TI0153 - Trabalho de Conclusão de Curso II
Department of Teleinformatics Engineering
Federal University of Ceará - UFC

Optimal Control: An application to a non-isothermal continuous reactor

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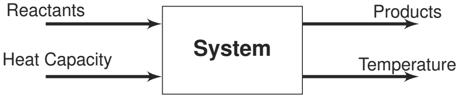
Summary

Introduction

- We are discussing the **optimal control** of **dynamical systems**.
- We are discussing **chemical reactor network systems**, for processes described as:



$$\begin{cases} \mathcal{R}^{(1)} & \xrightarrow{K_{11}(T)} & \dots & \xrightarrow{K_{1n}(T)} & \mathcal{P}^{(1)} & \Delta H_1 \\ \vdots & & \vdots & & \vdots & \\ \mathcal{R}^{(N)} & \xrightarrow{K_{n1}(T)} & \dots & \xrightarrow{K_{nn}(T)} & \mathcal{P}^{(N)} & \Delta H_n \end{cases}$$

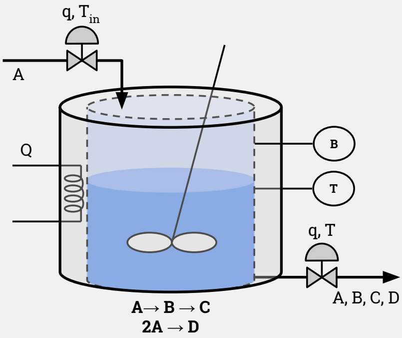
- Optimal control theory has been revisited from several innovative fields in the last years[1,2]. Furthermore, reactor systems are subject of active research, with several open challenges.[3]

[1] Xun Tang et al. "Optimal Feedback Controlled Assembly of Perfect Crystals". In: *ACS Nano* 10.7 (2016), pp. 6791–6798.

[2] Utku Eren et al. "Model Predictive Control in Aerospace Systems: Current State and Opportunities". In: *Journal of Guidance, Control, and Dynamics* 40.7 (2017), pp. 1541–1566.

[3] Michela Mulas et al. "Predictive control of an activated sludge process: An application to the Viikinmäki wastewater treatment plant". In: *Journal of Process Control* 35 (2015), pp. 89–100.

- The system used for the experiments was the *non-isothermal Continuous Stirred Tank Reactor* (CSTR) presented by [Klatt and Engell, 1998][4].



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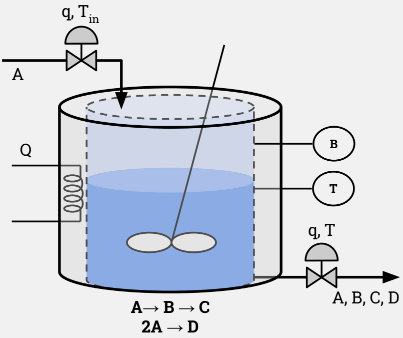

This system...

- class of system that represents a wide range of industrial applications.
- classical benchmark for multiple-input multiple-output (MIMO) control systems.
- nonlinear behavior, models with non-minimum phase behavior and unmeasurable states.

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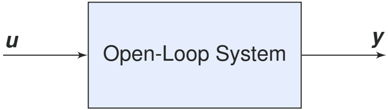
Dynamical System Analysis

Dynamical System Analysis

We desire to obtain a State-Space mathematical representation of the system.

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)) \\ \boldsymbol{y}(t) = \boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{u}(t)) \end{cases}$$

- $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the *state-vector*.
- $\boldsymbol{u}(t) \in \mathbb{R}^r$ is the *input-vector*.
- $\boldsymbol{y}(t) \in \mathbb{R}^p$ is the *output-vector*.
- $\boldsymbol{f}(\cdot) : \mathbb{R}^{n \times r} \to \mathbb{R}^n$ is a *state-transition function*.
- $\boldsymbol{g}(\cdot) : \mathbb{R}^{n \times r} \to \mathbb{R}^p$ is a *output observation function*.



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$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t) \end{cases}$$

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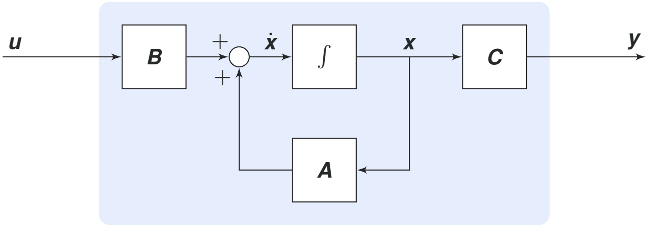
- $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is the *system matrix*.
- $\boldsymbol{B} \in \mathbb{R}^{n\times r}$ is the *input matrix*.
- $\boldsymbol{C} \in \mathbb{R}^{p\times n}$ is the *output matrix*.
- $\boldsymbol{D} \in \mathbb{R}^{p\times r}$ is the *feedthrough matrix*.

Open-Loop System



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This representation has several advantages.

- The *response* of the system has an analytical solution:

$$\begin{cases} \boldsymbol{x}(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t) + \int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau \\ \boldsymbol{y}(t) = \boldsymbol{C}e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t) + \boldsymbol{C}\int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau + \boldsymbol{D}\boldsymbol{u}(t) \end{cases}$$

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- The *stability* of the model is directly related to the eigenvalues ($\boldsymbol{\lambda}$) of $\boldsymbol{A}$.

$$\boldsymbol{x}(t) < \infty,\ t \to \infty \quad \text{if } \mathrm{Re}[\lambda_i] \leq 0,\ \forall i \in [1, n]$$

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- The *controllability* property is directly related to the *Controllability Matrix*:

$$\mathcal{C} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{A}\boldsymbol{B} & \boldsymbol{A}^2\boldsymbol{B} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix}$$

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- The *observability* property is directly related to the *Observability Matrix*:

$$\mathcal{O} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{C}\boldsymbol{A} & \boldsymbol{C}\boldsymbol{A}^2 & \cdots & \boldsymbol{C}\boldsymbol{A}^{n-1} \end{bmatrix}^T$$

Linearization by Taylor Series Expansion

Consider a nonlinear time-invariant system:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)) \\ \boldsymbol{y}(t) = \boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{u}(t)) \end{cases} .$$

