TI0153 - Trabalho de Conclusão de Curso II
Department of Teleinformatics Engineering
Federal University of Ceará - UFC

Optimal Control: An application to a non-isothermal continuous reactor

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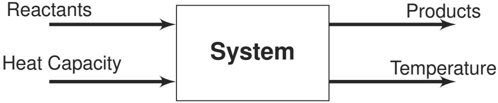
Summary

Introduction

- We are discussing the **automatic control** of **dynamical systems**;
- We are discussing **chemical reactor network systems**, for processes described as:

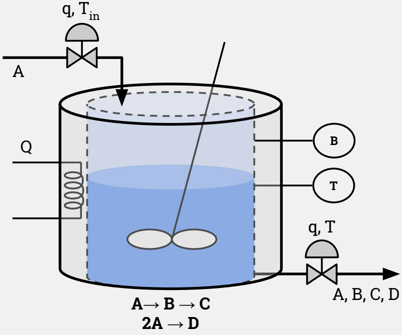


$$\begin{cases} \mathbb{R}^{(1)} \xrightarrow{K_{11}(T)} \cdots \xrightarrow{K_{1n}(T)} \mathcal{P}^{(1)} & \Delta H_1 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots & \\ \mathbb{R}^{(N)} \xrightarrow{K_{n1}(T)} \cdots \xrightarrow{K_{nn}(T)} \mathcal{P}^{(N)} & \Delta H_n \end{cases}$$

- Optimal control theory has been revisited from several innovative fields in the last years **[Cairano et al., 2014, Tang et al., 2016, Eren et al., 2017]**. Furthermore, reactor systems are subject of active research, with several open challenges **[Gupta et al., 2012, Mulas et al., 2015]**.

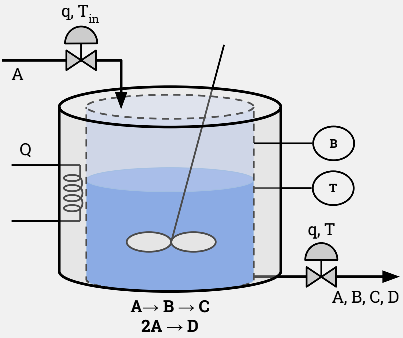
- The system used for the experiments was the non-isothermal Continuous Stirred Tank Reactor (CSTR) presented by **[Klatt and Engell, 1998]**.

Non-isothermal CSTR



- The system used for the experiments was the non-isothermal Continuous Stirred Tank Reactor (CSTR) presented by **[Klatt and Engell, 1998]**.

Non-isothermal CSTR



This system...

- characterizes a wide range of industrial applications;
- is a classical benchmark for multiple-input multiple-output (MIMO) control systems;
- is highly nonlinear, with non-minimum phase behavior and unmeasurable states;

Dynamical System Analysis

- Dynamical systems are modelled through the Conservation Laws from physics;

Mass Balance for Chemical Reactors

$$\begin{aligned}\begin{pmatrix}\text{Accumulation}\\ \text{of mass}\\ \text{in the system}\end{pmatrix} = &\left[\begin{pmatrix}\text{Mass flow}\\ \text{entering}\\ \text{system}\end{pmatrix} + \begin{pmatrix}\text{Mass}\\ \text{produced}\\ \text{by reactions}\end{pmatrix}\right] \\ &- \left[\begin{pmatrix}\text{Mass flow}\\ \text{leaving}\\ \text{system}\end{pmatrix} + \begin{pmatrix}\text{Mass}\\ \text{consumed}\\ \text{by reactions}\end{pmatrix}\right]\end{aligned} \tag{1}$$

Conservation of Energy for Chemical Reactors

$$\begin{pmatrix}\text{Accumulation}\\ \text{of thermal energy}\\ \text{in the system}\end{pmatrix} = \begin{pmatrix}\text{Heat flow}\\ \text{entering}\\ \text{the system}\end{pmatrix} - \begin{pmatrix}\text{Heat flow}\\ \text{leaving}\\ \text{the system}\end{pmatrix} + \begin{pmatrix}\text{Entropy}\\ \text{contribution}\\ \text{from reactions}\end{pmatrix} \tag{2}$$

- Assuming a continuous reactor of constant volume, comprised by a dilute solution of reactant and products...

Mass Balance for Chemical Reactors

$$\frac{d(\rho_A)}{dt} = q(\rho_{in}^{(A)} - \rho_{out}^{(A)}) + \left(\sum_{\alpha X \to \beta A} \frac{1}{\beta} K_{XA}(T)(\rho_X)^{\alpha} \right) - \left(\sum_{\alpha A \to \beta X} \frac{1}{\beta} K_{AX}(T)(\rho_A)^{\alpha} \right) \quad (3)$$

Conservation of Energy for Chemical Reactors

$$\begin{cases} \dfrac{d(T)}{dt} = q(T_{in} - T_{out}) + \eta(T_C - T) + \delta \displaystyle\sum_{\alpha A \to \beta X} K_{AX}(T)(\rho_A)^{\alpha} \Delta H_{AX} \\ \dfrac{d(T_C)}{dt} = \gamma Q + \beta(T - T_C) \end{cases} \quad (4)$$

- **Remark:** this model is highly nonlinear!

