# spmoran: An R package for Moran's eigenvector-based spatial regression analysis (version 2017/06)

#### Daisuke Murakami

Center for Global Environmental Research, National Institute for Environmental Studies 16-2 Onogawa, Tsukuba, Ibaraki, Japan

E-mail: murakami\_daisuke\_t@yahoo.co.jp

# **Table of Contents**

1.	Introduction2
2.	Extraction of Moran's eigenvectors
3.	ESF and RE-ESF models5
	3.1 ESF model5
	3.2 RE-ESF model
4.	Extended models9
	4.1 Spatially varying coefficients model
	4.2 Spatially filtered unconditional quantile regression13
5.	Tips for fast computation
	5.1 Eigen-decomposition
	5.2 Parameter estimation
6.	Future directions
Da	foreness 10

# Update: 2017/06

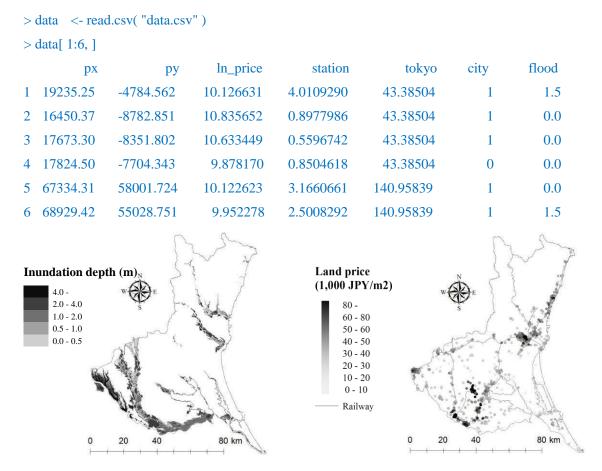
- Functions are modified for large data analysis (in the previous version, an error occurred when the sample size is large).
- Spatially filtered unconditional quantile regression is implemented.
- A bug in the esf function is fixed.

# 1. Introduction

Eigenvector spatial filtering (ESF; e.g., Griffith, 2003), which is also known as Moran's eigenvector mapping (MEM; e.g., Dray et al., 2006), is a regression approach to estimate and infer regression coefficients in the presence of spatial dependence. Recently, ESF is extended to random effects ESF (RE-ESF; Murakami and Griffith, 2015). RE-ESF increases the estimation accuracy of regression coefficients and their standard errors with shorter computational time. RE-ESF is also extended to spatially varying coefficient (SVC) modeling (Murakami et al., 2017). The package "spmoran" provides R functions for fast estimation of ESF and RE-ESF models with/without SVCs.

This tutorial applies ESF and RE-ESF to a land price analysis of flood hazard. The target area is Ibaraki prefecture, Japan. Explained variables are logged land prices in 2015 (JPY/m²; sample size: 647; Figure 1). Explanatory variables are as listed in Table 1. All these variables are downloaded from the National Land Numerical Information download service (NLNI; http://nlftp.mlit.go.jp/ksj-e/index.html).

The following is a data image, in which "px" and "py" are spatial coordinates:



**Figure 1.** Anticipated inundation depth (left) and officially assessed land prices in 2015 (right) in the Ibaraki prefecture

Table 1. Explanatory variables

Variables	Description
tokyo	Logarithm of the distance from the nearest railway station to Tokyo Station [km]
station	Logarithm of the distance to the nearest railway station [km]
flood	Anticipated inundation depth [m]
city	1 if the site is in an urban promotion land and 0 otherwise

ESF/RE-ESF are implemented in the following two steps:

- Extraction of Moran's eigenvectors (see Section 2);
- Parameter estimation of the ESF/RE-ESF model (see Sections 3 and 4).

Sections 2, 3 and 4 explain implementation of these steps, while Section 5 explains how to accelerate the computation.

# 2. Extraction of Moran's eigenvectors

Consider a doubly-centered spatial connectivity matrix, MCM, where C is a symmetric spatial proximity matrix whose diagonals are zeros,  $\mathbf{M} = \mathbf{I} - \mathbf{11}'/N$  is a centering operator, where I is an identity matrix, and 1 is a vector of ones, and N is the sample size. The eigenvectors,  $\mathbf{E} = \{\mathbf{e}_1, ..., \mathbf{e}_N\}$ , of MCM furnish all possible distinct map pattern descriptions of latent spatial dependence, with each level being indexed by the Moran coefficient (MC; Griffith, 2003; Tiefelsdorf and Griffith, 2007). Eigenvectors corresponding to large positive eigenvalue describe map patterns with greater positive spatial dependence (i.e. greater positive MC), whereas eigenvectors corresponding to negative eigenvalue describe map patterns with negative spatial dependence. As positive spatial dependence is dominant in most real-world cases, only eigenvectors with positive eigenvalues are considered in many applied studies.

