

TI0153 - Trabalho de Conclusão de Curso II
Department of Teleinformatics Engineering
Federal University of Ceará - UFC

Optimal Control: An application to a non-isothermal continuous reactor

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Summary

Introduction

Dynamical System Analysis

State-Feedback Controllers

Optimal Control

Optimal State Estimation

Conclusion



Introduction



- We are discussing the **optimal control** of **dynamical systems**.

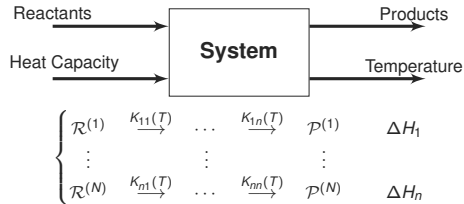
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- We are discussing the **optimal control** of **dynamical systems**.
- We are discussing **chemical reaction network systems**, for processes described as:



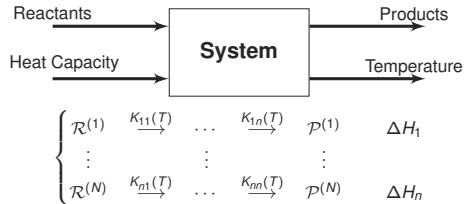
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- ▶ We are discussing the **optimal control** of **dynamical systems**.
- ▶ We are discussing **chemical reaction network systems**, for processes described as:



- ▶ Optimal control theory has been revisited from several innovative fields in the last years^{1,2}. Furthermore, reactor systems are subject of active research, with several open challenges.³

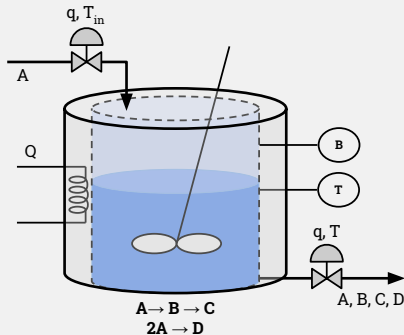
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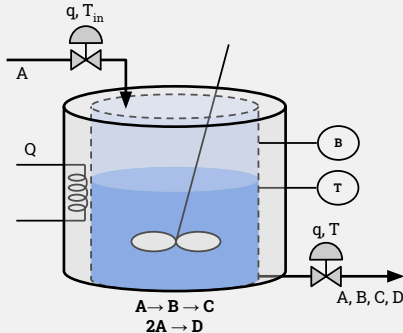


- The system used for the experiments was the *non-isothermal Continuous Stirred Tank Reactor* (CSTR) presented by [Klatt and Engell, 1998]⁴.



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This system...

- class of system that represents a wide range of industrial applications.
- classical benchmark for multiple-input multiple-output (MIMO) control systems.
- nonlinear behavior, models with non-minimum phase behavior and unmeasurable states.

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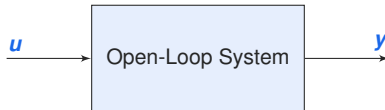
Dynamical System Analysis



We desire to obtain a State-Space mathematical representation of the system.

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)) \end{cases}$$

- $\mathbf{x}(t) \in \mathbb{R}^n$ is the *state-vector*.
- $\mathbf{u}(t) \in \mathbb{R}^r$ is the *input-vector*.
- $\mathbf{y}(t) \in \mathbb{R}^p$ is the *output-vector*.
- $\mathbf{f}(\cdot) : \mathbb{R}^{n \times r} \rightarrow \mathbb{R}^n$ is a *state-transition function*.
- $\mathbf{g}(\cdot) : \mathbb{R}^{n \times r} \rightarrow \mathbb{R}^p$ is a *output observation function*.



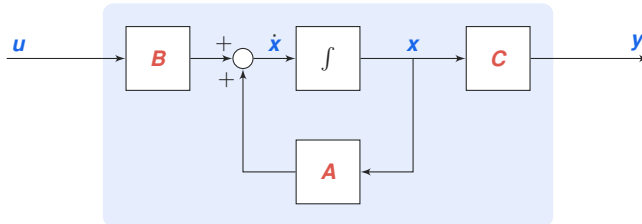


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$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

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- $\mathbf{u}(t) \in \mathbb{R}^r$ is the *input-vector*.
- $\mathbf{y}(t) \in \mathbb{R}^p$ is the *output-vector*.
- $\mathbf{A} \in \mathbb{R}^{n \times n}$ is the *system matrix*.
- $\mathbf{B} \in \mathbb{R}^{n \times r}$ is the *input matrix*.
- $\mathbf{C} \in \mathbb{R}^{p \times n}$ is the *output matrix*.
- $\mathbf{D} \in \mathbb{R}^{p \times r}$ is the *feedthrough matrix*.

Open-Loop System





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This representation has several advantages.

- The *response* of the system has an analytical solution:

$$\begin{cases} \mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \\ \mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \mathbf{C}\int_{t_0}^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t) \end{cases}$$



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- The *stability* of the model is directly related to the eigenvalues (λ) of \mathbf{A} .

$$\|\mathbf{x}(t)\| < \infty, \quad t \rightarrow \infty \quad \text{if } \operatorname{Re}[\lambda_i] < 0, \quad \forall i \in [1, n]$$



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- The *controllability* property is directly related to the *Controllability Matrix*:

$$\mathbf{C} = [\mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \mathbf{A}^2\mathbf{B} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}]$$



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- The *observability* property is directly related to the *Observability Matrix*:

$$\mathcal{O} = [\mathbf{C} \quad \mathbf{C}\mathbf{A} \quad \mathbf{C}\mathbf{A}^2 \quad \dots \quad \mathbf{C}\mathbf{A}^{n-1}]^T$$

Linearization by Taylor Series Expansion

Consider a nonlinear time-invariant system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)) \end{cases}.$$

Given steady-state operating points \mathbf{x}_o , \mathbf{y}_o and \mathbf{u}_o , the dynamics of the system in the neighborhood of these points can be represented by the linear model:

$$\begin{cases} \Delta \dot{\mathbf{x}}(t) = \mathbf{A} \Delta \mathbf{x}(t) + \mathbf{B} \Delta \mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C} \Delta \mathbf{x}(t) + \mathbf{D} \Delta \mathbf{u}(t) \end{cases},$$

