Novrianti Khairunnisa (1106170)
matematika - C

Tugas 13

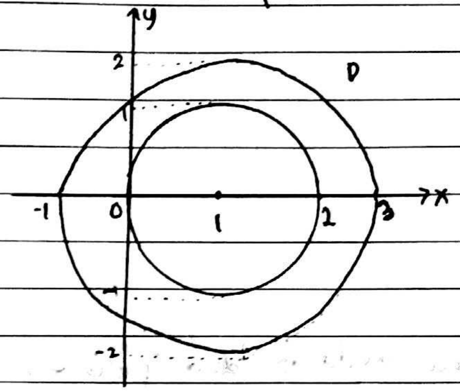
1) Hitunglah $\int_C f(z)\,dz$ dengan orientasi C positif

(a) $f(z) = z^3 - 1$, $C : |z-1| = 1$

Jawab:

$C : |z-1| = 1 \Rightarrow |z-(1+0i)| = 1$

Lingkaran dengan pusat $(1,0)$ dan $r = 1$

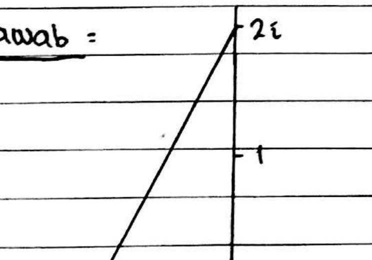


$f(z) = z^3 - 1$ polinomial $\Rightarrow f(z)$ analitik

Karena daerahnya terhubung sederhana dan C lintasan tertutup sederhana, $f(z)$ analitik. Maka berdasarkan Teorema Cauchy - Goursat $\int_C f(z)\,dz = \int_C z^3 - 1 = 0$

(c) $f(z) = \dfrac{z^2}{z-2}$, C : segitiga dengan titik sudut $-1, 0$, dan $2i$

Jawab:



$z = 2$ titik singular

$z = 2 \notin C$

f analitik di dalam dan pada C.

f' kontinu di dalam dan pada C.

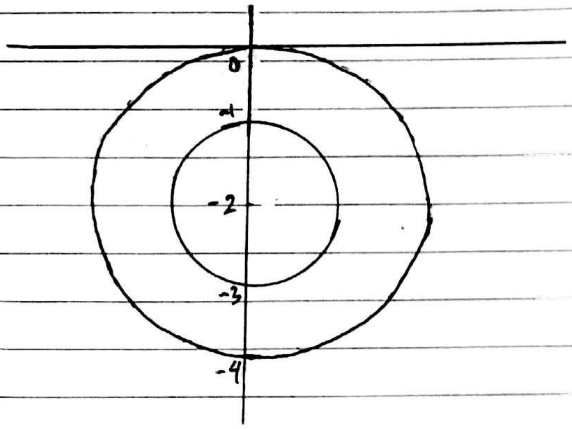
Karena daerahnya terhubung sederhana, dan C lintasan tertutup sederhana, $f(z)$ analitik, maka berdasarkan teorema Cauchy - Goursat, $\int_C f(z)\,dz = \int_C \dfrac{z^2}{z-2}\,dz = 0$

d) $f(z) = \frac{\cos z}{z^3}$, $C : |z+2i| = 1$

Jawab:

$C : |z+2i| = |z-(0-2i)| = 1$

lingkaran dengan pusat $(0,-2)$ dan $r = 1$



$z = 0$ titik singular

$z = 0 \notin C$

$\therefore f$ analitik didalam dan pada C. f' kontinu di dalam dan pada C.

Karena daerahnya terhubung sederhana, dan C lintasan tertutup sederhana, $f(z)$ analitik, maka berdasarkan teorema Cauchy-Goursat, $\oint_C f(z)\,dz$

$$= \oint_C \frac{\cos z}{z^3}\,dz = 0$$

b) $f(z) = \frac{3}{z} - \frac{2}{z-2i}$, $C : |z-2i| = 1$

Jawab:

Misalkan $z = x+iy$

$$\frac{3}{z} = \frac{3}{x+iy} \cdot \frac{x-iy}{x-iy}$$

$$= \frac{3x - i3y}{x^2+y^2}$$

$u(x,y) = \frac{3x}{x^2+y^2}$; $v(x,y) = \frac{-3y}{x^2+y^2}$

$* \; u_x(x,y) = \frac{3(x^2+y^2) - (2x)(3x)}{(x^2+y^2)^2} = \frac{3x^2+3y^2-6x^2}{(x^2+y^2)^2} = \frac{-3x^2+3y^2}{(x^2+y^2)^2}$