The function meigen extracts eigenvectors corresponding to positive eigenvalue (i.e.  $\lambda_l > 0$ , where  $\lambda_l$  is the l-th eigenvalue)<sup>1</sup>. The command is as follows:

```
> coords <- data[ ,c( "px", "py" ) ]
> meig <- meigen( coords = coords )</pre>
```

Calculated eigenvectors and eigenvalues are displayed by commanding meig\$sf and meig\$ev, respectively. By default,  $\mathbf{C}$  is given by the matrix whose (i, j)-th element equals  $\exp(-d_{i,j}/r)$ , where

<sup>&</sup>lt;sup>1</sup> For the distance-based C, it is standard to set the threshold by  $\lambda_l > 0$ , which attempts to consider all elements describing positive spatial dependence.

 $d_{ij}$  is the Euclidean distance between sites i and j, and r is the longest distance in the minimum spanning tree covering the sample sites (Dray et al., 2006; Murakami and Griffith, 2015).

The distance-based  $\mathbb{C}$  may be replaced with other types of spatial connectivity matrix. In this case, user must construct the matrix *a priori*. For example, the following command employs the 4-nearest-neighbor-based  $\mathbb{C}$ :

If the spatial connectivity matrix is not symmetric like the 4-nearest neighbor-based  $\mathbb{C}$ , meigen symmetrizes it by taking{ $\mathbb{C} + t(\mathbb{C})$ }/2. In cases with binary connectivity-based  $\mathbb{C}$  (e.g. proximity-based  $\mathbb{C}$ ; k-nearest-neighbor-based  $\mathbb{C}$ ),  $\lambda_l / \lambda_1 > 0.25$  is a standard threshold for the eigenvector extraction<sup>2</sup>. The thresholding is implemented by the following command:

```
> meigB <- meigen( cmat = cmat, threshold = 0.25 )
```

The eigen-decomposition can be very slow for large samples. To accelerate the computation, the function meigen\_f approximates the eigenvectors by applying the Nystrom extension, which is a dimension reduction technique (Murakami and Griffith, 2017)<sup>3</sup>. The command is as follows:

```
> meig_f <- meigen_f( coords = coords )
```

Just like meigen, meig\_f\$sf and meig\_f\$ev return approximated eigenvectors and eigenvalues, respectively. By default, the first 200 eigenvectors are approximated<sup>4</sup>. While meigen takes 243.79 seconds for the exact eigen-decomposition, meigen\_f takes only 0.38 seconds (see Section 5 for further details).

<sup>&</sup>lt;sup>2</sup> The threshold  $\lambda_l / \lambda_1 > 0.25$  attempts to capture roughly 5% of the variance in explained variables attributable to positive spatial dependence (Griffith and Chun, 2014).

<sup>&</sup>lt;sup>3</sup> This approximation is available only for the distance-based **C**.

<sup>&</sup>lt;sup>4</sup> Consideration of 200 eigenvectors is recommended because Murakami and Griffith (2017) show that the approximation error in regression coefficients is quite small when 200 (or more) eigenvectors are considered while the error increases in cases with fewer than 200 eigenvectors.

#### 3. ESF and RE-ESF models

#### 3.1. ESF model

The linear ESF model is formulated as follows:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{E}\boldsymbol{\gamma} + \boldsymbol{\varepsilon}, \qquad \qquad \boldsymbol{\varepsilon} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}),$$

where **E** is a matrix whose *l*-th column is the *l*-th eigenvector,  $\mathbf{e}_l$ , and  $\gamma$  is a vector of coefficients. This model is identical to the standard linear regression model.

The ESF model is estimated using the following steps: (i) eigenvectors whose eigenvalue exceeds a threshold are extracted from **MCM**; (ii) stepwise eigenvector selection is performed; (iii) the ESF model with selected eigenvectors is estimated by ordinary least squares.

The following command estimates the linear ESF model. In step (i), following many ESF studies (Griffith, 2003; Tiefelsdorf and Griffith, 2007), eigenvectors whose eigenvalue fulfills  $\lambda_l/\lambda_1 > 0.25$  are extracted from a binary connectivity-based **C** (4-nearest-neighbor-based **C**; see Section 2):

To cope with possible multicollinearity, eigenvectors are selected so that the variance inflation factor (VIF), which is an indicator of multicollinearity, does not exceed 10. It is implemented by setting vif = 10, whereas VIF is not considered by default. The eigenvector selection is performed by the adjusted  $R^2$  maximization (fn = "r2"; default). Akaike information criterion (AIC) minimization (fn = "aic") or Bayesian Information criterion (BIC) minimization (fn = "bic"). Alternatively, all eigenvectors are considered without selecting them by setting fn = "all".

