(1a)

$$\underbrace{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}_{A(n)}$$

legyen $n=1$

$$A(1): \frac{1(2)}{2} = 1 \quad \text{igaz}$$

$n \to n+1$

$$A(n+1): \frac{(n+1)(n+2)}{2} \stackrel{?}{=} \sum_{k=1}^{n+1} k$$

$$\sum_{k=1}^{n} k + (n+1) = \frac{(n+1)(n+2)}{2}$$

$$\frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2} \quad /:(n+1)$$

$$\frac{n}{2} + 1 = \frac{n+2}{2}$$

$$n+2 = n+2 \quad \checkmark \quad \text{igaz}$$

1b $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

$n=1$ $\quad A(1): \frac{1(2)(3)}{6} = \frac{6}{6} = 1$ ✓ igaz

$n \to n+1$

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)(n+2)(2(n+1)+1)}{6} \quad /:n+1$$

$$\frac{n(2n+1)}{6} + n+1 = \frac{(n+2)(2n+3)}{6} \quad /\cdot 6$$

$$2n^2+n+6n+6 = 2n^2+3n+4n+6$$ ✓ igaz

1c $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$

$n=1$ $\quad A(1): \frac{1}{1(2)} = \frac{1}{1+1} = \frac{1}{2}$ ✓ igaz

$n \to n+1$

$$\frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2} \quad /\cdot(n+1)(n+2)$$

$$n(n+2)+1 = (n+1)^2$$

$$n^2+2n+1 = n^2+2n+1$$ ✓ igaz

4a $2\sqrt{n+1}-2<\sum_{k=1}^{n}\frac{1}{\sqrt{k}}\leq 2\sqrt{n}-1$

$2\sqrt{n+1}-2<\sum_{k=1}^{n}\frac{1}{\sqrt{k}}$ ← ugyan az so magic →

$n=1$

$2\sqrt{2}-2<1$

$\sqrt{2}<\frac{3}{2}$

$2<\frac{9}{4}$ igaz

$n\to n+1$

$2\sqrt{n+2}-2<\sum_{k=1}^{n}\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{n+1}}$

$2\sqrt{n+2}-\cancel{2}<2\sqrt{n+1}-\cancel{2}+\frac{1}{\sqrt{n+1}}$

$2\sqrt{(n+2)(n+1)}<2n+2+1 \quad /:2 \quad /()^2$

$n^2+3n+2<n^2+2\cdot\frac{3}{2}n+9$

$2<9$ ✓ igaz

$\sum_{k=1}^{n}\frac{1}{\sqrt{k}}\leq 2\sqrt{n}-1$

$n=1$ $\quad 1\leq 2\cdot 1-1$

$1\leq 1$

✓ igaz

$n\to n+1$

$\sum_{k=1}^{n}\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{n+1}}\leq 2\sqrt{n+1}-1$

$2\sqrt{n}-\cancel{1}+\frac{1}{\sqrt{n+1}}\leq 2\sqrt{n+1}-\cancel{1}$

$2\sqrt{n(n+1)}+1\leq 2n+2 \quad /-1 \quad /:2$

$\sqrt{n^2+n}\leq n+\frac{1}{2}$

$n^2+n\leq n^2+2\cdot\frac{1}{2}n+\frac{1}{4}$

$0\leq\frac{1}{4}$ ✓ igaz

$\Longrightarrow$ IGAZ

magic

4b

$(2n)! < 2^{2n} \cdot (n!)^2$

$n=1$

$2! < 2^2 \cdot (1!)^2$

$2 < 4$ igaz

$n \to n+1$

$(2n+2)! < 2^{2n+2} \cdot ((n+1)!)^2$

$(2n)! \cdot (2n+1) \cdot (2n+2) < 2^{2n+2} \cdot ((n+1)!)^2$

$2^{2n} \cdot (n!)^2 \cdot (2n+1)(2n+2) < 2^{2n+2} \cdot ((n+1)!)^2$

$\cancel{2^{2n}} \cdot \cancel{n!} \cdot \cancel{n!} \cdot (2n+1)(2n+2) < \cancel{2^{2n}} \cdot 2^2 \cdot \cancel{n!}(n+1) \cdot \cancel{n!}(n+1)$

$4n^2+6n+2 < 4(n^2+2n+1)$

$4n^2+6n+2 < 4n^2+8n+4$

$0 < 2n+4$ ✓ igaz

[redacted] [redacted]

