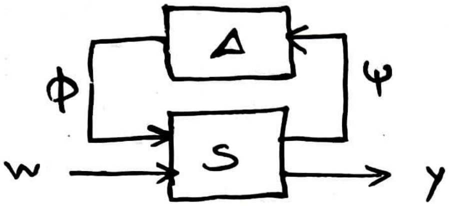


$$\dot{x} = Ax + K\phi$$
$$\psi = Mx + H\phi$$
$$\phi = \Delta\psi$$

1. $$\dot{x} = \begin{bmatrix} -4+\delta & 2 \\ 1+\delta & -7 \end{bmatrix} x \quad ; \quad \Delta = \delta$$

$$= \left(\underbrace{\begin{bmatrix} -4 & 2 \\ 1 & -7 \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}}_{\underset{K}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \underset{M}{\begin{bmatrix} 1 & 0 \end{bmatrix}}} \underbrace{\delta}_{\Delta} \right) x$$

$$= Ax + \underbrace{K\phi}$$
$$\phi = \underbrace{\Delta\psi}$$
$$\psi = Mx + \cancelto{0}{H\phi}$$

$$A = \begin{bmatrix} -4 & 2 \\ 1 & -7 \end{bmatrix} \qquad K = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad H = 0$$

2. To find maximum bound for δ that guarantees stability:

$$\min \; \gamma$$

$$\text{s.t.} \begin{bmatrix} PA + A^T P & PK & M^T \\ K^T P & -\frac{1}{\gamma} I & H^T \\ M & H & -\frac{1}{\gamma} I \end{bmatrix} < 0 \qquad \ldots (\text{SGT LMI})$$

$$P > 0$$

$$\equiv \min \; \bar{\gamma}$$

$$\text{s.t.} \begin{bmatrix} PA + A^T P & PK & M^T \\ K^T P & -\bar{\gamma} I & H^T \\ M & H & -\bar{\gamma} I \end{bmatrix} < 0$$

$$P > 0$$

↓

Solve (Python)

↓

$\bar{\gamma}^*$

For guaranteed stability, $|\delta| < \frac{1}{\bar{\gamma}^*}$

i.e. $|\delta| < \gamma^*$ $\quad \ldots \left(\gamma^* = \frac{1}{\bar{\gamma}^*}\right)$

4. $\dot{x} = \begin{bmatrix} 0 & 1 \\ -1+0.5\delta & -0.2 \end{bmatrix} x$

$$= \left(\underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -0.2 \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0.5 & 0 \end{bmatrix}}_{\underbrace{\begin{bmatrix} 0 \\ 0.5 \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{M}} \underbrace{\delta}_{\Delta} \right) x$$

$$= A x + K \phi$$

$$\phi = \Delta \psi$$

$$\psi = M x + \overset{0}{\cancel{H}} \phi$$

$A = \begin{bmatrix} 0 & 1 \\ -1 & -0.2 \end{bmatrix}$ $\quad K = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$ $\quad M = \begin{bmatrix} 1 & 0 \end{bmatrix}$ $\quad H = 0$

