Robust Stability with Structured Real Parameter Perturbations

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Abstract—**This paper considers the problem of robust stabilization of a linear time-invariant system subject to variations of a real parameter vector. For a given controller the radius of the largest stability hypersphere in this parameter space is calculated. This radius is a measure of the stability margin of the closed-loop system. The results developed are applicable to all systems where the closed-loop characteristic polynomial coefficients are linear functions of the parameters of interest. In particular, this always occurs for single-input (multioutput) or single-output (multiinput) systems where the transfer function coefficients are linear or affine functions of the parameters. Many problems with transfer function coefficients which are nonlinear functions of physical parameters can be cast into this mathematical framework by suitable weighting and redefinition of functions of physical parameters as new parameters. The largest stability hyperellipsoid for the case of weighted perturbations**

known, a multiinput multioutput system can always be reduced to a single-input or single-output one by a preliminary constant output feedback. Although this may lead to conservative results in some cases, it is clear that a broad class of practical problems can be effectively treated by this approach.

In Section II we formulate the problem of calculating the largest stability hypersphere when the parameter consists of the transfer function coefficients of a single-output or single-input system. This calculation is given in Section III, and extended in Section IV to the case where the transfer function coefficients are an affine function of the primary parameters. This allows us to treat the case of interdependent parameters. In Section V the case of weighted perturbations is considered and the largest stability hyperellipsoid is determined. In Section VI a prescribed polytype

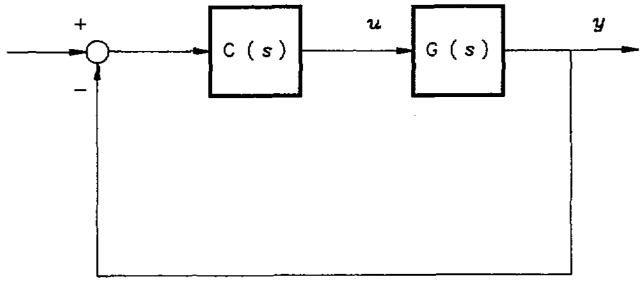


Fig. 1. Feedback system.

Let

$$p := [n_0^T, d_0, \cdots, n_q^T, d_q]^T \tag{2.3}$$

denote the plant parameter vector in R^k, $k = (1 + m)(1 + q)$,

$$p^0 := [n_0^{0T}, d_0^0, \cdots, n_q^{0T}, d_q^0]^T \tag{2.4}$$

its nominal value, and

$$\Delta p := [\Delta n_0^T, \Delta d_0, \cdots, \Delta n_q^T, \Delta d_q]^T \tag{2.5}$$

its perturbation, so that

$$p = p^0 + \Delta p. \tag{2.6}$$

The size of the real perturbation vector Δp is measured by its Euclidean length, denoted by $\|\Delta p\|_2$ and is given by

$$\|\Delta p\|_2^2 = \|\Delta n_0\|_2^2 + \|\Delta n_1\|_2^2 + \cdots + \|\Delta n_q\|_2^2 + (\Delta d_0)^2 + \cdots + (\Delta d_q)^2.$$

Clearly, larger (smaller) values of $\|\Delta p\|_2$ correspond to larger (smaller) perturbations of the transfer function coefficients n_i and d_i. For a given stabilizing controller there exists a largest value $\rho(p^0)$ of $\|\Delta p\|_2$ for which closed-loop stability is preserved. To rule out trivial cases we make the standing assumption that $n(s)$ and $d(s)$ continue to be right coprime under these perturbations. This value therefore serves as a measure of *stability margin*. Based on these considerations we formulate several problems to be solved in this paper.

Problem A Determining the Stability Hypersphere: For a given stabilizing controller $C(s)$ determine the radius $\rho(p^0)$ of the stability hypersphere centered at p^0 defined by the condition that whenever $\|\Delta p\|_2 < \rho(p^0)$ the closed-loop system with plant parameter $p^0 + \Delta p$ is stable and there exists at least one perturbation $\overline{\Delta p}$ with $\|\overline{\Delta p}\|_2 = \rho(p^0)$ such that the closed-loop system with the parameter $p^0 + \overline{\Delta p}$ is not stable.

Problem B Robust Controller Design: Let $C(s)$ denote a stabilizing controller with an adjustable parameter vector $x \in R^s$. Determine x so that the radius $\rho(p^0, x)$ of the stability hypersphere is maximized as a function of x.

Solutions of the above two problems will also allow us to treat the following types of perturbation classes.

i) Let Δp_i, the ith component of Δp, be bounded by

$$-\gamma_i < \Delta p_i < \epsilon_i \tag{2.7}$$

for given positive numbers γ_i, ϵ_i, $i = 1, \cdots, k$.

ii) Let the perturbation bounds be given by

$$-w_i\epsilon < \Delta p_i < w_i\epsilon, \qquad i = 1, \cdots, k \tag{2.8}$$

where w_i are weights and ϵ is a positive constant.

A natural solution to the problem with the above perturbation classes is obtained within the framework of our approach by determining the largest stability *ellipsoid* in parameter space.

A special case of (2.8) was treated in [7] where p was considered to be the characteristic polynomial coefficient vector and Kharitonov's theorem [8] was used to determine the maximum value of ϵ. Kharitonov's theorem is considered to be one of the most important recent results on stability of sets of polynomials. This result has been given an interesting system theoretic treatment in [9] where an alternative proof is given. The same type of problem has been treated in several recent papers, on the stability of sets of polynomials (see [7]–[18]). Of these the papers [8]–[10] and [13]–[17] deal with independent perturbations of the coefficients of the characteristic polynomial and various sufficient conditions for stability are given. In [11] and [12] the geometry of Hurwitz polynomials for discrete systems has been studied again in the space of coefficients of the characteristic polynomial. The maximization of the general "box-type" kind of perturbations described in (2.7) has been dealt with in [18] and a reviewer has pointed out to us that the problem of maximizing ϵ in (2.8) corresponds to the multiloop stability margin formulated in [32] and [33].

Kharitonov's theorem, which is the basic tool in some of the above papers, does not hold in the space of the plant parameter vector p. One-dimensional convexity of the stability region in the parameter space is required for the existence of Kharitonov like theorems. This would require convexity of the stability region in the space of the characteristic polynomial coefficients, which does not hold in general.

