# Lines and Planes in 3 Space

**Motivation:** The vector equations of lines are a preview of more general vector valued functions and planes are examples of more general multivariable functions. Lines and planes in 3 space also help us to better understand the geometry of multivariable functions as well as the solutions of systems of linear algebraic equations.

**Outcomes** In today's lecture we will learn how to:

- Identify a line from its equation and vice versa.
- Identify a plane from its equation and vice versa.
- Determine distances between points, lines and planes.

## Contents

- Vector and parametric equations of a line.
- Cartesian equations of a line.
- Distance between a line and a point.
- Distance between skew lines.
- The general equation of a plane and geometric interpretation.
- Distance between a line and a plane.
- Parallel planes.
- Line of intersection of two planes.
- Distance from a point to a plane.

## Exercises

1. Find the vector and parametric equations for the line passing through the point $(0, 14, -10)$ and is parallel to the line $x = -1 + 2t$, $y = 6 - 3t$, $z = 3 + 9t$.
2. Find the parametric and Cartesian equations for the line passing through the points $(1.0, 2.4, 4.6)$ and $(2.6, 1.2, 0.3)$.
3. Find the parametric equations of the line which goes through the point $(1, 0, -3)$ and which is parallel to the vector $2\boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k}$. Then find the point where this line crosses the $yz$ plane.

4. Find the parametric equations of the line which passes through the origin and which is parallel to the line $x = 2t$, $y = 1 - t$, $z = 4 + 3t$. Then calculate the shortest distance between the line you've found and the point $(1, 1, 1)$.

5. Show that the line through the points $(0, 1, 1)$ and $(1, -1, 6)$ is perpendicular to the line through the points $(-4, 2, 1)$ and $(-1, 6, 2)$.

6. Find the Cartesian equations of the line passing through the point $(0, 2, -1)$ which is parallel to the line with parametric equations $x = 1 + 2t$, $y = 3t$, $z = 5 - 7t$.

7. Determine if the following lines $L_1$ and $L_2$ are parallel, skew or intersecting.

$$L_1 : x = 5 - 12t, y = 3 + 9t, z = 1 - 3t \qquad L_2 : x = 3 + 8s, y = -6s, z = 7 + 2s$$

8. Show that $L_1 : \dfrac{x-2}{2} = \dfrac{y+5}{4} = \dfrac{z-1}{-3}$ and $L_2 : \dfrac{x-2}{1} = \dfrac{y+1}{3} = \dfrac{z}{2}$ are skew lines and find the shortest distance between them.

9. Find the equations of the line through the point $(1, 0, 6)$ and perpendicular to the plane $x + 3y + z = 5$.

10. Find the equation of the plane through the point $(6, 3, 2)$ and perpendicular to the vector $[-2, 1, 5]$.

11. Find the equation of the plane containing the point $(-2, 8, 10)$ and perpendicular to the line $x = 1 + t$, $y = 2t$, $z = 4 - 3t$.

12. Find the equation of the plane containing the point $(4, -2, 3)$ and parallel to the plane $3x - 7z = 12$.

13. Find the equation of the plane that passes through the point $(-1, 2, 1)$ and contains the line of intersection of the planes $x + y - z = 2$ and $2x - y + 3z = 1$.

14. Find the point at which the line $x = 1 + t$, $y = 2t$, $z = 3t$ intersects the plane $x + y + z = 1$.

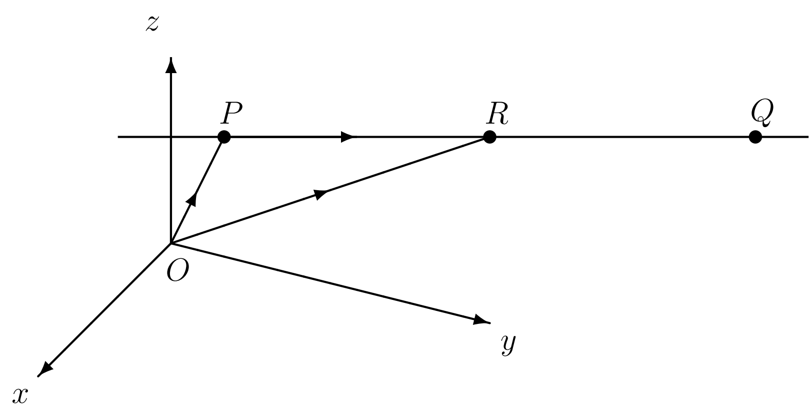
15. Find the distance from the point $(2, 8, 5)$ to the plane $x - 2y - 2z = 1$.

