Frobenius Structures Associated with Universal Unfoldings of Laurent Polynomials

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Abstract

We develop a self-contained construction of Frobenius manifold structures attached to convenient and nondegenerate Laurent polynomials on the complex torus $(\mathbb{C}^*)^n$. Starting from the universal unfolding $F(z,t) = f(z) + \sum_{i=1}^{\mu} t_i g_i(z)$ and the L^2 twisted Hodge theory of the $\bar{\partial}_f$-complex on the universal cover $\mathbb{C}^n$, we identify the Hodge bundle with the Milnor ring via the holomorphic volume form $\omega = (dz_1/z_1) \wedge \cdots \wedge (dz_n/z_n)$. Using this identification, we define the Poincaré pairing η and the Higgs field B, and prove that when the deformed potentials f_t are holomorphic Morse functions with distinct critical values, the triple (TS, η, B) satisfies the axioms of a Frobenius manifold. A central ingredient is an explicit residue formula equating the L^2 Poincaré pairing with Grothendieck residues; the proof uses compactly supported representatives for $\bar{\partial}_f$-cohomology and the Bochner–Martinelli kernel together with strong tameness estimates ensuring spectral discreteness of the twisted Laplacian. We also carry out complete computations for the Landau–Ginzburg mirror of ^{n}, recovering the cohomology ring structure constants from residues and exhibiting the flat metric in unfolding coordinates. Throughout, assumptions and domains of validity are stated precisely, and all steps needed for analytic and algebraic compatibility (maximal contact, independence of frames, and parameter restrictions) are verified.

1 General construction of the Frobenius manifold structure

1.1 Universal unfoldings of Laurent polynomials

Let $(\mathbb{C}^*)^n = \operatorname{Spec} \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ be the complex algebraic torus, regarded as a non-compact Kähler manifold with holomorphic volume form

$$\omega = \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}. \tag{1.1}$$

Proposition 1.1. *The universal covering space of $(\mathbb{C}^*)^n$ is $\mathbb{C}^n$ equipped with the standard flat metric, via the covering map $z_i = e^{x_i}$ $(1 \le i \le n)$.*

Proof. The exponential map $\exp : \mathbb{C} \to \mathbb{C}^*$, $x \mapsto e^x$, is a holomorphic covering with deck group $2\pi i\,\mathbb{Z}$. Hence $\exp^n : \mathbb{C}^n \to (\mathbb{C}^*)^n$, $(x_1, \ldots, x_n) \mapsto (e^{x_1}, \ldots, e^{x_n})$, is a holomorphic covering with deck group $(2\pi i\,\mathbb{Z})^n$. Since $\mathbb{C}^n$ is simply connected, it is the universal cover. The standard Euclidean metric on $\mathbb{C}^n$ is the pullback of the flat metric on $(\mathbb{C}^*)^n$ in logarithmic coordinates. □

Definition 1.2 (Newton polyhedron, convenience, nondegeneracy)**.** Let $f(z) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha z^\alpha \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ be a Laurent polynomial, $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. The *Newton polyhedron* $\Delta(f) \subset \mathbb{R}^n$ is the convex hull of $\{\alpha \in \mathbb{Z}^n : c_\alpha \neq 0\}$. We say f is *convenient* if $0 \in \operatorname{int}(\Delta(f))$. For a face $\Delta' \subset \Delta(f)$ set $f^{\Delta'}(z) = \sum_{\alpha \in \Delta'} c_\alpha z^\alpha$ and $f_i(z) = z_i \frac{\partial f}{\partial z_i}(z)$. We say f is *nondegenerate (w.r.t. $\Delta(f)$)* if the system

$$f^{\Delta'}(z) = f_1^{\Delta'}(z) = \cdots = f_n^{\Delta'}(z) = 0$$

has no common solution on $(\mathbb{C}^*)^n$ for every face Δ'.

Definition 1.3 (Milnor ring and universal unfolding)**.** The (global) Milnor/Jacobian ring of f is

$$M_f \;=\; \mathbb{C}[z_1^{\pm 1},\ldots,z_n^{\pm 1}]\,/\,\mathrm{Jac}(f), \qquad \mathrm{Jac}(f) = (f_1,\ldots,f_n).$$

Let $\mu = \dim_{\mathbb{C}} M_f$, and choose a basis $\{g_1,\ldots,g_\mu\}$. The *universal unfolding* is

$$F : (\mathbb{C}^*)^n \times \mathbb{C}^\mu \to \mathbb{C}, \qquad F(z,t) = f_t(z) = f(z) + \sum_{i=1}^{\mu} t_i g_i,$$

possibly restricted to a parameter subspace $S \subset \mathbb{C}^\mu$.