Given steady-state operating points $\boldsymbol{x}_o$, $\boldsymbol{y}_o$ and $\boldsymbol{u}_o$, the dynamics of the system in the neighborhood of these points can be represented by the linear model:

$$\begin{cases} \Delta\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\Delta\boldsymbol{x}(t) + \boldsymbol{B}\Delta\boldsymbol{u}(t) \\ \boldsymbol{y}(t) = \boldsymbol{C}\Delta\boldsymbol{x}(t) + \boldsymbol{D}\Delta\boldsymbol{u}(t) \end{cases} ,$$

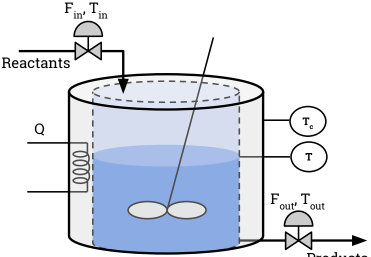
where

$$\boldsymbol{A} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}_o,\boldsymbol{u}_o} ; \quad \boldsymbol{B} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right|_{\boldsymbol{x}_o,\boldsymbol{u}_o} ; \quad \boldsymbol{C} = \left.\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}_o,\boldsymbol{u}_o} ; \quad \boldsymbol{D} = \left.\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{u}}\right|_{\boldsymbol{x}_o,\boldsymbol{u}_o}$$

and

$$\Delta\boldsymbol{x}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_o; \qquad \Delta\boldsymbol{u}(t) = \boldsymbol{u}(t) - \boldsymbol{u}_o.$$

- The first principle models are obtained through Conservation Laws from physics.



Mass Balance for Chemical Compounds

$$\begin{pmatrix}\text{Accumulation}\\ \text{of mass}\\ \text{in the system}\end{pmatrix} = \begin{pmatrix}\text{Mass flow}\\ \text{entering}\\ \text{system}\end{pmatrix} - \begin{pmatrix}\text{Mass flow}\\ \text{leaving}\\ \text{system}\end{pmatrix} \pm \begin{pmatrix}\text{Mass flow}\\ \text{from}\\ \text{reactions}\end{pmatrix}$$

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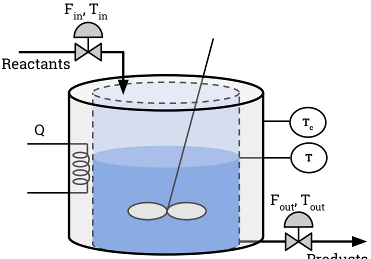

Mass Balance for Chemical Reactors

$$\frac{d(\rho_A)}{dt} = q(\rho_{in}^{(A)} - \rho_{out}^{(A)}) + \left(\sum_{\alpha X \to \beta A} \frac{1}{\beta} K_{XA}(T)(\rho_X)^{\alpha} \right) - \left(\sum_{\alpha A \to \beta X} \frac{1}{\beta} K_{AX}(T)(\rho_A)^{\alpha} \right)$$

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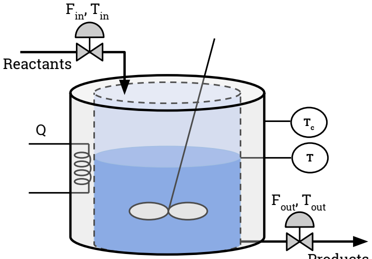

Conservation of Energy for Chemical Reactors

$$\begin{pmatrix}\text{Accumulation}\\ \text{of thermal energy}\\ \text{in the system}\end{pmatrix} = \begin{pmatrix}\text{Heat flow}\\ \text{entering}\\ \text{the system}\end{pmatrix} - \begin{pmatrix}\text{Heat flow}\\ \text{leaving}\\ \text{the system}\end{pmatrix} \pm \begin{pmatrix}\text{Entropy}\\ \text{contribution}\\ \text{from reactions}\end{pmatrix}$$

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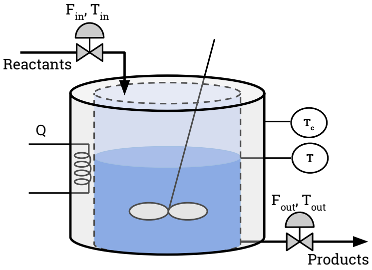

Conservation of Energy for Chemical Reactors

$$\begin{cases} \dfrac{d(T)}{dt} = q(T_{in} - T_{out}) + \eta(T_C - T) + \delta \displaystyle\sum_{\alpha A \to \beta X} K_{AX}(T)(\rho_A)^{\alpha} \Delta H_{AX} \\ \dfrac{d(T_C)}{dt} = \gamma Q + \beta(T - T_C) \end{cases}$$

- In the case of the reactor system in discussion, the models becomes...

Mathematical Model of Non-Isothermal CSTR

$$
\begin{cases}
\dfrac{d(\rho_A)}{dt} = q(\rho_{in}^{(A)} - \rho_A) - (K_1(T)\rho_A + K_3(T)\rho_A^2) \\[2ex]
\dfrac{d(\rho_B)}{dt} = -q\rho_B + K_1(T)\rho_A - K_2(T)\rho_B \\[2ex]
\dfrac{d(T)}{dt} = q(T_{in} - T) + \dfrac{k_W A_r}{\varrho C_p V_r}(T_C - T) \\
\qquad\qquad - \dfrac{1}{\varrho C_p}(K_1(T)\rho_A \Delta H_{AB} + K_2(T)\rho_B \Delta H_{BC} + K_1(T)\rho_A^2 \Delta H_{AC}) \\[2ex]
\dfrac{d(T_C)}{dt} = \dfrac{1}{m_K C_{pK}} Q + \dfrac{k_W A_r}{m_K C_{pK}}(T - T_C)
\end{cases}
$$

- We choose $\boldsymbol{x} = [\rho_A, \rho_B, T, T_C]^T$, $\boldsymbol{u} = [q, Q]^T$ and $\boldsymbol{y} = [\rho_B, T]^T$.
- We consider the steady-state point $\boldsymbol{x}_o = [1.23, 0.90, 134.14, 128.95]^T$ and $\boldsymbol{u}_o = [18.83, -4495.7]^T$.