16. Find an equation for the plane consisting of all points which are equidistant from the points $(1, 1, 0)$ and $(0, 1, 1)$.

These exercises should take around 2 hours to complete.

(Answers: 1.$\boldsymbol{r} = 2t\boldsymbol{i} + (14 - 3t)\boldsymbol{j} + (-10 + 9t)\boldsymbol{k}$, $x = 2t$, $y = 14 - 3t$, $z = -10 + 9t$; 2. $x = 1.0 + 1.6t$, $y = 2.4 - 1.2t$, $z = 4.6 - 4.3t$, $\frac{x-1.0}{1.6} = \frac{y-2.4}{-1.2} = \frac{z-4.6}{-4.3}$; 3. $x = 1 + 2t$, $y = -4t$, $z = -3 + 5t$, $(0, 2, -\frac{11}{2})$; 4. $x = 2t$, $y = -t$, $z = 3t$, $d = \frac{\sqrt{91}}{7} \approx 1.36$ (2 d.p.); 6. $\frac{x}{2} = \frac{y-2}{3} = \frac{z+1}{-7}$; 7. parallel lines; 8. $d = \frac{30}{\sqrt{342}} \approx 1.62$ (2 d.p.); 9. $x = 1 + t$, $y = 3t$, $z = 6 + t$; 10. $-2x + y + 5z = 1$; 11. $x + 2y - 3z = -16$; 12. $3x - 7z = -9$; 13. $x - 2y + 4z = -1$; 14. $(1, 0, 0)$; 15. $\frac{25}{3}$; 16. $x - z = 0$)

## Equations of Lines




We want to find equation of the line passing through the two points $P(x_0, y_0, z_0)$ and $Q(x_1, y_1, z_1)$ in $\mathbb{R}^3$. The direction of the line is given by

$$\overrightarrow{PQ} = [x_1 - x_0, y_1 - y_0, z_1 - z_0].$$

Putting $a_1 = x_1 - x_0$, $a_2 = y_1 - y_0$ and $a_3 = z_1 - z_0$, then $\boldsymbol{a} = [a_1, a_2, a_3]$ is a vector parallel to the line. For this reason, we call $\boldsymbol{a}$ a direction vector of the line.

Let $R(x, y, z)$ be an arbitrary point on the line. Then $\overrightarrow{PR}$ is clearly parallel to$\overrightarrow{PQ}$, *i.e.*

$$\overrightarrow{PR} = t\overrightarrow{PQ} = t\boldsymbol{a}$$

for some scalar $t$.

Clearly, $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR}$, *i.e.*

$$[x, y, z] = [x_0, y_0, z_0] + t[a_1, a_2, a_3]$$

Putting $\boldsymbol{r} = [x, y, z]$ and $\boldsymbol{r}_0 = [x_0, y_0, z_0]$, we have

$$\boxed{\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{a},}$$

which is called the vector equation of a line. In component form,

$$\boxed{\begin{array}{rcl} x & = & x_0 + ta_1 \\ y & = & y_0 + ta_2 \\ z & = & z_0 + ta_3, \end{array}}$$

called the parametric equations of a line. Note that, as $t$ goes from $-\infty$ to $+\infty$, these equations generate all points on the line.

Finally, eliminating $t$ from each of these equations, we get

$$\boxed{\frac{x-x_0}{a_1}=\frac{y-y_0}{a_2}=\frac{z-z_0}{a_3}}$$

known as the cartesian equations of a line.

Note how in either form of the equations of a line, we can immediately identify $(x_0, y_0, z_0)$ (a point on the line) and $\boldsymbol{a}$ (a direction vector of the line).

**Ex:** Find the equation of the line passing through $P(3, 1, -1)$ and $Q(-2, 7, -4)$. At which point does the line intersect the $xy$-plane?

