Advanced Engineering Mathematics (I)

工程數學（一）

Homework Assignment 2

Higher-Order Differential Equations and Systems

請貢獻冷笑話一則

有一天小明他爸很渴 就叫小明幫他倒水 但小明遲遲沒去倒 小明爸就說:「你是要逼爸渴死嗎？」

於是小明就開始 B Box 了

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[Problem 1]

1.

(a)

1\. $y'' + 2y' + 10y = 0,\quad y(0) = 2,\quad y'(0) = -1$

(a) Aux. eq: $m^2 + 2m + 10 = 0$

$$m = \frac{-2 \pm \sqrt{4-40}}{2} = \frac{-2 \pm 6i}{2} = -1 \pm 3i \quad (\alpha = -1, \beta = 3)$$

Case 3: $y = e^{-x}(C_1\cos(3x) + C_2\sin(3x))$

$y(0) = 0 \xrightarrow{\text{代入}} y(0) = e^0(C_1\cos(0) + C_2\sin(0)) = C_1 = 2 \Rightarrow C_1 = 2$ 代入

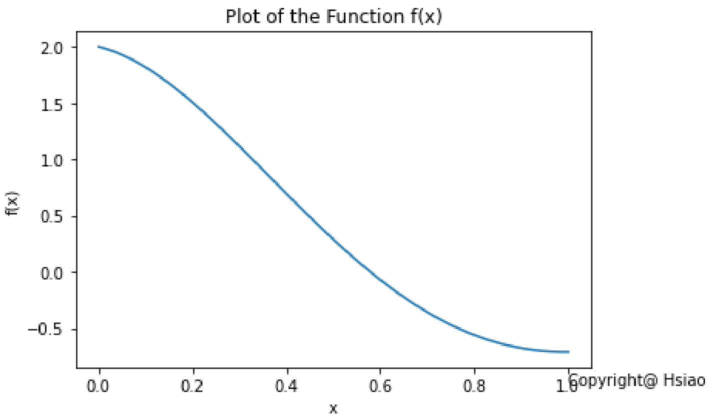
$y'(0) = -1 \xrightarrow{\text{代入}} \begin{cases} y = e^{-x}(C_1\cos(3x) + C_2\sin(3x)) \\ y' = -e^{x}(C_1\cos(3x) + C_2\sin(3x)) + e^{-x}(-3\sin(3x) + 3C_2\cos(3x)) \end{cases}$

$$y'(0) = -e^0(C_1\cos(0) + C_2\sin(0)) + e^0(-3C_1\sin(0) + 3C_2\cos(0))$$

$$= -C_1 + 3C_2 = -1 \Rightarrow C_2 = \frac{1}{3}$$

$$\Rightarrow y = e^{-x}\left(2\cos(3x) + \frac{1}{3}\sin(3x)\right)_{\#}$$

(b)



2.

(a)

2\.

Aux eq: $m^2 + 2m + 10 = 0$

$$m = \frac{-2 \pm 6i}{2} = -1 \pm 3i \quad (\alpha = -1, \beta = 3)$$

Case 3 共轭: $y = e^{-x} \cdot (C_1\cos(3x) + C_2\sin(3x))$

$y(0) = 0 \xrightarrow{\text{代入}} y(0) = e^0(C_1\cos(0) + C_2\sin(0)) = C_1 = 1 \Rightarrow C_1 = 1$

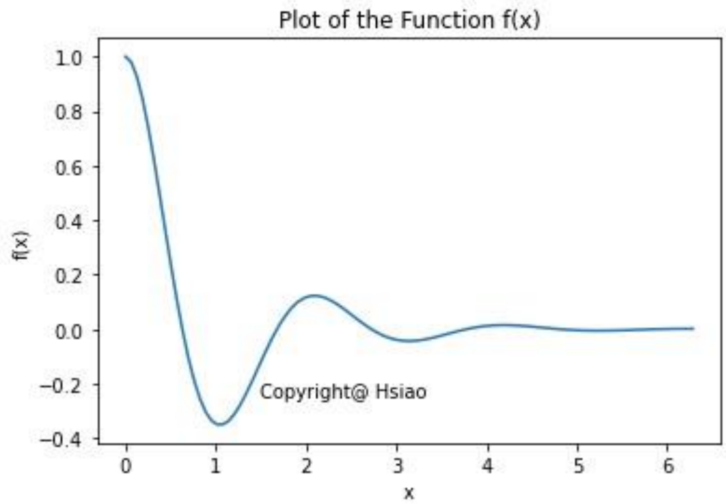
$y'(0) = 0 \xrightarrow{\text{代入}} y' = -e^{-x}(C_1\cos(3x) + C_2\sin(3x)) + e^{-x}(-3C_1\sin(3x) + 3C_2\cos(3x))$

$$y'(0) = -e^0(\cancel{C_1\cos(0)} + C_2\sin(0)) + e^0(-3C_1\sin 0 + 3C_2\cos(0))$$

$$= -C_1 + 3C_2 = 0 \Rightarrow C_2 = \frac{1}{3}$$

$$\Rightarrow y = e^{-x} \cdot \left(\cos(3x) + \frac{1}{3}\sin(3x)\right)$$

(b)



3.