An important recent result by C. B. Soh *et al.* [16] has determined the largest stability hypersphere in the space of coefficients of the characteristic polynomial. The concept of the largest stability hypersphere in the space of coefficients of the characteristic polynomial also occurs in [19] and [20] where it is used in the synthesis of a controller which is robust with respect to transfer function coefficient perturbations. It is restrictive, however, to assume that all coefficients of the closed-loop characteristic polynomial are subject to independent perturbations and this is why the results of [16] and of [19] and [20] are limited especially when used for synthesis. In the present paper we consider the more realistic situation where a real parameter vector originating in the plant is subject to perturbation. By exploiting the methodology introduced in [16] we develop a new approach to the problem of robustification of a given system against real parameter variations.

The justification for using $\rho(p^0, x)$ as a stability margin is that if x_1 and x_2 are two controllers with

$$\rho(p^0, x_1) > \rho(p^0, x_2) \tag{2.9}$$

then clearly x_1 is more robust than x_2 because the stability hypersphere $S_\rho(p^0, x_1)$ is larger than $S_\rho(p^0, x_2)$ and in fact

$$S_\rho(p^0, x_2) \subset S_\rho(p^0, x_1) \tag{2.10}$$

so that the family of perturbed plants stabilized by x_1 contains the family stabilized by x_2. From this it follows that x^*, a maximally robust controller, is

$$\rho(p^0, x^*) \geq \rho(p^0, x) \qquad \forall x \in R^s \text{ which is stabilizing.} \tag{2.11}$$

In the following sections the above problems are solved, first in the space of coefficients of the plant transfer function. These results are then used to extend the solutions to the space of primary parameters and this allows us to treat the important practical case where the transfer function coefficients are interdependent. The results are illustrated by examples.

III. Calculation of the Stability Hypersphere

Let

$$C(s) = \frac{1}{d_c(s)}(n_{c1}(s), \cdots, n_{cm}(s)) := d_c^{-1}(s) n_c^T(s) \tag{3.1}$$

and write

$$d_c(s)=d_{cp}(s)s^p+\cdots+d_{c0}$$
$$n_c^T(s)=n_{cp}^Ts^p+\cdots+n_{c0}^T \tag{3.2}$$

where d_{ci} are real scalars, n_{ci} are $1 \times m$ constant real vectors, and d_c, n_c^T are left coprime.

The characteristic polynomial of the closed-loop system of Fig. 1 is given from (2.1) and (3.1) by the $p+q$ degree polynomial

$$\delta(s)=d_c(s)d(s)+n_c^T(s)n(s). \tag{3.3}$$

Write

$$\delta(s) := \delta_0+\delta_1 s+\cdots+\delta_n s^n,\ n := p+q \tag{3.4}$$

and define the closed-loop *characteristic vector* by

$$\delta := [\delta_n, \delta_{n-1}, \cdots, \delta_0]^T \in R^{n+1}. \tag{3.5}$$

Definition: The characteristic vector $\delta \in R^{n+1}$ is said to be Hurwitz if and only if the corresponding polynomial $\delta(s)$ in (3.4) is strictly Hurwitz (i.e., has roots in the open left half-plane). The set of Hurwitz vectors in R^{n+1} corresponding to nth degree Hurwitz polynomials is denoted by $H_n \subset R^{n+1}$.

The controller $C(s)$ may be viewed as an operator that maps the plant parameter vector p into the closed-loop characteristic vector δ. Let δ^0 denote the image under this mapping of the nominal plant parameter p^0, as defined in (2.4). If $\Delta := \{\Delta p\}$ denotes a given class of perturbation of p^0 against which stabilization is required, then the controller $C(s)$ must map $p^0+\Delta p$ into $\delta \in H_n$ for every $\Delta p \in \Delta$. These ideas can be sharpened by the following lemma which makes the above-mentioned mapping specific.

Lemma 3.1: The closed-loop characteristic vector δ satisfies

$$Xp=\delta \tag{3.6}$$

as well as

$$Px=\delta \tag{3.7}$$

where $X \in R^{(p+q+1)\times[(1+m)(1+q)]}$, $P \in R^{(p+q+1)\times[(1+m)(p+1)]}$, $x \in R^{(1+m)(p+1)}$ and $p \in R^{(1+m)(1+q)}$

$$X=\begin{bmatrix} & & & & & & n_{cp} & d_{cp}\\ & & n_{cp} & d_{cp} & \cdot & \cdot & \cdot & \cdot\\ n_{cp} & d_{cp} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & n_{c0} & d_{c0}\\ \cdot & \cdot & n_{c0} & d_{c0} & & & & \\ n_{c0} & d_{c0} & & & & & & \end{bmatrix} \tag{3.8}$$

$$P=\begin{bmatrix} n_q^T & d_q & & & & & & \\ \cdot & \cdot & n_q^T & d_q & & & & \\ \cdot & \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & n_q^T & d_q\\ n_0^T & d_0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ & & n_0^T & d_0 & \cdot & \cdot & \cdot & \cdot\\ & & & & & & \cdot & \cdot\\ & & & & & & \cdot & \cdot\\ & & & & & & n_0^T & d_0 \end{bmatrix} \tag{3.9}$$

$$x=(n_{cp}\ d_{cp}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ n_{c0}\ d_{c0})^T. \tag{3.10}$$

The proof of this lemma is a simple calculation obtained by equating coefficients in (3.3) and is omitted. The lemma shows that the compensator maps the parameter p into the characteristic vector δ via the *linear* transformation X in (3.8). Analogously, the plant maps the compensator parameter vector x into δ via the linear transformation P in (3.9). Without loss of generality we make the standing assumption that the matrix X has full rank since otherwise the vector p or δ can be suitably redefined to make it so.

Now, define the following sets in the space of the characteristic vector δ:

$$\Delta_0 := \{\delta \mid \delta \in R^{n+1},\ \delta_0=0\} \tag{3.11}$$

$$\Delta_n := \{\delta \mid \delta \in R^{n+1},\ \delta_n=0\} \tag{3.12}$$

and for any real ω, $0 \le \omega < \infty$,

$$\Delta(\omega) := \{\delta \mid \delta \in R^{n+1},\ \delta(s)=(s^2+\omega^2)l(s),\ l(s) \text{ arbitrary}\}. \tag{3.13}$$

The inverse image of each of these sets with respect to the compensator map X in the parameter space of p is defined next:

$$\Pi_0 := X^{-1}(\Delta_0) := \{p \mid p \in R^k,\ Xp \in \Delta_0\} \tag{3.14}$$

$$\Pi_n := X^{-1}(\Delta_n) := \{p \mid p \in R^k,\ Xp \in \Delta_n\} \tag{3.15}$$

$$\Pi(\omega) := X^{-1}(\Delta(\omega)) := \{p \mid p \in R^k,\ Xp \in \Delta(\omega)\}. \tag{3.16}$$

Note that Π_0, Π_n, and $\Pi(\omega)$ are not empty as $p=\mathbf{0} \in \Pi_0 \cap \Pi_n \cap \Pi(\omega)$.

