Embedded Resolution of Singularities in Characteristic Zero:

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Abstract

We give a detailed, proof-oriented exposition of embedded resolution of singularities over fields of characteristic zero. Building on Hironaka and later constructive algorithms (Bierstone–Milman, Villamayor, Encinas–Hauser, Wlodarczyk), we prove upper semicontinuity of order, existence of hypersurfaces of maximal contact in characteristic zero, construction and functoriality of coefficient ideals, controlled transforms, and the lexicographic descent of a canonical invariant ensuring termination. Complete low-dimensional cases (curves, surfaces) and a toric resolution for nondegenerate hypersurfaces are included.

Keywords: resolution of singularities; embedded desingularization; maximal contact; coefficient ideal; toric modification; Newton polyhedron.
MSC (2020): 14E15; 14B05; 14M25.

1 Introduction

Resolution of singularities in characteristic zero was proved by Hironaka [1, 2]. Constructive, functorial versions were developed in [3, 4, 5, 6, 7]. We present a self-contained, implementation-friendly account with full proofs of the central steps, keeping strict citation discipline. When we follow a classical argument (e.g., differential operators for maximal contact), we attribute the method and rephrase in our own words.

2 Preliminaries: blow-ups and SNC divisors

Definition 2.1 (Blow-up)**.** Let W be a smooth variety over a characteristic-zero field k and $C \subset W$ a smooth closed subscheme. The blow-up $\pi : \widetilde{W} = \mathrm{Bl}_C(W) \to W$ is Proj of the Rees algebra $\oplus_{m \geq 0} \mathcal{I}_C^m$. The exceptional divisor is $\mathrm{Exc}(\pi) = V(\mathcal{I}_C \cdot \mathcal{O}_{\widetilde{W}})$.

Definition 2.2 (Simple normal crossings (SNC))**.** A divisor E on smooth W has SNC if etale-locally there are coordinates $(x_1, \ldots, x_d)$ with $E = \{x_1 \cdots x_r = 0\}$.

Lemma 2.3 (SNC stability)**.** *If C is smooth and has normal crossings with E, then $\widetilde{W}$ is smooth and $\widetilde{E} := \mathrm{Exc}(\pi) + \pi_*^{-1} E$ is SNC.*

Proof. Etale-locally choose coordinates so that C and the components of E are coordinate subspaces/hyperplanes. In each affine chart of the blow-up, the total transform is again a normal crossings union of coordinate hyperplanes, and Jacobian rank shows smoothness; see [8, Ch. 1]. $\square$

3 Order and its upper semicontinuity

Definition 3.1 (Order)**.** For a coherent ideal $\subset \mathcal{O}_W$ and $a \in W$,

$$\operatorname{ord}_a() := \max\{b \in \mathbb{Z}_{\geq 0} \mid {}_a \subset \mathfrak{m}_a^b\}. \tag{3.1}$$

The top locus is $\operatorname{Top}() := \{a \mid \operatorname{ord}_a() = \max_W \operatorname{ord}()\}$.

Proposition 3.2 (Upper semicontinuity)**.** *The map $a \mapsto \operatorname{ord}_a()$ is upper semicontinuous; hence $\operatorname{Top}()$ is closed.*

Proof. Locally, fix generators $f_1, \ldots, f_t$ of . The condition $\operatorname{ord}_a() \geq b$ holds iff all partials $\partial^\alpha f_i$ with $|\alpha| < b$ vanish at a. This is the common zero locus of finitely many regular functions, hence Zariski-closed. Therefore $a \mapsto \operatorname{ord}_a()$ is upper semicontinuous. □

4 Maximal contact and coefficient ideals

4.1 Existence of hypersurfaces of maximal contact in characteristic zero

Definition 4.1 (Maximal contact)**.** Let $\operatorname{ord}_a() = b > 0$. A smooth hypersurface $H \subset W$ through a has maximal contact at a if, in etale-local coordinates $(x_1, \ldots, x_{n-1}, x_n)$ with $H = (x_n = 0)$, the initial forms of elements of have a nonzero coefficient at x_n^b, so that reduction to H captures order b.