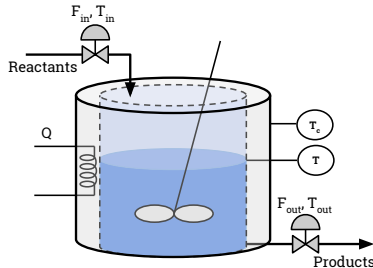
where

$$\mathbf{A} = \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|_{\mathbf{x}_o, \mathbf{u}_o}; \quad \mathbf{B} = \left. \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \right|_{\mathbf{x}_o, \mathbf{u}_o}; \quad \mathbf{C} = \left. \frac{\partial \mathbf{g}}{\partial \mathbf{x}} \right|_{\mathbf{x}_o, \mathbf{u}_o}; \quad \mathbf{D} = \left. \frac{\partial \mathbf{g}}{\partial \mathbf{u}} \right|_{\mathbf{x}_o, \mathbf{u}_o}$$

and

$$\Delta \mathbf{x}(t) = \mathbf{x}(t) - \mathbf{x}_o; \quad \Delta \mathbf{u}(t) = \mathbf{u}(t) - \mathbf{u}_o.$$

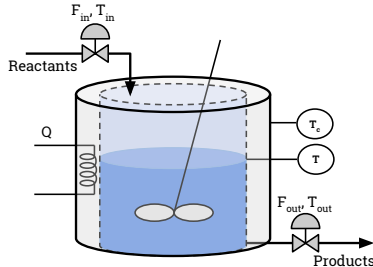
- The first principle models are obtained through Conservation Laws from physics.



Mass Balance for Chemical Compounds

$$\left(\begin{array}{c} \text{Accumulation} \\ \text{of mass} \\ \text{in the system} \end{array} \right) = \left(\begin{array}{c} \text{Mass flow} \\ \text{entering} \\ \text{system} \end{array} \right) - \left(\begin{array}{c} \text{Mass flow} \\ \text{leaving} \\ \text{system} \end{array} \right) \pm \left(\begin{array}{c} \text{Mass flow} \\ \text{from} \\ \text{reactions} \end{array} \right)$$

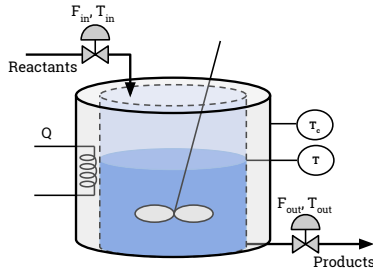
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Mass Balance for Chemical Reactors

$$\frac{d(\rho A)}{dt} = q(\rho_{in}^{(A)} - \rho_{out}^{(A)}) + \left(\sum_{\alpha X \rightarrow \beta A} \frac{1}{\beta} K_{XA}(T) (\rho_X)^\alpha \right) - \left(\sum_{\alpha A \rightarrow \beta X} \frac{1}{\beta} K_{AX}(T) (\rho_A)^\alpha \right)$$

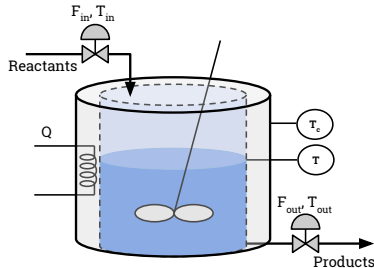
- The first principle models are obtained through Conservation Laws from physics.



Conservation of Energy for Chemical Reactors

$$\left(\begin{array}{c} \text{Accumulation} \\ \text{of thermal energy} \\ \text{in the system} \end{array} \right) = \left(\begin{array}{c} \text{Heat flow} \\ \text{entering} \\ \text{the system} \end{array} \right) - \left(\begin{array}{c} \text{Heat flow} \\ \text{leaving} \\ \text{the system} \end{array} \right) \pm \left(\begin{array}{c} \text{Entropy} \\ \text{contribution} \\ \text{from reactions} \end{array} \right)$$

- The first principle models are obtained through Conservation Laws from physics.



Conservation of Energy for Chemical Reactors

$$\begin{cases} \frac{d(T)}{dt} = q(T_{in} - T_{out}) + \eta(T_C - T) + \delta \sum_{\alpha A \rightarrow \beta X} K_{AX}(T)(\rho_A)^\alpha \Delta H_{AX} \\ \frac{d(T_C)}{dt} = \gamma Q + \beta(T - T_C) \end{cases}$$



- In the case of the reactor system in discussion, the model becomes...

Mathematical Model of Non-Isothermal CSTR

$$\left\{ \begin{array}{l} \frac{d(\rho_A)}{dt} = q(\rho_{in}^{(A)} - \rho_A) - (K_1(T)\rho_A + K_3(T)\rho_A^2) \\ \frac{d(\rho_B)}{dt} = -q\rho_B + K_1(T)\rho_A - K_2(T)\rho_B \\ \frac{d(T)}{dt} = q(T_{in} - T) + \frac{k_W A_r}{\varrho C_p V_r}(T_C - T) \\ \quad - \frac{1}{\varrho C_p} (K_1(T)\rho_A \Delta H_{AB} + K_2(T)\rho_B \Delta H_{BC} + K_1(T)\rho_A^2 \Delta H_{AC}) \\ \frac{d(T_C)}{dt} = \frac{1}{m_K C_{pK}} Q + \frac{k_W A_r}{m_K C_{pK}} (T - T_C) \end{array} \right.$$



- We choose $\mathbf{x} = [\rho_A, \rho_B, T, T_C]^T$, $\mathbf{u} = [q, Q]^T$ and $\mathbf{y} = [\rho_B, T]^T$.
- We consider the steady-state point $\mathbf{x}_o = [1.23, 0.90, 134.14, 128.95]^T$ and $\mathbf{u}_o = [18.83, -4495.7]^T$.

