Statistical Sciences Honours
Matrix Methods
Lecture 2 – Determinants

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2.1 Definitions

- First, note that other more complex definitions of determinants exist than what will be covered here, but they imply the definitions we will use.
- We will start with the base definition for a 2×2 matrix and illustrate a simple and useful geometric interpretation thereof, to give some sense of understanding to its meaning as opposed to just an arbitrary formula:

2.1 Definitions

- First, note that other more complex definitions of determinants exist than what will be covered here, but they imply the definitions we will use.
- We will start with the base definition for a 2×2 matrix and illustrate a simple and useful geometric interpretation thereof, to give some sense of understanding to its meaning as opposed to just an arbitrary formula:

$$det(\boldsymbol{A}) = |\boldsymbol{A}| = a_{11}a_{22} - a_{12}a_{21}$$

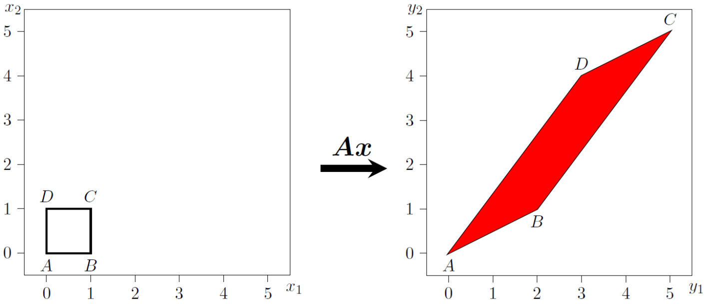
2.1 Definitions

- Suppose that $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$, where both $\boldsymbol{y}$ and $\boldsymbol{x}$ are 2-dimensional vectors.
- Consider the unit square in $\boldsymbol{x}$-space defined by the points $(0,0); (0,1); (1,0)$ and $(1,1)$.
- The matrix $\boldsymbol{A}$ linearly transforms ("pulls") the coordinates in $\boldsymbol{x}$-space to a new set in $\boldsymbol{y}$-space:

$$y_1 = a_{11}x_1 + a_{12}x_2$$
$$y_2 = a_{21}x_1 + a_{22}x_2$$

- If we specify that $a_{11} > a_{21} > 0$ and $a_{22} > a_{12} > 0$, then the unit square gets pulled into a parallelogram that looks as follows.

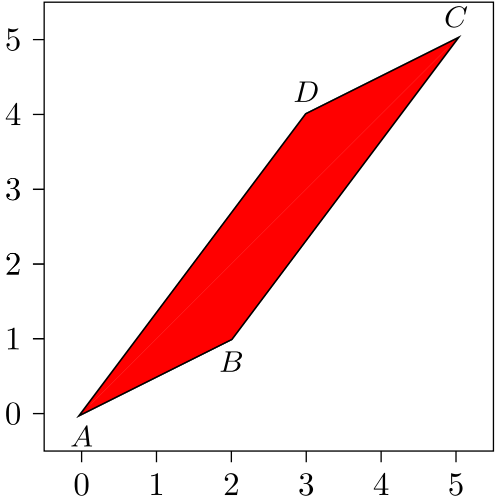
2.1 Definitions



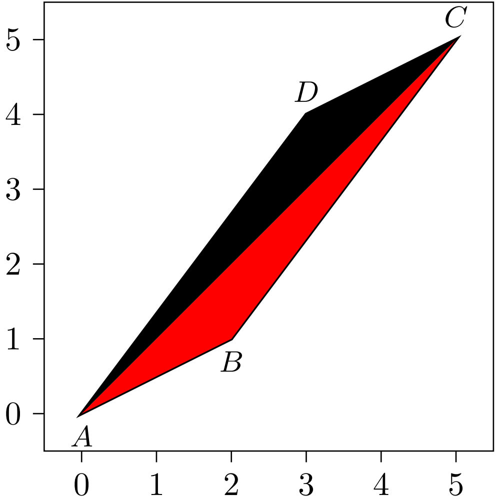
$$\boldsymbol{y} = \boldsymbol{Ax} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}$$