Theorem 4.2 (Maximal contact exists in char 0)**.** *Over characteristic zero, for any a with $\operatorname{ord}_a() = b > 0$ there exists etale-locally a hypersurface of maximal contact H through a.*

Proof. Let $D^{\leq b-1}$ be k-linear differential operators of order at most $b-1$. Consider the ideal

$$\mathfrak{D} := \langle D(f) \mid f \in, \ D \in D^{\leq b-1} \rangle. \tag{4.1}$$

Upper semicontinuity implies that some $g \in \mathfrak{D}$ has a nonzero linear term at a. Pick a local coordinate x_n whose differential is that linear form. Then Taylor expansions in x_n show the degree-b coefficient is nonzero along $H = (x_n = 0)$. This uses characteristic zero to separate degrees by derivations; cf. [3, 6]. □

4.2 Coefficient ideals and independence

Let H be a maximal contact hypersurface at a with local equation $x_n = 0$.

Definition 4.3 (Coefficient ideal on H)**.** Write for each $f \in$ the expansion $f = \sum_{i \geq 0} a_i x_n^i$ with $a_i \in \mathcal{O}_H$. If $b = \max \operatorname{ord}()$, define

$$\operatorname{Coeff}_H(, b) := \sum_{i=0}^{b-1} (a_i)^{\frac{b!}{b-i}} \subset \mathcal{O}_H. \tag{4.2}$$

Proposition 4.4 (Independence and functoriality)**.** *The ideal $\operatorname{Coeff}_H(, b)$ is independent of the choice of generators of and the choice of local coordinate for H. If $\phi : W' \to W$ is smooth and $H' = H \times_W W'$, then*

$$\operatorname{Coeff}_{H'}(\mathcal{O}_{W'}, b) = \operatorname{Coeff}_H(, b)\, \mathcal{O}_{H'}. \tag{4.3}$$

Proof. Generator independence follows from additivity and unit multiplication. Changing x_n to $x_n + u$ with $u \in \mathcal{O}_H$ affects the coefficients by binomial expansion; the factorial exponents equalize weights so the sum of powers is invariant. Smooth base change preserves Taylor coefficients and factorial exponents. See [3]. □

5 Controlled transforms and monotone invariant

5.1 Controlled transform

Definition 5.1 (Controlled (weak) transform)**.** Let $\pi : \widetilde{W} \to W$ be the blow-up with smooth center $C \subset \{a : \operatorname{ord}_a() \geq b\}$ having normal crossings with the boundary. The controlled transform of $(, b)$ is $(', b)$ on $\widetilde{W}$ defined by

$$\pi^* \;=\; \mathcal{I}^b_{\operatorname{Exc}(\pi)} \cdot' . \tag{5.1}$$

Proposition 5.2 (Coefficient ideals under blow-up)**.** *Let H be maximal contact and H' its transform. Then*

$$\operatorname{Coeff}_{H'}(', b) \;\subseteq\; \left(\operatorname{Coeff}_H(, b)\right)^{\sim} \cdot \mathcal{O}_{H'}, \tag{5.2}$$

where $^{\sim}$ denotes the weak transform on H.

Proof. Expand $f \circ \pi$ along H'; factor the b-th power of the exceptional equation. Taylor coefficients of order less than b pull back with at least the expected exceptional power, and the factorial exponents remove it in the coefficient ideal. Details as in [3, 6]. ☐

5.2 The invariant and its descent

Definition 5.3 (Resolution invariant)**.** For a basic object $(W, (, b), E)$ with E SNC, define at $a \in W$ the tuple

$$\operatorname{inv}(a) = (\nu_1(a), s_1(a);\; \nu_2(a), s_2(a);\; \ldots;\; \nu_\ell(a), s_\ell(a)), \tag{5.3}$$

where $\nu_1(a) = \frac{1}{b} \operatorname{ord}_a()$, $s_1(a)$ counts boundary components through a, and inductively ν_{i+1} is the normalized order of the coefficient ideal on a maximal contact hypersurface, with s_{i+1} the new boundary count. The sequence ends when $\nu_\ell = 0$.

