

Robust Stability with Structured Real Parameter Perturbations

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Abstract—This paper considers the problem of robust stabilization of a linear time-invariant system subject to variations of a real parameter vector. For a given controller the radius of the largest stability hypersphere in this parameter space is calculated. This radius is a measure of the stability margin of the closed-loop system. The results developed are applicable to all systems where the closed-loop characteristic polynomial coefficients are linear functions of the parameters of interest. In particular, this always occurs for single-input (multioutput) or single-output (multiinput) systems where the transfer function coefficients are linear or affine functions of the parameters. Many problems with transfer function coefficients which are nonlinear functions of physical parameters can be cast into this mathematical framework by suitable weighting and redefinition of functions of physical parameters as new parameters. The largest stability hyperellipsoid for the case of weighted perturbations and a stability polytope in parameter space are also determined.

Based on these calculations a design procedure is proposed to robustify a given stabilizing controller. This algorithm iteratively enlarges the stability hypersphere or hyperellipsoid in parameter space and can be used to design a controller to stabilize a plant subject to given ranges of parameter excursions. These results are illustrated by an example.

I. INTRODUCTION

THE present paper deals with robust stability of a linear time-invariant feedback control system where the plant contains a vector of real parameters subject to perturbation. The main new results, Theorems 3.1 and 4.1 give a simple, exact calculation of the radius of the largest stability hypersphere in this parameter space for the special case in which the parameters enter the closed-loop characteristic polynomial coefficients linearly. This radius serves as a measure of the system's stability margin and the calculation can be used to determine the stability of the closed-loop system for prescribed ranges of parameter variations as well as to robustify a given stabilizing controller. Although linearity or affinity of the closed-loop characteristic polynomial coefficients as a function of the parameter is essential for the method proposed, it is emphasized that this is not necessarily a severe restriction. In particular, this assumption is always satisfied when the plant is single-input (multioutput) or single-output (multiinput) and the plant transfer function coefficients are linear or affine functions of the parameters. In the general case, functions of physical parameters can always be redefined as the primary parameters to satisfy this assumption. Also, as is well

known, a multiinput multioutput system can always be reduced to a single-input or single-output one by a preliminary constant output feedback. Although this may lead to conservative results in some cases, it is clear that a broad class of practical problems can be effectively treated by this approach.

In Section II we formulate the problem of calculating the largest stability hypersphere when the parameter consists of the transfer function coefficients of a single-output or single-input system. This calculation is given in Section III, and extended in Section IV to the case where the transfer function coefficients are an affine function of the primary parameters. This allows us to treat the case of interdependent parameters. In Section V the case of weighted perturbations is considered and the largest stability hyperellipsoid is determined. In Section VI a prescribed polytope of parameter perturbations is considered and using the previous results a sufficient condition for robust stability under this class of perturbations is given. In Section VII we discuss a procedure for robustification of a given stabilizing controller and present an illustrative example in Section VIII.

The class of problems treated here falls outside the scope of the robust control literature dealing with unstructured perturbations (see [1]–[5] and references therein). There is a growing literature [6]–[37] dealing with structured perturbations in both the state space and frequency domains. Some of these are briefly discussed in Section II and also in the concluding section.

II. PROBLEM FORMULATION

Consider the feedback system of Fig. 1 where $G(s)$ is the plant transfer function and $C(s)$ is the controller transfer function.

We shall develop our results specifically for single-input (multioutput) or single-output (multiinput) plants. Since the theories for the two cases are analogous, we restrict our considerations to the single-input case.

Let

$$G(s) = \begin{pmatrix} \frac{\hat{n}_1(s)}{\hat{d}_1(s)} \\ \vdots \\ \frac{\hat{n}_m(s)}{\hat{d}_m(s)} \end{pmatrix} = \frac{1}{d(s)} \begin{pmatrix} n_1(s) \\ \vdots \\ n_m(s) \end{pmatrix} := n(s)d^{-1}(s) \quad (2.1)$$

where $d(s)$ is the least common denominator of all elements of $G(s)$ and write

$$\begin{aligned} d(s) &= d_q s^q + \cdots + d_0, \\ n(s) &= n_q s^q + \cdots + n_0 \end{aligned} \quad (2.2)$$

where d_i are real scalars, n_i are $m \times 1$ constant real vectors, and n and d are right coprime.

The coefficients \hat{d}_i and \hat{n}_i of a physical system are generally subject to perturbation, and therefore we introduce the nominal values denoted by d_i^0 , n_i^0 and the perturbations denoted by Δd_i , Δn_i , $i = 0, 1, \dots, q$.

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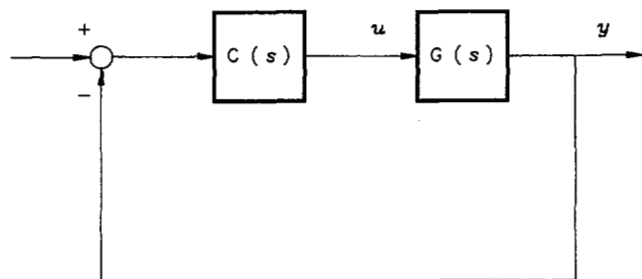


Fig. 1. Feedback system.

Let

$$p := [n_0^T, d_0, \dots, n_q^T, d_q]^T \quad (2.3)$$

denote the plant parameter vector in R^k , $k = (1+m)(1+q)$,

$$p^0 := [n_0^{0T}, d_0^0, \dots, n_q^{0T}, d_q^0]^T \quad (2.4)$$

its nominal value, and

$$\Delta p := [\Delta n_0^T, \Delta d_0, \dots, \Delta n_q^T, \Delta d_q]^T \quad (2.5)$$

its perturbation, so that

$$p = p^0 + \Delta p. \quad (2.6)$$

The size of the real perturbation vector Δp is measured by its Euclidean length, denoted by $\|\Delta p\|_2$ and is given by

$$\|\Delta p\|_2^2 = \|\Delta n_0\|_2^2 + \|\Delta n_1\|_2^2 + \dots + \|\Delta n_q\|_2^2 + (\Delta d_0)^2 + \dots + (\Delta d_q)^2.$$

Clearly, larger (smaller) values of $\|\Delta p\|_2$ correspond to larger (smaller) perturbations of the transfer function coefficients n_i and d_i . For a given stabilizing controller there exists a largest value $\rho(p^0)$ of $\|\Delta p\|_2$ for which closed-loop stability is preserved. To rule out trivial cases we make the standing assumption that $n(s)$ and $d(s)$ continue to be right coprime under these perturbations. This value therefore serves as a measure of *stability margin*. Based on these considerations we formulate several problems to be solved in this paper.

Problem A Determining the Stability Hypersphere: For a given stabilizing controller $C(s)$ determine the radius $\rho(p^0)$ of the stability hypersphere centered at p^0 defined by the condition that whenever $\|\Delta p\|_2 < \rho(p^0)$ the closed-loop system with plant parameter $p^0 + \Delta p$ is stable and there exists at least one perturbation Δp with $\|\Delta p\|_2 = \rho(p^0)$ such that the closed-loop system with the parameter $p^0 + \Delta p$ is not stable.

Problem B Robust Controller Design: Let $C(s)$ denote a stabilizing controller with an adjustable parameter vector $x \in R^s$. Determine x so that the radius $\rho(p^0, x)$ of the stability hypersphere is maximized as a function of x .

Solutions of the above two problems will also allow us to treat the following types of perturbation classes.

i) Let Δp_i , the i th component of Δp , be bounded by

$$-\gamma_i < \Delta p_i < \epsilon_i \quad (2.7)$$

for given positive numbers $\gamma_i, \epsilon_i, i = 1, \dots, k$.

ii) Let the perturbation bounds be given by

$$-w_i \epsilon < \Delta p_i < w_i \epsilon, \quad i = 1, \dots, k \quad (2.8)$$

where w_i are weights and ϵ is a positive constant.

A natural solution to the problem with the above perturbation classes is obtained within the framework of our approach by determining the largest stability *ellipsoid* in parameter space.