The linearized state-space model is described by matrices:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \begin{bmatrix} -86.1 & 0 & -4.2 & 0 \\ 50.6 & -69.4 & 1.0 & 0 \\ 172.2 & 198.0 & -36.7 & 30.8 \\ 0 & 0 & 86.8 & -86.7 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 3.9 & 0 \\ -0.9 & 0 \\ -4.1 & 0 \\ 0 & 0.1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \\ \boldsymbol{y} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \end{cases}$$

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- All the eigenvalues are real and negative. The system is **stable**.

$$\boldsymbol{\lambda} = [-16.79, -54.84, -86.33, -121.01]$$

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- The Controllability Matrix has full-row rank. The system is **controllable**.

$$\texttt{rank}\left(\mathcal{C}\right) = \texttt{rank}\left(\begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} & \boldsymbol{A}^2\boldsymbol{B} & \boldsymbol{A}^3\boldsymbol{B} \end{bmatrix}\right) = 4$$

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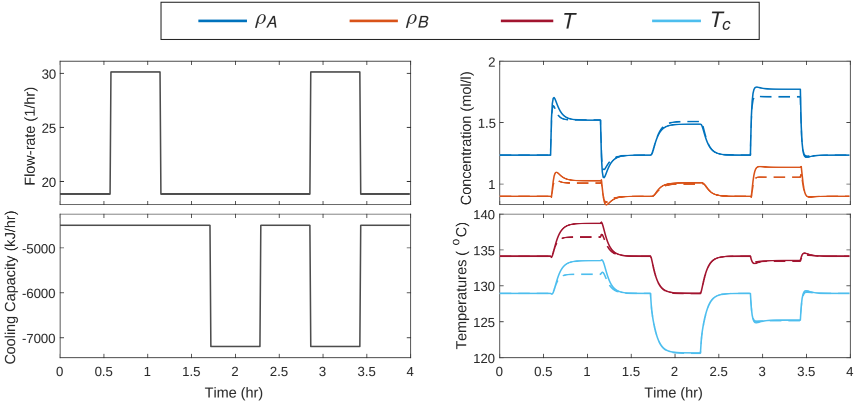
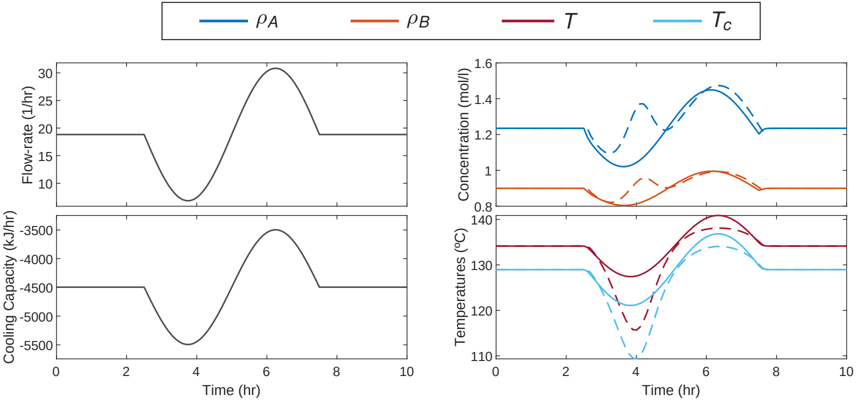
- The Observability Matrix has full-column rank. The system is **observable**.

$$\texttt{rank}\,(\mathcal{O}) = \texttt{rank}\left(\begin{bmatrix} \boldsymbol{C} & \boldsymbol{CA} & \boldsymbol{CA}^2 & \boldsymbol{CA}^3 \end{bmatrix}\right) = 4$$


ρ_A
ρ_B
T
T_c
Flow-rate (1/hr)
Cooling Capacity (kJ/hr)
Concentration (mol/l)
Temperatures (^{o}C)
Time (hr)
Time (hr)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T
T_c
Flow-rate (1/hr)
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Temperatures (ºC)
Time (hr)
Time (hr)

State-Feedback Controllers

Full State-Feedback Controller

The input action $\boldsymbol{u}(t)$ is calculated by the linear control law $\pi(\cdot)$ through state-feedback as:

$$\boldsymbol{u}(t) = \pi(\boldsymbol{r}(t), \boldsymbol{x}(t)) = \boldsymbol{r}(t) - \boldsymbol{K}\boldsymbol{x}(t),$$

where $\boldsymbol{r} : \mathbb{R} \to \mathbb{R}^n$ is a state reference signal and $\boldsymbol{K} \in \mathbb{R}^{r \times n}$ is the *feedback gain matrix*.

- Substituting this control law to the model results in the closed-loop representation:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}(\boldsymbol{r}(t) - \boldsymbol{K}\boldsymbol{x}(t)) \\ &= (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{r}(t) \\ &= \boldsymbol{A}_{cl}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{r}(t) \end{aligned}$$

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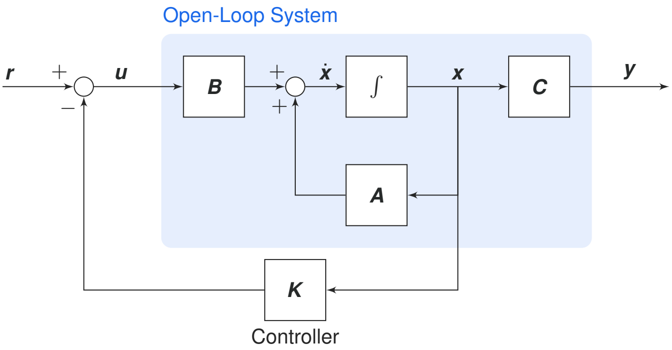

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- **Pole-Placement Property:** If $(\boldsymbol{A}, \boldsymbol{B})$ is *controllable*, we can assign the eigenvalues of $\boldsymbol{A}_{cl}$ anywhere in the complex plane by means of designing $\boldsymbol{K}$.

We discuss the two modes of operation for state-feedback controllers.

Regulation

- **Objective:** maintain the system in some position (the zero-state) and reject disturbances.
- Special case of the state-feedback when $\boldsymbol{r}(t) = 0$.
- Direct from the controller formulation.

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- **Objective:** maintain the system in some position (the zero-state) and reject disturbances.
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- Direct from the controller formulation.

Reference Tracking

- **Objective:** drive the states of the system towards a non-constant reference signal.
- Special case of the state-feedback when $\boldsymbol{r}(t)$ is non-constant.
- Raises the need for *integral action*.

We discuss the two modes of operation for state-feedback controllers.