	Initial coordinates	Transformed coordinates
A	$\boldsymbol{x}' = [0\ 0]$	$\boldsymbol{y}' = [0\ 0]$
B	$\boldsymbol{x}' = [1\ 0]$	$\boldsymbol{y}' = [a_{11}\ a_{21}]$
C	$\boldsymbol{x}' = [1\ 1]$	$\boldsymbol{y}' = [a_{11} + a_{12}\ a_{21} + a_{22}]$
D	$\boldsymbol{x}' = [0\ 1]$	$\boldsymbol{y}' = [a_{12}\ a_{22}]$

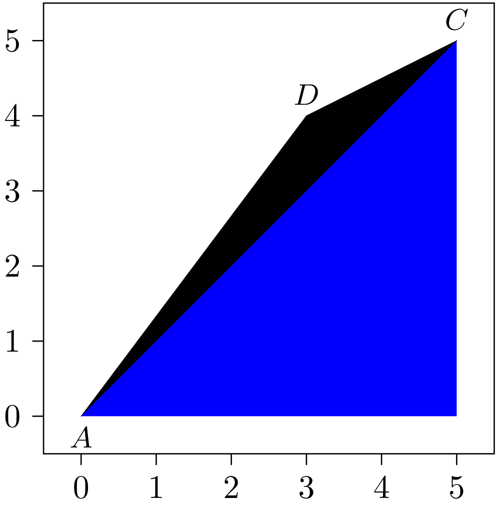
Now let's find the **area** of the red parallelogram:



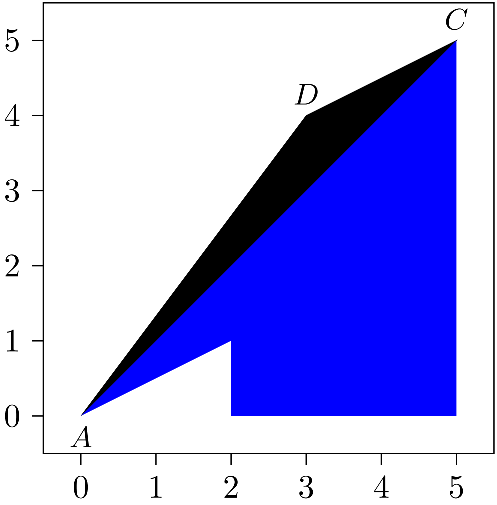
The area of the parallelogram is twice the area of the red triangle



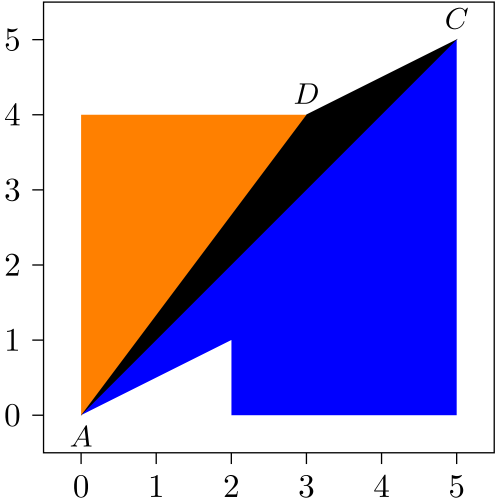
Start with the area of the blue triangle



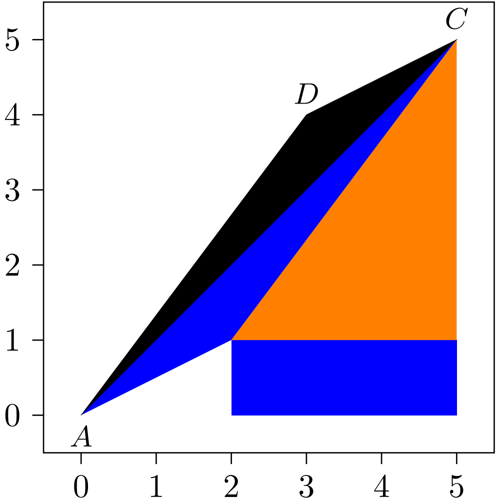
Remove the white triangle



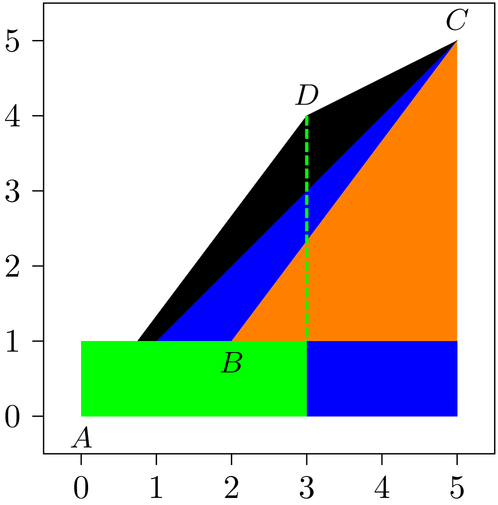
Consider the size of the orange triangle