Python Code:

```
import numpy as np
import cvxpy as cp
import matplotlib.pyplot as plt

#################################################################################

# (1) Define the State-Space Model of System

A = np.array([[-4,  2],
              [1,  -7]])
K = np.array([[1],
              [1]])
M = np.array([[1, 0]])
H = np.array([[0]])
print('System:\nA = \n{} \nK = \n{}\nM = \n{} \nH = \n{}'.format(A, K, M, H))

#################################################################################

# (2) Find Bounds on Disturbance for Uncertain System

# Define variables
P = cp.Variable((2, 2), symmetric=True)
gamma_bar = cp.Variable(1)
M11 = P@A + A.T@P
M12 = P@K
M13 = M.T
M21 = K.T@P
M22 = cp.multiply(-gamma_bar,np.eye(1))
M23 = H.T
M31 = M
M32 = H
M33 = cp.multiply(-gamma_bar,np.eye(1))
# LMI Problem in Small Gain Theorem (SGT)
LMI = cp.vstack([
    cp.hstack([M11[0][0], M11[0][1], M12[0][0], M13[0][0]]),
    cp.hstack([M11[1][0], M11[1][1], M12[1][0], M13[1][0]]),
    cp.hstack([M21[0][0], M21[0][1], M22[0], M23[0][0]]),
    cp.hstack([M31[0][0], M31[0][1], M32[0][0], M33[0]])
])
constraints = [LMI << 0, P >> 0]
# Set up the optimization problem
objective = cp.Minimize(gamma_bar)
```

```
problem = cp.Problem(objective, constraints)
# Solve the LMI problem
problem.solve()
# Get the optimal value of gamma_bar
gamma_bar_star = gamma_bar.value[0]
gamma_star = 1/gamma_bar_star
delta_bounds = gamma_star
print(f'Stability Guaranteed for |δ| < {delta_bounds:.4f} i.e. -
{delta_bounds:.4f} < δ < {delta_bounds:.4f}')

###############################################################################

# (3) Plot Root Locus of the System as a Function of δ

delta_values = np.linspace(-3, 3, 100000) # Range of δ values
real_parts = [] # List to store real parts of eigenvalues for each δ
# Loop through each δ value
for delta in delta_values:
    A_delta = A+(K*M*delta) # Update the system matrix with the current δ
    eigenvalues = np.linalg.eigvals(A_delta) # Calculate eigenvalues
    real_parts.append(np.real(eigenvalues)) # Store the real part of the
eigenvalues
# Plot the root locus
plt.figure()
plt.plot(delta_values, real_parts)
plt.title('Root Locus as a Function of δ')
plt.xlabel('δ')
plt.ylabel('Re(λi)')
plt.legend(['λ1', 'λ2'])
plt.grid(False)
plt.show()
# Find the maximum value of |δ| for negative real part eigenvalues
max_delta = max(delta_values[np.max(real_parts, axis=1) < 0])
print(f"Maximum |δ| for Negative Real Part of Eigenvalues is: {max_delta:.4f}")
# Determine if this solution is consistent with SGT LMI result
tolerance = 0.001
print(f'Is this Solution Consistent with SGT LMI Result? {abs(max_delta-
delta_bounds) <= tolerance}')
print('\n----------------------------------------------------------------------
-----------\n')

###############################################################################

# (4) Repeat (1)-(3) for Different Uncertain System
# Define the state-space model of system
A = np.array([[0,      1],
              [-1,  -0.2]])
K = np.array([[0],
              [0.5]])
M = np.array([[1, 0]])
H = np.array([[0]])
print('System:\nA = \n{} \nK = \n{}\nM = \n{} \nH = \n{}'.format(A, K, M, H))

# Find bounds on disturbance for uncertain system
# Define variables
P = cp.Variable((2, 2), symmetric=True)
gamma_bar = cp.Variable(1)
M11 = P@A + A.T@P
M12 = P@K
M13 = M.T
M21 = K.T@P
M22 = cp.multiply(-gamma_bar,np.eye(1))
M23 = H.T
M31 = M
M32 = H
M33 = cp.multiply(-gamma_bar,np.eye(1))
# LMI Problem in Small Gain Theorem (SGT)
LMI = cp.vstack([
    cp.hstack([M11[0][0], M11[0][1], M12[0][0], M13[0][0]]),
    cp.hstack([M11[1][0], M11[1][1], M12[1][0], M13[1][0]]),
    cp.hstack([M21[0][0], M21[0][1], M22[0], M23[0][0]]),
    cp.hstack([M31[0][0], M31[0][1], M32[0][0], M33[0]])
])
constraints = [LMI << 0, P >> 0]
# Set up the optimization problem
objective = cp.Minimize(gamma_bar)
problem = cp.Problem(objective, constraints)
# Solve the LMI problem
problem.solve()
# Get the optimal value of gamma_bar
gamma_bar_star = gamma_bar.value[0]
gamma_star = 1/gamma_bar_star
delta_bounds = gamma_star