Let r_0, r_n, and $r(\omega)$ denote the Euclidean distances between p^0 and Π_0, Π_n, and $\Pi(\omega)$, respectively. Then

$$r_0 := \|p^0-t_0^*\|_2 \tag{3.17}$$

where $t_0^* \in \Pi_0$ and

$$\|t_0^*-p^0\|_2 \le \|t_0-p^0\|_2,\ \forall t_0 \in \Pi_0. \tag{3.18}$$

Similarly,

$$r_n := \|p^0-t_n^*\|_2 \tag{3.19}$$

where $t_n^* \in \Pi_n$ and

$$\|t_n^*-p^0\|_2 \le \|t_n-p^0\|_2,\ \forall t_n \in \Pi_n \tag{3.20}$$

and

$$r(\omega) := \|p^0-t^*(\omega)\|_2 \tag{3.21}$$

where $t^*(\omega) \in \Pi(\omega)$ and

$$\|t^*(\omega)-p^0\|_2 \le \|t(\omega)-p^0\|_2,\ \forall t(\omega) \in \Pi(\omega). \tag{3.22}$$

Let

$$r := \inf_{0 \le \omega < \infty} r(\omega). \tag{3.23}$$

We are now ready to state one of the main results of this paper.

Theorem 3.1: Let $C(s)$ be a fixed stabilizing controller as in (3.1) and (3.2). Then the radius of the largest stability hypersphere in the space of p, centered at p^0, is given by,

$$\rho(p^0)=\min\{r_0, r_n, r\}. \tag{3.24}$$

The proof of this theorem is given in Appendix A.

Next we give formulas for the calculation of the distances r_0, r_n, and r. Let $\delta \in \Delta_0$, $t \in \Pi_0$ and let $w_1=[0, 0, \cdots, 0, 1]^T \in R^{n+1}$. Then from (3.6)

$$\delta=Xt \tag{3.25}$$

and from (3.11),

$$w_1^T\delta=\delta_0=0=w_1^TXt. \tag{3.26}$$

Let X_l denote the last row of X. Then (3.26) can be rewritten

$$Xt=0 \tag{3.27}$$

which shows that the shortest distance from p^0 to Π_0 must lie on the normal X_l^T to Π_0. Therefore, the shortest vector is given by

$$p^0-t=\alpha X_l^T. \tag{3.28}$$

To determine α, premultiply (3.28) by X_l and use (3.27)

$$X_lp^0-X_lt=\alpha X_lX_l^T \tag{3.29}$$

$$X_lp^0=\alpha X_lX_l^T \tag{3.30}$$

so that

$$\alpha=\frac{X_lp^0}{X_lX_l^T}. \tag{3.31}$$

so that

$$t_I(\omega)=X_I^{-1}\Phi(\omega)-X_I^{-1}X_Jt_J(\omega).$$

Then a representative vector $t(\omega)\in\Pi(\omega)$ is given by,

$$t(\omega)=\underbrace{\begin{pmatrix} X_I^{-1}\Phi(\omega) & -X_I^{-1}X_J \\ \mathbf{0} & I \end{pmatrix}}_{P(\omega)}\underbrace{\begin{pmatrix} 1 \\ t_J \end{pmatrix}}_{l_t} \tag{3.38}$$

$$:=\ P(\omega)l_t \tag{3.39}$$

where $P(\omega)$ is a fixed real matrix for each ω and l_t is an arbitrary real vector. Note that the dependence of t_J on ω can be dropped since t_J can be any vector. By letting l_t sweep over all real vectors in (3.39) we generate all solutions of (3.36). Now

$$t(\omega)-p^0=P(\omega)l_t-p^0 \tag{3.40}$$

and

$$\|t(\omega)-p^0\|^2=p^{0^T}p^0-2l_t^TP^T(\omega)p^0+l_t^TP^T(\omega)P(\omega)l_t. \tag{3.41}$$

and since $\Delta(\infty) \subset \Delta_n$, $r(\infty) \geq r_n$. Similarly, since $\Delta(0) \subset \Delta_0$, if the minimum of $r(\omega)$ occurs at $\omega = 0$, we have $r(0) \geq r_0$. Therefore, the global minimum of $r(\omega)$ need be found only in the interior of the interval $0 \leq \omega \leq \infty$.

The above calculations show that Theorem 1 provides a constructive procedure for calculating the stability hypersphere. In the next section we show how this calculation can be extended to handle situations where the transfer function coefficients are interdependent.

IV. Calculations in the Space of Primary Parameters

Let

$$a := [a_1, a_2, \cdots, a_l]^T \tag{4.1}$$

denote the vector of primary parameters in R^l,

$$a^0 := [a_1^0, a_2^0, \cdots, a_l^0]^T \tag{4.2}$$

its nominal value and

$$\Delta a := [\Delta a_1, \Delta a_2, \cdots, \Delta a_l]^T \tag{4.3}$$

its perturbation, so that

$$a = a^0 + \Delta a. \tag{4.4}$$

Let us assume that the vector p in (2.3) of the plant transfer function coefficients depends linearly on a as

$$p = Aa + b, \tag{4.5}$$

where $A \in R^{k \times l}$ and $b \in R^k$. Without loss of generality we can assume that A is of full column rank and therefore $l \leq k$ (otherwise the parameters a could be redefined).

According to (3.6) and (4.5) the closed-loop characteristic polynomial vector δ is now expressed as

$$XAa + Xb = \delta \tag{4.6}$$

which shows the *affine* transformation mapping the parameter vector a into the characteristic vector δ. As before, let us consider the sets (3.11)–(3.13) and denote the inverse images of Δ_0, Δ_n, and $\Delta(\omega)$ [analogous to (3.14)–(3.16)] in the space of a as Π_0, Π_n, and $\Pi(\omega)$. It is to be noted, however, that now some of these sets may be empty. Therefore, definitions (3.17), (3.19), and (3.21) with a substituted for p have to be augmented by

$$r_0 = \infty \qquad \text{if } \Pi_0 = \emptyset \tag{4.7}$$

$$r_n = \infty \qquad \text{if } \Pi_n = \emptyset \tag{4.8}$$

$$r(\omega) = \infty \qquad \text{if } \Pi(\omega) = \emptyset. \tag{4.9}$$

Defining r as in (3.23) we can generalize Theorem 3.1 as follows.

