Matrix Methods – Statistics Honours

Chapter 1 Exercises Solutions

Exercise 1.1

$$\boldsymbol{D}_{(p\times p)} = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_p \end{bmatrix}, \text{ then}$$

$$_p\boldsymbol{D}_p\boldsymbol{A}_m = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_p \end{bmatrix} \begin{bmatrix} a_{11} & \dots & a_{1m} \\ a_{21} & \dots & a_{2m} \\ \vdots & & \vdots \\ a_{p1} & \dots & a_{pm} \end{bmatrix} = \begin{bmatrix} d_1a_{11} & \dots & d_1a_{1m} \\ d_2a_{21} & \dots & d_2a_{2m} \\ \vdots & & \vdots \\ d_pa_{p1} & \dots & d_pa_{pm} \end{bmatrix} \text{ so that}$$

The ij-th element of $\boldsymbol{DA}$ is given by d_ia_{ij}, or $\boldsymbol{DA} = \{d_ia_{ij}\}$.

Exercise 1.2

a)

$$(_m\boldsymbol{A}_n\boldsymbol{B}_p)' = \left(\left\{\sum_{k=1}^n a_{ik}b_{kj}\right\}\right)' = \begin{bmatrix} \sum_{k=1}^n a_{1k}b_{k1} & \sum_{k=1}^n a_{1k}b_{k2} & \dots & \sum_{k=1}^n a_{1k}b_{kp} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^n a_{mk}b_{k1} & \sum_{k=1}^n a_{mk}b_{k2} & \dots & \sum_{k=1}^n a_{mk}b_{kp} \end{bmatrix}'$$

$$= \begin{bmatrix} \sum_{k=1}^n a_{1k}b_{k1} & \dots & \sum_{k=1}^n a_{mk}b_{k1} \\ \sum_{k=1}^n a_{1k}b_{k2} & \dots & \sum_{k=1}^n a_{mk}b_{k2} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^n a_{1k}b_{kp} & \dots & \sum_{k=1}^n a_{mk}b_{kp} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^n b_{k1}a_{1k} & \dots & \sum_{k=1}^n b_{k1}a_{mk} \\ \sum_{k=1}^n b_{k2}a_{1k} & \dots & \sum_{k=1}^n b_{k2}a_{mk} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^n b_{kp}a_{1k} & \dots & \sum_{k=1}^n b_{kp}a_{mk} \end{bmatrix}$$

$$= \left\{\sum_{k=1}^n b_{ki}a_{jk}\right\} = \boldsymbol{B}'\boldsymbol{A}'$$

b)

$$
\begin{aligned}
tr({}_m\boldsymbol{A}_n\boldsymbol{B}_m) &= tr\left(\left\{\sum_{k=1}^{n} a_{ik}b_{kj}\right\}\right) \\
&= \sum_{i=1}^{m}\left[\sum_{k=1}^{n} a_{ik}b_{ki}\right] \\
&= \sum_{k=1}^{n}\sum_{i=1}^{m} b_{ki}a_{ik} \\
&= tr\left(\left\{\sum_{i=1}^{m} b_{ki}a_{ih}\right\}\right) \\
&= tr(\boldsymbol{BA})
\end{aligned}
$$

Exercise 1.3

Let $\boldsymbol{B} = \boldsymbol{A}'$, then

$$
tr(\boldsymbol{A}'\boldsymbol{A}) = tr\left(\left\{\sum_{k=1}^{n} b_{ik}a_{kj}\right\}\right) = \sum_{i=1}^{n}\left[\sum_{k=1}^{n} b_{ik}a_{ki}\right] = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ki}a_{ki} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2
$$

Exercise 1.4

a)

$$
\mathbf{1}'\mathbf{1} = [1 \dots 1]\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 1 + \cdots + 1 = n
$$

b)

$$
\mathbf{1}'\boldsymbol{x} = [1 \dots 1]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 + \cdots + x_n = \sum_{i=1}^{n} x_i
$$

c)

$$
\left(\boldsymbol{I} - \frac{1}{n}\boldsymbol{J}\right)\boldsymbol{x} = \boldsymbol{x} - \frac{1}{n}\mathbf{1}\mathbf{1}'\boldsymbol{x} = \boldsymbol{x} - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\mathbf{1} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} - \bar{x}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ \vdots \\ x_n - \bar{x} \end{bmatrix}
$$

Exercise 1.5

$$\begin{aligned}
\left(\boldsymbol{I} - \frac{1}{n}\boldsymbol{J}\right)\left(\boldsymbol{I} - \frac{1}{n}\boldsymbol{J}\right) &= \boldsymbol{I} - \frac{1}{n}\boldsymbol{I}\boldsymbol{J} - \frac{1}{n}\boldsymbol{J}\boldsymbol{I} + \frac{1}{n^2}\boldsymbol{J}\boldsymbol{J} \\
&= \boldsymbol{I} - \frac{1}{n}\boldsymbol{J} - \frac{1}{n}\boldsymbol{J} + \frac{1}{n^2}\mathbf{1}\mathbf{1}'\mathbf{1}\mathbf{1}' \\
&= \boldsymbol{I} - \frac{2}{n}\boldsymbol{J} + \frac{n}{n^2}\mathbf{1}\mathbf{1}' \\
&= \boldsymbol{I} - \frac{2}{n}\boldsymbol{J} + \frac{1}{n}\boldsymbol{J} = \boldsymbol{I} - \frac{1}{n}\boldsymbol{J}
\end{aligned}$$

Exercise 1.6

$$\begin{aligned}
\boldsymbol{x}'\left(\boldsymbol{I} - \frac{1}{n}\boldsymbol{J}\right)\left(\boldsymbol{I} - \frac{1}{n}\boldsymbol{J}\right)\boldsymbol{x} &= \boldsymbol{x}'\left(\boldsymbol{I} - \frac{1}{n}\boldsymbol{J}\right)\boldsymbol{x} \\
&= \boldsymbol{x}'\boldsymbol{x} - \frac{1}{n}\boldsymbol{x}'\boldsymbol{J}\boldsymbol{x} \\
&= \boldsymbol{x}'\boldsymbol{x} - \frac{1}{n}\boldsymbol{x}'\mathbf{1}\mathbf{1}'\boldsymbol{x} \\
&= \sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2 \\
&= \sum_{i=1}^{n} x_i^2 - n\bar{x}^2 \\
&= \sum_{i=1}^{n}(x_i - \bar{x})^2
\end{aligned}$$