(a)

3.

Aux eq $m^3 - m^2 + 2 = 0$, 根 $1, -1, 2, -2$

$= (m+1)(m^2 - 2m + 2)$

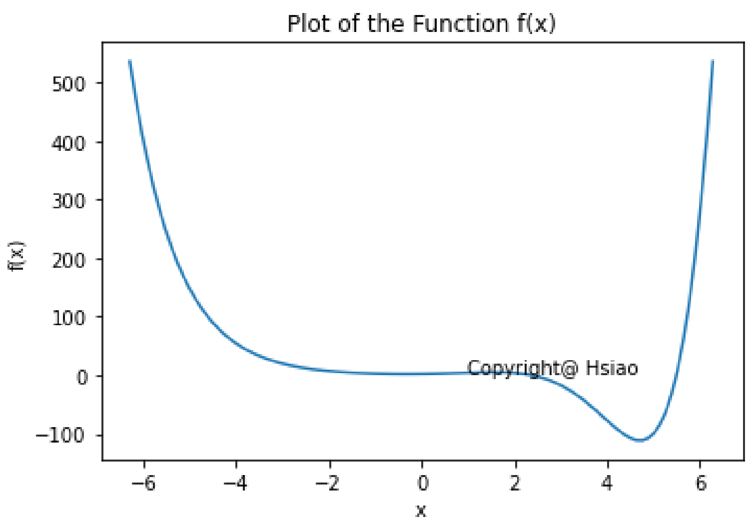
$m_{1,2,3} = -1, \frac{2 \pm \sqrt{4-4\cdot1\cdot2}}{2}$

$= -1, \quad 1 \pm i \quad (\alpha = 1, \beta = 1)$

Case I & III. $y = c_1 e^{-1x} + e^{x}(c_2 \cos x + c_3 \sin x)$

$c_1 = c_2 = c_3 = 1, \quad y = e^{-x} + e^{x}(\cos x + \sin x)$

(b)



4.

(a)

4、 Auxeq， $m^2-16 = e^{-4x}$

求 $y_h \Rightarrow (m+4)(m-4) = e^{-4x}$

$m_{1,2} = -4、4$， Case I： $y_h = c_1 e^{-4x} + c_2 e^{4x}$

求 $y_p \Rightarrow$ 設 $y_p = Ae^{-4x}$

$y_p' = -4Ae^{-4x}$， $y_p'' = 16Ae^{-4x}$ 代入DE

$16Ae^{-4x} - 16Ae^{-4x} = e^{-4x} \Rightarrow 0 = e^{-4x}$ (無解)

改設 $y_p = Axe^{-4x}$， $y_p' = Ae^{-4x} + (-4)Axe^{-4x}$

$y_p'' = Ae^{-4x} - 4Axe^{-4x} + (-4)Ae^{-4x} + 16Axe^{-4x}$

$= -3Ae^{-4x} + 12Axe^{-4x}$ 代入DE

$-3Ae^{-4x} + 12Axe^{-4x} - 16Axe^{-4x} = e^{-4x}$

$\Rightarrow (-3A-1)e^{-4x} - 4Axe^{-4x} = 0$

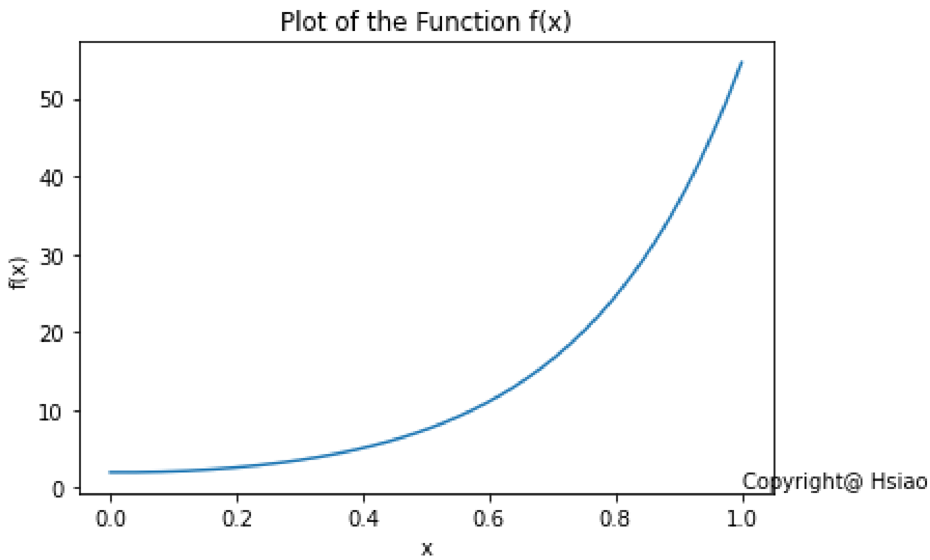
代入 (0.1) $-3A-1 = 1 \Rightarrow A = -\frac{2}{3}$