Theorem 4.1: Let $C(s)$ be a fixed stabilizing controller as in (3.1) and (3.2). Then the radius of the largest stability hypersphere in the space of primary parameters centered at a^0, is given by

$$\rho(a^0) = \min\{r_0, r_n, r\}. \tag{4.10}$$

The proof of this theorem is similar to the proof of Theorem 3.1 and is omitted.

We now give formulas for the calculation of the distances r_0, r_n, and r in the space of a. We note that $t \in \Pi_0$ if and only if

$$X_l A t + X_l b = 0. \tag{4.11}$$

The above equation fails to hold if and only if the vector $X_l A = 0$, $X_l b \neq 0$ and then Π_0 is empty ($r_0 = \infty$). Otherwise, if $X_l A \neq 0$, the distance r_0 of the point a^0 from the hyperplane (4.11) is given by the formula

$$r_0 = \frac{1}{\|X_l A\|_2} |X_l A a^0 + X_l b| \tag{4.12}$$

or

$$r_0^2 = \frac{1}{X_l A A^T X_l^T} [a^{0T} A^T X_l^T X_l A a^0 + 2a^{0T} A^T X_l^T X_l b + b^T X_l^T X_l b]. \tag{4.13}$$

The distance r_n is calculated similarly as

$$r_n = \begin{cases} \infty & \text{if } X_f A = 0 \text{ and } X_f b \neq 0, \\ \dfrac{1}{\|X_f A\|_2} |X_f A a^0 + X_f b & \text{if } X_f A \neq 0 \end{cases} \tag{4.14}$$

It should be mentioned that it is not possible that $X_l A = 0$ and $X_l b = 0$ (or $X_f A = 0$ and $X_f b = 0$) simultaneously if the nominal point a^0 is stabilized by $C(s)$.

The calculation of the distance $r(\omega)$ is now given. After exactly analogous derivations as in (3.34)–(3.38) the formula (3.38) can be written as

$$t(\omega) = At_a(\omega) + b = P(\omega) l_t \tag{4.15}$$

where now $t_a(\omega) \in \Pi(\omega)$ in the space of a.

Since A is of full column rank it can be, after some possible row interchanges, partitioned as

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \tag{4.16}$$

where A_1 is a square nonsingular matrix, $A_1 \in R^{l \times l}$, $A_2 \in R^{(k-l)\times l}$. Equation (4.15), after the same row interchanges, can be expressed as

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} t_a(\omega) = \begin{pmatrix} P_1(\omega) \\ P_2(\omega) \end{pmatrix} l_t - \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{4.17}$$

From the first part of (4.17) we get

$$t_a(\omega) = A_1^{-1} P_1(\omega) l_t - A_1^{-1} b_1 \tag{4.18}$$

which substituted to the second part of (4.17) gives

$$\underbrace{[A_2 A_1^{-1} P_1(\omega) - P_2(\omega)]}_{B(\omega)}\, l_t = \underbrace{A_2 A_1^{-1} b_1 - b_2}_{c} \tag{4.19}$$

or

$$B(\omega) l_t = c. \tag{4.20}$$

Equation (4.20) is of primary importance in determining whether $\Pi(\omega)$ is empty or not. Let

$$\Omega := \{\omega \mid \text{rank}\,[B(\omega)] = \text{rank}\,[B(\omega)|c],\ 0 \leq \omega < \infty\}. \tag{4.21}$$

Then $\Pi(\omega) \neq \emptyset$ if and only if $\omega \in \Omega \neq \emptyset$ and therefore we take

$$r(\omega) = \infty \qquad \text{for } \omega \notin \Omega. \tag{4.22}$$

For $\omega \in \Omega$ (if $\Omega \neq \emptyset$) the general solution of (4.20) is of the form

$$l_t = D(\omega) \bar{l}_t + e(\omega) \tag{4.23}$$

and therefore

$$t_a(\omega) = \tilde{P}(\omega) \bar{l}_t + \tau(\omega) \tag{4.24}$$

is the general solution of (4.15) with $\bar{l}_t$ an arbitrary real vector.

Now, in a manner analogous to (3.40)–(3.43), we obtain

$$r^2(\omega)=(a^0-\tau(\omega))^T\tilde{Q}(\omega)(a^0-\tau(\omega)) \tag{4.25}$$

and finally

$$r^2=\inf_{\omega\in\Omega} r^2(\omega). \tag{4.26}$$

The above derivation allows us to effectively deal with a more general class of perturbations than that covered in Section III, i.e., some plant transfer function coefficient can be interdependent and some can be fixed.

V. The Largest Stability Ellipsoid

In this section we extend the method presented in the previous sections to provide the solution to a more general problem, namely, that of finding the largest stability hyperellipsoid in the parameter space of $\boldsymbol{a}$ centered at the nominal point $\boldsymbol{a}^0$. As will be seen later, this is useful in applications where weighted perturbations occur.

By "largest" in the above we mean that the shape of the ellipsoid is fixed by specifying the ratios of the principal axes as

$$\alpha_1 : \alpha_2 : \cdots : \alpha_l \tag{5.1}$$

where

$$\alpha_i>0, \quad i=1, 2, \cdots, l \tag{5.2}$$

and such an ellipsoid is enlarged to the maximum possible extent. Obviously, the largest stability hypersphere is a particular case of the largest stability hyperellipsoid if all α_i in (5.1) are equal.

