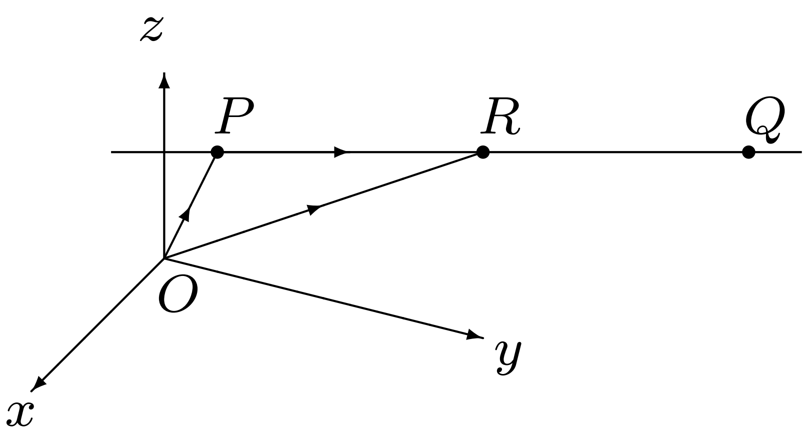
Lecture 10

Lines and Planes in 3 Space

Equations of Lines



Want to find equation of the line passing through $P(x_0, y_0, z_0)$ and $Q(x_1, y_1, z_1)$ in $\mathbb{R}^3$. The direction is given by

$$\overrightarrow{PQ} = [x_1 - x_0, y_1 - y_0, z_1 - z_0].$$

Putting $a_1 = x_1 - x_0$, $a_2 = y_1 - y_0$ and $a_3 = z_1 - z_0$, then $\boldsymbol{a} = [a_1, a_2, a_3]$ is a direction vector of the line.

Let $R(x, y, z)$ be an arbitrary point on the line. Then $\overrightarrow{PR} \parallel \overrightarrow{PQ}$, *i.e.*

$$\overrightarrow{PR} = t\overrightarrow{PQ} = t\boldsymbol{a}$$

for some scalar t.

Clearly, $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR}$, *i.e.*

$$[x, y, z] = [x_0, y_0, z_0] + t[a_1, a_2, a_3]$$

Putting $\boldsymbol{r} = [x, y, z]$ and $\boldsymbol{r}_0 = [x_0, y_0, z_0]$, we have

$$\boxed{\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{a},}$$

which is called the vector equation of a line. In component form,

$$\boxed{\begin{array}{rcl} x & = & x_0 + ta_1 \\ y & = & y_0 + ta_2 \\ z & = & z_0 + ta_3, \end{array}}$$

called the parametric equations of a line. As t goes from $-\infty$ to $+\infty$, these equations generate all points on the line.

Finally, eliminating t, we get

$$\boxed{\frac{x-x_0}{a_1}=\frac{y-y_0}{a_2}=\frac{z-z_0}{a_3}}$$

known as the cartesian equations of a line.

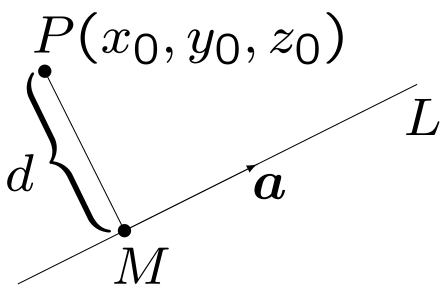
Can immediately identify (x_0, y_0, z_0) and $\boldsymbol{a}$.

Ex: Find the equation of the line passing through $P(2, 0, -3)$ and $Q(-1, 4, 2)$.

Finally note that points A, B and C are co-linear if $\overrightarrow{AB} \parallel \overrightarrow{BC}$, *i.e.* $\overrightarrow{AB} = m\overrightarrow{BC}$ for some scalar m.

Distance from Point to a Line

Want to find the shortest (*i.e.* perpendicular) distance between $P(x_0, y_0, z_0)$ and the line L given by $\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{a}t$.



Let M be the point on L which is closest to P. Then,

$$\boxed{d = ||\overrightarrow{PM}||}$$

where $\overrightarrow{PM} \perp L$ (and therefore $\perp \boldsymbol{a}$), so that $\overrightarrow{PM}.\boldsymbol{a} = 0$.

