

EMTH1019 Linear Algebra & Statistics for Engineers

Tutorial 9 Determinants

SOLUTIONS

1. $A = \begin{bmatrix} 2 & -3 \\ 6 & -9 \end{bmatrix}$

(i) $\det(A) = (2)(-9) - (-3)(6) = -18 + 18 = 0$

(ii) Singular as $\det(A) = 0$, no inverse exists.

$$B = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$$

(i) $\det(B) = (2)(-7) - (5)(-3) = -14 + 15 = 1$

(ii) Non-singular as $\det(B) \neq 0$, inverse exists.

$$C = \begin{bmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{bmatrix}$$

(i) $\det(C) = \begin{vmatrix} 5^{+} & 0^{-} & -1^{+} \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{vmatrix}$ Cofactor expansion along 1st row:

$$\begin{aligned} \det(C) &= (5) \begin{vmatrix} -3 & -2 \\ 5 & 3 \end{vmatrix} - (0) + (1) \begin{vmatrix} 1 & -3 \\ 0 & 5 \end{vmatrix} \\ &= 5(-9 - (-10)) - 1(5 - 0) = 5(1) - 1(5) = 5 - 5 = 0. \end{aligned}$$

(ii) Singular as $\det(C) = 0$, no inverse exists.

$$D = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

(i) $\det(D) = \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0^{+} & -2^{-} & 0^{+} \end{vmatrix}$ Cofactor expansion along 3rd row:

$$\begin{aligned} \det(D) &= (0) - (-2) \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + (0) \\ &= 2(-1 - 0) = 2(-1) = -2. \end{aligned}$$

(ii) Non-singular as $\det(D) \neq 0$, inverse exists.

$$E = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 4 \\ 0 & 0 & -2 \end{bmatrix}$$

(i) $\det(E) = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 4 & 4 \\ 0 & 0 & -2 \end{vmatrix}$ Since it's an upper triangular matrix:

$$\det(E) = (1)(4)(-2) = -8$$

(ii) Non-singular as $\det(E) \neq 0$, inverse exists.

$$2. \left[\begin{array}{cccc|l} 2 & 1 & 3 & 1 & \\ -2 & 3 & -1 & 2 & R_2 \rightarrow R_2 + R_1 \\ 2 & 1 & 2 & 3 & R_3 \rightarrow R_3 - R_1 \\ -4 & -2 & -6 & -1 & R_4 \rightarrow R_4 + 2R_1 \end{array} \right] = \left[\begin{array}{cccc|l} 2 & 1 & 3 & 1 & \\ 0 & 4 & 2 & 3 & \\ 0 & 0 & -1 & 2 & \\ 0 & 0 & 0 & 1 & \end{array} \right]$$

Since it's an upper triangular matrix:

$$= (2)(4)(-1)(1) = -8$$

$$3. \quad (i) \left[\begin{array}{cc} 5 & 3 \\ 7 & 4 \end{array} \right]^{-1} = \frac{1}{(5)(4) - (3)(7)} \left[\begin{array}{cc} 4 & -3 \\ -7 & 5 \end{array} \right] = \frac{1}{-1} \left[\begin{array}{cc} 4 & -3 \\ -7 & 5 \end{array} \right] = \left[\begin{array}{cc} -4 & 3 \\ 7 & -5 \end{array} \right]$$

$$(ii) \left[\begin{array}{cc} -2 & 4 \\ -3 & 6 \end{array} \right]^{-1} = \frac{1}{(-2)(6) - (4)(-3)} \left[\begin{array}{cc} 6 & -4 \\ 3 & -2 \end{array} \right] = \frac{1}{0} \left[\begin{array}{cc} 6 & -4 \\ 3 & -2 \end{array} \right]$$

Singular matrix, no inverse exists.

$$(iii) \left[\begin{array}{cc} 3 & 5 \\ 2 & 4 \end{array} \right]^{-1} = \frac{1}{(3)(4) - (5)(2)} \left[\begin{array}{cc} 4 & -5 \\ -2 & 3 \end{array} \right] = \frac{1}{2} \left[\begin{array}{cc} 4 & -5 \\ -2 & 3 \end{array} \right] = \left[\begin{array}{cc} 2 & -\frac{5}{2} \\ -1 & \frac{3}{2} \end{array} \right]$$

$$4. \quad (i) A = \left[\begin{array}{cc} 3 & -2 \\ -5 & 4 \end{array} \right], \det(A) = (3)(4) - (-2)(-5) = 12 - 10 = 2.$$

$$A_1 = \left[\begin{array}{cc} 6 & -2 \\ 8 & 4 \end{array} \right], \det(A_1) = (6)(4) - (-2)(8) = 24 + 16 = 40.$$