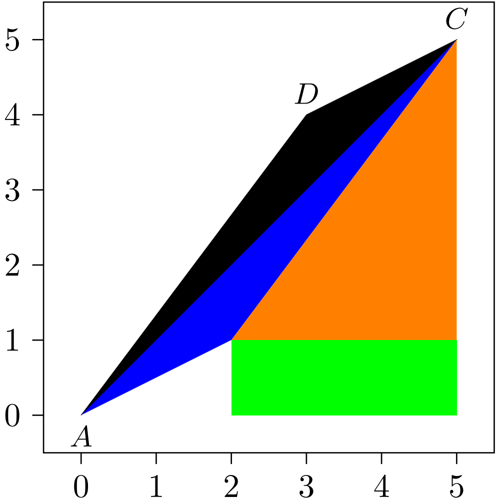
The two orange triangles in the two plots above are the same size

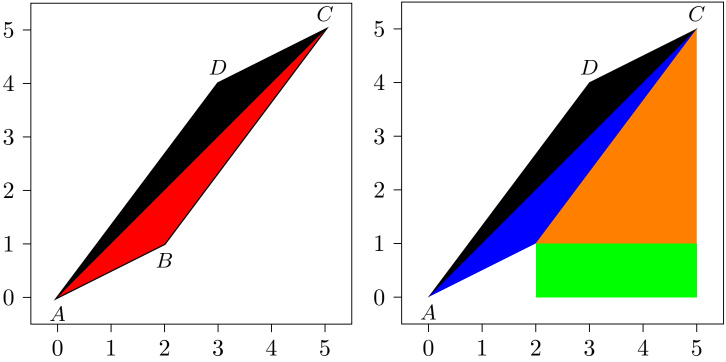


Consider the size of the green rectangle



Then shift it to the right edge

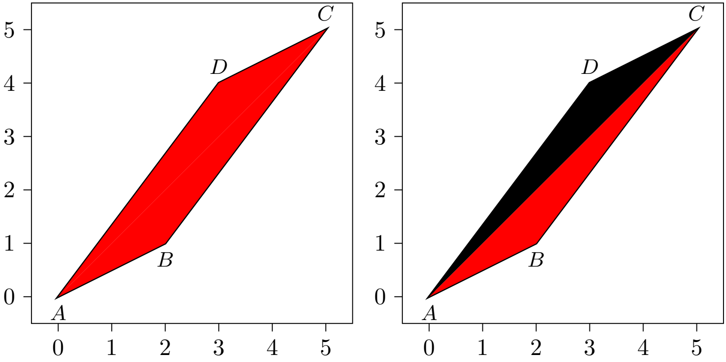




Red triangle = blue − white − orange − green

$$= \frac{1}{2}(a_{11} + a_{12})(a_{21} + a_{22}) - \frac{1}{2}a_{11}a_{21} - \frac{1}{2}a_{12}a_{22} - a_{12}a_{21}$$

2.1 Definitions



$$
\begin{aligned}
\text{Pgram} &= 2 \times \text{Red triangle} \\
&= (a_{11} + a_{12})(a_{21} + a_{22}) - a_{11}a_{21} - a_{12}a_{22} - 2a_{12}a_{21} \\
&= a_{11}a_{21} + a_{12}a_{21} + a_{11}a_{22} + a_{12}a_{22} - a_{11}a_{21} - a_{12}a_{22} - 2a_{12}a_{21} \\
&= a_{11}a_{22} - a_{12}a_{21}
\end{aligned}
$$

2.1 Definitions

- This area is also the determinant of a 2×2 matrix:

$$|\boldsymbol{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- The determinant is the factor by which the area (hypervolume) is scaled in the linear transformation described by the matrix.

