Seminars on Process Systems Engineering
Aalto University, September 25, 2025

Predictive control and feedback equilibrium seeking for sustainable water resource recovery

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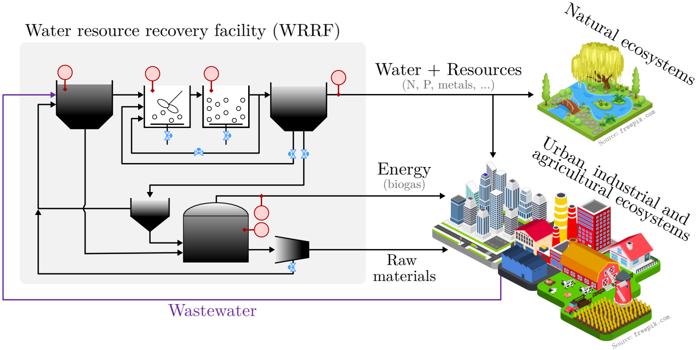
Predictive control, operating WWTPs as self-sufficient WRRFs

Paradigm shift: Wastewater as a sustainable source of water, energy, and raw materials



Proposal $\mathcal{A}$**:** Predictive control to enable self-sufficient water resource recovery facilities

Predictive control, operating WWTPs as self-sufficient WRRFs

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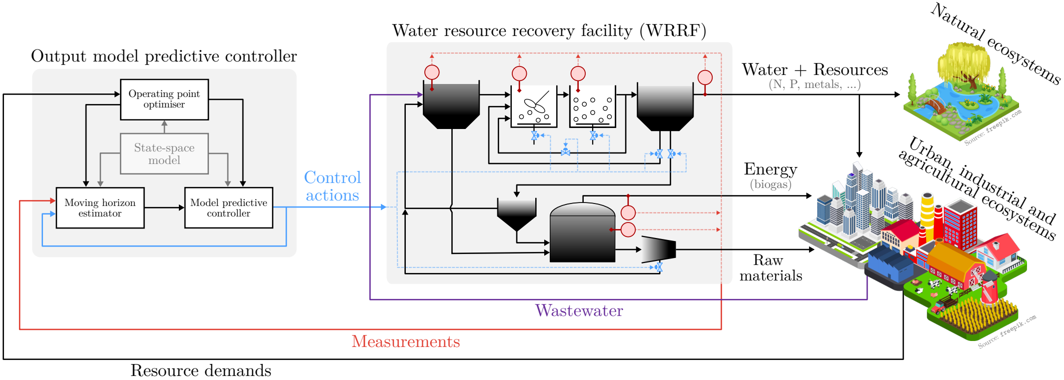
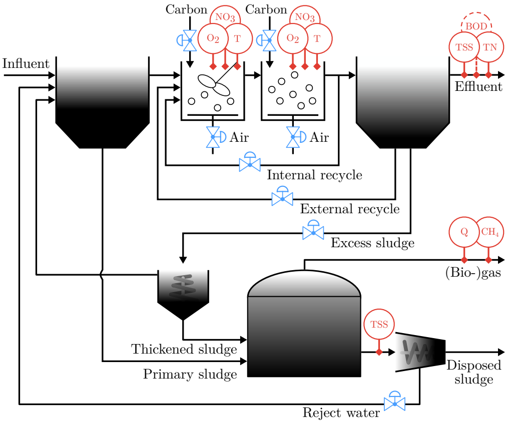


Proposal *A*: Predictive control to enable self-sufficient water resource recovery facilities

Predictive control, operating WWTPs as self-sufficient WRRFs (cont.)

We consider the task of operating a **biological wastewater treatment plant** (WWTP, secondary treatment) as a **water resource recovery facility** (WRRF)



- **Primary objective:**
  on demand, produce effluent water of specific quality

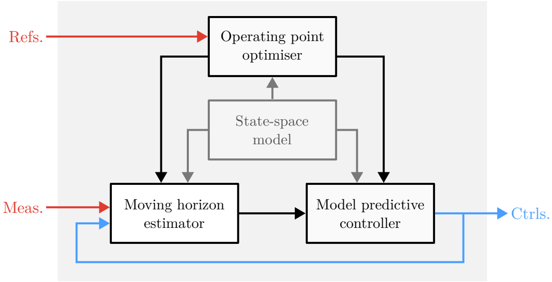
Water class	Biochemical profile		
	TSS	BOD	TN
A	≤ 30 g/m^3	≤ 10 g/m^3	≤ 15 g/m^3
B	≤ 30 g/m^3	≤ 15 g/m^3	≤ 30 g/m^3
C	≤ 30 g/m^3	≤ 20 g/m^3	≤ 45 g/m^3