When fn = "r2", the coefficient estimates yield:

#### $> e_res$b$

	Estimate	SE	t_value	p_value
(Intercept)	9.932080e+00	0.0587240255	169.13146372	0.000000e+00
station	-6.911515e-02	0.0065601988	-10.53552610	5.070594e-24
tokyo	-2.846888e-05	0.0004214075	-0.06755664	9.461599e-01
city	6.738630e-01	0.0360500253	18.69244166	2.121536e-62
flood	2.795299e-02	0.0142681894	1.95911280	5.053884e-02

Station (-) and city (+) are statistically significant at the 0.1% level. It is verified that urban areas

with good access to a railway station have a higher land price than other areas. We can see that flood is positively significant at the 10% level. This result suggests that influence from flood disaster, which is expected to be negative, is not appropriately reflected to land price.

VIF values are displayed by the following command:

#### > e\_res\$vif VIF station 1.367917 tokyo 1.225594 city 1.282930 flood 1.208189 sf4 1.167728 sf9 1.017697 sf12 1.142611 sf31 1.084662 sf33 1.032077 sf45 1.035118 sf32 1.095973 sf26 1.012234 sf6 1.059948

sf20

The following command displays error statistics, including residual standard error (residual\_SE), adjusted  $R^2$  (adjR2), log-likelihood (logLik), AIC, BIC, and degrees of freedom (DF):

#### 

1.016059

While we have discussed ESF with binary connectivity-based C, which is popular in regional science, ESF with distance-based C, which is popular in ecology, is implemented as follows:

```
> meig <- meigen( coords=coords ) #Moran's eigenvectors (distance-based C) 
> e_res <- esf( y=y, x=x, meig=meig, fn = "r2" )
```

The distance-based ESF is often referred to as MEM or a principal coordinate neighborhood matrix (PCNM) (see Legendre and Legendre, 2012).

A major disadvantage of ESF is the computational cost. To cope with this problem, Murakami and Griffith (2017) develops a fact approximation. It is implemented by the following command:

```
> meig_f <- meigen_f( coords = coords )
> e_res <- esf( y = y, x = x, meig = meig_f, fn = "all" )
```

Here, all eigenvectors in meig\_f are considered without selecting them by setting fn = "all". It is sufficient for thousands or more samples (Murakami and Griffith, 2017).

#### 3.2.RE-ESF model

The RE-ESF model is formulated as follows:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{E}\boldsymbol{\gamma} + \boldsymbol{\varepsilon}, \quad \boldsymbol{\gamma} \sim N(\mathbf{0}, \sigma_{\boldsymbol{\gamma}}^2 \boldsymbol{\Lambda}(\boldsymbol{\alpha})), \quad \boldsymbol{\varepsilon} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}).$$

Unlike ESF,  $\gamma$  is given by a vector of random coefficients:  $\gamma \sim N(\mathbf{0}, \sigma_{\gamma}^2 \Lambda(\alpha))$ .  $\Lambda(\alpha)$  is a diagonal matrix whose elements are the eigenvalues, which are multiplied by  $\alpha$ .  $\sigma_{\gamma}^2$  and  $\alpha$  represent the variance and the scale of the spatially dependent component; large  $\alpha$  implies global-scale spatial variation, while small  $\alpha$  implies local variation. These parameters act as shrinkage parameters controlling variance inflation.

The RE-ESF model is estimated using the following steps: (i) eigenvectors whose eigenvalue exceeds a threshold are extracted from MCM; (ii) parameters are estimated by the maximum likelihood (ML) method or the restricted maximum likelihood (REML) method. REML estimation is preferable because it accounts for the degrees of freedom lost by estimating the regression coefficients.

The REML estimation is implemented by the following command:

```
> meig <- meigen( coords = coords ) #Moran's eigenvectors (distance-based C)
> r_res <- resf( y = y, x = x, meig = meig, method = "reml" )
```

ML is implemented by replacing method = "reml" with method = "ml".

