1 Chapter 11, Sequences

Sequences are a kind of summation, written in the following form:

$$\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$$

They can be alternating, meaning that x_n is on the opposite side of x_{n+1} for any n. Sequences are basically an ordered list of computed values, kinda like a function, but discrete instead of possibly continuous. Sequences can also depend on the previous term, which would make them recursive. A popular sequence like this is the fibionacci sequence:

$$x_1 = 0$$
$$x_2 = 1$$
$$x_n = \{x_{n-1} + x_{n+2}\}_{n=1}$$

which gives

$$\{0, 1, 1, 2, 3, 5, 8, \ldots\}$$

A sequence $\{a_n\}_{n=0}$ has a limit L and we write

$$\lim_{n\to\infty} \{a_n\}_{n=0} = L$$
$$or$$
$$a_n \to L \text{ as } n \to \infty$$

If a sequence has a limit L, it can be said it converges to L, else it diverges. If $\{a_n\}_{n=0}$ converges to L, then each a_{n+1} is closer to L than a_n

Theorem (Th 5)**.**

$$\lim_{n\to\infty} = \infty$$

For every positive number M, there is an integer N such that if $n > N$, then $a_n > M$

Example 1: $\{2n\}_{n=0}, M = 1000, N = 500$ Dan is er voor elke $M > 1000, N \geq 500$

Theorem (Th 6)**.** *If* $\{a_n\}_{n=0}, \{b_n\}_{n=0}$ *are convergent sequences and c is a constant, then the following limit laws apply*

$$\begin{aligned}
\lim_{n\to\infty} a_n \pm b_m &= \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n \\
\lim_{n\to\infty} ca_n &= c \lim_{n\to\infty} a_n \\
\lim_{n\to\infty} (a_n \cdot b_n) &= \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n \\
\lim_{n\to\infty} \frac{a_n}{b_n} &= \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} if \ \lim b_n \neq 0 \\
\lim_{n\to\infty} (a_n)^p &= \left(\lim_{n\to\infty} a_n\right)^p \text{ if n and p } > 0
\end{aligned}$$

All the normal limit laws apply as well, so review ana1.

Theorem (Squeeze th.)**.** *Squeeze theorem also applies to sequences. If* $a_n \leq b_n \leq c_n$ *and* $\lim a_n = \lim c_n = L$*, then* $\lim b_n = L$

Theorem (Th 6)**.** *If lim* $|a_n|$ *= 0, then lim* a_n *= 0*

See examples in notebook.

Theorem (Th 7)**.** *If* $\lim a_n = L$*, and if* $f(x)$ *is continuous at* L*, then lim* $f(a_n) = f(L)$

Theorem. $\{a_n\}_{n=0}$ *is increasing if* $a_n < a_{n+1}$ *for all* $n > 0$*,*
is decreasing if $a_n < a_{n-1}$ *for all* $n > 0$*,*
is monotonic if it's only decreasing or increasing

Definition. $\{a_n\}_{n=0}$ *is bounded* $\frac{above}{below}$ *if there is a number* M *such that* a_n *is always* $\frac{below}{above}M$ *for all* n

Theorem (Th 12, Monotonic sequence theorem)**.** *Every bounded monotone sequence is convergent*

2 Chapter 11.2, Sequences

A series is just a summation of a sequence. An infinite series is a summation of an infinite sequence, usually noted in the following forms:

$$\sum_{n=1}^{\infty} a_n \; or \; \sum a_n$$

Furthermore, we can also have partial sums, denoted as S_n. This is defined as

$$S_n = \sum_{n=1}^{n} = a_1 + a_2 + \ldots + a_n$$

If given a series S_n, we can compute a_n by subtracting S_n with S_{n-1}. Example:

$$\begin{aligned}
S_n &= \frac{n+1}{n+10} \\
S_{n-1} &= \frac{n}{n+9} \\
a_n &= S_n - S_{n-1} \\
&= \frac{n+1}{n+10} - \frac{n}{n+9} \\
&= \frac{9}{(n+9)(n+10)}
\end{aligned}$$

We can also compute a_k for any integer constant k, given an S_n by subtracting S_k by S_{k-1}.