Let $E_\epsilon(\boldsymbol{a}^0, \alpha)$ denote an ellipsoid centered at $\boldsymbol{a}^0$ with principal axes parallel to the coordinate axes and of lengths $\epsilon\alpha_1, \cdots, \epsilon\alpha_l$. Consider the family of ellipsoids

$$\mathcal{E}(\boldsymbol{a}^0, \alpha) := \{E_\epsilon(\boldsymbol{a}^0, \alpha) | 0\le\epsilon<\infty\}. \tag{5.3}$$

Let

$$Q := \begin{pmatrix} \alpha_1 & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \alpha_l \end{pmatrix} \tag{5.4}$$

and define $\tilde{\boldsymbol{a}}$ by

$$\boldsymbol{a}=Q\tilde{\boldsymbol{a}} \tag{5.5}$$

so that

$$\tilde{\boldsymbol{a}}=Q^{-1}\boldsymbol{a}. \tag{5.6}$$

Clearly, the linear transformation (5.4) maps the set of all hyperspheres in the $\tilde{\boldsymbol{a}}$ space, centered at $\tilde{\boldsymbol{a}}^0$, onto $\mathcal{E}(\boldsymbol{a}^0, \alpha)$ and the mapping is one-to-one. Therefore, the largest stability hyperellip-

Theorem 5.1: Let $C(s)$ be a given stabilizing controller as in (3.1) and (3.2). Then the largest stability hyperellipsoid $E_{\epsilon^*}(\boldsymbol{a}^0, \alpha)$ in the class $\mathcal{E}(\boldsymbol{a}^0, \alpha)$ is given by

$$\epsilon^*=\tilde{\rho}. \tag{5.10}$$

Remark: The case where the ellipsoid axes are not parallel to the coordinate axes can be handled analogously and is omitted.

VI. The Stability Polytope in the Space of Primary Parameters

In some applications the plant parameters are known to lie within given bounds

$$a_i^0-\gamma_i<a_i<a_i^0+\epsilon_i, \qquad i=1, 2, \cdots, l \tag{6.1}$$

or

$$-\gamma_i<\Delta a_i<\epsilon_i, \qquad i=1, 2, \cdots l \tag{6.2}$$

and closed-loop stability is required for all such values of the parameter vector. Equation (6.1) determines a rectangular polytope in the $\boldsymbol{a}$ space. It should be noted that whenever the parameters are perturbed independently the stability polytope, rather than the stability hypersphere (or hyperellipsoid), is of primary interest. A procedure for treating the above problem within the framework of this paper is to find the stability hypersphere (or hyperellipsoid) and ensure that it inscribes the polytope (6.1).

Since it is desirable to center the stability hypersphere (ellipsoid) at the center of the polytope we redefine the nominal point and the tolerances as follows. Let

$$\epsilon := [\epsilon_1, \epsilon_2, \cdots, \epsilon_l]^T, \; \gamma=[\gamma_1, \gamma_2, \cdots, \gamma_l]^T \tag{6.3}$$

and introduce the new nominal parameter vector

$$\bar{\boldsymbol{a}}^0=\boldsymbol{a}^0+\frac{1}{2}(\epsilon-\gamma) \tag{6.4}$$

and new tolerances

$$\bar{\epsilon}=\frac{1}{2}(\epsilon+\gamma). \tag{6.5}$$

Then (6.1) and (6.2) are equivalent to

$$\bar{a}_i^0-\bar{\epsilon}_i<a_i<\bar{a}_i^0+\bar{\epsilon}_i, \qquad i=1, \cdots l \tag{6.6}$$

and

$$-\bar{\epsilon}_i<\Delta\bar{a}_i<\bar{\epsilon}_i, \qquad i=1, \cdots l. \tag{6.7}$$

This shows that, for the *fixed* polytope problem (6.1), the perturbation classes in i) and ii) of Section II can be both treated within the same mathematical framework. Therefore, without loss of generality, we will consider the class ii), i.e., we assume to Theorem 5.1. Let us also define the vector

$$w' := \left[\frac{w_1}{\alpha_1}, \frac{w_2}{\alpha_2}, \cdots, \frac{w_l}{\alpha_l}\right]^T \tag{6.10}$$

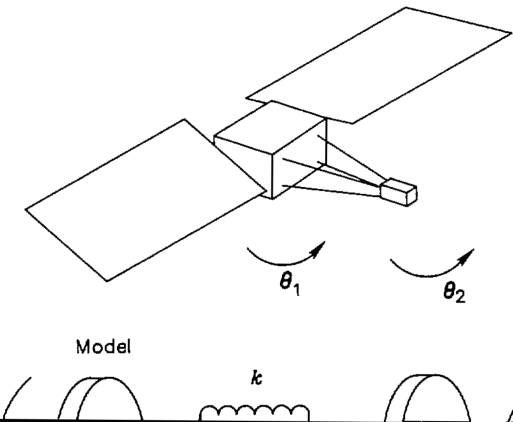
where α_i, $i = 1, 2, \cdots, l$ are the parameters characterizing the ellipsoid $E_{\tilde{\rho}}(a^0, \alpha)$ as in Section V.

Theorem 6.1: Let $C(s)$ be a controller that stabilizes the plant with nominal parameter a^0 and let $E_{\tilde{\rho}}(a^0$ be the largest stability hyperellipsoid. Then the controller $C(s)$ stabilizes the closed-loop system for all parameters lying in the polytope (6.8) if

$$\epsilon \| w' \|_2 \leq \tilde{\rho}(\tilde{a}^0) \tag{6.11}$$

where w' is given by (6.10).

The proof of this theorem is obtained by applying transforma-



We select $k^0 = 0.245$ and $d^0 = 0.0218973$ which are the middle points of the variation ranges of (8.2) and (8.3) as nominal values. Therefore, the nominal vector $\boldsymbol{a}^0$ of physical plant parameters which are subject to perturbation is

$$\boldsymbol{a}^0 = \begin{pmatrix} k^0 \\ d^0 \end{pmatrix}.$$

We are going to get a low-order controller $\boldsymbol{x}$ which will generate a stability hyperellipsoid inscribing the perturbation bound given by (8.2) and (8.3). We start with the stability hypersphere of radius $\rho(\boldsymbol{a}^0)$ in the space of physical plant parameters $\boldsymbol{a}$ which is centered at $\boldsymbol{a}^0$ and with a 0th-order controller. The controller is

$$C(s) = \frac{n_{c0}}{d_{c0}}$$

and the closed-loop characteristic vector is

$$\underbrace{\begin{pmatrix} 0 & d_4 \\ 0 & d_3 \\ n_2 & d_2 \\ n_1 & 0 \\ n_0 & 0 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} n_{c0} \\ d_{c0} \end{pmatrix}}_{x} = \delta$$

or

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & d_{c0} & 0 \\ 0 & 0 & 0 & d_{c0} & 0 & 0 \\ 0 & 0 & d_{c0} & 0 & 0 & n_{c0} \\ 0 & n_{c0} & 0 & 0 & 0 & 0 \\ n_{c0} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{X} \underbrace{\begin{pmatrix} n_0 \\ n_1 \\ d_2 \\ d_3 \\ d_4 \\ n_2 \end{pmatrix}}_{p} = \delta.$$

$$X_I = \begin{pmatrix} 0 & 0 & 0 & 0 & d_{c0} \\ 0 & 0 & 0 & d_{c0} & 0 \\ 0 & 0 & d_{c0} & 0 & 0 \\ 0 & n_{c0} & 0 & 0 & \\ n_{c0} & 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_J = \begin{pmatrix} 0 \\ 0 \\ n_{c0} \\ 0 \\ 0 \end{pmatrix}.$$