#### Estimated coefficients are displayed as follows:

#### $> r_res$b$

	Estimate	SE	t_value	p_value
(Intercept)	9.9902998898	0.169833051	58.8242385	0.000000e+00
station	-0.0792859163	0.009598674	-8.2600901	8.881784e-16
tokyo	-0.0003715008	0.001795810	-0.2068709	8.361807e-01
city	0.6857752216	0.036926493	18.5713608	0.000000e+00
flood	-0.0043670379	0.014784271	-0.2953841	7.678025e-01

Just like the estimates for ESF, station (-) and city (+) are statistically significant, and tokyo is not. In contrast, unlike ESF, flood is not statistically significant. Because RE-ESF tends to outperform ESF in terms of the estimation accuracy of regression coefficients and their standard errors (Murakami and Griffith, 2015), the results of RE-ESF might be more reliable. Error statistics are extracted by the following command:

#### > r\_res\$e

	stat
resid_SE	0.3116825
adjR2(cond)	0.7649824
rlogLik	-262.9627231
AIC	543.9254462
BIC	584.1765628

where adjR2(cond) is the adjusted conditional  $R^2$ , and rlogLik is the restricted log-likelihood. rlogLik is replaced with loglik, which denotes log-likelihood, if method = "ml". It is important to note that, when REML is used, AIC and BIC are comparable only with models with the same explanatory variables. resf also returns the estimated shrinkage parameters as follows:

#### $> r_res$s$

```
par
shrink_sf_SE 0.4337118
shrink_sf_alpha 0.2449076
```

where shrink\_sf\_SE and shrink\_sf\_alpha are  $\sigma_{\gamma}$  and  $\alpha$ , respectively. The standard error of the

spatially dependent component (shrink\_sf\_SE = 0.4337118)<sup>5</sup> is greater than the residual standard error (resid\_SE = 0.3116825). In other words, substantial spatial dependent variations, which are ignored if the linear regression model is estimated, are captured by  $E\gamma$ . shrink\_sf\_alpha is smaller than one. This implies that coefficients on each eigenvector are shrunk comparatively equally, irrespective of their corresponding eigenvalues. The resulting  $E\gamma$  has local-scale spatial variations relative to  $E\gamma$  with large shrink\_sf\_alpha.

resf performs the computationally efficient ML/REML estimation of Murakami and Griffith (2017). The command is as follows:

```
> meig_f <- meigen_f( coords = coords )
> r_res <- resf( y = y, x = x, meig = meig_f, method = "reml" )
```

#### 4. Extended models

#### 4.1. Spatially varying coefficients (SVC) model

Murakami et al. (2017) suggest that a RE-ESF-based SVC modeling outperforms geographically weighted regression (GWR), which is the standard approach for SVC modeling, in terms of coefficient estimation accuracy and computational time.

The RE-ESF-based SVC model is formulated as follows:

$$\mathbf{y} = \Sigma_k \mathbf{x}_k \otimes \mathbf{\beta}_k + \mathbf{E} \mathbf{y} + \mathbf{\varepsilon}, \quad \mathbf{\beta}_k = \beta_{k,0} \mathbf{1} + \mathbf{E} \mathbf{y}_k, \quad \mathbf{y}_k \sim N(\mathbf{0}, \sigma_{y,k}^2 \mathbf{\Lambda}(\alpha_k)), \quad \mathbf{\varepsilon} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}),$$

where  $\beta_k$  is the vector of SVCs on the *k*th explanatory variables,  $\mathbf{x}_k$ .  $\beta_k$  consists of the constant component,  $\beta_{k,0}\mathbf{1}$ , and the spatially varying component,  $\mathbf{E}\gamma_k$ . The latter is modeled by Moran's eigenvectors,  $\mathbf{E}$ , and their random coefficients,  $\gamma_k \sim N(\mathbf{0}, \sigma_{\gamma,k}^2\mathbf{\Lambda}(\alpha_k))$ .  $\mathbf{\Lambda}(\alpha_k)$  is a diagonal matrix whose elements are the eigenvalues, which are multiplied by  $\alpha_k$ .  $\sigma_{\gamma,k}^2$  denotes the variance of the spatially dependent component,  $\mathbf{E}\gamma_k$ , whereas  $\alpha_k$  denotes the spatial scale of the component; large/small  $\alpha_k$  implies global/local-scale spatial variation explained by  $\mathbf{E}\gamma_k$ . These parameters act as shrinkage parameters controlling variance inflation. An interesting point is that, unlike GWR, the RE-ESF-based approach estimates the spatial scale of each SVC using  $\alpha_k$ .

\_

<sup>&</sup>lt;sup>5</sup> The following relationship holds:  $Var[\mathbf{E}\gamma] = \mathbf{E}\gamma\gamma' \mathbf{E}' = \sigma_{\gamma}^{2}\mathbf{E}\Lambda(\alpha) \mathbf{E}' = \sigma_{\gamma}^{2}\hat{\mathbf{C}}_{M}^{\alpha}$ , where  $\hat{\mathbf{C}}_{M}^{\alpha}$  is  $\mathbf{M}\mathbf{C}^{\alpha}\mathbf{M}$  approximated by the eigenvectors in  $\mathbf{E}$ . Hence,  $\sigma_{\gamma}^{2}$  denotes the variance of the spatially dependent component.