A special case of (2.8) was treated in [7] where p was considered to be the characteristic polynomial coefficient vector and Kharitonov's theorem [8] was used to determine the maximum value of ϵ . Kharitonov's theorem is considered to be one of the most important recent results on stability of sets of polynomials. This result has been given an interesting system theoretic treatment in [9] where an alternative proof is given. The same type of problem has been treated in several recent papers, on the stability of sets of polynomials (see [7]–[18]). Of these the papers [8]–[10] and [13]–[17] deal with independent perturbations of the coefficients of the characteristic polynomial and various sufficient conditions for stability are given. In [11] and [12] the geometry of Hurwitz polynomials for discrete systems has been studied again in the space of coefficients of the characteristic polynomial. The maximization of the general "box-type" kind of perturbations described in (2.7) has been dealt with in [18] and a reviewer has pointed out to us that the problem of maximizing ϵ in (2.8) corresponds to the multiloop stability margin formulated in [32] and [33].

Kharitonov's theorem, which is the basic tool in some of the above papers, does not hold in the space of the plant parameter vector p . One-dimensional convexity of the stability region in the parameter space is required for the existence of Kharitonov like theorems. This would require convexity of the stability region in the space of the characteristic polynomial coefficients, which does not hold in general.

An important recent result by C. B. Soh *et al.* [16] has determined the largest stability hypersphere in the space of coefficients of the characteristic polynomial. The concept of the largest stability hypersphere in the space of coefficients of the characteristic polynomial also occurs in [19] and [20] where it is used in the synthesis of a controller which is robust with respect to transfer function coefficient perturbations. It is restrictive, however, to assume that all coefficients of the closed-loop characteristic polynomial are subject to independent perturbations and this is why the results of [16] and of [19] and [20] are limited especially when used for synthesis. In the present paper we consider the more realistic situation where a real parameter vector originating in the plant is subject to perturbation. By exploiting the methodology introduced in [16] we develop a new approach to the problem of robustification of a given system against real parameter variations.

The justification for using $\rho(p^0, x)$ as a stability margin is that if x_1 and x_2 are two controllers with

$$\rho(p^0, x_1) > \rho(p^0, x_2) \quad (2.9)$$

then clearly x_1 is more robust than x_2 because the stability hypersphere $S_p(p^0, x_1)$ is larger than $S_p(p^0, x_2)$ and in fact

$$S_p(p^0, x_2) \subset S_p(p^0, x_1) \quad (2.10)$$

so that the family of perturbed plants stabilized by x_1 contains the family stabilized by x_2 . From this it follows that x^* , a maximally robust controller, is

$$\rho(p^0, x^*) \geq \rho(p^0, x) \quad \forall x \in R^s \text{ which is stabilizing.} \quad (2.11)$$

In the following sections the above problems are solved, first in the space of coefficients of the plant transfer function. These results are then used to extend the solutions to the space of primary parameters and this allows us to treat the important practical case where the transfer function coefficients are interdependent. The results are illustrated by examples.

III. CALCULATION OF THE STABILITY HYPERSPHERE

Let

$$C(s) = \frac{1}{d_c(s)} (n_{c1}(s), \dots, n_{cm}(s)) := d_c^{-1}(s) n_c^T(s) \quad (3.1)$$

and from (3.11),

$$w_I^T \delta = \delta_0 = 0 = w_I^T X t. \quad (3.26)$$

Let X_I denote the last row of X . Then (3.26) can be rewritten

$$X t = 0 \quad (3.27)$$

which shows that the shortest distance from p^0 to Π_0 must lie on the normal X_I^T to Π_0 . Therefore, the shortest vector is given by

$$p^0 - t = \alpha X_I^T. \quad (3.28)$$

To determine α , premultiply (3.28) by X_I and use (3.27)

$$X_I p^0 - X_I t = \alpha X_I X_I^T \quad (3.29)$$

$$X_I p^0 = \alpha X_I X_I^T \quad (3.30)$$

so that

$$\alpha = \frac{X_I p^0}{X_I X_I^T}. \quad (3.31)$$

Therefore, the distance r_0 from p^0 to the hyperplane Π_0 is given by

$$r_0^2 = \frac{1}{X_I X_I^T} [p^{0T} X_I^T X_I p^0]. \quad (3.32)$$

By exactly analogous arguments the distance r_n is calculated as

$$r_n^2 = \frac{1}{X_f X_f^T} [p^{0T} X_f^T X_f p^0] \quad (3.33)$$

where X_f denotes the first row of X .

The calculation of the distance $r(\omega)$ is now given. A representative vector $\delta \in \Delta(\omega)$ is given by

$$\delta = \Phi(\omega) l \quad (3.34)$$

where

$$\Phi(\omega) = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ \omega^2 & 0 & \cdot & \\ & \omega^2 & \cdot & \\ & & 1 & \\ & & 0 & \\ & & & \omega^2 \end{pmatrix} \in R^{(n+1) \times (n-1)} \quad (3.35)$$

and $l \in R^{n-1}$ is arbitrary.

If $t(\omega) \in \Pi(\omega)$, then

$$X t(\omega) = \Phi(\omega) l. \quad (3.36)$$

The matrix X is of size $(p + q + 1) \times [(m + 1)(1 + q)]$. Consider first the case where X has at least as many columns as rows and then the case where it has more rows. When the plant order q is greater than or equal to the controller order p , X will have full row rank.

Case I: $m + mq \geq p$: We partition X and $t(\omega)$ as follows:

$$X = [X_I, X_f] \\ t(\omega) = \begin{bmatrix} t_I(\omega) \\ t_f(\omega) \end{bmatrix}$$

where X_I is square and nonsingular. Such a partition can be made without loss of generality. This is proved in Appendix B where a specific way of constructing the nonunique matrices X_I and X_f is also given. Now,

$$X_I t_I(\omega) = \Phi(\omega) l - X_f t_f(\omega) \quad (3.37)$$

so that

$$t_I(\omega) = X_I^{-1} \Phi(\omega) l - X_I^{-1} X_f t_f(\omega).$$

Then a representative vector $t(\omega) \in \Pi(\omega)$ is given by,

$$t(\omega) = \underbrace{\begin{pmatrix} X_I^{-1} \Phi(\omega) & -X_I^{-1} X_f \\ 0 & I \end{pmatrix}}_{P(\omega)} \underbrace{\begin{pmatrix} l \\ t_f \end{pmatrix}}_{l_t} \quad (3.38)$$

$$:= P(\omega) l_t \quad (3.39)$$

where $P(\omega)$ is a fixed real matrix for each ω and l_t is an arbitrary real vector. Note that the dependence of t_f on ω can be dropped since t_f can be any vector. By letting l_t sweep over all real vectors in (3.39) we generate all solutions of (3.36). Now

$$t(\omega) - p^0 = P(\omega) l_t - p^0 \quad (3.40)$$

and

$$\|t(\omega) - p^0\|_2^2 = p^{0T} p^0 - 2 l_t^T p^T(\omega) p^0 + l_t^T P^T(\omega) P(\omega) l_t. \quad (3.41)$$

The vector $l_t = l_t^*$ which minimizes the distance (3.41) for a fixed ω is given by setting the corresponding gradient to zero. This gives

$$l_t^* = (P^T(\omega) P(\omega))^{-1} P^T(\omega) p^0 \quad (3.42)$$

and

$$r^2(\omega) = p^{0T} \underbrace{(I - P(\omega)(P^T(\omega) P(\omega))^{-1} P^T(\omega))}_{Q(\omega)} p^0. \quad (3.43)$$

Case II: $p > m + mq$: In this case the equation (3.36) may or may not have a solution for a fixed ω and l . If a solution exists, it is unique and is given by

$$t(\omega) = \underbrace{(X^T X)^{-1} X^T \Phi(\omega)}_{\bar{P}(\omega)} l \quad (3.44)$$

where l satisfies

$$[X(X^T X)^{-1} X^T - I] \Phi(\omega) l = [X \bar{P}(\omega) - \Phi(\omega)] l = 0. \quad (3.45)$$

Let

$$l = N(\omega) l_t \quad (3.46)$$

denote the solutions of (3.45) with l_t arbitrary so that

$$t(\omega) = \underbrace{\bar{P}(\omega) N(\omega) l_t}_{P(\omega)} := P(\omega) l_t. \quad (3.47)$$

Now (3.47) gives the solution set of (3.36) as l_t ranges over all real vectors. Since (3.47) and (3.39) are of the same form we can apply the preceding development (3.39)–(3.42) to obtain the expression (3.43) for $r^2(\omega)$.