**Soln:** $\boldsymbol{a} = \overrightarrow{PQ} = [-5, 6, -3]$. Hence, the parametric equations are

$$\begin{aligned} x &= 3-5t \\ y &= 1+6t \\ z &= -1-3t \end{aligned}$$

On the $xy$ plane, $z = 0$. Hence $-1-3t = 0$ and $t = -\frac{1}{3}$. This in turn gives $x = \frac{14}{3}$ and $y = -1$. Thus, the point of intersection is $(\frac{14}{3}, -1, 0)$.

**Ex:** Does the point $(-2, 6, 6)$ lie on the line

$$L\begin{cases} x = 1-t \\ y = 2t \\ z = 3+t \end{cases} \quad ?$$

**Soln:** If the point does lie on the line, then there must be a value of $t$ such that

$$\begin{aligned} x &= 1-t &= -2 \\ y &= 2t &= 6 \\ z &= 3+t &= 6 \end{aligned}$$

Clearly, $t = 3$ satisfies all the three equations, so the point must lie on the line. (If the solution for any one of the equations did not match that of another equation, we could have concluded that the point is not on the line.)

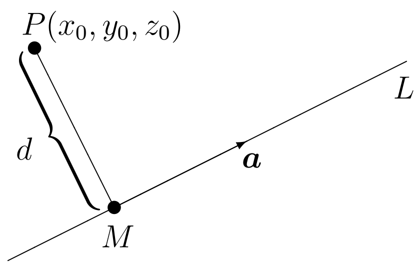
Finally note that points $A$, $B$ and $C$ are co-linear (ie. lie on a straight line) if $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel, *i.e.* $\overrightarrow{AB} = m\overrightarrow{BC}$ for some scalar $m$.

**Ex:** Show that $P(1, 0, 3)$, $Q(0, 2, 4)$ and $R(-2, 6, 6)$ are co-linear.

**Soln:** We have $\overrightarrow{PQ} = [-1, 2, 1]$ and $\overrightarrow{QR} = [-2, 4, 2] = 2[-1, 2, 1] = 2\overrightarrow{PQ}$, so the three points must be collinear.

## Distance from Point to a Line

We want to find the shortest (*i.e.* perpendicular) distance between $P(x_0, y_0, z_0)$ and the line $L$ given by $\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{a}t$.




Let $M$ be the point on $L$ which is closest to $P$. Then, the required distance is clearly given by

$$\boxed{d = ||\overrightarrow{PM}||}$$

where $\overrightarrow{PM}$ is perpendicular to $L$ (and therefore $\boldsymbol{a}$), so that $\overrightarrow{PM}.\boldsymbol{a} = 0$. This latter equation is important in analyzing the problem, as we will see in the example below.

**<u>Ex:</u>** Find the distance between $P(1, 0, 1)$ and the line

$$L \begin{cases} x &= 1 - t \\ y &= 3 + 2t \\ z &= -2 + 5t \end{cases}$$

**<u>Soln:</u>** Clearly, $\boldsymbol{a} = [-1, 2, 5]$. Let $t_1$ be the value of $t$ such that the coordinates of $M$ are $(1 - t_1, 3 + 2t_1, -2 + 5t_1)$. (We know the coordinates must take this form, since $M$ is on the line.) Then $\overrightarrow{PM} = [-t_1, 3 + 2t_1, -3 + 5t_1]$ and $\overrightarrow{PM}.\boldsymbol{a} = 0$ then gives

$$\begin{aligned} [-t_1, 3 + 2t_1, -3 + 5t_1].[-1, 2, 5] &= 0, \\ i.e. \quad t_1 + 6 + 4t_1 - 15 + 25t_1 &= 0 \\ i.e. \quad t_1 = \frac{3}{10}. \end{aligned}$$

Therefore $\overrightarrow{PM} = [-\frac{3}{10}, \frac{18}{5}, -\frac{3}{2}]$ and

$$d = ||\overrightarrow{PM}|| = \sqrt{\left(-\frac{3}{10}\right)^2 + \left(\frac{18}{5}\right)^2 + \left(-\frac{3}{2}\right)^2} = \frac{\sqrt{1530}}{10} \approx 3.9115.$$

**Skew Lines**

Consider two lines $L_1$ and $L_2$ with direction vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, respectively. If $L_1$ and $L_2$ are parallel, then $\boldsymbol{a}_1 = m\boldsymbol{a}_2$ for some scalar $m$. If they are not parallel, they may or may not intersect. Skew lines are those which are not parallel and do not intersect.