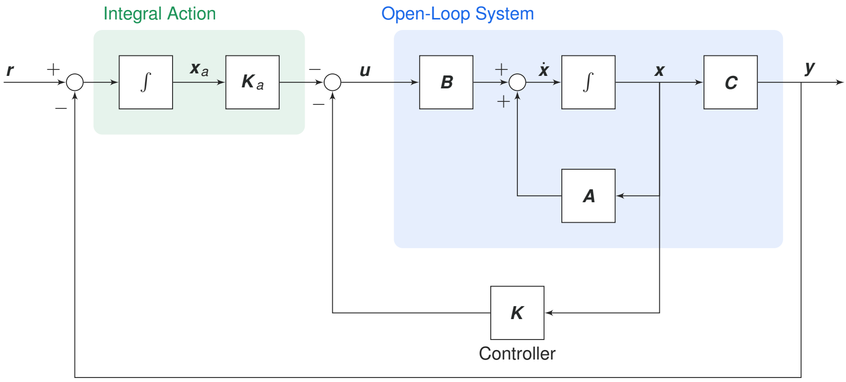
State-Feedback with Integral Action

Consider a State-space system and augmented state $\boldsymbol{x}_a : \mathbb{R} \to \mathbb{R}^p$ defined by:

$$\boldsymbol{x}_a(t) = \int_0^t \boldsymbol{r}(\tau) - \boldsymbol{y}(\tau) d\tau \Longrightarrow \dot{\boldsymbol{x}}_a(t) = \boldsymbol{r}(t) - \boldsymbol{y}(t).$$

A robust tracking (or servo) controller, defined by the gain $\tilde{\boldsymbol{K}} = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{K}_a \end{bmatrix}$, for $\boldsymbol{K} \in \mathbb{R}^{r \times n}$ and $\boldsymbol{K}_a \in \mathbb{R}^{r \times p}$, operates on the following augmented version of the original system:

$$\begin{cases} \begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{x}}_a(t) \end{bmatrix} & = \begin{bmatrix} \boldsymbol{A} - \boldsymbol{B}\boldsymbol{K} & -\boldsymbol{B}\boldsymbol{K}_a \\ -\boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}_a(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}_p \end{bmatrix} \boldsymbol{r}(t) \\ \boldsymbol{y}(t) & = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}_a(t) \end{bmatrix} \end{cases}.$$


Integral Action
Open-Loop System
r
+
−
∫
x_a
K_a
−
−
u
B
+
+
$\dot{x}$
∫
x
C
y
A
K
Controller

Optimal Control

Optimal Controller - General Formulation

The input action $\boldsymbol{u}(t) \in \mathbb{R}^r$ is *optimal* if an *optimal control law* $\pi^* : \mathbb{R}^{n \times n \times 1} \rightarrow \mathbb{R}^r$ can be found as:

$$\boldsymbol{u}(t) = \pi^*(\boldsymbol{x}, \boldsymbol{r}, t) = \arg\min_{\boldsymbol{u}} J(\boldsymbol{x}, \boldsymbol{r}, t),$$

where $J : \mathbb{R}^{n \times n \times 1} \rightarrow \mathbb{R}$ is known as the *cost function*.

- **This is still a state-feedback controller**: the cost function is a function of past values of the state and reference signals.

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- Reduces the need for hand-designing the feedback gain or any other control law for the system, given some specifications.
- The type of the optimal controller depends on the choice of $J(\cdot)$ and (possibly) on how the optimization is solved.

Finite-Horizon Optimal Regulators

A *Finite-Horizon Optimal Regulator* is defined as any controller whose optimal policy over a time interval $t \in [t_0, T]$ minimizes the cost functional:

$$J(\boldsymbol{x}, \boldsymbol{u}, t_0) = \int_{t_0}^{T} l(\boldsymbol{x}, \boldsymbol{u}, \tau) d\tau + l_f(\boldsymbol{x}, T),$$

where $l(\cdot) : \mathbb{R}^{n \times r \times 1} \rightarrow \mathbb{R}$ and $l_f(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$ are, respectively, the *trajectory* and *terminal loss functions*. In the case that $t_0 = 0$, T is also known as the *control horizon*.

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- If $l(\boldsymbol{x}, \boldsymbol{u}, \tau) \geq 0$ and $l_f(\boldsymbol{x}, T) \geq 0$, $\forall \boldsymbol{x}, \boldsymbol{u}, \tau$, then the cost function penalizes deviations from the zero-state.

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- Characterizes an entire class of optimal controllers based on finite horizon optimization.

Hamilton-Jacobi-Bellman (HJB) equation

Consider a finite-horizon cost function for a system described by $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t)$:

$$V(\boldsymbol{x}, \boldsymbol{u}, t_0) = \int_{t_0}^{T} l(\boldsymbol{x}, \boldsymbol{u}, \tau) d\tau + l_f(\boldsymbol{x}(T))$$

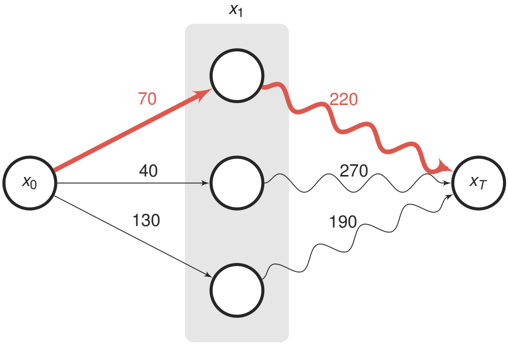
Minimizing any functional in the form of $V(\cdot)$ is equivalent to determining the solution of the *Hamilton-Jacobi equation**:

$$\frac{\partial V^*}{\partial t} = -\min_{u(t)} \left[l(\boldsymbol{x}, \boldsymbol{u}, t) + \left[\frac{\partial V^*}{\partial x} \right]^T \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t) \right]$$

and the boundary condition:

$$V^*(\boldsymbol{x}, T) = l_f(\boldsymbol{x}(T)).$$

- The HJB equation provides a *sufficient condition* for finite-horizon optimal controllers.
- This class of optimal controllers is related to the *Bellman's Principle of Optimality*.



Linear Quadratic Regulator (LQR)

Given a linear State-Space system in the form:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t) \end{cases} .$$

A *Linear Quadratic Regulator* (LQR) for this system is an optimal controller defined by the quadratic cost function:

$$J(\boldsymbol{x}, \boldsymbol{u}, t_0) = \int_{t_0}^{T} \left(\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u}\right) dt + \boldsymbol{x}^T(T)\boldsymbol{Q}_f\boldsymbol{x}(T),$$

where is assumed that $\boldsymbol{Q}, \boldsymbol{Q}_f \succ 0$ and $\boldsymbol{R} \succ 0$.