From (8.1) there is the following linear relation between the transfer function coefficients of plant n_i and d_i, and physical parameters, d and k.

$$\underbrace{\begin{pmatrix} n_0 \\ n_1 \\ d_2 \\ d_3 \\ d_4 \\ n_2 \end{pmatrix}}_{p} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} k \\ d \end{pmatrix}}_{a} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}}_{b}.$$

From (4.12) and (4.14), we obtain

$$r_0 = k^0, \; r_n = \infty$$

since $X_I A = [n_{c0}, 0]$, $X_I \boldsymbol{b} = 0$, and $X_J A = [0, 0]$. Now, following (4.15)–(4.20), we get the solution (4.17)

$$t_a(\omega) = \begin{pmatrix} t_k(\omega) \\ t_d(\omega) \end{pmatrix} = \begin{pmatrix} l_0\omega^2/n_{c0} \\ l_1\omega^2/n_{c0} \end{pmatrix}$$

and then we formulate (4.20) whose solution is

$$l_2 = d_{c0}, \; l_1 = 0, \; l_0 = n_{c0} \frac{d_{c0}\omega^2 - n_{c0}}{2d_{c0}\omega^2 - n_{c0}}, \; t_J = 1$$

if $\omega \neq \sqrt{n_{c0}/2d_{c0}}$. For $\omega = \sqrt{n_{c0}/2d_{c0}}$ (4.20) is inconsistent and therefore

$$\Omega = [0, \infty) \setminus \{\sqrt{n_{c0}/2d_{c0}}\}.$$

For $\omega \in \Omega$, $t(\omega) \in \Pi(\omega)$ takes the form (4.24)

$$\begin{pmatrix} t_k(\omega) \\ t_d(\omega) \end{pmatrix} = \begin{pmatrix} \omega^2 \dfrac{d_{c0}\omega^2 - n_{c0}}{2d_{c0}\omega^2 - n_{c0}} \\ 0 \end{pmatrix}.$$

Note that $\tilde{P}(\omega)$ and $\bar{l}_t$ of (4.24) do not appear here so finally

$$r^2(\omega) = \begin{cases} \infty & \text{if } \omega = \sqrt{n_{c0}/2d_{c0}}, \\ \left(k^0 - \omega^2 \dfrac{d_{c0}\omega^2 - n_{c0}}{2d_{c0}\omega^2 - n_{c0}}\right)^2 + (d^0)^2 & \text{otherwise.} \end{cases}$$

It can be easily shown that $r^2 = \min_\omega r^2(\omega) = (d^0)^2$, so $r = d^0$ and, since $d^0 < k^0$,

$$\rho(\boldsymbol{a}^0) = d^0.$$

It is interesting that the above result does not depend on the controller $\boldsymbol{x}$, i.e., it holds for any stabilizing controller $C(s)$. From Fig. 3 we see that the stability hypersphere S_0, with radius $\rho(\boldsymbol{a}^0)$, does not inscribe the given perturbation bound. Because of the oblong range of the perturbation region, we now consider an ellipsoid $E_\epsilon(\boldsymbol{a}^0, \alpha)$ in the space of k and d, centered at (k^0, d^0), with $\alpha = [1, \alpha_d]$. The polytope which is to be inscribed into the ellipse is given by

$$-\epsilon < \Delta k < \epsilon$$

$$-0.1168\epsilon < \Delta d < 0.1168\epsilon$$

where $\epsilon = 0.155$ and $\boldsymbol{w} = [1, 0.1168]$. Now, from Theorem 6.2 we have the condition

$$\epsilon \leq \min \left\{ k^0, \infty, \frac{\tilde{r}}{\sqrt{1 + \left(\dfrac{0.1168}{\alpha_d}\right)^2}} \right\}$$

where it can be shown that $\tilde{r} = \tilde{d}^0 = d^0/\alpha_d$. The above condition can be satisfied by any $\alpha_d < 0.079$. For example, for $\alpha_d = 0.07$ we obtain the ellipse with the semiaxes 0.3128186 and 0.0218973 containing the polytope given, as shown in Fig. 3. In fact, a slightly larger polytope can be inscribed into the ellipse since $\epsilon_{\max} = 0.1608$. Note that the above solution is independent of the controller as long as $C(s)$ is of 0th-order and stabilizes the plant. Therefore, every 0th-order stabilizing controller will be robust for the perturbations given and the value n_{c0}/d_{c0} can be used to satisfy other design requirements. In fact, it can be shown for the plant considered that for any 0th-order stabilizing controller, i.e., such that $n_{c0}/d_{c0} > 0$, the parameters k and d can be perturbed anywhere in the first quadrant.

We have also calculated the largest stability hypersphere obtained for a maximally robust first-order controller using the algorithm of Section VII. This is shown in Fig. 3.

IX. Concluding Remarks

In this paper we have considered the problem of stabilizing a linear time-invariant system subject to perturbations of a real parameter vector $\boldsymbol{a}$. The focus has been on systems where the characteristic vector components (i.e., characteristic polynomial coefficients) are linear functions of the plant parameter $\boldsymbol{a}$ for any fixed controller parameter $\boldsymbol{x}$. This linearity has allowed us to calculate the largest stability hypersphere exactly and to use it to

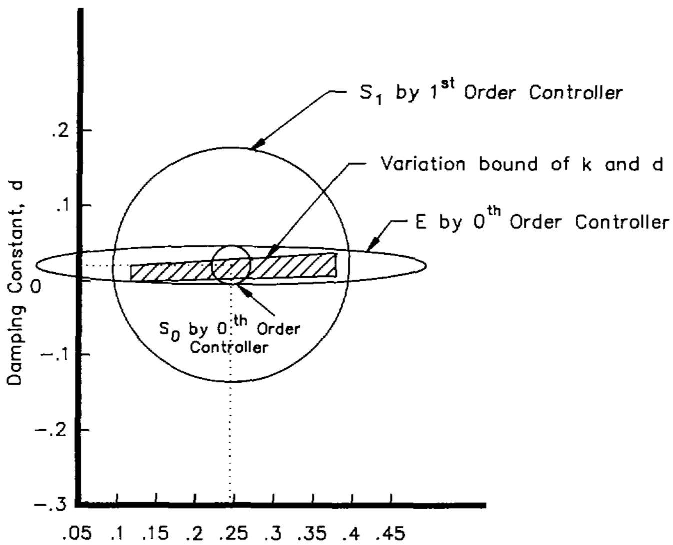


Fig. 3. Perturbation bounds and stability regions in k and d space.

robustify a given controller. On the other hand, this assumption can be a serious limitation in multivariable problems where the replacement of nonlinear functions of physical parameters by independent parameters to linearize the problem leads to overly conservative results.

