) - 3 + 0 = 9.$$

Cofactor expansion along the third row:

$$\det(A) = (0)\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} - (3)\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + (0)\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 0 - 3(1 - 4) + 0 = 9.$$

This confirms the statement we made above. Note that expansion along the third row was the easiest, since it contained the most number of zeros. More generally, we choose the row or column in a matrix which has the most number of zeros.

Ex: Find $\det A$, if $A = \begin{bmatrix} 2^+ & 1^- & 0^+ & -1 \\ -5 & 0 & 4^- & 2 \\ 1 & -3 & 0^+ & 4 \\ 0 & -1 & 0^- & -2 \end{bmatrix}$.

Soln: Taking the cofactor expansion along the third column, we have

$$\det(A) = -(4)\begin{vmatrix} 2 & 1 & -1 \\ 1 & -3 & 4 \\ 0 & -1 & -2 \end{vmatrix}$$

$$
\begin{aligned}
&= -4\left((0)\begin{vmatrix} 1 & -1 \\ -3 & 4 \end{vmatrix} - (-1)\begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} + (-2)\begin{vmatrix} 2 & 1 \\ 1 & -3 \end{vmatrix}\right) \\
&= -4((8+1) - 2(-6-1)) \\
&= -4(9+14) \\
&= -92
\end{aligned}
$$

Rules for Calculating Determinants

There are a number of simple rules with regards to the calculation and application of determinants. The first 9 help us to calculate determinants more efficiently. Rule 10 helps us to understand the meaning of the determinant. Consider an $n \times n$ matrix A:

1. If A has a row or a column of zeros, $\det(A) = 0$.

   e.g. $\begin{vmatrix} 5 & 0 & 6 & -1 \\ 0 & 0 & 8 & -2 \\ 1 & 0 & -3 & 4 \\ 3 & 0 & 0 & 1 \end{vmatrix} = 0$ and $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 3 & -1 \end{vmatrix} = 0.$

2. Multiply any one row of A by a scalar k to obtain A'. Then

$$\det(A') = k\det(A)$$

   (makes sense when you consider taking a cofactor expansion along that row) *e.g.* $\begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} = 5$ and $\begin{vmatrix} 3 & 9 \\ -1 & 2 \end{vmatrix} = 15$, as expected

3. Interchange any two rows in A to obtain A'. Then

$$\det(A') = -\det(A)$$

   e.g. $\begin{vmatrix} 13 & 1 \\ 2 & -1 \end{vmatrix} = -15$ and $\begin{vmatrix} 2 & -1 \\ 13 & 1 \end{vmatrix} = 15$, as expected.

4. Add a multiple of one row in A to another to obtain A'. Then

$$\det(A') = \det(A)$$

   e.g. let $A = \begin{bmatrix} 1 & 3 \\ -3 & 5 \end{bmatrix}$, then $\det(A) = 14$. Consider $\begin{bmatrix} 1 & 3 \\ -3 & 5 \end{bmatrix}$ $\mathrm{R}_2 \to \mathrm{R}_2 + 3\mathrm{R}_1$ $\sim \begin{bmatrix} 1 & 3 \\ 0 & 14 \end{bmatrix} = A'$ and $\det(A') = 14$, as expected.

Note that Rules 2, 3 and 4 deal in particular with how elementary row operation affect the determinant of a matrix. Rule 4 is particularly useful as it allows us to induce additional zeros in a matrix to simplify the determinant calculation.

Ex: Evaluate $\begin{vmatrix} 2 & -1 & 3 \\ 1 & 2 & 6 \\ -3 & 5 & 1 \end{vmatrix}$.

Soln: $\begin{vmatrix} 2 & -1 & 3 \\ 1 & 2 & 6 \\ -3 & 5 & 1 \end{vmatrix} \begin{matrix} R_1 \to R_1 - 2R_2 \\ \\ R_3 \to R_3 + 3R_2 \end{matrix} = \begin{vmatrix} 0^{+} & -5 & -9 \\ 1^{-} & 2 & 6 \\ 0^{+} & 11 & 19 \end{vmatrix} = -\begin{vmatrix} -5 & -9 \\ 11 & 19 \end{vmatrix} = -4.$

5. If one row in A is a scalar multiple of another, then $\det(A) = 0$. This makes sense, since we can then apply an e.r.o. to get a zero row before applying Rule 1. *e.g.* $\begin{vmatrix} 3 & -1 & 4 & 7 \\ 2 & 2 & 3 & -1 \\ -3 & 1 & -4 & -7 \\ 1 & 6 & 2 & 1 \end{vmatrix} = 0$, since $R_3 = -R_1$.

6. $\det(A^{\top}) = \det(A)$. This rule basically allows us to apply Rules 1-5 to columns as well as rows.