- **Secondary objective:**
  produce biogas to ensure nonpositive energy cost index,

$$\mathrm{ECI} = \mathrm{AE} + \mathrm{PE} + \mathrm{ME} - \eta_E \mathrm{MP} + \max(0, \mathrm{HE} - \eta_H \mathrm{MP})$$

Predictive control, general architecture and specific configuration

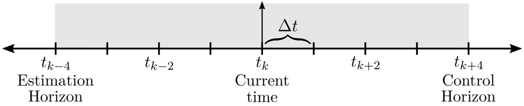
Output model predictive controller



We design a **model-based** output-feedback controller

$$\frac{d}{dt}x(t) = f(x(t), u(t), w(t)) \qquad (dynamics)$$
$$y(t) = g(x(t), u(t)) \qquad (measurements)$$
$$z(t) = h(x(t), u(t)) \qquad (performance)$$

which autonomously operates the plant in cycles



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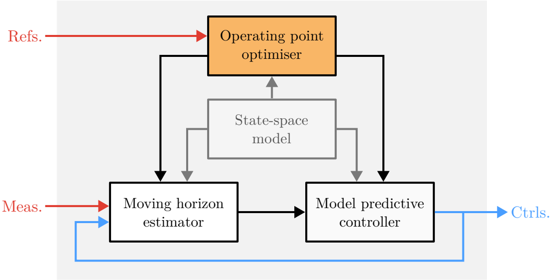

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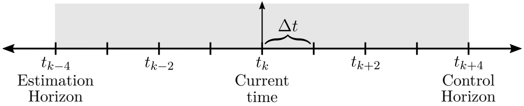

Operating point optimiser

Determines an *operating point* which yields the *desired key performance indicators*

$\Longrightarrow$

Moving horizon estimator

Determines an *estimate of the current state* based on *measurement data*

$\Longrightarrow$

Model predictive control

Determines a *sequence of control actions* to regulate the plant to the *operating point*

Predictive control, general architecture and specific configuration

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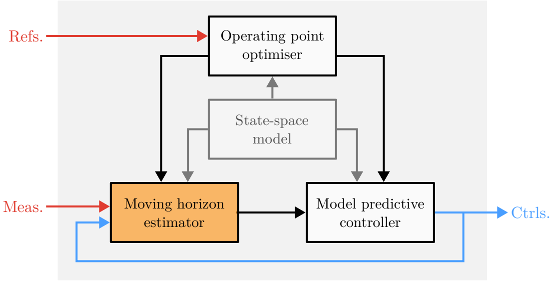

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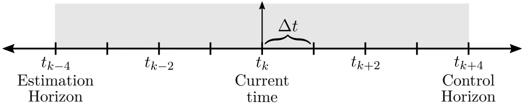

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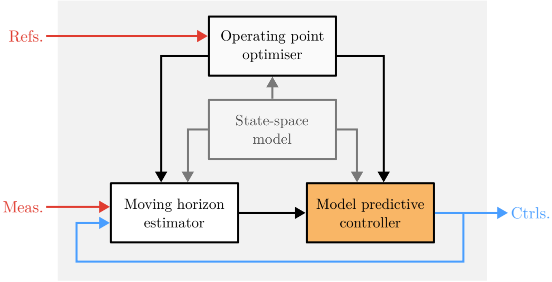

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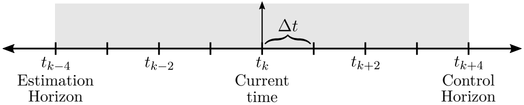