The linearized state-space model is described by matrices:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} -86.1 & 0 & -4.2 & 0 \\ 50.6 & -69.4 & 1.0 & 0 \\ 172.2 & 198.0 & -36.7 & 30.8 \\ 0 & 0 & 86.8 & -86.7 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 3.9 & 0 \\ -0.9 & 0 \\ -4.1 & 0 \\ 0 & 0.1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \\ \mathbf{y} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \end{cases}$$



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- All the eigenvalues are real and negative. The system is **stable**.

$$\lambda = [-16.79, -54.84, -86.33, -121.01]$$



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- The Controllability Matrix has full-row rank. The system is **controllable**.

$$\text{rank}(\mathcal{C}) = \text{rank} \left(\begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \mathbf{A}^3\mathbf{B} \end{bmatrix} \right) = 4$$



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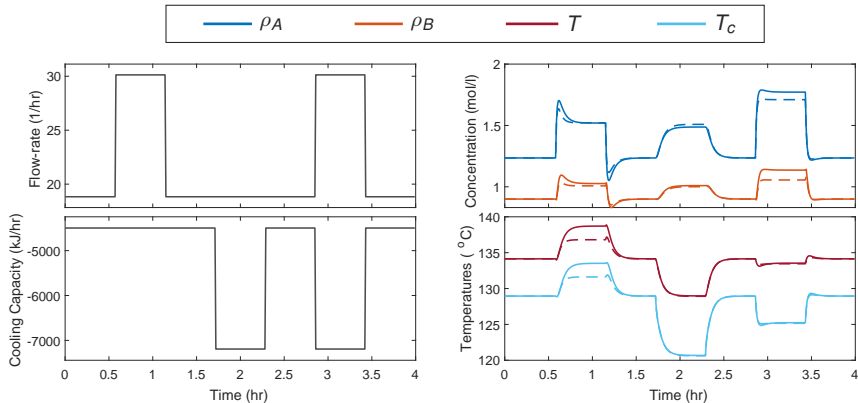
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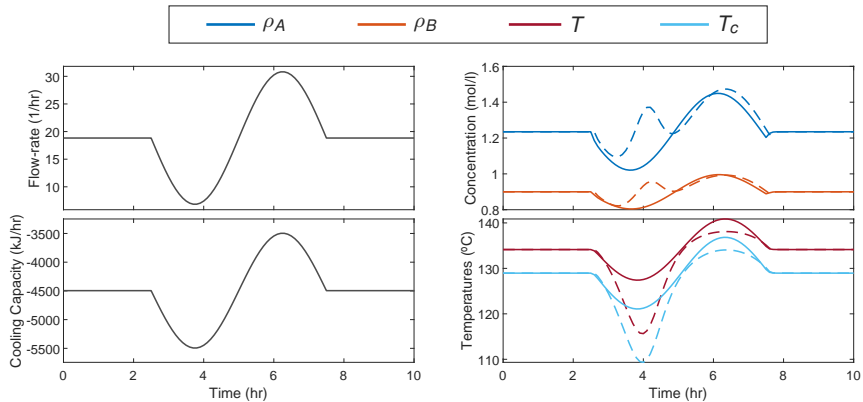
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- The Observability Matrix has full-column rank. The system is **observable**.

$$\text{rank}(\mathcal{O}) = \text{rank} \left(\begin{bmatrix} \mathbf{C} & \mathbf{CA} & \mathbf{CA}^2 & \mathbf{CA}^3 \end{bmatrix} \right) = 4$$







State-Feedback Controllers



Full State-Feedback Controller

The input action $\mathbf{u}(t)$ is calculated by the linear control law $\pi(\cdot)$ through state-feedback as:

$$\mathbf{u}(t) = \pi(\mathbf{r}(t), \mathbf{x}(t)) = \mathbf{r}(t) - \mathbf{K}\mathbf{x}(t),$$

where $\mathbf{r} : \mathbb{R} \rightarrow \mathbb{R}^n$ is a state reference signal and $\mathbf{K} \in \mathbb{R}^{r \times n}$ is the *feedback gain matrix*.

- Substituting this control law to the model results in the closed-loop representation:

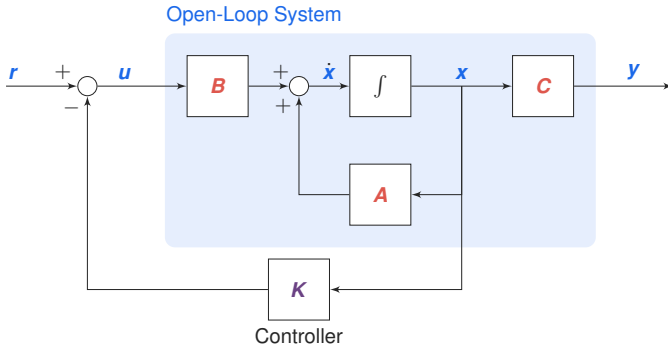
$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}(\mathbf{r}(t) - \mathbf{K}\mathbf{x}(t)) \\ &= (\mathbf{A} - \mathbf{BK})\mathbf{x}(t) + \mathbf{B}\mathbf{r}(t) \\ &= \mathbf{A}_{cl}\mathbf{x}(t) + \mathbf{B}\mathbf{r}(t)\end{aligned}$$

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$$u(t) = \pi(r(t), x(t)) = r(t) - Kx(t),$$

where $r : \mathbb{R} \rightarrow \mathbb{R}^n$ is a state reference signal and $K \in \mathbb{R}^{r \times n}$ is the *feedback gain matrix*.





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- **Pole-Placement Property:** If (\mathbf{A}, \mathbf{B}) is *controllable*, we can assign the eigenvalues of \mathbf{A}_{cl} anywhere in the complex plane by means of designing \mathbf{K} .



We discuss the two modes of operation for state-feedback controllers.

Regulation

- ▶ **Objective:** maintain the system in some position (the zero-state) and reject disturbances.
- ▶ Special case of the state-feedback when $r(t) = 0$.
- ▶ Direct from the controller formulation.



