$$x_p = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

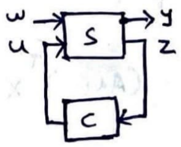
$$\dot{x}_p = \begin{bmatrix} 0 & 10 & 2 \\ -1 & 1 & 0 \\ 0 & 2 & -5 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} w$$

$$y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^T x_p + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$z = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T x_p + 2w$$

Now, we know the general form

$$\dot{x}_p = A_p x_p + B_p u + D_p w$$
$$y = C_p x_p + B_y u + D_y w$$
$$z = M_p x_p + D_z w$$



Comparing, we get

$$A_p = \begin{bmatrix} 0 & 10 & 2 \\ -1 & 1 & 0 \\ 0 & 2 & -5 \end{bmatrix}$$

Since we want CL poles to left of $\delta = -0.2$, we shift A_p to $A_p + 0.2I$

$$\Rightarrow \bar{A}_p = A_p + 0.2I = \begin{bmatrix} 0 & 10 & 2 \\ -1 & 1 & 0 \\ 0 & 2 & -5 \end{bmatrix} + \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 & 10 & 2 \\ -1 & 1.2 & 0 \\ 0 & 2 & -4.8 \end{bmatrix}$$

$$B_p = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$C_p = [1\ 0\ 0]$$

$$D_p = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$B_y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$D_y = 0$$

$$M_p = [0\ 1\ 0]$$

$$D_z = 2$$

There exists an H_∞ controller of order $n_c \leq n_p$: $\begin{cases} \dot{x}_c = A_c x_c + B_c z \\ u = C_c x_c + D_c z \end{cases}$ such that CL system is stable and

$$\Gamma_{ee} = \max_{w \neq 0} \frac{\|y\|_{L_2}}{\|w\|_{L_2}} < \gamma \quad \text{iff}$$

$$\begin{bmatrix} B_p \\ B_y \end{bmatrix}^{\perp} \begin{bmatrix} A_p\mathbf{X} + \mathbf{X}A_p^T + D_pD_p^T & \mathbf{X}C_p^T + D_pD_y^T \\ * & D_yD_y^T - \gamma^2 I \end{bmatrix} \begin{bmatrix} B_p \\ B_y \end{bmatrix}^{\perp^T} < 0 \qquad (C_1)$$

$$\begin{bmatrix} M_p^T \\ D_z^T \end{bmatrix}^{\perp} \begin{bmatrix} \mathbf{Y}A_p + A_p^T\mathbf{Y} + C_p^TC_p & \mathbf{Y}D_p + C_p^TD_y \\ * & D_y^TD_y - \gamma^2 I \end{bmatrix} \begin{bmatrix} M_p^T \\ D_z^T \end{bmatrix}^{\perp^T} < 0 \qquad (C_2)$$

$$\begin{bmatrix} \mathbf{X} & \gamma I \\ \gamma I & \mathbf{Y} \end{bmatrix} \geq 0 \qquad (C_3)$$

$$\text{rank} \begin{bmatrix} \mathbf{X} & \gamma I \\ \gamma I & \mathbf{Y} \end{bmatrix} \leq n_p + n_c \qquad (C_4)$$

- For static state feedback (i.e. $n_c = 0$, $M_p = I$), $C_2 - C_4$ are redundant and need not be checked.
- For full order controller (i.e. $n_p = n_c$) condition (C_4) is redundant and need not be checked.

MATLAB Notation (LMI Control Toolbox)

$$\dot{x} = Ax + B_1 w + B_2 u$$
$$z = C_1 x + D_{11} w + D_{12} u$$
$$y = C_2 x + D_{21} w + D_{22} u$$

Comparing with the earlier notation:

$A = A_p$, $B_1 = D_p$, $B_2 = B_p$

$C_1 = C_p$, $D_{11} = D_y$, $D_{12} = B_y$

$C_2 = M_p$, $D_{21} = D_z$, $D_{22} = 0$

let $A = \bar{A}_p$

$B = \begin{bmatrix} D_p & B_p \end{bmatrix}$

$C = \begin{bmatrix} C_p \\ M_p \end{bmatrix}$

$D = \begin{bmatrix} D_y & B_y \\ D_z & 0 \end{bmatrix}$

By lumping matrices together:

$$\dot{x}_p = A_p x_p + \begin{bmatrix} D_p & B_p \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} C_p \\ M_p \end{bmatrix} x_p + \begin{bmatrix} D_y & B_y \\ D_z & 0 \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

H_∞ Dynamic Controller Design in MATLAB

```
s = ltisys(A, B, C, D)
r = [nz, nu]
[gopt, G] = hinflmi(s, r)
```