Now, in both cases we have

$$r^2 = \inf_{\omega} r^2(\omega) = \inf_{\omega} p^{0T} Q(\omega) p^0. \quad (3.48)$$

Because $\Phi(\omega)$, and hence $P(\omega)$ and $Q(\omega)$ are known the function $r^2(\omega)$ can be determined and the minimization in (3.48) can be carried out numerically [for instance by graphing $r^2(\omega)$] and the global minimum in (3.48) can be found over $0 \leq \omega < \infty$. Since $r^2(\omega)$ is a continuous function of ω^2 the minimum of $r^2(\omega)$ occurs either for a finite value of ω or at $\omega = \infty$. The latter case corresponds to $\delta_n = 0$, $\delta_{n-1} = 0$ (i.e., a double pole at infinity)

and since $\Delta(\infty) \subset \Delta_n$, $r(\infty) \geq r_n$. Similarly, since $\Delta(0) \subset \Delta_0$, if the minimum of $r(\omega)$ occurs at $\omega = 0$, we have $r(0) \geq r_0$. Therefore, the global minimum of $r(\omega)$ need be found only in the interior of the interval $0 \leq \omega \leq \infty$.

The above calculations show that Theorem 1 provides a constructive procedure for calculating the stability hypersphere. In the next section we show how this calculation can be extended to handle situations where the transfer function coefficients are interdependent.

IV. CALCULATIONS IN THE SPACE OF PRIMARY PARAMETERS

Let

$$\mathbf{a} := [a_1, a_2, \dots, a_l]^T \quad (4.1)$$

denote the vector of primary parameters in R^l ,

$$\mathbf{a}^0 := [a_1^0, a_2^0, \dots, a_l^0]^T \quad (4.2)$$

its nominal value and

$$\Delta \mathbf{a} := [\Delta a_1, \Delta a_2, \dots, \Delta a_l]^T \quad (4.3)$$

its perturbation, so that

$$\mathbf{a} = \mathbf{a}^0 + \Delta \mathbf{a}. \quad (4.4)$$

Let us assume that the vector \mathbf{p} in (2.3) of the plant transfer function coefficients depends linearly on \mathbf{a} as

$$\mathbf{p} = \mathbf{A}\mathbf{a} + \mathbf{b}, \quad (4.5)$$

where $\mathbf{A} \in R^{k \times l}$ and $\mathbf{b} \in R^k$. Without loss of generality we can assume that \mathbf{A} is of full column rank and therefore $l \leq k$ (otherwise the parameters \mathbf{a} could be redefined).

According to (3.6) and (4.5) the closed-loop characteristic polynomial vector δ is now expressed as

$$\mathbf{X}\mathbf{A}\mathbf{a} + \mathbf{X}\mathbf{b} = \delta \quad (4.6)$$

which shows the *affine* transformation mapping the parameter vector \mathbf{a} into the characteristic vector δ . As before, let us consider the sets (3.11)–(3.13) and denote the inverse images of Δ_0 , Δ_n , and $\Delta(\omega)$ [analogous to (3.14)–(3.16)] in the space of \mathbf{a} as Π_0 , Π_n , and $\Pi(\omega)$. It is to be noted, however, that now some of these sets may be empty. Therefore, definitions (3.17), (3.19), and (3.21) with \mathbf{a} substituted for \mathbf{p} have to be augmented by

$$r_0 = \infty \quad \text{if } \Pi_0 = \emptyset \quad (4.7)$$

$$r_n = \infty \quad \text{if } \Pi_n = \emptyset \quad (4.8)$$

$$r(\omega) = \infty \quad \text{if } \Pi(\omega) = \emptyset. \quad (4.9)$$

Defining r as in (3.23) we can generalize Theorem 3.1 as follows.

Theorem 4.1: Let $C(s)$ be a fixed stabilizing controller as in (3.1) and (3.2). Then the radius of the largest stability hypersphere in the space of primary parameters centered at \mathbf{a}^0 , is given by

$$\rho(\mathbf{a}^0) = \min \{r_0, r_n, r\}. \quad (4.10)$$

The proof of this theorem is similar to the proof of Theorem 3.1 and is omitted.

We now give formulas for the calculation of the distances r_0 , r_n , and r in the space of \mathbf{a} . We note that $t \in \Pi_0$ if and only if

$$\mathbf{X}_l \mathbf{A} t + \mathbf{X}_l \mathbf{b} = \mathbf{0}. \quad (4.11)$$

The above equation fails to hold if and only if the vector $\mathbf{X}_l \mathbf{A} = \mathbf{0}$, $\mathbf{X}_l \mathbf{b} \neq \mathbf{0}$ and then Π_0 is empty ($r_0 = \infty$). Otherwise, if $\mathbf{X}_l \mathbf{A} \neq \mathbf{0}$, the distance r_0 of the point \mathbf{a}^0 from the hyperplane (4.11) is given

by the formula

$$r_0 = \frac{1}{\|\mathbf{X}_l \mathbf{A}\|_2} |\mathbf{X}_l \mathbf{A} \mathbf{a}^0 + \mathbf{X}_l \mathbf{b}| \quad (4.12)$$

or

$$r_0^2 = \frac{1}{\mathbf{X}_l \mathbf{A} \mathbf{A}^T \mathbf{X}_l^T} [\mathbf{a}^{0T} \mathbf{A}^T \mathbf{X}_l^T \mathbf{X}_l \mathbf{A} \mathbf{a}^0 + 2 \mathbf{a}^{0T} \mathbf{A}^T \mathbf{X}_l^T \mathbf{X}_l \mathbf{b} + \mathbf{b}^T \mathbf{X}_l^T \mathbf{X}_l \mathbf{b}]. \quad (4.13)$$

The distance r_n is calculated similarly as

$$r_n = \begin{cases} \infty & \text{if } \mathbf{X}_f \mathbf{A} = \mathbf{0} \text{ and } \mathbf{X}_f \mathbf{b} \neq \mathbf{0}, \\ \frac{1}{\|\mathbf{X}_f \mathbf{A}\|_2} |\mathbf{X}_f \mathbf{A} \mathbf{a}^0 + \mathbf{X}_f \mathbf{b}| & \text{if } \mathbf{X}_f \mathbf{A} \neq \mathbf{0} \end{cases} \quad (4.14)$$

It should be mentioned that it is not possible that $\mathbf{X}_l \mathbf{A} = \mathbf{0}$ and $\mathbf{X}_f \mathbf{b} = \mathbf{0}$ (or $\mathbf{X}_f \mathbf{A} = \mathbf{0}$ and $\mathbf{X}_f \mathbf{b} = \mathbf{0}$) simultaneously if the nominal point \mathbf{a}^0 is stabilized by $C(s)$.

The calculation of the distance $r(\omega)$ is now given. After exactly analogous derivations as in (3.34)–(3.38) the formula (3.38) can be written as

$$\mathbf{t}(\omega) = \mathbf{A} \mathbf{t}_a(\omega) + \mathbf{b} = \mathbf{P}(\omega) \mathbf{l}_t \quad (4.15)$$

where now $\mathbf{t}_a(\omega) \in \Pi(\omega)$ in the space of \mathbf{a} .

