Linear Regression

Art Tay

Gradient Derivation

a)

$$\begin{aligned}\frac{\partial}{\partial \beta_0}\left[\frac{1}{2m}\sum_{i=0}^{m}(\beta_0+\beta_1 x^{(i)}-y^{(i)})^2\right] &= \frac{\partial}{\partial \beta_0}\left[\frac{1}{2m}\sum_{i=0}^{m}(\beta_0+\beta_1 x^{(i)}-y^{(i)})^2\right] \\ &= \frac{1}{2m}\sum_{i=0}^{m}\left[2(\beta_0+\beta_1 x^{(i)}-y^{(i)})\cdot 1\right] \\ &= \frac{1}{m}\sum_{i=0}^{m}\beta_0+\beta_1\frac{1}{m}\sum_{i=0}^{m}x^{(i)}-\frac{1}{m}\sum_{i=0}^{m}y^{(i)} \\ &= \beta_0+\beta_1\bar{x}-\bar{y}\end{aligned}$$

b)

$$\begin{aligned}\frac{\partial}{\partial \beta_1}\left[\frac{1}{2m}\sum_{i=0}^{m}(\beta_0+\beta_1 x^{(i)}-y^{(i)})^2\right] &= \frac{1}{2m}\sum_{i=0}^{m}\frac{\partial}{\partial \beta_1}\left[(\beta_0+\beta_1 x^{(i)}-y^{(i)})^2\right] \\ &= \frac{1}{2m}\sum_{i=0}^{m}\left[2x^{(i)}(\beta_0+\beta_1 x^{(i)}-y^{(i)})\right] \\ &= \frac{1}{m}\sum_{i=0}^{m}\left[\beta_0 x^{(i)}+\beta_1 {x^{(i)}}^2-x^{(i)}y^{(i)}\right] \\ &= \beta_0\frac{1}{m}\sum_{i=0}^{m}x^{(i)}+\beta_1\frac{1}{m}\sum_{i=0}^{m}{x^{(i)}}^2-\frac{1}{m}\sum_{i=0}^{m}x^{(i)}y^{(i)} \\ &= \beta_0\bar{x}+\beta_1\bar{x^2}-\bar{xy}\end{aligned}$$

Linear Regression by Gradient Decent

```
# Generates linear data with normal residuals
set.seed(123)
x <- rnorm(n = 30)

epsilon <- rnorm(n = 30)

y <- 5*x + 1 + epsilon
```

```
summary(lm(y ~ x))
```

```
##
## Call:
## lm(formula = y ~ x)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -1.6085 -0.5056 -0.2152  0.6932  2.0118
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   1.1720     0.1534   7.639 2.54e-08 ***
## x             4.8660     0.1589  30.629  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.8393 on 28 degrees of freedom
## Multiple R-squared:  0.971,  Adjusted R-squared:   0.97
## F-statistic: 938.1 on 1 and 28 DF,  p-value: < 2.2e-16
```

```r
# Calculates the mean squared error for a simple linear regression model.
# @param x - a vector of the explainatory variable.
# @param y - a vector of the response variable.
# @param beta_0 - the intercept value for the current SLR model.
# @param beta_1 - the slope value for the current SLR model.
# @return - sum total mean squared error (y_hat - y)^2
slr_mse <- function(x, y, beta_0, beta_1){
    cost <- ((beta_1 * x + beta_0) - y)^2
    return(sum(cost))
}
```

```r
# Calculates the slope and intercept values for SLR
# or simple linear regression.
# @param x - a vector of the explanatory variable.
# @param y - a vector of the response variable.
# @param alpha - the learning rate.
# @return betas - a vector containing the calculated betas.
slr_gradient_desc <- function(x, y, alpha){

    # Summary statistic calculations.
    # Helps to calculate the gradient faster.
    x_bar <- mean(x)
    y_bar <- mean(y)
    xy_bar <- mean(x*y)
    x_sqbar <- mean(x^2)

    # initial guess for beta_0 and beta_1.
    beta_0 <- y_bar
    beta_1 <- 0

    # A counter to determine is the error is unchanging.
    # This is the Loop-Control-Variable (LCV).
    count_same <- 0

    # Iterate 100 times or until the cost remains unchanged for 10 iterations.
```

```
    for(i in 1:1000){

        # Stop the loop if the LCV >= 10.
        if(count_same >= 10){
            break
        }

        # Cost prior to beta adjustment.
        cost_start <- slr_mse(x, y, beta_0, beta_1)

        # Calculate gradient values.
        g_0 <- beta_0 + (beta_1 * x_bar) - y_bar
        g_1 <- (beta_0 * x_bar) + (beta_1 * x_sqbar) - xy_bar

        # Update betas.
        beta_0 <- beta_0 - (alpha * g_0)
        beta_1 <- beta_1 - (alpha * g_1)

        # If the cost is unchanged add 1 to the LCV.
        if(cost_start == slr_mse(x, y, beta_0, beta_1)){
            count_same <- count_same + 1
        }
    }

    return(c(beta_0 = round(beta_0, 4),
             beta_1 = round(beta_1, 4),
             iterations = i))
}
```

```
slr_gradient_desc(x, y, alpha = 0.1)
```

```
##     beta_0     beta_1 iterations
##      1.172      4.866    256.000
```

