CS101 Advanced Engineering Mathematics (I)

工程數學(一)

[Guidelines]

- All the homework in this course will involve solving advanced engineering mathematics problems (differential equations in particular) by hand and computer.
- While discussion with other classmates is allowed, you MUST work independently to generate your own solutions to the problems.
- Python programming will be used for plotting solutions. You should reference the Python Tutorial (課程教學影片) for detail information.
- For each homework, you must submit a ***written report* (書面報告)**.

[General Instructions]

To get a good grading in homework assignments, you are advised to do the following:

- Do not copy other classmate's works! (請遵守學術倫理，嚴禁抄襲)
- Provide correct answers in details. (詳細推導過程與標明正確答案)
- Prepare your written reports in good quality (使用 Template 檔並書寫工整).
- Meet the deadline! Late homework will **not** be collected. (按時繳交，逾時不候)

指導教授：張元翔

Homework Assignment 1

Review of Calculus & First-Order Differential Equations

Deadline: 11 / 11 / 2021 (星期四)

(期中考當天下班前繳交至電學 603 計算機視覺研究室)

[Instructions]

The concept of Taylor series is very important in the study of mathematics. The Taylor series of a function is an infinite sum of terms that are expressed in terms of the function's derivatives at a single point, i.e.,

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

Python is useful for generating 2D plots using Numpy and Matplotlib. For example, the following Python source codes can be used to plot the function (程式範例請參考講義附錄).

Python 程式範例

```
import numpy as np
import matplotlib.pyplot as plt

def f1 ( x ) :
   y = np.exp ( -x ) * np.sin ( 2 * x )
   return y

def f2 ( x ) :
   y = np.exp ( -x ) * np.cos ( 3 * x )
   return y

x = np.linspace ( 0, 2 * np.pi, 100 )     #  於 [0, 2π] 產生 100 個點
y1 = f1 ( x )
y2 = f2 ( x )
plt.plot ( x, y1, '-', x, y2, '--' )              #  繪製函數曲線
plt.xlabel ( 'x' )
plt.ylabel ( 'f(x) & g(x)' )
plt.title ( 'Plot of Two Functions' )
plt.text ( 4, -0.2, 'Copyright @ Chang' )   #  請加上你的數位簽章
plt.show ( )
```

[Problems]

1. The Taylor series for the function $f(x) = \sin(x)$ at $x = 0$ is given by:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - ...$$

(assuming $-\pi \le x \le \pi$).

Please do the following: (20%)

(a) Plot both the functions $\sin x$ and $x - \frac{x^3}{3!}$ in one plot.

(b) Plot both the functions $\sin x$ and $x - \frac{x^3}{3!} + \frac{x^5}{5!}$ in one plot.

(c) Plot both the functions $\sin x$ and $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$ in one plot.

(d) Plot both the functions $\sin x$ and $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$ in one plot.

(e) Compare the results and discuss your findings (請用中文解釋).

Note: The figures must be carefully ***labeled***, ***titled***, and with your own ***copyright*** for full credits. However, it’s not necessary to include your Python source codes in the written report.

[Instructions]

Calculus is useful for solving optimization problems. In this homework assignment, our goal is to learn the ***method of least squares***, also known as ***Linear Regression*** (線性迴歸).

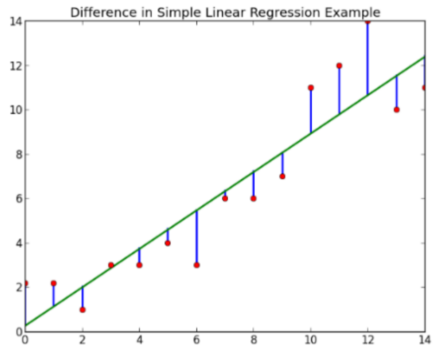
[Problems]