In this tutorial, coefficients on station, city, and flood are allowed to vary across geographical space whereas coefficients on Tokyo are not. The command for the SVC modeling is as

The constant coefficient estimate for tokyo is returned by the following command:

```
> rv_res$b

Estimate SE t_value p_value
V1 -0.0009924332 0.001782719 -0.5566962 0.5779578
```

As with the output from the basic RE-ESF model, tokyo is statistically insignificant. Considering computational cost and stability, it might be preferable to employ SVCs on at most around four explanatory variables, and constant coefficients on the other explanatory variables (see Section 5).

Estimated SVCs and their *p*-values are displayed by the following command:

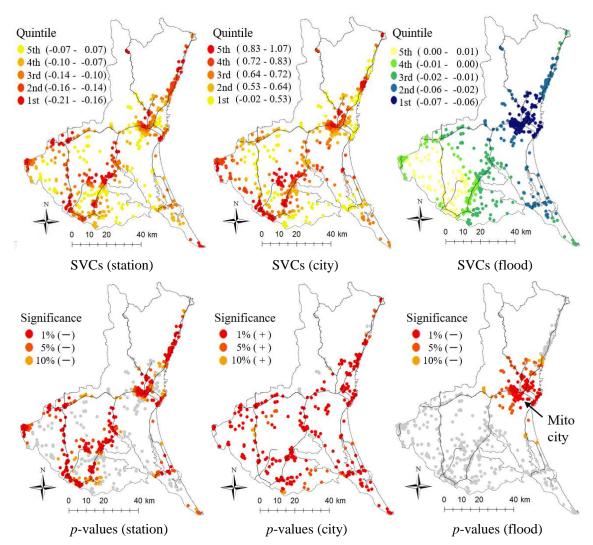
```
> rv_res$b_vc[ 1:6, ]
  (Intercept)
                    station
                                   city
                                                 flood
     9.875385
               -0.06311678
                            0.5690735
                                          0.006637360
   10.278009
               -0.11503321
                             0.8255947
                                          0.005446833
 3 10.173544
               -0.10025270
                            0.7743310
                                          0.006120595
 4 10.138267
               -0.09395701
                             0.7445319
                                         0.006262945
   10.207279
               -0.10122246
                             0.5212322
                                         -0.058020901
   10.258219
               -0.08688370
                             0.5006614
                                         -0.059386765
> rv_res$p_vc[ 1:6, ]
  (Intercept)
                                                    flood
                    station
                                      city
 1
          0
               0.288107321
                             1.324063e-06
                                              0.76500129
 2
          0
               0.006000605
                             6.344747e-11
                                              0.79813709
 3
          0
               0.012536479
                             6.735013e-10
                                              0.77200809
 4
          0
               0.019306686
                             3.431317e-10
                                              0.76597759
 5
          0
               0.058577563
                             5.667240e-05
                                              0.04008828
 6
          0
               0.124107522
                             1.405513e-03
                                              0.15243568
```

They can be summarized as follows:

```
> summary( rv_res$b_vc )
                                                                                   flood
        (Intercept)
                                  station
                                                          city
   Min.:
             8.909
                       Min.:
                               -0.21020
                                             Min.:
                                                     -0.02115
                                                                    Min.:
                                                                              -0.066797
                               -0.15448
                                                      0.57226
  1st Qu.:
             9.831
                      1st Qu.:
                                           1st Qu.:
                                                                   1st Qu.:
                                                                              -0.049578
Median:
                               -0.12184
            10.062
                     Median:
                                          Median:
                                                      0.68319
                                                                  Median:
                                                                              -0.013046
  Mean:
            10.061
                      Mean:
                               -0.11572
                                            Mean:
                                                      0.67039
                                                                   Mean:
                                                                              -0.021668
 3rd Qu.:
            10.242
                               -0.07764
                                                                               0.003591
                     3rd Qu.:
                                           3rd Qu.:
                                                      0.81286
                                                                  3rd Qu.:
   Max.:
            10.946
                       Max.:
                                0.06522
                                             Max.:
                                                      1.06872
                                                                    Max.:
                                                                               0.009442
> summary( rv_res$p_vc )
        (Intercept)
                                                                                   flood
                                  station
                                                          city
                       Min.:
                               0.000001
                                             Min.: 0.0000000
                                                                               0.003934
   Min.:
                 0
                                                                    Min.:
  1st Ou.:
                 0
                      1st Ou.:
                               0.001426
                                           1st Qu.: 0.0000001
                                                                   1st Qu.:
                                                                               0.086623
 Median:
                 0
                    Median:
                               0.010549
                                          Median: 0.0000068
                                                                  Median:
                                                                               0.585556
  Mean:
                 0
                      Mean:
                               0.123792
                                            Mean: 0.0171369
                                                                               0.495177
                                                                   Mean:
 3rd Qu.:
                     3rd Qu.:
                               0.175201
                                           3rd Qu.: 0.0006345
                                                                               0.853780
                 0
                                                                  3rd Qu.:
   Max.:
                 0
                               0.945239
                                             Max.: 0.9582583
                                                                               0.995845
                       Max.:
                                                                    Max.:
```