We discuss the two modes of operation for state-feedback controllers.

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- ▶ Special case of the state-feedback when $r(t) = 0$.
- ▶ Direct from the controller formulation.

Reference Tracking

- ▶ **Objective:** drive the states of the system towards a non-constant reference signal.
- ▶ Special case of the state-feedback when $r(t)$ is non-constant.
- ▶ Raises the need for *integral action*.



We discuss the two modes of operation for state-feedback controllers.

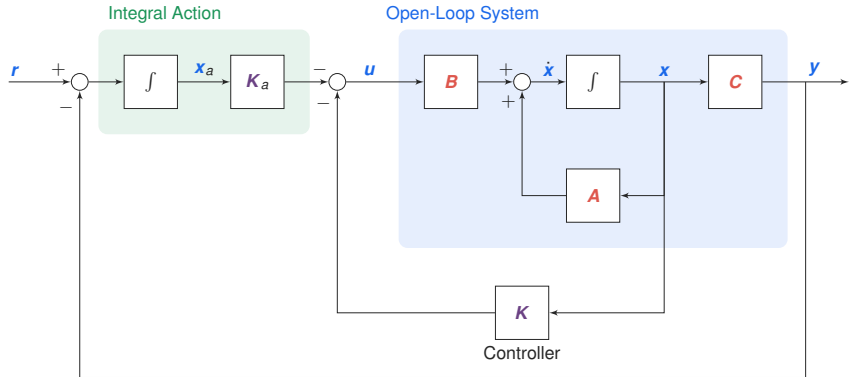
State-Feedback with Integral Action

Consider a State-space system and augmented state $\mathbf{x}_a : \mathbb{R} \rightarrow \mathbb{R}^p$ defined by:

$$\mathbf{x}_a(t) = \int_0^t (\mathbf{r}(\tau) - \mathbf{y}(\tau)) d\tau \implies \dot{\mathbf{x}}_a(t) = \mathbf{r}(t) - \mathbf{y}(t).$$

A robust tracking (or servo) controller, defined by the gain $\tilde{\mathbf{K}} = [\mathbf{K} \quad \mathbf{K}_a]$, for $\mathbf{K} \in \mathbb{R}^{r \times n}$ and $\mathbf{K}_a \in \mathbb{R}^{r \times p}$, operates on the following augmented version of the original system:

$$\begin{cases} \begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{x}}_a(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & -\mathbf{BK}_a \\ -\mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}_a(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_p \end{bmatrix} \mathbf{r}(t) \\ \mathbf{y}(t) = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}_a(t) \end{bmatrix} \end{cases}.$$





Optimal Control



Optimal Controller - General Formulation

The input action $\mathbf{u}(t) \in \mathbb{R}^r$ is *optimal* if an *optimal control law* $\pi^* : \mathbb{R}^{n \times n \times 1} \rightarrow \mathbb{R}^r$ can be found as:

$$\mathbf{u}(t) = \pi^*(\mathbf{x}, \mathbf{r}, t) = \arg \min_{\mathbf{u}} J(\mathbf{x}, \mathbf{r}, t),$$

where $J : \mathbb{R}^{n \times n \times 1} \rightarrow \mathbb{R}$ is known as the *cost function*.

- **This is still a state-feedback controller:** the cost function is a function of past values of the state and reference signals.



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- ▶ **This is still a state-feedback controller:** the cost function is a function of past values of the state and reference signals.
- ▶ Reduces the need for hand-designing the feedback gain or any other control law for the system, given some specifications.
- ▶ The type of the optimal controller depends on the choice of $J(\cdot)$ and (possibly) on how the optimization is solved.



Finite-Horizon Optimal Regulators

A *Finite-Horizon Optimal Regulator* is defined as any controller whose optimal policy over a time interval $t \in [t_0, T]$ minimizes the cost functional:

$$J(\mathbf{x}, \mathbf{u}, t_0) = \int_{t_0}^T l(\mathbf{x}, \mathbf{u}, \tau) d\tau + l_f(\mathbf{x}, T),$$

where $l(\cdot) : \mathbb{R}^{n \times r \times 1} \rightarrow \mathbb{R}$ and $l_f(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$ are, respectively, the *trajectory* and *terminal loss functions*. In the case that $t_0 = 0$, T is also known as the *control horizon*.

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- If $l(\mathbf{x}, \mathbf{u}, \tau) \geq 0$ and $l_f(\mathbf{x}, T) \geq 0, \forall \mathbf{x}, \mathbf{u}, \tau$, then the cost function penalizes deviations from the zero-state.



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- Characterizes an entire class of optimal controllers based on finite horizon optimization.



Hamilton-Jacobi-Bellman (HJB) equation

Consider a finite-horizon cost function for a system described by $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$:

$$V(\mathbf{x}, \mathbf{u}, t_0) = \int_{t_0}^T l(\mathbf{x}, \mathbf{u}, \tau) d\tau + l_f(\mathbf{x}(T))$$

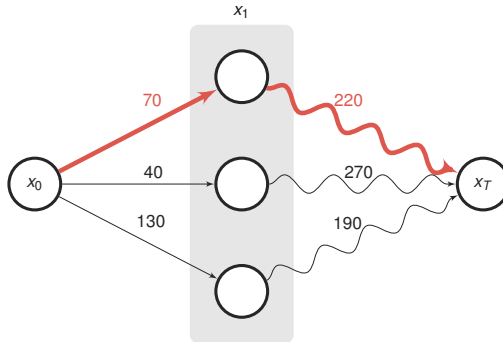
Minimizing any functional in the form of $V(\cdot)$ is equivalent to determining the solution of the *Hamilton-Jacobi equation**:

$$\frac{\partial V^*}{\partial t} = - \min_{\mathbf{u}(t)} \left[l(\mathbf{x}, \mathbf{u}, t) + \left[\frac{\partial V^*}{\partial \mathbf{x}} \right]^T \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \right]$$

and the boundary condition:

$$V^*(\mathbf{x}, T) = l_f(\mathbf{x}(T)).$$

- ▶ The HJB equation provides a *sufficient condition* for finite-horizon optimal controllers.
- ▶ This class of optimal controllers is related to the *Bellman's Principle of Optimality*.