Linearization by Taylor Series Expansion

Consider a nonlinear time-invariant system:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)) \\ \boldsymbol{y}(t) = \boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{u}(t)) \end{cases} . \tag{5}$$

Given steady-state operating points $\boldsymbol{x_o}$, $\boldsymbol{y_o}$ and $\boldsymbol{u_o}$, the dynamics of the system in the neighborhood of these points can be represented by the linear model:

$$\begin{cases} \Delta\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\Delta\boldsymbol{x}(t) + \boldsymbol{B}\Delta\boldsymbol{u}(t) \\ \boldsymbol{y}(t) = \boldsymbol{C}\Delta\boldsymbol{x}(t) + \boldsymbol{D}\Delta\boldsymbol{u}(t) \end{cases} , \tag{6}$$

where

$$\boldsymbol{A} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x_o},\boldsymbol{u_o}} ; \quad \boldsymbol{B} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right|_{\boldsymbol{x_o},\boldsymbol{u_o}} ; \quad \boldsymbol{C} = \left.\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x_o},\boldsymbol{u_o}} ; \quad \boldsymbol{D} = \left.\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{u}}\right|_{\boldsymbol{x_o},\boldsymbol{u_o}} \tag{7}$$

and

$$\Delta\boldsymbol{x}(t) = \boldsymbol{x}(t) - \boldsymbol{x_o}; \qquad \Delta\boldsymbol{u}(t) = \boldsymbol{u}(t) - \boldsymbol{u_o}. \tag{8}$$

- A linear and time-invariant model in State-Space representation has the form:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\ \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u} \end{cases} \tag{9}$$

This representation has several analysis and control design advantages...

- The response of the system has a practical analytical solution:

$$\begin{cases} \boldsymbol{x}(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t) + \int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau \\ \boldsymbol{y}(t) = \boldsymbol{C}e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t) + \boldsymbol{C}\int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau + \boldsymbol{D}\boldsymbol{u}(t) \end{cases} ; \tag{10}$$

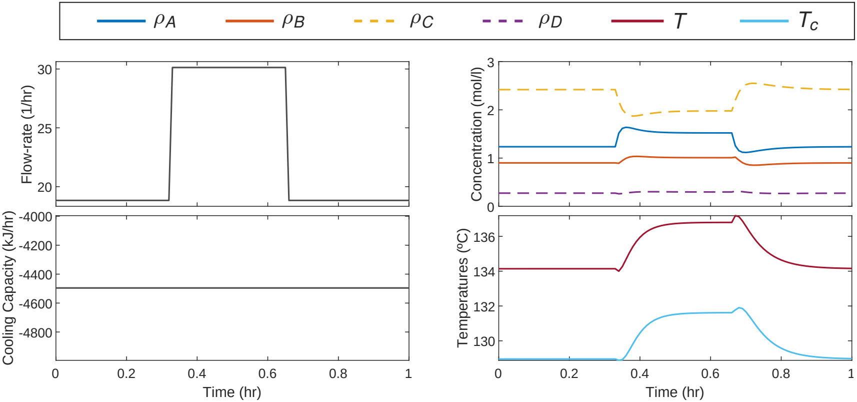
- The controllability property is directly related to the *Controllability Matrix*:

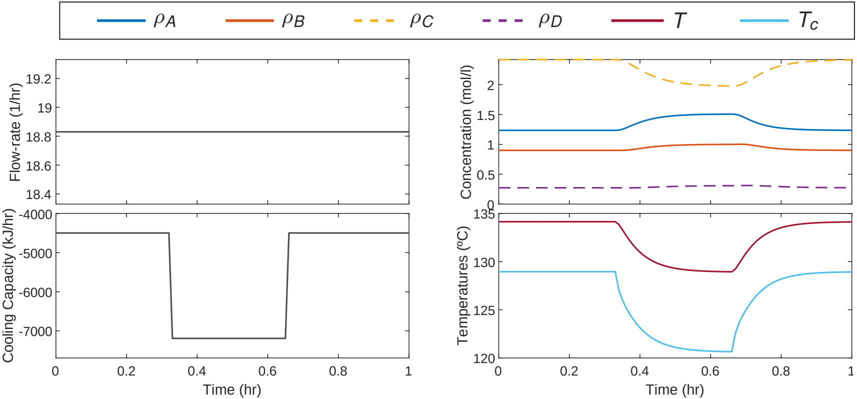
$$\mathcal{C} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{A}\boldsymbol{B} & \boldsymbol{A}^2\boldsymbol{B} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix} ; \tag{11}$$

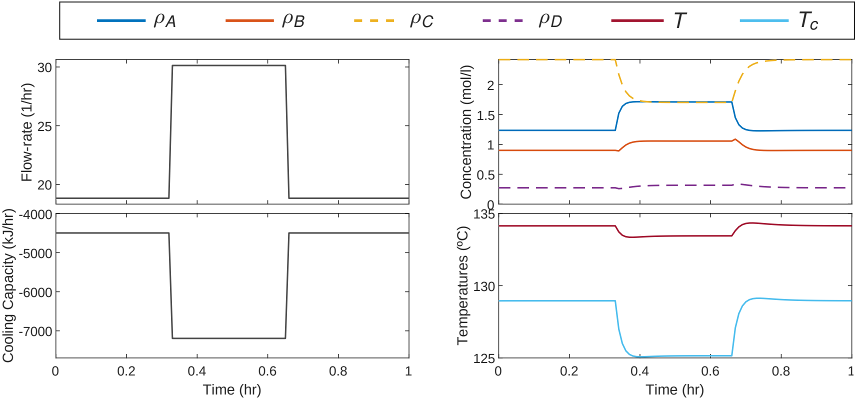
- The observability property is directly related to the *Observability Matrix*:

$$\mathcal{O} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{C}\boldsymbol{A} & \boldsymbol{C}\boldsymbol{A}^2 & \cdots & \boldsymbol{C}\boldsymbol{A}^{n-1} \end{bmatrix}^T ; \tag{12}$$