Linear Model on Airbnb Data

```
# Libraries
library(tidyverse)
library(tidymodels)
tidymodels_prefer()
```

Modeling Bulding

```
# Import raw data.
data_raw <- read.csv(file = "AB_NYC_2019.csv", stringsAsFactors = T,
                     header = T)
```

```r
# 70/30 train/test data split
# Data was split prior to cleaning to prevent data leakage.
set.seed(123)
data_split <- initial_split(data_raw, prop = 0.7)

train_data <- training(data_split)
test_data <- testing(data_split)
```

```r
# Define a cleaning recipe from the data.
# Assigns price the role of response.
# Recipe is used to create a pipline that can works with new raw data
# assuming that the format matches the training data.
cleaning_recipe <- recipe(price ~ ., data = train_data)
```

```r
cleaning_recipe <- cleaning_recipe %>% step_rm(id, host_id, name, host_name)
```

```r
cleaning_recipe <- cleaning_recipe %>%
    step_mutate(
        # numeric recodes
        latitude = ifelse(latitude == 0, NA, latitude),
        longitude = ifelse(longitude == 0, NA, longitude),
        minimum_nights = ifelse(minimum_nights == 0, NA, minimum_nights),

        #Factor recodes
        # It was done using replace to maintain factor coding
        room_type = replace(room_type,
            room_type == "" | room_type == " ", NA),
        neighbourhood_group = replace(neighbourhood_group,
            neighbourhood_group == "" | neighbourhood_group == " ",  NA),
        neighbourhood = replace(neighbourhood,
            neighbourhood == "" | neighbourhood == " ", NA),
        last_review = replace(last_review,
            last_review == "" | room_type == " ", NA)
    )
```

```r
# Skip = T tell the fit to ignore the step if it can't be done.
# Necessary because we assume the test_data does not have the response.
cleaning_recipe <- cleaning_recipe %>%
                    step_mutate(price = ifelse(price == 0, NA, price), skip = T)
```

```r
# Geodist (?)
cleaning_recipe <- cleaning_recipe %>%
    step_mutate(

            # Change NA reviews per month to be 0.
            reviews_per_month = ifelse(is.na(reviews_per_month),
                0, reviews_per_month),

            # Create a new variable last_review_year to reduce dimensionality.
            last_review_year = substring(last_review, 1, 4),

            # Modify NA last_review to be a new level "none".
```

```
            last_review_year = replace(last_review_year,
            is.na(last_review_year), "none"),

            # Cast to be factor.
            last_review_year = as.factor(last_review_year),

            # Remove old last review variable
            last_review = NULL
    )
```

```
plot_skew <- train_data %>%
             select(where(is.numeric)) %>%
             select(-c(id, host_id)) %>%
             pivot_longer(cols = everything()) %>%
             ggplot(aes(value)) +
             geom_histogram() +
             facet_wrap(~name, scales = "free_x") +
             theme_bw() +
             labs(x = "", y = "Frequency")
```

```
cleaning_recipe <- cleaning_recipe %>%
                   step_YeoJohnson(all_numeric_predictors()) %>%
                   # price was logged instead of YeoJohnson to maintain
                   # reversibility and interpretability.
                   step_log(price, skip = T)
```

```
cleaning_recipe <- cleaning_recipe %>%
                   step_normalize(all_numeric_predictors()) %>%
                   step_dummy(all_nominal_predictors()) %>%
                   step_impute_knn(everything(), neighbors = 10,
                                   impute_with = imp_vars(everything()),
                                   skip = T)
```

```
cleaning_recipe <- cleaning_recipe %>%
                   # removes predictor with less than 10% unique values and
                   # a greater than 95/5 ratio of most prevalent to next most
                   step_nzv(all_predictors()) %>%
                   step_corr(all_numeric_predictors())
```