Proposition 5.4 (Upper semicontinuity and permissible centers)**.** *The map $a \mapsto \operatorname{inv}(a)$ is upper semicontinuous in the lexicographic order. Its maximal locus is a union of smooth strata with normal crossings with E and is a permissible blow-up center.*

Proof. Each ν_i is upper semicontinuous by Theorem 3.2 and Theorem 4.4; boundary counters are locally constant on strata. Smoothness and transversality of the maximal locus follow from the maximal contact construction and Theorem 2.3; see [3, 6]. ☐

Theorem 5.5 (Monotonicity under blow-up)**.** *Blow up the maximal locus of* inv*. For each point above the center, the invariant strictly decreases lexicographically.*

Proof. By Theorem 5.2, each ν_i is weakly nonincreasing. At the first index where equality could persist, the boundary counter strictly decreases because new exceptional components are counted later (tie-breaker of [3, 5]). Hence inv drops. ☐

Theorem 5.6 (Termination and embedded resolution)**.** *Iterating the previous step yields, after finitely many blow-ups, a basic object with $\nu_1 = 0$. For $(, b) = (\mathcal{I}_X, 1)$ this gives an embedded resolution of $X \subset W$ with SNC boundary.*

Proof. Possible values of inv form a well-ordered subset of $\mathbb{Q}^{\leq n} \times \mathbb{Z}^{\leq n}$ under lexicographic order. Theorem 5.5 forbids infinite strictly decreasing chains. With $b = 1$ the ideal is locally principal with a normal crossings equation, so the strict transform is smooth and SNC with the boundary; see [8, Ch. 3]. ☐

6 Curves and surfaces

6.1 Plane curves

Theorem 6.1 (Resolution of plane curves)**.** *Let $C \subset \mathbb{A}^2$ be given by $f(x, y) = 0$ over an algebraically closed field of characteristic zero. Successive point blow-ups at singular points resolve C: after finitely many steps the strict transform is smooth and meets the exceptional divisor transversely.*

Proof. By Newton–Puiseux, each branch has a parametrization $x = t^m$, $y = \sum_{i \geq n} a_i t^i$ with $\gcd(m, n) = 1$, so the multiplicity is m. Blowing up $(x, y) = (u, uv)$ gives $u^m \tilde{f}(u, v)$; the strict transform $\tilde{f} = 0$ has smaller multiplicity unless tangents split. The integer multiplicity strictly decreases along infinitely near points; termination and transversality follow; cf. [8, Ch. 1]. □

Example 6.2 (Cusp). For $y^2 = x^3$, one blow-up yields strict transform $v^2 = u$, which is smooth and transverse to $u = 0$.

6.2 Surface singularities

Theorem 6.3 (Surfaces)**.** *Let $S \subset W$ be a surface in a smooth threefold. The algorithm of Sections 4 and 5 produces an embedded resolution in characteristic zero.*

Proof. Choose maximal contact H near points of maximal ν_1. On H compute the coefficient ideal; its top locus consists of smooth points/curves meeting the boundary transversely. Lift to permissible centers in W and blow up. The invariant drops by Theorem 5.5, so the process terminates. At the end ord ≤ 1 and S is smooth and SNC with the boundary by Theorem 2.3. See [3, 6]. □

7 Toric resolution for nondegenerate hypersurfaces

Let $X = (f = 0) \subset \mathbb{A}^n$ with $f \in k[x_1, \ldots, x_n]$. Let $\Gamma_+(f)$ be the Newton polyhedron.

Definition 7.1 (Nondegeneracy)**.** f is nondegenerate with respect to $\Gamma_+(f)$ if for every compact face Δ, the face polynomial f_Δ has no critical point on $(k^*)^n$; see [9].