- Defined for a *linear* system with *quadratic* cost function.

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- Defined for a *linear* system with *quadratic* cost function.
- The weights $\boldsymbol{Q}$, $\boldsymbol{Q}_f$ and $\boldsymbol{R}$ are chosen to reflect how the designer wish to penalize state-deviation or control effort.
- **Optimization:** closed-form solution for $\pi^*(\boldsymbol{x}, \boldsymbol{u}, t_0)$.

LQR Control Action from Dynamic Programming

Given a Linear Quadratic Regulator, the optimal action produced is given by:

$$\boldsymbol{u}^*(t) = -\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}(t)\boldsymbol{x}(t),$$

where $\boldsymbol{P}(t)$ is the solution of the matrix Riccati differential equation:

$$-\dot{\boldsymbol{P}}(t) = \boldsymbol{A}^T\boldsymbol{P}(t) + \boldsymbol{P}(t)\boldsymbol{A} - \boldsymbol{P}(t)\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}(t) + \boldsymbol{Q},$$

with terminal condition $\boldsymbol{P}(T) = \boldsymbol{Q}_f$.

- Notice that:

$$\begin{aligned}
\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}^*(t) \\
&= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}(-\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}(t)\boldsymbol{x}(t)) \\
&= (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}(t))\boldsymbol{x}(t) \\
&= (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}(t))\boldsymbol{x}(t)
\end{aligned}$$

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- The optimization results in a *linear* and *time-varying* control law:

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- **“Open-Loop" Optimization:** the matrix $\boldsymbol{K}(t)$ does not depend on past (or actual) states. The Ricatti differential equation can be solved off-line.
- **Remark:** nevertheless, the optimal controller performs corrective action.

LQR with Integral Action

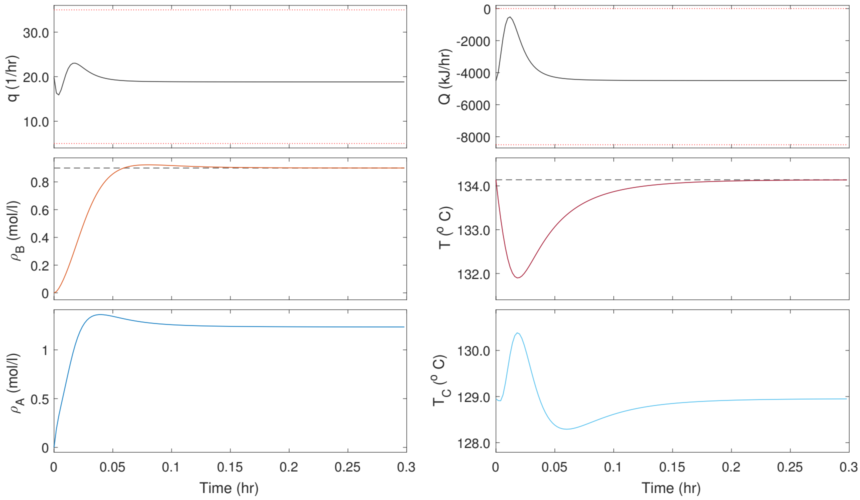
Given a linear State-Space system augmented with state $\dot{\boldsymbol{x}}_a(t) = \boldsymbol{r}(t) - \boldsymbol{C}\boldsymbol{x}(t)$:

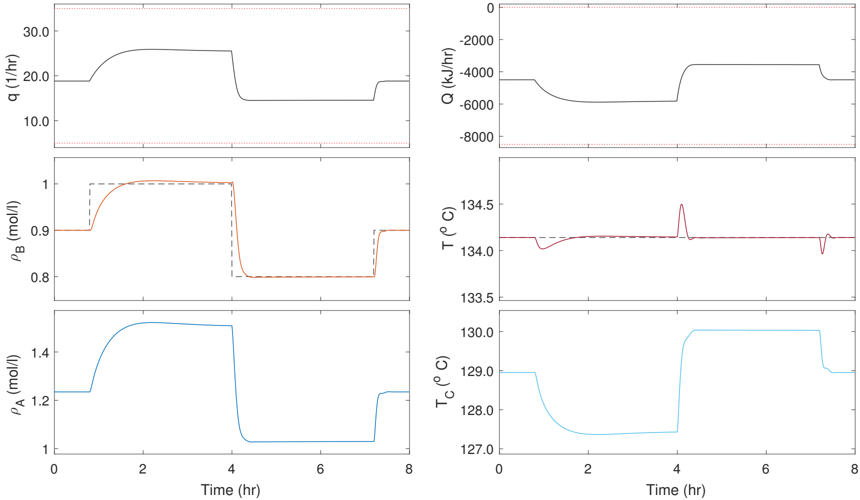
$$\begin{cases} \begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{x}}_a(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ -\boldsymbol{C} & \boldsymbol{0} \end{bmatrix}}_{\tilde{\boldsymbol{A}}} \underbrace{\begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}_a(t) \end{bmatrix}}_{\tilde{\mathbf{x}}(t)} + \underbrace{\begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{bmatrix}}_{\tilde{\boldsymbol{B}}} \boldsymbol{u}(t) + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I} \end{bmatrix} \boldsymbol{r}(t) \\ \boldsymbol{y}(t) = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}_a(t) \end{bmatrix} \end{cases}.$$

A *Linear Quadratic Servo* (LQ-Servo) for this system is an optimal controller defined by the quadratic cost function:

$$J(\boldsymbol{x}, \boldsymbol{u}, t_0) = \int_{t_0}^{T} \left(\tilde{\mathbf{x}}^T \tilde{\boldsymbol{Q}} \tilde{\mathbf{x}} + \boldsymbol{u}^T \boldsymbol{R} \boldsymbol{u} \right) dt + \tilde{\mathbf{x}} \tilde{\boldsymbol{Q}}_f \tilde{\mathbf{x}}(T),$$

where is assumed that $\tilde{\boldsymbol{Q}}, \tilde{\boldsymbol{Q}}_f \succ 0$ and $\boldsymbol{R} \succ 0$.


q (1/hr)
30.0
20.0
10.0
Q (kJ/hr)
0
-2000
-4000
-6000
-8000
ρ_B (mol/l)
0.8
0.6
0.4
0.2
0
T (^{o} C)
134.0
133.0
132.0
ρ_A (mol/l)
1
0.5
0
T_C (^{o} C)
130.0
129.0
128.0
0
0.05
0.1
0.15
0.2
0.25
0.3
Time (hr)
Time (hr)


