AuE-6600: Dynamic Performance of Vehicles

Capstone Project

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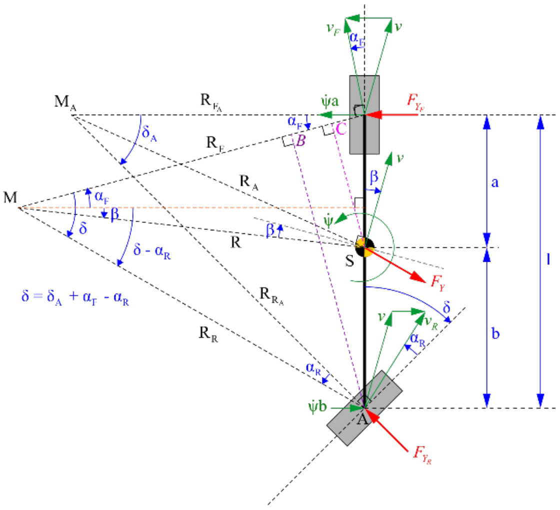
PART I – MODEL DEVELOPMENT



Transport Theorem:

$$v = \begin{bmatrix} v * cos\beta \\ v * sin\beta \\ 0 \end{bmatrix}$$

$$\dot{v} = \begin{bmatrix} \dot{v} * cos\beta - v * \dot{\beta} * sin\beta \\ -\dot{v} * sin\beta - v * \dot{\beta} * cos\beta \\ 0 \end{bmatrix}$$

$$\omega = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$$

$$\omega \times v = \begin{bmatrix} \dot{\psi} * v * sin\beta \\ \dot{\psi} * v * cos\beta \\ 0 \end{bmatrix}$$

$$a = \begin{bmatrix} \dot{v} * cos\beta - v * \dot{\beta} * sin\beta + \dot{\psi} * v * sin\beta \\ -\dot{v} * sin\beta - v * \dot{\beta} * cos\beta + \dot{\psi} * v * cos\beta \\ 0 \end{bmatrix}$$

Vehicle Dynamics:

Summation of forces along X and Y and moment about CG.

$\sum \boldsymbol{F_{x^B}}$:

$F_{XF} + F_{XR} * cos\delta + F_{YR} * sin\delta$

$= F_Y * sin\beta$

$= m * a_x^{I/B}$

$= m * [\dot{v} * cos\beta - v * \dot{\beta} * sin\beta + \dot{\psi} * v * sin\beta]$

$\sum \boldsymbol{F_{Y^B}}$:

$F_{YF} + F_{YR} * cos\delta - F_{XR} * sin\delta$

$= F_Y * cos\beta$

$= m * a_y^{I/B}$

$= m * [-\dot{v} * sin\beta - v * \dot{\beta} * cos\beta + \dot{\psi} * v * cos\beta]$

$\sum \boldsymbol{M_{CG}}$:

$F_{YF} * a + I_Z\ddot{\psi}$

$= F_{YR} * cos\delta * b + F_{XR} * sin\delta * b$

Small angle approx: $sin\theta \approx \theta, cos\theta \approx 1$

Constant velocity approx: $|v| \approx const$

$\Rightarrow \dot{v} \approx 0 \Rightarrow F_{XF} \approx 0$

Simplified lateral dynamics (linear)

$\sum \boldsymbol{F}$: $F_{YF} + F_{YR} - F_Y = 0 \Rightarrow F_Y = F_{YF} + F_{YR}$

$\sum \boldsymbol{M}$: $I_Z\ddot{\psi} = F_{YF} * a - F_{YF} * b$ (+ CCW)

From figure:

$|-\alpha_F| + |-\beta| = \left|\frac{\dot{\psi}*a}{v}\right| \Rightarrow \alpha_F = \frac{-\dot{\psi}*a}{v} - \beta$ --(*)

$|-\delta| = |-\alpha_R| + \left|\frac{\dot{\psi}*b}{v}\right| + |-\beta| \Rightarrow \alpha_R = \delta + \frac{\dot{\psi}*b}{v} - \beta$ --(**)

Combine with known quantities and linear tire model (for $a_y \leq 4\, m/s^2$)

$F_{YF} = C_{\alpha F} * \alpha_F$

$F_{YR} = C_{\alpha R} * \alpha_R$

$\Rightarrow C_{\alpha F} * \alpha_F + C_{\alpha R} * \alpha_R - m * v * (\dot{\psi} - \dot{\beta}) = 0$

$\Rightarrow I_Z * \ddot{\psi} - (C_{\alpha F} * \alpha_F) * a + (C_{\alpha R} * \alpha_R) * b = 0$

With angles from (*) and (**)

I. $C_{\alpha F}\left(\frac{-\dot{\psi}*a}{v} - \beta\right) + C_{\alpha R}\left(\delta + \frac{\dot{\psi}*b}{v} - \beta\right) - m * v * (\dot{\psi} - \dot{\beta}) = 0$

II. $I_Z * \ddot{\psi} - C_{\alpha F}\left(\frac{-\dot{\psi}*a}{v} - \beta\right) * a + C_{\alpha R}\left(\delta + \frac{\dot{\psi}*b}{v} - \beta\right) * b = 0$

$$\Rightarrow \dot{\beta} = \underbrace{\left(\frac{C_{\alpha F}+C_{\alpha R}}{m*v}\right)}_{a_{11}}\beta + \underbrace{\left(\frac{C_{\alpha F}*a - C_{\alpha R}*b}{m*v^2}+1\right)}_{a_{12}}\dot{\psi} - \underbrace{\left(\frac{C_{\alpha R}}{m*v}\right)}_{b_1}\delta$$

$$\Rightarrow \ddot{\psi} = \underbrace{\left(\frac{b*C_{\alpha R} - a*C_{\alpha F}}{I_Z}\right)}_{a_{21}}\beta - \underbrace{\left(\frac{C_{\alpha F}*a^2 + C_{\alpha R}*b^2}{I_Z*v}\right)}_{a_{22}}\dot{\psi} - \underbrace{\left(\frac{C_{\alpha R}*b}{I_Z}\right)}_{b_2}\delta$$