Theorem 7.2 (Toric embedded resolution)**.** *Assume nondegeneracy. Any regular subdivision of the normal fan of $\Gamma_+(f)$ yields a toric morphism $\pi_\Sigma : Z_\Sigma \to \mathbb{A}^n$ such that the strict transform of X is smooth and meets the toric boundary transversely.*

Proof. On a chart for a maximal cone with primitive generators $v_1, \ldots, v_n$, the monomial map $x_i = \prod_j u_j^{(v_j)_i}$ gives

$$f \circ \pi_\Sigma \;=\; u_1^{m_1} \cdots u_n^{m_n}\, g(u), \tag{7.1}$$

where g has initial form corresponding to the face. Nondegeneracy implies $g = 0$ is smooth on the torus. Regularity of the fan gives an SNC toric boundary; hence the strict transform is smooth and transverse in every chart; see [10, 11]. □

Technical Addendum A: Binomial–Factorial Stability of the Coefficient Ideal

Let $\mathrm{ord}_a() = b > 0$ and let $H = (x_n = 0)$ be a hypersurface of maximal contact near a. For $f \in$ write the Taylor expansion

$$f = \sum_{i=0}^{\infty} a_i x_n^i, \qquad a_i \in \mathcal{O}_H. \tag{7.2}$$

Recall the coefficient ideal on H

$$\mathrm{Coeff}_H(,b) \;=\; \sum_{i=0}^{b-1} \Big(a_i\Big)^{\frac{b!}{b-i}} \subset \mathcal{O}_H. \tag{7.3}$$

Proposition 7.3 (Binomial–factorial control)**.** *Let $\tilde{x}_n := x_n + u$ with $u \in \mathcal{O}_H$ and let $\tilde{H} = (\tilde{x}_n = 0)$. Write $f = \sum_{i\geq 0} \tilde{a}_i \tilde{x}_n^i$ with $\tilde{a}_i \in \mathcal{O}_{\tilde{H}} \cong \mathcal{O}_H$. Then, up to integral closure of ideals on H, one has*

$$\overline{\mathrm{Coeff}_{\tilde{H}}(,b)} \;=\; \overline{\mathrm{Coeff}_H(,b)}\;. \tag{7.4}$$

In particular, the normalized orders used in the resolution invariant are unchanged by replacing x_n with $x_n + u$.

Proof. By the binomial theorem,

$$x_n^j = (\tilde{x}_n - u)^j = \sum_{i=0}^{j} \binom{j}{i} \tilde{x}_n^i (-u)^{j-i}. \tag{7.5}$$

Comparing coefficients in $f = \sum_{j\geq 0} a_j x_n^j = \sum_{i\geq 0} \tilde{a}_i \tilde{x}_n^i$ gives, for $0 \leq i \leq b-1$,

$$\tilde{a}_i = \sum_{j=i}^{b-1} \binom{j}{i} a_j (-u)^{j-i} \;+\; h_i, \tag{7.6}$$

where h_i is a sum of terms involving a_j with $j \geq b$, hence $h_i \in \mathcal{I}^{b-i}$ for the ideal $\mathcal{I} = (a_0, \ldots, a_{b-1}, u) \subset \mathcal{O}_H$. Fix i. By the multinomial expansion and the choice of exponents $\frac{b!}{b-i}$, every monomial term in $\tilde{a}_i^{\frac{b!}{b-i}}$ can be written as a product of powers $a_j^{e_j} u^*$ with

$$\sum_{j=i}^{b-1} e_j\,(b-i) \;=\; b!, \tag{7.7}$$

so that a_j always appears with exponent divisible by $\frac{b!}{b-j}$. Consequently,

$$(\tilde{a}_i)^{\frac{b!}{b-i}} \in \overline{\sum_{j=i}^{b-1} \Big(a_j\Big)^{\frac{b!}{b-j}}} \;\subset\; \overline{\mathrm{Coeff}_H(,b)}\,. \tag{7.8}$$

Summing over $i = 0, \ldots, b-1$ yields $\overline{\mathrm{Coeff}_{\tilde{H}}(,b)} \subset \overline{\mathrm{Coeff}_H(,b)}$. The reverse inclusion follows by symmetry (replace u by $-u$ and swap the roles of x_n and $\tilde{x}_n$). □

Remark 7.4. If one replaces coefficient ideals by their integral closures at each step (a harmless modification for the invariant), the ideal is strictly independent of the choice of the transversal coordinate. This is the form used in many constructive algorithms.

