Assignment 5: ME 8930 (LMIs in Optimal and Robust Control)

Due on Dec. 11, 2023 by midnight (NO EXTENSION WILL BE GRANTED)

IMPORTANT NOTE: For Problems 1 and 3, you may use MATLAB commands for design but for Problem 2, you must code LMIs and submit your codes and outputs.

Problem 1 Consider the following spring-mass system model:

$$\begin{aligned} m_1\ddot{q}_1 + k_1 q_1 - k_1 q_2 &= f(t) + w(t) \\ m_2\ddot{q}_2 - k_1 q_1 + (k_1 + k_2) q_2 - k_2 q_3 &= 0 \\ m_3\ddot{q}_3 - k_2 q_2 + k_2 q_3 &= 0 \end{aligned}$$

where $q_i(t)$ is the displacement of the ith mass. Suppose that the monimal system parameters are $m_1 = 20$ Kg, $m_2 = 40$ Kg, $m_3 = 30$ Kg and $k_1 = 2000$ N/m, $k_2 = 4000$ N/m The actuation force $f(t)$ is used to control this system in the presence of the disturbance force $w(t)$. We are interested to design a state feedback control law of the form

$$u = Gx$$

to minimize the effects of the disturbance force w on the position variable $y = q_3$.

1. Write the equations of motion of the system in state-space form

$$\begin{aligned} \dot{x} &= A_p x + B_p u + D_p w \\ y &= C_p x \end{aligned}$$

where the state-vector x contains the positions and a the veclocities of the three masses.

2. Design a state-feedback controller G_1 to minimize the energy-to-peak gain Γ_{ep} of the system from the disturbance input w to the system output y. Compute the energy-to-peak gain of the closed-loop system to verify your result.

3. Consider the disturbance force

$$w(t) = \begin{cases} 100 & 0 \le t \le 2 \\ 0 & t > 2 \end{cases}$$

Plot the response of the open-loop (no control u) and the closed-loop system to this excitation. Verify that closed-loop system response is consistent with your result in #1.

4. Suppose that there is a 10% uncertainty in the value of the spring constant k_2, that is $3600 \leq k_2 \leq 4400$. Plot the energy-to-peak gain of the closed-loop system (i.e., the spring-mass system with the controller G_1) when k_2 is varying in the above range.

5. Design a *robust* state-feedback controller G_2 to minimize the energy-to-peak gain of the system when k_2 is varying in the above range (use a polytopic of affine representation of the system and the LMI Toolbox command `msfsys`). What is the optimal guaranteed energy-to-peak gain for the ubcertain system?

6. Plot the energy-to-peak gain of the closed-loop system (i.e., the spring-mass system with the controller G_2) when k_2 is varying in the above range.

Problem 2 Consider the following system that corresponds to a double integrator

$$\begin{aligned}
\dot{x}_p &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} w \\
y &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
z &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_p + \begin{bmatrix} 0 & 1 \end{bmatrix} w
\end{aligned}$$

We seek to obtain a reduced order controller of order 1 to guarantee a energy-to-energy gain constraint of $\Gamma_{ee} < \gamma$.

1. To obtain a reduce-order controller, solve the following optimization problem for X and Y

$$\underset{X,Y}{\text{minimize}} \ \text{trace}(X+Y)$$

subject to the constraints:

$$\begin{aligned}
\begin{bmatrix} B_p \\ B_y \end{bmatrix}^{\perp} \begin{bmatrix} A_pX + XA_p^T + D_pD_p^T & XC_p^T + D_pD_y^T \\ C_pX + D_yD_p^T & D_yD_y^T - \gamma^2 I \end{bmatrix} \begin{bmatrix} B_p \\ B_y \end{bmatrix}^{\perp T} &< 0 \\
\begin{bmatrix} M_p^T \\ D_z^T \end{bmatrix}^{\perp} \begin{bmatrix} YA_p + A_p^TY + C_p^TC_p & YD_p + C_p^TD_y \\ D_p^TY + D_y^TC_p & D_y^TD_y - \gamma^2 I \end{bmatrix} \begin{bmatrix} M_p^T \\ D_z^T \end{bmatrix}^{\perp T} &< 0 \\
\begin{bmatrix} X & \gamma I \\ \gamma I & Y \end{bmatrix} &\geq 0
\end{aligned}$$

for $\gamma = 3$, 4 and 5.

2. Confirm that

$$\text{rank}\begin{bmatrix} X & \gamma I \\ \gamma I & Y \end{bmatrix} = 3$$

that guarantees the existence of a 1st order controller. Use the SVD of $Y - \gamma^2 X^{-1}$ to obtain matrices $Y_{12} \in \mathbb{R}^{2\times 1}$ and $Y_{22} \in \mathbb{R}^{1\times 1}$ such that $Y - \gamma^2 X^{-1} = Y_{12} Y_{22}^{-1} Y_{12}^T$

3. Define the augmented matrix

$$P = \begin{bmatrix} Y & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix}.$$

Find the unknown controller parameters by solving the General Matrix Inequality

$$\Gamma G \Lambda + (\Gamma G \Lambda)^T + Q < 0$$

where

$$\Gamma = \begin{bmatrix} PB \\ 0 \\ H \end{bmatrix}, \Lambda = \begin{bmatrix} M & E & 0 \end{bmatrix}, Q = \begin{bmatrix} PA + A^T P & PD & C^T \\ D^T P & -\gamma^2 I & F^T \\ C & F & -I \end{bmatrix}$$

and

$$G = \begin{bmatrix} d_c & c_c \\ b_c & a_c \end{bmatrix}$$

4. Now, take a different approach to design the reduced (first)-order controller. This time, solve the LMIs derived in class and implement the alternating projection (AP) method between (C1)-(C3) and (C4).

Problem 3 Consider the following state-space model of the longitudinal dynamics of an F-8 aircraft:

$$
\dot{x} = \begin{bmatrix} 0 & 0 & 1.0 & 0 & 0 & 0 & 0 & 0 \\ 1.5 & -1.5 & 0 & 0.0057 & 1.5 & 0 & 0 & 0 \\ -12 & 12 & -0.6 & -0.0344 & -12 & 0 & 0 & 0 \\ -0.852 & 0.29 & 0 & -0.014 & -0.29 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.73 & 2.82940625 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1000 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0.16 & 0.8 \\ -19 & -3 \\ -0.0115 & -0.0087 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.1149 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 1024 & 0 \\ 0 & 0 & 1024 \end{bmatrix} w
$$

$$
y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.01 & 0 \\ 0 & 0.01 \end{bmatrix} u + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} w
$$

$$
z = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -139.020647321 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -139.020647321 \end{bmatrix} x + \begin{bmatrix} 0 & 142.857142857 & 0 \\ 0 & 0 & 142.857142857 \end{bmatrix} w
$$

We are interested to solve the optimal H_∞ control problem that corresponds to a wind gust disturbance rejection with the above state-space pepresentation. Find a full-order optimal H_∞ controller and the optimal energy-to-energy gain. Plot the response $y_1(t)$ (pitch angle) of the system for the disturbance input

$$
w(t) = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} \delta(t)
$$

where $\delta(t)$ is the an impulse disturbance.