Since \mathbf{A} is of full column rank it can be, after some possible row interchanges, partitioned as

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{pmatrix} \quad (4.16)$$

where \mathbf{A}_1 is a square nonsingular matrix, $\mathbf{A}_1 \in R^{l \times l}$, $\mathbf{A}_2 \in R^{(k-l) \times l}$. Equation (4.15), after the same row interchanges, can be expressed as

$$\begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{pmatrix} \mathbf{t}_a(\omega) = \begin{pmatrix} \mathbf{P}_1(\omega) \\ \mathbf{P}_2(\omega) \end{pmatrix} \mathbf{l}_t - \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix}. \quad (4.17)$$

From the first part of (4.17) we get

$$\mathbf{t}_a(\omega) = \mathbf{A}_1^{-1} \mathbf{P}_1(\omega) \mathbf{l}_t - \mathbf{A}_1^{-1} \mathbf{b}_1 \quad (4.18)$$

which substituted to the second part of (4.17) gives

$$\underbrace{[\mathbf{A}_2 \mathbf{A}_1^{-1} \mathbf{P}_1(\omega) - \mathbf{P}_2(\omega)]}_{\mathbf{B}(\omega)} \mathbf{l}_t = \underbrace{\mathbf{A}_2 \mathbf{A}_1^{-1} \mathbf{b}_1 - \mathbf{b}_2}_{\mathbf{c}} \quad (4.19)$$

or

$$\mathbf{B}(\omega) \mathbf{l}_t = \mathbf{c}. \quad (4.20)$$

Equation (4.20) is of primary importance in determining whether $\Pi(\omega)$ is empty or not. Let

$$\Omega := \{\omega | \text{rank} [\mathbf{B}(\omega)] = \text{rank} [\mathbf{B}(\omega) | \mathbf{c}], 0 \leq \omega < \infty\}. \quad (4.21)$$

Then $\Pi(\omega) \neq \emptyset$ if and only if $\omega \in \Omega \neq \emptyset$ and therefore we take

$$r(\omega) = \infty \quad \text{for } \omega \notin \Omega. \quad (4.22)$$

For $\omega \in \Omega$ (if $\Omega \neq \emptyset$) the general solution of (4.20) is of the form

$$\mathbf{l}_t = \mathbf{D}(\omega) \bar{\mathbf{l}}_t + \mathbf{e}(\omega) \quad (4.23)$$

and therefore

$$\mathbf{t}_a(\omega) = \bar{\mathbf{P}}(\omega) \bar{\mathbf{l}}_t + \boldsymbol{\tau}(\omega) \quad (4.24)$$

is the general solution of (4.15) with $\bar{\mathbf{l}}_t$ an arbitrary real vector.

Now, in a manner analogous to (3.40)–(3.43), we obtain

$$r^2(\omega) = (a^0 - \tau(\omega))^T \tilde{Q}(\omega) (a^0 - \tau(\omega)) \quad (4.25)$$

and finally

$$r^2 = \inf_{\omega \in \Omega} r^2(\omega). \quad (4.26)$$

The above derivation allows us to effectively deal with a more general class of perturbations than that covered in Section III, i.e., some plant transfer function coefficient can be interdependent and some can be fixed.

V. THE LARGEST STABILITY ELLIPSOID

In this section we extend the method presented in the previous sections to provide the solution to a more general problem, namely, that of finding the largest stability hyperellipsoid in the parameter space of a centered at the nominal point a^0 . As will be seen later, this is useful in applications where weighted perturbations occur.

By "largest" in the above we mean that the shape of the ellipsoid is fixed by specifying the ratios of the principal axes as

$$\alpha_1 : \alpha_2 : \cdots : \alpha_l \quad (5.1)$$

where

$$\alpha_i > 0, \quad i = 1, 2, \dots, l \quad (5.2)$$

and such an ellipsoid is enlarged to the maximum possible extent. Obviously, the largest stability hypersphere is a particular case of the largest stability hyperellipsoid if all α_i in (5.1) are equal.

Let $E_i(a^0, \alpha)$ denote an ellipsoid centered at a^0 with principal axes parallel to the coordinate axes and of lengths $\epsilon\alpha_1, \dots, \epsilon\alpha_l$. Consider the family of ellipsoids

$$\mathcal{E}(a^0, \alpha) := \{E_i(a^0, \alpha) | 0 \leq \epsilon < \infty\}. \quad (5.3)$$

Let

$$Q := \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_l \end{pmatrix} \quad (5.4)$$

and define \tilde{a} by

$$a = Q\tilde{a} \quad (5.5)$$

so that

$$\tilde{a} = Q^{-1}a. \quad (5.6)$$

Clearly, the linear transformation (5.4) maps the set of all hyperspheres in the \tilde{a} space, centered at \tilde{a}^0 , onto $\mathcal{E}(a^0, \alpha)$ and the mapping is one-to-one. Therefore, the largest stability hyperellipsoid in the a space can be found by determining the largest stability hypersphere in the subsidiary space of \tilde{a} . This can be carried out by the method described in Section IV with the matrix

$$\tilde{A} = AQ \quad (5.8)$$

substituted for A , a replaced by \tilde{a} , and a^0 by \tilde{a}^0 .

Let

$$\tilde{\rho} = \tilde{\rho}(\tilde{a}^0) = \min \{\tilde{r}_0, \tilde{r}_n, \tilde{r}\} \quad (5.9)$$

denote the radius of the largest stability hypersphere in the \tilde{a} space. The above considerations lead to the following theorem.

Theorem 5.1: Let $C(s)$ be a given stabilizing controller as in (3.1) and (3.2). Then the largest stability hyperellipsoid $E_i^*(a^0, \alpha)$ in the class $\mathcal{E}(a^0, \alpha)$ is given by

$$\epsilon^* = \tilde{\rho}. \quad (5.10)$$

Remark: The case where the ellipsoid axes are not parallel to the coordinate axes can be handled analogously and is omitted.

VI. THE STABILITY POLYTOPE IN THE SPACE OF PRIMARY PARAMETERS

In some applications the plant parameters are known to lie within given bounds

$$a_i^0 - \gamma_i < a_i < a_i^0 + \epsilon_i, \quad i = 1, 2, \dots, l \quad (6.1)$$

or

$$-\gamma_i < \Delta a_i < \epsilon_i, \quad i = 1, 2, \dots, l \quad (6.2)$$

and closed-loop stability is required for all such values of the parameter vector. Equation (6.1) determines a rectangular polytope in the a space. It should be noted that whenever the parameters are perturbed independently the stability polytope, rather than the stability hypersphere (or hyperellipsoid), is of primary interest. A procedure for treating the above problem within the framework of this paper is to find the stability hypersphere (or hyperellipsoid) and ensure that it inscribes the polytope (6.1).

Since it is desirable to center the stability hypersphere (ellipsoid) at the center of the polytope we redefine the nominal point and the tolerances as follows. Let

$$\epsilon := [\epsilon_1, \epsilon_2, \dots, \epsilon_l]^T, \quad \gamma := [\gamma_1, \gamma_2, \dots, \gamma_l]^T \quad (6.3)$$

and introduce the new nominal parameter vector

$$\tilde{a}^0 = a^0 + \frac{1}{2}(\epsilon - \gamma) \quad (6.4)$$

and new tolerances

$$\tilde{\epsilon} = \frac{1}{2}(\epsilon + \gamma). \quad (6.5)$$

Then (6.1) and (6.2) are equivalent to

$$\tilde{a}_i^0 - \tilde{\epsilon}_i < a_i < \tilde{a}_i^0 + \tilde{\epsilon}_i, \quad i = 1, \dots, l \quad (6.6)$$

and

$$-\tilde{\epsilon}_i < \Delta \tilde{a}_i < \tilde{\epsilon}_i, \quad i = 1, \dots, l. \quad (6.7)$$

This shows that, for the *fixed* polytope problem (6.1), the perturbation classes in i) and ii) of Section II can be both treated within the same mathematical framework. Therefore, without loss of generality, we will consider the class ii), i.e., we assume

$$a_i^0 - w_i\epsilon < a_i < a_i^0 + w_i\epsilon \quad (6.8)$$

where ϵ is a positive constant and $w_i, i = 1, 2, \dots, l$ are given positive weights defined in the vector form as:

$$w := [w_1, w_2, \dots, w_l]^T. \quad (6.9)$$

For a fixed controller $C(s)$ that stabilizes the nominal plant with parameter a^0 let $\tilde{\rho}(\tilde{a}^0)$ denote the radius of the largest stability hypersphere in the space \tilde{a} calculated according to (5.9) and let $E_p^*(a^0, \alpha)$ denote the largest hyperellipsoid determined according

to Theorem 5.1. Let us also define the vector

$$\mathbf{w}' := \left[\frac{w_1}{\alpha_1}, \frac{w_2}{\alpha_2}, \dots, \frac{w_l}{\alpha_l} \right]^T \quad (6.10)$$

where α_i , $i = 1, 2, \dots, l$ are the parameters characterizing the ellipsoid $E_{\tilde{\rho}}(\mathbf{a}^0, \alpha)$ as in Section V.