The problem of calculating the multivariable stability margin for structured perturbations in the general case is an active area of research (see [32]–[36] and references therein). References [32]–[35] deal with a frequency domain approach to this problem where plant uncertainty from various blocks are arranged into a canonical block diagonal feedback perturbation system with norm bounded perturbations, for which a stability margin is determined. The calculation of the exact stability margin in this general setting seems to be a difficult problem. The only similarity between the approach of the above references and those of our paper is that in each case the stability domain must be such that each element of the family of perturbed characteristic polynomials obtained by continuously perturbing the original system does not vanish at any frequency. In the linear case this condition leads to a simple procedure for calculating the stability margin exactly as we have shown, but the general case is unsolved. The results of this paper have been extended in [36] to the general case but the results obtained are also conservative in the sense that the stability hypersphere determined is not the largest. A reviewer has informed us that the unpublished paper [37] contains new results on this problem.

Finally, we mention that an entirely different approach to stability under structured perturbations is being developed in the state space framework in [21]–[28], [31] and [36]. Several of these approaches use Lyapunov methods and the stability margins evaluated are in general, also conservative, indicating that much work remains to be done in this domain as well.

Appendix A
Proof of Theorem 3.1

Let $C = C^- \cup C^-$ denote the complex plane with

$$C^+ = \{s \mid \text{Re } s \geq 0\},\ C^- = \{s \mid \text{Re } s < 0\}$$

and let $C_I \in C^+$ denote the imaginary axis. Let $Z(\delta)$ denote the zeros of the polynomial $\delta(s) = \delta_0 + \delta_1 s + \cdots \delta_n s^n$ and introduce the function

$$\delta(s) \to [\delta_n, \delta_{n-1}, \cdots, \delta_0]^T := \delta \in R^{n+1}.$$

As in Section III let

$$\Delta_0 := \{\delta \mid 0 \in Z(\delta)\} \tag{A.1}$$

$$\Delta_n := \{\delta \mid \delta_n = 0\} \tag{A.2}$$

and for $0 \leq \omega < \infty$

$$\Delta(\omega) := \{\delta \mid \delta(s) = (s^2 + \omega^2) l(s),\ l(s) \text{ arbitrary}\}. \tag{A.3}$$

Define

$$\Delta_I := \{\delta \mid Z(\delta) \cap C_I \neq \emptyset\} \tag{A.4}$$

$$\Delta^- := \{\delta \mid Z(\delta) \subset C^-\} \tag{A.5}$$

$$\Delta^+ := \{\delta \mid Z(\delta) \cap C^+ \neq \emptyset\} \tag{A.6}$$

and let H_n denote the set of nth degree polynomials with zeros in C^-:

$$H_n := \{\delta \mid \delta \in R^{n+1},\ \delta_n \neq 0,\ Z(\delta) \in C^-\} \tag{A.7}$$

or

$$H_n = \Delta^- \setminus \Delta_n. \tag{A.8}$$

We also note that

$$\Delta_I = \left(\bigcup_{0 \leq \omega < \infty} \Delta(\omega) \right) \cup \Delta_0. \tag{A.9}$$

Now consider the closed-loop system of Fig. 1 and the equation

$$Xp = \delta$$

for the characteristic vector. With the compensator, i.e., X fixed and the plant parameter $p = p^0 + \Delta p$ we have $\delta = \delta(X, p^0 + \Delta p)$ and closed-loop stability is equivalent to

$$\delta(X, p^0 + \Delta p) \in H_n. \tag{A.10}$$

Let

$$\rho(p^0) = \min\{r_0, r_n, r\} \tag{A.11}$$

as in (3.24) and let $S_\rho(p^0)$ denote the interior of the hypersphere of radius $\rho(p^0)$ centered at p^0 in parameter space:

$$S_\rho(p^0) := \{p \mid p \in R^k,\ p = p^0 + \Delta p,\ \|\Delta p\|_2 < \rho(p^0)\}. \tag{A.12}$$

Let $S_\rho^B(p^0)$ denote the boundary of this hypersphere:

$$S_\rho^B(p^0) := \{p \mid p \in R^k,\ p = p^0 + \Delta p,\ \|\Delta p\|_2 = \rho(p^0)\}. \tag{A.13}$$

The proof of the theorem now consists of showing that

$$\delta(X, p) \in H_n\ \forall p \in S_\rho(p^0) \tag{A.14}$$

and

$$\delta(X, p^*) \notin H_n \qquad \text{for some } p^* \in S_\rho^B(p^0) \tag{A.15}$$

which together show that $S_\rho(p^0)$ is the largest stability hypersphere. Note that $S_\rho(p^0)$ cannot intersect Π_n as otherwise $\delta(s)$ has a root at $s = \infty$ which causes instability of the closed-loop system.

For clarity of presentation let us assume that

$$\rho = \rho(p^0) = r_0. \tag{A.16}$$

Then there exists $t_0^* \in \Pi_0$ (see 3.14) such that

$$r_0 = \|t_0^* - p^0\| \leq \|t_0 - p^0\|_2\ \forall t_0 \in \Pi_0. \tag{A.17}$$

Then

$$t_0^* \in S_{r_0}^B(p^0) \tag{A.18}$$

and

$$Xt_0^* := \delta_0^* \in \Delta_0 \tag{A.19}$$

and therefore

$$Xt_0^* \notin H_n. \tag{A.20}$$

With $p^* := t_0^*$ and $\delta(X, p^*) = \delta_0^*$ this proves (A.15), which shows that at least one point on the boundary of $S_{r_0}(p^0)$ corresponds to an unstable system.