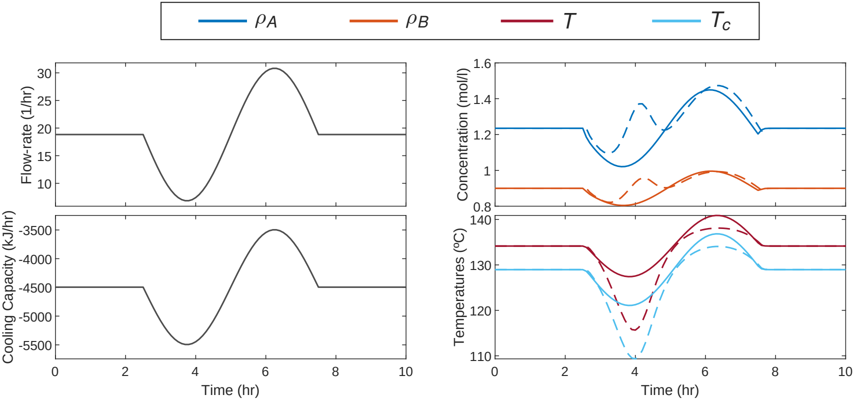
- The stability of the system is directly related to the eigenvalues of $\boldsymbol{A}$;


ρ_A
ρ_B
ρ_C
ρ_D
T
T_c
Flow-rate (1/hr)
30
25
20
Cooling Capacity (kJ/hr)
-4000
-4200
-4400
-4600
-4800
0
0.2
0.4
0.6
0.8
1
Time (hr)
Concentration (mol/l)
3
2
1
0
Temperatures (ºC)
136
134
132
130
0
0.2
0.4
0.6
0.8
1
Time (hr)


ρ_A
ρ_B
ρ_C
ρ_D
T
T_c
Flow-rate (1/hr)
19.2
19
18.8
18.6
18.4
Cooling Capacity (kJ/hr)
-4000
-5000
-6000
-7000
Concentration (mol/l)
2
1.5
1
0.5
0
Temperatures (ºC)
135
130
125
120
0
0.2
0.4
0.6
0.8
1
Time (hr)



ρ_A
ρ_B
ρ_C
ρ_D
T
T_c
Flow-rate (1/hr)
Cooling Capacity (kJ/hr)
Concentration (mol/l)
Temperatures (ºC)
Time (hr)
Time (hr)


ρA
ρB
T
Tc
Flow-rate (1/hr)
30
25
20
15
10
Cooling Capacity (kJ/hr)
-3500
-4000
-4500
-5000
-5500
0
2
4
6
8
10
Time (hr)
Concentration (mol/l)
1.6
1.4
1.2
1
0.8
Temperatures (ºC)
140
130
120
110
0
2
4
6
8
10
Time (hr)

State-Feedback Controllers

Full State-Feedback Controller

Given a linear system in State-Space representation, an input action $\boldsymbol{u}(t)$ is calculated by the linear control law $\pi(\cdot)$ through state-feedback as:

$$\boldsymbol{u}(t) = \pi(\boldsymbol{r}(t), \boldsymbol{x}(t)) = \boldsymbol{r}(t) - \boldsymbol{K}\boldsymbol{x}(t), \tag{13}$$

where $\boldsymbol{r} : \mathbb{R} \to \mathbb{R}^n$ is a state reference signal that the system must follows and $\boldsymbol{K} \in \mathbb{R}^{r \times n}$ is the *feedback gain matrix*.

Full State-Feedback Controller

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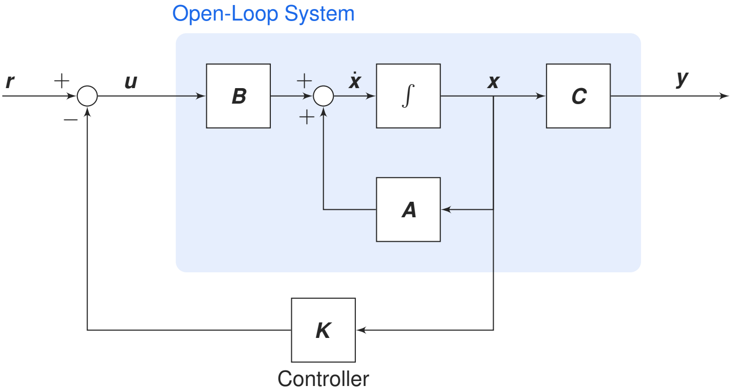
where $\boldsymbol{r} : \mathbb{R} \rightarrow \mathbb{R}^n$ is a state reference signal that the system must follows and $\boldsymbol{K} \in \mathbb{R}^{r \times n}$ is the *feedback gain matrix*.

Pole-Placement Property

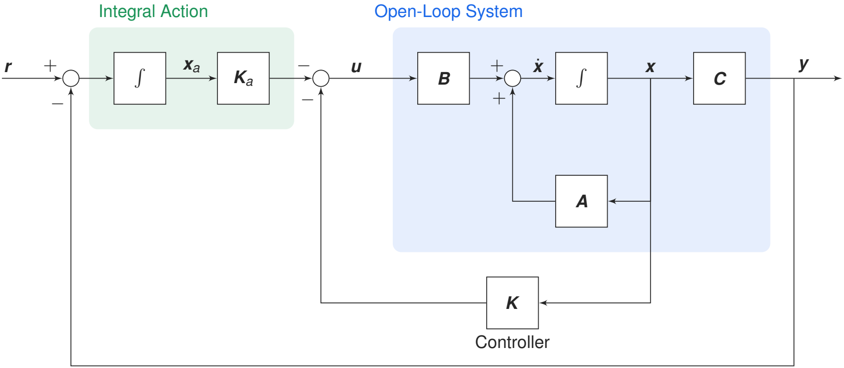
If a system in State-Space representation is controllable, then by state feedback using a gain matrix $\boldsymbol{K} \in \mathbb{R}^{r \times n}$ the eigenvalues of $\boldsymbol{A}_{cl} = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}$, the poles of the closed-loop system, can be placed arbitrarily in the complex plane, as long as complex conjugate eigenvalues are assigned in pairs.


Open-Loop System
r
+
−
u
B
+
+
ẋ
∫
x
C
y
A
K
Controller

During the semester, we have discussed Conservation Laws of fluid elements...