Let $\dot{\psi} = r \Rightarrow \ddot{\psi} = \dot{r}$

State-space form:

$$\dot{X} = \begin{bmatrix}\dot{\beta}\\ \dot{r}\end{bmatrix} = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}X + \begin{bmatrix}b_1\\ b_2\end{bmatrix}u$$

where, $X = \begin{bmatrix}\beta\\ r\end{bmatrix}$ $and\ u = \delta$

Eigen Values:

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Eigen values of the system are:
  -0.5633 + 3.9821i
  -0.5633 - 3.9821i
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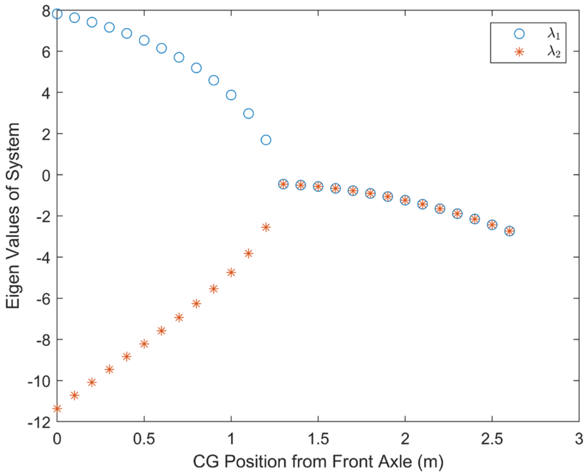


Non-Linear Tire Model with Load Transfer and Roll Dynamics:

$_* =$ Front/Rear and $_\# =$ Left/Right

$$a_y = v * (\dot{\psi} - \dot{\beta})$$

$$\ddot{\phi} = \frac{1}{I_\phi}[m * a_y * h_{cr} + m * g * h_{cr} * \phi - B_{SR} * \dot{\phi} - K_{SR} * \phi]$$

$$\dot{\phi}_{t+1} = \dot{\phi}_t + \ddot{\phi}_t * \Delta t$$

$$\Delta W_* = \frac{K_{\phi *} * K_{SR} * \phi}{w} + \frac{m_* * a_y * h_*}{w}$$

$$F_{Z*\#} = S_{*\#} \pm \Delta W_*$$

$$\alpha_F = -\frac{\dot{\psi} * a}{v} - \beta$$

$$\alpha_R = \delta + \frac{\dot{\psi} * b}{v} - \beta$$

$$F_{Y*\#} = -\text{nonlintire}(\alpha_*, F_{Z*\#}, v)$$

$$F_{YF} = F_{YFL} + F_{YFR}$$

$$F_{YR} = F_{YRL} + F_{YRR}$$

$$\dot{\boldsymbol{X}} = \begin{bmatrix}\dot{\beta}\\ \ddot{\psi}\end{bmatrix} = \begin{bmatrix}\dot{\psi} - \frac{F_{YF} + F_{YR}}{m * v}\\ \frac{a * F_{YF} - b * F_{YR}}{I_Z}\end{bmatrix}$$

$$\boldsymbol{X}_{t+1} = \boldsymbol{X}_t + \dot{\boldsymbol{X}}_t * \Delta t$$

$$\psi_{t+1} = \psi_t + \dot{\psi}_t * \Delta t$$

$$v_{x_t} = v * \cos(\psi - \beta)$$

$$v_{y_t} = v * \sin(\psi - \beta)$$

$$p_{x_{t+1}} = p_{x_t} + v_{x_t} * \Delta t$$

$$p_{y_{t+1}} = p_{y_t} + v_{y_t} * \Delta t$$

Geometric Analysis:

I. Force-free rolling:

$\boxed{\boldsymbol{\delta}}$ $\tan(\delta_A) = \frac{l}{R_f} \Rightarrow \delta_A = tan^{-1}\left(\frac{l}{R_f}\right) \approx \frac{l}{R}$

$\boxed{\boldsymbol{\beta}}$ $\tan(\beta_o) = \frac{a}{Rf} \Rightarrow \beta_o = tan^{-1}\left(\frac{a}{R_f}\right) \approx \frac{a}{R}$

II. Rolling with lateral forces:

$\boxed{\boldsymbol{\delta}}$ $\tan(\delta - \alpha_R + \alpha_F) = \frac{l*cos\alpha_F}{R_f}$

Small angle approximation:

$\delta - \alpha_R + \alpha_F = \underbrace{\frac{l}{R}}_{\delta_A} \Rightarrow \delta = \delta_A + \alpha_R - \alpha_F$

$\boxed{\boldsymbol{\beta}}$ $\overline{SC} = a * cos\alpha_F$

$\Rightarrow R * \sin(\alpha_F - \beta) = a * cos\alpha_F$

Small angle approximation:

$R * (\alpha_F - \beta) = a$

$\Rightarrow \beta = \alpha_F - \frac{a}{R}$

$\boxed{\boldsymbol{UG}}$ $F_y \sim a_y$ and $\alpha_F \sim F_y, \alpha_R \sim F_y$ (for small angles) with $\delta \sim (\alpha_F - \alpha_R)$

$$\Rightarrow \delta \sim a_y \Rightarrow \delta = \delta_A + UG * a_y$$

We know that

$$\delta = \delta_A + \alpha_R - \alpha_F$$

with $\alpha_F = \frac{F_{YF}}{C_{\alpha F}} = \left(\frac{b}{l}\right) * \left(\frac{mv^2}{R}\right) * \frac{1}{C_{\alpha F}}$

and $\alpha_R = \frac{F_{YR}}{C_{\alpha_R}} = \left(\frac{a}{l}\right) * \left(\frac{mv^2}{R}\right) * \frac{1}{C_{\alpha_R}}$

$$\Rightarrow \delta = \delta_A + \underbrace{\left[\frac{m*(-C_{\alpha_R}*b + C_{\alpha_F}*a)}{l*C_{\alpha_R}*C_{\alpha_F}}\right]}_{UG} a_y$$

Stability Derivatives:

$\alpha_F = -\frac{\dot{\psi}*a}{v} - \beta \Rightarrow C_{\alpha_F} * \alpha_F = -\frac{C_{\alpha_F}*\dot{\psi}*a}{v} - \beta * C_{\alpha_F}$