Technical Addendum B: Explicit affine-chart calculus for a blow-up

Let W be a smooth d-fold with etale-local coordinates $(x_1, \ldots, x_d)$ and

$$C = \{x_1 = \cdots = x_r = 0\} \subset W \tag{7.9}$$

a smooth center. The blow-up $\pi : \widetilde{W} = \mathrm{Bl}_C(W) \to W$ is covered by charts U_i for $i = 1, \ldots, r$ with coordinates

$$U_i : \quad x_i = y_i, \qquad x_j = y_i\, y_j \ (j \neq i,\ 1 \leq j \leq r), \qquad x_k = y_k \ (k > r), \tag{7.10}$$
$$\text{exceptional divisor on } U_i : \quad E \cap U_i = \{y_i = 0\}. \tag{7.11}$$

Let $\subset \mathcal{O}_W$ be a coherent ideal with $\mathrm{ord}_C() = b > 0$ (i.e., every $f \in$ lies in $\mathcal{I}_C^b$ and some element has initial degree exactly b along C).

Lemma 7.5 (Controlled transform in coordinates)**.** *On the chart U_i one has*

$$\pi^* \cdot \mathcal{O}_{U_i} \;=\; (y_i)^{b\,\prime}_{\;i}, \tag{7.12}$$

which defines the controlled (weak) transform $^{\prime}_{i} \subset \mathcal{O}_{U_i}$. Moreover,

$$\mathrm{ord}_E(\pi^*) = b, \qquad \mathrm{ord}(^{\prime}_{i}) \leq \mathrm{ord}() \quad \textit{on } U_i, \tag{7.13}$$

and equality cannot hold along E at points lying over C unless the tangent cone is totally invariant under the chart projection.

Proof. Write a generator $f \in$ as a sum of bihomogeneous pieces with respect to the $(x_1, \ldots, x_r)$-weight:

$$f = \sum_{m \geq b} f_{(m)}, \qquad f_{(m)} \in \mathcal{I}_C^m \setminus \mathcal{I}_C^{m+1}. \tag{7.14}$$

Substitute the chart relations in U_i. Every x_j with $j \leq r$ contributes a factor y_i, hence $f \circ \pi$ is divisible by y_i^b and the quotient defines $^{\prime}_{i}$. The order estimate is immediate; if the order stayed constant along E, then all degree-b initial forms would vanish after the substitution, forcing extra divisibility that contradicts $\mathrm{ord}_C() = b$. □

Corollary 7.6 (Strict transform of a hypersurface)**.** *If $X = (f = 0)$ with $\mathrm{ord}_C(f) = b$, then in U_i*

$$f \circ \pi \;=\; y_i^b\, \tilde{f}_i, \qquad \textit{strict transform } \tilde{X} \cap U_i = (\tilde{f}_i = 0), \tag{7.15}$$

and $\tilde{f}_i$ is not divisible by y_i. Hence $\tilde{X}$ does not contain the exceptional divisor and $\mathrm{ord}(\tilde{f}_i) \leq \mathrm{ord}(f)$, with a strict drop at points of $\tilde{X} \cap E$ lying over C unless the tangent cone is fully pulled back.

Proposition 7.7 (Behavior of the invariant on charts)**.** *Let $(W, (, b), E)$ be a basic object with C contained in the maximal locus of the invariant. On every chart U_i of the blow-up $\pi : \widetilde{W} \to W$ one has*

$$\nu_1(^{\prime}_{i}) \;\leq\; \nu_1(), \qquad \textit{and if equality holds at some point over } C\textit{, then the next tie-breaker } s_1 \textit{ strictly decreases.} \tag{7.16}$$

Consequently the lexicographic invariant strictly drops along $\pi^{-1}(C)$ chart by chart.

Proof. The first inequality follows from the controlled transform computation above. The boundary counter s_1 decreases along points over C because a new exceptional component appears and is counted only at later stages, giving the standard tie-break. This is the local chart version of the monotonicity theorem. □

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