The result suggests that the spatially varying intercept and SVCs on city are positively significant across the target area. station is negatively significant in many sample sites, and flood is statistically insignificant in most sample sites.

Figure 2 displays the estimated coefficients and their statistical significance. Estimated SVCs on station demonstrate that the distance to a railway station has a significant influence on land price in areas along railways. SVCs on city are positively significant across the target area. SVCs on flood suggest that flood risk is negatively significant around Mito city, which is the prefectural capital. Mito city has a long history as a castle town. The negative sign on flood might be because Mito city has adapted to flood disaster in its long history.



**Figure 2.** Estimated SVCs and their *p*-values (the spatially varying intercept is omitted)

Just like resf, resf\_vc returns shrinkage parameter estimates for SVCs. In our case, the estimates are as follows:

#### > rv\_res\$s

	(Intercept)	station	city	flood
Shrink_sf_SE	0.4562311	0.0820578615	0.30293756	0.04153263
Shrink sf alpha	0 1472938	0.0001043149	0.04450748	1 59444026

Shrink\_sf\_SE summarizes the estimated standard errors,  $\sigma_{\gamma,k}$ , for each SVC, and Shrink\_sf\_alpha summarizes the estimated  $\alpha_k$  parameters. Large  $\alpha_k$  values imply strong shrinkage for local variations. For example, SVCs on flood have a global map pattern due to the large  $\alpha_k$  value while SVCs on station have a local pattern due to the small  $\alpha_k$  value. Thus, the  $\alpha_k$  parameter controls the spatial scale

of the *k*-th SVCs.

Error statistics for the SVC model are displayed by the following command although logLik and AIC are reference values because we apply the REML estimation:

#### $> r_res\$e$

	stat
resid_SE	0.2637017
adjR2(cond)	0.8312410
rlogLik	-230.4469132
AIC	482.8938264
BIC	532.0896357

# 4.2. Spatially filtered unconditional quantile regression (SF-UQR)

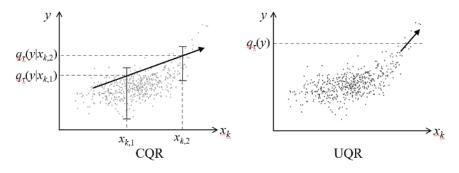
While the conventional conditional quantile regression (CQR) estimates the influence of  $x_k$  on the  $\tau$ -th "conditional" quantile of y,  $q_{\tau}(y|x_k)$ , the unconditional quantile regression (UQR; Firpo et al., 2009) estimates the influence of  $x_k$  on the "unconditional" quantile of y,  $q_{\tau}(y)$ .

Suppose y and  $x_k$  represent land price and accessibility, respectively. UQR estimates the influence of accessibility on land price in each price range. This interpretation does not hold for CQR, because it quantifies the influence of accessibility on land prices conditional on  $x_k$  (see, **Figure 3**). Thus, UQR coefficients are more interpretable than CQR coefficients.

In this context, Murakami and Seya (2017) developed the spatial filter UQR (SF-UQR). The SF-UQR model is formulated as follows:

$$\mathbf{r}_{\tau} = \mathbf{X}\boldsymbol{\beta}_{\tau} + \mathbf{E}\boldsymbol{\gamma}_{\tau} + \boldsymbol{\epsilon}, \quad \boldsymbol{\gamma}_{\tau} \sim N(\mathbf{0}, \, \sigma_{\gamma, \tau}^{2}\boldsymbol{\Lambda}(\alpha_{\tau})), \quad \boldsymbol{\epsilon} \sim N(\mathbf{0}, \, \sigma_{\tau}^{2}\mathbf{I}),$$

where  $\mathbf{r}_{\tau}$  is a vector whose *i*-th element equals the re-centered influence function (RIF) for the *i*-th explained variable,  $y_i$ . The SF-UQR is a UQR considering spatial dependence.