print(f'Stability Guaranteed for |δ| < {delta_bounds:.4f} i.e. -
{delta_bounds:.4f} < δ < {delta_bounds:.4f}')

# Plot root locus of the system as a function of δ
delta_values = np.linspace(-3, 3, 100000) # Range of δ values
real_parts = [] # List to store real parts of eigenvalues for each δ
```

```
# Loop through each δ value
for delta in delta_values:
    A_delta = A+(K*M*delta) # Update the system matrix with the current δ
    eigenvalues = np.linalg.eigvals(A_delta) # Calculate eigenvalues
    real_parts.append(np.real(eigenvalues)) # Store the real part of the
eigenvalues
# Plot the root locus
plt.figure()
plt.plot(delta_values, real_parts)
plt.title('Root Locus as a Function of δ')
plt.xlabel('δ')
plt.ylabel('Re(λi)')
plt.legend(['λ1', 'λ2'])
plt.grid(False)
plt.show()
# Find the maximum value of |δ| for negative real part eigenvalues
max_delta = max(delta_values[np.max(real_parts, axis=1) < 0])
print(f"Maximum |δ| for Negative Real Part of Eigenvalues is: {max_delta:.4f}")
# Determine if this solution is consistent with SGT LMI result
tolerance = 0.005
print(f'Is this Solution Consistent with SGT LMI Result? {abs(max_delta-
delta_bounds) <= tolerance}')
print('\n------------------------------------------------------------------------
-----------\n')

#################################################################################

# (5) Simulate the System with Time-Varying Uncertainty

# Plot root locus of the system as a function of δ(t)
time_values = np.linspace(0, 10, 100000) # Time values
delta_values = np.cos(2*time_values) # Range of δ values
real_parts = [] # List to store real parts of eigenvalues for each δ
# Loop through each δ value
for delta in delta_values:
    A_delta = A+(K*M*delta) # Update the system matrix with the current δ
    eigenvalues = np.linalg.eigvals(A_delta) # Calculate eigenvalues
    real_parts.append(np.real(eigenvalues)) # Store the real part of the
eigenvalues
# Plot the root locus
plt.figure()
plt.plot(delta_values, real_parts)
plt.title('Root Locus as a Function of δ(t)')
plt.xlabel('δ(t)')
plt.ylabel('Re(λi)')
plt.legend(['λ1', 'λ2'])
plt.grid(False)
plt.show()
# Find the maximum value of |δ| for negative real part eigenvalues
max_delta = max(delta_values[np.max(real_parts, axis=1) < 0])
print(f"Maximum |δ| for Negative Real Part of Eigenvalues is: {max_delta:.4f}")
# Is the system stable or unstable for the given δ(t)?
print('Is the System Stable or Unstable for Given δ(t)=cos(2t)?\n'
      '    In general, the system stability cannot be guaranteed for given
δ(t).\n'
      '    However, for specific time-frozen instances where |δ(t)| < 0.3980,\n'
      '    and withhout prior destabilization, the system may exhibit stable\n'
      '    behavior.')
# Determine if this solution is consistent with SGT LMI result
tolerance = 0.005
print(f'Is this Solution Consistent with SGT LMI Result? {abs(max_delta-
delta_bounds) <= tolerance}')
if not abs(max_delta-delta_bounds) <= tolerance:
    print('WHY?:\n    The results from eigenvalue test and SGT LMI are not
consistent\n'
          '    since the former cannot be applied to time-varying systems
while\n'
          '    the latter can. Hence in this case, results of SGT analysis
should\n'
          '    be trusted.')
print('\n-----------------------------------------------------------------------
-----------\n')
##################################################################################

# (6) Comment on Stability of the System to Time-Invariant and Time-Varying
Perturbations

print('The results from stability analysis indicate that the system is
guaranteed\n'
          'to be stable for time-invariant perturbations (where |δ| < 2.8890
i.e.,\n'
          '-2.8890 < δ < 2.8890), but is NOT guaranteed to be stable for the
given\n'
          'time-varying perturbations of δ(t) = cos(2t).\n'
          'In case of systems with time-varying perturbations (uncertainty),
the\n'
          'results from eigenvalue test and SGT LMI may/will NOT be
consistent.\n'
          'This is because the former (i.e., eigenvalue test) can ONLY be
applied\n'
```

