

$$(i). \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

*Proof.* Suppose  $n = 1$ , we have  $1^2 = \frac{1(1+1)(2+1)}{6}$

Assume that for some positive integer  $k$ ,  $\sum_{i=1}^k i = \frac{k(k+1)(2k+1)}{6}$

$$\begin{aligned} \sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^k i^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= (k+1) \left[ \frac{k(2k+1) + 6(k+1)}{6} \right] = (k+1) \left[ \frac{2k^3 + 7k + 6}{6} \right] \\ &= \frac{(k+1)(k+2)(2k+3)}{6}, \end{aligned}$$

giving us the desired result. By the Principle of Mathematical Induction,

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

□

$$(ii). 1^3 + \dots + n^3 = (1 + \dots + n)^2$$

Do note that  $1^3 + \dots + n^3$  is the same as  $\sum_{i=1}^n i^3$  and that

$$(1 + \dots + n)^2 = \left( \sum_{i=1}^n i \right)^2 = \left( \frac{n(n+1)}{2} \right)^2 = \frac{n^2(n+1)^2}{4}$$

So we can simplify the problem into;

$$\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$$

*Proof.* Let  $n = 1$ , then  $1^3 = \frac{1^2(1+1)^2}{4}$ . Hence the proposition is true.

Now assume that,  $\sum_{i=1}^k i^3 = \frac{k^2(k+1)^2}{4}$  is true for some positive integer  $k$ . Then,

$$\begin{aligned} \sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^k i^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= (k+1)^2 \left[ \frac{k^2 + 4(k+1)}{4} \right] = (k+1)^2 \left[ \frac{k^2 + 4k + 4}{4} \right] \\ &= \frac{(k+1)^2 \cdot (k+2)^2}{4} \end{aligned}$$

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By the principle of induction, the formula holds for all positive integers  $n$ .  $\square$