$\therefore y_p = -\frac{2}{3}xe^{-4x}$

通解 $\Rightarrow y = y_h + y_p$ or $y = c_1 e^{-4x} + c_2 e^{4x} + (-\frac{2}{3}x e^{-4x})$

$c_1 = c_2 = 1$， $y = e^{-4x} + e^{4x} - \frac{2}{3}x e^{-4x}$ ※

(b)



5.

(a)

5、 $y''+4y=\cos(2x)$， $y(0)=1$， $y'(0)=0$

$(I=[0,4\pi])$

求 $y_h \Rightarrow$ Aux eq $m^2+4=0$， $m=\dfrac{\pm\sqrt{4\cdot1\cdot4}}{2}$

$=\pm 2i \ (\alpha=0, \beta=2)$

$y_h = c_1\cos(2x)+c_2\sin(2x)$

求 $y_p \Rightarrow$ 設 $y_p = A\cos 2x + B\sin 2x$

$$\left.\begin{aligned} y_p' &= -2A\sin 2x + 2B\cos 2x \\ y_p'' &= -4A\cos 2x - 4B\sin 2x \end{aligned}\right\} \text{代入 DE}$$

$y''+4y=\cos 2x$

$\Rightarrow -4A\cos 2x - 4B\sin 2x + 4A\cos 2x + 4B\cos 2x = \cos 2x$

$\Rightarrow 0 = \cos 2x$ (無解)

改設 $y_p = Ax\cos 2x + Bx\sin 2x$

$y_p' = -2Ax\sin 2x + A\cos 2x + 2Bx\cos 2x + B\sin 2x$

$y_p'' = -2A\sin 2x + (-4)Ax\cos 2x + (-2)A\sin 2x + 2B\cos 2x - 4Bx\sin 2x + 2B\cos 2x$

$= (-4A-4Bx)\sin 2x + (4B-4Ax)\cos 2x$ 代入 DE

$(-4A-4Bx)\sin 2x + (4B-4Ax)\cos 2x + 4Ax\cos 2x + 4Bx\sin 2x = \cos(2x)$

$x=0$， $B=\frac{1}{4}$， $A=0$. $y_p = \frac{1}{4}x\sin 2x$

通解 $\Rightarrow y = y_h + y_p$ or $y = c_1\cos(2x) + c_2\sin(2x) + \frac{1}{4}x\sin(2x)$

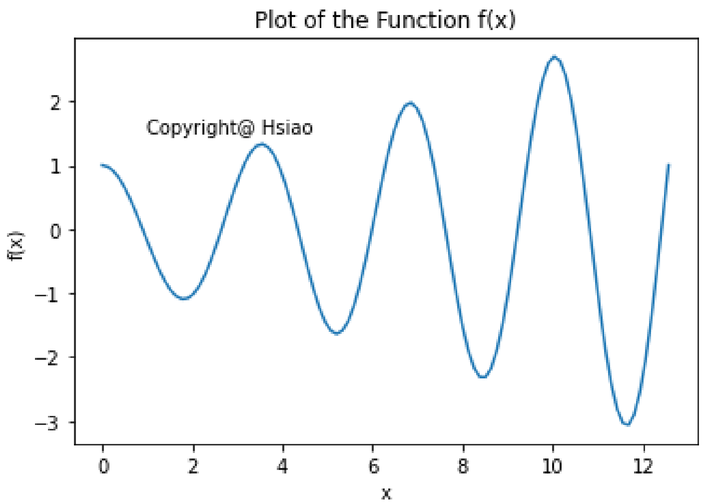
$y(0)=1$， $y = c_1\cos(2x) + c_2\sin(2x) + \frac{1}{4}\cdot 1\cdot \sin(2x) = 1 \Rightarrow c_1 = 1$

$y'(0)=0$， $y = -2c_1\sin(2x) + 2(2\cos(2x)) + \frac{1}{4}x\cos(2x) + \frac{1}{4}\sin(2x) = 0$

$\Rightarrow 2c_2 + 0 = 0 \Rightarrow c_2 = 0$

$\therefore y = \cos(2x) + \frac{1}{4}x\sin(2x)$ ※

(b)



6.