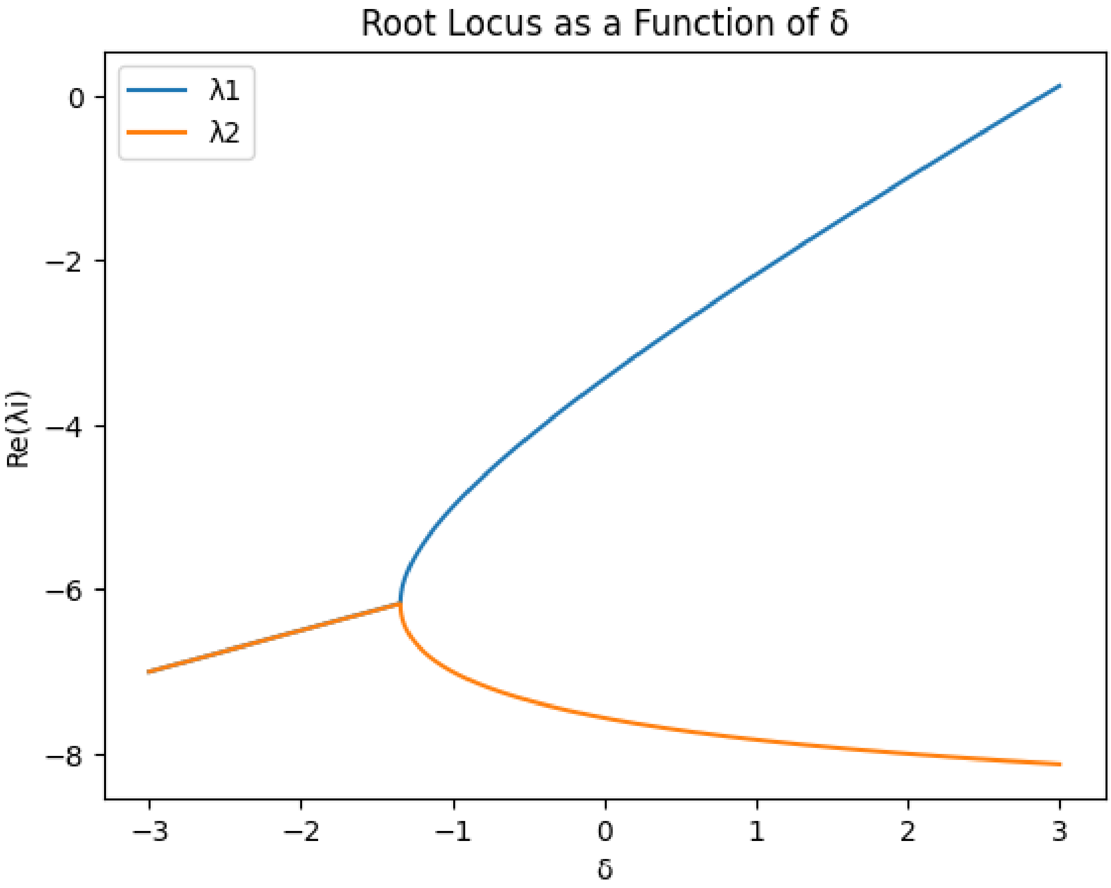
```
          'to time-invariant systems or if the system is slowly varying (i.e., 
if\n'
          'the uncertainty is much slower than system dynamics) while the 
latter\n'
          '(i.e., small gain theorem) being generalization of the Nyquist 
criterion\n'
          'to non-linear time-varying MIMO systems, can be applied to systems 
with\n'
          'time-invariant as well as time-varying perturbations (uncertainty). 
Hence,\n'
          'in general, results of SGT analysis can be trusted but the results 
of\n'
          'eigenvalue test cannot be trusted blindly.')
```