Conservation of Mass

$$\begin{pmatrix}\text{Time rate of}\\ \text{change of mass}\\ \text{in the system}\end{pmatrix} = \begin{pmatrix}\text{Mass}\\ \text{entering}\\ \text{the system}\end{pmatrix} - \begin{pmatrix}\text{Mass}\\ \text{leaving}\\ \text{the system}\end{pmatrix}$$


Integral Action
Open-Loop System
r
+
−
∫
x_a
K_a
−
−
u
B
+
+
$\dot{x}$
∫
x
C
y
A
K
Controller

Optimal Control

Optimal Controller

Given a system in State-Space formulation, with state signal $\boldsymbol{x} : \mathbb{R} \to \mathbb{R}^n$, and a reference signal $\boldsymbol{r} : \mathbb{R} \to \mathbb{R}^n$, the input signal $\boldsymbol{u}(t) \in \mathbb{R}^r$, for any time t, is optimal if an optimal control law $\pi^* : \mathbb{R}^{n \times n \times 1} \to \mathbb{R}^r$ can be found as:

$$\boldsymbol{u}(t) = \pi^*(\boldsymbol{x}, \boldsymbol{r}, t) = \min_{\boldsymbol{u}} J(\boldsymbol{x}, \boldsymbol{r}, t), \tag{15}$$

where $J : \mathbb{R}^{n \times n \times 1} \to \mathbb{R}$ is known as a *cost function* of the states and reference signals.

Optimal Controller

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Finite-Horizon Optimal Regulators

A *Finite-Horizon Optimal Regulator* is defined as any controller whose optimal policy over a time interval $t \in [t_0, T]$ minimizes the cost functional:

$$J(\boldsymbol{x}, \boldsymbol{u}, t_0) = \int_{t_0}^{T} l(\boldsymbol{x}, \boldsymbol{u}, \tau) d\tau + l_f(\boldsymbol{x}, T), \tag{17}$$

where $l(\cdot) : \mathbb{R}^{n \times r \times 1} \to \mathbb{R}$ and $l_f(\cdot) : \mathbb{R}^n \to \mathbb{R}$ are, respectively, the *trajectory* and *terminal loss functions*. In the case that $t_0 = 0$, T is also known as the *control horizon*.

Hamilton-Jacobi-Bellman Equation

Consider a finite-horizon cost function for a system described by $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t)$:

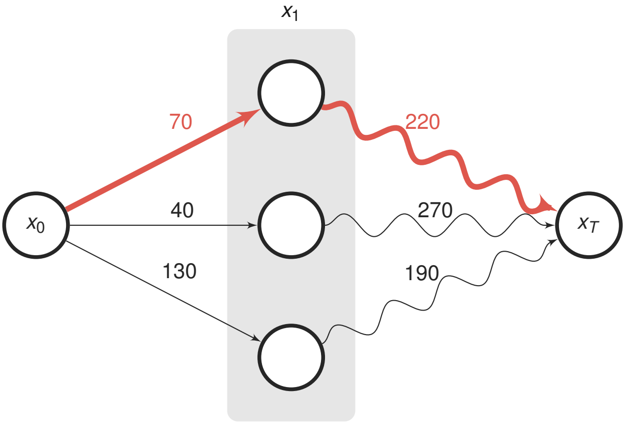
$$V(\boldsymbol{x}, \boldsymbol{u}, t_0) = \int_{t_0}^{T} l(\boldsymbol{x}, \boldsymbol{u}, \tau) d\tau + l_f(\boldsymbol{x}(T)) \tag{18}$$

Consider also that the loss $l(\cdot)$ and state function $\boldsymbol{f}$ are smooth on their parameters. Then, minimizing any functional in the form of $V(\cdot)$ is equivalent to determining the solution of the *Hamilton-Jacobi equation*, which is given by the partial differential equation:

$$\frac{\partial V^*}{\partial t} = -\min_{u(t)} \left[l(\boldsymbol{x}, \boldsymbol{u}, t) + \left[\frac{\partial V^*}{\partial x}\right]^T \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t) \right] \tag{19}$$

and the boundary condition:

$$V^*(\boldsymbol{x}, T) = l_f(\boldsymbol{x}(T)). \tag{20}$$


x_1
70
220
40
270
130
190
x_0
x_T

Linear Quadratic Regulator (LQR)

Given a linear State-Space system in the form:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t) \end{cases} . \tag{21}$$

A *Linear Quadratic Regulator* (LQR) for this system is an optimal controller defined by the quadratic cost function:

$$J(\boldsymbol{x}, \boldsymbol{u}, t_0) = \int_{t_0}^{T} \left(\boldsymbol{x}^T \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^T \boldsymbol{R}\boldsymbol{u} \right) dt + \boldsymbol{x}^T(T)\boldsymbol{Q}_f \boldsymbol{x}(T), \tag{22}$$

where is assumed that $\boldsymbol{Q}$, $\boldsymbol{Q}_f \succ 0$ and $\boldsymbol{R} \succ 0$ are matrices penalizing, respectively, the state-vector magnitude and the control effort.

LQR Control Action from Dynamic Programming

Given a Linear Quadratic Regulator, the optimal action produced by this optimal controller at any time $t \in [t_0, T]$ is given by:

$$\boldsymbol{u}^*(t) = -\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}(t)\boldsymbol{x}(t), \tag{23}$$

where $\boldsymbol{P}(t)$ is the solution of the matrix Riccati differential equation:

$$-\dot{\boldsymbol{P}}(t) = \boldsymbol{A}^T\boldsymbol{P}(t) + \boldsymbol{P}(t)\boldsymbol{A} - \boldsymbol{P}(t)\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}(t) + \boldsymbol{Q}, \tag{24}$$

with terminal condition $\boldsymbol{P}(T) = \boldsymbol{Q}_f$.

