1 Chapter 1, differential equations

Differential equations(DEs) are equations in the form of

$$y'' + 2ty' + sin(t)y = ln(t)$$

Basically, an equation that has one or more deratives in it.

we can use the following conventions to write the derative:

$$\begin{aligned} y' &= y \\ y'(x) &= y(x) \\ y'(t) &= y(t) \\ \frac{dy}{dt} &= y(t) \end{aligned}$$

DEs have an order, and the order is the highest derative. Also note that the n'th derative for n¿2 is written like $y^{(n)}$ an example would be the following: $ay^{(5)} + by^{(3)} + y'' + 2y = y$, where the order is 5.

Differential equations can be linear or not. to be linear, a function has to be in the following form:

$$ay'' + by' = cy$$

where a,b,c can be any sin, cos, tan, ln, etc of any variable. As long as the y part is linear, meaning no ln(y), sin(y), y^2, etc

2 Solving DEs

First, we have to determine the type of DE. There are lots, but here are the basic ones:

2.1 Seperable DEs

If we can split the DE in the following form:

$$(ay + by)\frac{dy}{dx} = (at + bt)$$

Basically, if you can seperate the y's and the t's to different sides. The strategy is then to just integrate both sides to y or t (depending on the side)

2.2 Linear DEs

If the equation is or can be written in the standard form

$$\frac{dy}{dt} + p(t)y = g(t)$$

Then we can multiply both sides by $u(t)$, where u(t) is the following:

$$u(t) = e^{\int p(t)dt}$$

Which should deliver

$$u(t)\frac{dy}{dt} + u(t)p(t)y = u(t)g(t)$$

If done correctly, then the following holds true:

$$u(t)\frac{dy}{dt} + u(t)p(t)y = [p(t)y]'$$

The next step is to integrate the right side to t, and bring u(t) y back. like this:

$$u(t)y = \int u(t)g(t)dt$$

The final step is to make it an explicit function of y if asked, and to determine what C is if given a starting value.

2.3 Homogenous DEs

If the equation can be written in the form of

$$\frac{dy}{dx} = f(\frac{y}{x})$$

This is most often done by factoring out an x^n We then substitute every $\frac{y}{x}$ with **v**, and **y'** with **v' x + v** We then make it into a seperable DE with the Vs on one side, and the Xs on the other, integrate both sides and then substitute the v back with $\frac{y}{x}$

2.4 Exact DEs

A DE is exact if we can put it in the following form:

$$Mdx + Ndy = 0$$
$$\text{where } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

When that's done, ***NOTE, ASK JOSH FOR CORRECT NOTES***

2.5 Uniqueness and existence of a solution

How do we know if a DE has a solution, and better yet, that it's unique? For linear DEs, we do the following:

Theorem 1. *given a DE $y' + p(t)y = g(t)$ and a starting value $y(t_0) = y_0$ If p(t) and g(t) are continuous on interval I where $t_0 \in I$, then there is an unique solution in this interval.*

For non linear DEs, we use the following:

Theorem 2. *Given the following: $y' = f(t, y), y(t_0) = y_0$ If f and $\frac{\partial f}{\partial y}$ are continuous in a rectangle $S : \alpha < t < \beta, j < y < k$ where $t_0 \in S$, then there exists a unique solution in some interval I where $t_0 \in I, \alpha < I < \beta$*

3 2nd order DEs

The standard form for these is

$$y" + p(t)y' + q(t)y = g(t)$$

If g(t) = 0, then the DE is said to be homogenous. This should not be confused with homogenous 1st order DEs, where it means something completely different. Anyway, when not homogenous, it is inhomogenous.

To solve homogenous n^{th} order DEs, we do this:

1. From the DE, we derive the characteristic equation:

$$\begin{aligned} ay" + by' + cy &= 0 \\ a\lambda^2 + b\lambda + c &= 0 \end{aligned}$$

2. Solve for λ

$$\begin{aligned} a\lambda^2 + b\lambda + c &= 0 \\ (\lambda + d)(\lambda - f) &= 0 \\ \lambda = -d \vee \lambda &= f \end{aligned}$$

3. from the solutions, we get the solution for y

$$\begin{aligned} \lambda = -d \vee \lambda &= f \\ y(t) = Ae^{-d} + Be^{f} \end{aligned}$$

4. To solve for the unknown constants, we use the starting values.

3.1 Imaginary lambdas

We can get values for lambda such as $\lambda = 1 \pm 2i$. For these, the solution would be

$$\begin{aligned} y(t) &= ae^{1+2i} + be^{1-2i} \\ y(t) &= ae^{1}e^{2i} + be^{1}e^{-2i} \\ y(t) &= ae^{1}cos(2t) + be^{1}sin(2t) \end{aligned}$$

3.2 Repeated roots

When solving for lambda, you could get an equation like this: $(\lambda - 2)^3(\lambda - 5)^2 = 0$ The solution would then be

$$y(t) = Ae^{2t} + Bte^{2t} + Ct^2e^{2t} + De^{5t} + Ete^{5t}$$

3.3 existence and uniqueness for 2nd order linear DEs

Given a 2nd order DE y" + p(t)y' + q(t) y = g(t), then there exists a unique solution if p(t), g(t), g(t) are continuous on interval I where $t_n \in I$