1) $A = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \end{array} \in \mathbb{R}^{2\times 2}$

a)

$\det(A) = \overset{11}{1} \cdot \overset{22}{4} - \overset{12}{2} \cdot \overset{21}{3} = 4 - 6 = -2$ ← első sor módszer

$\det(A) = \overset{12}{2} \cdot (-1)^{1+2} \cdot \det(\underset{3}{\overset{21}{[3]}}) + \overset{22}{4} \cdot (-1)^{2+2} \cdot \det(\underset{1}{\overset{11}{[1]}}) = -6 + 4 = -2$ ← utolsó oszlop módszer

$\det(A) \neq 0 \Rightarrow A$ reguláris

$$A^{-1} = \frac{1}{\det(A)} \cdot \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{-2} & \frac{-2}{-2} \\ \frac{-3}{-2} & \frac{1}{-2} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$$A \cdot A^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1(-2) + 2 \cdot \frac{3}{2} & 1 \cdot 1 + 2 \cdot (-\frac{1}{2}) \\ 3(-2) + 4 \cdot \frac{3}{2} & 3 \cdot 1 + 4 \cdot (-\frac{1}{2}) \end{bmatrix} = \begin{bmatrix} -2+3 & 1-1 \\ -6+6 & 3-2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$ ✓ igaz

b) $A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix} \in \mathbb{R}^{3\times 3}$

$$\det(A) = 3 \cdot (-1)^2 \cdot \det\left(\begin{bmatrix} 5 & 6 \\ 4 & 8 \end{bmatrix}\right) + 1 \cdot (-1)^3 \cdot \det\left(\begin{bmatrix} 2 & 6 \\ 1 & 8 \end{bmatrix}\right) - 4(-1)^4 \cdot \det\left(\begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}\right)$$

$$= 3 \cdot (5 \cdot 8 - 6 \cdot 4) - 1 \cdot (2 \cdot 8 - 1 \cdot 6) - 4(2 \cdot 4 - 1 \cdot 5) = 3 \cdot 16 - 10 - 12 = 26$$

$$\det(A) = -4(-1)^4 \cdot \det\left(\begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}\right) + 6(-1)^5 \cdot \det\left(\begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}\right) + 8(-1)^6 \cdot \det\left(\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}\right)$$

$$= -4 \cdot (2 \cdot 4 - 1 \cdot 5) - 6(3 \cdot 4 - 1) + 8(3 \cdot 5 - 2) = -12 - 66 + 104 = 26$$

3x3!!

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow aei+bfg+cdh-ceg-afh-bdi$$

$$\begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

$$\det(A) = 3\cdot5\cdot8 + 1\cdot6\cdot1 + (-4)\cdot2\cdot4 - (-4)\cdot5\cdot1 - 3\cdot6\cdot4 - 1\cdot2\cdot8$$
$$= 120 + 6 - 32 + 20 - 72 - 16 = \underline{\underline{26}}$$

$$a'_{11} = (-1)^2\cdot\det(A_{11}) = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 1\cdot(5\cdot8-6\cdot4) = 16$$
$$a'_{12} = (-1)^3\cdot\det(A_{12}) = -\begin{vmatrix} 2 & 6 \\ 1 & 8 \end{vmatrix} = -(2\cdot8-6) = -10$$
$$a'_{13} = (-1)^4\cdot\det(A_{13}) = \begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = 8-5 = 3$$
$$a'_{21} = (-1)^3\cdot\det(A_{21}) = -\begin{vmatrix} 1 & -4 \\ 4 & 8 \end{vmatrix} = -(8+16) = -24$$
$$a'_{22} = (-1)^4\cdot\det(A_{22}) = \begin{vmatrix} 3 & -4 \\ 1 & 8 \end{vmatrix} = 24+4 = 28$$
$$a'_{23} = (-1)^5\cdot\det(A_{23}) = -\begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix} = -(12-1) = -11$$
$$a'_{31} = (-1)^4\cdot\det(A_{31}) = \begin{vmatrix} 1 & -4 \\ 5 & 6 \end{vmatrix} = +(6+20) = 26$$
$$a'_{32} = (-1)^5\cdot\det(A_{32}) = -\begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = -(18+8) = -26$$
$$a'_{33} = (-1)^6\cdot\det(A_{33}) = \begin{vmatrix} 3 & 1 \\ 2 & 5 \end{vmatrix} = 15-2 = 13$$

$\det(A) \neq 0 \Rightarrow$ reguláris

$$A^{-1} = \frac{1}{\det(A)}\cdot\begin{bmatrix} a'_{11} & a'_{12} & a'_{13} \\ a'_{21} & a'_{22} & a'_{23} \\ a'_{31} & a'_{32} & a'_{33} \end{bmatrix}^T = \frac{1}{26}\cdot\begin{bmatrix} 16 & -10 & 3 \\ -24 & 28 & -11 \\ 26 & -26 & 13 \end{bmatrix}^T = \frac{1}{26}\cdot\begin{bmatrix} 16 & -24 & 26 \\ -10 & 28 & -26 \\ 3 & -11 & 13 \end{bmatrix} = \begin{bmatrix} \frac{16}{26} & -\frac{24}{26} & \frac{26}{26} \\ \frac{-10}{26} & \frac{28}{26} & -\frac{26}{26} \\ \frac{3}{26} & -\frac{11}{26} & \frac{13}{26} \end{bmatrix} = \begin{bmatrix} \frac{8}{13} & -\frac{12}{13} & 1 \\ \frac{-5}{13} & \frac{14}{13} & -1 \\ \frac{3}{26} & -\frac{11}{26} & \frac{13}{26} \end{bmatrix}$$