(a)

6. $y''+y=\cos^2 x$, $y(0)=4/3$, $y'(0)=-1$, $I=[0,4\pi]$

求 $y_h \Rightarrow$ Aux eq: $m^2+1=0$,

Case III $\Rightarrow y_h = C_1 \underbrace{\cos x}_{y_1} + C_2 \underbrace{\sin x}_{y_2}$.

求 $y_p \Rightarrow$ 設 $y_p = u_1 y_1 + u_2 y_2$

$$W = \begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0$$

$$W_1 = \begin{vmatrix}0 & \sin x\\ \cos^2 x & \cos x\end{vmatrix} = -\sin x\cos^2 x$$

$$W_2 = \begin{vmatrix}\cos x & 0\\ -\sin x & \cos^2 x\end{vmatrix} = \cos^3 x$$

$$u_1' = \frac{W_1}{W} = -\sin x\cos^2 x \xrightarrow{積分} u_1 = \cos x - \frac{3}{4}\cos x + \frac{1}{12}\cos 3x + C$$

$$u_2' = \frac{W_2}{W} = \cos^3 x \xrightarrow{積分} u_2 = \frac{3}{4}\sin x + \frac{1}{12}\sin 3x + C$$

$$\Rightarrow y_p = \left(\frac{1}{4}\cos x + \frac{1}{12}\cos 3x\right)\cos x + \left(\frac{3}{4}\sin x + \frac{1}{12}\sin 3x\right)\sin x$$

$$\therefore y = C_1\cos x + C_2\sin x + \left(\frac{1}{4}\cos x + \frac{1}{12}\cos 3x\right)\cos x + \left(\frac{3}{4}\sin x + \frac{1}{12}\sin 3x\right)\sin x$$

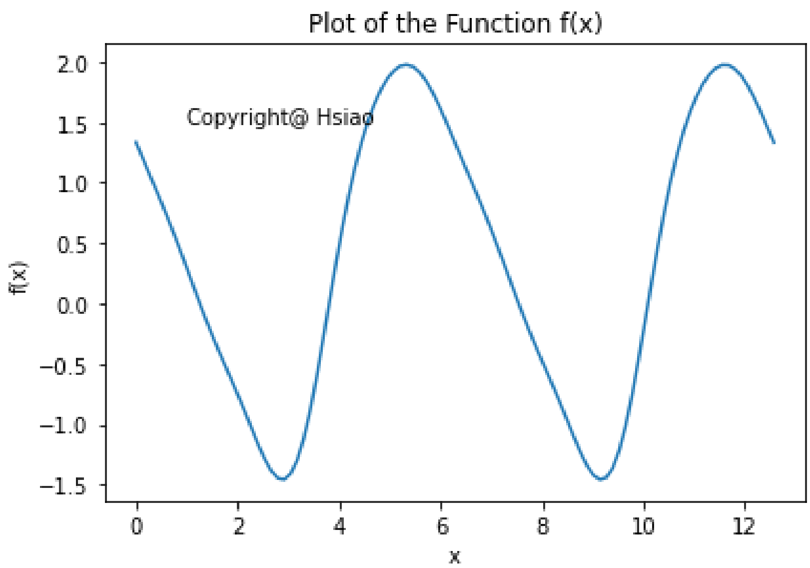
$$y(0) = \frac{4}{3} \Rightarrow y(0) = C_1 + \frac{1}{4} + \frac{1}{12} = \frac{4}{3} \Rightarrow C_1 = 1$$

$$y'(0) = -1 \Rightarrow y' = -C_1\sin x + C_2\cos x + \left(-\frac{1}{4}\sin x - \frac{1}{4}\sin 3x\right)\cos x - \left(\frac{1}{4}\cos x + \frac{1}{12}\cos 3x\right)\sin x + \left(\frac{3}{4}\cos x + \frac{1}{4}\cos 3x\right)\sin x + \left(\frac{3}{4}\sin x + \frac{1}{12}\sin 3x\right)\cos x$$

$$y'(0) = C_2 = -1$$

$$\Rightarrow y = \cos x - \sin x + \left(\frac{1}{4}\cos x + \frac{1}{12}\cos 3x\right) + \left(\frac{3}{4}\sin x + \frac{1}{12}\sin 3x\right)\sin x \quad ※$$

(b)



7.