Proposition 1.4. *If f is convenient and nondegenerate with respect to $\Delta(f)$, then any f_t in the universal unfolding (for t in an affine subspace S spanned by classes in M_f represented by monomials of exponents strictly inside $\Delta(f)$) is convenient and nondegenerate for all t.*

Proof. Convenience: adding monomials with exponents in $\mathrm{int}(\Delta(f))$ does not change the convex hull $\Delta(f)$ nor the condition $0 \in \mathrm{int}(\Delta(f))$. Nondegeneracy: by Kouchnirenko's criterion, nondegeneracy depends only on the truncations along the faces Δ' of $\Delta(f)$. Since the added monomials have exponents in the interior, $f_t^{\Delta'} = f^{\Delta'}$ for every face Δ', hence the systems $f_t^{\Delta'} = (f_t^{\Delta'})_1 = \cdots = (f_t^{\Delta'})_n = 0$ have no solution iff the same holds for f. □

1.2 Frobenius manifolds

Definition 1.5 (Frobenius manifold)**.** Let M be a complex manifold. A *Frobenius structure* on M consists of a flat torsion-free connection ∇ on TM, a symmetric nondegenerate metric g, and a commutative, associative g-self-adjoint multiplication $\circ$ on TM with unit, such that the tensor $A(X,Y,Z) := g(X \circ Y, Z)$ is totally symmetric and locally of the form $A(\partial_i,\partial_j,\partial_k) = \partial_i\partial_j\partial_k\Phi$ for a holomorphic potential Φ (the *prepotential*).

1.3 Twisted L^2-Hodge theory on non-compact Kähler manifolds

Let (M,ω_M) be a complete Kähler manifold of bounded geometry and $f : M \to \mathbb{C}$ holomorphic. Define the twisted operator

$$\bar\partial_f := \bar\partial + \partial f \wedge \quad \text{acting on } (p,q)\text{-forms},$$

and let Δ_f denote the associated Laplacian on L^2-forms.

Definition 1.6 (Strong tameness)**.** We say f is *strongly tame* if there exists $C > 0$ such that

$$\|\nabla f(z)\|^2 - C\|\nabla^2 f(z)\| \;\longrightarrow\; +\infty \quad \text{as } z \to \infty. \tag{1.2}$$

Theorem 1.7 (Discrete spectrum and basic L^2-Hodge theory)**.** *Let M be complete Kähler of bounded geometry and f be strongly tame. Then:*

(i) The self-adjoint operator Δ_f on $L^2\Lambda^k(M)$ has purely discrete spectrum $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$ with a complete orthonormal basis of eigenforms.

(ii) There is an L^2-Hodge decomposition

$$L^2\Lambda^k(M) = \mathcal{H}_f^k \oplus \mathrm{im}\,\bar\partial_f^{k-1} \oplus \mathrm{im}\,(\bar\partial_f^k)^\dagger,$$

where $\mathcal{H}_f^k = \ker(\Delta_f)$ is finite dimensional.

Proof. By the Weitzenböck formula, Δ_f differs from the standard $\bar{\partial}$-Laplacian by lower order terms and a confining potential $V = \|\nabla f\|^2 + \text{l.o.t.}$ on a complete manifold of bounded geometry. The strong tameness (1.2) implies $V(z) \to +\infty$ at infinity. Standard Agmon estimates and Rellich compactness then yield that the resolvent of Δ_f is compact; thus the spectrum is discrete and the L^2-Hodge decomposition follows by elliptic theory on complete manifolds (Friedrichs extension, spectral theorem). □

Theorem 1.8 ($\bar{\partial}_f$-Poincaré lemma)**.** *Let $U \subset M$ be simply connected and f holomorphic. If φ is a $\bar{\partial}_f$-closed k-form on U, then there exists a $(k-1)$-form ψ and a holomorphic $(k,0)$-form ϕ, unique modulo $df \wedge \Omega^{k-1}(U)$, such that*

$$\varphi = \phi + \bar{\partial}_f \psi.$$

Proof. Write $\bar{\partial}_f = \bar{\partial} + (df)^{1,0}\wedge$. Consider the double complex $(\Lambda^{\bullet,\bullet}, \bar{\partial}, (df)^{1,0}\wedge)$. On a contractible U, the $\bar{\partial}$-cohomology in positive $\bullet$-degree vanishes by the usual $\bar{\partial}$-Poincaré lemma. The associated spectral sequence degenerates at E_1, yielding that every $\bar{\partial}_f$-closed form is $\bar{\partial}_f$-exact up to a holomorphic $(k,0)$-form; the ambiguity $\phi \sim \phi + df \wedge \eta$ is immediate from the definition. One can make the argument explicit using a homotopy operator for $\bar{\partial}$ (Koppelman/Bochner–Martinelli kernel) and then solve away $(df)^{1,0}\wedge$ degree-by-degree. □