30.0
20.0
10.0
q (1/hr)
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-2000
-4000
-6000
-8000
Q (kJ/hr)
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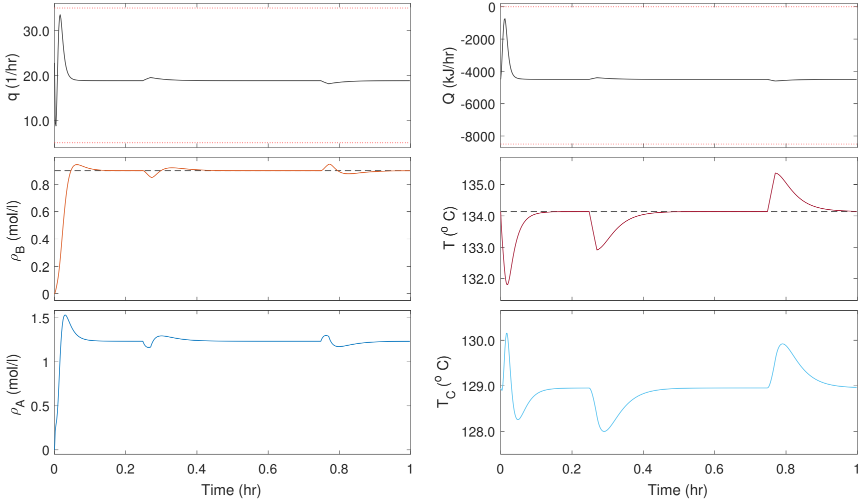
0.9
0.8
ρ_B (mol/l)
134.5
134.0
133.5
T (º C)
1.4
1.2
1
ρ_A (mol/l)
130.0
129.0
128.0
127.0
T_C (º C)
0
2
4
6
8
Time (hr)
Time (hr)


30.0
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q (1/hr)
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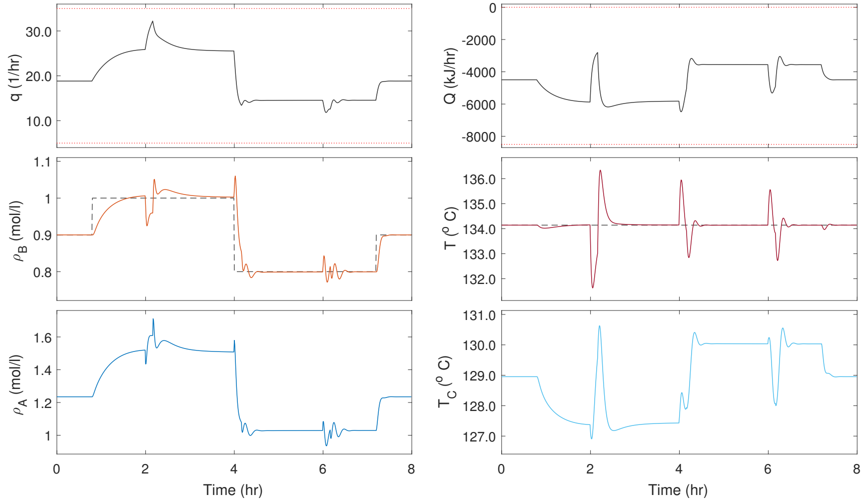
Time (hr)


q (1/hr)
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128.0
127.0
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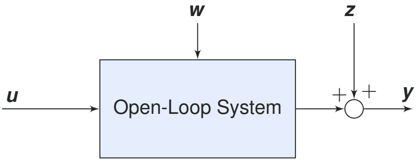
2
4
6
8
Time (hr)

Optimal State Estimation

- We can **never** access real state of a physical system.



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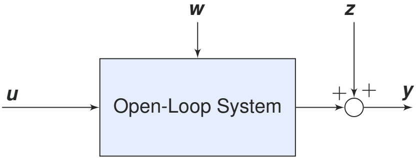

- **Stochastic State-Space:** formulation that accounts for process and measurements uncertainty ($\boldsymbol{w}(t)$ and $\boldsymbol{v}(t)$, respectively).

$$\begin{cases} \dot{\boldsymbol{x}}(t) & = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{w}(t) \\ \boldsymbol{y}(t) & = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{v}(t) \end{cases}$$

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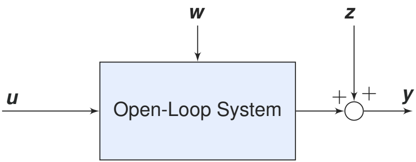
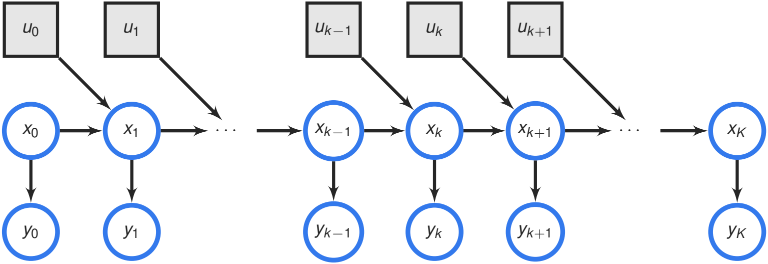


- In probabilistic terms, we interpret the system as a Hidden Markov Model (HMM).



Closed-Loop Observer

Given a system in State-Space with output signal $\boldsymbol{y}(t) : \mathbb{R} \rightarrow \mathbb{R}^p$ and an observer gain $\boldsymbol{L} \in \mathbb{R}^{n \times p}$, the estimated state-vector $\hat{\boldsymbol{x}}(t)$ is represented by the system:

$$\dot{\hat{\boldsymbol{x}}}(t) = \boldsymbol{A}\hat{\boldsymbol{x}}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{L}\left(\boldsymbol{y}(t) - \boldsymbol{C}\hat{\boldsymbol{x}}(t)\right),$$

or, equivalently:

$$\dot{\hat{\boldsymbol{x}}}(t) = \left(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}\right)\hat{\boldsymbol{x}}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{L}\boldsymbol{y}(t).$$

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- Consider a variable $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t)$ such that:

$$\begin{aligned}\dot{\boldsymbol{e}} &= \boldsymbol{x} - \hat{\boldsymbol{x}} \\ &= \left(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}\right)\left(\boldsymbol{x} - \hat{\boldsymbol{x}}\right) \\ &= \left(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}\right)\boldsymbol{e}\end{aligned}$$

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- **Control-Estimator Duality:** the design of $\boldsymbol{L}$ follows the same design procedure of $\boldsymbol{K}$.