**Ex:** Do the lines

$$L_1 \left\{ \begin{array}{rcl} x &=& 1+t \\ y &=& 2-2t \\ z &=& 3+5t \end{array} \right. \qquad L_2 \left\{ \begin{array}{rcl} x &=& 2-\tau \\ y &=& 3\tau \\ z &=& 8+\tau \end{array} \right.$$

intersect?

**Soln:** If there is a point $(x_0, y_0, z_0)$ of intersection of the two lines, then there must be a value of $t$ such that

$$\begin{array}{rcl} x_0 &=& 1+t \\ y_0 &=& 2-2t \\ z_0 &=& 3+5t \end{array}$$

and a value of $\tau$ such that

$$\begin{array}{rcl} x_0 &=& 2-\tau \\ y_0 &=& 3\tau \\ z_0 &=& 8+\tau \end{array}$$

*i.e.* there must be values of $t$ and $\tau$ such that

$$1+t = 2-\tau \tag{1}$$
$$2-2t = 3\tau \tag{2}$$
$$3+5t = 8+\tau \tag{3}$$

Note that we have three equations and 2 unknowns. If we can find a solution, the lines will intersect at the corresponding point. If there turns out to be no solution, they don't intersect. To check which case occurs, we proceed as follows. Using any two of the equations to solve for $t$ and $\tau$ (you will get a unique solution in most cases). Substitute the resulting values into the third equation and see if it satisfied or not. If it is, all the three equations are consistent and yield the same solution, thereby telling us that the lines intersect. If it is not, the set of three equations is inconsistent and there is no solution, *i.e.* the lines do not intersect.
In this case, from (1), $t = 1-\tau$. Substituting into (2), we have $2-2(1-\tau) = 3\tau$, *i.e.* $\tau = 0$ which in turn gives $t = 1$. Now to check the third equation. Substituting into the left hand side of (3), we have LHS$= 3+5(1) = 8$. Substituting into the right hand side, we have RHS$= 8+0 = 8 =$LHS, so (3) is satisfied. Thus, the two lines intersect at the point corresponding to $\tau = 0$, *i.e.* $(2, 0, 8)$.

**Ex:** Do the lines

$$L_1 \begin{cases} x &= 1-t \\ y &= 2t \\ z &= 3+t \end{cases} \qquad L_2 \begin{cases} x &= 2+\tau \\ y &= 3 \\ z &= 1-\tau \end{cases}$$

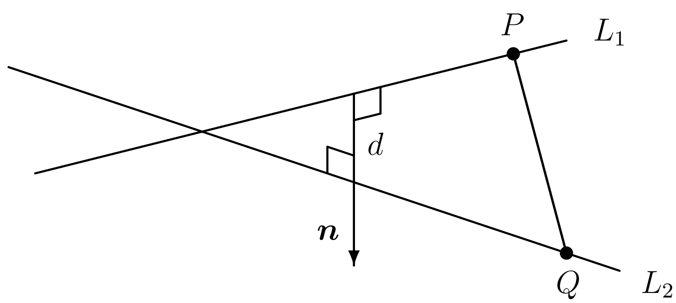
intersect ?

**Soln:** We require $t$ and $\tau$ such that

$$\begin{aligned} 1-t &= 2+\tau \\ 2t &= 3 \quad \to t=\frac{3}{2} \\ 3+t &= 1-\tau \quad \to 3+\frac{3}{2}=1-\tau \quad \to \tau=\frac{7}{2} \end{aligned}$$

Substituting these values into the first equation gives LHS$= 1-\frac{3}{2} = -\frac{1}{2}$ and RHS$= 2+\frac{7}{2} = \frac{11}{2} \neq$LHS. Hence the lines do not intersect. Note that they are also not parallel ($\boldsymbol{a}_1 = [-1,2,1] \neq [1,0,-1] = \boldsymbol{a}_2$), so they must be skew lines.