$\Gamma_{ee} < \gamma^*$ controller

```
[Ac, Bc, Cc, Dc] = ltiss(G)
```

Closed-loop (CL) system:

$$A_{cl} = \begin{bmatrix} A_p + B_p D_c M_p & B_p C_c \\ B_c M_p & A_c \end{bmatrix}$$

$$B_{cl} = \begin{bmatrix} D_p + B_p D_c D_z \\ B_c D_z \end{bmatrix}$$

$$C_{cl} = \begin{bmatrix} C_p + B_y D_c M_p & B_y C_c \end{bmatrix}$$

$$D_{cl} = D_y + B_y D_c D_z$$

Verification:

H_∞ norm:

$$\left.\begin{aligned} S_{cl} &= ss(A_{cl}, B_{cl}, C_{cl}, D_{cl}) \\ hinf\_norm &= hinfnorm(S_{cl}) \\ hinf\_norm &< \gamma^* \ (g_{opt}) \end{aligned}\right\} \text{MATLAB}$$

$\hookrightarrow T_{ee} < \gamma^*$

Eigenvalues (Poles): $Re(\lambda_i(A_{cl})) < 0$ ← stability guaranteed

$Re(\lambda_i(A_{cl})) < 0.2$ ← stability with desired performance constraint guaranteed

Problem 1

CODE:

```
% PROBLEM 1


close all
clear
clc


% Define the system matrices
Ap = [0, 10, 2; -1, 1, 0; 0, 2, -5];
Ap_bar = Ap + 0.2*eye(3);
Bp = [0; 1; 0];
Cp = [1, 0, 0; 0, 0, 0];
Dp = [1; 0; 1];
By = [0; 1];
Dy = [0; 0];
Mp = [0, 1, 0];
Dz = 2;


% Convert to MATLAB notation
A = Ap_bar;
B1 = Dp;
B2 = Bp;
C1 = Cp;
D11 = Dy;
D12 = By;
C2 = Mp;
D21 = Dz;
D22 = 0;


% LTI system
S = ltisys(A, [B1 B2], [C1; C2], [D11 D12; D21 D22]);


% H-infinity LMI
[gopt, G] = hinflmi(S,[1 1])


% Controller matrices
disp('H∞ controller:')
[Ac, Bc, Cc, Dc] = ltiss(G)
```

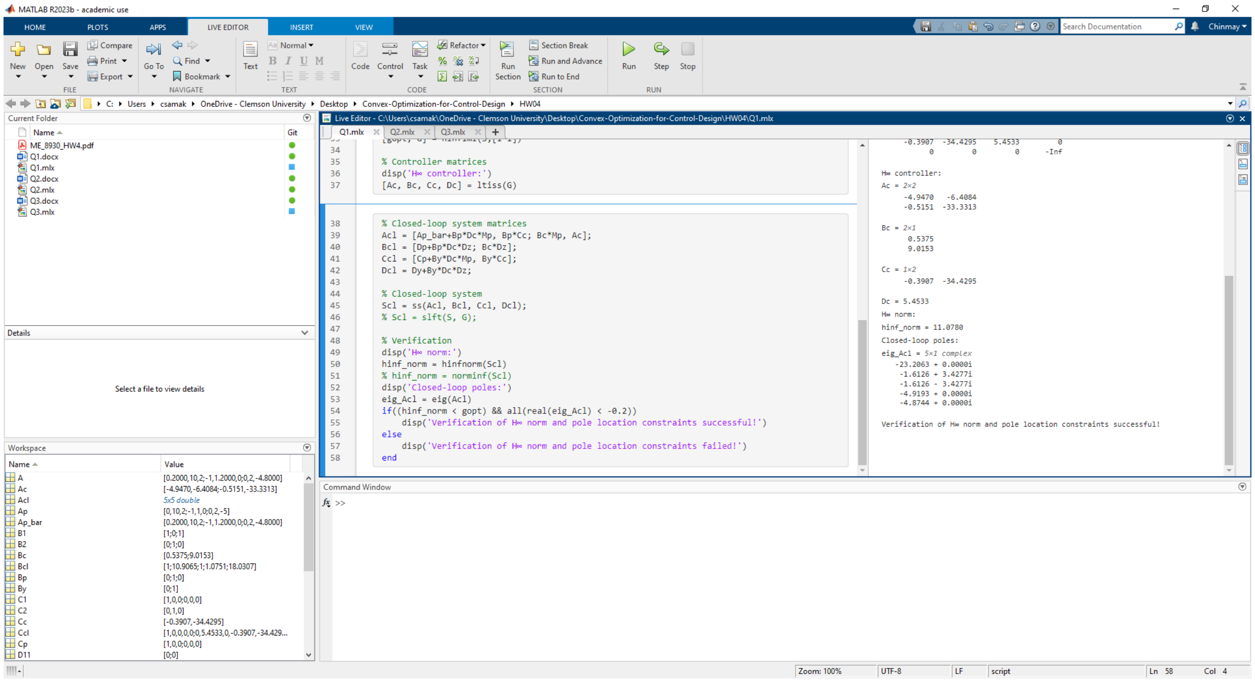
```matlab
% Closed-loop system matrices
Acl = [Ap_bar+Bp*Dc*Mp, Bp*Cc; Bc*Mp, Ac];
Bcl = [Dp+Bp*Dc*Dz; Bc*Dz];
Ccl = [Cp+By*Dc*Mp, By*Cc];
Dcl = Dy+By*Dc*Dz;


% Closed-loop system
Scl = ss(Acl, Bcl, Ccl, Dcl);
% Scl = slft(S, G);


% Verification
disp('H∞ norm:')
hinf_norm = hinfnorm(Scl)
% hinf_norm = norminf(Scl)
disp('Closed-loop poles:')
eig_Acl = eig(Acl)
if((hinf_norm < gopt) && all(real(eig_Acl) < -0.2))
    disp('Verification of H∞ norm and pole location constraints successful!')
else
    disp('Verification of H∞ norm and pole location constraints failed!')
end
```