$\alpha_R = \delta + \frac{\dot{\psi}*b}{v} - \beta \Rightarrow C_{\alpha_R} * \alpha_R = C_{\alpha_R} * \delta + \frac{C_{\alpha_R}*\dot{\psi}*b}{v} - C_{\alpha_R} * \beta$

$\sum \boldsymbol{F_y}$**:** using small angle approximation:

$F_{YF} + F_{YR}$

$= C_{\alpha_F} * \alpha_F + C_{\alpha_R} * \alpha_R$

$= \left(-\frac{\dot{\psi}*a*C_{\alpha_F}}{v}\right) - \beta * C_{\alpha_F} + C_{\alpha_R} * \delta + \frac{C_{\alpha_R}*\dot{\psi}*b}{v} - C_{\alpha_R} * \beta$

$= \left(-\frac{r*a*C_{\alpha_F}}{v}\right) - \beta * C_{\alpha_F} + C_{\alpha_R} * \delta + \frac{C_{\alpha_R}*r*b}{v} - C_{\alpha_R} * \beta$

$\sum \boldsymbol{M_z}$**:** using small angle approximation:

$a * F_{YF} - b * F_{YR}$

$= a * C_{\alpha_F} * \alpha_F - b * C_{\alpha_R} * \alpha_R$

$= -b * C_{\alpha_R} * \delta - \frac{b^2*C_{\alpha_R}*r}{v} + b * C_{\alpha_R} * \beta - \frac{a^2*C_{\alpha_F}*r}{v} - a * C_{\alpha_F} * \beta$

i. Damping in sideslip:

$$Y_\beta = \frac{\partial \sum F_y}{\partial \beta} = -(C_{\alpha_F} + C_{\alpha_R})$$

ii. Lateral force yaw coupling:

$$Y_r = \frac{\partial \sum F_y}{\partial r} = \frac{b * C_{\alpha_R} - a * C_{\alpha_F}}{v}$$

iii. Control force derivative:

$$Y_\delta = \frac{\partial \sum F_y}{\partial \delta} = C_{\alpha_R}$$

iv. Static directional stability:

$$N_\beta = \frac{\partial \sum M_z}{\partial \beta} = b * C_{\alpha_R} - a * C_{\alpha_F}$$

v. Yaw damping:

$$N_r = \frac{\partial \sum M_z}{\partial r} = \frac{-(a^2 * C_{\alpha_F} + b^2 * C_{\alpha_R})}{v}$$

vi. Control moment derivative:

$$N_\delta = \frac{\partial \sum M_z}{\partial \delta} = -b * C_{\alpha_R}$$

Critical Velocity:

$\frac{\dot{\psi}}{\delta} = \frac{r}{\delta} = \frac{v/R}{\delta_A + UG*a_y} = \frac{v}{l + UG*v^2} = \frac{v/l}{1+K*v^2}$

where, $K = \frac{UG}{l}$

as $\frac{\dot{\psi}}{\delta} \to \infty$, vehicle reacts with enormous yaw velocity change to a very small steering angle

i.e. @ $\frac{v}{l + UG*v^2} \to \infty$

$\Rightarrow l + UG * v_{critical}^2 = 0$

$\Rightarrow v_{critical} = \sqrt{\frac{-l}{UG}} = \sqrt{\frac{-1}{K}}$

Characteristic Velocity:

@ characteristic velocity the vehicle steers with maximum steering sensitivity

i.e. $\frac{d\left(\frac{\dot{\psi}}{\delta}\right)}{dv} = \frac{l - UG*v^2}{(l+UG*v^2)^2} := 0$

$\Rightarrow l - UG * v_{characteristic}^2 = 0$

$\Rightarrow v_{charcteristic} = \sqrt{\frac{l}{UG}} = \sqrt{\frac{1}{K}}$

Comments:

The equations of motion (EOM) of rear-wheel-steered vehicle configuration reveal that the terms corresponding to first state variable in the system and control matrices are opposite to those for front-wheel-steered vehicle configuration which is expected since the vehicle sideslip angle is opposite in this case.

The eigenvalues of system matrix contain imaginary component which is logical since the resultant transformation involves rotation and thus knocks off all the input vectors from their span.

Geometric analysis reveals expected behavior for Ackerman steering angle as well as vehicle sideslip angle (expressed in terms of front axle distance to CG since we have a rear-wheel-steered vehicle configuration instead of front-wheel-steered vehicle configuration).

The understeer gradient (UG) of rear-wheel-steered vehicle configuration comes out to be opposite as compared to that of front-wheel-steered vehicle configuration, which makes sense sine we have steering attached to rear wheel instead of front one.

The control force derivative comes in terms of rear cornering stiffness instead of front (which makes sense since we have a rear-wheel-steered vehicle configuration instead of front-wheel-steered vehicle configuration).

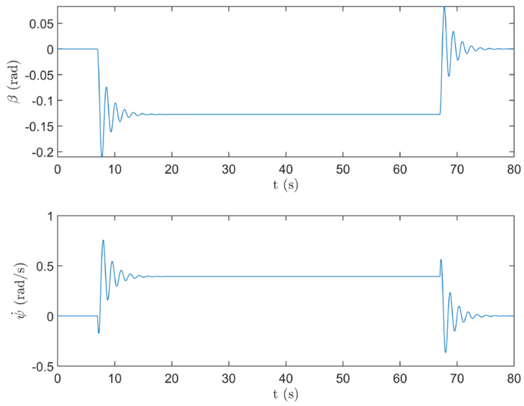
The static directional stability derivative for rear-wheel-steered vehicle configuration comes out to be opposite (negative) as compared to the one in case of front-wheel-steered vehicle configuration (which makes sense since directions of slip angles are opposite in rear-wheel-steered vehicle configuration as compared to front-wheel-steered vehicle configuration).