Finally, we want to find the shortest distance between two skew lines $L_1$ and $L_2$ (with directions $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, respectively).




Clearly, $\boldsymbol{n} = \boldsymbol{a}_1 \times \boldsymbol{a}_2$ is perpendicular to both lines. Also, the line joining the two closest points on $L_1$ and $L_2$ is parallel to $\boldsymbol{n}$. Let $P$ be any point on $L_1$ and $Q$ any point on $L_2$. Then the required distance is

$$\boxed{d = |\overrightarrow{PQ}.\hat{\boldsymbol{n}}|, \qquad \text{where } \hat{\boldsymbol{n}} = \frac{\boldsymbol{n}}{||\boldsymbol{n}||}}$$

(*i.e.* simply the scalar projection of $\overrightarrow{PQ}$ on $\boldsymbol{n}$)

**Ex:** Find the closest distance between

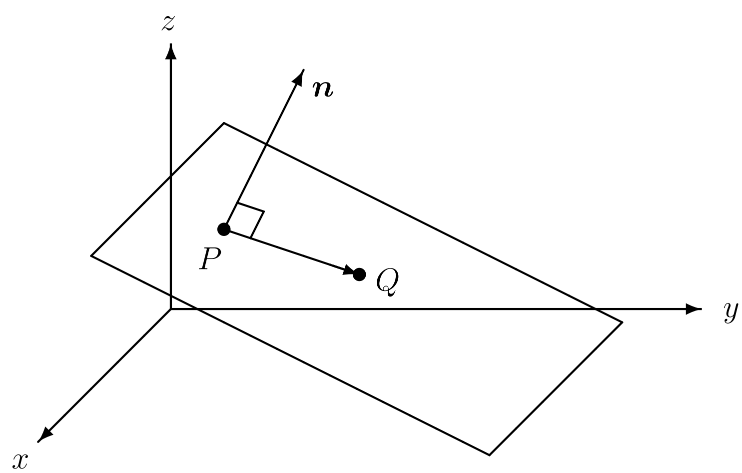
$$L_1 \begin{cases} x &= 1-t \\ y &= 2t \\ z &= 3+t \end{cases} \qquad L_2 \begin{cases} x &= 2+\tau \\ y &= 3 \\ z &= 1-\tau \end{cases}$$

**<u>Soln:</u>** $\boldsymbol{a}_1 = [-1,2,1]$, $\boldsymbol{a}_2 = [1,0,-1]$, $P(1,0,3)$, $Q(2,3,1)$. $\boldsymbol{n} = \boldsymbol{a}_1 \times \boldsymbol{a}_2 = [-2,0,-2]$, $||\boldsymbol{n}|| = \sqrt{8} = 2\sqrt{2}$, $\hat{\boldsymbol{n}} = [-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}]$, so

$$d = |\overrightarrow{PQ}.\hat{\boldsymbol{n}}| = |[1,3,-2].[-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}]| = \frac{1}{\sqrt{2}}.$$

## <u>Equation of a Plane in 3-Space</u>

Consider a plane containing a point $P(x_0, y_0, z_0)$. Let $\boldsymbol{n} = [a,b,c]$ be any vector perpendicular (or orthogonal) to the plane.




Let $Q(x,y,z)$ be any point on the plane. Then $\overrightarrow{PQ}$ is orthogonal to $\boldsymbol{n}$ and

$$\boldsymbol{n}.\overrightarrow{PQ} = 0$$

$$i.e. \quad [a,b,c].[x-x_0, y-y_0, z-z_0] = 0$$

$$i.e. \quad \boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$$

This is the *equation of a plane perpendicular to* $\boldsymbol{n} = [a,b,c]$ *and containing the point* $P(x_0, y_0, z_0)$. In this sense, $\boldsymbol{n}$ is called the *normal of the plane*. Note how this equation allows you to identify both a point on the plane and a normal vector to the plane.

We can easily re-arrange the above equation into $ax + by + cz = ax_0 + by_0 + cz_0$ or

$$\boxed{ax + by + cz = d}$$

where $d$ is a constant. This form is known as the the *general equation of a plane*. Note that we can still identify $\boldsymbol{n} = [a,b,c]$ from this equation, but it is no longer as easy to write down a point on the plane.