Output:

1) System:
   A =
     [[-4 2]
      [1 -7]]
   K =
     [[1]
      [1]]
   M =
     [[1 0]]
   H =
     [[0]]

2) Stability Guaranteed for |δ| < 2.8890 i.e. -2.8890 < δ < 2.8890

3) Maximum |δ| for Negative Real Part of Eigenvalues is: 2.8889
   Is this Solution Consistent with SGT LMI Result? True

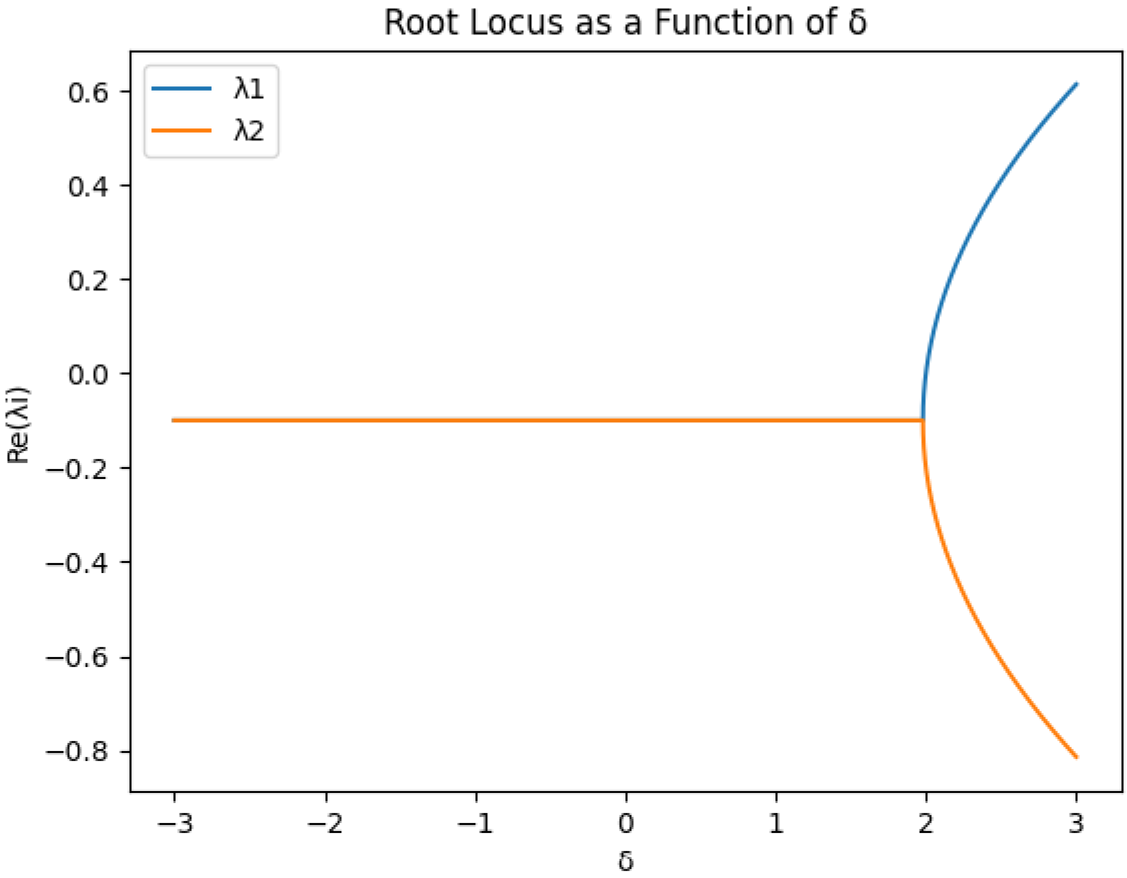


4) System:

```
A =
  [[ 0.   1. ]
   [-1.  -0.2]]
