3a)

$f(x) := x^2 - 6x + 5$

$$\mathbb{R}_f = \{y \in \mathbb{R} \mid \exists x \in D_f : y = f(x)\} = \{y \in \mathbb{R} \mid \exists x \in \mathbb{R} : y = x^2 - 6x + 5\}$$

$x^2 - 6x + 5 = y$

$x^2 - 6x + 5 - y = 0 \qquad \Delta = 36 - 4(5-y)$

$= 36 - 20 - 4y$

$= 16 + 4y$

$$x_{1,2} = \frac{6 \pm 2\sqrt{4+y}}{2}$$

$x_{1,2} = 3 \pm \sqrt{4+y} \in \mathbb{R} \iff D = 4 + y \geq 0 \iff y \geq -4$

$y \in [-4, +\infty) = R_f$

3b)

$f(x): x^2 - 6x + 5 \quad (-1 \leq x \leq 6)$

$$R_f = \{y \in \mathbb{R} \mid \exists x \in D_f : y = f(x)\} = \{y \in \mathbb{R} \mid \exists x \in [-1, 6] : y = x^2 - 6x + 5\}$$

$\Rightarrow$ az előző alpontban kijött

$y \geq -4 \quad (1)$

$x_1 \in [-1, 6] \iff -1 \leq 3 + \sqrt{y+4} \leq 6$

$-4 \leq \sqrt{y+4} \leq 3$

$0 \leq \sqrt{y+4} \leq 3 \quad /()^2$

$y + 4 \leq 9$

$y \leq 5$

$y \in [-4, 5] \quad (2)$

$x_2 \in [-1, 6] \iff -1 \leq 3 - \sqrt{y+4} \leq 6$

$-3 \leq \sqrt{y+4} \leq 4$

$0 \leq \sqrt{y+4} \leq 4 \quad /()^2$

$y + 4 \leq 16$

$y \leq 12$

$y \in [-4, 12] \quad (3)$

(1) és (2) és (3) $\Rightarrow R_f = [-4, 12]$

egyesíteni kell

(3c) $f(x) := 1-x^2 \quad (-2 \leq x \leq 3)$

$R_f = \{ y \in \mathbb{R} \mid \exists x \in D_f : y = f(x) \} = \{ y \in \mathbb{R} \mid \exists x \in [-2,3] : y = 1-x^2 \}$

$x^2 = 1-y$

$1-y \geq 0 \wedge x_{1,2} = \pm\sqrt{1-y} \in [-2,3]$

$y \leq 1 \wedge ((-2 \leq -\sqrt{1-y} \leq 3)$ vagy $(-2 \leq \sqrt{1-y} \leq 3))$

$-2 \leq -\sqrt{1-y} \leq 3$

$-3 \leq \sqrt{1-y} \leq 2$

$0 \leq \sqrt{1-y} \leq 2 \quad /()^2$

$1-y \leq 4$

$y \geq -3$

$-2 \leq \sqrt{1-y} \leq 3$

$0 \leq \sqrt{1-y} \leq 3 \quad /()^2$

$1-y \leq 9$

$y \geq -8$

$y \in [-8,1]$ (2)

$y \in [-3, 1]$ (1)

(1) és (2) $\Rightarrow R_f = [-8, 1]$

8d)

$f(x) := \frac{3x+2}{x-1} \quad (x \in (1,+\infty))$

I $\forall x,t \in D_f: x \neq t \Rightarrow f(x) \neq f(t)$

$x,t \in D_f = (1,+\infty)$

$f(x) - f(t) = \frac{3x+2}{x-1} - \frac{3t+2}{t-1} = \frac{(3x+2)(t-1)-(3t+2)(x-1)}{(x-1)(t-1)} = \frac{5(t-x)}{(x-1)(t-1)} \neq 0$ ha $x \neq t \Rightarrow f(x) \neq f(t)$

$\Rightarrow f$ invertálható

II $x,t \in D_f: f(x) = f(t) \Rightarrow x = t$

$f(x) = f(t)$

$\frac{3x+2}{x-1} = \frac{3t+2}{t-1}$

$(3x+2)(t-1) = (3t+2)(x-1)$

$3xt + 2t - 3x - 2 = 3xt - 3t + 2x - 2$

$5x = 5t$

$x = t$

$\hookrightarrow$ invertálható

$D_{f^{-1}} = R_f$

$R_{f^{-1}} = D_f = (1,+\infty)$

$R_f = \{y \in \mathbb{R} \mid \exists x \in D_f : y = f(x)\} = \{y \in \mathbb{R} \mid \exists x \in (1,+\infty) : y = \frac{3x+2}{x-1}\}$

$y = \frac{3x+2}{x-1}$

$y(x-1) = 3x+2$

$(y-3)\cdot x = y+2$

HA $y \neq 3$

$\exists x = \frac{y+2}{y-3} = f^{-1}(y)$

$x = \frac{y+2}{y-3} > 1$

$\frac{y+2}{y-3} - 1 > 0$

$\frac{5}{y-3} > 0 \quad \Rightarrow D_{f^{-1}} = R_f = (3,+\infty) \wedge f^{-1}(y) = \frac{y+2}{y-3} \quad (y>3)$

$y > 3$

86

$f(x) := x^2 - 2x + 2 \quad (x \in (-\infty, 1])$

$\forall x, t \in D_f : x \neq t \Rightarrow f(x) \neq f(t)$

$x, t \in D_f = (-\infty, 1]$

$f(x) - f(t) = (x^2 - 2x + 2) - (t^2 - 2t + 2)$

$= (x^2 - t^2) - 2(x - t)$

$= (x - t)(x + t - 2) \neq 0$

$x \leq 1 \quad t < 1$

$x + t - 2 < 1 + 1 - 2 = 0$

↳ invertálható

$D_{f^{-1}} = R_f$

$R_{f^{-1}} = D_f = (-\infty, 1]$

$R_f = \{y \in \mathbb{R} \mid \exists x \in D_f : y = f(x)\} = \{y \in \mathbb{R} \mid \exists x \in (-\infty, 1] : y = x^2 - 2x + 2\}$

$y = x^2 - 2x + 2$

$x^2 - 2x + 2 - y = 0$

$x_{1,2} = \dfrac{2 \pm \sqrt{4 - 4(2 - y)}}{2} = 1 \pm \sqrt{y - 1}$

↳ $y - 1 \geq 0$

$y \geq 1$

$y \in [1, +\infty)$

$x_{1,2} = 1 \pm \sqrt{y - 1} \in (-\infty, 1]$

$y \geq 1$

$x_1 = 1 + \sqrt{y - 1} \leq 1$

$y \leq 1$

$y = 1$

$x_2 = 1 - \sqrt{y - 1} \leq 1$

$y \geq 1$

$y = 1 \quad x_1 = x_2 = 1$

$D_{f^{-1}} = R_f = [1, +\infty) \wedge f^{-1}(y) = 1 - \sqrt{y - 1} \quad (y \geq 1)$