(1) $H = \left\{ \frac{2x^2+5}{x^2+2} \in \mathbb{R} \mid x > -1 \right\}$

$$\frac{2x^2+5}{x^2+2} = \frac{2(x^2+2)+1}{x^2+2} = 2 + \frac{1}{x^2+2} > 2$$

$$\Rightarrow \inf(H) = 2$$

$$2 + \frac{1}{x^2+2} \leq 2 + \frac{1}{0+2} = 2 + \frac{1}{2} = \frac{5}{2}$$

$$\Rightarrow \sup(H) = \frac{5}{2}$$

② $f(x) = \sqrt{x+3} \quad (x \geq 1)$

$g(x) = x^2 - 4x + 1 \quad (x < 1)$

$$D_{f\circ g} = \{x \in D_g : g(x) \in D_f\}$$

$$= \{x \in (-\infty, 1) : x^2 - 4x + 1 \in [1, +\infty)\}$$

$$= \{x \in (-\infty, 1) : x^2 - 4x + 1 \geq 1\}$$

$$x^2 - 4x + 1 \geq 1$$

$$x^2 - 4x \geq 0$$

$$x(x-4) \geq 0$$

$x \leq 0$ $\quad$ $x \geq 4$

$\hookrightarrow \quad x \in (-\infty, 0] \cup [4, +\infty)$

$$D_{f\circ g} = (-\infty, 0]$$

$$(f\circ g)(x) = f(g(x)) = \sqrt{x^2-4x+1+3} = \sqrt{(x-2)^2} = |x-2| = 2-x$$

mert $x<0$

$[-1,2]$ ősképe f-re

$$f^{-1}[[-1,2]] = \{x \in [1,+\infty) : \sqrt{x+3} \in [-1,2]\}$$

$$= \{x \in [1,+\infty) : -1 \le \sqrt{x+3} \le 2\}$$

$-1 \le \sqrt{x+3}$ igaz, $\sqrt{x+3} \ge 0$

$$0 \le \sqrt{x+3} \le 2 \quad /()^2$$

$$0 \le x+3 \le 4$$

$$-3 \le x \le 1$$

$$f^{-1}[[-1,2]] = \{x \in [1,+\infty) : x \in [-3,1]\} = \{1\}$$

③ $\lim\limits_{n \to +\infty} \frac{n^2+4n+1}{3n^2+6n+5} = \frac{1}{3}$

$$\left|\frac{n^2+4n+1}{3n^2+6n+5} - \frac{1}{3}\right| = \left|\frac{3(n^2+4n+1) - (3n^2+6n+5)}{(3n^2+6n+5)\cdot 3}\right|$$

$$= \frac{3n^2+12n+3-3n^2-6n-5}{9n^2+18n+15}$$

$$= \frac{6n-2}{9n^2+18n+15} < \frac{6n}{9n^2+18n+15} < \frac{6n}{9n^2} = \frac{2}{3n} < \frac{1}{n}$$

$$\frac{1}{n} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < n$$

$$\left|\frac{n^2+4n+1}{3n^2+6n+5} - \frac{1}{3}\right| < \varepsilon \Rightarrow \lim\left(\frac{n^2+4n+1}{3n^2+6n+5}\right) = \frac{1}{3}$$

(4) a) $\lim_{n\to+\infty} \frac{n\cdot(-2)^n+2^{2n+1}}{4^n+3^{n+1}} = \frac{n(-2)^n+4^n\cdot 2}{4^n+3^n\cdot 3}$

$$= \frac{n\left(-\frac{2}{4}\right)^n+\frac{4^n}{4^n}\cdot 2}{\frac{4^n}{4^n}+\left(\frac{3}{4}\right)^n\cdot 3}$$

$$= \frac{\overset{\to 0}{n\cdot\left(-\frac{1}{2}\right)^n}+2}{1+\underbrace{\left(\frac{3}{4}\right)^n\cdot 3}_{\to 0}}$$

$\to |q|<1 \Rightarrow q^n\to 0$ ÉS $n\cdot q^n\to 0$

$$= \frac{0+2}{1+0} = 2$$

b) $\lim_{n\to+\infty} \sqrt[n]{2\cdot 3^n+3\cdot 5^n+n} = \sqrt[n]{5^n\left(2\cdot\left(\frac{3}{5}\right)^n+3+\frac{n}{5^n}\right)}$

$$= 5\cdot\sqrt[n]{2\cdot\underbrace{\left(\frac{3}{5}\right)^n}_{0}+3+\underbrace{n\left(\frac{1}{5}\right)^n}_{0}} = 5$$

$\hookrightarrow \lim\left(2\cdot\left(\frac{3}{5}\right)^n+3+n\left(\frac{1}{5}\right)^n\right) = 0+3+0 = 3 \in (0,+\infty)$

c) $\lim\limits_{n\to+\infty} \left(\frac{2n+5}{2n+3}\right)^{4n+5} = \left(\frac{2n+3+2}{2n+3}\right)^{4n+5}$

$\left(= \left(1+\frac{2}{2n+3}\right)^{4n+5}\right)$ így kéne ha $2n+3$ jobban hasonlítana $4n+5$ höz

$$= \left(\left(\frac{\frac{2n}{2n}+\frac{5}{2n}}{\frac{2n}{2n}+\frac{3}{2n}}\right)^{n}\right)^{4} + \left(\frac{1+\overbrace{\frac{5}{2n}}^{\to 0}}{1+\underbrace{\frac{3}{2n}}_{\to 0}}\right)^{5}$$

$$= \left(\frac{\left(1+\frac{\frac{5}{2}}{n}\right)^{n}}{\left(1+\frac{\frac{3}{2}}{n}\right)^{n}}\right)^{4} \cdot \left(\frac{1}{1}\right)^{5}$$

$$= \left(\frac{e^{\frac{5}{2}}}{e^{\frac{3}{2}}}\right)^{4} = \left(e^{\frac{5-3}{2}}\right)^{4} = e^{4}$$

5 $a_0 = 0, \quad a_{n+1} = \dfrac{a_n^2 + 3}{4}$

$$a_0 = 0 < a_{n+1} = \frac{a_n^2 + 3}{4} < \frac{a_{n+1}^2 + 3}{4} = a_{n+2}$$

↳ szigorúan monoton nő

HA (a_n) konvergens $\Rightarrow A = \lim(a_n) \Rightarrow A = \lim(a_{n+1})$

$$A = \frac{A^2 + 3}{4}$$

$$A^2 - 4A + 3 = 0$$

$$A_{1,2} = \frac{4 \pm \sqrt{16 - 4 \cdot 3}}{2} \begin{cases} 1 \\ 3 \end{cases}$$

↳ (a_n) legkisebb felső korlátja 1

teljes indukció $n = 0$ esetén

$$a_0 = 0 < 1 \quad \checkmark \text{ igaz}$$

$n \in \mathbb{N}_0$, $a_n \leq 1$

$$a_{n+1} = \frac{a_n^2 + 3}{4} \leq \frac{1^2 + 3}{4} = 1$$

$\Rightarrow$ felülről korlátos

összefoglalva: monoton nő és felülről korlátos

$\Rightarrow$ konvergens sorozat $\quad \lim(a_n) = 1$