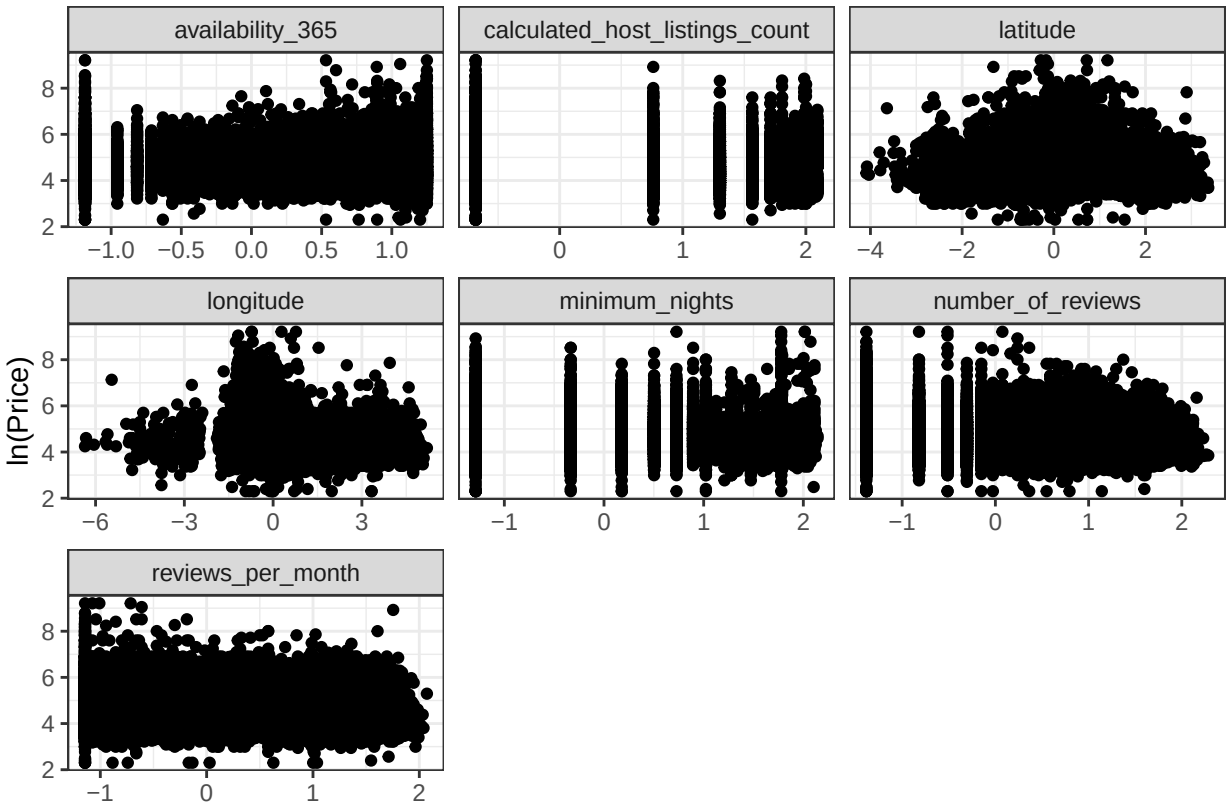
```
cleaning_recipe <- cleaning_recipe %>% prep(retain = T, verbose = T)
```

```
## oper 1 step rm [training]
## oper 2 step mutate [training]
## oper 3 step mutate [training]
## oper 4 step mutate [training]
## oper 5 step YeoJohnson [training]
## oper 6 step log [training]
## oper 7 step normalize [training]
## oper 8 step dummy [training]
## oper 9 step impute knn [training]
## oper 10 step nzv [training]
```

```
## oper 11 step corr [training]
## The retained training set is ~ 5.5 Mb  in memory.
```

```
train_scatter <- bake(cleaning_recipe, new_data = NULL)
```

```
# Matrix Scatterplot
plot_scatter <- train_scatter %>%
                select(-starts_with(c("neighbourhood",
                                      "room_type", "last_review"))) %>%
                pivot_longer(cols = -price) %>%
                ggplot(aes(x = value, y = price)) +
                geom_point() +
                facet_wrap(~name, scales = "free_x") +
                theme_bw() +
                labs(x = "", y = "ln(Price)")

plot_scatter
```




```
cleaning_recipe <- cleaning_recipe %>%
                   # makes sure not to accidentally spline your response variable
                   step_ns(-starts_with(c("neighbourhood",
                                "room_type", "last_review", "price")),
                           deg_free = 3) %>%
                   # called again to remove overdetermined splines
                   step_corr(all_numeric_predictors())
```

```r
mlr_mod <- linear_reg() %>% set_engine("lm")

mlr_wflow <- workflow() %>%
             add_model(mlr_mod) %>%
             add_recipe(cleaning_recipe)

mlr_wflow
```

