Multivariate Analysis – Honours 2024

Continuous Assessment 3

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Question 1

Using spectral decomposition $\Sigma \ = \ \sum_{i=1}^{m} e_i e_i^{'}$ but since Σ is posititve definite and symmetric,

$$\Sigma^{-1} \ = \ \sum_{i=1}^{m} \frac{1}{\lambda_i} e_i e_i^{'}$$

we have that:

$$\begin{aligned}
(X-\mu)^{'}\Sigma^{-1}(X-\mu) \ &= \ (X-\mu)^{'} \sum_{i=1}^{m} \frac{1}{\lambda_i} e_i e_i^{'}(X-\mu) \\
&= \ \sum_{i=1}^{m} \frac{1}{\lambda_i}(X-\mu)^{'} e_i e_i^{'}(X-\mu) \\
&= \ \sum_{i=1}^{m} [\frac{1}{\sqrt{\lambda_i}} e_i^{'}(X-\mu)]^2 \\
&= \ \sum_{i=1}^{m} Z_i^2
\end{aligned}$$

Note that Z is of the form $A(x-\mu)$ where, $A = \begin{bmatrix} \frac{1}{\sqrt{\lambda_1}} e_1^{'} \\ \frac{1}{\sqrt{\lambda_1}} e_2^{'} \\ . \\ . \\ \frac{1}{\sqrt{\lambda_m}} e_m^{'} \end{bmatrix}$ and $(X-\mu)$ is distributed as $(X-\mu) \sim \ N(0, \Sigma)$

$$\text{Therefore: } Z \ \sim \ N(0, A\Sigma A^{'})$$

Looking at $A\Sigma$ we see that it has the following form:

$$A\Sigma = \begin{bmatrix} \frac{1}{\sqrt{\lambda_1}} e_1^{'} \\ \frac{1}{\sqrt{\lambda_1}} e_2^{'} \\ . \\ . \\ \frac{1}{\sqrt{\lambda_m}} e_m^{'} \end{bmatrix} \sum_{i=1}^{m} \lambda_i e_i e_i^{'}$$

For each row j = 1, 2, ..., m in A, this gives us:

$$\frac{1}{\sqrt{\lambda_j}} e_j^{'} \lambda_i e_i e_i^{'} + \sum_{i \neq j}^{m} \frac{1}{\sqrt{\lambda_j}} e_j^{'} \lambda_i e_i e_i^{'} \ = \ \sqrt{\lambda_j} e_j^{'}$$

since the $e_i^{\prime}s$ are all mutually orthonormal.

Therefore:

$$
\begin{aligned}
A\Sigma A^{'} &= \begin{bmatrix} \frac{1}{\sqrt{\lambda_1}}e_1^{'} \\ \frac{1}{\sqrt{\lambda_2}}e_2^{'} \\ . \\ . \\ \frac{1}{\sqrt{\lambda_m}}e_m^{'} \end{bmatrix} \sum_{i=1}^{m} \lambda_i e_i e_i^{'} \left[\frac{1}{\sqrt{\lambda_1}}e_1 \frac{1}{\sqrt{\lambda_2}}e_2 .. \frac{1}{\sqrt{\lambda_m}}e_m \right] \\
&= \begin{bmatrix} \sqrt{\lambda_1}e_1^{'} \\ \sqrt{\lambda_2}e_2^{'} \\ . \\ . \\ \sqrt{\lambda_m}e_m^{'} \end{bmatrix} \left[\frac{1}{\sqrt{\lambda_1}}e_1 \frac{1}{\sqrt{\lambda_2}}e_2 .. \frac{1}{\sqrt{\lambda_m}}e_m \right] \\
&= \mathrm{I}
\end{aligned}
$$

Now, since $Z \sim N_p(0, I)$, we have a sum of independent standard normals.

Therefore:

$$(X-\mu)^{'}\Sigma^{-1}(X-\mu) = \sum_{i=1}^{m} Z_i^2 \sim \chi_m^2$$

Question 2

Consider $\mathbf{X} \sim N_5(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu} = \begin{bmatrix} 5 & 0 & -2 & 6 & 2 \end{bmatrix}'$, $\boldsymbol{\Sigma} = \begin{bmatrix} 8 & 3 & -1 & 0 & 5 \\ 3 & 12 & 2 & 2 & -2 \\ -1 & 2 & 9 & 0 & 1 \\ 0 & 2 & 0 & 8 & 2 \\ 5 & -2 & 1 & 2 & 10 \end{bmatrix}$.

