$$\begin{cases} \dot{x}_p = \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w \\ y = \begin{bmatrix} 3 & 1 \end{bmatrix} x_p + u + 2w \\ z = \begin{bmatrix} 1 & 0 \end{bmatrix} x_p + 2w \end{cases} \quad - (1)$$

$$\begin{cases} \dot{x}_c = -4x_c + 2z \\ u = x_c - 2z \end{cases} \quad - (2)$$

Closed loop system equations:

$$\dot{x}_{cl} = A_{cl} \cdot x_{cl} + B_{cl} \cdot w \quad - (3)$$

$$y = C_{cl} \cdot x_{cl} + D_{cl} \cdot w \quad - (4)$$

We know that:

$$\text{System } S: \begin{cases} \dot{x}_p = A_p x_p + B_p u + D_p w \\ y = C_p x_p + B_y u + D_y w \\ z = M_p x_p + \cancel{B_z}^{\,0} u + D_z w \end{cases} \quad - (5)$$

$$\text{Controller } C: \begin{cases} \dot{x}_c = A_c x_c + B_c z \\ u = C_c x_c + D_c z \end{cases} \quad - (6)$$

Comparing (1) & (2) with (5) and (6),

$$\left.\begin{aligned} & A_p = \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix} \quad B_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad D_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ & C_p = \begin{bmatrix} 3 & 1 \end{bmatrix} \quad B_y = 1 \quad D_y = 2 \\ & M_p = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad B_z = 0 \quad D_z = 2 \\ & A_c = -4 \quad B_c = 2 \\ & C_c = 1 \quad D_c = -2 \end{aligned}\right\} - (7)$$

$$
\left.
\begin{aligned}
A_{cl} &= \begin{bmatrix} A_p + B_p D_c M_p & B_p C_c \\ B_c M_p & A_c \end{bmatrix} \\
B_{cl} &= \begin{bmatrix} D_p + B_p D_c D_z \\ B_c D_z \end{bmatrix} \\
C_{cl} &= \begin{bmatrix} C_p + B_y D_c M_p & B_y C_c \end{bmatrix} \\
D_{cl} &= D_y + B_y D_c D_z
\end{aligned}
\right\} - 8
$$

Substitute 7 in 8 to find 3 and 4

To find H_∞ norm:

$$
\min \ \gamma
$$

$$
\text{s.t.} \quad \begin{bmatrix} PA + A^T P & PB & C^T \\ B^T P & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix}
$$

$$
P > 0
$$

$\downarrow$

Solve (Python)

$\downarrow$

γ^*

H_∞ norm of system
(energy to energy gain Γ_{ee})
(induced L_2 gain) $= \gamma^*$

Python Code:

```
import numpy as np
import control as ctrl
import cvxpy as cp

# Define plant matrices
Ap = np.array([[3, 1], [-2, 2]])
Bp = np.array([[0], [1]])
Dp = np.array([[0], [1]])
Cp = np.array([[3, 1]])
By = np.array([[1]])
Dy = np.array([[2]])
Mp = np.array([[1, 0]])
Bz = np.array([[0]])
Dz = np.array([[2]])

# Define controller matrices
Ac = np.array([[-4]])
Bc = np.array([[2]])
Cc = np.array([[1]])
Dc = np.array([[-2]])

# Compute closed-loop matrices
Acl = np.vstack((np.hstack((Ap+(Bp@Dc@Mp), Bp@Cc)),
                 np.hstack((Bc@Mp, Ac))))
Bcl = np.vstack((Dp+(Bp@Dc@Dz),
                 Bc@Dz))
Ccl = np.hstack((Cp+(By@Dc@Mp), By@Cc))
Dcl = Dy+(By@Dc@Dz)

# Display closed-loop matrices
print('Closed-loop system matrices:\n')
print("Acl:")
print(Acl)
print("\nBcl:")
print(Bcl)
print("\nCcl:")
print(Ccl)
print("\nDcl:")
print(Dcl)

# Convert to state space form
sys_cl = ctrl.ss(Acl, Bcl, Ccl, Dcl)
```

```python
# Check stability
eigenvalues = np.linalg.eigvals(Acl)
if all(np.real(eig) < 0 for eig in eigenvalues):
    print("\nThe closed-loop system is stable")
else:
    print("\nThe closed-loop system is unstable")

# Calculate H∞ norm
P = cp.Variable((3, 3), symmetric=True)
gamma = cp.Variable(1)
M11 = P@Acl + Acl.T@P
M12 = P@Bcl
M13 = Ccl.T
M21 = Bcl.T@P
M22 = cp.multiply(-gamma,np.eye(1))
M23 = Dcl.T
M31 = Ccl
M32 = Dcl
M33 = cp.multiply(-gamma,np.eye(1))
# LMI Problem
LMI = cp.vstack([
    cp.hstack([M11[0][0], M11[0][1], M11[0][2], M12[0][0], M13[0][0]]),
    cp.hstack([M11[1][0], M11[1][1], M11[1][2], M12[1][0], M13[1][0]]),
    cp.hstack([M11[2][0], M11[2][1], M11[2][2], M12[2][0], M13[2][0]]),
    cp.hstack([M21[0][0], M21[0][1], M21[0][2], M22[0],    M23[0][0]]),
    cp.hstack([M31[0][0], M31[0][1], M31[0][2], M32[0][0], M33[0]])
])
constraints = [LMI << 0, P >> 0]
# Set up the optimization problem
objective = cp.Minimize(gamma)
problem = cp.Problem(objective, constraints)
# Solve the LMI problem
problem.solve()

if problem.status == 'optimal':
    # Get the value of optimal gamma
    gamma_star = gamma.value[0]
    hinfinity_norm = gamma_star
else:
    hinfinity_norm = np.inf
print(f"\nThe H∞ norm of the closed-loop system is: {hinfinity_norm:.4f}")
```