OUTPUT:

```
Minimization of gamma:

 Solver for linear objective minimization under LMI constraints

 Iterations   :    Best objective value so far

     1
     2
     3
     4                  21.011488
     5                  17.864947
     6                  15.385893
     7                  14.718282
     8                  14.718282
     9                  13.310861
    10                  13.310861
    11                  11.988542
    12                  11.988542
    13                  11.352617
    14                  11.352617
    15                  11.146445
    16                  11.125254
    17                  11.110262
    18                  11.110262
```

```
    19                  11.110262
***                 new lower bound:    10.816366
    20                  11.081688
***                 new lower bound:    11.010885

 Result:  feasible solution of required accuracy
          best objective value:    11.081688
          guaranteed relative accuracy:  6.39e-03
          f-radius saturation:  0.188% of R =  1.00e+08

 Optimal Hinf performance:  1.108e+01
gopt = 11.0815
G = 4×4
   -4.9470   -6.4084    0.5375    2.0000
   -0.5151  -33.3313    9.0153         0
   -0.3907  -34.4295    5.4533         0
         0         0         0      -Inf
H∞ controller:
Ac = 2×2
   -4.9470   -6.4084
   -0.5151  -33.3313
Bc = 2×1
    0.5375
    9.0153
Cc = 1×2
   -0.3907  -34.4295
Dc = 5.4533
H∞ norm:
hinf_norm = 11.0780
Closed-loop poles:
eig_Acl = 5×1 complex
 -23.2063 + 0.0000i
  -1.6126 + 3.4277i
  -1.6126 - 3.4277i
  -4.9193 + 0.0000i
  -4.8744 + 0.0000i
Verification of H∞ norm and pole location constraints successful!
```

SCREENSHOT:


MATLAB R2023b - academic use
Live Editor - C:\Users\csamak\OneDrive - Clemson University\Desktop\Convex-Optimization-for-Control-Design\HW04\Q1.mlx
% Controller matrices
disp('H∞ controller:')
[Ac, Bc, Cc, Dc] = ltiss(G)
% Closed-loop system matrices
Acl = [Ap_bar+Bp*Dc*Mp, Bp*Cc; Bc*Mp, Ac];
Bcl = [Dp+Bp*Dc*Dz; Bc*Dz];
Ccl = [Cp+By*Dc*Mp, By*Cc];
Dcl = Dy+By*Dc*Dz;
% Closed-loop system
Scl = ss(Acl, Bcl, Ccl, Dcl);
% Scl = slft(S, G);
% Verification
disp('H∞ norm:')
hinf_norm = hinfnorm(Scl)
% hinf_norm = norminf(Scl)
disp('Closed-loop poles:')
eig_Acl = eig(Acl)
if((hinf_norm < gopt) && all(real(eig_Acl) < -0.2))
disp('Verification of H∞ norm and pole location constraints successful!')
else
disp('Verification of H∞ norm and pole location constraints failed!')
end
-0.3907 -34.4295 5.4533 0
0 0 0 -Inf
H∞ controller:
Ac = 2×2
-4.9470 -6.4084
-0.5151 -33.3313
Bc = 2×1
0.5375
9.0153
Cc = 1×2
-0.3907 -34.4295
Dc = 5.4533
H∞ norm:
hinf_norm = 11.0780
Closed-loop poles:
eig_Acl = 5×1 complex
-23.2063 + 0.0000i
-1.6126 + 3.4277i
-1.6126 - 3.4277i
-4.9193 + 0.0000i
-4.8744 + 0.0000i
Verification of H∞ norm and pole location constraints successful!