Lecture 8

More on Linear Systems

& Inverses

Homogeneous Systems

Homogeneous systems are simply linear systems where the right hand side of each of the equations is equal to zero. In general,

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n & = & 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n & = & 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n & = & 0 \\ & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn} & = & 0 \end{array}$$

where the variables are x_1, x_2, ..., x_n and the coefficients a_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$ are given. In matrix form,

$$A\boldsymbol{x} = \mathbf{0}$$

Note that $\boldsymbol{x} = \mathbf{0}$ is always a solution (we call this the trivial solution), *i.e.* a homogeneous system is always consistent. We only need to distinguish between two cases:

(i) If $\mathrm{r}(A) = n$, the trivial solution is the only solution.

(ii) If $\mathrm{r}(A) < n$, we get infinitely many solutions. Amongst these is the trivial (*i.e.* zero) solution as well as an infinite number of non trivial (*i.e.* nonzero) solutions.

In the case of infinitely many solutions, use parameters.

Ex: Solve the following homogeneous system:

$$\begin{array}{rrrcl} x_1 & +3x_2 & +2x_3 & = & 0 \\ x_1 & +2x_2 & +3x_3 & = & 0 \\ 2x_1 & +x_2 & -2x_3 & = & 0 \end{array}$$

Ex: Solve the following homogeneous system:

$$\begin{array}{rrrcl} 3x_1 & +5x_2 & -4x_3 & = & 0 \\ -3x_1 & -2x_2 & +4x_3 & = & 0 \\ 6x_1 & +x_2 & -8x_3 & = & 0 \end{array}$$

<u>The Gauss Jordan Method</u>

A reduced row echelon form is one where all leading entries in the matrix are 1, any leading entry occurs to the right of the leading entry in the row above, all zero rows are at the bottom of the matrix and any column containing a leading entry has only zeros in the remaining entries.

<u>Ex:</u> Solve
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 9 \\ 2x_1 - x_2 + x_3 &= 8 \\ 3x_1 - x_3 &= 3 \end{aligned}$$

Calculating Inverses

Let A be an invertible $n \times n$ matrix (we also say that A is non-singular). The technique to obtain the inverse of A is as follows. Form the augmented matrix $[A|I]$ (*i.e.* append the identity matrix to the right hand side of A) and then apply e.r.o.'s to this augmented matrix until the left hand side turns into the identity. The right hand side will then automatically become A^{-1}.

Ex: Find the inverse of $A = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix}$.

What happens if A is not invertible?

Ex: Find the inverse of $B = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 2 & -1 & 4 \end{bmatrix}$.

Invertibility and Solutions of Systems

Consider a system of n equations in n unknowns, *i.e.*

$$A\boldsymbol{x} = \boldsymbol{b}$$

where A is $n \times n$, $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.

Clearly, if A is invertible with inverse A^{-1}, we have

$$\begin{aligned} A^{-1}A\boldsymbol{x} &= A^{-1}\boldsymbol{b} \\ \textit{i.e. } I\boldsymbol{x} &= A^{-1}\boldsymbol{b} \\ \textit{i.e. } \boldsymbol{x} &= A^{-1}\boldsymbol{b} \end{aligned}$$

Ex: Solve the system $A\boldsymbol{x} = \boldsymbol{b}$, where
$A = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Why should we solve a system of n equations in n unknowns by Gaussian Elimination when we could just use the formula above?

When A is not invertible, we can only conclude that $A\boldsymbol{x} = \boldsymbol{b}$ is either inconsistent (no solution) or it has infinitely many solutions.

Finally, for the homogeneous system $A\boldsymbol{x} = \mathbf{0}$, if A is invertible, then $\boldsymbol{x} = A^{-1}\mathbf{0} = \mathbf{0}$, *i.e.* the zero solution is the only solution. In this case, if A is not invertible, we can conclude that $A\boldsymbol{x} = \mathbf{0}$ must have infinitely many solutions.

How can we tell whether a square matrix is invertible or not?