2. 給定一組資料點 (x_i, y_i), $i = 1\ldots n$，**最小平方法** (Method of Least Squares) 的目的是找一直線 $y = ax + b$，使得每一點至直線的垂直距離總和(又稱為**平方誤差和** Sum of Square Errors) 可以達到最小值：

$$\varepsilon = \sum_{i=1}^{n} \left[y_i - ax_i - b\right]^2$$

假設給定一組資料點 (2, 1)、(3, 2)、(4, 3)、(5, 2)，試利用最小平方法求得最佳之直線方程式 (10%，須列出手寫推導過程)

【提示】 分別**設微分為 0**，即 $\dfrac{\partial \varepsilon}{\partial a} = 0$ 與 $\dfrac{\partial \varepsilon}{\partial b} = 0$。



[Instructions]

Direction fields are particularly useful for solving first-order differential equations when analytic solutions can't be found. To plot a direction field for a first-order differential equation with Python programming, the equation must be in **normal form**. A first-order differential equation is in normal form if it is expressed as (程式範例請參考講義附錄):

$$\frac{dy}{dx} = f(x, y)$$

[Problems]

3. Following the aforementioned instructions, use the Python programming to obtain the direction field for each of the following differential equations (the interval I is given for (x, y) coordinates accordingly). The figures must be carefully ***labeled***, ***titled***, and with your own ***copyright*** for full credits. (20%)

(a) $\frac{dy}{dx} = x$; I: [-5 : 0.5 : 5, -5 : 0.5 : 5]

(b) $\frac{dy}{dx} = x + y$; I: [-5 : 0.5 : 5, -5 : 0.5 : 5]

(c) $\frac{dy}{dx} = xy$; I: [-5 : 0.5 : 5, -5 : 0.5 : 5]

(d) $\frac{dy}{dx} = \sin x \cos y$; I: [-5 : 0.5 : 5, -5 : 0.5 : 5]

[Instructions]

Given a first-order differential equation, the *explicit solution* of the differential equation can be defined as the form: $y = f(x)$. As the result, you may use Python programming to plot the integral curve (程式範例請參考講義附錄).

[Problems]

4. Solve the following *initial value problems*, and plot the solution curves. The interval I is given for the x-data in the plots. The figures must be carefully ***labeled***, ***titled***, and with your own ***copyright*** for full credits. (25%)

   注意：請先用手寫推導解題，再用 Python 畫圖，每一題的手寫推導過程與圖須放在同一頁面。

   (a) $\dfrac{dy}{dx} = -xy,\ y(0) = 1, \quad I: [-3, 3]$

   (b) $\dfrac{dy}{dx} = x\sqrt{1-y^2},\ y(0) = 1, \quad I: [0, 2\pi]$

   (c) $\dfrac{dy}{dx} + (\tan x)y = \cos x,\ y(0) = 1, \quad I: [0, 4\pi]$

   (d) $\dfrac{dy}{dx} = (x + y + 1)^2,\ y(0) = -1, \quad I: [0, 4\pi]$

[Instructions]

For many differential equations, *explicit* solutions may not be found. Instead, we may find the *implicit* solutions only. In these cases, we plot the level curves (contour plot) as the implicit solution curves. Therefore, if the implicit solution is defined by the relationship $f(x, y) = c$, you may plot the solution curves using Python programming (程式範例請參考講義附錄).

[Problems]

5. Solve the following differential equations, and plot the solution curves. The interval I is given for the xy-data in the plots. The figures must be carefully ***labeled***, ***titled***, and with your own ***copyright*** for full credits. (25%)

   注意：請先用手寫推導解題，再用 Python 畫圖，每一題的手寫推導過程與圖須放在同一頁面。

   (a) $(2xy + 2y^2)\,dx + (x^2 + 4xy)\,dy = 0$, I: [-2 ~ 2, -2 ~ 2], 30 contours

   (b) $(e^x + y)dx + (2 + x + ye^y)dy = 0$, I: [0 ~ 2π, 0 ~ 2π], 30 contours

   (c) $\cos x dx + \left(1 + \dfrac{2}{y}\right)\sin x dy = 0$, I: [-2 ~ 2, -2 ~ 2], 30 contours

   (d) $(\sin y - y\sin x)dx + (\cos x + x\cos y - y)dy = 0$, I: [-2π ~ 2π, -2π ~ 2π], 30 contours