Linear Quadratic Regulator (LQR)

Given a linear State-Space system in the form:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}.$$

A *Linear Quadratic Regulator* (LQR) for this system is an optimal controller defined by the quadratic cost function:

$$J(\mathbf{x}, \mathbf{u}, t_0) = \int_{t_0}^T \left(\mathbf{x}^T \mathbf{Q} \mathbf{x} + \mathbf{u}^T \mathbf{R} \mathbf{u} \right) dt + \mathbf{x}^T(T) \mathbf{Q}_f \mathbf{x}(T),$$

where is assumed that $\mathbf{Q}, \mathbf{Q}_f \succ 0$ and $\mathbf{R} \succ 0$.

- Defined for a *linear* system with *quadratic* cost function.

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- Defined for a *linear* system with *quadratic* cost function.
- The weights \mathbf{Q} , \mathbf{Q}_f and \mathbf{R} are chosen to reflect how the designer wish to penalize state-deviation or control effort.
- **Optimization:** closed-form solution for $\pi^*(\mathbf{x}, \mathbf{u}, t_0)$.

LQR Control Action from Dynamic Programming

Given a Linear Quadratic Regulator, the optimal action produced is given by:

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1} \mathbf{B}^T \mathbf{P}(t) \mathbf{x}(t),$$

where $\mathbf{P}(t)$ is the solution of the matrix Riccati differential equation:

$$-\dot{\mathbf{P}}(t) = \mathbf{A}^T \mathbf{P}(t) + \mathbf{P}(t) \mathbf{A} - \mathbf{P}(t) \mathbf{B} \mathbf{R}^{-1} \mathbf{B}^T \mathbf{P}(t) + \mathbf{Q},$$

with terminal condition $\mathbf{P}(T) = \mathbf{Q}_f$.

► Notice that:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A} \mathbf{x}(t) + \mathbf{B} \mathbf{u}^*(t) \\ &= \mathbf{A} \mathbf{x}(t) + \mathbf{B} (-\mathbf{R}^{-1} \mathbf{B}^T \mathbf{P}(t) \mathbf{x}(t)) \\ &= (\mathbf{A} - \mathbf{B} \mathbf{R}^{-1} \mathbf{B}^T \mathbf{P}(t)) \mathbf{x}(t) \\ &= (\mathbf{A} - \mathbf{B} \mathbf{K}(t)) \mathbf{x}(t) \end{aligned}$$



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- The optimization results in a *linear* and *time-varying* control law:

$$\mathbf{u}^*(t) = \pi^*(\mathbf{x}, \mathbf{u}, t) = \mathbf{r}(t) - \mathbf{K}(t) \mathbf{x}(t)$$



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- **Remark:** nevertheless, the optimal controller performs corrective action.

LQR with Integral Action

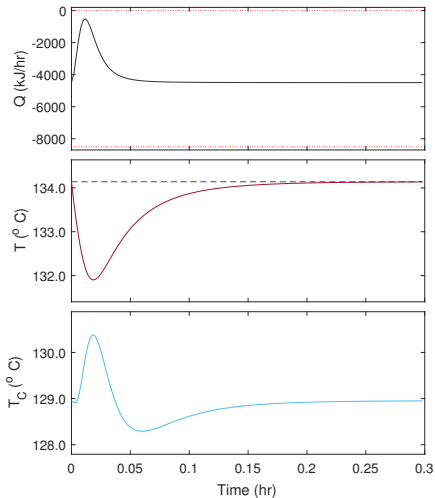
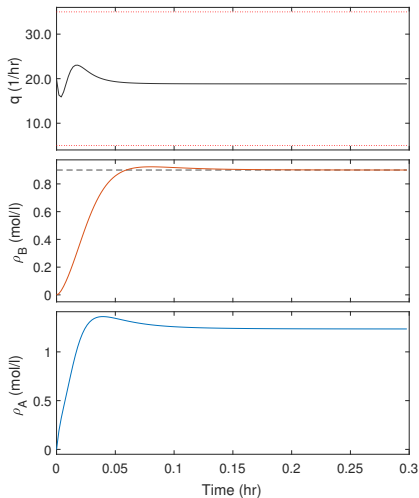
Given a linear State-Space system augmented with state $\dot{\mathbf{x}}_a(t) = \mathbf{r}(t) - \mathbf{C}\mathbf{x}(t)$:

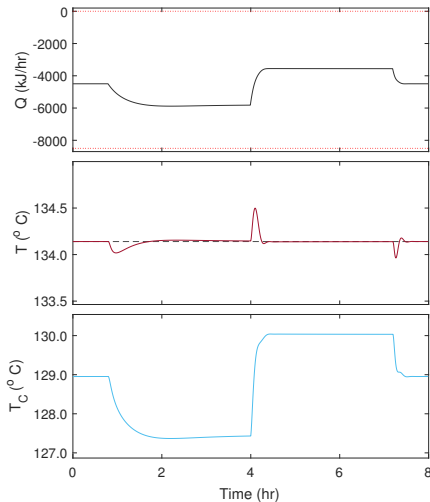
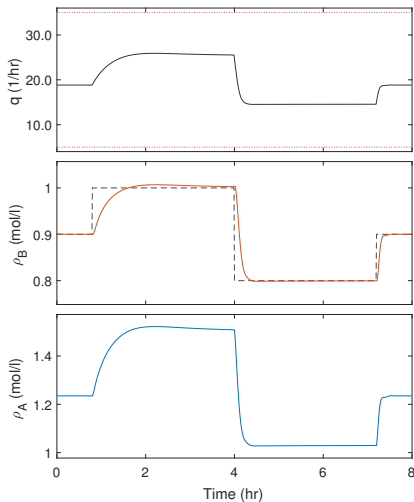
$$\begin{cases} \begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{x}}_a(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & \mathbf{0} \end{bmatrix}}_{\tilde{\mathbf{A}}} \underbrace{\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}_a(t) \end{bmatrix}}_{\tilde{\mathbf{x}}(t)} + \underbrace{\begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix}}_{\tilde{\mathbf{B}}} \mathbf{u}(t) + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \mathbf{r}(t) \\ \mathbf{y}(t) = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}_a(t) \end{bmatrix} \end{cases}$$