LQR with Integral Action

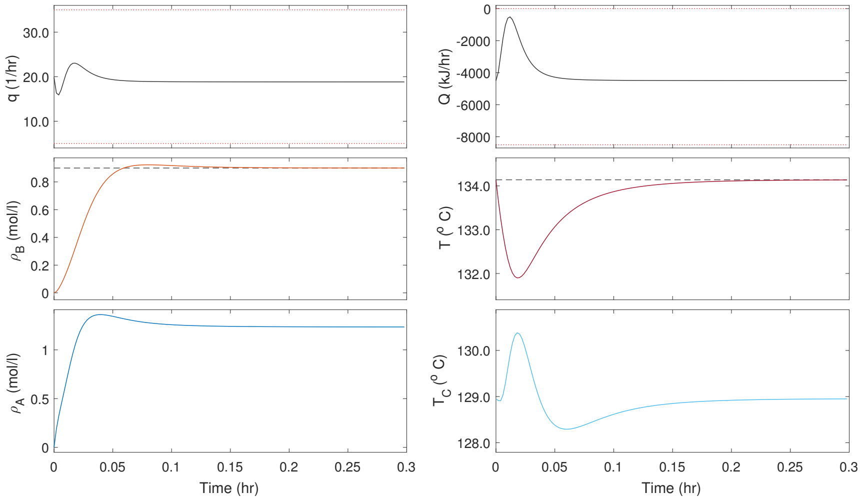
Given a linear State-Space system represented by matrices $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$, augmented with state $\dot{\boldsymbol{x}}_a(t) = \boldsymbol{r}(t) - \boldsymbol{C}\boldsymbol{x}(t)$:

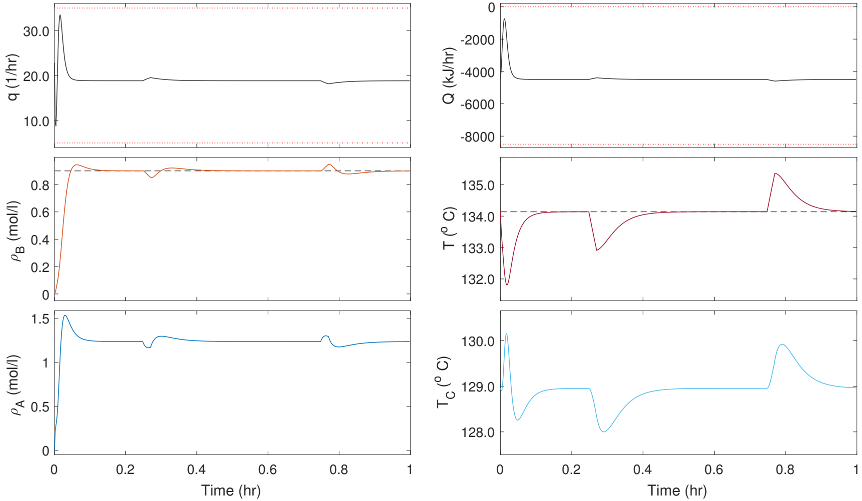
$$\begin{cases} \begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{x}}_a(t) \end{bmatrix} &= \underbrace{\begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ -\boldsymbol{C} & \boldsymbol{0} \end{bmatrix}}_{\tilde{\boldsymbol{A}}} \underbrace{\begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}_a(t) \end{bmatrix}}_{\tilde{\boldsymbol{x}}(t)} + \underbrace{\begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{bmatrix}}_{\tilde{\boldsymbol{B}}} \boldsymbol{u}(t) + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I} \end{bmatrix} \boldsymbol{r}(t) \\ \boldsymbol{y}(t) &= \begin{bmatrix} \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}_a(t) \end{bmatrix} \end{cases} . \tag{25}$$

A *Linear Quadratic Servo* (LQ-Servo) for this system is an optimal controller defined by the quadratic cost function:

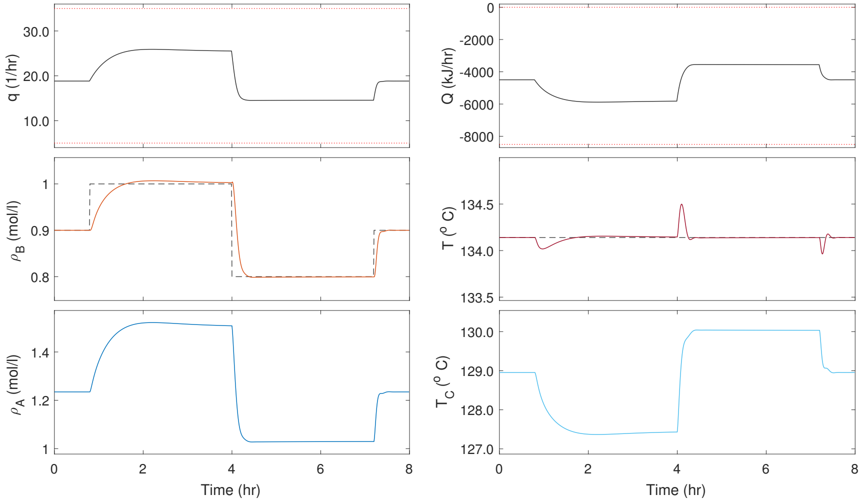
$$J(\boldsymbol{x}, \boldsymbol{u}, t_0) = \int_{t_0}^{T} \left(\tilde{\boldsymbol{x}}^T \tilde{\boldsymbol{Q}} \tilde{\boldsymbol{x}} + \boldsymbol{u}^T \boldsymbol{R} \boldsymbol{u} \right) dt + \tilde{\boldsymbol{x}} \tilde{\boldsymbol{Q}}_f \tilde{\boldsymbol{x}}(T), \tag{26}$$