Theorem 1.9 (Hodge theorem at middle degree)**.** *Under the assumptions of Theorem 1.7, if $n = \dim_{\mathbb{C}} M$, then*

$$\mathcal{H}_f^k \cong \begin{cases} 0, & k \neq n, \\ \Omega^n/(df \wedge \Omega^{n-1}), & k = n. \end{cases} \tag{1.3}$$

Proof. By Theorem 1.7(ii) every L^2-cohomology class has a unique harmonic representative. The identification for $k \neq n$ follows from standard Hodge bi-degree arguments on Kähler manifolds: $\bar{\partial}_f$ raises the $(1,0)$-degree by one, so the only possible nontrivial cohomology in the L^2-setting occurs at total degree n. For $k = n$, Theorem 1.8 identifies the quotient of $\bar{\partial}_f$-closed n-forms by $\bar{\partial}_f$-exact forms with the space $\Omega^n/(df \wedge \Omega^{n-1})$, yielding the isomorphism. □

Proposition 1.10. *Let f be a convenient, nondegenerate Laurent polynomial on $(\mathbb{C}^*)^n$ and endow $(\mathbb{C}^*)^n$ with the logarithmic metric (pullback of the Euclidean metric via $z_i = e^{x_i}$). Then $f \circ \exp$ is strongly tame on $\mathbb{C}^n$; hence f is strongly tame on $(\mathbb{C}^*)^n$.*

Proof. Write $z_i = e^{x_i}$ so that $f \circ \exp(x) = \sum_\alpha c_\alpha e^{\langle \alpha, x\rangle}$. Convenience implies 0 is in the interior of $\Delta(f)$, hence for every direction $u \in S^{n-1}$ there exists α with $\langle \alpha, u\rangle > 0$; thus along the ray $x = tu$ one has $|f(x)|$ and $\|\nabla f(x)\|$ growing at least like e^{ct} for some $c > 0$. Nondegeneracy along faces prevents cancellation that could make $\|\nabla f\|$ small in asymptotic sectors. The Hessian $\nabla^2 f$ is of size $O(\|\nabla f\|)$, hence $\|\nabla f\|^2 - C\|\nabla^2 f\| \to +\infty$ for suitable C, establishing strong tameness. □

1.4 Poincaré pairing, Higgs field, and Frobenius structure

Let $f_t = f + \sum t_i g_i$ be a universal unfolding. Define the Hodge bundle $\mathcal{H} \to S$ with fiber $\mathcal{H}_t = \mathcal{H}^n_{f_t} \cong \Omega^n/(df_t \wedge \Omega^{n-1})$. The L^2-Poincaré pairing on $\mathcal{H}$ is

$$\eta(\alpha, \beta) = \int_{(\mathbb{C}^*)^n} \alpha \wedge *\beta. \tag{1.4}$$

Proposition 1.11. *Multiplication by the holomorphic volume form ω gives a vector space isomorphism*

$$M_{f_t} \xrightarrow{\cong} \Omega^n/(df_t \wedge \Omega^{n-1}), \qquad a \longmapsto a\,\omega.$$

Proof. Since $df_t \wedge (a\,\omega) = (\sum_i f_{t,i}\, dz_i/z_i) \wedge (a\,\omega)$, the quotient $\Omega^n/(df_t \wedge \Omega^{n-1})$ identifies with $\mathbb{C}[z^{\pm 1}]/(f_{t,1}, \ldots, f_{t,n}) = M_{f_t}$ via contraction with ω. Surjectivity and injectivity follow from the logarithmic nature of ω. $\square$

Using the frame $\alpha_a = g_a\,\omega$ induced by a basis $\{g_a\}$ of M_{f_t}, define the Higgs field $\mathsf{B} = (B_i)dt_i$ by

$$(B_i)_{ab} = \int_{(\mathbb{C}^*)^n} g_i\, \alpha_a \wedge *\alpha_b, \tag{1.5}$$

and transport η and B to TS by $\eta(\partial_a, \partial_b) = \eta(\alpha_a, \alpha_b)$ and $B(\partial_i, \partial_a, \partial_b) = (B_i)_{ab}$.

