Wednesday, November 08, 2023 10:16 AM

$\varphi : [0,1] \to \mathbb{R}$

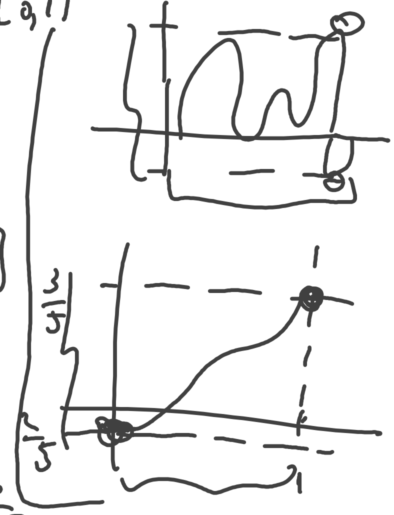
?

$\varphi : [0,1] \to [0,1]$

7.1. (i) $\varphi(x) = \dfrac{x^3+2}{5}$

$\varphi(x) = \frac{1}{5}x^3 + \frac{2}{5}$

polinom $\Rightarrow$ mon növ $[0,1]$



$\varphi(0) \rightsquigarrow$ min pont

$\varphi(1) \rightsquigarrow$ max p.

$\varphi(0) = \frac{2}{5}$ $\qquad \varphi(1) = \frac{3}{5}$

$\varphi : [0,1] \to \left[\frac{2}{5}, \frac{3}{5}\right]$

$\varphi\big[[0,1]\big] = \left[\frac{2}{5}, \frac{3}{5}\right] \subset [0,1]$ ✓

(ii) polin. $\Rightarrow$ folyt. diff.

$\varphi'(x) = \frac{3}{5}x^2 \qquad [0,1]$

$\hookrightarrow$ pol. $\Rightarrow$ mon növ.

$x \in [0,1]$ $\varphi'(x)$ max $\rightsquigarrow \varphi'(1) = \frac{3}{5} < 1 \to q$

$\Downarrow$

φ - kontr. $[0,1]$

konv ✓

(i), (ii) $\xRightarrow{\text{Banach f.p.t}}$ $\exists! x^*$

$x_{k+1} = \varphi(x_k)$

$|x_k - x^*| \leq q^k \, |x_0 - x^*|$

$x_k \in [0,1] \ni x^*$

$$\le \left(\frac{3}{5}\right)^{k} |1-0| \quad \hookrightarrow [0,1] \text{ hossz}$$

$$= \left(\frac{3}{5}\right)^{k}$$

$k = ?\ ;\ \varepsilon = \frac{1}{100}$

$$\left(\frac{3}{5}\right)^{k} = \frac{1}{100}$$

$$k \log\left(\frac{3}{5}\right) = \log\left(\frac{1}{100}\right)$$

$$k = \frac{\log \frac{1}{100}}{\log \frac{3}{5}} = 9{,}02\ldots$$

$k = 10$ elég s. it. szám

$$0 = x^3 - 5x + 2$$

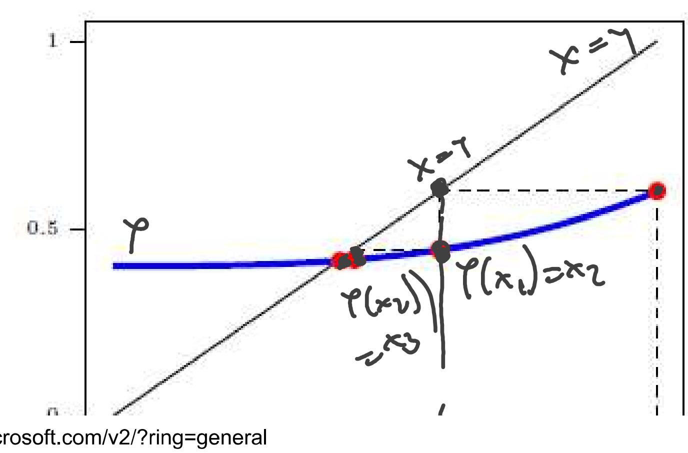
(I) $\quad 5x = x^3 + 2$

$$x = \frac{x^3+2}{5} \rightsquigarrow x_{k+1} = \frac{x_k^3+2}{5}$$

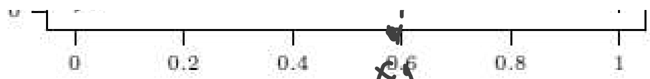
(II) $\quad 0 = x^3 - x - 4x + 2$

$$x = x^3 - 4x + 2$$

$$\rightsquigarrow x_{k+1} = x_k^3 - 4x_k + 2$$



$\varphi(x_0) = x_1$.



x_1 $\quad x_0 = 1$

$x_2 = \varphi(x_1)$

7.2.