Theorem 6.1: Let $C(s)$ be a controller that stabilizes the plant with nominal parameter \mathbf{a}^0 and let $E_{\tilde{\rho}}(\mathbf{a}^0)$ be the largest stability hyperellipsoid. Then the controller $C(s)$ stabilizes the closed-loop system for all parameters lying in the polytope (6.8) if

$$\epsilon \|\mathbf{w}'\|_2 \leq \tilde{\rho}(\mathbf{a}^0) \quad (6.11)$$

where \mathbf{w}' is given by (6.10).

The proof of this theorem is obtained by applying transformation (5.6) to the polytope (6.8) and then inscribing the corresponding polytope into the largest stability hypersphere in the space of $\tilde{\mathbf{a}}$ centered at $\tilde{\mathbf{a}}^0$. The result of Theorem 6.1 can be strengthened by taking advantage of the fact that Π_0 and Π_n , if nonempty, are $(l-1)$ -dimensional *hyperplanes* in the \mathbf{a} space. Thus, finding the conditions ensuring that the polytope does not intersect the hyperplanes is a relatively easy task.

Theorem 6.2: Let $C(s)$ be a controller that stabilizes the plant with nominal parameter \mathbf{a}^0 and let $E_{\tilde{\rho}}(\mathbf{a}^0, \alpha)$ be the largest stability hyperellipsoid. Then the controller $C(s)$ stabilizes the closed-loop system for all parameters lying in the polytope (6.8) if

$$\epsilon \leq \min \left\{ \frac{|X_l A \mathbf{a}^0 + X_l \mathbf{b}|}{\sum_{i=1}^l |X_l A_i| w_i}, \frac{|X_f A \mathbf{a}^0 + X_f \mathbf{b}|}{\sum_{i=1}^l |X_f A_i| w_i}, \frac{\tilde{r}}{\|\mathbf{w}'\|_2} \right\} \quad (6.12)$$

where X_l and X_f are the first and the last rows of the matrix in (3.8), respectively, A_i denotes the i th column of the matrix A in (4.5) and \tilde{r} is the same as in (5.9).

The proof of this theorem is omitted. The formula (6.12) is an improvement over (6.11) because the bound given by (6.11) can be rewritten as

$$\epsilon \leq \min \left\{ \frac{|X_l A \mathbf{a}^0 + X_l \mathbf{b}|}{\|X_l A Q\|_2 \|\mathbf{w}'\|_2}, \frac{|X_f A \mathbf{a}^0 + X_f \mathbf{b}|}{\|X_f A Q\|_2 \|\mathbf{w}'\|_2}, \frac{\tilde{r}}{\|\mathbf{w}'\|_2} \right\} \quad (6.13)$$

and it can be shown, using the Schwartz inequality, that the first two terms of (6.12) are greater than or equal to the corresponding terms in (6.13).

VII. CONTROLLER DESIGN VIA ROBUSTIFICATION

The stability margins ρ and $\tilde{\rho}$ calculated in the previous sections are useful for comparing the robustness against variations of the parameter \mathbf{a} of two given controllers. When the controller is not specified but is to be synthesized by choosing a vector \mathbf{x} of free parameters, we regard these margins as functions of \mathbf{x} , i.e., $\rho = \rho(\mathbf{x})$, $\tilde{\rho} = \tilde{\rho}(\mathbf{x})$. Robustification of a controller then consists of iteratively choosing successive values of \mathbf{x} to increase these margins until they are maximized or the stability hypersphere or ellipsoid obtained contains the given range of parameter variations that are to be tolerated. The controller order is successively increased until the above criteria are met. We note that only a local minimum can be found in general by such methods. The ascent direction for increasing $\rho(\mathbf{x})$ and robustification of the controller can be found by numerical methods. Such an algorithm has been implemented and applied to various examples in [29]. One of them is presented in the next section.

VIII. EXAMPLE

As an illustrative example a satellite control problem [30] is considered. A sketch of the satellite is shown in Fig. 2.

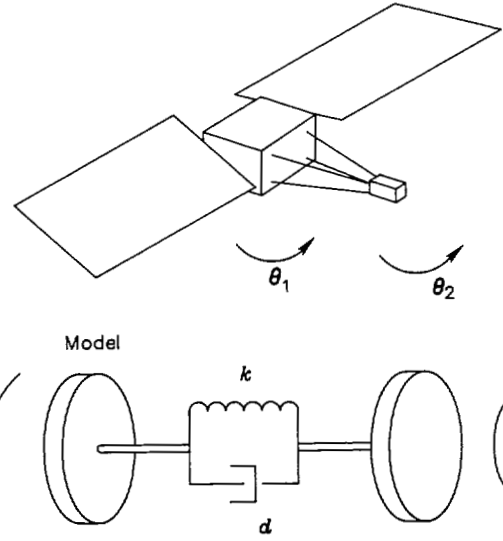


Fig. 2. A sketch of the satellite.

For the satellite model we assume two masses connected by a spring with torque constant k and viscous damping constant d . The equations of motion from Fig. 2 are

$$\begin{aligned} J_1 \ddot{\theta}_1 + d(\dot{\theta}_1 - \dot{\theta}_2) + k(\theta_1 - \theta_2) &= T_c \\ J_2 \ddot{\theta}_2 + d(\dot{\theta}_2 - \dot{\theta}_1) + k(\theta_2 - \theta_1) &= 0 \end{aligned}$$

where T_c is the control torque and J_1 and J_2 are inertias. If we choose, as the state vector,

$$\mathbf{x}_s^T = (\theta_2 \ \dot{\theta}_2 \ \theta_1 \ \dot{\theta}_1)$$

then the state equations with $T_c = u$ are

$$\dot{\mathbf{x}}_s = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{J_2} & -\frac{d}{J_2} & \frac{k}{J_2} & \frac{d}{J_2} \\ 0 & 0 & 0 & 1 \\ \frac{k}{J_1} & \frac{d}{J_1} & -\frac{k}{J_1} & -\frac{d}{J_1} \end{bmatrix} \mathbf{x}_s + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J_1} \end{bmatrix} u$$

$$y = (0 \ 0 \ 1 \ 0) \mathbf{x}_s.$$

Thus, the corresponding transfer function with $J_1 = 1$ and $J_2 = 1$ is

$$G(s) = \frac{s^2 + ds + k}{s^2(s^2 + 2ds + 2k)}. \quad (8.1)$$

Physical analysis of the boom leads us assume that the k and d parameters vary within bounds given by

$$0.09 \leq k \leq 0.4 \quad (8.2)$$

$$0.04 \sqrt{\frac{k}{10}} \leq d \leq 0.2 \sqrt{\frac{k}{10}}. \quad (8.3)$$

As a result, the vehicle resonance ω_n can vary between 1 and 2 rad/s, and the damping ratio ζ varies between 0.02 and 0.1. Our problem is to design a controller which stabilizes the closed-loop system for all perturbations given by (8.2) and (8.3).