**Ex:** Find the equation of the plane through the point $A(2, -1, 3)$ and normal to $\boldsymbol{n} = [5, -1, 2]$.

**Soln:** The general equation takes the form $5x - y + 2z = d$. To find $d$, we simply substitute the given point, since its coordinates must also satisfy the equation of the plane, *i.e.* $d = 5(2) - (-1) + 2(3) = 17$. Hence, the required equation of the plane is $5x - y + 2z = 17$.

**Ex:** Find the equation of the plane passing through $A(1, 0, 1)$, $B(0, 1, 2)$ and $C(1, 3, 2)$.

**Soln:** In this case, we first need to establish a normal vector for the plane. Note that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are both vectors on (or parallel to) the plane. Since their cross product is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$, it must be a normal vector to the plane as well. Thus, we have $\boldsymbol{n} = \overrightarrow{AB} \times \overrightarrow{AC} = [-1, 1, 1] \times [0, 3, 1] = [-2, 1, -3]$. Hence, the general equation takes the form $-2x + y - 3z = d$, where $d$ can be found by substituting the point $A$, *i.e.* $d = -2(1) + (0) - 3(1) = -5$. Hence, the equation of the plane is $-2x + y - 3z = -5$.

**Note the following:**

(i) Two planes are parallel if and only if they have parallel normal directions (*i.e.* $\boldsymbol{n}_1 = m\boldsymbol{n}_2$ for some scalar $m$).

(ii) A line is parallel to a plane if its direction is perpendicular to the normal of the plane.

**Ex:** Show that the line

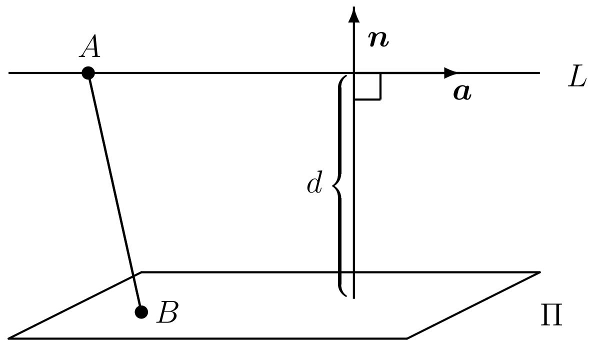
$$L: \qquad \frac{x-1}{3} = \frac{y-2}{-1} = z + 4$$

is parallel to the plane

$$\Pi: \qquad 2x + 2y - 4z = 7.$$

What is the distance between them?

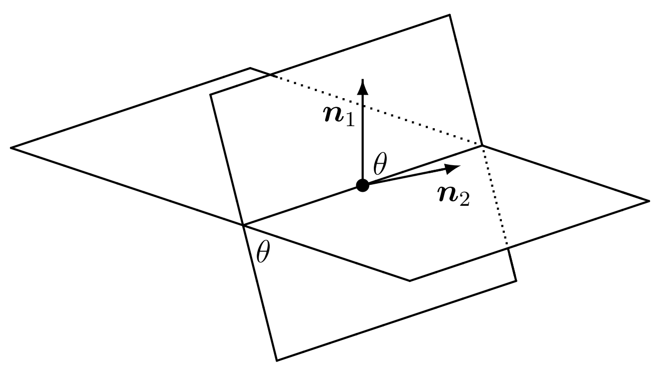
**Soln:** The direction of $L$ is $\boldsymbol{a} = [3, -1, 1]$. A normal vector to the plane is $\boldsymbol{n} = [2, 2, -4]$. Since $\boldsymbol{a}.\boldsymbol{n} = [3, -1, 1].[2, 2, -4] = 0$, $\boldsymbol{a} \perp \boldsymbol{n}$ and thus the line is parallel to the plane.