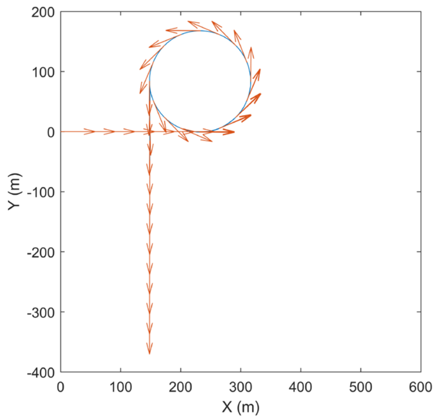
The control moment derivative comes in terms of rear cornering stiffness and distance from CG to rear axle instead of front (which makes sense since we have a rear-wheel-steered vehicle configuration instead of front-wheel-steered vehicle configuration).

PART II – MODEL SIMULATION

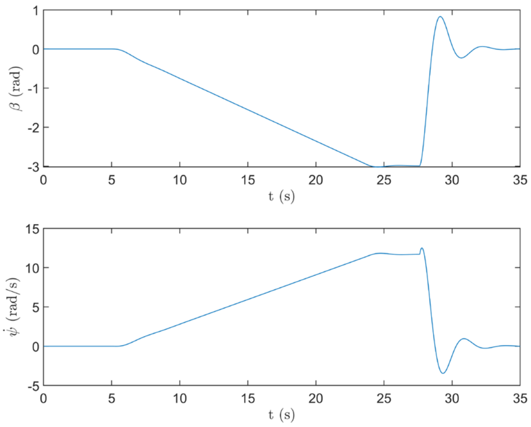
Linear Tire Model without Load Transfer and Roll Dynamics:

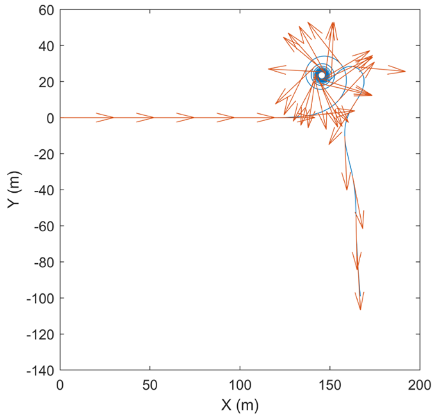
Step Maneuver:





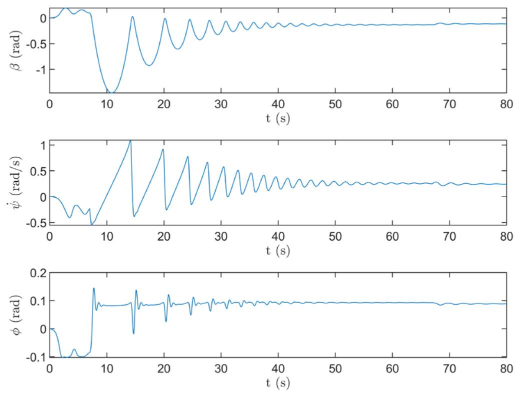
Fishhook Maneuver:

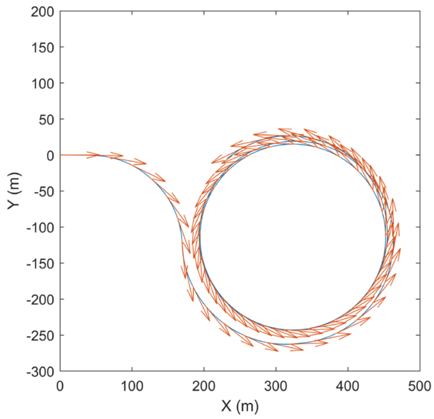




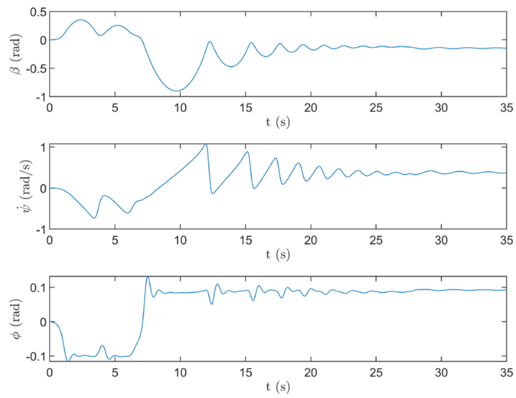
Non-Linear Tire Model with Load Transfer and Roll Dynamics:

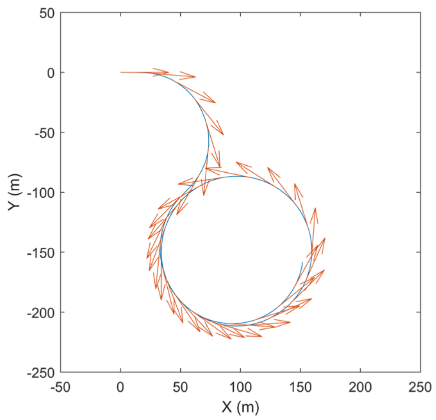
Step Maneuver:





Fishhook Maneuver:





UG for Various Weight/Roll Distributions:

Maneuver	(Front Weight Distribution, Front Roll Stiffness Distribution)		
	(60%, 60%)	(55%, 46%)	(50%, 60%)
Step	`UG = 0.0116 rad-s^2/m`	`UG = 0.0129 rad-s^2/m`	`UG = 0.0121 rad-s^2/m`
Fishhook	`UG = 0.0109 rad-s^2/m`	`UG = 0.0111 rad-s^2/m`	`UG = 0.0113 rad-s^2/m`

N.B. Refer Appendix for details and δ vs. a_y plots.

Comments:

The state variable plots and trajectory plots reveal expected behavior when using linear tire model.

When repeating simulation runs with non-linear tire model including roll dynamics and load transfer, the results show slightly off behavior since the nonlinear tire model that we have outputs non-zero lateral forces for zero slip angles.

The UG for step maneuver could not be computed through slope calculation for δ vs. a_y plot since neither the longitudinal velocity nor the steering input was varied for step maneuver. Hence, we calculated UG for step maneuver based on the following relation:

$$\alpha_F - \alpha_R = UG * a_y \Longrightarrow UG = \frac{\alpha_F - \alpha_R}{a_y}$$

The UG for fishhook maneuver was computed through slope calculation for δ vs. a_y plot.