**Figure 3**: CQR and UQR coefficients. Allows illustrate their coefficients on the 0.9 quantile. CQR coefficient equals  $\partial q_r(y|x_k)/\partial x_k$  whereas UQR coefficient equals  $\partial q_r(y)/\partial x_k$ .

The spmoran package provides the resf\_qr function to estimate the SF-UQR model. The command is as follows:

```
> qr_res < -resf_qr(y = y, x = x, meig = meig, boot = T)
```

If boot = T, a semiparametric bootstrapping is performed to estimate the standard errors of UQR coefficients, and these are not calculated if boot = F. This function returns parameters estimated at 0.1, 0.2, ..., 0.9 quantiles by default. The quantile(s) can be specified by using an argument tau; for example, parameters at the 0.22 quantile are estimated by assigning tau = 0.22.

The computational complexity for the bootstrap iterations does not depend on the sample size, N (see, Murakmai and Griffith, 2017), but it depends on the number of eigenpairs, L, which grows as N increases. Hence, for large samples, it is useful to restrict L as follows:

```
> meig <- meig( coords, enum = 200 )
> qr_res <- resf_qr( y = y, x = x, meig = meig, boot = T )
```

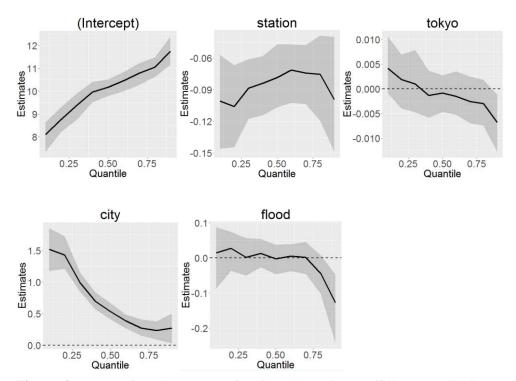
For very large N, which prohibits the eigen-decomposition, the following eigen-approximation would be useful:

```
> meig <- meig_f( coords ) #It approximates the first 200 eignen-pairs by default
```

UQR coefficients estimated by the resf\_qr function can be visualized by the plot\_qr function. The commands to plot estimated coefficients for the first five explanatory variables are as follows:

```
> plot_qr( qr_res, 1 )
> plot_qr( qr_res, 2 )
> plot_qr( qr_res, 3 )
> plot_qr( qr_res, 4 )
> plot_qr( qr_res, 5 )
```

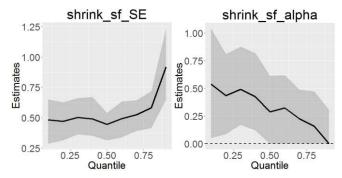
The numbers 1 to 5 specify which regression coefficients are plotted (1: intercept). The resulting plots are as follows:



**Figure 4.** Outputs from the plot\_qr function (regression coefficients). Solid lines are coefficient estimates and gray areas are their 95% confidential intervals.

On the other hand, the standard errors for the residual spatial dependent component (shrink\_sf\_SE) are plotted by assigning pnum = 1 and par = "s", while the scale (degree) parameters for the component (shrink\_sf\_alpha) are plotted by assigning pnum = 2 and par = "s". The commands and the outcomes are as follows:

```
> plot( qr_res, 1, "s" )
> plot( qr_res, 2, "s" )
```



**Figure 5.** Outputs from the plot\_qr function (shrinkage (variance) parameters). Solid lines are coefficient estimates and gray areas are their 95 % confidential intervals.

Parameter estimates are displayed by the following commands:

```
> res$b
> res$s
```

When boot = T, parameter estimates, lower and upper bounds for their 95% confidential intervals, and p-values are returned by the following command:

```
> res$B
```

> res\$S

Error statistics, including the residual standard error and the adjusted quasi conditional  $R^2$ , are displayed as follows:

# > res\$e

	tau=0.1	tau=0.2	tau=0.3	 tau=0.9
resid_SE	0.93164	0.67819	0.58475	 1.0025
quasi_adjR2(cond)	0.43749	0.57931	0.57318	 0.4258