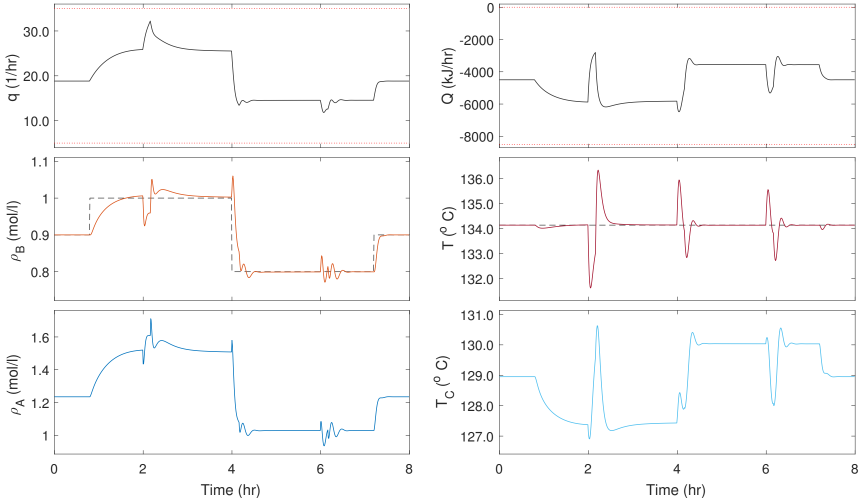
where is assumed that $\tilde{\boldsymbol{Q}}, \tilde{\boldsymbol{Q}}_f \succ 0$ and $\boldsymbol{R} \succ 0$ are matrices penalizing, respectively, the state-vector magnitude and the control effort.


q (1/hr)
30.0
20.0
10.0
Q (kJ/hr)
0
-2000
-4000
-6000
-8000
ρ_B (mol/l)
0.8
0.6
0.4
0.2
0
T (^{o} C)
134.0
133.0
132.0
ρ_A (mol/l)
1
0.5
0
T_C (^{o} C)
130.0
129.0
128.0
0
0.05
0.1
0.15
0.2
0.25
0.3
Time (hr)


q (1/hr)
30.0
20.0
10.0
Q (kJ/hr)
0
-2000
-4000
-6000
-8000
ρ_B (mol/l)
0.8
0.6
0.4
0.2
0
T (^{o} C)
135.0
134.0
133.0
132.0
ρ_A (mol/l)
1.5
1
0.5
0
T_C (^{o} C)
130.0
129.0
128.0
0
0.2
0.4
0.6
0.8
1
Time (hr)
Time (hr)

Optimal Control




q (1/hr)
30.0
20.0
10.0
Q (kJ/hr)
0
-2000
-4000
-6000
-8000
ρ_B (mol/l)
1.1
1
0.9
0.8
T (^{o} C)
136.0
135.0
134.0
133.0
132.0
ρ_A (mol/l)
1.6
1.4
1.2
1
T_C (^{o} C)
131.0
130.0
129.0
128.0
127.0
0
2
4
6
8
Time (hr)
Time (hr)

Optimal State Estimation

Closed-Loop Observer

Given a system in State-Space with output signal $\boldsymbol{y}(t) : \mathbb{R} \rightarrow \mathbb{R}^p$ and an observer gain $\boldsymbol{L} \in \mathbb{R}^{n \times p}$, the estimated state-vector $\hat{\boldsymbol{x}}(t)$ is represented by the system:

$$\dot{\hat{\boldsymbol{x}}}(t) = \boldsymbol{A}\hat{\boldsymbol{x}}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{L}\left(\boldsymbol{y}(t) - \boldsymbol{C}\hat{\boldsymbol{x}}(t)\right), \tag{27}$$

or, equivalently:

$$\dot{\hat{\boldsymbol{x}}}(t) = \left(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}\right)\hat{\boldsymbol{x}}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{L}\boldsymbol{y}(t). \tag{28}$$

- The observer system works as a parallel system that is simulated alongside the actual system;
- Alternatively, it is possible to create a variable $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t)$ such that:

$$\begin{aligned} \dot{\boldsymbol{e}} &= \boldsymbol{x} - \hat{\boldsymbol{x}} \\ &= \left(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}\right)\left(\boldsymbol{x} - \hat{\boldsymbol{x}}\right) \\ &= \left(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}\right)\boldsymbol{e}, \end{aligned} \tag{29}$$

implying that the observer tracks the actual state-vector if $\boldsymbol{e}(t) = \boldsymbol{0}$ as $t \rightarrow \infty$.

Pole-Placement Property of Observers

If a system in State-Space representation is observable, then by a closed-loop observer with gain matrix $\boldsymbol{L} \in \mathbb{R}^{n \times p}$ the eigenvalues of $\boldsymbol{A}_{obs} = \boldsymbol{A} - \boldsymbol{LC}$ can arbitrarily be assigned anywhere in the complex plane, as long as that complex conjugate eigenvalues are assigned in pairs.

- a

The Separation Principle

Given a system in State-Space with a Luenberger observer of gain $\boldsymbol{L}$ and state-feedback controller of gain $\boldsymbol{K}$, the closed-loop eigenvalues contributions of $(\boldsymbol{A} - \boldsymbol{BK})$ are independent from those of $(\boldsymbol{A} - \boldsymbol{LC})$.

- a

Kalman-Bucy Optimal Filter

Consider a continuous-time State-Space linear system subject to additive process noise variable $\boldsymbol{w}_k \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q}_{kf})$ and measurement noise variable $\boldsymbol{v}_k \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{R}_{kf})$, where the covariances $\boldsymbol{Q}_{kf} \in \mathbb{R}^{n \times n}$ and $\boldsymbol{R}_{kf} \in \mathbb{R}^{p \times p}$ represents the *power spectral density* of the noises. In this case, for an estimated state $\bar{\boldsymbol{x}}(t)$ at time t, the error covariance:

$$\boldsymbol{J}(\boldsymbol{x}, \bar{\boldsymbol{x}}, t) = \mathbb{E}\left\{[\boldsymbol{x}(t) - \bar{\boldsymbol{x}}(t)][\boldsymbol{x}(t) - \bar{\boldsymbol{x}}(t)]^T\right\} \tag{30}$$

is minimized by $\bar{\boldsymbol{x}}(t)$ obtained through the system:

$$\dot{\bar{\boldsymbol{x}}}(t) = \boldsymbol{A}\bar{\boldsymbol{x}}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{K}_e(t)\left(\boldsymbol{y}(t) - \boldsymbol{C}\bar{\boldsymbol{x}}(t)\right), \tag{31}$$

where $\boldsymbol{K}_e(t) = \boldsymbol{P}_e(t)\boldsymbol{C}\boldsymbol{R}^{-1}$, being $\boldsymbol{P}_e(t)$ the solution of the Riccati differential matrix equation:

$$\dot{\boldsymbol{P}}_e(t) = \boldsymbol{A}\boldsymbol{P}_e(t) + \boldsymbol{P}_e(t)\boldsymbol{A}^T - \boldsymbol{P}_e(t)\boldsymbol{C}^T\boldsymbol{R}_{kf}^{-1}\boldsymbol{C}\boldsymbol{P}_e(t) + \boldsymbol{Q}_{kf}, \tag{32}$$

with initial condition $\boldsymbol{P}_e(t_0) = \mathbb{E}\left\{[\boldsymbol{x}(t_0) - \bar{\boldsymbol{x}}(t_0)][\boldsymbol{x}(t_0) - \bar{\boldsymbol{x}}(t_0)]^T\right\}$ for $t_0 > -\infty$.

Linear Quadratic Gaussian (LQG) Controller

Consider a stochastic system in State-Space representation:

$$\begin{cases} \dot{\boldsymbol{x}}(t) & = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{w}(t) \\ \boldsymbol{y}(t) & = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{v}(t) \end{cases}, \tag{33}$$

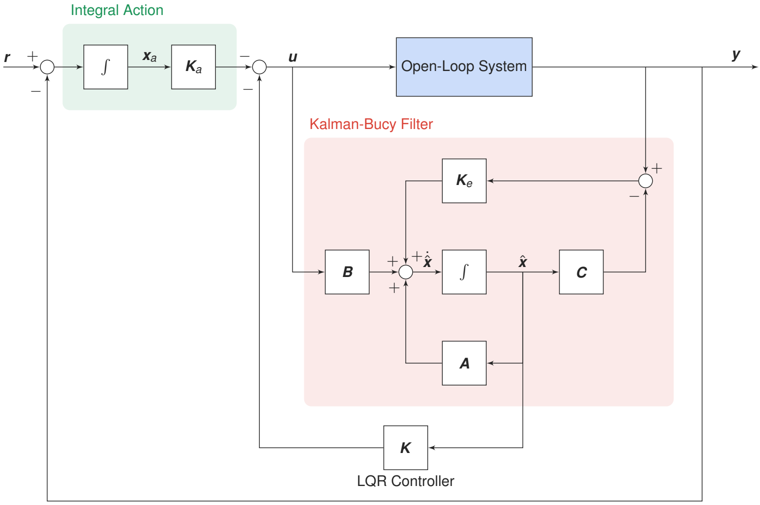
whose estimated state-vector $\hat{\boldsymbol{x}}(t)$ is determined by a Kalman-Bucy filter and whose optimal input signal $\boldsymbol{u}(t)$ is calculated through a finite-horizon LQR. The Linear Quadratic Gaussian (LQG) control for the horizon $t \in [t_0, T]$, with $-\infty < t_0 \leq T < \infty$, is defined as:

$$\dot{\hat{\boldsymbol{x}}}(t) = [\boldsymbol{A} - \boldsymbol{K}_e(t)\boldsymbol{C} - \boldsymbol{B}\boldsymbol{K}(t)]\,\hat{\boldsymbol{x}}(t) + \boldsymbol{K}_e(t)\boldsymbol{y}(t), \tag{34}$$

where $\boldsymbol{K}(t) = \boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}(t)$ and $\boldsymbol{K}_e(t) = \boldsymbol{P}_e(t)\boldsymbol{C}\boldsymbol{R}^{-1}$ are, respectively, the LQR and Kalman-Bucy gains for matrices $\boldsymbol{P}(t)$ and $\boldsymbol{P}_e(t)$ that solve the Riccati differential equations:

$$\begin{cases} -\dot{\boldsymbol{P}}(t) = \boldsymbol{A}^T\boldsymbol{P}(t) + \boldsymbol{P}(t)\boldsymbol{A} - \boldsymbol{P}(t)\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}(t) + \boldsymbol{Q} \\ \quad\dot{\boldsymbol{P}}_e(t) = \boldsymbol{A}\boldsymbol{P}_e(t) + \boldsymbol{P}_e(t)\boldsymbol{A}^T - \boldsymbol{P}_e(t)\boldsymbol{C}^T\boldsymbol{R}_{kf}^{-1}\boldsymbol{C}\boldsymbol{P}_e(t) + \boldsymbol{Q}_{kf} \end{cases} \tag{35}$$

for boundary conditions $\boldsymbol{P}(T) = \boldsymbol{Q}_f$ and $\boldsymbol{P}_e(t_0) = \mathbb{E}\left\{[\boldsymbol{x}(t_0) - \bar{\boldsymbol{x}}(t_0)][\boldsymbol{x}(t_0) - \bar{\boldsymbol{x}}(t_0)]^T\right\}$, respectively.