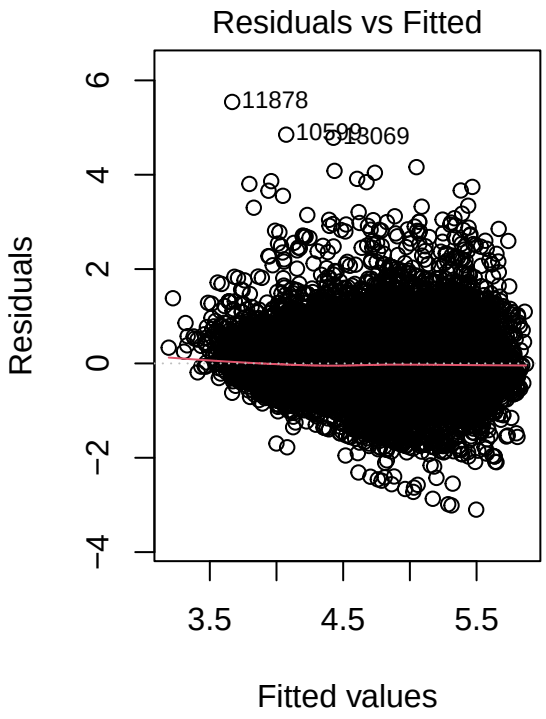
```
## == Workflow ====================================================================
## Preprocessor: Recipe
## Model: linear_reg()
##
## -- Preprocessor ----------------------------------------------------------------
## 13 Recipe Steps
##
## * step_rm()
## * step_mutate()
## * step_mutate()
## * step_mutate()
## * step_YeoJohnson()
## * step_log()
## * step_normalize()
## * step_dummy()
## * step_impute_knn()
## * step_nzv()
## * ...
## * and 3 more steps.
##
## -- Model -----------------------------------------------------------------------
## Linear Regression Model Specification (regression)
##
## Computational engine: lm
```

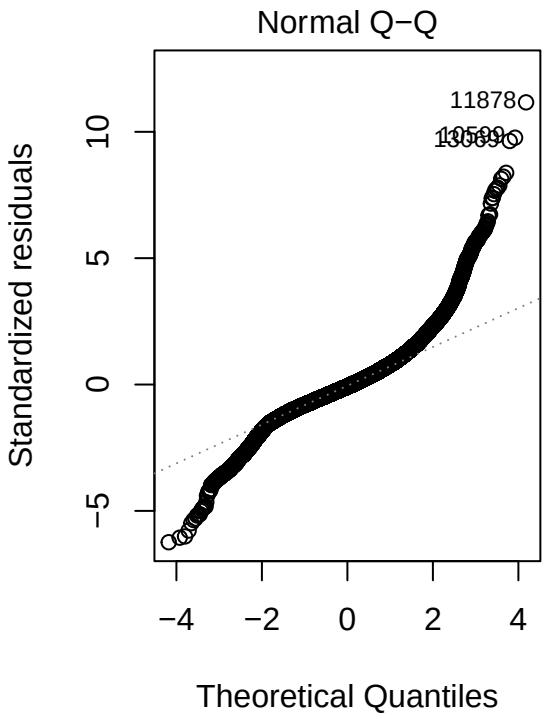
```r
mlr_fit <- mlr_wflow %>%
           fit(data = train_data)

mlr_fit_lm <- mlr_fit %>% extract_fit_engine()

summary(mlr_fit_lm)
```

```
##
## Call:
## stats::lm(formula = ..y ~ ., data = data)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -3.0993 -0.2811 -0.0383  0.2333  5.5426
##
## Coefficients:
##                                 Estimate Std. Error  t value Pr(>|t|)
## (Intercept)                     3.598496   0.102484   35.113  < 2e-16 ***
## neighbourhood_group_Brooklyn    0.017221   0.025884    0.665 0.505863
```

```
## neighbourhood_group_Manhattan        0.158845   0.022715    6.993 2.74e-12 ***
## neighbourhood_group_Queens          -0.117120   0.025326   -4.624 3.77e-06 ***
## neighbourhood_Bedford.Stuyvesant    -0.036826   0.012519   -2.942 0.003267 **
## neighbourhood_Bushwick              -0.052567   0.015845   -3.318 0.000909 ***
## neighbourhood_Harlem                -0.101567   0.014018   -7.246 4.40e-13 ***
## neighbourhood_Williamsburg           0.118317   0.012558    9.422  < 2e-16 ***
## room_type_Private.room              -0.666184   0.005806 -114.750  < 2e-16 ***
## last_review_year_X2016              -0.098130   0.017791   -5.516 3.50e-08 ***
## last_review_year_X2017              -0.096721   0.018090   -5.347 9.02e-08 ***
## last_review_year_X2018              -0.095461   0.018332   -5.207 1.93e-07 ***
## last_review_year_X2019              -0.149531   0.019646   -7.611 2.78e-14 ***
## latitude_ns_1                        0.761940   0.032911   23.151  < 2e-16 ***
## latitude_ns_2                        0.818374   0.108689    7.529 5.22e-14 ***
## latitude_ns_3                       -0.257714   0.044796   -5.753 8.84e-09 ***
## longitude_ns_1                      -0.274553   0.065781   -4.174 3.00e-05 ***
## longitude_ns_2                       2.111494   0.236104    8.943  < 2e-16 ***
## longitude_ns_3                       0.343268   0.064950    5.285 1.26e-07 ***
## minimum_nights_ns_1                 -0.011806   0.013877   -0.851 0.394918
## minimum_nights_ns_2                 -0.124602   0.014705   -8.473  < 2e-16 ***
## minimum_nights_ns_3                 -0.378582   0.013341  -28.377  < 2e-16 ***
## number_of_reviews_ns_1              -0.085484   0.018736   -4.563 5.07e-06 ***
## number_of_reviews_ns_2              -0.313913   0.039953   -7.857 4.05e-15 ***
## number_of_reviews_ns_3              -0.194295   0.021876   -8.882  < 2e-16 ***
## reviews_per_month_ns_1               0.053901   0.020394    2.643 0.008221 **
## reviews_per_month_ns_2               0.156019   0.049282    3.166 0.001548 **
## reviews_per_month_ns_3              -0.024813   0.020468   -1.212 0.225410
## calculated_host_listings_count_ns_1  0.032294   0.010035    3.218 0.001291 **
## calculated_host_listings_count_ns_3 -0.125242   0.011395  -10.991  < 2e-16 ***
## availability_365_ns_1                0.051268   0.010436    4.913 9.03e-07 ***
## availability_365_ns_3                0.309121   0.008771   35.244  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.4968 on 34194 degrees of freedom
## Multiple R-squared:  0.4903, Adjusted R-squared:  0.4898
## F-statistic:  1061 on 31 and 34194 DF,  p-value: < 2.2e-16
```