# 5. Tips for fast computation

### 5.1. Eigen-decomposition

As discussed, meigen\_f performs a fact eigen-approximation, and extracts the first 200 eigenvectors by default. The computation is further accelerated by reducing number of approximated eigenvectors. It is achieved by setting enum by a positive integer less than 200. For example, in the case with 5000 samples and enum = 200 (default), 100, and 50, computational times are as follows:

```
------CP time (with approximation) ------
> system.time( meig_test200 <- meigen_f( coords = coords_test )
                     elapsed
   user
          system
   0.37
             0.00
                        0.38
> system.time( meig_test100 <- meigen_f( coords = coords_test, enum = 100 ) )
          system
                     elapsed
   0.15
             0.00
                        0.16
> system.time( meig_test50 <- meigen_f( coords = coords_test, enum = 50 ) )
   user
          system
                     elapsed
   0.08
             0.00
                        0.08
```

Figure 3 maps the calculated 1st, 10th, and 100th eigenvectors. It is important to note that, while approximated and exact eigenvectors can have different map patterns respectively, both of them describe patterns in similar spatial scales. In other words, in both cases, 1st eigenvectors describe global map patterns, 10th medium-scale patterns, and 100th local patterns.

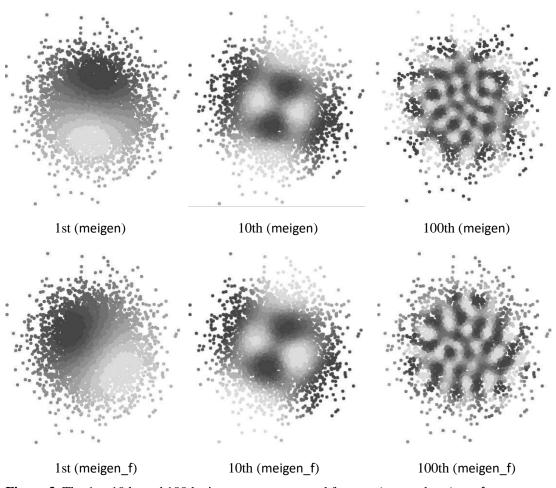


Figure 3. The 1st, 10th, and 100th eigenvectors extracted from meigen and meigen\_f

#### 5.2.Parameter estimation

The basic ESF model is estimated computationally efficiently by setting fn = "all" in the function esf. The RE-ESF model is estimated by small computational cost by the function resf, by default.

The RE-ESF-based SVC model can also be estimated computationally efficiently by the resf\_vc function. To achieve this, (i) the number of eigen-pairs in meig must not be large. It is achieved by setting enum = 200 that is sufficiently small and approximation error is sufficiently small (Murakami and Griffith, 2017). Besides, the number of SVCs must also be small. It is fulfilled by including at most about 4 explanatory variables into x, and the others into xconst. After all, the following command implements the SVC model computationally efficiently:

The SF-UQR model requires a bootstrapping to estimate confidential intervals for the coefficients. However, computational cost for the iteration does not dependent on sample size, but only on the number of eigenvectors in meig (see, Murakami and Seya, 2017). That is, the SF-UQR is applicable to large data if only meig is defined just as mentioned above.

#### 6. Future directions

I plan to enrich functions relating Moran's eigenvector-based regression approach gradually.

# References

- Dray, S., Legendre, P., and Peres-Neto, P.R. (2006) Spatial modelling: a comprehensive framework for principal coordinate analysis of neighbour matrices (PCNM). *Ecological Modelling*, 196 (3), 483-493.
- Firpo, S., Fortin, N.M., and Lemieux, T. (2009) Unconditional quantile regressions. *Econometrica*, 77 (3), 953-973.
- Griffith, D.A. (2003) Spatial autocorrelation and spatial filtering: gaining understanding through theory and scientific visualization. Springer Science & Business Media.
- Griffith, D., & Chun, Y. (2014). Spatial autocorrelation and spatial filtering. In: *Handbook of regional science*, pp. 1477-1507. Springer.
- Legendre, P. and Legendre, L.F. (2012) Numerical Ecology. Elsevier.
- Murakami, D. and Griffith, D.A. (2015) Random effects specifications in eigenvector spatial filtering: a simulation study. *Journal of Geographical Systems*, 17 (4), 311-331.
- Murakami, D. and Griffith, D.A. (2017) Eigenvector spatial filtering for large data sets: fixed and random effects approaches. *Arxiv*, 1702.06220.
- Murakami, D. and Seya, H. (2017) Spatially filtered unconditional quantile regression, Arxiv.
- Murakami, D., Yoshida, T., Seya, H., Griffith, D.A., and Yamagata, Y. (2017) A Moran coefficient-based mixed effects approach to investigate spatially varying relationships. *Spatial Statistics*, 19, 68-89.
- Tiefelsdorf, M., and Griffith, D. A. (2007). Semiparametric filtering of spatial autocorrelation: the eigenvector approach. *Environment and Planning A*, 39 (5), 1193-1221.