Output:

Closed-loop system matrices:

Acl:

```
[[ 3  1  0]
 [-4  2  1]
 [ 2  0 -4]]
```

Bcl:

```
[[ 0]
 [-3]
 [ 4]]
```

Ccl:

```
[[1 1 1]]
```

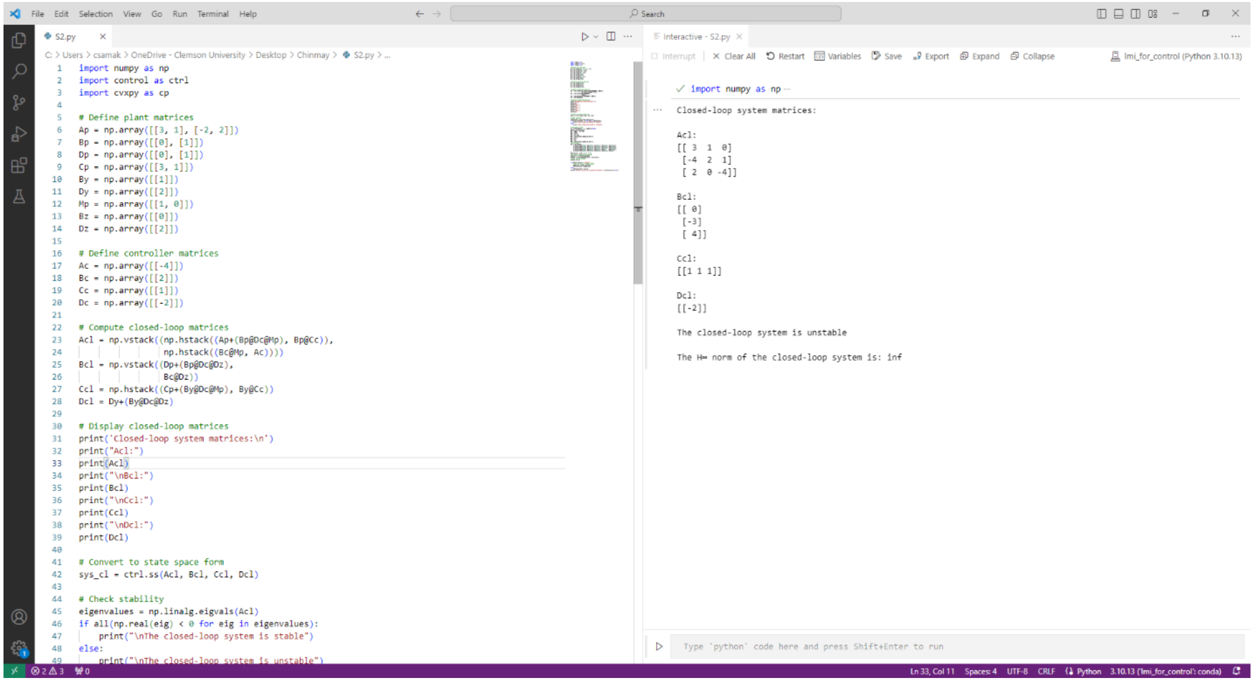
Dcl:

```
[[-2]]
```

The closed-loop system is unstable

The H∞ norm of the closed-loop system is: inf

Screenshot:


File Edit Selection View Go Run Terminal Help
Search
S2.py
C: > Users > csamak > OneDrive - Clemson University > Desktop > Chinmay > S2.py > ...
import numpy as np
import control as ctrl
import cvxpy as cp
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Dp = np.array([[0], [1]])
Cp = np.array([[3, 1]])
By = np.array([[1]])
Dy = np.array([[2]])
Mp = np.array([[1, 0]])
Bz = np.array([[0]])
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Acl = np.vstack((np.hstack((Ap+(Bp@Dc@Mp), Bp@Cc)),
np.hstack((Bc@Mp, Ac))))
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print("\nThe closed-loop system is stable")
else:
print("\nThe closed-loop system is unstable")
Interactive - S2.py
Interrupt
Clear All
Restart
Variables
Save
Export
Expand
Collapse
lmi_for_control (Python 3.10.13)
import numpy as np ...
Closed-loop system matrices:
Acl:
[[3 1 0]
[-4 2 1]
[2 0 -4]]
Bcl:
[[0]
[-3]
[4]]
Ccl:
[[1 1 1]]
Dcl:
[[-2]]
The closed-loop system is unstable
The H∞ norm of the closed-loop system is: inf
Type 'python' code here and press Shift+Enter to run
Ln 33, Col 11
Spaces: 4
UTF-8
CRLF
Python
3.10.13 ('lmi_for_control': conda)