(a)

7、 $x^2y''-2xy'+2y=x^3e^x$

求 y_h ⇒ 設 $y=x^m$ ⇒ $y'=mx^{m-1}$, $y''=m(m-1)y^{m-2}$ 代入 DE

⇒ $x^2(m(m-1))x^{m-2}-2xmx^{m-1}+2x^m=0$

⇒ $\underbrace{[m(m-1)-2m+2]}_{=0}\underbrace{x^m}_{\neq 0}=0$ ⇒ Auxeq $m^2-3m+2=0$

⇒ $(m-1)(m-2)=0$

$m_{1,2}=1,2$ $y_h=c_1x+c_2x^2$ ※

求 y_p ⇒ 設 $y_p=u_1x+u_2x^2$

$W=\begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}=2x^2-x^2=x^2\neq 0$

$W_1=\begin{vmatrix} 0 & x^2 \\ xe^x & 2x \end{vmatrix}=-x^3e^x$

$W_2=\begin{vmatrix} x & 0 \\ 1 & xe^x \end{vmatrix}=x^2e^x$

$u_1'=-xe^x$ $u_2'=e^x$

積分 $u_1=-xe^x+e^x$ $u_2=e^x$

⇒ $y_p=u_1y_1+u_2y_2$

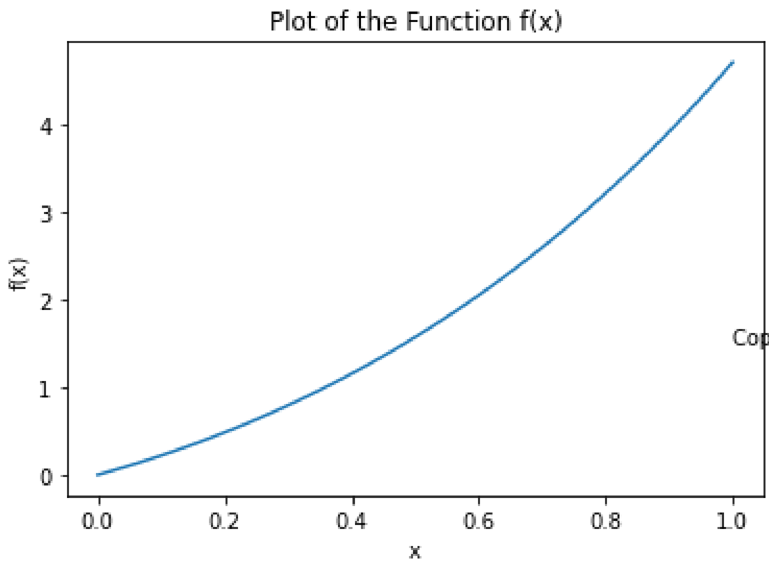
$=(-xe^x+e^x)x+e^x\cdot x^2$

$=xe^x$

通解 ⇒ $y=y_h+y_p$ or $y=c_1x+c_2x^2+xe^x$,

$c_1=c_2=1$, ∴ $y=x+x^2+xe^x$ ※

(b)



8.

(a)

8. $L\frac{di}{dt} + Ri + \frac{1}{C}q = E(t)$, $q(0) = 0$, $u(0) = 0$.

$\Rightarrow L = 1,\ R = 20,\ C = 0.001,\ E(t) = 10\sin(60t)$

$\Rightarrow \frac{d^2q}{dt^2} + 20\frac{dq}{dt} + 1000q = 10\sin(60t)$

Aux eq $\Rightarrow m^2 + 20m + 1000 = 0$

$m_{1,2} = \frac{-20 \pm \sqrt{-3600}}{2} = -10 \pm 30i \quad (\alpha = 10, \beta = 30)$

Case III $\Rightarrow y = e^{-10t}(c_1\cos 30t + c_2\sin 30t)$

設 $y_p = A\cos 60t + B\sin 60t$

$y_p' = -60A\sin 60t + 60B\cos 60t$

$y_p'' = -3600A\cos 60t - 3600B\sin 60t$

代入 $\Rightarrow -3600A\cos 60t + (-3600B\sin 60t) - 1200A\sin 60t + 1200B\cos 60t + 1000A\cos 60t$
$+ 1000B\sin 60t = 10\sin 60t$

$(-2600A + 1200B)\cos 60t + (-1200A - 2600B)\sin 60t = 10\sin 60t$

$\begin{cases} -2600A + 1200B = 0 \\ -1200A - 2600B = 10 \end{cases} \Rightarrow \begin{cases} A = -0.0015 \\ B = -0.0032 \end{cases}$

