1. (i). $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Proof. Suppose $n = 1$, we have $1^2 = \frac{1(1+1)(2+1)}{6}$

Assume that for some positive integer k, $\sum_{i=1}^{k} i = \frac{k(k+1)(2k+1)}{6}$

$$\begin{aligned}\sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= (k+1)\left[\frac{k(2k+1)+6(k+1)}{6}\right] = (k+1)\left[\frac{2k^3+7k+6}{6}\right] \\ &= \frac{(k+1)(k+2)(2k+3)}{6},\end{aligned}$$

giving us the desired result. By the Principle of Mathematical Induction,

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

□

(ii). $1^3 + \cdots + n^3 = (1 + \cdots + n)^2$

Do note that $1^3 + \cdots + n^3$ is the same as $\sum_{i=1}^{n} i^3$ and that

$(1 + \cdots + n)^2 = (\sum_{i=1}^{n} i)^2 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^2(n+1)^2}{4}$

So we can simplify the problem into;
$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Proof. Let $n = 1$, then $1^3 = \frac{1^2\,(1+1)^2}{4}$. Hence the proposition is true.

Now assume that, $\sum_{i=1}^{k} i^3 = \frac{k^2\,(k+1)^2}{4}$ is true for some positive integer k. Then,

$$\begin{aligned}\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= (k+1)^2\left[\frac{k^2+4(k+1)}{4}\right] = (k+1)^2\left[\frac{k^2+4k+4}{4}\right] \\ &= \frac{(k+1)^2/,(k+2)^2}{2}\end{aligned}$$

By the principle of induction, the formula holds for all positive integers n. □

2. Find a formula for

$$\sum_{i=1}^{n}(2i-1) = 1+3+5+\cdots(2i-1) \tag{1}$$