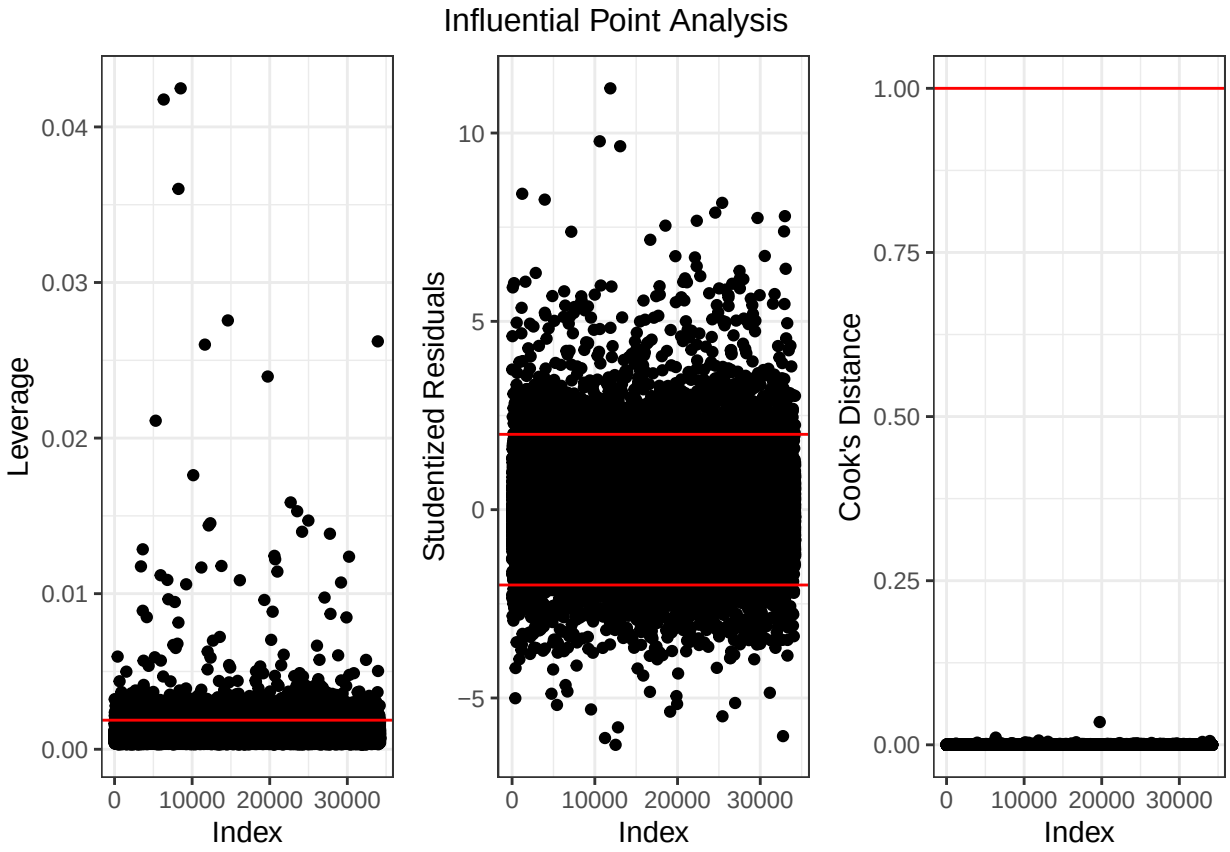
```
par(mfrow = c(1,2))
plot(mlr_fit_lm, which = 1:2)
```






```
library(MASS)
library(ggpubr)

##Influential Point Analysis ##

##Calculate Leverage, Studentized Residuals, and Cook's Distance.
Id <- 1:length(train_data$id)
Leverage <- hatvalues(mlr_fit_lm)
StudRes <- studres(mlr_fit_lm)
CookD <- cooks.distance(mlr_fit_lm)

inful_data <- cbind(Id, Leverage, StudRes, CookD)
inful_data <- as.data.frame(inful_data)

##Plots
##Leverage
lev <- ggplot(data = inful_data, aes(x = Id, y = Leverage)) + geom_point() +
            geom_hline(yintercept = 2 * length(mlr_fit_lm$coefficients) /
                length(inful_data$Id), col = "red") +
            labs(x = "Index") +
            theme_bw()

##Studentized Residuals
studres <- ggplot(data = inful_data, aes(x = Id, y = StudRes)) + geom_point() +
                geom_hline(yintercept = 2, col = "red") +
                geom_hline(yintercept = -2, col = "red") +
```