$$A_2 = \left[\begin{array}{cc} 3 & 6 \\ -5 & 8 \end{array} \right], \det(A_2) = (3)(8) - (6)(-5) = 24 + 30 = 54.$$

$$\text{Hence, } x_1 = \frac{\det(A_1)}{\det(A)} = \frac{40}{2} = 20 \text{ and } x_2 = \frac{\det(A_2)}{\det(A)} = \frac{54}{2} = 27$$

$$(ii) A = \left[\begin{array}{cc} 1 & 2 \\ 3 & 1 \end{array} \right], \det(A) = (1)(1) - (2)(3) = 1 - 6 = -5.$$

$$A_1 = \left[\begin{array}{cc} 3 & 2 \\ -1 & 1 \end{array} \right], \det(A_1) = (3)(1) - (2)(-1) = 3 + 2 = 5.$$

$$A_2 = \left[\begin{array}{cc} 1 & 3 \\ 3 & -1 \end{array} \right], \det(A_2) = (1)(-1) - (3)(3) = -1 - 9 = -10.$$

$$\text{Hence, } x_1 = \frac{\det(A_1)}{\det(A)} = \frac{5}{-5} = -1 \text{ and } x_2 = \frac{\det(A_2)}{\det(A)} = \frac{-10}{-5} = 2$$

$$5. A = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 2 & -5 & -3 \\ 4 & 8 & 2 \end{array} \right], \mathbf{b} = \left[\begin{array}{c} 0 \\ 10 \\ 4 \end{array} \right], A_3 = \left[\begin{array}{ccc} 1 & 1 & 0 \\ 2 & -5 & 10 \\ 4 & 8 & 4 \end{array} \right]$$

$$\det(A) = \left| \begin{array}{ccc} 1^+ & 1^- & 1^+ \\ 2 & -5 & -3 \\ 4 & 8 & 2 \end{array} \right| \quad \text{Cofactor expansion along 1st row:}$$

$$\begin{aligned} \det(A) &= (1) \left| \begin{array}{cc} -5 & -3 \\ 8 & 2 \end{array} \right| - (1) \left| \begin{array}{cc} 2 & -3 \\ 4 & 2 \end{array} \right| + (1) \left| \begin{array}{cc} 2 & -5 \\ 4 & 8 \end{array} \right| \\ &= 1(-10 - (-24)) - 1(4 - (-12)) + 1(16 - (-20)) = 1(14) - 1(16) + 1(36) \\ &= 14 - 16 + 36 = 34 \end{aligned}$$

$$\det(A_3) = \begin{vmatrix} 1^+ & 1^- & 0^+ \\ 2 & -5 & 10 \\ 4 & 8 & 4 \end{vmatrix} \quad \text{Cofactor expansion along 1st row:}$$

$$\det(A_3) = (1) \begin{vmatrix} -5 & 10 \\ 8 & 4 \end{vmatrix} - (1) \begin{vmatrix} 2 & 10 \\ 4 & 4 \end{vmatrix} + (0)$$

$$= 1(-20 - 80) - 1(8 - 40) + 0 = 1(-100) - 1(-32) = -100 + 32 = -68$$

$$\text{Hence, } x_3 = \frac{\det(A_3)}{\det(A)} = \frac{-68}{34} = -2$$

6. (i) $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ -1 & 1 & 4 \end{vmatrix}$, doing a cofactor expansion along the first row we get:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \mathbf{i} \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} \\ &= (8 - 1)\mathbf{i} - (12 - (-1))\mathbf{j} + (3 - (-2))\mathbf{k} \\ &= (7)\mathbf{i} - (13)\mathbf{j} + (5)\mathbf{k} \\ &= [7, -13, 5] \end{aligned}$$

(ii) $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix}$, doing a cofactor expansion along the first row we get:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \mathbf{i} \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} \\ &= (0 - 1)\mathbf{i} - (-2 - 1)\mathbf{j} + (2 - 0)\mathbf{k} \\ &= (-1)\mathbf{i} - (-3)\mathbf{j} + (2)\mathbf{k} \\ &= [-1, 3, 2] \end{aligned}$$

7. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 2 & 6 & -2 \\ -3 & 2 & 0 \\ 0 & 1 & 5 \end{vmatrix}$, doing a cofactor expansion along the first row we get,

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= 2 \begin{vmatrix} 2 & 0 \\ 1 & 5 \end{vmatrix} - 6 \begin{vmatrix} -3 & 0 \\ 0 & 5 \end{vmatrix} + (-2) \begin{vmatrix} -3 & 2 \\ 0 & 1 \end{vmatrix} \\ &= 2(10 - 0) - 6(-15 - 0) - 2(-3 - 0) \\ &= 20 + 90 + 6 \\ &= 116 \end{aligned}$$

Since the volume of the parallelepiped is given by the absolute value of the scalar triple product, we have $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = |116| = 116$.