The variation in vehicle (model) parameters such as weight distribution and roll-stiffness distribution causes expected variations in UG (as explained in PART I comments).

PART III – TWO-AXLE STEERING

Modifying (*) and (**) from previous derivation to arrive at model for 2-axle steered vehicle:

$$\alpha_F = \delta_F - \frac{\dot\psi * a}{v} - \beta \quad \text{--(*)}$$

$$\alpha_R = \delta_R + \frac{\dot\psi * b}{v} - \beta \quad \text{--(**)}$$

Combine with known quantities and linear tire model (for $a_y \le 4m/s^2$)

$$F_{YF} = C_{\alpha F} * \alpha_F$$

$$F_{YR} = C_{\alpha R} * \alpha_R$$

$$\Rightarrow C_{\alpha F} * \alpha_F + C_{\alpha R} * \alpha_R - m * v * (\dot\psi + \dot\beta) = 0$$

$$\Rightarrow I_Z * \ddot\psi - C_{\alpha F} * \alpha_F * a + C_{\alpha R} * \alpha_R * b = 0$$

With angles from (*) and (**)

I. $C_{\alpha F}\left[\delta_F - \frac{\dot\psi * a}{v} - \beta\right] + C_{\alpha R}\left[\delta_R + \frac{\dot\psi * b}{v} - \beta\right] - m * v * (\dot\psi + \dot\beta) = 0$

II. $I_Z * \ddot\psi - C_{\alpha F}\left[\delta_F - \frac{\dot\psi * a}{v} - \beta\right] * a + C_{\alpha R}\left[\delta_R + \frac{\dot\psi * b}{v} - \beta\right] * b = 0$

$$\Rightarrow \dot\beta = \underbrace{-\left(\frac{C_{\alpha F} + C_{\alpha R}}{m * v}\right)}_{a_{11}}\beta + \underbrace{\left(\frac{-C_{\alpha F} * a + C_{\alpha R} * b}{m * v^2} - 1\right)}_{a_{12}}\dot\psi + \underbrace{\left(\frac{C_{\alpha F}}{m * v}\right)}_{b_{11}}\delta_f + \underbrace{\left(\frac{C_{\alpha R}}{m * v}\right)}_{b_{12}}\delta_r$$

$$\Rightarrow \ddot\psi = \underbrace{\left(\frac{b * C_{\alpha R} - a * C_{\alpha F}}{I_Z}\right)}_{a_{21}}\beta - \underbrace{\left(\frac{C_{\alpha F} * a^2 + C_{\alpha R} * b^2}{I_Z * v}\right)}_{a_{22}}\dot\psi + \underbrace{\left(\frac{a * C_{\alpha F}}{I_Z}\right)}_{b_{21}}\delta_f - \underbrace{\left(\frac{b * C_{\alpha R}}{I_Z}\right)}_{b_{22}}\delta_r$$

State-space form:

$$\dot X = \begin{bmatrix}\dot\beta \\ \dot r\end{bmatrix} = \begin{bmatrix}a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix} X + \begin{bmatrix}b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix} u$$

$$\text{where, } X = \begin{bmatrix}\beta \\ r\end{bmatrix} \text{ and } u = \begin{bmatrix}\delta_f \\ \delta_r\end{bmatrix}$$

For neutral steering behavior, $\alpha_F = \alpha_R$

$$\Rightarrow \delta_F - \frac{\dot\psi * a}{v} - \beta = \delta_R - \frac{\dot\psi * b}{v} - \beta$$

$$\Rightarrow \delta_R = \delta_F - \frac{\dot\psi * a}{v} - \frac{\dot\psi * b}{v}$$

$$\Rightarrow \delta_R = \delta_F - \frac{\dot\psi * (a+b)}{v} = \delta_F - \frac{\dot\psi * l}{v}$$

Non-Linear Tire Model with Load Transfer and Roll Dynamics:

$_*$ = Front/Rear and $_\#$ = Left/Right

$$a_y = v * (\dot\psi + \dot\beta)$$

$$\ddot\phi = \frac{1}{I_\phi}[m * a_y * h_{cr} + m * g * h_{cr} * \phi - B_{SR} * \dot\phi - K_{SR} * \phi]$$

$$\dot\phi_{t+1} = \dot\phi_t + \ddot\phi_t * \Delta t$$

$$\Delta W_* = \frac{K_{\phi*} * K_{SR} * \phi}{w} + \frac{m_* * a_y * h_*}{w}$$

$$F_{Z*\#} = S_{*\#} \pm \Delta W_*$$

$$\delta_R = \delta_F - \frac{\dot{\psi} * l}{v}$$

$$\alpha_F = \delta_F - \frac{\dot{\psi} * a}{v} - \beta$$

$$\alpha_R = \delta_R + \frac{\dot{\psi} * b}{v} - \beta$$

$$F_{Y*\#} = -\text{nonlintire}(\alpha_*, F_{Z*\#}, v)$$

$$F_{YF} = F_{YFL} + F_{YFR}$$

$$F_{YR} = F_{YRL} + F_{YRR}$$

$$\dot{\boldsymbol{X}} = \begin{bmatrix} \dot{\beta} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} \dfrac{F_{YF} + F_{YR}}{m * v} - \dot{\psi} \\ \dfrac{a * F_{YF} - b * F_{YR}}{I_Z} \end{bmatrix}$$