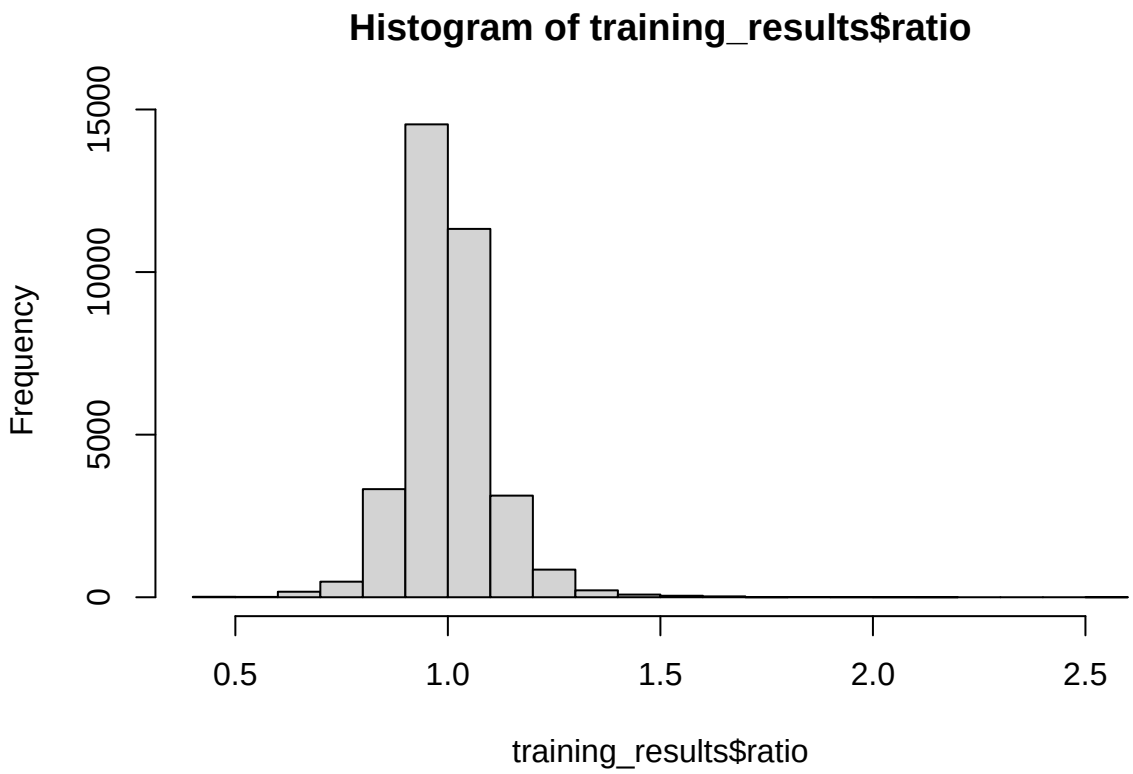
```
                labs(y = "Studentized Residuals", x = "Index") +
                theme_bw()

##Cooks distance
cooks <- ggplot(data = inful_data, aes(x = Id, y = CookD)) + geom_point() +
             geom_hline(yintercept = 1, col = "red") +
             labs(y = "Cook's Distance", x = "Index") +
             theme_bw()

inful <- ggarrange(lev, studres, cooks, ncol = 3, nrow = 1)

annotate_figure(inful, top = text_grob("Influential Point Analysis"))
```