We select $k^0 = 0.245$ and $d^0 = 0.0218973$ which are the middle points of the variation ranges of (8.2) and (8.3) as nominal values. Therefore, the nominal vector a^0 of physical plant parameters which are subject to perturbation is

$$a^0 = \begin{pmatrix} k^0 \\ d^0 \end{pmatrix}.$$

We are going to get a low-order controller x which will generate a stability hyperellipsoid inscribing the perturbation bound given by (8.2) and (8.3). We start with the stability hypersphere of radius $\rho(a^0)$ in the space of physical plant parameters a which is centered at a^0 and with a 0th-order controller. The controller is

$$C(s) = \frac{n_{c0}}{d_{c0}}$$

and the closed-loop characteristic vector is

$$\underbrace{\begin{pmatrix} 0 & d_4 \\ 0 & d_3 \\ n_2 & d_2 \\ n_1 & 0 \\ n_0 & 0 \end{pmatrix}}_P \underbrace{\begin{pmatrix} n_{c0} \\ d_{c0} \end{pmatrix}}_x = \delta$$

or

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & d_{c0} & 0 \\ 0 & 0 & 0 & d_{c0} & 0 & 0 \\ 0 & 0 & d_{c0} & 0 & 0 & n_{c0} \\ 0 & n_{c0} & 0 & 0 & 0 & 0 \\ n_{c0} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_X \underbrace{\begin{pmatrix} n_0 \\ n_1 \\ d_2 \\ d_3 \\ d_4 \\ n_2 \end{pmatrix}}_p = \delta.$$

$$X_I = \begin{pmatrix} 0 & 0 & 0 & 0 & d_{c0} \\ 0 & 0 & 0 & d_{c0} & 0 \\ 0 & 0 & d_{c0} & 0 & 0 \\ 0 & n_{c0} & 0 & 0 & 0 \\ n_{c0} & 0 & 0 & 0 & 0 \end{pmatrix}, X_J = \begin{pmatrix} 0 \\ 0 \\ n_{c0} \\ 0 \\ 0 \end{pmatrix}.$$

From (8.1) there is the following linear relation between the transfer function coefficients of plant n_i and d_i , and physical parameters, d and k .

$$\underbrace{\begin{pmatrix} n_0 \\ n_1 \\ d_2 \\ d_3 \\ d_4 \\ n_2 \end{pmatrix}}_p = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}_A \underbrace{\begin{pmatrix} k \\ d \end{pmatrix}}_a + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}}_b.$$

From (4.12) and (4.14), we obtain

$$r_0 = k^0, r_n = \infty$$

since $X_I A = [n_{c0}, 0]$, $X_I b = 0$, and $X_J A = [0, 0]$. Now, following (4.15)–(4.20), we get the solution (4.17)

$$t_a(\omega) = \begin{pmatrix} t_k(\omega) \\ t_d(\omega) \end{pmatrix} = \begin{pmatrix} l_0 \omega^2 / n_{c0} \\ l_1 \omega^2 / n_{c0} \end{pmatrix}$$

and then we formulate (4.20) whose solution is

$$l_2 = d_{c0}, l_1 = 0, l_0 = n_{c0} \frac{d_{c0} \omega^2 - n_{c0}}{2 d_{c0} \omega^2 - n_{c0}}, t_J = 1$$

if $\omega \neq \sqrt{n_{c0}/2d_{c0}}$. For $\omega = \sqrt{n_{c0}/2d_{c0}}$ (4.20) is inconsistent and therefore

$$\Omega = [0, \infty) \setminus \{\sqrt{n_{c0}/2d_{c0}}\}.$$

For $\omega \in \Omega$, $t(\omega) \in \Pi(\omega)$ takes the form (4.24)

$$\begin{pmatrix} t_k(\omega) \\ t_d(\omega) \end{pmatrix} = \begin{pmatrix} \omega^2 \frac{d_{c0} \omega^2 - n_{c0}}{2 d_{c0} \omega^2 - n_{c0}} \\ 0 \end{pmatrix}.$$

Note that $\tilde{P}(\omega)$ and \tilde{I}_i of (4.24) do not appear here so finally

$$r^2(\omega) = \begin{cases} \infty & \text{if } \omega = \sqrt{n_{c0}/2d_{c0}}, \\ \left(k^0 - \omega^2 \frac{d_{c0} \omega^2 - n_{c0}}{2 d_{c0} \omega^2 - n_{c0}} \right)^2 + (d^0)^2 & \text{otherwise.} \end{cases}$$

It can be easily shown that $r^2 = \min_{\omega} r^2(\omega) = (d^0)^2$, so $r = d^0$ and, since $d^0 < k^0$,

$$\rho(a^0) = d^0.$$

It is interesting that the above result does not depend on the controller x , i.e., it holds for any stabilizing controller $C(s)$. From Fig. 3 we see that the stability hypersphere S_0 , with radius $\rho(a^0)$, does not inscribe the given perturbation bound. Because of the oblong range of the perturbation region, we now consider an ellipsoid $E_c(a^0, \alpha)$ in the space of k and d , centered at (k^0, d^0) , with $\alpha = [1, \alpha_d]$. The polytope which is to be inscribed into the ellipse is given by

$$-\epsilon < \Delta k < \epsilon$$

$$-0.1168\epsilon < \Delta d < 0.1168\epsilon$$

where $\epsilon = 0.155$ and $w = [1, 0.1168]$. Now, from Theorem 6.2 we have the condition

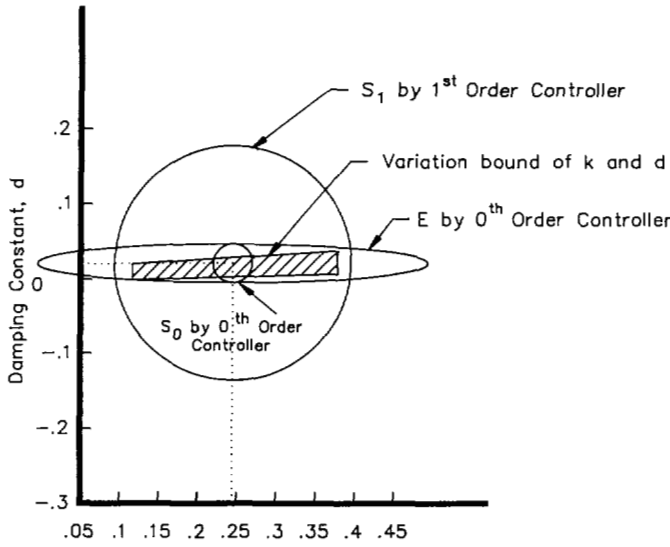
$$\epsilon \leq \min \left\{ k^0, \infty, \frac{\tilde{r}}{\sqrt{1 + \left(\frac{0.1168}{\alpha_d} \right)^2}} \right\}$$

where it can be shown that $\tilde{r} = \tilde{d}^0 = d^0/\alpha_d$. The above condition can be satisfied by any $\alpha_d < 0.079$. For example, for $\alpha_d = 0.07$ we obtain the ellipse with the semiaxes 0.3128186 and 0.0218973 containing the polytope given, as shown in Fig. 3. In fact, a slightly larger polytope can be inscribed into the ellipse since $\epsilon_{\max} = 0.1608$. Note that the above solution is independent of the controller as long as $C(s)$ is of 0th-order and stabilizes the plant. Therefore, every 0th-order stabilizing controller will be robust for the perturbations given and the value n_{c0}/d_{c0} can be used to satisfy other design requirements. In fact, it can be shown for the plant considered that for any 0th-order stabilizing controller, i.e., such that $n_{c0}/d_{c0} > 0$, the parameters k and d can be perturbed anywhere in the first quadrant.

We have also calculated the largest stability hypersphere obtained for a maximally robust first-order controller using the algorithm of Section VII. This is shown in Fig. 3.

IX. CONCLUDING REMARKS

In this paper we have considered the problem of stabilizing a linear time-invariant system subject to perturbations of a real parameter vector a . The focus has been on systems where the characteristic vector components (i.e., characteristic polynomial coefficients) are linear functions of the plant parameter a for any fixed controller parameter x . This linearity has allowed us to calculate the largest stability hypersphere exactly and to use it to

Fig. 3. Perturbation bounds and stability regions in k and d space.

robustify a given controller. On the other hand, this assumption can be a serious limitation in multivariable problems where the replacement of nonlinear functions of physical parameters by independent parameters to linearize the problem leads to overly conservative results.

The problem of calculating the multivariable stability margin for structured perturbations in the general case is an active area of research (see [32]–[36] and references therein). References [32]–[35] deal with a frequency domain approach to this problem where plant uncertainty from various blocks are arranged into a canonical block diagonal feedback perturbation system with norm bounded perturbations, for which a stability margin is determined. The calculation of the exact stability margin in this general setting seems to be a difficult problem. The only similarity between the approach of the above references and those of our paper is that in each case the stability domain must be such that each element of the family of perturbed characteristic polynomials obtained by continuously perturbing the original system does not vanish at any frequency. In the linear case this condition leads to a simple procedure for calculating the stability margin exactly as we have shown, but the general case is unsolved. The results of this paper have been extended in [36] to the general case but the results obtained are also conservative in the sense that the stability hypersphere determined is not the largest. A reviewer has informed us that the unpublished paper [37] contains new results on this problem.