Ex: Find the distance between $P(0, 1, 2)$ and the line

$$L \begin{cases} x &= 2 + t \\ y &= 6 - 2t \\ z &= 1 - 2t \end{cases}$$

Skew Lines

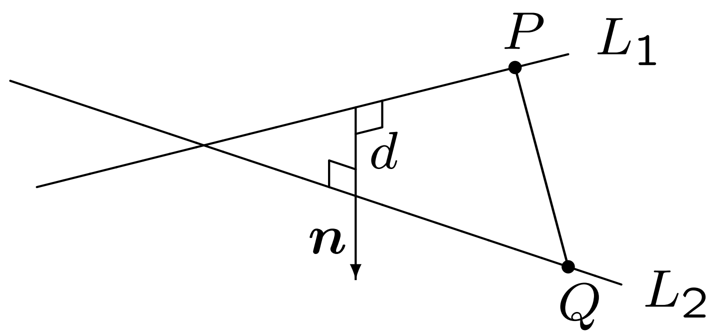
Consider L_1 and L_2 with direction vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, respectively. If $L_1 \parallel L_2$, then $\boldsymbol{a}_1 = m\boldsymbol{a}_2$ for some scalar m. If they are not parallel, they may or may not intersect. Skew lines are those which are not parallel and do not intersect.

Ex: Do the lines

$$L_1 \begin{cases} x &= 2t \\ y &= 3 + t \\ z &= 1 + 2t \end{cases} \qquad L_2 \begin{cases} x &= 4 + \tau \\ y &= -2 - 3\tau \\ z &= 3 + 2\tau \end{cases}$$

intersect?

Finally, we want to find the shortest distance between two skew lines L_1 and L_2 (with directions $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, respectively).



Clearly, $\boldsymbol{n} = \boldsymbol{a}_1 \times \boldsymbol{a}_2$ is perpendicular to both lines. Also, the line joining the two closest points on L_1 and L_2 is parallel to $\boldsymbol{n}$. Let P be any point on L_1 and Q any point on L_2. Then

$$\boxed{d = |\overrightarrow{PQ}.\hat{\boldsymbol{n}}|, \qquad \text{where } \hat{\boldsymbol{n}} = \frac{\boldsymbol{n}}{||\boldsymbol{n}||}}$$

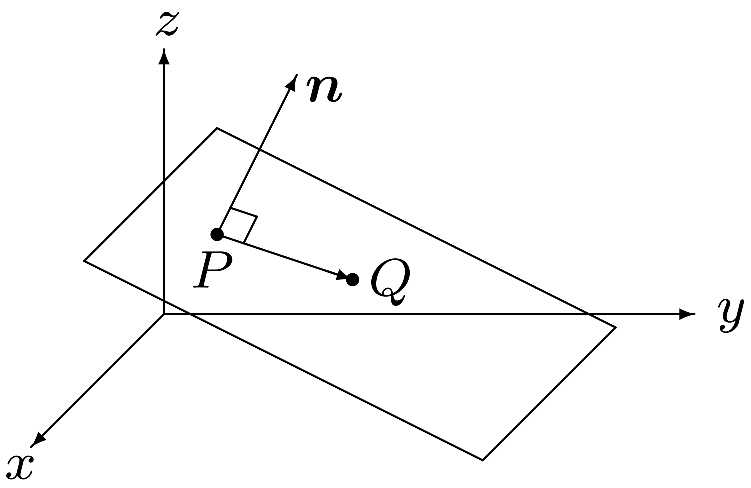
(*i.e.* simply the scalar projection of $\overrightarrow{PQ}$ on $\boldsymbol{n}$)

Ex: Find the closest distance between

$$L_1 \left\{ \begin{array}{rcl} x & = & 2t \\ y & = & 3+t \\ z & = & 1+2t \end{array} \right. \qquad L_2 \left\{ \begin{array}{rcl} x & = & 4+\tau \\ y & = & -2-3\tau \\ z & = & 3+2\tau \end{array} \right.$$

Equation of a Plane in 3-Space

Consider a plane containing $P(x_0, y_0, z_0)$. Let $\boldsymbol{n} = [a, b, c]$ be any vector perpendicular to the plane.



