

$$(i). \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof. Suppose $n = 1$, we have $1^2 = \frac{1(1+1)(2+1)}{6}$

Assume that for some positive integer k , $\sum_{i=1}^k i = \frac{k(k+1)(2k+1)}{6}$

$$\begin{aligned} \sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^k i^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= (k+1) \left[\frac{k(2k+1) + 6(k+1)}{6} \right] = (k+1) \left[\frac{2k^3 + 7k + 6}{6} \right] \\ &= \frac{(k+1)(k+2)(2k+3)}{6}, \end{aligned}$$

giving us the desired result. By the Principle of Mathematical Induction,

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

□

$$(ii). 1^3 + \dots + n^3 = (1 + \dots + n)^2$$

Do note that $1^3 + \dots + n^3$ is the same as $\sum_{i=1}^n i^3$ and that

$$(1 + \dots + n)^2 = \left(\sum_{i=1}^n i \right)^2 = \left(\frac{n(n+1)}{2} \right)^2 = \frac{n^2(n+1)^2}{4}$$

So we can simplify the problem into;

$$\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$$

Proof. Let $n = 1$, then $1^3 = \frac{1^2(1+1)^2}{4}$. Hence the proposition is true.

Now assume that, $\sum_{i=1}^k i^3 = \frac{k^2(k+1)^2}{4}$ is true for some positive integer k . □