Finally, we mention that an entirely different approach to stability under structured perturbations is being developed in the state space framework in [21]–[28], [31] and [36]. Several of these approaches use Lyapunov methods and the stability margins evaluated are in general, also conservative, indicating that much work remains to be done in this domain as well.

APPENDIX A PROOF OF THEOREM 3.1

Let $C = C^+ \cup C^-$ denote the complex plane with

$$C^+ = \{s | \operatorname{Re} s \geq 0\}, \quad C^- = \{s | \operatorname{Re} s < 0\}$$

and let $C_I \in C^+$ denote the imaginary axis. Let $Z(\delta)$ denote the zeros of the polynomial $\delta(s) = \delta_0 + \delta_1 s + \cdots + \delta_n s^n$ and introduce the function

$$\delta(s) \rightarrow [\delta_n, \delta_{n-1}, \dots, \delta_0]^T := \delta \in R^{n+1}.$$

As in Section III let

$$\Delta_0 := \{\delta | 0 \in Z(\delta)\} \quad (\text{A.1})$$

$$\Delta_n := \{\delta | \delta_n = 0\} \quad (\text{A.2})$$

and for $0 \leq \omega < \infty$

$$\Delta(\omega) := \{\delta | \delta(s) = (s^2 + \omega^2)I(s), I(s) \text{ arbitrary}\}. \quad (\text{A.3})$$

Define

$$\Delta_I := \{\delta | Z(\delta) \cap C_I \neq \emptyset\} \quad (\text{A.4})$$

$$\Delta^- := \{\delta | Z(\delta) \subset C^-\} \quad (\text{A.5})$$

$$\Delta^+ := \{\delta | Z(\delta) \cap C^+ \neq \emptyset\} \quad (\text{A.6})$$

and let H_n denote the set of n th degree polynomials with zeros in C^- :

$$H_n := \{\delta | \delta \in R^{n+1}, \delta_n \neq 0, Z(\delta) \in C^-\} \quad (\text{A.7})$$

or

$$H_n = \Delta^- \setminus \Delta_n. \quad (\text{A.8})$$

We also note that

$$\Delta_I = \left(\bigcup_{0 \leq \omega < \infty} \Delta(\omega) \right) \cup \Delta_0. \quad (\text{A.9})$$

Now consider the closed-loop system of Fig. 1 and the equation

$$Xp = \delta$$

for the characteristic vector. With the compensator, i.e., X fixed and the plant parameter $p = p^0 + \Delta p$ we have $\delta = \delta(X, p^0 + \Delta p)$ and closed-loop stability is equivalent to

$$\delta(X, p^0 + \Delta p) \in H_n. \quad (\text{A.10})$$

Let

$$\rho(p^0) = \min \{r_0, r_n, r\} \quad (\text{A.11})$$

as in (3.24) and let $S_\rho(p^0)$ denote the interior of the hypersphere of radius $\rho(p^0)$ centered at p^0 in parameter space:

$$S_\rho(p^0) := \{p | p \in R^k, p = p^0 + \Delta p, \|\Delta p\|_2 < \rho(p^0)\}. \quad (\text{A.12})$$

Let $S_\rho^B(p^0)$ denote the boundary of this hypersphere:

$$S_\rho^B(p^0) := \{p | p \in R^k, p = p^0 + \Delta p, \|\Delta p\|_2 = \rho(p^0)\}. \quad (\text{A.13})$$

The proof of the theorem now consists of showing that

$$\delta(X, p) \in H_n \quad \forall p \in S_\rho(p^0) \quad (\text{A.14})$$

and

$$\delta(X, p^*) \notin H_n \quad \text{for some } p^* \in S_\rho^B(p^0) \quad (\text{A.15})$$

which together show that $S_\rho(p^0)$ is the largest stability hypersphere. Note that $S_\rho(p^0)$ cannot intersect Π_n as otherwise $\delta(s)$ has a root at $s = \infty$ which causes instability of the closed-loop system.

For clarity of presentation let us assume that

$$\rho = \rho(p^0) = r_0. \quad (\text{A.16})$$

Then there exists $t_0^* \in \Pi_0$ (see 3.14) such that

$$r_0 = \|t_0^* - p^0\|_2 \leq \|t_0 - p^0\|_2 \quad \forall t_0 \in \Pi_0. \quad (\text{A.17})$$

Then

$$t_0^* \in S_{r_0}^B(p^0) \quad (A.18)$$

and

$$Xt_0^* := \delta_0^* \in \Delta_0 \quad (A.19)$$

and therefore

$$Xt_0^* \notin H_n. \quad (A.20)$$

With $p^* := t_0^*$ and $\delta(X, p^*) = \delta_0^*$ this proves (A.15), which shows that at least one point on the boundary of $S_{r_0}(p^0)$ corresponds to an unstable system.

To prove (A.14) we note that

$$\|p - p^0\|_2 < r_0 \leq \min\{r_n, r\} \quad \forall p \in S_{r_0}(p^0). \quad (A.21)$$

Define

$$\Pi := \bigcup_{0 \leq \omega < \infty} \Pi(\omega). \quad (A.22)$$

Now (A.21) implies that

$$S_{r_0}(p^0) \cap \Pi_0 = \emptyset \quad (A.23)$$

$$S_{r_0}(p^0) \cap \Pi_n = \emptyset \quad (A.24)$$

and

$$S_{r_0}(p^0) \cap \Pi = \emptyset. \quad (A.25)$$

Let

$$XS_{r_0}(p^0) := \{\delta | \delta = Xp, p \in S_{r_0}(p^0)\}. \quad (A.26)$$

Then (A.23)–(A.25) and the definition (3.14)–(3.16) of Π_0 , Π_n , and $\Pi(\omega)$ imply that

$$XS_{r_0}(p^0) \cap \Delta_0 = \emptyset \quad (A.27)$$

$$XS_{r_0}(p^0) \cap \Delta_n = \emptyset \quad (A.28)$$

and

$$XS_{r_0}(p^0) \cap \Delta_I = \emptyset. \quad (A.29)$$

Now, since the function $Z(\delta)$ is continuous the following well-known result will hold.

Fact A.1: If $\Delta \subset R^{n+1}$ is a simply connected region, then $\Delta \subset H_n$ if and only if there exists $\delta^* \in \Delta$ such that $\delta^* \in H_n$ and $\Delta \cap H_n^B = \emptyset$, where H_n^B is the boundary of H_n . From the definition of H_n^B

$$H_n^B \subset \Delta_I \cup \Delta_n = \Delta_0 \cup \Delta_n \cup \Delta_I \quad (A.30)$$

and (A.27)–(A.30) imply that

$$XS_{r_0}(p^0) \cap H_n^B = \emptyset. \quad (A.31)$$

Since $C(s)$ stabilizes, by assumption, the nominal system we have

$$Xp^0 \in H_n. \quad (A.32)$$

We also note that $XS_{r_0}(p^0)$ is a simply connected region in R^{n+1} since $S_{r_0}(p^0)$ is simply connected, δ is a continuous function of p and the number of roots of $\delta(s)$ does not change for $p \in S_{r_0}(p^0)$.

Therefore, the conditions (A.31), (A.32) and Fact A.1 imply that

$$XS_{r_0}(p^0) \subset H_n. \quad (A.33)$$

This shows that all parameter points p inside the open hypersphere $S_{r_0}(p^0)$ of radius r_0 centered at p^0 result in stable closed-loop systems and completes the proof for the case $\rho = r_0$. When $\rho = r_n$ or $\rho = r$ exactly analogous arguments apply. These details are omitted.

APPENDIX B

CONSTRUCTION OF X_I AND PROOF OF ITS INVERTIBILITY

In this section we consider the equations (3.6) and (3.36) and show how to construct the nonsingular matrix X_I in (3.36). In (3.36)

$$X = \begin{bmatrix} n_{cp} & d_{cp} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & n_{c0} & d_{c0} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$X = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

and

$$p^T = (n_0^T, d_0 \cdots n_q^T, d_q)$$

where $X \in R^{(q+p+1) \times [(1+m)(q+1)]}$ and $p \in R^{[(1+m)(q+1)]}$.