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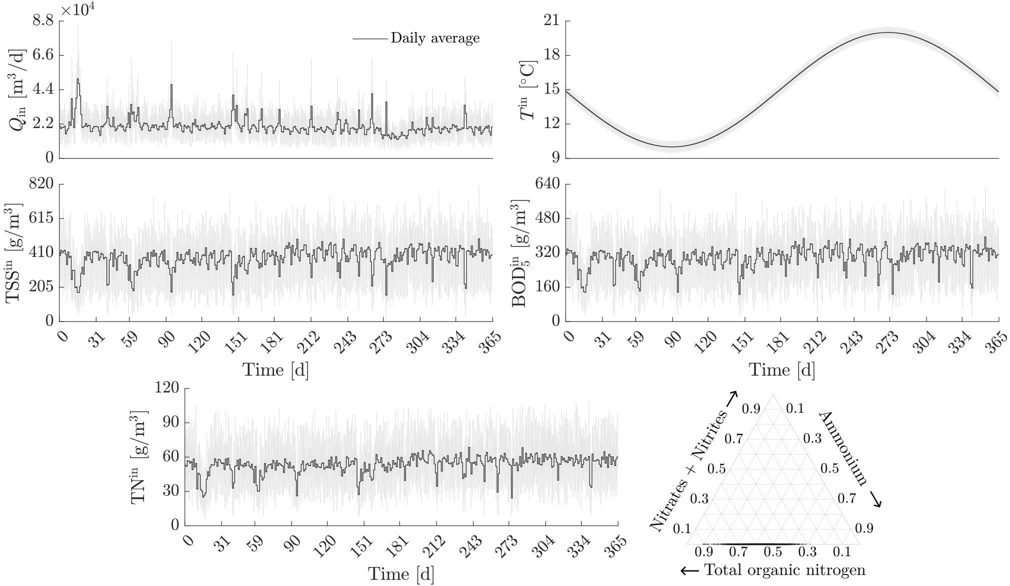
Predictive control, experimental results



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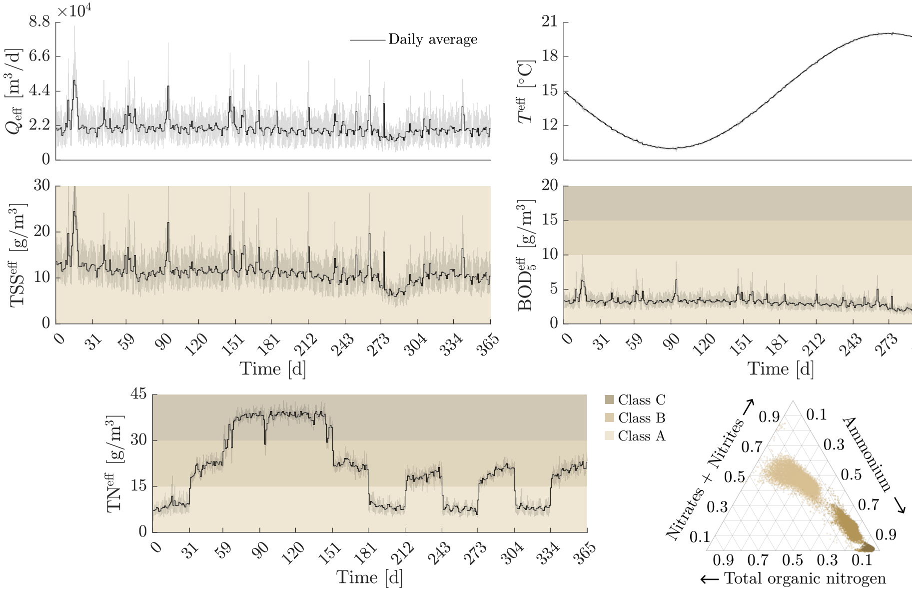
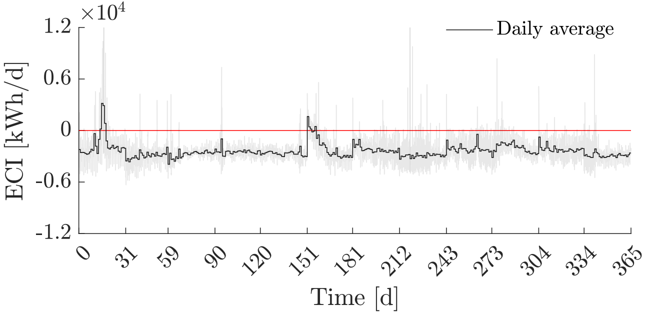


Predictive control, experimental results

(On average) the controller renders the WRRF energetically self-sufficient



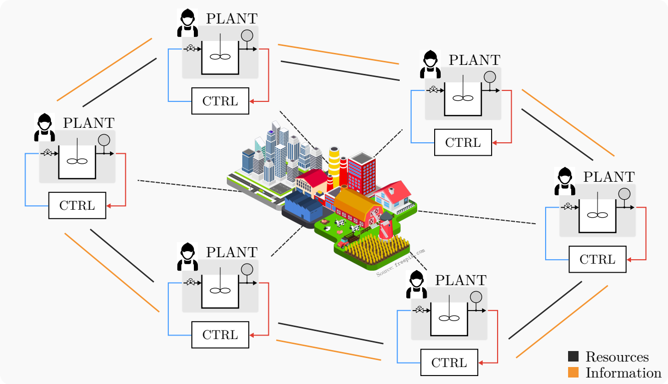
Summary

Total production: 83 GWh

Energy output: 2385 ± 1312 kWh

Intro, control in (large-scale) cyber-physical systems

Decentralised Market/Infrastructure



Goal:

⇝ Optimal control in multi-agent settings

Challenges:

⇝ Large-scale network of subsystems

⇝ Non-cooperative decision-making agents

⇝ Subsystems are only partially observed

⇝ Information exchange often asymmetric

Feedback controllers (**control theory**)
+
Equilibrium-seeking (**game theory**)

We propose a systematic approach to **seeking feedback equilibrium strategies** in N_P-players non-cooperative dynamic games

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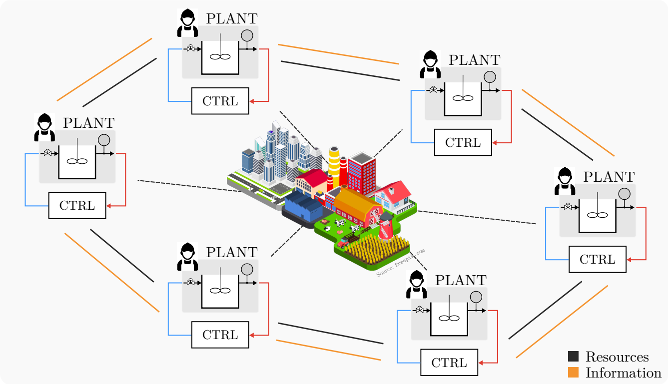

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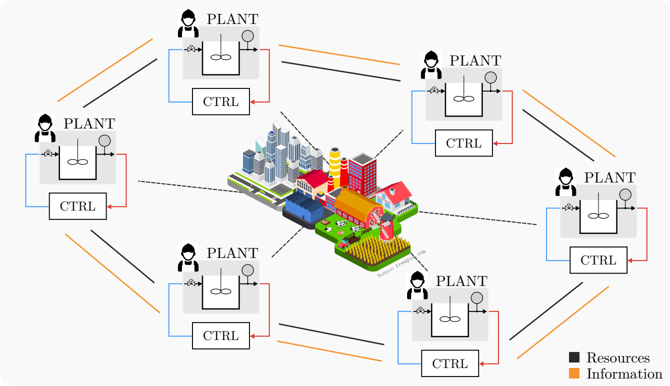

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Intro, non-cooperative games and equilibrium-seeking

A non-cooperative game is defined as the tuple

$$\mathcal{G} := (\underbrace{\{1,\ldots,N_P\}}_{\text{set of players } \mathcal{P}}, \underbrace{\{L^1,\ldots,L^{N_P}\}}_{\text{objective functions}}, \underbrace{\{\mathcal{S}^1,\ldots,\mathcal{S}^{N_P},\mathcal{S}^{\text{global}}\}}_{\text{admissible strategies}})$$

ASSUMPTION: Players prefer best-response strategies

$$BR^p(s^{-p}) := \arg\min_{s^p \in \mathcal{S}^p}\{\, L^p(s^p \mid s^{-p}) : (s^p, s^{-p}) \in \mathcal{S}^{\text{global}} \,\}$$

$$\Downarrow$$

SOLUTION CONCEPT: generalized Nash equilibrium (GNE)

$$(s^{1^\star},\ldots,s^{N_P^\star}) \in BR^1(s^{-1^\star}) \times \cdots \times BR^{N_P}(s^{-N_P^\star})$$

$$\Downarrow$$

PROBLEM STATEMENT: Fixed point problem

Find $s^\star \in \mathbb{R}^{N_s}$ such that $s^\star \in BR(s^\star)$

Algorithm: GNE-Seeking via BRD

for $k = 0, 1, 2, \ldots$ **do**
  for *each player* $p \in \mathcal{P}$ **do**
    $s^p_{k+1} := (1-\eta)s^p_k + \eta BR^p(s^p_k, s^{-p}_k)$;

- ✓ Realistic routine
- ✓ Fully decentralised
- × Scalability
- × Robustness to errors
- × Convergence guarantees

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SOLUTION CONCEPT (REFINED): variational GNE (vGNE)

$$\langle F(s^\star), s - s^\star \rangle \geq 0, \quad s \in \mathcal{S}_{\mathcal{G}}$$

$$\Downarrow$$

PROBLEM STATEMENT: Monotone inclusion / VI problem*

Find $s^\star \in \mathbb{R}^{N_s}$ such that $0 \in F(s^\star) + N_{\mathcal{S}_{\mathcal{G}}}(s^\star)$

