1a

a) $A = \begin{bmatrix} 2 & -1 & -1 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{bmatrix}$

$$A - \lambda I = A - \lambda \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & -1 & -1 \\ 3 & -2-\lambda & -3 \\ -1 & 1 & 2-\lambda \end{bmatrix}$$

$\det(A - \lambda I) = 0$

$$\det(A - \lambda I) = (2-\lambda) \begin{vmatrix} -2-\lambda & -3 \\ 1 & 2-\lambda \end{vmatrix} + 1 \begin{vmatrix} 3 & -3 \\ -1 & 2-\lambda \end{vmatrix} - 1 \begin{vmatrix} 3 & -2-\lambda \\ -1 & 1 \end{vmatrix}$$

$$= (2-\lambda)\left((-2-\lambda)(2-\lambda) - (-3)\right) + 3(2-\lambda) - 3 - (3 - 2 - \lambda)$$

$$= (2-\lambda)(-4 + 2\lambda - 2\lambda + \lambda^2 + 3) + 6 - 3\lambda - 3 - 1 + \lambda$$

$$= (2-\lambda)(\lambda^2 - 1) + 2 - 2\lambda$$

$$= (2-\lambda)(\lambda^2 - 1) - 2(\lambda - 1)$$

$$= (\lambda - 1)\left((2-\lambda)(\lambda+1) - 2\right)$$

$$= (\lambda - 1)(2\lambda - \lambda^2 + 2 - \lambda - 2)$$

$$= (\lambda - 1)(-\lambda^2 + \lambda)$$

$$= (\lambda - 1)\lambda(-\lambda + 1) = 0$$

$\lambda_1 = 1$ $\lambda_2 = 0$ $\lambda_3 = 1$

$\lambda_1 = 1$: $a(1) = 2$ $\Leftarrow$ 2x jöH bi 1

$\lambda_2 = 0$: $a(c) = 1$ $\Leftarrow$ 1x jöH bi 0

$\lambda = 0 \Rightarrow \begin{bmatrix} 2 & -1 & -1 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{bmatrix} \cdot x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$2x_1 - x_2 - x_3 = 0 \Rightarrow x_2 = 2x_1 - x_3$

$3x_1 - 2x_2 - 3x_3 = 0$

$-x_1 + x_2 + 2x_3 = 0$

$$\left.\begin{array}{l} 3x_1 - 4x_1 + 2x_3 - 3x_3 = 0 \Rightarrow -x_1 - x_3 = 0 \\ -x_1 + 2x_1 - x_3 + 2x_3 = 0 \Rightarrow x_1 + x_3 = 0 \end{array}\right\} \Rightarrow x_1 = -x_3$$

$\hookrightarrow x_1 = -x_3$

$x_2 = -3x_3$

$x_3 \in \mathbb{R}$

$$W_0 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ -3x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} : x_3 \in \mathbb{R} \right\}$$

$\hookrightarrow$ sajátvektor pl. $(-1, -3, 1)$

$\dim W_0 = 1 \Rightarrow g(0) = 1$

$\lambda = 1 \quad \begin{bmatrix} 2-1 & -1 & -1 \\ 3 & -2-1 & -3 \\ -1 & 1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 3 & -3 & -3 \\ -1 & 1 & 1 \end{bmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left.\begin{array}{l} x_1 - x_2 - x_3 = 0 \\ 3x_1 - 3x_2 - 3x_3 = 0 \\ -x_1 + x_2 + x_3 = 0 \end{array}\right\} \Rightarrow x_1 = x_2 + x_3$$

$\hookrightarrow x_1 = x_2 + x_3$

$x_2 \in \mathbb{R}$

$x_3 \in \mathbb{R}$

$$W_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 + x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} : x_2, x_3 \in \mathbb{R} \right\}$$

sajátvektor pl. $(1, 1, 0)$ és $(1, 0, 1)$

$\dim W_1 = 2 \Rightarrow g(1) = 2$

$a(1) = g(1) = 2$

$a(0) = g(0) = 1$ $\Rightarrow$ VAN sajátbázis

1b

$$A=\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$A-\lambda I = A-\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & -1 & 1 \\ 1 & 1-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{bmatrix}$$