Example: Calculate a_7 for the given S_n:

$$\begin{aligned}
S_n &= \frac{n+1}{n+10} \\
a_7 &= S_7 - S_6 = \frac{8}{17} - \frac{7}{16} = \frac{9}{272}
\end{aligned}$$

Theorem. *Given a series $S_n = \sum a_n$, we can compute if it diverges or converges, and where it does so by computing the limit of s_n as n goes to infinity. Or more math-y:*

$$\lim_{n\to\infty} S_n = L$$

if $L \neq \infty, S_n$ converges to L.

2.1 Geometric series

Geometric series are a special kind of series written in the following form:

$$\sum_{n=0}^{\infty} a(r)^k$$

Example:

$$S_n = 8 + \frac{8}{3} + \frac{8}{9} + \frac{8}{27} + \ldots S_n = 8(\frac{1}{3})^n$$

The special property of these is that for any $r \in < -1, 1 >$,

$$\sum^{\infty} a(r)^k = \frac{a}{1-r}$$

If r $\notin < -1, 1 >$, then S_n diverges.

Theorem. *if $\sum a_n, \sum b_n$ are convergent, then so are the following:*

$$\sum c_a n = c \sum a_n$$
$$\sum (a_n + b_n) = \sum a_n + \sum b_n$$

2.2 Divergence/convergence tests

There are several tests to see whether a series diverges or not. The first one is the n'th term test. This one does not guarantee that if a series passes this test, that it converges. But it does guarantee that if it fails the test, that it diverges.

Theorem. *Divergence test*
If

$$\lim_{n\to\infty} a_n \neq 0, \sum^{\infty} a_n$$

will diverge.Furthermore, if $\sum a_n$ converges, $lim a_n = 0$

This makes sense, because if a_n diverges to a nonzero value, as n approaches infinity, you'd be summing up that value infinity times, which does reach infinity. And if there is no limit, it obviously can't converge.

Theorem. *P-series test*
A series in the form $\sum \frac{1}{n^p}$ *is called a p-series, and converges for all* $p > 1$

Theorem. *Integral test*
If f(n) is a cont, decreasing function on [1, ∞], then $\sum_{n=k}^{\infty} f(n)$ *converges iff* $\int_k^{\infty} f(x)dx$ *converges(returns a real, non infinite number)*
Furthermore, if $R_n = S - S_n$*, if given a continuous, decreasing function f(n) and* $\sum f(n)$ *converges, then*

$$\int_{n+1}^{\infty} f(n)dn \leq R_n \leq \int_{n}^{\infty} f(n)dn$$

Theorem. *Comparison test*
Given an $\sum a_n$ *and* $\sum b_n$ *where* $a_n, b_n > 0$ *and* $a_n < b_n$*, if* b_n *converges,* a_n *converges too.*
Conversely, if a_n *diverges, so does* b_n*.*

Theorem. *Limit comparison test*
Given an a_n, b_n*, and both are positive for all* $n \in \mathbb{N}$
if $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ *where* $L \in \mathbb{R}^+$*, then they both converge or diverge.*

Theorem. *Alternating series test*
Given a series $\sum a_n$ *where* $a_n = (-1)^n \cdot b_n$ *or* $(-1)^{n+1} \cdot b_n$
where $b_n > 0$ *and* $\lim b_n = 0$ *and* b_n *is decreasing,* $\sum a_n$ *converges*

Theorem. *Ratio test*
Given a series $\sum a_n$, if

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = L$$
$$thenif$$
$$L < 1 \to series\ converges$$
$$L > 1 \to series\ diverges$$
$$L = 1 \to inconclusive$$

Conditional and absolute convergence

Given a series $\sum a_n$,
if it converges but $\sum |a_n|$ diverges, we say that it **converges conditionally**.
If $\sum |a_n|$ converges too, we speak of **absolute convergence**.