$$\boldsymbol{X}_{t+1} = \boldsymbol{X}_t + \dot{\boldsymbol{X}}_t * \Delta t$$

$$\psi_{t+1} = \psi_t + \dot{\psi}_t * \Delta t$$

$$v_{x_t} = v * \cos(\psi + \beta)$$

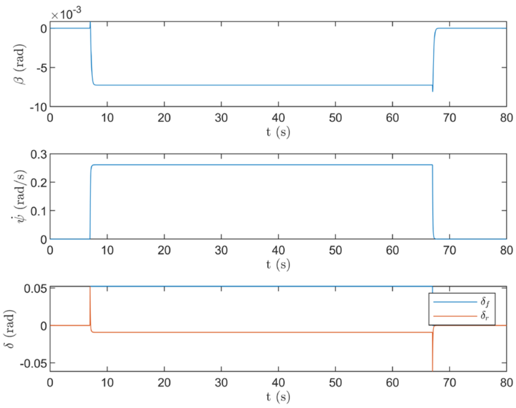
$$v_{y_t} = v * \sin(\psi + \beta)$$

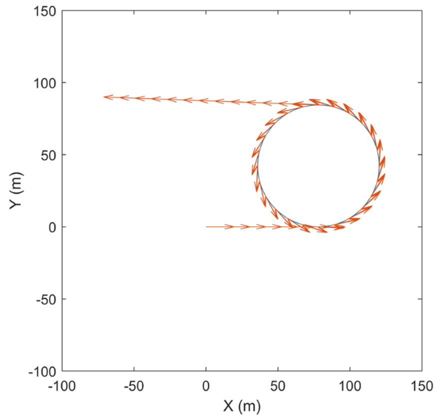
$$p_{x_{t+1}} = p_{x_t} + v_{x_t} * \Delta t$$

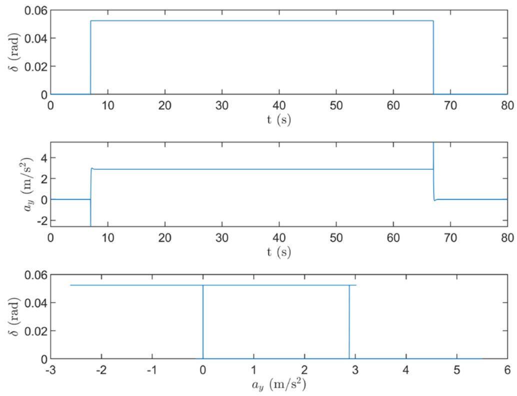
$$p_{y_{t+1}} = p_{y_t} + v_{y_t} * \Delta t$$

Simulation Results for Linear Tire Model without Load Transfer and Roll Dynamics:

Step Maneuver:

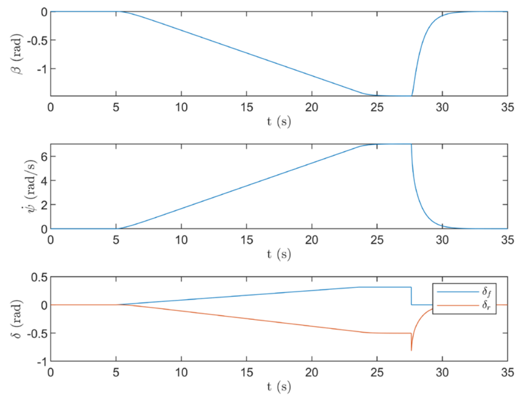


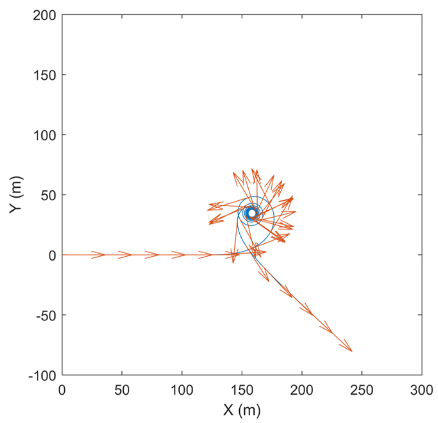


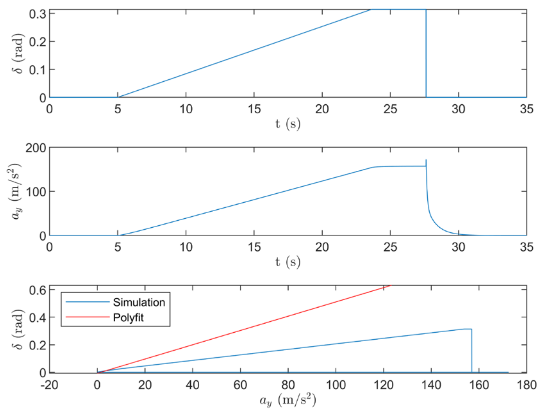


`UG = 0.0 rad-s^2/m`

Fishhook Maneuver:



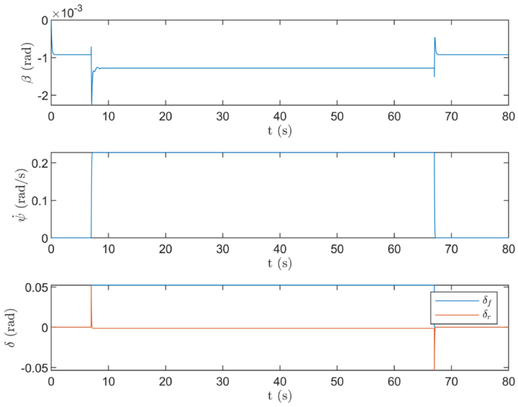


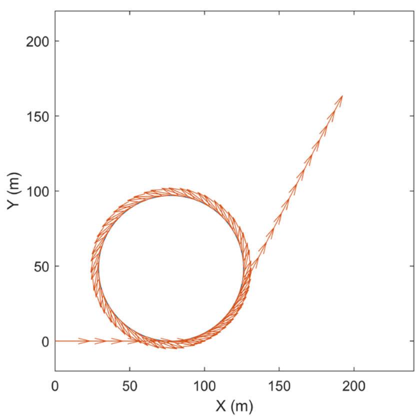


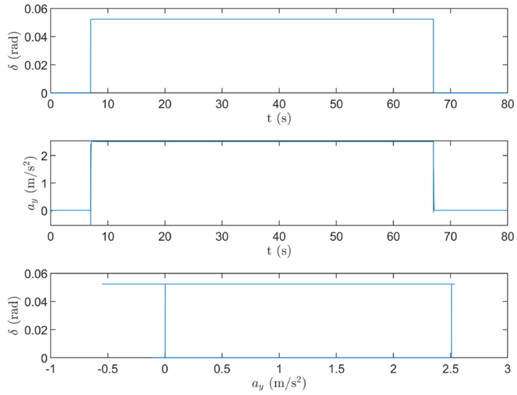
UG = 0.0000 rad-s^2/m

Simulation Results for Non-Linear Tire Model with Load Transfer and Roll Dynamics:

Step Maneuver:

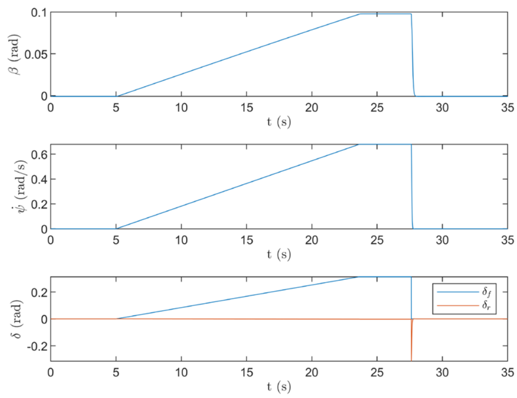


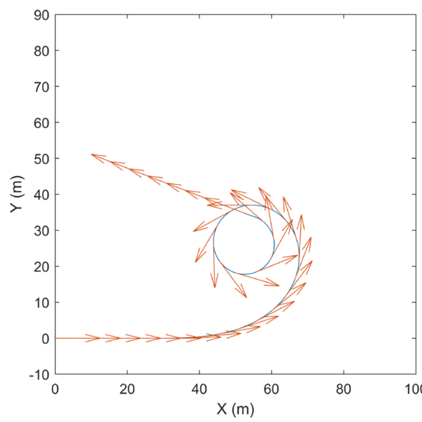


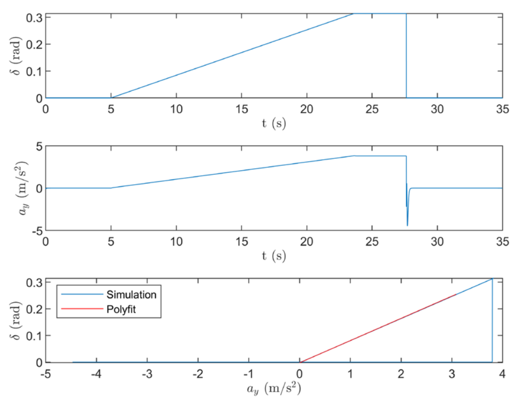


UG = 0.0 rad-s^2/m

Fishhook Maneuver:







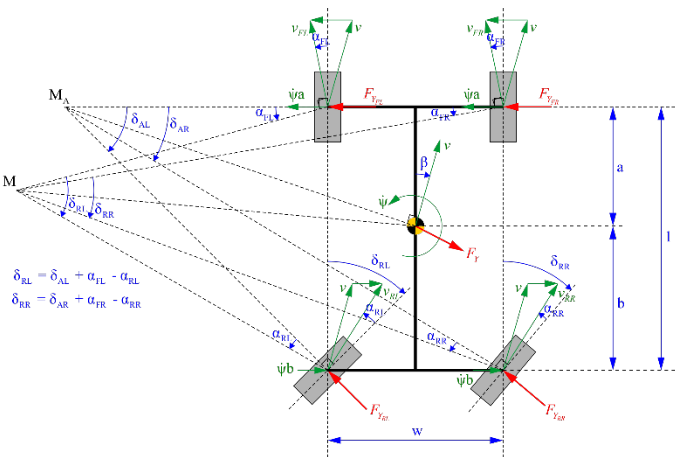
UG = 0.0000 rad-s^2/m

Comments:

The assistive rear-steering angle for Part A (linear tire model without considering load transfer and roll dynamics) was quite significant as compared to Part B (non-linear tire model considering load transfer and roll dynamics). This could be explained by the fact that the non-linear tire model coupled with load transfer and roll dynamics by itself drove the vehicle closer to neutral-steer behavior and hence less correction was required from the rear wheel steering.

The UG was verified to be close to zero for both Part A and Part B.

PART IV – FOUR-WHEEL MODEL



Vehicle Model:

$_{*}$ = Front/Rear and $_{\#}$ = Left/Right

$$a_y = v * (\dot{\psi} - \dot{\beta})$$

$$\ddot{\phi} = \frac{1}{I_\phi}[m * a_y * h_{cr} + m * g * h_{cr} * \phi - B_{SR} * \dot{\phi} - K_{SR} * \phi]$$

$$\dot{\phi}_{t+1} = \dot{\phi}_t + \ddot{\phi}_t * \Delta t$$

$$\Delta W_* = \frac{K_{\phi *} * K_{SR} * \phi}{w} + \frac{m_* * a_y * h_*}{w}$$

$$F_{Z_{*\#}} = S_{*\#} \pm \Delta W_*$$

From Ackerman geometry,

$$\delta_{RL} = \frac{l}{\left(\frac{l}{\delta}\right) - \left(\frac{w}{2}\right)}$$

$$\delta_{RR} = \frac{l}{\left(\frac{l}{\delta}\right) + \left(\frac{w}{2}\right)}$$

$$\alpha_{F\#} = -\frac{\dot{\psi} * a}{v} - \beta \pm \delta_{toe}$$

$$\alpha_{R\#} = \delta_{R\#} + \frac{\dot{\psi} * b}{v} - \beta$$

$$F_{Y_{*\#}} = -\text{nonlintire}(\alpha_{*\#}, F_{Z_{*\#}}, v)$$

$$F_{YF} = F_{YFL} + F_{YFR}$$

$$F_{YR} = F_{YRL} + F_{YRR}$$

$$\dot{\boldsymbol{X}} = \begin{bmatrix} \dot{\beta} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} \dot{\psi} - \frac{F_{YF} + F_{YR}}{m * v} \\ \frac{a * F_{YF} - b * F_{YR}}{I_Z} \end{bmatrix}$$

$$\boldsymbol{X}_{t+1} = \boldsymbol{X}_t + \dot{\boldsymbol{X}}_t * \Delta t$$

$$\psi_{t+1} = \psi_t + \dot{\psi}_t * \Delta t$$

$$v_{x_t} = v * \cos(\psi - \beta)$$

$$v_{y_t} = v * \sin(\psi - \beta)$$

$$p_{x_{t+1}} = p_{x_t} + v_{x_t} * \Delta t$$

$$p_{y_{t+1}} = p_{y_t} + v_{y_t} * \Delta t$$

Wheel Velocities:

$$v_{x_{FL}} = v * cos\beta$$

$$v_{x_{FR}} = v * cos\beta$$

$$v_{x_{RL}} = v * \cos(|\delta_{RL}| - |\beta|)$$

$$v_{x_{RR}} = v * \cos(|\delta_{RR}| - |\beta|)$$

Ackermann Steering Angle:

$$\delta_{RL} = \tan^{-1}\left(\frac{l}{R - \frac{w}{2}}\right) = \tan^{-1}\left(\frac{l}{\frac{l}{\tan(\delta)} - \frac{w}{2}}\right)$$