$$A\cdot A^{-1} = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}\cdot\frac{1}{26}\cdot\begin{bmatrix} 16 & -24 & 26 \\ -10 & 28 & -26 \\ 3 & -11 & 13 \end{bmatrix} =$$

$$= \frac{1}{26}\cdot\begin{bmatrix} 3\cdot16+1(-10)+(-4)3 & 3(-24)+1\cdot28+(-4)(-11) & 3\cdot26+1(-26)+(-4)13 \\ 2\cdot16+5(-10)+6\cdot3 & 2(-24)+5\cdot28+6\cdot(-11) & 2\cdot26+5\cdot(-26)+6\cdot13 \\ 1\cdot16+4(-10)+8\cdot3 & 1\cdot(-24)+4\cdot28+8\cdot(-11) & 1\cdot26+4\cdot(-26)+8\cdot13 \end{bmatrix}$$

$$= \frac{1}{26}\begin{bmatrix} 48-10-12 & -72+28+44 & 78-26-52 \\ 32-50+18 & -48+140-66 & 52-130+78 \\ 16-40+24 & -24+112-88 & 26-104+104 \end{bmatrix} = \frac{1}{26}\cdot\begin{bmatrix} 26 & 0 & 0 \\ 0 & 26 & 0 \\ 0 & 0 & 26 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \checkmark$$

igaz

(2)

$$A = \begin{bmatrix} -2 & 5 \\ -3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & 3 \\ -4 & 6 \end{bmatrix}$$

$\det(A) = -2\cdot1+3\cdot5 = 13$ ⇐ reguláris

$\det(B) = -2\cdot6+3\cdot4 = 0$ ⇐ szinguláris

$$A^{-1} = \frac{1}{13}\cdot\begin{bmatrix} 1 & -5 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{13} & -\frac{5}{13} \\ \frac{3}{13} & -\frac{2}{13} \end{bmatrix}$$

③

- ha egyik sorban/oszlopban csak 0 van $\Rightarrow$ determináns = 0

$$\begin{vmatrix} 0 & 0 \\ 2 & 7 \end{vmatrix} = 0\cdot 5 - 0\cdot 2 = 0$$

- ha egy mátrix 2 sora/oszlopa helyet cserél $\Rightarrow$ változik a determináns előjele

$$\begin{vmatrix} 1 & 2 \\ 2 & 7 \end{vmatrix} = 1\cdot 7 - 2\cdot 2 = 3$$

$$\begin{vmatrix} 2 & 7 \\ 1 & 2 \end{vmatrix} = 4 - 7 = -3$$

- ha egy mátrix 2 sora/oszlopa azonos $\Rightarrow$ determináns = 0

$$\begin{vmatrix} 2 & 2 \\ 7 & 7 \end{vmatrix} = 2\cdot 7 - 2\cdot 7 = 0$$

- mátrix sor/oszlop $\cdot c \Rightarrow$ determináns $\cdot c$

$$\begin{vmatrix} 1 & 2 \\ 2 & 7 \end{vmatrix} = 7 - 4 = 3$$

$$\begin{vmatrix} 2 & 4 \\ 2 & 7 \end{vmatrix} = 14 - 8 = 6$$

- ha $A \in K^{n\times n}, c \in K \Rightarrow \det(c\cdot A) = c^n \cdot \det(A)$

$$\begin{vmatrix} 1 & 2 \\ 2 & 7 \end{vmatrix} = 7 - 4 = 3$$

$$\begin{vmatrix} 2 & 4 \\ 4 & 14 \end{vmatrix} = 28 - 16 = 12$$

$\hookrightarrow 2^2 \cdot 3 = 12$

- ha egy mátrix 2 sora/oszlopa arányos $\Rightarrow$ determináns = 0

$$\begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 6 - 6 = 0$$

- a determináns sor/oszlop szerint additív

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$$

$$\begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix} = 6 - 10 = -4$$

$$\begin{vmatrix} 1 & 2 \\ 8 & 10 \end{vmatrix} = -6$$

- ha egy mátrix valamelyik sor/oszlop + másik sor $\cdot x \Rightarrow$ determináns nem változik

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$$

$$\begin{vmatrix} 1 & 2 \\ 6 & 10 \end{vmatrix} = 10 - 12 = -2$$

- szorzat determináns = determinánsok szorzata

$$\begin{vmatrix} 1 & 2 \\ 2 & 7 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (7-4)\cdot(4-6) = -6$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 2\cdot 3 & 1\cdot 2 + 2\cdot 4 \\ 2\cdot 1 + 7\cdot 3 & 2\cdot 2 + 7\cdot 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 23 & 32 \end{bmatrix} \Rightarrow \begin{vmatrix} 7 & 10 \\ 23 & 32 \end{vmatrix} = 7\cdot 32 - 10\cdot 23 = 224 - 230 = -6$$