$\det(A)=0$

$$\det(A) = (1-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} +1 \begin{vmatrix} 1 & -1 \\ 0 & (2-\lambda) \end{vmatrix} +1 \begin{vmatrix} 1 & 1-\lambda \\ 0 & -1 \end{vmatrix}$$

$$= (1-\lambda)\Big((1-\lambda)(2-\lambda)-1\Big) + 2-\lambda -1$$

$$= (1-\lambda)(2-\lambda-2\lambda+\lambda^2-1) + 1-\lambda$$

$$= (1-\lambda)(\lambda^2-3\lambda+1+1)$$

$$= (1-\lambda)(\lambda^2-3\lambda+2)$$

$\lambda_1=1$ $\qquad$ $\Delta=9-8=1$

$$\lambda_{2,3}=\frac{3\pm 1}{2} < \begin{matrix}1\\2\end{matrix}$$

$a(1)=2 \Leftarrow 2\times$ jött ki a 1

$a(2)=1 \Leftarrow 1\times$ jött ki a 2

$$\lambda=1 \quad \begin{bmatrix} 1-1 & -1 & 1 \\ 1 & 1-1 & -1 \\ 0 & -1 & 2-1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} -x_2 + x_3 &= 0 \\ x_1 - x_3 &= 0 \\ -x_2 + x_3 &= 0 \end{aligned}\right\} \quad \begin{aligned} x_2 &= x_3 \\ x_1 &= x_3 \end{aligned}$$

$$W_1 = \left\{ x \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} : x_3 \in \mathbb{R} \right\}$$

↳ sajátvektor pl $(1,1,1)$

$\dim W_1 = g(1) = 1$

$$\lambda=2 \quad \begin{bmatrix} 1-2 & -1 & 1 \\ 1 & 1-2 & -1 \\ 0 & -1 & 2-2 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} -x_1 - x_2 + x_3 &= 0 \\ x_1 - x_2 - x_3 &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} -x_1 + x_3 &= 0 \\ x_1 - x_3 &= 0 \end{aligned}\right\} \Rightarrow x_1 = x_3$$

$-x_2 = 0 \Rightarrow x_2 = 0$

$$W_2 = \left\{ x \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} : x_1 \in \mathbb{R} \right\}$$

↳ sajátvektor pl. $(1,0,1)$

$\dim W_2 = g(2) = 1$

$a(1) \neq g(1)$
$a(2) = g(2)$ $\Rightarrow$ nincs sajátbázis

1c)

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -1 & 1 \\ 1 & 1-\lambda & -1 \\ 2 & -1 & 0-\lambda \end{bmatrix}$$

$\det(A - \lambda I) = 0$

$$\det(A-\lambda I) = (1-\lambda) \begin{vmatrix} 1-\lambda & -1 \\ -1 & -\lambda \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 \\ 2 & -\lambda \end{vmatrix} + 1 \begin{vmatrix} 1 & 1-\lambda \\ 2 & -1 \end{vmatrix}$$

$$= (1-\lambda)\left(-\lambda(1-\lambda) - 1\right) + (-\lambda + 2) + \left(-1 - 2(1-\lambda)\right)$$

$$= (1-\lambda)(\lambda^2 - \lambda - 1) - \lambda + 2 - 1 - 2 + 2\lambda$$

$$= (1-\lambda)(\lambda^2 - \lambda - 1) - (1-\lambda)$$

$$= (1-\lambda)(\lambda^2 - \lambda - 2)$$

$\lambda_1 = 1$

$\Delta = 1 + 8 = 9$

$$\lambda_{2,3} = \frac{1 \pm 3}{2} \begin{cases} 2 \\ -1 \end{cases}$$

$a(1) = 1$
$a(2) = 1$
$a(-1) = 1$

$\lambda = -1$

$$\begin{bmatrix} 1+1 & -1 & 1 \\ 1 & 1+1 & -1 \\ 2 & -1 & 0+1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$2x_1 - x_2 + x_3 = 0 \Rightarrow x_2 = 2x_1 + x_3$
$x_1 + 2x_2 - x_3 = 0$
$2x_1 - x_2 + x_3 = 0$

$x_1 + 4x_1 + 2x_3 - x_3 = 0 \Rightarrow 5x_1 + x_3 = 0$
$x_3 = -5x_1$

$\hookrightarrow$ $x_1 \in \mathbb{R}$
$x_2 = -3x_1$
$x_3 = -5x_1$

$$W_{-1} = \left\{ x = \begin{pmatrix} x_1 \\ -3x_1 \\ -5x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix} : x_1 \in \mathbb{R} \right\}$$

↳ sajátvektor $(1,-3,-5)$

$\dim W_{-1} = g(-1) = 1$

$\lambda = 1$

$$\begin{bmatrix} 1-1 & -1 & 1 \\ 1 & 1-1 & -1 \\ 2 & -1 & 0-1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 2 & -1 & -1 \end{bmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-x_2 + x_3 = 0 \Rightarrow x_3 = x_2$

$x_1 - x_3 = 0$

$2x_1 - x_2 - x_3 = 0$

$x_1 - x_2 = 0 \Rightarrow x_1 = x_2$

$2x_1 - x_2 - x_2 = 0$

$0 = 0$

↳ $x_2 \in \mathbb{R}$

$x_2 \in \mathbb{R}$

$x_2 \in \mathbb{R}$

$$W_1 = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} : x_2 \in \mathbb{R} \right\}$$