Using small angle approximation

$$\delta_{RL} = \left(\frac{l}{\frac{l}{\delta} - \frac{w}{2}}\right)$$

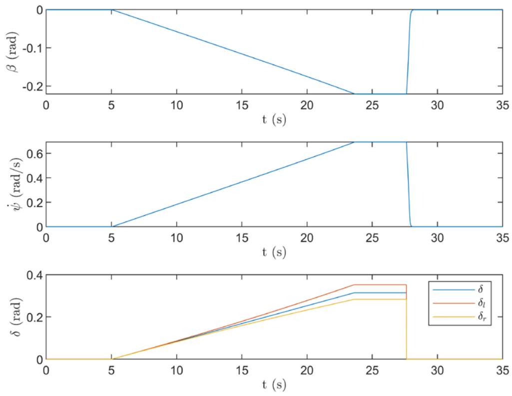
Similarly,

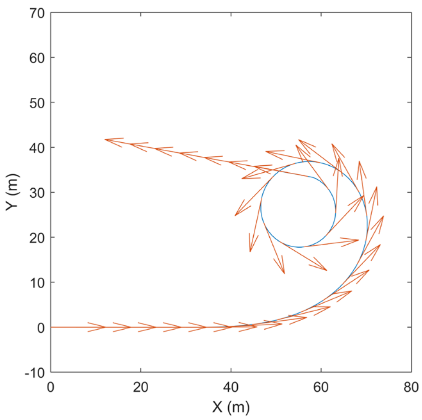
$$\delta_{RR} = \tan^{-1}\left(\frac{l}{R + \frac{w}{2}}\right) = \tan^{-1}\left(\frac{l}{\frac{l}{\tan(\delta)} + \frac{w}{2}}\right)$$

Using small angle approximation

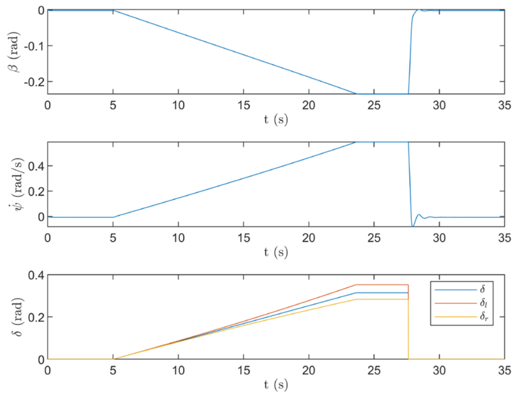
$$\delta_{RR} = \left(\frac{l}{\frac{l}{\delta} + \frac{w}{2}}\right)$$

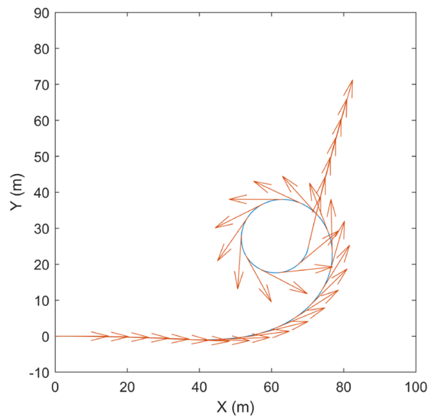
Simulation for Fishhook Maneuver with Zero Toe Angle ($\delta_{toe} = 0°$):



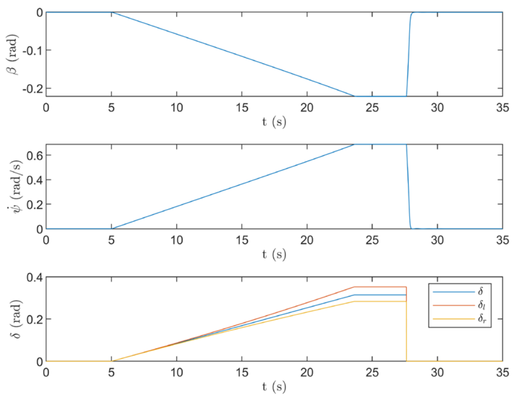


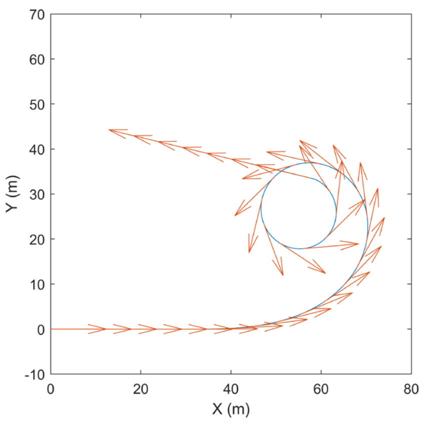
Simulation for Fishhook Maneuver with Large Toe Angle ($\delta_{toe} = 5°$):





Simulation for Fishhook Maneuver with Small Toe Angle ($\delta_{toe} = 0.5°$):





Comments:

Since the given toe angle (0.5°) was quite small, the differences in vehicle behavior were not very apparent.

We increased the toe angle to 5° (which is still decently small) to notice/observe the difference in vehicle behavior with and without toe angle. The state variables changed resulting in a different vehicle trajectory. This is because the rate of change of state variables depends on the lateral force, which in turn depends on the tire slip angle – and this is influenced by the toe-in/toe-out nature of the vehicle.