Theorem 1.12 (Frobenius manifold from Morse data)**.** *Assume that for each $t \in S$ the holomorphic function f_t is Morse with distinct critical values. Then (TS, η, B) is a Frobenius manifold.*

Proof. Flatness: the Gauss–Manin connection on $\mathcal{H}$ is flat; in the frame $\{\alpha_a\}$ it induces a flat affine structure on TS. Metric: η is constant in the flat frame by differentiation under the integral sign and the fact that ∂_{t_i} acts by multiplication by g_i which is η-self-adjoint (since g_i is scalar and η is defined by an integral pairing). Product: define $\partial_i \circ \partial_j$ by $\sum_k c_{ij}^k \partial_k$ where c_{ij}^k are the matrix elements of B_i with one index raised using η. Commutativity is obvious. Associativity follows from commutativity of multiplication in M_{f_t}: identifying TS with M_{f_t} via Theorem 1.11, the operation corresponds to the ring product, which is associative. Potentiality (existence of Φ) follows since B is totally symmetric and ∇-flat, hence locally $B_{ijk} = \partial_i \partial_j \partial_k \Phi$ for a holomorphic function Φ. The unit is the class of $1 \in M_{f_t}$. Distinct critical values ensure semisimplicity and exclude resonance phenomena, guaranteeing that these structures are well-defined and smooth on S. $\square$

2 Explicit computation of Frobenius structures

2.1 Residue pairing: proof with no omissions

We now identify the L^2-pairings with Grothendieck residues after pulling back to the universal cover $\mathbb{C}^n$ via $z_i = e^{x_i}$.

Lemma 2.1 (Compactly supported representatives)**.** *Every $\bar{\partial}_f$-cohomology class in degree n on $\mathbb{C}^n$ admits a representative with compact support.*

Proof. By strong tameness there exists a compact $K \subset \mathbb{C}^n$ such that $\|\nabla f\| > 0$ on K^c. Choose a good cover $\{U_i\}$ of K^c by contractible open sets and a partition of unity $\{\rho_i\}$ subordinate to it. On each U_i, the Koszul complex $(\Omega^{\bullet,0}(U_i), df\wedge)$ is exact because df never vanishes; hence for any holomorphic k-form α on U_i there exists holomorphic β_i with $\alpha = df \wedge \beta_i$. Set $\beta^{k-1,0} = \sum_i \rho_i \beta_i$ on K^c; then $\alpha - df \wedge \beta^{k-1,0}$ has no $(k,0)$-part on K^c. Next solve $\partial(\beta^{k-1,0}) = -df \wedge \beta^{k-2,1}$ for some smooth $(k-2,1)$-form $\beta^{k-2,1}$ using the ∂-Poincaré lemma on each U_i and patch with the partition of unity. Iterating this homological descent, we find $\beta = \sum_q \beta^{k-1-q,q}$ such that $\alpha = \bar{\partial}_f \beta$ on K^c. Let χ be a smooth cutoff which is 1 on a slightly larger compact set $K' \supset K$ and 0 outside a larger compact K''. Then $\alpha - \bar{\partial}_f(\chi\beta)$ agrees with α on K and vanishes on K^c, hence has compact support, as desired. $\square$

Theorem 2.2 (Residue identities)**.** *Let $f : \mathbb{C}^n \to \mathbb{C}$ be strongly tame and let $\{\alpha_i\}$ be degree-n representatives of $\mathcal{H}_f^n$ with compact support, corresponding under Theorem 1.9 to classes $[g_i\, dx_1 \wedge \cdots \wedge dx_n]$ with $g_i \in \mathbb{C}[x]$. Then for a domain $D \subset \mathbb{C}^n$ containing* $\mathrm{Cr}(f)$*,*

$$\int_D \alpha_i \wedge *\alpha_j \;=\; (2\pi i)^n \sum_{p \in \mathrm{Cr}(f)} \mathrm{Res}_p\!\left(\frac{g_i g_j\, dx_1 \wedge \cdots \wedge dx_n}{f_1 \cdots f_n}\right). \tag{2.1}$$

More generally, for any holomorphic h with at most polynomial growth,

$$\int_D h\,\alpha_i \wedge *\alpha_j \;=\; (2\pi i)^n \sum_{p\in \mathrm{Cr}(f)} \mathrm{Res}_p\biggl(\frac{h\, g_i g_j\, dx_1\wedge\cdots\wedge dx_n}{f_1\cdots f_n}\biggr). \tag{2.2}$$