Kalman-Bucy Optimal Filter

Consider a model subject to a additive process noise variable $\boldsymbol{w}(t) \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q}_{kf})$ and a measurement noise variable $\boldsymbol{v}(t) \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{R}_{kf})$. In this case, for an estimated state $\hat{\boldsymbol{x}}(t)$ at time t, the error covariance:

$$\boldsymbol{J}(\boldsymbol{x}, \hat{\boldsymbol{x}}, t) = \mathbb{E}\left\{[\boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t)][\boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t)]^T\right\}$$

is minimized by a $\hat{\boldsymbol{x}}(t)$ obtained through the system:

$$\dot{\hat{\boldsymbol{x}}}(t) = \boldsymbol{A}\hat{\boldsymbol{x}}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{K}_e(t)\left(\boldsymbol{y}(t) - \boldsymbol{C}\hat{\boldsymbol{x}}(t)\right),$$

where $\boldsymbol{K}_e(t) = \boldsymbol{P}_e(t)\boldsymbol{C}\boldsymbol{R}^{-1}$, being $\boldsymbol{P}_e(t)$ the solution of the Riccati differential matrix equation:

$$\dot{\boldsymbol{P}}_e(t) = \boldsymbol{A}\boldsymbol{P}_e(t) + \boldsymbol{P}_e(t)\boldsymbol{A}^T - \boldsymbol{P}_e(t)\boldsymbol{C}^T\boldsymbol{R}_{kf}^{-1}\boldsymbol{C}\boldsymbol{P}_e(t) + \boldsymbol{Q}_{kf},$$

with initial condition $\boldsymbol{P}_e(t_0) = \mathbb{E}\left\{[\boldsymbol{x}(t_0) - \bar{\boldsymbol{x}}(t_0)][\boldsymbol{x}(t_0) - \bar{\boldsymbol{x}}(t_0)]^T\right\}$ for $t_0 > -\infty$.

The Separation Principle

Given a system in State-Space with a Luenberger observer of gain $\boldsymbol{L}$ and state-feedback controller of gain $\boldsymbol{K}$, the closed-loop eigenvalues contributions of $(\boldsymbol{A} - \boldsymbol{BK})$ are independent from those of $(\boldsymbol{A} - \boldsymbol{LC})$.

- It can be shown that the closed-loop dynamics of the system and the error reduces to:

$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{e}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} - \boldsymbol{BK} & -\boldsymbol{BK} \\ \boldsymbol{0} & \boldsymbol{A} - \boldsymbol{LC} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{e} \end{bmatrix} + \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{r}$$

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- **Direct Result:** the controller and the estimator can be designed separately, and they are *dual problems*.

Linear Quadratic Gaussian (LQG) Controller

Consider a stochastic system in State-Space representation:

$$\begin{cases} \dot{\boldsymbol{x}}(t) & = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{w}(t) \\ \boldsymbol{y}(t) & = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{v}(t) \end{cases},$$

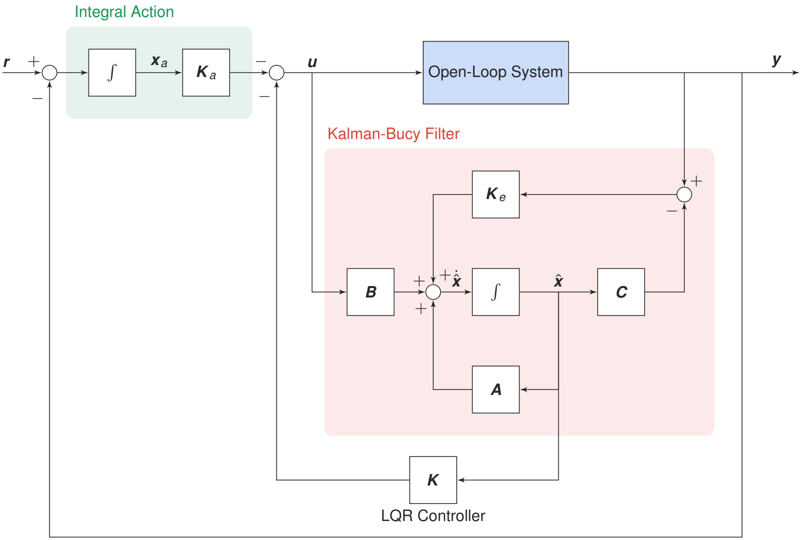
whose state-vector is determined by a Kalman-Bucy filter and whose input signal is calculated by a finite-horizon LQR. The Linear Quadratic Gaussian (LQG) is defined as:

$$\dot{\hat{\boldsymbol{x}}}(t) = [\boldsymbol{A} - \boldsymbol{K}_e(t)\boldsymbol{C} - \boldsymbol{B}\boldsymbol{K}(t)]\,\hat{\boldsymbol{x}}(t) + \boldsymbol{K}_e(t)\boldsymbol{y}(t),$$

where $\boldsymbol{K}(t) = \boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}(t)$ and $\boldsymbol{K}_e(t) = \boldsymbol{P}_e(t)\boldsymbol{C}\boldsymbol{R}^{-1}$ are, respectively, the LQR and Kalman-Bucy gains for matrices $\boldsymbol{P}(t)$ and $\boldsymbol{P}_e(t)$ that solve the Riccati differential equations:

$$\begin{cases} -\dot{\boldsymbol{P}}(t) = \boldsymbol{A}^T\boldsymbol{P}(t) + \boldsymbol{P}(t)\boldsymbol{A} - \boldsymbol{P}(t)\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}(t) + \boldsymbol{Q} \\ \dot{\boldsymbol{P}}_e(t) = \boldsymbol{A}\boldsymbol{P}_e(t) + \boldsymbol{P}_e(t)\boldsymbol{A}^T - \boldsymbol{P}_e(t)\boldsymbol{C}^T\boldsymbol{R}_{kf}^{-1}\boldsymbol{C}\boldsymbol{P}_e(t) + \boldsymbol{Q}_{kf} \end{cases}$$

for boundary conditions $\boldsymbol{P}(T) = \boldsymbol{Q}_f$ and $\boldsymbol{P}_e(t_0) = \mathbb{E}\left\{[\boldsymbol{x}(t_0) - \bar{\boldsymbol{x}}(t_0)][\boldsymbol{x}(t_0) - \bar{\boldsymbol{x}}(t_0)]^T\right\}$, respectively.