$\Rightarrow y_p = -0.0015\cos 60t - 0.0032\sin 60t$

$q(t) = e^{-10t}(c_1\cos 30t + c_2\sin 30t) + (-0.0015\cos 60t - 0.0032\sin 60t)$

$q(0) = 0 \Rightarrow C_1 - 0.0015 = 0 \Rightarrow C_1 = 0.0015$

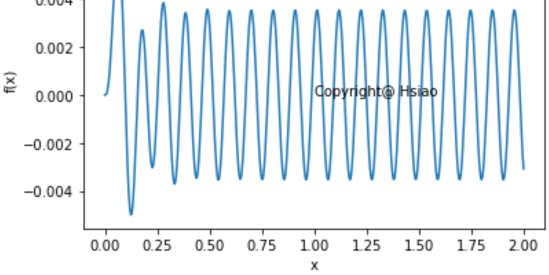
$q'(0) = i(0) = 0 \Rightarrow q' = -10e^{-10t}(C_1\cos 30t + C_2\sin 30t) + e^{-10t}(-30C_1\sin 30t + 30C_2\cos 30t)$
$+ 0.09\sin 60t - 0.192\cos 60t$

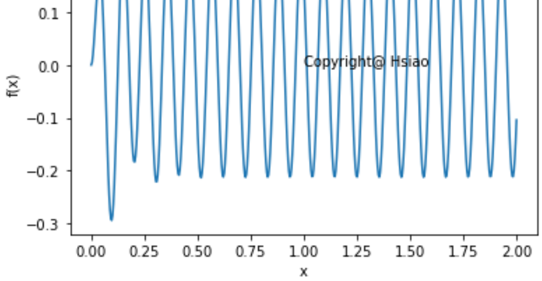
$\Rightarrow q'(0) = -10C_1 + 30C_2 - 0.192 = 0 \Rightarrow C_2 = 0.0069$

$q(t) = e^{-10t}(0.0015\cos 30t + 0.0069\sin 30t) - 0.0015\cos 60t - 0.0032\sin 60t$

$i(t) = -10e^{-10t}(0.0015\cos 30t + 0.0069\sin 30t) + e^{-10t}(-0.045\sin 30t + 0.207\cos 30t)$
$+ 0.09\sin 60t - 0.192\cos 60t$ ※

(b) q(t) i(t)





[Problem 2]

9.

(a)

9. $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$, $q(0) = 50$, $i(0) = 0$

$L=1$, $R=0$, $C=0.001$, $E(t) = 0$

$\Rightarrow \frac{d^2q}{dt^2} + 1000 = 0$. $\underline{\text{Aux eq}}$ $m^2 + 1000 = 0$

$m_{1,2} = \pm 10\sqrt{10}\, i \; (\alpha = 0, \beta = 10\sqrt{10})$

$\Rightarrow q(t) = C_1\cos(10\sqrt{10}t) + C_2\sin(10\sqrt{10}t)$

$q(0) = 50 \Rightarrow q(0) = C_1 = 50$

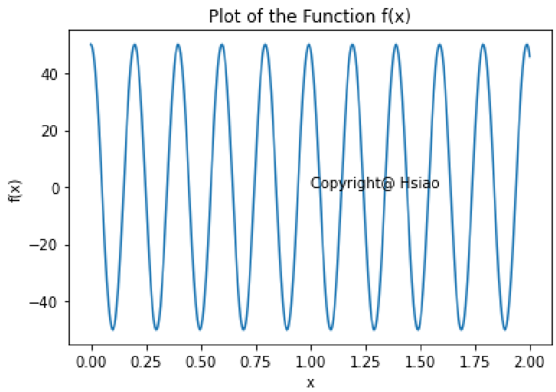
$i(0) = q'(0) = 0 \Rightarrow q' = -10\sqrt{10}\sin(10\sqrt{10}t) + 10\sqrt{10}\cos(10\sqrt{10}t)$

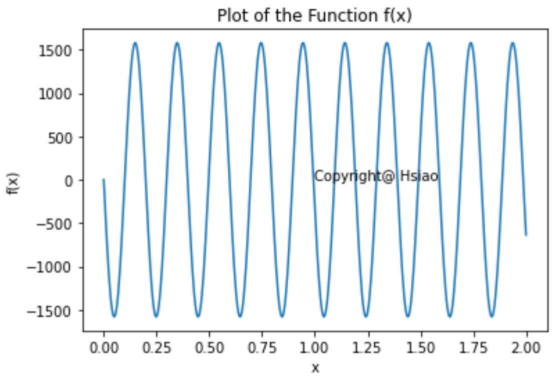
$q'(0) = 10\sqrt{10}\, C_2 = 0 \Rightarrow C_2 = 0$

$\Rightarrow q(t) = 50\cos(10\sqrt{10}t)$

$i(t) = -500\sqrt{10}\sin(10\sqrt{10}t)$ #

(b) q(t) i(t)





(c)

10.