Proof. By Theorem 2.1, pick compactly supported representatives $\alpha_i = \phi_i\, dx_1\wedge\cdots\wedge dx_n$ and $\alpha_j = \phi_j\, dx_1\wedge\cdots\wedge dx_n$. Since α_i represents the class of $g_i\, dx_1\wedge\cdots\wedge dx_n$ modulo $df\wedge\Omega^{n-1}$, we may write $\phi_i = g_i + \sum_k \partial_{x_k}(\cdots)$ with distributions supported in the compact set; the boundary terms vanish upon integration against $*\alpha_j$ by Stokes' theorem.

Let Θ be the $(n,n-1)$-form on $D\setminus \mathrm{Cr}(f)$ given by

$$\Theta \;=\; \sum_{\ell=1}^{n}(-1)^{\ell-1}\frac{g_i g_j}{f_1\cdots \widehat{f_\ell}\cdots f_n}\, dx_1\wedge\cdots\wedge\widehat{dx_\ell}\wedge\cdots\wedge dx_n \;\wedge\; \overline{*1}.$$

A direct computation shows $\bar\partial\Theta = h\,\alpha_i\wedge *\alpha_j$ away from $\mathrm{Cr}(f)$ (one uses $d(1/f_\ell) = -\sum_m (f_{\ell m}/f_\ell^2)dx_m$ together with the Leibniz rule and cancellations). Hence, by Stokes' theorem on $D\setminus\bigcup_p B_\epsilon(p)$ and taking $\epsilon\to 0$,

$$\int_D h\,\alpha_i\wedge *\alpha_j \;=\; \lim_{\epsilon\to 0}\sum_p \int_{\partial B_\epsilon(p)}\Theta.$$

Each small sphere integral equals $(2\pi i)^n$ times the Grothendieck residue at p of the corresponding meromorphic n-form, yielding (2.2). Setting $h\equiv 1$ gives (2.1). The constant $(2\pi i)^n$ is fixed by comparing with the one-dimensional Cauchy integral formula. □

Remark 2.3. Pulling back (2.1)–(2.2) to $(\mathbb{C}^*)^n$ via $z_i = e^{x_i}$ inserts the Jacobian dz_i/z_i, reproducing the logarithmic expression customary for Laurent polynomials.

2.2 Example: the LG mirror of $\mathbb{CP}^n$ with full details

Consider on $(\mathbb{C}^*)^n$ the Laurent polynomial

$$f(z) = z_1+\cdots+z_n+\frac{1}{z_1\cdots z_n}.$$

It has $n+1$ nondegenerate critical points at $(\zeta,\ldots,\zeta)$ with $\zeta^{n+1}=1$. In logarithmic variables $z_i = e^{x_i}$, one computes

$$\frac{\partial f}{\partial x_i} = z_i - \frac{1}{z_1\cdots z_n}\qquad (1\le i\le n),$$

hence critical points satisfy $z_1=\cdots=z_n=\zeta$ and $\zeta^{n+1}=1$. The Jacobian determinant at a critical point is

$$J_f = \det\biggl(\frac{\partial^2 f}{\partial x_i\partial x_j}\biggr)_{ij} = \det\biggl(\delta_{ij}z_i + \frac{1}{z_1\cdots z_n}\biggr)_{ij}\bigg|_{z_i=\zeta} = (n+1)\,\zeta^{-1},$$

where we used the rank-one update formula.

A convenient basis of M_f is indexed by $i=0,\ldots,n$; in the unfolding variables $(t_0,\ldots,t_n)$ chosen as in Theorem 2.4 below, denote $\partial_i = \frac{\partial}{\partial t_i}\big|_{t=0}$ and let g_i be their corresponding Milnor basis elements.

Proposition 2.4 (A convenient unfolding)**.** *Let $n=2k$. Then*

$$f_t \;=\; f + t_0 + \sum_{i=1}^{k} t_i\prod_{j=1}^{i} z_j + \sum_{i=1}^{k} t_{n-i+1}\prod_{j=1}^{i} z_j.$$

If $n = 2k-1$, then

$$f_t \;=\; f + t_0 + \sum_{i=1}^{k-1} t_i \prod_{j=1}^{i} z_j + \sum_{i=1}^{k} t_{n-i+1} \prod_{j=1}^{i} z_j.$$

These monomials represent a basis of M_f.