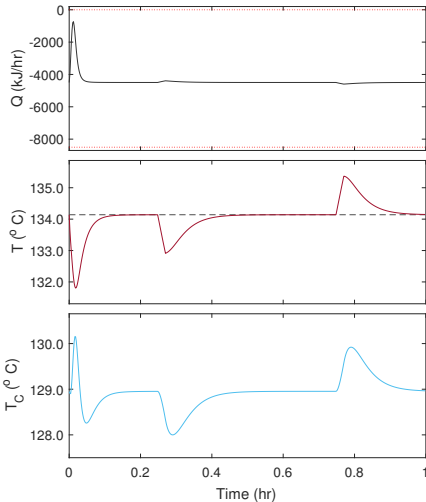
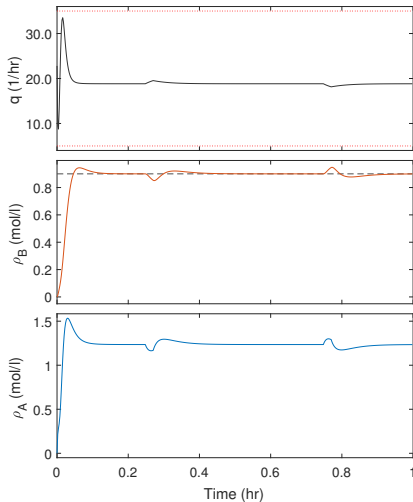
A *Linear Quadratic Servo* (LQ-Servo) for this system is an optimal controller defined by the quadratic cost function:

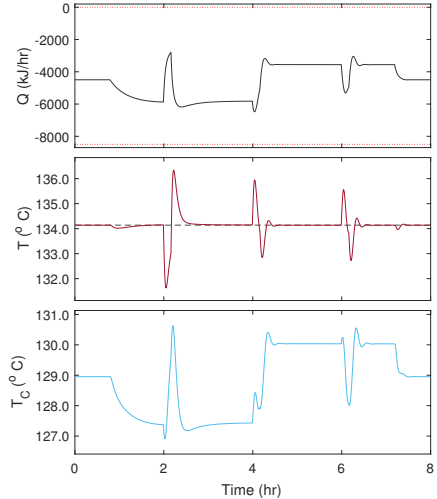
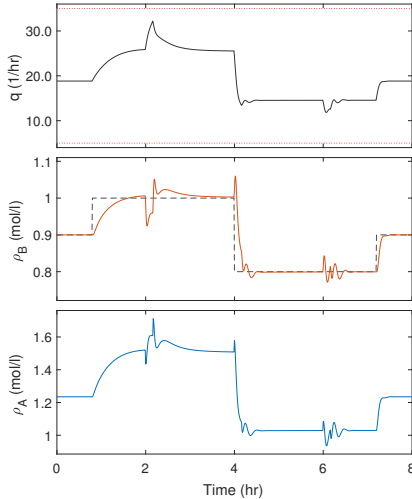
$$J(\mathbf{x}, \mathbf{u}, t_0) = \int_{t_0}^T \left(\tilde{\mathbf{x}}^T \tilde{\mathbf{Q}} \tilde{\mathbf{x}} + \mathbf{u}^T \mathbf{R} \mathbf{u} \right) dt + \tilde{\mathbf{x}}^T \tilde{\mathbf{Q}}_f \tilde{\mathbf{x}}(T),$$

where is assumed that $\tilde{\mathbf{Q}}, \tilde{\mathbf{Q}}_f \succ 0$ and $\mathbf{R} \succ 0$.







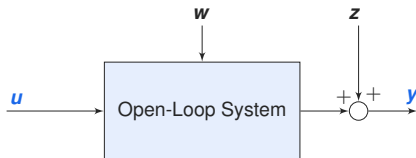




Optimal State Estimation

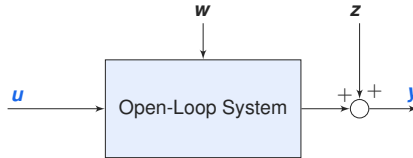


- We can **never** access real state of a physical system.





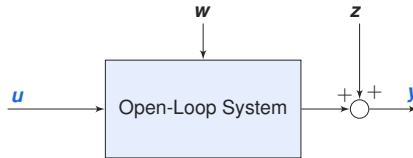
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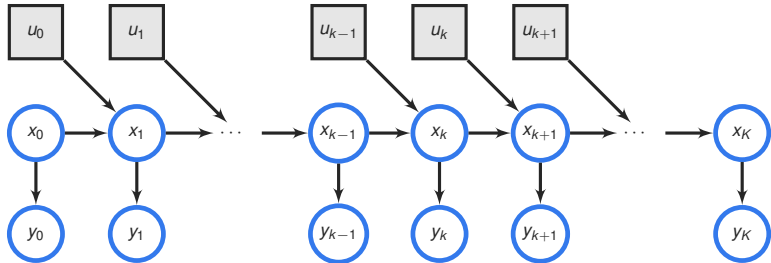
- **Stochastic State-Space:** formulation that accounts for process and measurements uncertainty ($\mathbf{w}(t)$ and $\mathbf{v}(t)$, respectively).

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{w}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{v}(t) \end{cases}$$

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- In probabilistic terms, we interpret the system as a Hidden Markov Model (HMM).