(a)

10、
$$\begin{cases} \dfrac{dx}{dt} + y = t \\ \dfrac{dy}{dt} - x = -t \end{cases} \Rightarrow \begin{cases} Dx + y = t \\ Dy - x = -t \end{cases}$$

消去 y：
$$\begin{array}{rl} & D^2x + Dy = 1 \\ -) & -x + Dy = -t \\ \hline & D^2x + x = 1 + t \end{array}$$

Aux eq $m^2 + 1 = 0$

$m_{1,2} = \pm i \ (\alpha = 0, \beta = 1)$

Case III $\Rightarrow x = C_1\cos t + C_2\sin t$

設 $x_p = At + B$

$x_p' = A$,

$x_p'' = 0$ 代入 DE

$\rightarrow 0 + At + B = 1 + t \Rightarrow A = 1, B = 1 \Rightarrow x(t) = C_1\cos t + C_2\sin t + t + 1$

消去 x：
$$\begin{array}{rl} & y + Dx = t \\ +) & D^2y - Dy = -1 \\ \hline & D^2y + y = t - 1 \end{array}$$

Aux eq : $m^2 + 1 = 0$

$m_{1,2} = \pm i \ (\alpha = 0, \beta = 1)$

Case III $\Rightarrow y = C_3\cos t + C_4\sin t$

設 $y_p = At + B$. $y_p' = A$, $y_p'' = 0$ 代入 DE

$\Rightarrow 0 + At + B = t - 1 \Rightarrow A = 1, B = -1$

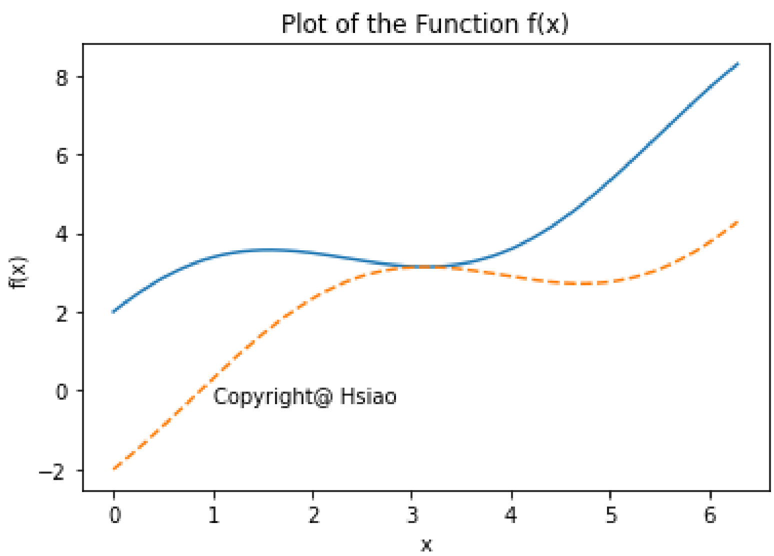
$y(t) = C_3\cos t + C_4\sin t + t - 1$

$$\begin{cases} \dfrac{dx}{dt} + y = t \\ \dfrac{dy}{dt} - x = -t \end{cases} \Rightarrow -C_1\sin t + C_2\cos t + 1 = t - C_3\cos t - C_4\sin t - t + 1$$

$\Rightarrow C_1 = C_4, \ C_2 = -C_3$

$$\Rightarrow \begin{cases} x(t) = C_1\cos t + C_2\sin t + t + 1 \\ y(t) = -C_2\cos t + C_1\sin t + t - 1 \end{cases} \#$$

(b)



11.

(a)

11.
$$\begin{cases} L_1\frac{di_1}{dt}+R_1(i_1-i_2)=E(t) \\ R_1(i_2-i_1)+L_2\frac{di_2}{dt}+R_2\cdot i_2=0 \end{cases} \Rightarrow \begin{cases} 10\frac{di}{dt}+20(i_1-i_2)=20 \\ 20(i_2-i_1)+10\frac{di_2}{dt}+30i_2=0 \end{cases}$$

令 $i_1=x,\ i_2=y$

$$\Rightarrow \begin{cases} 10Dx+20x-20y=20 & -① \\ 50y-20x+10Dy=0 & -② \end{cases} \Rightarrow \begin{cases} 5D^2x+10Dx-10Dy=0 & -③ \\ 50y-20x+10Dy=0 & -④ \end{cases}$$