K =
   [[0. ]
    [0.5]]
M =
   [[1 0]]
H =
   [[0]]
```

Stability Guaranteed for |δ| < 0.3980 i.e. -0.3980 < δ < 0.3980

Maximum |δ| for Negative Real Part of Eigenvalues is: 2.0000
Is this Solution Consistent with SGT LMI Result? False



5) Maximum |δ| for Negative Real Part of Eigenvalues is: 1.0000

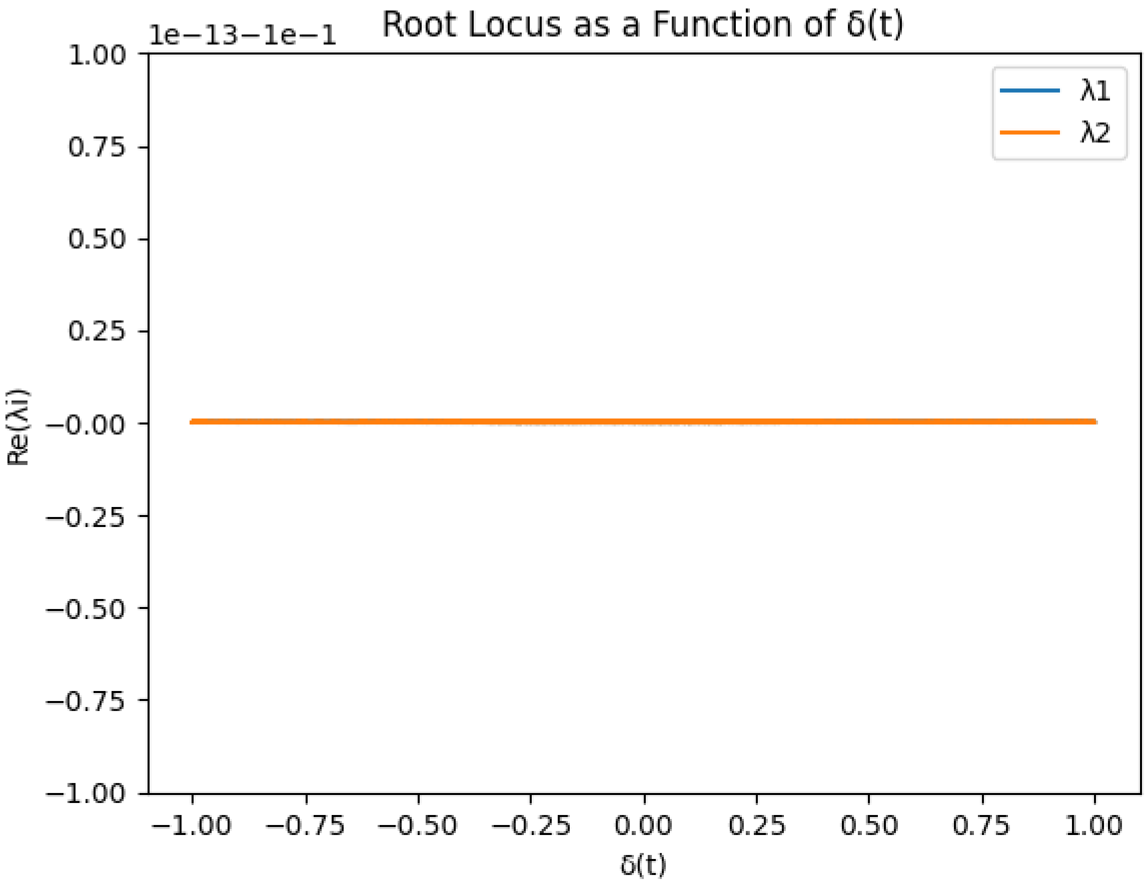
Is the System Stable or Unstable for Given δ(t)=cos(2t)?