4c

$\frac{1}{2\sqrt{n}} < \prod_{k=1}^{n} \frac{2k-1}{2k} < \frac{1}{\sqrt{3n+1}}$ $(2 \le n \in N)$

$\frac{1}{2\sqrt{n}} < \prod_{k=1}^{n} \frac{2k-1}{2k}$

$n=2$ $\frac{1}{2\sqrt{2}} < \frac{2 \cdot 2-1}{2 \cdot 2}$

$\frac{1}{2\sqrt{2}} < \frac{3}{4} \cdot \frac{1}{2}$

$1 < \frac{3\sqrt{2}}{4}$

$4 < 3\sqrt{2}$

$16 < 9 \cdot 2$

$n \to n+1$

$\frac{1}{2\sqrt{n+1}} < \prod_{k=1}^{n} \frac{2k-1}{2k} \cdot \frac{2n+2-1}{2n+2}$

$\frac{1}{2\sqrt{n+1}} < \frac{1}{2\sqrt{n}} \cdot \frac{2n+1}{2(n+1)}$ $k\sqrt{n+1}$

$1 < \frac{1}{2\sqrt{n}} \cdot \frac{2n+1}{\sqrt{n+1}}$ $/ \cdot 2\sqrt{n(n+1)}$

$2\sqrt{n^2+n} < 2n+1$ $/ :2$ $/ ()^2$

$n^2+n < n^2 + 2 \cdot \frac{1}{2} n + \frac{1}{4}$

$0 < \frac{1}{4}$ ✓ igaz

$\prod_{k=1}^{n} \frac{2k-1}{2k} < \frac{1}{\sqrt{3n+1}}$

$n=2$ $\frac{1}{2} \cdot \frac{3}{4} < \frac{1}{\sqrt{7}}$

$\frac{3}{8} < \frac{1}{\sqrt{7}}$

$3\sqrt{7} < 8$

$3 \cdot 7 < 64$ ✓ igaz

$n \to n+1$

$\prod_{k=1}^{n} \frac{2k-1}{2k} \cdot \frac{2(n+1)-1}{2(n+1)} < \frac{1}{\sqrt{3(n+1)+1}}$

$\frac{1}{\sqrt{3n+1}} \cdot \frac{2n+1}{2n+2} < \frac{1}{\sqrt{3n+4}}$ $/\sqrt{3n+4}$

$\frac{\sqrt{3n+4}}{\sqrt{3n+1}} \cdot \frac{2n+1}{2n+2} < 1$ $/ \cdot \sqrt{3n+1}\,(2n+2)$

$\sqrt{3n+4} \cdot (2n+1) < \sqrt{3n+1} \cdot (2n+2)$ $/ ()^2$

$(3n+4)(4n^2+4n+1) < (3n+1)(4n^2+8n+4)$

$\cancel{12n^3}+\cancel{12n^2}+3n+\cancel{16n^2}+16n+\cancel{4} < \cancel{12n^3}+\cancel{24n^2}+12n+\cancel{4n^2}+8n+\cancel{4}$

$3n < 12n$

$3 < 12$ ✓ igaz

4d

$\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots \frac{1}{3n+1} > 1$

$n=1$

$\frac{1}{1+1} + \frac{1}{1+2} + \frac{1}{1+3} > 1$

$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1$ $/ \cdot 12$

$6+4+3 > 12$ ✓ igaz

$n \to n+1$

$\frac{1}{n+2} + \frac{1}{n+3} + \frac{1}{n+4} + \dots \frac{1}{3n+4} > 1$ $/ + \frac{1}{n+1}$

$\underbrace{\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \frac{1}{n+4} + \dots \frac{1}{3n+1}}_{\text{magic}} + \frac{1}{3n+2} + \frac{1}{3n+3} + \frac{1}{3n+4} > \frac{1}{n+1} + 1$

$\cancel{1} + \frac{1}{3n+2} + \frac{1}{3n+3} + \frac{1}{3n+4} > \cancel{1} + \frac{1}{n+1}$ $/ (3n+2)(3n+3)(3n+4)$ $\hookrightarrow 3(n+1)$

$(3n+3)(3n+4) + (3n+2)(3n+4) + (3n+2)(3n+3) > 2(3n+2)(3n+4)$

$(3n+3)(3n+4) + (3n+2)(3n+4) + (3n+2)(3n+3)$ ... $9n^2+12n+9n+12+9n^2+12n+6n+8+9n^2+9n+6n+6 > 3(9n^2+12n+6n+8)$

$27n^2+54n+26 > 27n^2+54n+8$ ✓ igaz