$f(x) = x^3 - 3x + 1$

$$\left.\begin{array}{l} x=0: f(0) = 0^3 - 3\cdot 0 + 1 = 1 > 0 \\ x=1: f(1) = 1^3 - 3\cdot 1 + 1 = -1 < 0 \\ f \text{ pol.} \Rightarrow \text{folyt} \end{array}\right\} \Rightarrow$$

$\xrightarrow{\text{Bolzano-t.}}$ $[0,1]$ létezik f gyöke

(i) $x_{k+1} = \dfrac{x_k^3 + 1}{3} := \varphi(x_k)$

$\varphi : [0,1] \to ?$

φ pol $\Rightarrow$ mon. növ.

$$\left.\begin{array}{l} \text{min}: \varphi(0) = \frac{1}{3} \\ \text{max}: \varphi(1) = \frac{2}{3} \end{array}\right\} \varphi : [0,1] \to \left[\tfrac{1}{3}, \tfrac{2}{3}\right]$$

$$\varphi\big[[0,1]\big] = \left[\tfrac{1}{3}, \tfrac{2}{3}\right] \subset [0,1]$$

✓

(ii) φ pol $\Rightarrow$ folyt. diff.

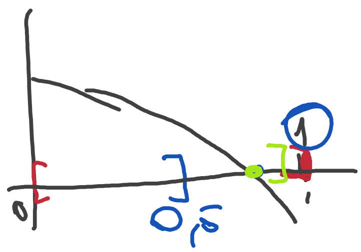
$$\varphi'(x) = \left(\frac{1}{3}x^3 + \frac{1}{3}\right)' = 3\cdot\frac{1}{3}x^2 = x^2$$

pol $\Leftrightarrow$ mon növ.

max $[0,1]$-en: $1^2 = 1$

$[0,1]$; $|\varphi'(x)| \leq 1 \Rightarrow \varphi$ nem kontr. a $[0,1]$-en

$\varphi(1) = \frac{2}{3}$



$\varphi\left(\frac{2}{3}\right) = \dfrac{\frac{8}{27}+1}{3}$

$= \frac{35}{\cancel{27}_9} \cdot \cancel{3}$

> 1

$[0,1]$ helyett $[0, 0,9]$

gyök van-e? ...

$f(0) > 0$

$f(0,9) < 0$

folyt f.

$\Bigg\}$ $\overset{\text{Bolzano}}{\Longrightarrow}$ $(0, 0,9]$ -ben van gyök

(i) (ii) ...

7.3. $f(x) = x - \sqrt{x+1} = 0$

$0 = x - \sqrt{x+1}$

$x = \sqrt{x+1}$

$x_{k+1} = \sqrt{x_k + 1}$

$\varphi(x) = \sqrt{x+1}$

(ii) $\varphi'(x) = \frac{1}{2} \frac{1}{\sqrt{x+1}}$

$= \dfrac{1}{2\sqrt{x+1}}$

... nem csökk

$$\max [0,3]: \ \varphi'(0) = \frac{1}{2}$$

$$|\varphi'(x)| \leq \frac{1}{2} = q < 1$$

$$\Rightarrow \varphi \text{ kontr } [0,3]$$

$$(i), (ii) \Rightarrow \text{konv} \quad [0,3]$$

$$|x_k - x^*| \leq \left(\frac{1}{2}\right)^k |x_0 - x^*|$$

$$\leq \left(\frac{1}{2}\right)^k 3 = \frac{3}{2^k}$$

$$k = ? \qquad \varepsilon = \frac{1}{1000}$$

$$\frac{3}{2^k} < \frac{1}{1000}$$

$$3000 < 2^k$$

$2^{10} \longmapsto 1024$
$2^{11} \longmapsto 2048$
$2^{12} \longmapsto 4096$

$k = 12$ elégséges