In general, the system stability cannot be guaranteed for given δ(t). However, for specific time-frozen instances where |δ(t)| < 0.3980, and withhout prior destabilization, the system may exhibit stable behavior.

Is this Solution Consistent with SGT LMI Result? False

WHY?:

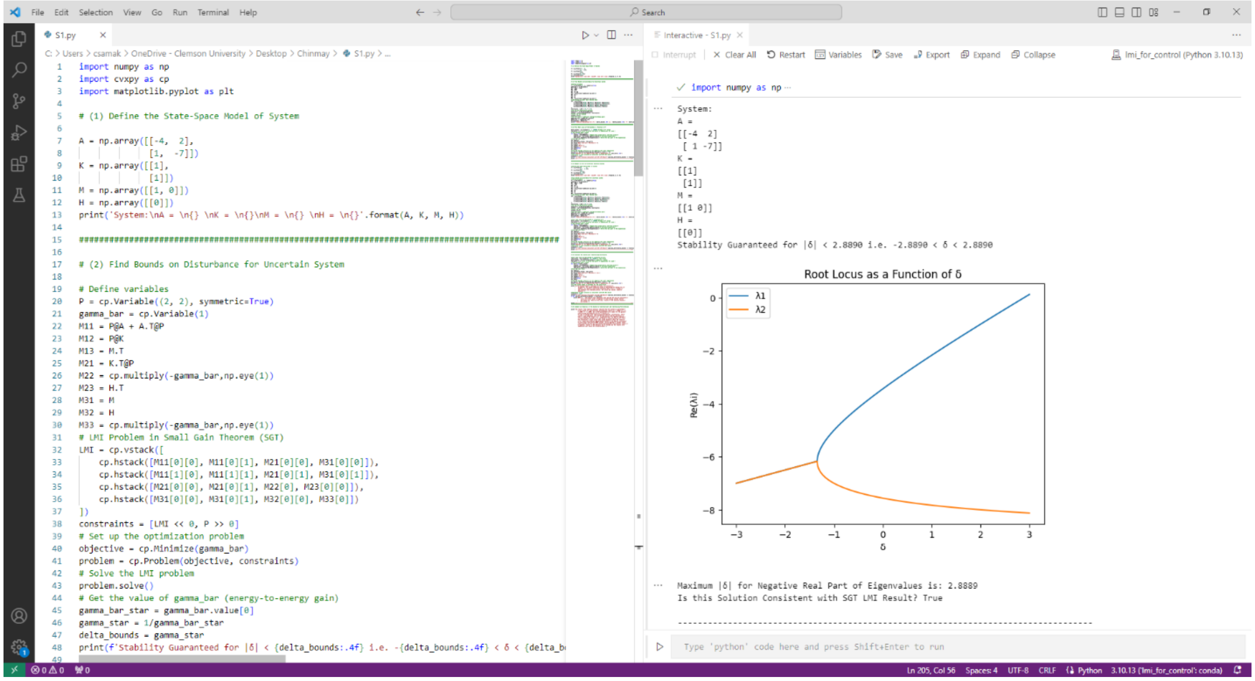
The results from eigenvalue test and SGT LMI are not consistent since the former cannot be applied to time-varying systems while the latter can. Hence in this case, results of SGT analysis should be trusted.