Ex: $\begin{vmatrix} 2 & 1 & 0 & -1 \\ -5 & 0 & 4 & 2 \\ 1 & -3 & 0 & 4 \\ 0 & 0 & -1 & -2 \end{vmatrix} R_3 \to R_3 + 3R_1 = \begin{vmatrix} 2^{+} & 1^{-} & 0 & -1 \\ -5 & 0^{+} & 4 & 2 \\ 7 & 0^{-} & 0 & 1 \\ 0 & 0^{+} & 1 & -2 \end{vmatrix} = -\begin{vmatrix} -5 & 4 & 2 \\ 7 & 0 & 1 \\ 0 & -1 & -2 \end{vmatrix} C_3 \to C_3 - 2C_2$

$$= -\begin{vmatrix} -5^{+} & 4 & -6 \\ 7^{-} & 0 & 1 \\ 0^{+} & -1^{-} & 0^{+} \end{vmatrix} = -(-(-1))\begin{vmatrix} -5 & -6 \\ 7 & 1 \end{vmatrix} = -(-5+42) = -37.$$

7. $\det(k\,A) = k^n \det(A)$ (This is simply an expanded version of Rule 2.)

8. $\det(AB) = \det(A)\det(B)$ This is not an obvious rule, but a very important one which is used frequently in practice. *e.g.* let $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$. Then $\det(A) = 2 - 12 = -10$, $\det(B) = -1 - 6 = -7$, $AB = \begin{bmatrix} 7 & 7 \\ -1 & 9 \end{bmatrix}$ and $\det(AB) = 63 - (-7) = 70 = \det(A)\det(B)$, as expected.

9. An upper triangular matrix is square with all entries below the main diagonal equal to zero. A lower triangular matrix is square with all entries above the main diagonal

equal to zero. Finally, the determinant of a lower triangular or upper triangular matrix is the product of all the diagonal elements. For example,

$$\text{(i)} \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 2 \end{vmatrix} = (1)\begin{vmatrix} 4 & 2 \\ 0 & 2 \end{vmatrix} = (1)((4)(2) - (0)(2)) = (1)(4)(2) = 8.$$

$$\text{(ii)} \begin{vmatrix} 2 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 3 & 0 & 5 & 0 \\ -1 & 1 & 2 & 4 \end{vmatrix} = (2)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 1 & 2 & 4 \end{vmatrix} = (2)(1)\begin{vmatrix} 5 & 0 \\ 2 & 4 \end{vmatrix} = (2)(1)(5)(4) = 40.$$

In particular, note that the identity matrix is both upper and lower triangular, so $\det(I) = 1^n = 1$ for any order n.

10. A square matrix A is invertible if and only if $\det(A) \neq 0$. This is the most important rule of determinants.

From Rule 10, it follows that:

(i) $\det(A) = 0$ shows that A is singular, *i.e.* A^{-1} does not exist. Compare this with the case of a scalar a. If $|a| = 0$ (*i.e.* if $a = 0$), then $a^{-1} = \frac{1}{a}$ also does not exist.

$\det(A) \neq 0$ shows that A is non-singular, *i.e.* A^{-1} does exist.

(ii) Note that if A is non-singular, then $AA^{-1} = I$. Hence,

$$\begin{array}{lrcl} i.e. & \det(AA^{-1}) & = & \det(I) \\ i.e. & \det(A)\det(A^{-1}) & = & 1 \\ i.e. & \det(A^{-1}) & = & \dfrac{1}{\det(A)} \end{array}$$

Calculating the Inverse of a 2×2 Matrix

Consider a general 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. To find the inverse A^{-1} of the matrix we can use the Gauss Jordan method.

$$[A|I] = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \quad \mathrm{R}_2 \to a\mathrm{R}_2 - c\mathrm{R}_1$$

$$\sim \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & ad - bc & -c & a \end{array}\right] \quad \mathrm{R}_2 \to \mathrm{R}_2 \div (ad - bc)$$

$$\sim \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \quad \mathrm{R}_1 \to \mathrm{R}_1 - b\mathrm{R}_2$$

$$\sim \left[\begin{array}{cc|cc} a & 0 & 1+\frac{bc}{ad-bc} & \frac{-ab}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} a & 0 & \frac{ad-bc+bc}{ad-bc} & \frac{-ab}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} a & 0 & \frac{ad}{ad-bc} & \frac{-ab}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \quad \mathrm{R}_1 \to \mathrm{R}_1 \div a$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] = [I|A^{-1}],$$

i.e. $A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$.

$$\boxed{A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

This is quite a useful formula to keep in mind.

$$e.g. \quad \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}^{-1} = \frac{1}{(2)(-2)-(3)(-1)}\begin{bmatrix} -2 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$$

<u>Cramer's Rule</u>

This is another way to solve systems of linear equations. Compared to Gaussian elimination, it is not very efficient because of the high computational cost of calculating determinants. However, many engineering texts refer to it and it can be useful when we try to find the value of just one variable in a solution without determining the others (see exercises).