Closed-Loop Observer

Given a system in State-Space with output signal $\mathbf{y}(t) : \mathbb{R} \rightarrow \mathbb{R}^p$ and an observer gain $\mathbf{L} \in \mathbb{R}^{n \times p}$, the estimated state-vector $\hat{\mathbf{x}}(t)$ is represented by the system:

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}(\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{x}}(t)),$$

or, equivalently:

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}\mathbf{y}(t).$$



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- ▶ The observer is operated in parallel to the actual system;
- ▶ Consider a variable $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ such that:

$$\begin{aligned}\dot{\mathbf{e}} &= \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} \\ &= (\mathbf{A} - \mathbf{L}\mathbf{C})(\mathbf{x} - \hat{\mathbf{x}}) \\ &= (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}\end{aligned}$$



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- **Control-Estimator Duality:** the design of \mathbf{L} follows the same design procedure of \mathbf{K} .



Kalman-Bucy Optimal Filter

Consider a model subject to a additive process noise variable $\mathbf{w}(t) \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_{kf})$ and a measurement noise variable $\mathbf{v}(t) \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_{kf})$. In this case, for an estimated state $\hat{\mathbf{x}}(t)$ at time t , the error covariance:

$$\mathbf{J}(\mathbf{x}, \hat{\mathbf{x}}, t) = \mathbb{E} \left\{ [\mathbf{x}(t) - \hat{\mathbf{x}}(t)][\mathbf{x}(t) - \hat{\mathbf{x}}(t)]^T \right\}$$

is minimized by a $\hat{\mathbf{x}}(t)$ obtained through the system:

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{K}_e(t)(\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{x}}(t)),$$

where $\mathbf{K}_e(t) = \mathbf{P}_e(t)\mathbf{C}\mathbf{R}^{-1}$, being $\mathbf{P}_e(t)$ the solution of the Riccati differential matrix equation:

$$\dot{\mathbf{P}}_e(t) = \mathbf{A}\mathbf{P}_e(t) + \mathbf{P}_e(t)\mathbf{A}^T - \mathbf{P}_e(t)\mathbf{C}^T\mathbf{R}_{kf}^{-1}\mathbf{C}\mathbf{P}_e(t) + \mathbf{Q}_{kf},$$

with initial condition $\mathbf{P}_e(t_0) = \mathbb{E} \left\{ [\mathbf{x}(t_0) - \bar{\mathbf{x}}(t_0)][\mathbf{x}(t_0) - \bar{\mathbf{x}}(t_0)]^T \right\}$ for $t_0 > -\infty$.



The Separation Principle

Given a system in State-Space with a Luenberger observer of gain L and state-feedback controller of gain K , the closed-loop eigenvalues contributions of $(A - BK)$ are independent from those of $(A - LC)$.

- It can be shown that the closed-loop dynamics of the system and the error reduces to:

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & -BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r$$



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$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & -BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r$$

- The eigenvalues of the augmented system is the direct contribution of the eigenvalues of $(A - BK)$ and $(A - LC)$.
- **Direct Result:** the controller and the estimator can be designed separately, and they are *dual problems*.



Linear Quadratic Gaussian (LQG) Controller

Consider a stochastic system in State-Space representation:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{w}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{v}(t) \end{cases},$$

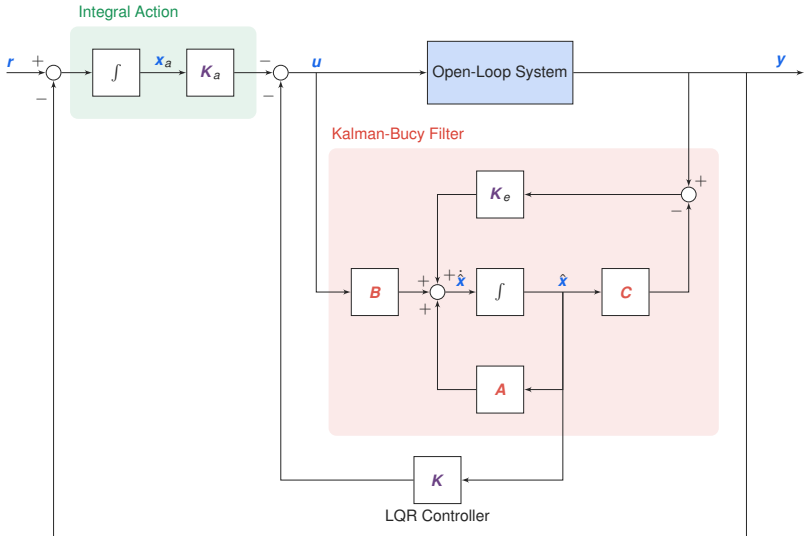
whose state-vector is determined by a Kalman-Bucy filter and whose input signal is calculated by a finite-horizon LQR. The Linear Quadratic Gaussian (LQG) is defined as:

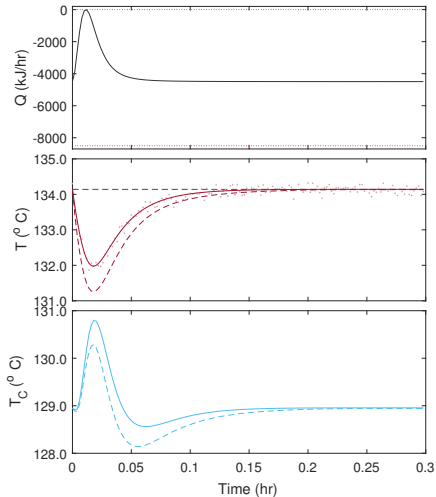
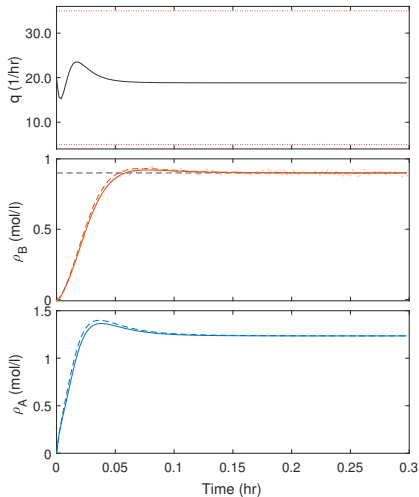
$$\dot{\hat{\mathbf{x}}}(t) = [\mathbf{A} - \mathbf{K}_e(t)\mathbf{C} - \mathbf{B}\mathbf{K}(t)]\hat{\mathbf{x}}(t) + \mathbf{K}_e(t)\mathbf{y}(t),$$