- Our example was chosen such that $|\boldsymbol{A}| > 0$. When it is negative, the sign indicates a flip/rotation into another quadrant as well.

- All this extends to higher dimensions.

2.1 Definitions

For $n > 2$, we define the determinant of $\boldsymbol{A}$: $n \times n$ by the recursive formula:

$$|\boldsymbol{A}| = \sum_{j=1}^{n} a_{ij}(-1)^{i+j} \left| \boldsymbol{M}_{(ij)} \right|$$

for any arbitrary $i = 1, \ldots, n$, where the $(n-1) \times (n-1)$ matrix $\boldsymbol{M}_{(ij)}$ is obtained from $\boldsymbol{A}$ by deleting row i and column j.

Class exercise

Calculate $\begin{vmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{vmatrix}$

2.2 Some properties of determinants

- $|\boldsymbol{A}| = |\boldsymbol{A}'|$
- Swapping two rows (or two columns) of a matrix $\boldsymbol{A}$ changes the sign of the determinant.
- The addition of a multiple of one row (or column) of $\boldsymbol{A}$ to another row (or column) leaves the determinant unchanged. This can help simplify calculations:

$$\begin{vmatrix} 1 & -1 & 0 \\ 2 & 1 & 2 \\ 4 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 2 \\ 4 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 8 & \frac{11}{3} \end{vmatrix} = 1 \times 3 \times \frac{11}{3} = 11$$

- $|\boldsymbol{AB}| = |\boldsymbol{A}||\boldsymbol{B}| = |\boldsymbol{BA}|$
- If $\boldsymbol{A}$ is orthogonal, then $|\boldsymbol{A}|^2 = |\boldsymbol{A}||\boldsymbol{A}'| = |\boldsymbol{AA}'| = |\boldsymbol{I}| = 1$ such that $|\boldsymbol{A}| = \pm 1$

2.3 Elementary operators

- Elementary operators are matrices obtained from making one alteration to the identity matrix.
- Multiplying a matrix with these elementary operators applies specific changes to the matrix.
- First consider $\boldsymbol{P}_{(ij,\lambda)}$, which is identical to $\boldsymbol{I}$, except that $p_{ij} = \lambda,\ i \neq j$.
- What is the effect of multiplying a matrix with this operator?

2.3 Elementary operators

- Pre-multiplication:

$$
\boldsymbol{P}_{(13,\lambda)}\boldsymbol{A} = \begin{bmatrix} 1 & 0 & \lambda & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{24} & a_{34} & a_{44} \end{bmatrix}
$$

- This adds a multiple $(\times\lambda)$ of one row (j^{th}) to another row (i^{th})
- Note that $\left|\boldsymbol{P}_{(ij,\lambda)}\right| = 1$

2.3 Elementary operators

- Post-multiplication:

$$
\boldsymbol{AP}_{(13,\lambda)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{24} & a_{34} & a_{44} \end{bmatrix} \begin{bmatrix} 1 & 0 & \lambda & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

- This adds a multiple ($\times\lambda$) of one column (i^{th}) to another column (j^{th})

2.3 Elementary operators

- Consider now $\boldsymbol{E}_{(ij)}$, which is $\boldsymbol{I}$, but with the i^{th} and j^{th} rows (or columns, same thing) interchanged.
- Pre-multiplication:

$$\boldsymbol{E}_{(ij)}\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

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- This swaps the i^{th} and j^{th} rows (pre-) or columns (post-)
- Note that $\left|\boldsymbol{E}_{(ij)}\right| = -1$

2.3 Elementary operators

- Finally, consider $\boldsymbol{R}_{(i,\lambda)}$, which is $\boldsymbol{I}$, but with the i^{th} diagonal element replaced by λ.
- Pre-multiplication:

$$\boldsymbol{R}_{(i,\lambda)}\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

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- This multiplies the i^{th} row (pre-) or column (post-) by λ
- Note that $\left|\boldsymbol{R}_{(i,\lambda)}\right| = \lambda$

2.3 Elementary operators

Summarising these determinants, we have shown what we stated earlier:

- Adding a multiple of one row (or column) to another row (or column) does not change the determinant.
- Swapping two rows (or columns) changes the sign of the determinant.
- Multiplying a row (or column) with a constant increases the determinant by that same factor.