Integral Action
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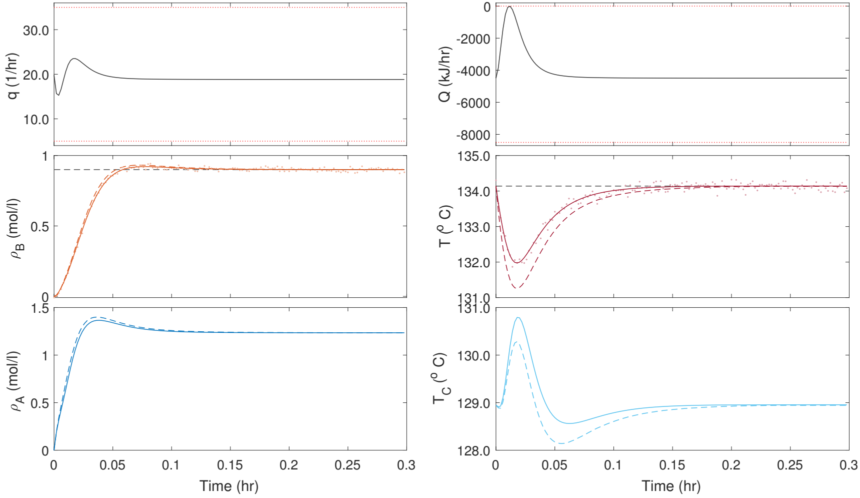
LQR Controller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0
T (^{o} C)
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134.0
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132.0
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ρ_A (mol/l)
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0.5
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T_C (^{o} C)
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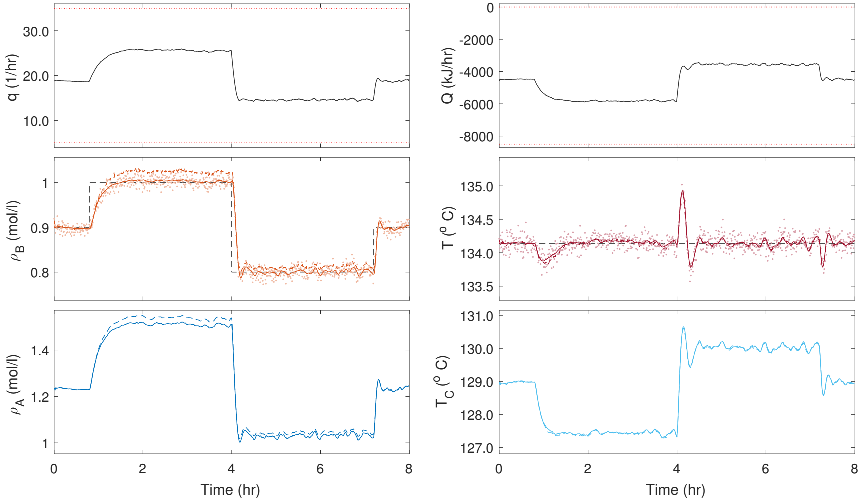
Time (hr)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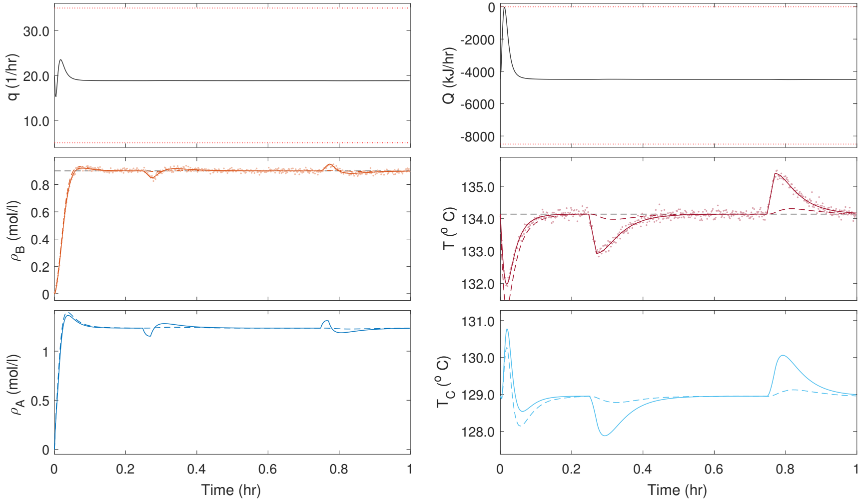
Time (hr)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T_C (^{o} C)
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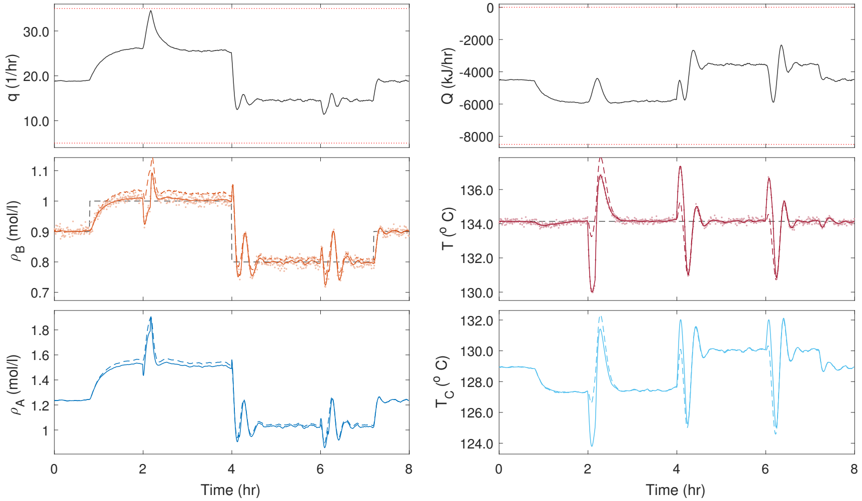
Time (hr)
Time (hr)

Optimal State Estimation



Conclusion

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- We have demonstrated the theoretical foundations of optimal control and its application to a specific challenging problem.
- There were noted some limitations:
  - The state estimator has not able to reconstruct the state information when far from the steady-state point.
  - The integral action directly from noisy measurements demonstrated a concerning behavior.
  - The optimization was performed in a unconstrained fashion, which is unpractical.
- This work opens the possibility of evaluating more advanced control and estimation techniques using the same mathematical framework.

Thank you!

Questions?