$$* \; V_y(x,y) = \frac{-3(x^2+y^2) - (2y)(-3y)}{(x^2+y^2)^2}$$

$$= \frac{-3x^2 - 3y^2 + 6y^2}{(x^2+y^2)^2}$$

$$= \frac{-3x^2+3y^2}{(x^2+y^2)^2}$$

$$* \; U_y(x,y) = \frac{0(x^2+y^2) - (3x)(2y)}{(x^2+y^2)^2}$$

$$= \frac{-6xy}{(x^2+y^2)^2}$$

$$* \; V_x(x,y) = \frac{0(x^2+y^2) - (-3y)(2x)}{(x^2+y^2)^2}$$

$$= \frac{6xy}{(x^2+y^2)^2}$$

Karena $U_x(x,y) = V_y(x,y)$ dan $U_y(x,y) = -V_x(x,y)$, maka PCR berlaku

$\therefore f$ analitik di C

$$\oint_C \frac{3}{z}\,dz = 0 \quad \ldots. (1)$$

Pandang : $\dfrac{2}{z-2i}$

Misalkan $z = x+iy$

$$\frac{2}{z-2i} = \frac{2}{x+iy-2i}$$

$$= \frac{2}{x+i(y-2)} \cdot \frac{x-i(y-2)}{x-i(y-2)}$$

$$= \frac{2x-2i(y-2)}{x^2+(y-2)^2}$$

$$= \frac{2x}{x^2+(y-2)^2} + \frac{-2i(y-2)}{x^2+(y-2)^2}$$

$$u(x,y) = \frac{2x}{x^2+(y-2)^2} \; ; \; v(x,y) = \frac{-2y+4}{x^2+(y-2)^2}$$

$$U_x(x,y) = \frac{2(x^2+(y-2)^2) - 2x(2x)}{(x^2+(y-2)^2)^2}$$

$$= \frac{2(x^2+y^2-4y+4) - 4x^2}{(x^2+(y-2)^2)^2} = \frac{2x^2+2y^2-8y+8-4x^2}{(x^2+(y-2)^2)^2} = \frac{-2x^2+2y^2-8y+8}{(x^2+(y-2)^2)^2}$$

$$u_y(x,y) = \frac{0 - 2x(2(y-2))}{(x^2+(y-2)^2)^2}$$

$$= \frac{-2x(2y-4)}{(x^2+(y-2)^2)^2}$$

$$= \frac{-4xy+8x}{(x^2+(y-2)^2)^2} = \frac{-(4xy-8x)}{(x^2+(y-2)^2)^2}$$

$$v_x(x,y) = \frac{0 - (-2y+4)(2x)}{(x^2+(y-2)^2)^2}$$

$$= \frac{-(-4xy+8x)}{(x^2+(y-2)^2)^2}$$

$$= \frac{4xy-8x}{(x^2+(y-2)^2)^2}$$

$$v_y(x,y) = \frac{-2(x^2+(y-2)^2) - ((-2y+4)\cdot 2(y-2))}{(x^2+(y-2)^2)^2}$$

$$= \frac{-2(x^2+y^2-4y+4) - ((-2y+4)(2y-4))}{(x^2+(y-2)^2)^2}$$

$$= \frac{-2x^2-2y^2+8y-8-(-4y^2+8y+8y-16)}{(x^2+(y-2)^2)^2}$$

$$= \frac{-2x^2-2y^2+8y-8+4y^2-16y+16}{(x^2+(y-2)^2)^2}$$

$$= \frac{-2x^2+2y^2-8y+8}{(x^2+(y-2)^2)^2}$$

Karena $u_x(x,y) = v_y(x,y)$ dan $u_y(x,y) = -v_x(x,y)$, maka PCR berlaku

$\therefore f$ analitik di C

$$\int_C \frac{2}{z-2i}\,dz = 0 \quad \ldots (2)$$

Dari (1) dan (2)

$$\int_C f(z) = \int_C \left(\frac{3}{z} - \frac{2}{z-2i}\right) dz$$

$$= 0 - 0$$

$$= 0$$

2) Jika γ lintasan tertutup dalam C. Tunjukkan $\int_\gamma e^{-z^2}\,dz = 0$

Jawab :

Misalkan : $z = x+iy$

$$f(z) = e^{-z^2}$$
$$= e^{-(x+iy)^2}$$
$$= e^{-(x^2+2xiy-y^2)}$$
$$= e^{(y^2-x^2)+i(-2xy)}$$
$$= e^{(y^2-x^2)} \operatorname{cis}(-2xy)$$
$$= \underbrace{(e^{y^2-x^2}.\cos(2xy))}_{u(x,y)} + i\underbrace{(-e^{y^2-x^2}.\sin(2xy))}_{v(x,y)}$$

$$u(x,y) = e^{y^2-x^2}\cos(2xy)$$
$$u_x = -2x e^{y^2-x^2}\cos(2xy) + e^{y^2-x^2}(2y)(-\sin(2xy))$$
$$= -2e^{y^2-x^2}(x\cos(2xy) + y\sin(2xy))$$

$$u_y = (2y)e^{y^2-x^2}\cos(2xy) + e^{y^2-x^2}(2x)(-\sin(2xy))$$
$$= 2e^{y^2-x^2}(y.\cos(2xy) - x\sin(2xy))$$

$$v(x,y) = -e^{y^2-x^2}.\sin(2xy)$$
$$v_x = (2x)e^{y^2-x^2}\sin(2xy) + (-e^{y^2-x^2})(2y)(\cos.(2xy))$$
$$= 2.e^{y^2-x^2}(x.\sin(2xy) - y.\cos(2xy))$$
$$v_y = (-2y)e^{y^2-x^2}\sin(2xy) + (-e^{y^2-x^2})(2x)(\cos(2xy))$$
$$= -2e^{y^2-x^2}(y.\sin(2xy) + x.\cos(2xy))$$

Karena $z = x+iy \in \mathbb{C}$ dipilih sembarang

1) u, v, u_x, v_x, u_y, v_y kontinu
2) PCR berlaku ($u_x = v_y$ dan $u_y = -v_x$)

Maka f analitik pada C

Sehingga, berdasarkan Teorema Cauchy-Goursat, $\int_c f(z)\,dz = \int_c e^{-z^2}\,dz = 0$

3) Misalkan γ sebarang lintasan tertutup yang tidak memuat nol.

Carilah $\int_\gamma \frac{\sin z}{z^2}\, dz$

Jawab:

$z=0$ titik singular

$z=0 \notin C$

$\therefore$ f analitik di dalam dan pada c. f' kontinu di dalam dan pada c

Karena daerahnya terhubung sederhana, dan c lintasan tertutup sederhana, $f(z)$ analitik, maka berdasarkan teorema Cauchy-Goursat, $\int_c f(z)\, dz = \int_c \frac{\sin z}{z^2}\, dz = 0$