To prove (A.14) we note that

$$\|p-p^0\|_2 < r_0 \le \min\{r_n, r\} \ \forall p \in S_{r_0}(p^0). \tag{A.21}$$

Define

$$\Pi := \bigcup_{0 \le \omega < \infty} \Pi(\omega). \tag{A.22}$$

Now (A.21) implies that

$$S_{r_0}(p^0) \cap \Pi_0 = \emptyset \tag{A.23}$$

$$S_{r_0}(p^0) \cap \Pi_n = \emptyset \tag{A.24}$$

and

$$S_{r_0}(p^0) \cap \Pi = \emptyset. \tag{A.25}$$

Let

$$XS_{r_0}(p^0) := \{\delta \mid \delta = Xp, \ p \in S_{r_0}(p^0)\}. \tag{A.26}$$

Then (A.23)–(A.25) and the definition (3.14)–(3.16) of Π_0, Π_n, and $\Pi(\omega)$ imply that

$$XS_{r_0}(p^0) \cap \Delta_0 = \emptyset \tag{A.27}$$

$$XS_{r_0}(p^0) \cap \Delta_n = \emptyset \tag{A.28}$$

and

$$XS_{r_0}(p^0) \cap \Delta_I = \emptyset. \tag{A.29}$$

Now, since the function $Z(\delta)$ is continuous the following well-known result will hold.

Fact A.1: If $\Delta \subset R^{n+1}$ is a simply connected region, then $\Delta \subset H_n$ if and only if there exists $\delta^* \in \Delta$ such that $\delta^* \in H_n$ and $\Delta \cap H_n^B = \emptyset$, where H_n^B is the boundary of H_n. From the definition of H_n

$$H_n^B \subset \Delta_I \cup \Delta_n = \Delta_0 \cup \Delta_n \cup \Delta_I \tag{A.30}$$

and (A.27)–(A.30) imply that

$$XS_{r_0}(p^0) \cap H_n^B = \emptyset. \tag{A.31}$$

Since $C(s)$ stabilizes, by assumption, the nominal system we have

$$Xp^0 \in H_n. \tag{A.32}$$

We also note that $XS_{r_0}(p^0)$ is a simply connected region in R^{n+1} since $S_{r_0}(p^0)$ is simply connected, δ is a continuous function of p and the number of roots of $\delta(s)$ does not change for $p \in S_{r_0}(p^0)$. Therefore, the conditions (A.31), (A.32) and Fact A.1 imply that

$$XS_{r_0}(p^0) \subset H_n. \tag{A.33}$$

This shows that all parameter points p inside the open hypersphere $S_{r_0}(p^0)$ of radius r_0 centered at p^0 result in stable closed-loop systems and completes the proof for the case $\rho = r_0$. When $\rho = r_n$ or $\rho = r$ exactly analogous arguments apply. These details are omitted.

Appendix B
Construction of X_I and Proof of its Invertibility

In this section we consider the equations (3.6) and (3.36) and show how to construct the nonsingular matrix X_I in (3.36). In (3.36)

$$X = \begin{array}{cccccccccc}
 & & & & & & & & n_{cp} & d_{cp} \\
 & & & & & & n_{cp} & d_{cp} & \cdot & \cdot \\
 & & n_{cp} & d_{cp} & \cdot & \cdot & \cdot & \cdot & & \\
n_{cp} & d_{cp} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & n_{c0} & d_{c0} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & &
\end{array}$$

$$X = \begin{bmatrix}
 & & & & & & & & n_{cp} & d_{cp} \\
 & & & & & & n_{cp} & d_{cp} & \cdot & \cdot \\
 & & n_{cp} & d_{cp} & \cdot & \cdot & \cdot & \cdot & & \\
n_{cp} & d_{cp} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & n_{c0} & d_{c0} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & n_{c0} & d_{c0} & & \\
\cdot & \cdot & n_{c0} & d_{c0} & & & & & & \\
n_{c0} & d_{c0} & & & & & & & &
\end{bmatrix}$$

and

$$p^T = (n_0^T, d_0 \cdots n_q^T, d_q)$$

where $X \in R^{(q+p+1) \times [(1+m)(q+1)]}$ and $p \in R^{[(1+m)(q+1)]}$.

Consider first the case $m = 1$. In this case we may define

$$X_I = \begin{bmatrix}
 & & & & & & & & d_{cp} \\
 & & & & & & & & \cdot \\
 & & & & & & d_{cp} & \cdot & \cdot \\
 & & & & & & \cdot & \cdot & \cdot \\
 & & & & n_{cp} & d_{cp} & \cdot & \cdot & d_{c0} \\
 & & n_{cp} & d_{cp} & \cdot & \cdot & \cdot & \cdot & \cdot \\
n_{cp} & d_{cp} & \cdot & \cdot & \cdot & \cdot & d_{c0} & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & \cdot & n_{c0} & d_{c0} & & & \\
\cdot & \cdot & n_{c0} & d_{c0} & & & & & \\
n_{c0} & d_{c0} & & & & & & &
\end{bmatrix} \in R^{(q+p+1) \times (q+p+1)}.$$

It is easily shown using the eliminant matrix that X_I is nonsingular if $n_c(s)$, $d_c(s)$ are coprime as assumed.

For the general case we let $n_{cj}(s)$ be coprime with $d_c(s)$ for some $1 \le j \le m$. Then the matrix

$$X_j = \begin{bmatrix}
 & & & & & & & & n_{cp}^j & d_{cp} \\
 & & & & n_{cp}^j & d_{cp} & \cdot & \cdot & \cdot & \cdot \\
n_{cp}^j & d_{cp} & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & n_{c0}^j & d_{c0} \\
\cdot & \cdot & \cdot & \cdot & n_{c0}^j & d_{c0} & & & & \\
n_{c0}^j & d_{c0} & & & & & & & &
\end{bmatrix} \in R^{2p \times 2p}$$

is nonsingular as before. Now let

$$X_I=\begin{bmatrix} & & & & & d_{cp}\\ & & & & \iddots & \vdots\\ & & & d_{cp} & & \vdots\\ & & X_j & \vdots & & d_{c0}\\ & & \iddots & \vdots & & \\ & & & d_{c0} & & \\ & X_j & & & & \\ X_j & & & & & \end{bmatrix} \in R^{(q+p-1)\times(q+p+1)}.$$

Clearly X_I is nonsingular because X_j is nonsingular and $d_{cp} \neq 0$. Now from the form of X we can write

$$Xp = X_I p_I + X_J p_J$$

by permuting the components of p to form p_I and p_J.

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