6) The results from stability analysis indicate that the system is guaranteed to be stable for time-invariant perturbations (where |δ| < 2.8890 i.e., -2.8890 < δ < 2.8890), but is NOT guaranteed to be stable for the given time-varying perturbations of δ(t) = cos(2t).

In case of systems with time-varying perturbations (uncertainty), the results from eigenvalue test and SGT LMI may/will NOT be consistent. This is because the former (i.e., eigenvalue test) can ONLY be applied to time-invariant systems or if the system is slowly varying (i.e., if the uncertainty is much slower than system dynamics) while the latter (i.e., small gain theorem) being generalization of the Nyquist criterion to non-linear time-varying MIMO systems, can be applied to systems with time-invariant as well as time-varying perturbations (uncertainty). Hence, in general, results of SGT analysis can be trusted but the results of eigenvalue test cannot be trusted blindly.

Screenshot:


File
Edit
Selection
View
Go
Run
Terminal
Help
Search
S1.py
C: > Users > csamak > OneDrive - Clemson University > Desktop > Chinmay > S1.py > ...
Interactive - S1.py
Interrupt
Clear All
Restart
Variables
Save
Export
Expand
Collapse
lmi_for_control (Python 3.10.13)
Root Locus as a Function of δ
λ1
λ2
Re(λ)
δ
0
−2
−4
−6
−8
−3
−2
−1
0
1
2
3
Type 'python' code here and press Shift+Enter to run
Ln 205, Col 56
Spaces: 4
UTF-8
CRLF
Python
3.10.13 ('lmi_for_control': conda)

```python
import numpy as np
import cvxpy as cp
import matplotlib.pyplot as plt

# (1) Define the State-Space Model of System

A = np.array([[-4,  2],
              [1,  -7]])
K = np.array([[1],
              [1]])
M = np.array([[1, 0]])
H = np.array([[0]])
print('System:\nA = \n{} \nK = \n{}\nM = \n{} \nH = \n{}'.format(A, K, M, H))

##############################################################################################

# (2) Find Bounds on Disturbance for Uncertain System

# Define variables
P = cp.Variable((2, 2), symmetric=True)
gamma_bar = cp.Variable(1)
M11 = P@A + A.T@P
M12 = P@K
M13 = M.T
M21 = K.T@P
M22 = cp.multiply(-gamma_bar,np.eye(1))
M23 = H.T
M31 = M
M32 = H
M33 = cp.multiply(-gamma_bar,np.eye(1))
# LMI Problem in Small Gain Theorem (SGT)
LMI = cp.vstack([
    cp.hstack([M11[0][0], M11[0][1], M21[0][0], M31[0][0]]),
    cp.hstack([M11[1][0], M11[1][1], M21[0][1], M31[0][1]]),
    cp.hstack([M21[0][0], M21[0][1], M22[0], M23[0][0]]),
    cp.hstack([M31[0][0], M31[0][1], M32[0][0], M33[0]])
])
constraints = [LMI << 0, P >> 0]
# Set up the optimization problem
objective = cp.Minimize(gamma_bar)
problem = cp.Problem(objective, constraints)
# Solve the LMI problem
problem.solve()
# Get the value of gamma_bar (energy-to-energy gain)
gamma_bar_star = gamma_bar.value[0]
gamma_star = 1/gamma_bar_star
delta_bounds = gamma_star
print(f'Stability Guaranteed for |δ| < {delta_bounds:.4f} i.e. -{delta_bounds:.4f} < δ < {delta_b
```

```
✓ import numpy as np ···
```

```
System:
A =
[[-4  2]
 [ 1 -7]]
K =
[[1]
 [1]]
M =
[[1 0]]
H =
[[0]]
Stability Guaranteed for |δ| < 2.8890 i.e. -2.8890 < δ < 2.8890
```

```
Maximum |δ| for Negative Real Part of Eigenvalues is: 2.8889
Is this Solution Consistent with SGT LMI Result? True

------------------------------------------------------------------------------------
```