Integral Action
r
+
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A
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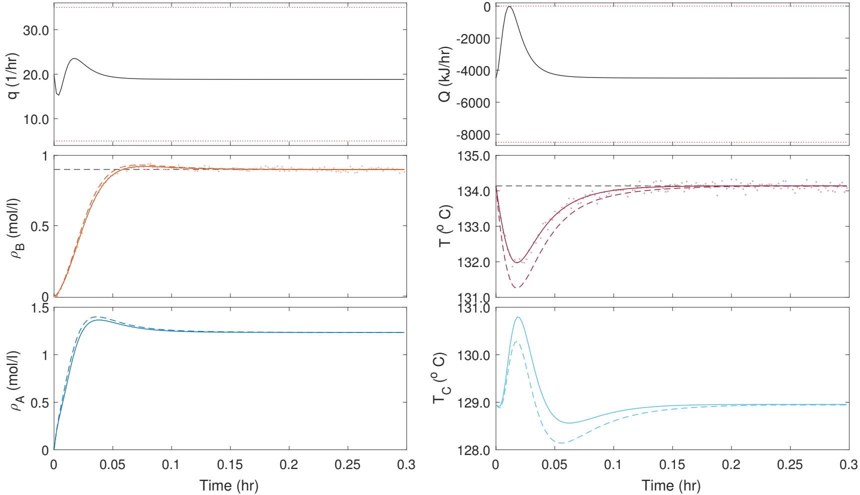
LQR Controller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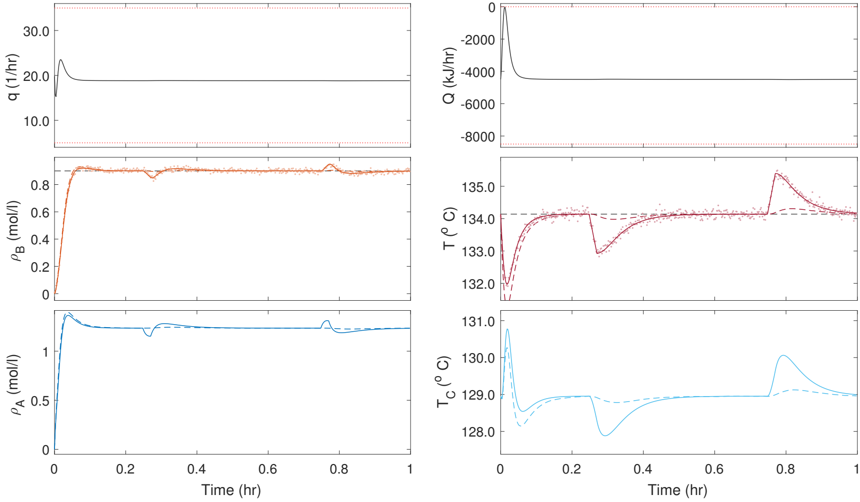
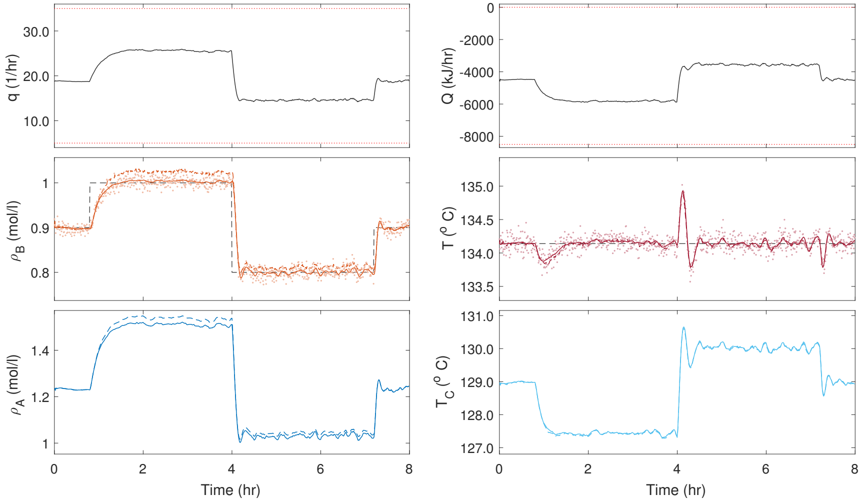
30.0
20.0
10.0
Q (kJ/hr)
0
-2000
-4000
-6000
-8000
ρ_B (mol/l)
1
0.5
0
T (^{o} C)
135.0
134.0
133.0
132.0
131.0
ρ_A (mol/l)
1.5
1
0.5
0
T_C (^{o} C)
131.0
130.0
129.0
128.0
0
0.05
0.1
0.15
0.2
0.25
0.3
Time (hr)

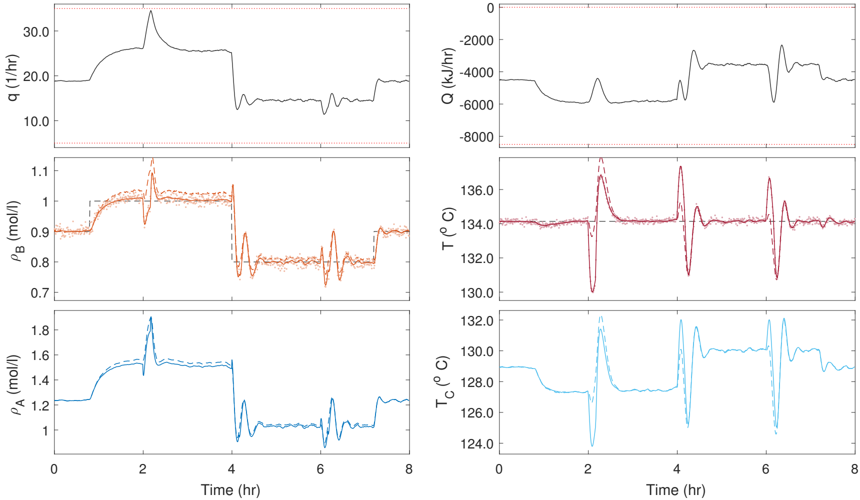


q (1/hr)
30.0
20.0
10.0
Q (kJ/hr)
0
-2000
-4000
-6000
-8000
ρ_B (mol/l)
0.8
0.6
0.4
0.2
0
T (^{o} C)
135.0
134.0
133.0
132.0
ρ_A (mol/l)
1
0.5
0
T_C (^{o} C)
131.0
130.0
129.0
128.0
0
0.2
0.4
0.6
0.8
1
Time (hr)

Optimal State Estimation



Optimal State Estimation



Conclusion

Conclusion

- Past;
- Present;
- Future;

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Thank you!

Questions?