↳ sajátvektor $(1,1,1)$

$\dim W_1 = g(1) = 1$

$\lambda = 2$

$$\begin{bmatrix} 1-2 & -1 & 1 \\ 1 & 1-2 & -1 \\ 2 & -1 & 0-2 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 2 & -1 & -2 \end{bmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-x_1 - x_2 + x_3 = 0 \Rightarrow x_3 = x_1 + x_2$

$x_1 - x_2 - x_3 = 0$

$2x_1 - x_2 - 2x_3 = 0$

$$\left.\begin{array}{l} x_1 - x_2 - x_1 - x_2 = 0 \\ 2x_1 - x_2 - 2x_1 - 2x_2 = 0 \end{array}\right\} \Rightarrow x_2 = 0$$

$\hookrightarrow x_1 \in \mathbb{R}$
$x_2 = 0$
$x_3 = x_1$

$$W_2 = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} : x_1 \in \mathbb{R} \right\}$$

$\hookrightarrow$ sajátvektor $(1,0,1)$

$\dim W_2 = g(2) = 1$

$a(1) = g(1)$
$a(-1) = g(-1)$
$a(2) = g(2)$ $\Rightarrow$ VAN saját bázisa

d) $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 1-\lambda & -1 & -1 \\ 1 & 1-\lambda & 0 \\ 3 & 0 & 1-\lambda \end{bmatrix}$

$\det(A - \lambda I) = 0$

$\det(A-\lambda I) = (1-\lambda) \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 3 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1-\lambda \\ 3 & 0 \end{vmatrix}$

$= (1-\lambda)\left((1-\lambda)^2 + (1-\lambda) + 3(1-\lambda)\right)$

$= (1-\lambda)\left(1 - 2\lambda + \lambda^2 + 1 + 3\right)$

$= (1-\lambda)\left(\lambda^2 - 2\lambda\ 5\right)$

$\lambda_1 = 1$

$\Delta = 4 - 20 = 16i^2$

$\lambda_{2,3} = \frac{2 \pm 4i}{2} < \begin{matrix} 1+2i \\ 1-2i \end{matrix}$

$a(1) = 1$
$a(1+2i) = 1$
$a(1-2i) = 1$

$\lambda = 1$

$\begin{bmatrix} 1-1 & -1 & -1 \\ 1 & 1-1 & 0 \\ 3 & 0 & 1-1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$-x_2 - x_3 = 0 \Rightarrow x_2 = -x_3$
$x_1 = 0$
$3x_1 = 0$

$\hookrightarrow x_1 = 0$
$x_2 = -x_3$
$x_3 \in \mathbb{R}$

$W_1 = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} : x_3 \in \mathbb{R} \right\}$

$\hookrightarrow$ Sajátvektor $(0, -1, 1)$

$\lambda = 1+2i$

$$\begin{bmatrix} 1-1-2i & -1 & -1 \\ 1 & 1-1-2i & 0 \\ 3 & 0 & 1-1-2i \end{bmatrix} \Rightarrow \begin{bmatrix} -2i & -1 & -1 \\ 1 & -2i & 0 \\ 3 & 0 & -2i \end{bmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2ix_1 - x_2 - x_3 = 0$$
$$x_1 - 2ix_2 = 0 \quad \Rightarrow x_1 = 2ix_2$$
$$3x_1 - 2ix_3 = 0$$

$$\left.\begin{array}{l} -2i(2ix_2) - x_2 - x_3 = 0 \Rightarrow 3x_2 - x_3 \\ 6ix_2 - 2ix_3 = 0 \end{array}\right\} = x_3 = 3x_2$$

$\hookrightarrow$ $x_1 = -2ix_2$

$x_2 \in \mathbb{R}$

$x_3 = 3x_2$

$$W_{(1+2i)} = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2ix_2 \\ x_2 \\ 3x_2 \end{pmatrix} = x_2 \begin{pmatrix} -2i \\ 1 \\ 3 \end{pmatrix} : x_2 \in \mathbb{R} \right\}$$

$\hookrightarrow$ sajátvektor $(-2i, 1, 3)$

$\dim W_{(1+2i)} = g(1+2i) = 1$

$\lambda = 1-2i$

$$\begin{bmatrix} 2i & -1 & -1 \\ 1 & 2i & 0 \\ 3 & 0 & 2i \end{bmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2ix_1 - x_2 - x_3 = 0$$
$$x_1 + 2ix_2 = 0 \Rightarrow x_1 = -2ix_2$$
$$3x_1 + 2ix_3 = 0$$

$$-5x_2 - x_3 = 0$$
$$6ix_2 + 2ix_3 = 0$$