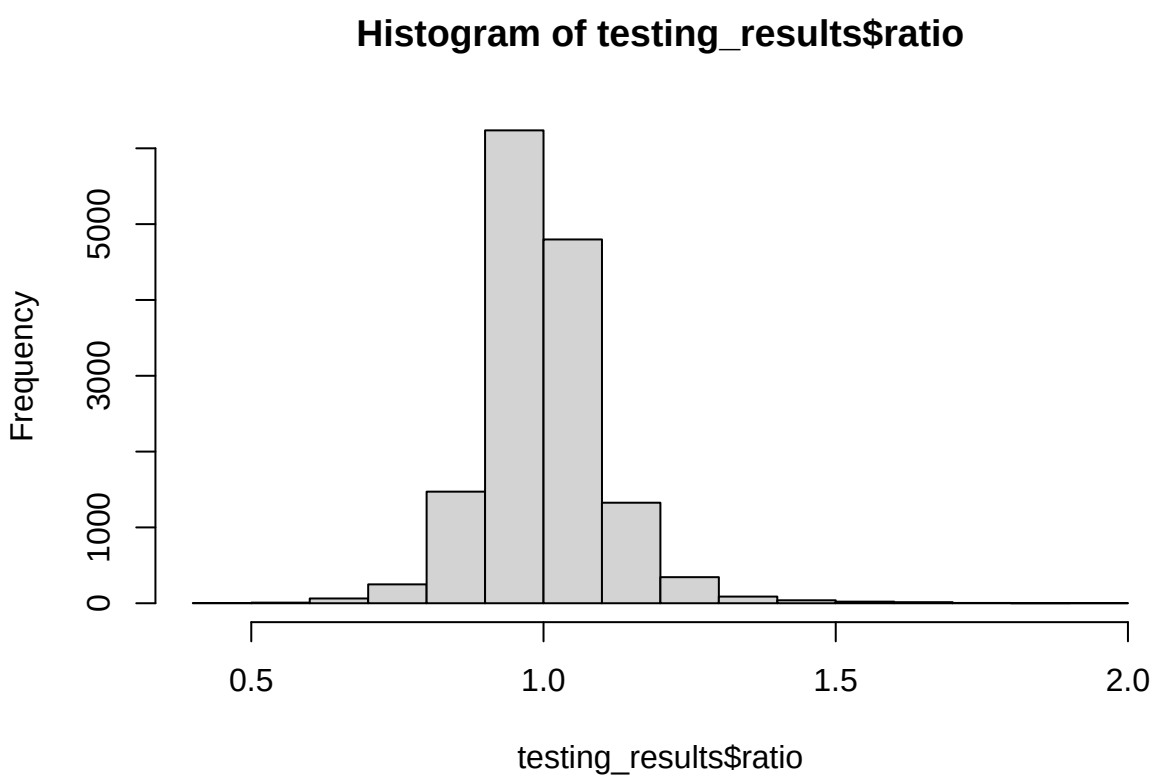

```
# Group prediction results raw v. training error v. testing error.
# metric r = Predicted / Actually
# Looking for the mean closest to 1 with the smallest spread.

# Extract transformed pricing variables
train_baked <- bake(prep(cleaning_recipe), new_data = NULL)

# Fresh = T tell the recipe to apply all the steps again
# training = test_data tell the recipe to treat the test_data
# as if it was the training data used in the preprocessing step
# this avoids the modeling problem of steps having to be skipped
# when the response is absent from the dataset.
```

```
test_baked <- bake(
    prep(cleaning_recipe, fresh = T, training = test_data, retain = T),
    new_data = NULL)

train_price <- train_baked %>% select(price)
test_price <- test_baked %>% select(price)
train_pred <- predict(mlr_fit, new_data = train_data)
test_pred <- predict(mlr_fit, new_data = test_data)

training_results <- cbind(train_price, train_pred)
testing_results <- cbind(test_price, test_pred)

training_results <- training_results %>%
                    mutate(ratio = price / .pred)

testing_results <- testing_results %>%
                    mutate(ratio = price / .pred)
```

```
train_rss <- sum((training_results$price - training_results$.pred)^2)
test_rss <- sum((testing_results$price - testing_results$.pred)^2)

train_n <- nrow(training_results)
test_n <- nrow(testing_results)
model_p <- length(coef(mlr_fit_lm)) - 1 #subtract 1 to ignore the intercept

train_tss <- var(training_results$price) * train_n
test_tss <- var(testing_results$price) * test_n

train_rsq <- 1 - (train_rss / train_tss)
test_rsq <- 1 - (test_rss / test_tss)

train_adj_rsq <- 1 - (1 - train_rsq) * (train_n - 1) /
                 (train_n - model_p - 1)
test_adj_rsq <- 1 - (1 - test_rsq) * (test_n - 1) /
                 (test_n - model_p - 1)

hist(training_results$ratio)
```