Let $A$ be a point on $L$, *e.g.* $A(1, 2, -4)$ will do. Let $B$ be a point on $\Pi$. For example, putting $y = z = 0$ in the equation of $\Pi$, $x = \frac{7}{2}$, *i.e.* $B(\frac{7}{2}, 0, 0)$ will do. Then the required distance can be easily calculated as the absolute value of the scalar projection of $\overrightarrow{AB}$ on $\boldsymbol{n}$, *i.e.*

$$d = |\overrightarrow{AB}.\hat{\boldsymbol{n}}| = \left|[\frac{5}{2}, -2, 4].\frac{[2, 2, -4]}{\sqrt{(2)^2 + (2)^2 + (-4)^2}}\right| = \left|-\frac{15}{\sqrt{24}}\right| = \frac{15}{2\sqrt{6}} \approx 3.06 \text{ (2 d.p.)}$$

Consider two planes which are not parallel. Clearly, they must intersect along a line:




Note that the angle between the planes is the same as the angle between their respective normal vectors. The direction of the line of intersection is clearly orthogonal to both $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$, *i.e.* $\boldsymbol{a} = \boldsymbol{n}_1 \times \boldsymbol{n}_2$.

**Ex:** Find the line of intersection of the planes

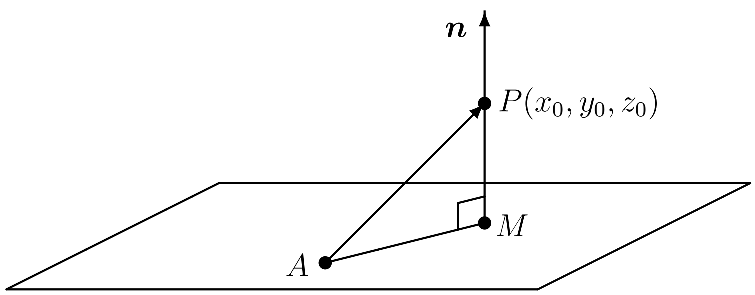
$$x + 2y - 2z = 5 \quad \text{and} \quad 6x - 3y + 2z = 8.$$

**Soln:** $\boldsymbol{n}_1 = [1, 2, -2]$, $\boldsymbol{n}_2 = [6, -3, 2]$. The direction of the line is $\boldsymbol{a} = \boldsymbol{n}_1 \times \boldsymbol{n}_2 = [-2, -14, -15]$. Now all we need is a point on the line. Note that it must satisfy both plane equations simultaneously, which gives us 2 equations in 3 unknowns, giving infinitely many solutions, since there are infinitely many points on the line of intersection. To get just one of these, let's put $x = 0$, then

$$\begin{aligned} 2y - 2z &= 5 \\ -3y + 2z &= 8 \end{aligned}$$

which gives $-y = 13$, *i.e.* $y = -13$ and $2z = 8 + 3y = 8 + 3(-13) = -31$, *i.e.* $z = -\frac{31}{2}$. Hence, a point on the line is $(0, -13, -\frac{31}{2})$. Finally, the parametric equations of the line are $x = -2t$, $y = -13 - 14t$, $z = -\frac{31}{2} - 15t$.

## Distance from a Point to a Plane




We want to find the distance between a plane $ax + by + cz = d$ and a point $P(x_0, y_0, z_0)$. Let $A$ be any point on the plane and let $\boldsymbol{n} = [a, b, c]$ be the normal vector to the plane. Then the distance is easily found as the absolute value of the scalar projection of $\overrightarrow{AP}$ on $\boldsymbol{n}$, *i.e.*

$$\boxed{d = |\overrightarrow{AP}.\hat{\boldsymbol{n}}|, \qquad \hat{\boldsymbol{n}} = \frac{\boldsymbol{n}}{||\boldsymbol{n}||}}$$

**Ex:** Find the distance of $P(1, -1, 0)$ from the plane $x + y - z = 2$.

**Soln:** Put $y = z = 0$ in the equation of the plane to get $x = 2$, *i.e.* a point on the plane is $A(2, 0, 0)$. We have $\boldsymbol{n} = [1, 1, -1]$, $||\boldsymbol{n}|| = \sqrt{3}$, and

$$d = |\overrightarrow{AP}.\hat{\boldsymbol{n}}| = \left| [-1, -1, 0].\frac{[1, 1, -1]}{\sqrt{3}} \right| = \frac{2}{\sqrt{3}} \approx 1.15 \text{ (2 d.p.)}$$