$②+③ \Rightarrow D^2x+2Dx-4x+10y=0$

$①+④\times2 \Rightarrow 2D^2x+14Dx+12x=20,\ D^2x+7Dx+6x=10$

求 $x_h \Rightarrow m^2+7m+6=0,\quad m_{1,2}=-6,-1 \Rightarrow x_h=C_1e^{-x}+C_2e^{-6x}$

求 $x_p \Rightarrow$ 設 $x_p=A,\ x_p'=x_p''=0$ 代入 DE

$6A=10 \Rightarrow A=\frac{5}{3} \Rightarrow x=C_1e^{-x}+C_2e^{-6x}+\frac{5}{3}$

消去 x:
$$\begin{cases} 10Dx+20x-20y=20 & -(1) \\ 50Dy-20Dx+10D^2y=0 & -(2) \end{cases}$$

$(1)\times2+(2) \Rightarrow 40x-40y+50Dy+10D^2y=40$

$$\Rightarrow \begin{cases} D^2y+5Dy-4y+4x=4 & -(3) \\ 50y-20x+10Dy=10 & -(4) \end{cases}$$

$(3)\times5+(4) \Rightarrow 5D^2y+35Dy+30y=20$

$\Rightarrow D^2y+7Dy+6y=4 \quad -(5)$

$y_h = m^2+7m+6=0,\ m_{1,2}=-6,-1,\ y_h=C_3e^{-x}+C_4e^{-6x}$

求 y_p = 設 $y_p=B,\ y_p'=0,\ y_p''=0$ 代入 $6B=4 \Rightarrow B=\frac{2}{3}$

$\Rightarrow y=C_3e^{-x}+C_4e^{-6x}+\frac{2}{3}$

$$\begin{cases} i_1: C_1e^{-x}+C_2e^{-6x}+\frac{5}{3} \\ i_2: C_3e^{-x}+C_4e^{-6x}+\frac{2}{3} \end{cases} \Rightarrow i_2=\frac{1}{2}C_1e^{-x}-2C_2e^{-6x}+\frac{2}{3}$$

$\frac{di}{dt}=-C_1e^{-x}-6C_2e^{-6x}$

代入 $\Rightarrow -10C_1e^{-x}-60C_2e^{-6x}+20(C_1e^{-x}+C_2e^{-6x}+\frac{5}{3}-C_3e^{-x}-C_4e^{-6x}+\frac{2}{3})=0$

$\Rightarrow 10C_1e^{-x}-40C_2e^{-6x}-20C_3e^{-x}-20C_4e^{-6x}=0$

$\Rightarrow (10C_1-20C_3)e^{-x}+(-40C_3-20C_4)e^{-6x}=0$

$\Rightarrow C_1=2C_2,\ C_2=-\frac{1}{2}C_4$

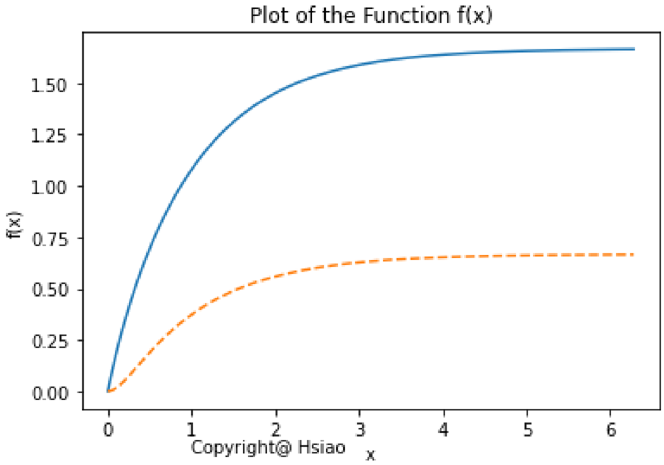
$i_1(0)=0 \Rightarrow C_1+C_2+\frac{5}{3}=0$

$i_2(0)=0 \Rightarrow C_3+C_4+\frac{2}{3} \Rightarrow \frac{1}{2}C_1+(-2)C_2+\frac{2}{3}=0$

$\Rightarrow \begin{cases} C_1=-\frac{8}{5}, \\ C_2=-\frac{1}{15} \end{cases}$

$$\Rightarrow \begin{cases} i_1(t)=-\frac{8}{5}e^{-t}-\frac{1}{15}e^{-6t}+\frac{5}{3} \\ i_2(t)=-\frac{4}{5}e^{-t}+\frac{2}{15}e^{-6t}+\frac{2}{3} \end{cases} \#$$

(b)



Plot of the Function f(x)
f(x)
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0
1
2
3
4
5
6
x
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(c)