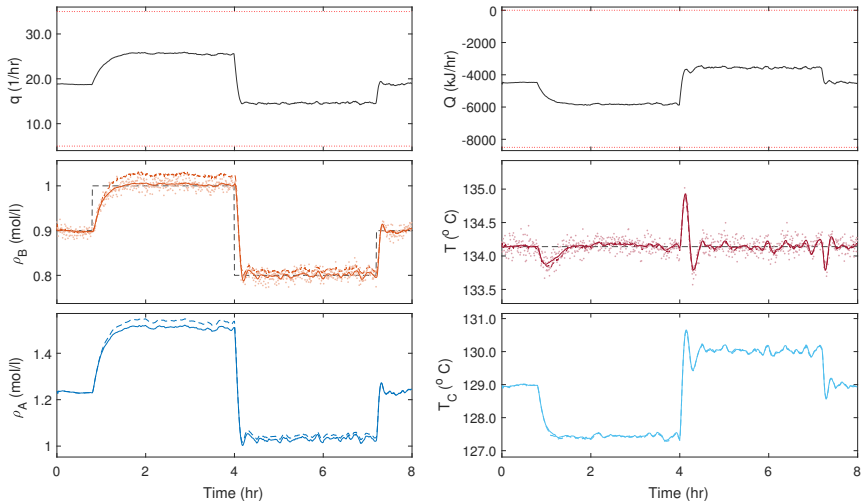
where $\mathbf{K}(t) = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(t)$ and $\mathbf{K}_e(t) = \mathbf{P}_e(t)\mathbf{C}\mathbf{R}^{-1}$ are, respectively, the LQR and Kalman-Bucy gains for matrices $\mathbf{P}(t)$ and $\mathbf{P}_e(t)$ that solve the Riccati differential equations:

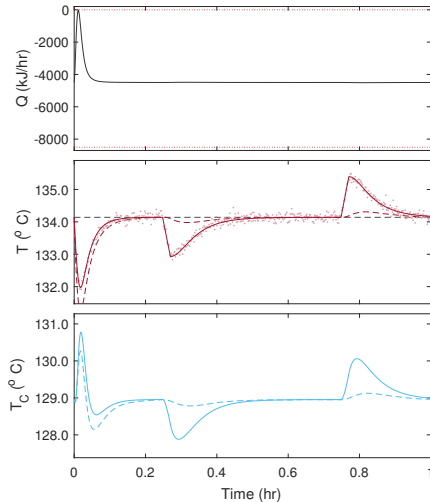
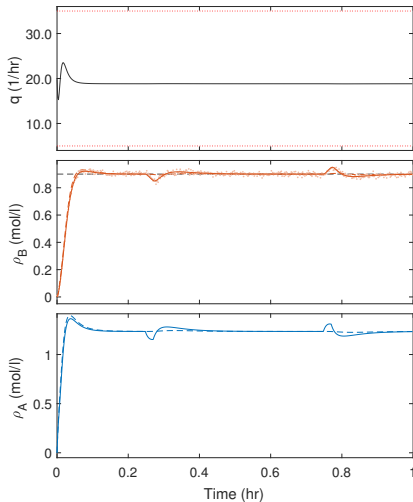
$$\begin{cases} -\dot{\mathbf{P}}(t) = \mathbf{A}^T\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A} - \mathbf{P}(t)\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(t) + \mathbf{Q} \\ \dot{\mathbf{P}}_e(t) = \mathbf{A}\mathbf{P}_e(t) + \mathbf{P}_e(t)\mathbf{A}^T - \mathbf{P}_e(t)\mathbf{C}^T\mathbf{R}_{kf}^{-1}\mathbf{C}\mathbf{P}_e(t) + \mathbf{Q}_{kf} \end{cases}$$

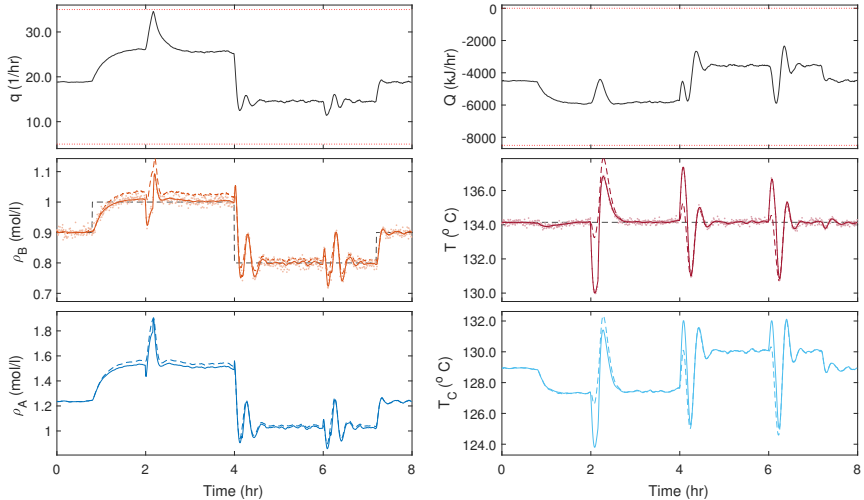
for boundary conditions $\mathbf{P}(T) = \mathbf{Q}_f$ and $\mathbf{P}_e(t_0) = \mathbb{E} \{ [\mathbf{x}(t_0) - \bar{\mathbf{x}}(t_0)][\mathbf{x}(t_0) - \bar{\mathbf{x}}(t_0)]^T \}$, respectively.











Conclusion

- ▶ We have demonstrated the theoretical foundations of optimal control and its application to a specific challenging problem.
- ▶ There were noted some limitations:
 - The state estimator has not able to reconstruct the state information when far from the steady-state point.
 - The integral action directly from noisy measurements demonstrated a concerning behavior.
 - The optimization was performed in a unconstrained fashion, which is unpractical.
- ▶ This work opens the possibility of evaluating more advanced control and estimation techniques using the same mathematical framework.

Thank you!



Questions?