Consider a system of n linear equations in n unknowns:

$$A\boldsymbol{x} = \boldsymbol{b},$$

where $\det(A) \neq 0$.
Let A_i be the matrix obtained from A by replacing the i-th column with $\boldsymbol{b}$. Then the solution of the system is given by

$$\boxed{x_i = \frac{\det(A_i)}{\det(A)}, \qquad i = 1, 2, \ldots, n}$$

Ex: Solve the following system of equations by Cramer's Rule:

$$\begin{array}{rcrcl} 2x_1 & + & x_2 & = & 1 \\ 3x_1 & - & 2x_2 & = & 2 \end{array}$$

Soln: $A = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$, $\det(A) = -4 - 3 = -7$.
$A_1 = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$, $\det(A_1) = -2 - 2 = -4$.
$A_2 = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, $\det(A_2) = 4 - 3 = 1$.
Hence, $x_1 = \dfrac{\det(A_1)}{\det(A)} = \dfrac{-4}{-7} = \dfrac{4}{7}$ and $x_2 = \dfrac{\det(A_2)}{\det(A)} = \dfrac{1}{-7} = -\dfrac{1}{7}$.

Cross Product using a determinant

As demonstrated in an earlier lecture the cross product of two vectors in 3 space results in an orthogonal vector also in 3 space. Two methods were presented that allowed us to calculate the cross product but a third method which uses a determinant was suggested but not explored, we will now look at this method.

For two vectors $\boldsymbol{a} = [a_1, a_2, a_3]$ and $\boldsymbol{b} = [b_1, b_2, b_3]$ their cross product $\boldsymbol{a} \times \boldsymbol{b}$ can be expressed as a determinant,

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

We can then do a cofactor expansion along the first row to get,

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} &= \boldsymbol{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= (a_2b_3 - a_3b_2)\boldsymbol{i} - (a_1b_3 - a_3b_1)\boldsymbol{j} + (a_1b_2 - a_2b_1)\boldsymbol{k} \\ &= [a_2b_3 - a_3b_2, a_1b_3 - a_3b_1, a_1b_2 - a_2b_1] \end{aligned}$$

Which gives us our known formula for the cross product.

Ex: Find $\boldsymbol{a} \times \boldsymbol{b}$, where $\boldsymbol{a} = [1, 2, 3]$ and $\boldsymbol{b} = [4, 5, 6]$.

Soln:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

Doing a cofactor expansion along the first row we get,

$$\begin{aligned}
\boldsymbol{a} \times \boldsymbol{b} &= \boldsymbol{i}\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \\
&= (12 - 15)\boldsymbol{i} - (6 - 12)\boldsymbol{j} + (5 - 8)\boldsymbol{k} \\
&= (-3)\boldsymbol{i} - (-6)\boldsymbol{j} + (-3)\boldsymbol{k} \\
&= [-3, 6, -3]
\end{aligned}$$

<u>Scalar triple product using a determinant</u>

The scalar triple product can also be calculated using the determinant of an appropriate matrix. Given three vectors $\boldsymbol{a} = [a_1, a_2, a_3]$, $\boldsymbol{b} = [b_1, b_2, b_3]$ and $\boldsymbol{c} = [c_1, c_2, c_3]$ the scalar triple product $\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c})$ can be expressed as the following determinant,

$$\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

We then do a cofactor expansion along the first row which results in,

$$\begin{aligned}
\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) &= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\
&= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)
\end{aligned}$$

<u>Ex:</u> Calculate the volume of the parallelepiped with edges given by the vectors $\boldsymbol{a} = [3, 4, -4]$, $\boldsymbol{b} = [1, -2, -1]$ and $\boldsymbol{c} = [5, -3, 2]$.

<u>Soln:</u>

$$\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} 3 & 4 & -4 \\ 1 & -2 & -1 \\ 5 & -3 & 2 \end{vmatrix}$$

Doing a cofactor expansion along the first row we get,

$$\begin{aligned}
\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) &= 3\begin{vmatrix} -2 & -1 \\ -3 & 2 \end{vmatrix} - 4\begin{vmatrix} 1 & -1 \\ 5 & 2 \end{vmatrix} + (-4)\begin{vmatrix} 1 & -2 \\ 5 & -3 \end{vmatrix} \\
&= 3(-4 - 3) - 4(2 - (-5)) - 4(-3 - (-10)) \\
&= -21 - 28 - 28 \\
&= -77
\end{aligned}$$

Since the volume of the parallelepiped is given by the absolute value of the scalar triple product, we have $|\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c})| = |-77| = 77$.