Let $Q(x, y, z)$ be any point on the plane. Then $\overrightarrow{PQ} \perp \boldsymbol{n}$ and

$$\boldsymbol{n}.\overrightarrow{PQ} = 0$$

i.e. $$[a, b, c].[x - x_0, y - y_0, z - z_0] = 0$$

i.e. $$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

This is the *equation of a plane perpendicular to* $\boldsymbol{n} = [a, b, c]$ *and containing the point*

$P(x_0, y_0, z_0)$. In this sense, $\boldsymbol{n}$ is called the *normal of the plane*. Can identify both a point and a normal vector...

We can easily re-arrange the above equation into $ax + by + cz = ax_0 + by_0 + cz_0$ or

$$\boxed{ax + by + cz = d}$$

where d is a constant. This form is known as the the *general equation of a plane*. Can still identify $\boldsymbol{n} = [a, b, c]$ ...

Ex: Find the equation of the plane through the point $A(4, 1, -2)$ and is perpendicular to the line: $\dfrac{x+1}{2} = \dfrac{y}{3} = \dfrac{z-1}{-5}$.

Ex: Find the equation of the plane passing through $A(1, 1, -2)$, $B(3, -1, 0)$ and $C(2, 1, 2)$.

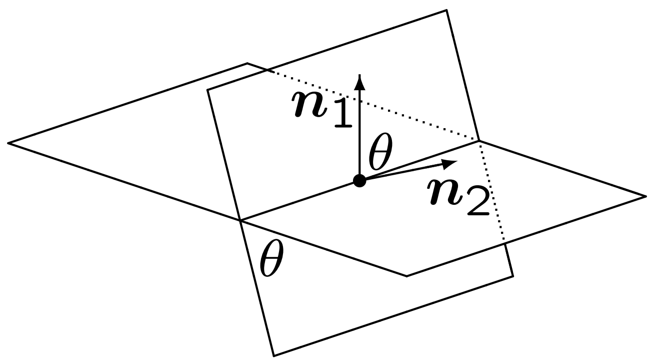
Note the following:

(i) Two planes are parallel if an only if they have parallel normal directions (*i.e.* $\boldsymbol{n}_1 = m\boldsymbol{n}_2$ for some scalar m).

(ii) A line is parallel to a plane if its direction is perpendicular to the normal of the plane.

Ex: Determine if the planes $2x - 3y + z = 1$ and $x + 4y + 10z = 2$ are parallel, perpendicular or neither.

Consider two planes which are not parallel.



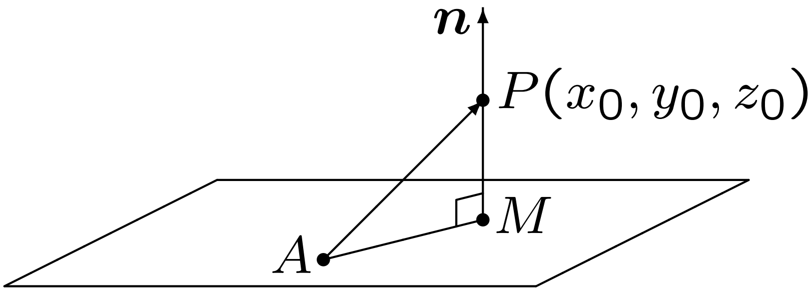
Note that the angle between the planes is the same as the angle between their respective normal vectors. Hence the angle between two planes can be determined by working out the angle between the normal vectors, i.e.,

$$\boxed{\theta = \cos^{-1}\left(\frac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{||\boldsymbol{n}_1||\,||\boldsymbol{n}_2||}\right)}$$

<u>Ex:</u> Find the angle between the planes

$$x + 2y - 2z = 5 \quad \text{and} \quad 6x - 3y + 2z = 8.$$

Distance from a Point to a Plane



We want to find the distance between a plane $ax + by + cz = d$ and a point $P(x_0, y_0, z_0)$. Let A be any point on the plane and let $\boldsymbol{n} = [a, b, c]$ be the normal vector to the plane. Then the distance is easily found as the absolute value of the scalar projection of $\overrightarrow{AP}$ on $\boldsymbol{n}$, *i.e.*

$$\boxed{d = |\overrightarrow{AP}.\hat{\boldsymbol{n}}|, \qquad \hat{\boldsymbol{n}} = \frac{\boldsymbol{n}}{||\boldsymbol{n}||}}$$

Ex: Find the distance of $P(-2, 3, -5)$ from the plane $2x + y + 4z = 6$.