Now define $\mathbf{X}_1 = \begin{bmatrix} X_1 \\ X_2 \\ X_4 \end{bmatrix}$, $\mathbf{X}_2 = \begin{bmatrix} X_3 \\ X_5 \end{bmatrix}$. Find the conditional (joint) distribution of $\mathbf{X}_1|\mathbf{X}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Answer:

Given $\mathbf{X}_1 = \begin{bmatrix} X_1 \\ X_2 \\ X_4 \end{bmatrix}$ and $\mathbf{X}_2 = \begin{bmatrix} X_3 \\ X_5 \end{bmatrix}$. we can partition $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ as:

$\boldsymbol{\mu}_1 = \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix}$, $\boldsymbol{\mu}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, $\boldsymbol{\Sigma}_{11} = \begin{bmatrix} 8 & 3 & 0 \\ 3 & 12 & 2 \\ 0 & 2 & 8 \end{bmatrix}$, $\boldsymbol{\Sigma}_{12} = \boldsymbol{\Sigma}_{21}' = \begin{bmatrix} -1 & 5 \\ 2 & -2 \\ 0 & 2 \end{bmatrix}$, $\boldsymbol{\Sigma}_{22} = \begin{bmatrix} 9 & 1 \\ 1 & 10 \end{bmatrix}$

$$\boldsymbol{\Sigma}_{22}^{-1} = \begin{bmatrix} 9 & 1 \\ 1 & 10 \end{bmatrix}^{-1} = \frac{1}{10 \times 9 - 1} \begin{bmatrix} 10 & -1 \\ -1 & 9 \end{bmatrix} = \begin{bmatrix} \frac{10}{89} & \frac{-1}{89} \\ \frac{-1}{89} & \frac{9}{89} \end{bmatrix}$$

The conditional distribution is multivariate normal with:

$$\mathbf{X_1}|\mathbf{X_2} \sim N(\boldsymbol{\mu}_{1|2}, \boldsymbol{\Sigma}_{1|2})$$

$E(\mathbf{X}_1|\mathbf{X}_2) = \boldsymbol{\mu}_{1|2} = \boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{X}_2 - \boldsymbol{\mu}_2)$, $\quad \mathrm{Cov}(\mathbf{X}_1|\mathbf{X}_2) = \boldsymbol{\Sigma}_{1|2} = \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}$

Therefore:

$$E(\mathbf{X}_1|\mathbf{X}_2) = \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix} + \begin{bmatrix} 8 & 3 & 0 \\ 3 & 12 & 2 \\ 0 & 2 & 8 \end{bmatrix} \begin{bmatrix} 9 & 1 \\ 1 & 10 \end{bmatrix}^{-1} \left(\begin{bmatrix} X_3 \\ X_5 \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right)$$

$$E(\mathbf{X}_1|\mathbf{X}_2) = \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix} + \begin{bmatrix} 8 & 3 & 0 \\ 3 & 12 & 2 \\ 0 & 2 & 8 \end{bmatrix} \begin{bmatrix} \frac{10}{89} & \frac{-1}{89} \\ \frac{-1}{89} & \frac{9}{89} \end{bmatrix} \left(\begin{bmatrix} -1 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right)$$

$$E(\mathbf{X}_1|\mathbf{X}_2) = \boldsymbol{\mu}_{1|2} = \begin{bmatrix} \frac{430}{89} \\ \frac{22}{89} \\ \frac{532}{89} \end{bmatrix} = \begin{bmatrix} 4.83 \\ 0.25 \\ 5.98 \end{bmatrix}$$

and covariance matrix

$$\mathrm{Cov}(\mathbf{X}_1|\mathbf{X}_2) = \begin{bmatrix} 8 & 3 & 0 \\ 3 & 12 & 2 \\ 0 & 2 & 8 \end{bmatrix} - \begin{bmatrix} -1 & 5 \\ 2 & -2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 9 & 1 \\ 1 & 10 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 2 & 0 \\ 5 & -2 & 2 \end{bmatrix}$$

$$\mathrm{Cov}(\mathbf{X}_1|\mathbf{X}_2) = \begin{bmatrix} 8 & 3 & 0 \\ 3 & 12 & 2 \\ 0 & 2 & 8 \end{bmatrix} - \begin{bmatrix} -1 & 5 \\ 2 & -2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{10}{89} & \frac{-1}{89} \\ \frac{-1}{89} & \frac{9}{89} \end{bmatrix} \begin{bmatrix} -1 & 2 & 0 \\ 5 & -2 & 2 \end{bmatrix}$$

$$\mathrm{Cov}(\mathbf{X}_1|\mathbf{X}_2) = \boldsymbol{\Sigma}_{1|2} = \frac{1}{89} \begin{bmatrix} 467 & 389 & -92 \\ 389 & 984 & 218 \\ -92 & 218 & 676 \end{bmatrix} = \begin{bmatrix} 5.25 & 4.37 & -1.03 \\ 4.37 & 11.05 & 2.45 \\ -1.03 & 2.45 & 7.6 \end{bmatrix}$$

The conditional (joint) distribution:

$$\mathbf{X}_1|\mathbf{X}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \sim N_3(\boldsymbol{\mu}_{1|2}, \boldsymbol{\Sigma}_{1|2})$$

$$\mathbf{X}_1|\mathbf{X}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \sim N_3(\begin{bmatrix} 4.83 \\ 0.25 \\ 5.98 \end{bmatrix}, \begin{bmatrix} 5.25 & 4.37 & -1.03 \\ 4.37 & 11.05 & 2.45 \\ -1.03 & 2.45 & 7.6 \end{bmatrix})$$