Consider first the case $m = 1$. In this case we may define

$$X_I = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \in R^{(q+p+1) \times (q+p+1)}.$$

It is easily shown using the eliminant matrix that X_I is nonsingular if $n_c(s)$, $d_c(s)$ are coprime as assumed.

For the general case we let $n_{c_j}(s)$ be coprime with $d_c(s)$ for some $1 \leq j \leq m$. Then the matrix

$$X_j = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \in R^{2p \times 2p}$$

is nonsingular as before. Now let

$$X_I = \begin{pmatrix} & & & d_{cp} \\ & & & \vdots \\ & & X_j & d_{cp} \\ & & \vdots & \vdots \\ & & d_{c0} & \\ & & \vdots & \\ X_j & & & d_{c0} \end{pmatrix} \in R^{(q+p-1) \times (q+p+1)}.$$

Clearly X_I is nonsingular because X_j is nonsingular and $d_{cp} \neq 0$. Now from the form of X we can write

$$Xp = X_I p_I + X_j p_j$$

by permuting the components of p to form p_I and p_j .

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REFERENCES

- [1] M. J. Chen and C. A. Desoer, "Necessary and sufficient conditions for robust stability of linear distributed feedback systems," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 880-888, Oct. 1984.
- [2] H. Kimura, "Robust stability of a class of transfer functions," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 788-793, Sept. 1984.
- [3] N. A. Lehtomaki, N. R. Sandell, and M. Athans, "Robustness results in linear quadratic Gaussian based multivariable control designs," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 75-92, Feb. 1981.
- [4] J. C. Doyle and G. Stein, "Multivariable feedback design: Concepts for a classical/modern synthesis," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 4-16, Feb. 1981.
- [5] B. Francis and J. Doyle, "Linear control theory with an H^∞ optimality criterion," Dep. Elec. Eng., Univ. Toronto, Toronto, Ont., Canada, Tech. Rep. 8501, Oct. 1985.
- [6] J. D. Ackermann, "Design of robust controllers by multi model methods," in *Proc. 7th Int. MTNS Conf.*, Stockholm, Sweden, June 1985.
- [7] B. R. Barmish, "Invariance of the strict Hurwitz property for polynomials with perturbed coefficients," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 935-936, Oct. 1984.
- [8] V. L. Kharitonov, "Asymptotic stability of an equilibrium position of a family of systems of linear differential equations," *Differentsial. Uravnen.*, vol. 14, no. 14, pp. 2086-2088, 1978.
- [9] N. K. Bose, "A system-theoretic approach to stability of sets of polynomials," *Contemporary Math.*, vol. 47, pp. 25-34, 1985.
- [10] J. P. Guiver and N. K. Bose, "Strictly Hurwitz property invariance of quartics under coefficient perturbation," *IEEE Trans. Automat. Contr.*, vol. AC-28, pp. 106-107, 1983.
- [11] A. T. Fam and J. S. Meditch, "A canonical parameter space for linear systems design," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 454-458, June 1978.
- [12] A. T. Fam, "On the geometry of stable polynomials of one and two variables," in *Information Linkage between Applied Mathematics and Industry*. New York: Academic, 1979, pp. 397-407.
- [13] K. S. Yeung, "Linear system stability under parameter uncertainties," *Int. J. Contr.*, vol. 38, no. 2, pp. 459-464, 1983.
- [14] M. A. Argoun, "Allowable coefficient perturbations with preserved stability of a Hurwitz polynomial," *Int. J. Contr.*, vol. 44, no. 4, pp. 927-934, 1986.
- [15] S. Bialas and J. Garloff, "Stability of polynomials under coefficient perturbations," *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 310-312, Mar. 1985.
- [16] C. B. Soh, C. S. Berger, and K. P. Dabke, "On the stability properties of polynomials with perturbed coefficients," *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 1033-1036, Oct. 1985.

- [17] N. K. Bose, E. I. Jury, and E. Zeheb, "On robust Hurwitz and Schur polynomials," in *Proc. IEEE 1986 Conf. Decision Contr.*, Athens, Greece, Dec. 1986.
- [18] R. M. Biernacki, "Sensitivities of stability constraints and their applications," *IEEE Trans. Automat. Contr.*, vol. AC-31, pp. 639-642, July 1986.
- [19] L. H. Keel, J. W. Howze, and S. P. Bhattacharyya, "Robust compensation via pole placement," in *Proc. 1985 Amer. Contr. Conf.*, Boston, MA, June 1985.
- [20] L. H. Keel and S. P. Bhattacharyya, "Low order robust stabilizer design using Hurwitz conditions," in *Proc. IEEE Conf. Decision Contr.*, Ft. Lauderdale, FL, Dec. 1985.
- [21] B. R. Barmish, "Necessary and sufficient conditions for quadratic stabilizability of an uncertain linear systems," *J. Optimiz. Theory Appl.*, vol. 46, 1985.
- [22] B. R. Barmish and G. Leitmann, "On ultimate boundedness control of uncertain systems in the absence of matching conditions," *IEEE Trans. Automat. Contr.*, vol. AC-27, pp. 253-258, Jan. 1982.
- [23] S. Gutman, "Uncertain dynamical systems, A Lyapunov min-max approach," *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 437-443, 1979.
- [24] J. S. Thorp and B. R. Barmish, "On guaranteed stability of uncertain linear systems via linear control," *J. Optimiz. Theory Appl.*, vol. 35, pp. 559-579, 1981.
- [25] S. Gutman and Z. Palmor, "Properties of min-max controllers in uncertain dynamical systems," *SIAM J. Contr. Optimiz.*, vol. 20, pp. 850-861, 1982.
- [26] B. R. Barmish, "Stabilization of uncertain systems via linear control," *IEEE Trans. Automat. Contr.*, vol. AC-28, pp. 848-850, 1983.
- [27] R. K. Yedavalli, "Reduced conservatism in testing for Hurwitz invariance of state space models," in *Proc. IEEE Conf. Decision Contr.*, Ft. Lauderdale, FL, Dec. 1985.
- [28] C. V. Hollot, "Matrix uncertainty structures for robust stability," in *Proc. Amer. Contr. Conf.*, Boston, MA, 1985.
- [29] H. Hwang, "Robust stabilization of plants subject to structured real parameter perturbations," M.S. thesis, Dep. Elec. Eng., Texas A&M Univ., July 1986.
- [30] G. J. Franklin, J. D. Powell, and A. Emami-Naeini, *Feedback Control of Dynamic Systems*. Reading, MA: Addison-Wesley, 1986.
- [31] M. A. Argoun, "On sufficient conditions for the stability of interval matrices," *Int. J. Contr.*, vol. 44, no. 5, pp. 1245-1250, 1986.
- [32] J. Doyle, "Analysis of feedback systems with structured uncertainties," *Proc. IEEE, Part D*, no. 6, pp. 242-250, Nov. 1982.
- [33] R. R. E. de Gaston and M. G. Safonov, "Calculation of the multiloop stability margin," in *Proc. Amer. Contr. Conf.*, Seattle, WA, June 1986.
- [34] R. R. E. de Gaston and A. Sideris, "Multivariable stability margin calculation with uncertain correlated parameters," in *Proc. IEEE Conf. Decision Contr.*, Athens, Greece, Dec. 1986, pp. 766-771.
- [35] M. Sacki, "A method of robust stability analysis with highly structured uncertainties," *IEEE Trans. Automat. Contr.*, vol. AC-31, pp. 925-940, Oct. 1986.
- [36] L. H. Keel, S. P. Bhattacharyya, and J. W. Howze, "Robust control with structured perturbations," Dep. Elec. Eng., Texas A&M Univ., College Station, TX, Tech. Rep. 8602, Aug. 1986.
- [37] R. R. E. de Gaston and M. G. Safonov, "Exact calculation of the multivariable stability margin," *IEEE Trans. Automat. Contr.*, to be published.



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