①

$u_1 = (1,1,1,1)$

$u_2 = (1,-1,-1,1)$

$u_3 = (-1,0,0,1)$

$x = (2,1,3,1) \qquad W := \mathrm{Span}(u_1,u_2,u_3)$

mivel u_1,u_2,u_3 O.R.

$$\hookrightarrow P(x) = \frac{\langle x,u_1\rangle}{\|u_1\|^2}\cdot u_1 + \frac{\langle x,u_2\rangle}{\|u_2\|^2}\cdot u_2 + \frac{\langle x,u_3\rangle}{\|u_3\|^2}\cdot u_3$$

$$= \frac{2+1+3+1}{4}\cdot(1,1,1,1) + \frac{2-1-3+1}{4}(1,-1,-1,1) + \frac{-2+0+0+1}{2}(-1,0,0,1)$$

$$= \frac{7}{4}(1,1,1,1) - \frac{1}{4}(1,-1,-1,1) - \frac{1}{2}(-1,0,0,1)$$

$$= \begin{pmatrix} \frac{7}{4}-\frac{1}{4}+\frac{2}{4} \\ \frac{7}{4}+\frac{1}{4}-0 \\ \frac{7}{4}+\frac{1}{4}-0 \\ \frac{7}{4}-\frac{1}{4}-\frac{2}{4} \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 1 \end{pmatrix}$$

$\hookrightarrow$ a merőleges komponens: $Q(x) = x - P(x) = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$

②

$b_1 = (1,1,1,1)$

$b_2 = (3,3,-1,-1)$

$b_3 = (-2,0,6,8)$

$v_1 = b_1 = (1,1,1,1)$

$$v_2 = b_2 - \frac{\langle b_2,v_1\rangle}{\langle v_1,v_1\rangle}\cdot v_1 = (3,3,-1,-1) - \frac{3+3-1-1}{4}\cdot(1,1,1,1)$$

$$= (2,2,-2,-2) \sim (1,1,-1,-1)$$

$\hookrightarrow$ egyszerűbb alak

$$v_3 = b_3 - \frac{\langle b_3,v_1\rangle}{\langle v_1,v_1\rangle}\cdot v_1 - \frac{\langle b_3,v_2\rangle}{\langle v_2,v_2\rangle}\cdot v_2 = (-2,0,6,8) - \frac{-2+0+6+8}{4}(1,1,1,1) - \frac{-2,0,-6-8}{4}(1,1,-1,-1)$$

$$= (-2,0,6,8) - (3,3,3,3) - (-4,-4,4,4)$$

$$= (-2-3+4,\ 0-3+4,\ 6-3-4,\ 8-3-4) = (-1,1,-1,1)$$

$\Rightarrow$ ekvivalens O.R

$v_1 = (1,1,1,1) \quad v_2 = (1,1,-1,-1) \quad v_3 = (-1,1,-1,1)$

$\|v_1\| = \sqrt{4} = 2$

$\|v_2\| = \sqrt{4} = 2$

$\|v_3\| = \sqrt{4} = 2$

$\Rightarrow$ ekvivalens O.N.R

$e_1 = \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) \quad e_2 = \left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right) \quad e_3 = \left(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$

(3)

$b_1 = (1,1,1,1)$

$b_2 = (3,3,-1,-1)$

$b_3 = (-2,0,6,8)$

$u_1 = (1,1,1,1)$ $u_2 = (1,1,-1,-1)$ $u_3 = (-1,1,-1,1)$ O.R. generátorrendszer

$e_1 = \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ $e_2 = \left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$ $e_3 = \left(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$ O.N.R ortonormált bázis

(4)

a) $W := \{ y \in \mathbb{R}^4 \mid 3y_1 + 2y_2 + y_3 - 2y_4 = 0 ,\ 5y_1 + 4y_2 + 3y_3 + 2y_4 = 0 \}$

$3y_1 + 2y_2 + y_3 - 2y_4 = 0 \Rightarrow y_3 = -3y_1 - 2y_2 + 2y_4$

$5y_1 + 4y_2 + 3y_3 + 2y_4 = 0$

$5y_1 + 4y_2 - 9y_1 - 6y_2 + 6y_4 + 2y_4 = 0$

$-4y_1 - 2y_2 + 8y_4 = 0$

$y_2 = -2y_1 + 4y_4 \Rightarrow y_3 = -3y_1 + 4y_1 - 8y_4 + 2y_4$

$= y_1 - 6y_4$

$$y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} y_1 \\ -2y_1 + 4y_4 \\ y_1 - 6y_4 \\ y_4 \end{pmatrix} = y_1 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + y_4 \begin{pmatrix} 0 \\ 4 \\ -6 \\ 1 \end{pmatrix}$$

$\Rightarrow$ $W = M_4$ altér egy bázisa $b_1 = (1,-2,1,0)$, $b_2 = (0,4,-6,1)$

$u_1 = b_1 = (1,-2,1,0)$

$$u_2 = b_2 - \frac{\langle b_2, u_1 \rangle}{\langle u_1, u_1 \rangle} \cdot u_1 = (0,4,-6,1) - \frac{0-8-6+1}{1+4+1+0}(1,-2,1,0)$$

$$= (0,4,-6,1) - \left(-\frac{7}{3}\right)(1,-2,1,0)$$

$$= \frac{1}{3}(7,-2,-11,3) \sim (7,-2,-11,3)$$

$\Rightarrow$ W egy ortogonális bázisa $u_1 = (1,-2,1,0)$, $u_2 = (7,-2,-11,3)$

$\|u_1\| = 1+4+1 = \sqrt{6}$

$\|u_2\| = 49+4+121+9 = \sqrt{183}$

$\Rightarrow$ O.N.B

$e_1 = \frac{1}{\sqrt{6}}(1,-2,1,0)$ $e_2 = \frac{1}{\sqrt{183}}(7,-2,-11,3)$

b) $x = (3,4,-3,5)$

$$P(x) = \frac{\langle x, u_1 \rangle}{\langle u_1, u_1 \rangle} \cdot u_1 + \frac{\langle x, u_2 \rangle}{\langle u_2, u_2 \rangle} \cdot u_2$$

$$= \frac{3-8-3+0}{6} \cdot (1,-2,1,0) + \frac{21-12+33+15}{183} (7,-2,-11,3)$$

$$= -\frac{8}{6}(1,-2,1,0) + \frac{67}{183}(7,-2,-11,3)$$

$$= (1, 2, -5, 1)$$

$$Q_2(x) = x - P(x) = \begin{pmatrix} 3 \\ 4 \\ -3 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 4 \end{pmatrix}$$

c) $3y_1 + 2y_2 + y_3 - 2y_4 = 0$

$5y_1 + 4y_2 + 3y_3 + 2y_4 = 0$

$\hookrightarrow$ homogén lineáris egyenletrendszer

$$W = \ker\left(\begin{bmatrix} 3 & 2 & 1 & -2 \\ 5 & 4 & 3 & 2 \end{bmatrix}\right)$$