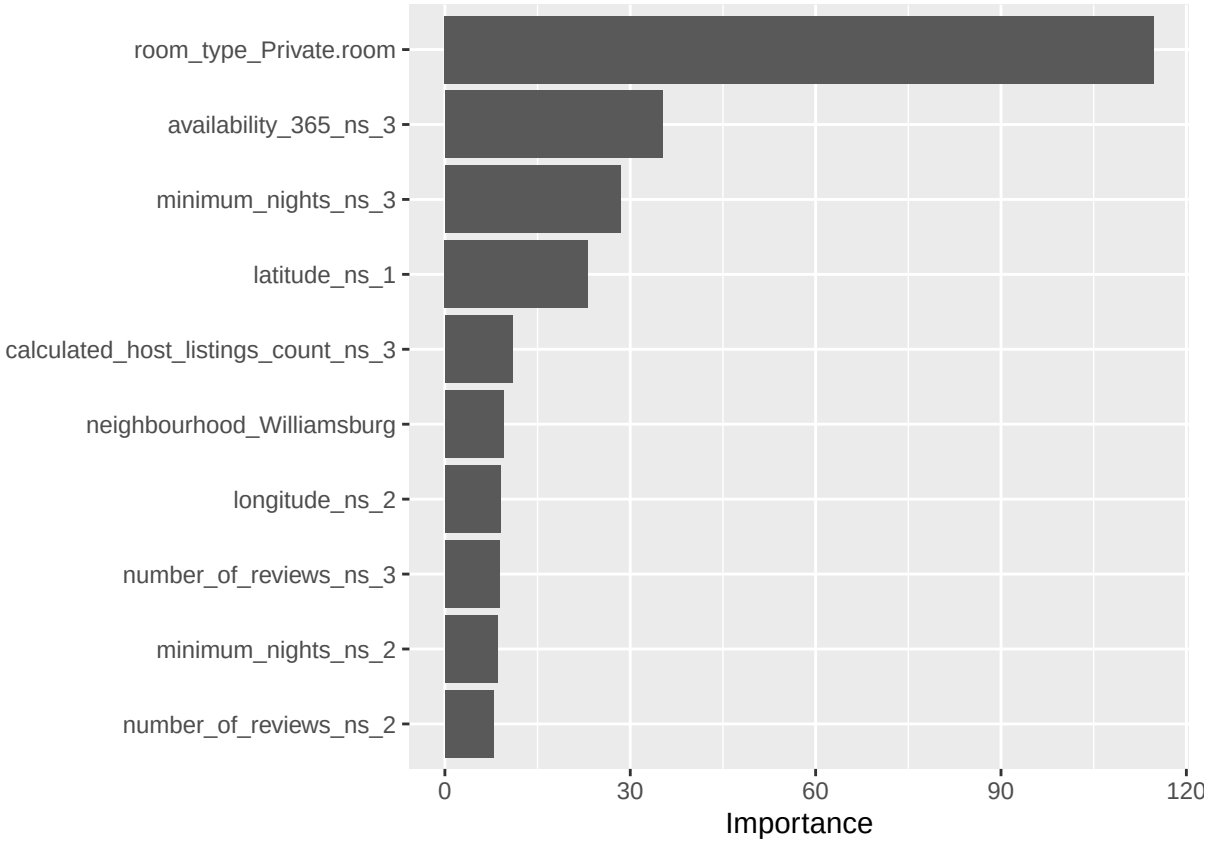



```
hist(testing_results$ratio)
```




```
train_ratio_mean <- mean(training_results$ratio)
train_ratio_sd <- sd(training_results$ratio)
test_ratio_mean <- mean(testing_results$ratio)
test_ratio_sd <- sd(testing_results$ratio)
```

```
library(vip)
mlr_fit %>% extract_fit_parsnip %>% vip()
```


room_type_Private.room
availability_365_ns_3
minimum_nights_ns_3
latitude_ns_1
calculated_host_listings_count_ns_3
neighbourhood_Williamsburg
longitude_ns_2
number_of_reviews_ns_3
minimum_nights_ns_2
number_of_reviews_ns_2
0
30
60
90
120
Importance