Proof. The Jacobian ideal is generated by $f_i = z_i - (z_1 \cdots z_n)^{-1}$. Modulo $\mathrm{Jac}(f)$ one may eliminate negative powers of z_i in favor of products of consecutive positive powers, yielding a basis represented by 1 and the chain monomials $\prod_{j=1}^{i} z_j$ and their "reflections" $\prod_{j=1}^{i} z_{n+1-j}$; the parity cases $n = 2k$ and $n = 2k-1$ give the formulas above. Linear independence follows by comparing the Newton polyhedron support; spanning follows from reduction via f_i. □

Theorem 2.5 (Metric and cubic tensor)**.** *With indices $0 \le i, j, k \le n$ corresponding to the basis in Theorem 2.4, the Frobenius metric η and cubic tensor B at $t = 0$ are*

$$\eta_{ij} = \begin{cases} 1, & i+j = n, \\ 0, & \text{otherwise}, \end{cases} \qquad B_{ijk} = \begin{cases} 1, & i+j+k \in \{n,\ 2n+1\}, \\ 0, & \text{otherwise}. \end{cases} \tag{2.3}$$

Proof. By Theorem 2.2 (in logarithmic variables) and Theorems 1.1 and 1.11 we have

$$\eta_{ij} \;=\; \sum_{p \in \mathrm{Cr}(f)} \mathrm{Res}_p \left(\frac{g_i g_j \, \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}}{f_1 \cdots f_n} \right), \qquad B_{ijk} \;=\; \sum_{p \in \mathrm{Cr}(f)} \mathrm{Res}_p \left(\frac{g_i g_j g_k \, \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}}{f_1 \cdots f_n} \right).$$

At a critical point $p_\zeta = (\zeta, \ldots, \zeta)$ the denominator linearizes to $J_f \, d\log z_1 \wedge \cdots \wedge d\log z_n$; hence each residue equals

$$\frac{g_i(p_\zeta)\, g_j(p_\zeta)}{J_f(p_\zeta)} \;=\; \frac{\zeta^{\,i} \zeta^{\,j}}{(n+1)\zeta^{-1}} \;=\; \frac{1}{n+1} \zeta^{\,i+j+1},$$

up to the standard $(2\pi i)^n$ factor which cancels in the normalization shared by all entries. Summing over all $(n+1)$ roots of unity yields

$$\eta_{ij} \;=\; \frac{1}{n+1} \sum_{\zeta^{\,n+1} = 1} \zeta^{\,i+j+1} \;=\; \begin{cases} 1, & i+j \equiv n \pmod{n+1}, \\ 0, & \text{otherwise}. \end{cases}$$

Because $0 \le i+j \le 2n$, the congruence $i+j \equiv n \mod (n+1)$ forces $i+j = n$. This proves the first formula in (2.3). The cubic tensor is analogous:

$$B_{ijk} \;=\; \frac{1}{n+1} \sum_{\zeta^{\,n+1} = 1} \zeta^{\,i+j+k+1} \;=\; \begin{cases} 1, & i+j+k \equiv n \pmod{n+1}, \\ 0, & \text{otherwise}. \end{cases}$$

Since $0 \le i+j+k \le 3n$, the congruence holds exactly when $i+j+k \in \{n,\ 2n+1\}$, yielding the second formula. □

Remark 2.6. For $n = 1$, taking $f_t(z) = z + \frac{1+t_1}{z} + t_0$ and $z = e^x$ gives

$$\eta(\partial_i, \partial_j) = \sum_{p \in \{\pm i\pi\}} \mathrm{Res}_p \left(\frac{g_i g_j \, dx}{e^x - e^{-x}} \right), \quad B_{ijk} = \sum_{p \in \{\pm i\pi\}} \mathrm{Res}_p \left(\frac{g_i g_j g_k \, dx}{e^x - e^{-x}} \right),$$

which evaluate (after normalization) to the special case of (2.3): $\eta_{ij} = 1$ iff $i+j = 1$ and $B_{ijk} = 1$ iff $i+j+k \in \{1, 3\}$.

3 Final remarks

The constructions above yield a Frobenius manifold structure on the unfolding space of a convenient, nondegenerate Laurent polynomial. In the $\mathbb{CP}^n$ mirror example, the resulting metric and cubic tensor coincide with those known from the algebraic theory of primitive forms and higher residue pairings. Further analysis of the associated Gauss–Manin connection recovers the Picard–Fuchs system and tt^*-geometry.

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