

1 Chapter 11, Sequences

Sequences are a kind of summation, written like

$$\left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty}$$

They can be alternating, meaning that x_n is on the opposite side of x_{n+1} for any n . Sequences are basically an ordered list of computed values, kinda like a function, but discrete instead of possibly continuous. Sequences can also depend on the previous term, which would make them recursive. A popular sequence like this is the fibonacci sequence:

$$x_1 = 0$$

$$x_2 = 1$$

$$x_n = \{x_{n-1} + x_{n+2}\}_{n=1}$$

which gives

$$\{0, 1, 1, 2, 3, 5, 8, \dots\}$$

A sequence $\{a_n\}_{n=0}$ has a limit L and we write

$$\lim_{n \rightarrow \infty} \{a_n = L\}_{n=0}$$

or

$$a_n \rightarrow L \text{ as } n \rightarrow \infty$$

If a sequence has a limit L , it can be said it converges to L , else it diverges. If $\{a_n\}_{n=0}$ converges to L , then each a_{n+1} is closer to L than a_n

Theorem (Th 5).

$$\lim_{n \rightarrow \infty} = \infty$$

for every positive number M , there is an integer N such that if $n > N$, then $a_n > M$

Example 1: $\{2n\}_{n=0}, M = 1000, N = 500$ Dan is er voor elke $M > 1000, N \geq 500$

Theorem (Th 6). *if $\{a_n\}_{n=0}, \{b_n\}_{n=0}$ are convergent sequences and c is a constant, then the following limit laws apply*

$$\begin{aligned}\lim_{n \rightarrow \infty} a_n \pm b_n &= \lim_{n \rightarrow \infty} a_n \pm \lim_{n \rightarrow \infty} b_n \\ \lim_{n \rightarrow \infty} ca_n &= c \lim_{n \rightarrow \infty} a_n \\ \lim_{n \rightarrow \infty} (a_n \cdot b_n) &= \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n \\ \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n} \text{ if } \lim_{n \rightarrow \infty} b_n \neq 0 \\ \lim_{n \rightarrow \infty} (a_n)^p &= \left(\lim_{n \rightarrow \infty} a_n \right)^p \text{ if } n \text{ and } p > 0\end{aligned}$$

All the normal limit laws apply as well, so review anal.

Theorem (Squeeze th.). *Squeeze theorem also applies to sequences. of $a_n \leq b_n \leq c_n$ and $\lim a_n = \lim c_n = L$, then $\lim b_n = L$*

Theorem (Th 6). *If $\lim |a_n| = 0$, then $\lim a_n = 0$*

See examples in notebook.

Theorem (Th 7). *if $\lim a_n = L$, and if $f(x)$ is continuous at L , then $\lim f(a_n) = f(L)$*

Theorem. $\{a_n\}_{n=0}$ *is increasing if $a_n < a_{n+1}$ for all $n > 0$,
is decreasing if $a_n > a_{n+1}$ for all $n > 0$,
is monotonic if it's only decreasing or increasing*

Definition. $\{a_n\}_{n=0}$ *is bounded $\frac{\text{above}}{\text{below}}$ if there is a number M such that a_n is always $\frac{\text{below}}{\text{above}} M$ for all n*

Theorem (Th 12, Monotonic sequence theorem). *Every bounded monotone sequence is convergent*

2 Chapter 11.2, Sequences

A series is just a summation of a sequence. An infinite series is a summation of an infinite sequence, usually noted like

$$\sum_{n=1}^{\infty} a_n \text{ or } \sum a_n$$

Furthermore, we can also have partial sums, denoted as S_n . This is defined as

$$S_n = \sum_{n=1}^n = a_1 + a_2 + \dots + a_n$$

If given a series S_n , we can compute a_n by subtracting S_n with S_{n-1} .

Example:

$$\begin{aligned} S_n &= \frac{n+1}{n+10} \\ S_{n-1} &= \frac{n}{n+9} \\ a_n &= S_n - S_{n-1} \\ &= \frac{n+1}{n+10} - \frac{n}{n+9} \\ &= \frac{9}{(n+9)(n+10)} \end{aligned}$$

We can also compute a_k for any integer constant k , given an S_n by subtracting S_k by S_{k-1} .

Example: Calculate a_7 for the given S_n :

$$\begin{aligned} S_n &= \frac{n+1}{n+10} \\ a_7 &= S_7 - S_6 = \frac{8}{17} - \frac{7}{16} = \frac{9}{272} \end{aligned}$$

Theorem. *given a series $S_n = \sum a_n$, we can compute if it diverges or converges, and where it does so by computing the limit of s_n as n goes to infinity. Or more math-y:*

$$\lim_{n \rightarrow \infty} S_n = L$$

if $L \neq \infty$, S_n converges to L .

2.1 Geometric series

Geometric series are a special kind of series written in the following form:

$$\sum_{n=0}^{\infty} a(r)^k$$

Example:

$$S_n = 8 + \frac{8}{3} + \frac{8}{9} + \frac{8}{27} + \dots S_n = 8\left(\frac{1}{3}\right)^n$$

The special property of these is that for any $r \in (-1, 1)$,

$$\sum_{n=0}^{\infty} a(r)^k = \frac{a}{1-r}$$

If $r \notin (-1, 1)$, then S_n diverges.

Theorem. if $\sum a_n, \sum b_n$ are convergent, then so are the following:

$$\begin{aligned} \sum c_n a_n &= c \sum a_n \\ \sum (a_n + b_n) &= \sum a_n + \sum b_n \end{aligned}$$

2.2 Divergence/convergence tests

There are several tests to see whether a series diverges or not. The first one is the n'th term test. This one does not guarantee that if a series passes this test, that it converges. But it does guarantee that if it fails the test, that it diverges.

Theorem. *Divergence test*

If

$$\lim_{n \rightarrow \infty} a_n \neq 0, \sum_{n=0}^{\infty} a_n$$

will diverge. Furthermore, if $\sum a_n$ converges, $\lim a_n = 0$

This makes sense, because if a_n diverges to a nonzero value, as n approaches infinity, you'd be summing up that value infinity times, which does reach infinity. And if there is no limit, it obviously can't converge.