Algorithm: vGNE-Seeking via FB-Splitting

for $k = 0, 1, 2, \dots$ **do**
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  for *coordinator* **do**
    $s_{k+1} \coloneqq \text{proj}_{\mathcal{S}_{\mathcal{G}}}(s_+)$;

- ✓ Realistic routine
- ✓ (Often) scalable
- ✓ Convergence guarantees
- × Semi-decentralised
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* Assuming $F(s) = \left(\nabla L^p(s^p \mid s^{-p}) \right)_{p \in \mathcal{P}}$ is maximal monotone and $\mathcal{S}_{\mathcal{G}} = (\mathcal{S}^1 \times \cdots \times \mathcal{S}^{N_P}) \cap \mathcal{S}^{\text{global}}$ is closed convex

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GFNE, problem formulation for (stochastic) dynamic games

(Stochastic) Dynamic games

In *(stationary) feedback Nash equilibrium* problems $\mathcal{G}_{\infty}^{\mathrm{LQ}}$, strategies are feedback policies

$$s^p \rightsquigarrow \qquad K^p : x_t \mapsto u_t := \Phi_K^p * x_t$$

computing actions $(u = (u^1, \ldots, u^{N_P}))$ to stabilize the state (x) of the stochastic system

$$x_{t+1} = Ax_t + \sum_p B_u^p u_t^p + w_t$$

while satisfying both **structural** (on K^p) and **operational** (on $\{x, u\}$) constraints

GFNE, problem formulation and system level parametrisation

$$
BR^p(K^{-p}) : \begin{cases} \underset{K^p}{\text{minimize}} & \mathrm{E}\left[\sum_{t=0}^{\infty}\left(\|W_x^p x_t\|_2^2 + \|W_u^p u_t^p\|_2^2\right)\right] \\ \underset{\substack{\forall t\,\in\,\mathbb{N} \\ \forall \bar{p}\,\in\,\mathcal{P}}}{\text{subject to}} & x_{t+1} = A x_t + \sum_{\bar{p}} B_u^{\bar{p}} u_t^{\bar{p}} + w_t, \quad x_0 = 0 \\ & u_t^{\bar{p}} = \Phi_K^{\bar{p}} * x_t, \\ & K^p \in \{\text{Stability constraints}\} \cap \{\text{Information constraints}\}, \\ & G_x x_t + G_u^p u_t \preceq 1 \end{cases}
$$

w is a AWGN process, that is, $\mathbb{E}w_t = 0$ and $\mathbb{E}(w_t w_{t-\tau}^{\mathsf{T}}) = \delta_\tau B_w B_w^{\mathsf{T}}$

[dynamics]

[control policy]

[struct. constraints]

[operat. constraints]

Problem:

The fixed-points of $BR(\cdot)$ cannot be searched through monotone inclusion problems

$\Downarrow$

!! No systematic method to compute GFNEs !!

Solution:

A parametrisation of K^p to design an alternative but equivalent $BR(\cdot)$ mapping

$\Downarrow$

System level synthesis (SLS)

GFNE, problem formulation and system level parametrisation

$$BR^p(K^{-p}): \begin{cases} \underset{K^p}{\text{minimize}} & \mathrm{E}\left[\sum_{t=0}^{\infty}\left(\|W_x^p x_t\|_2^2 + \|W_u^p u_t^p\|_2^2\right)\right] \\ \underset{\forall \bar{p} \in \mathcal{P}}{\underset{\forall t \in \mathbb{N}}{\text{subject to}}} & x_{t+1} = Ax_t + \sum_{\bar{p}} B_u^{\bar{p}} u_t^{\bar{p}} + w_t, \quad x_0 = 0 \qquad \text{[dynamics]} \\ & u_t^{\bar{p}} = \Phi_K^{\bar{p}} * x_t, \qquad \text{[control policy]} \\ & K^p \in \{\text{Stability constraints}\} \cap \{\text{Information constraints}\}, \qquad \text{[struct. constraints]} \\ & G_x x_t + G_u^p u_t \preceq 1 \qquad \text{[operat. constraints]} \end{cases}$$

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Step 1) System-level responses (frequency domain)

$$\boldsymbol{x} = \left(zI - A + \sum_p B_u^p \boldsymbol{K}^p\right)^{-1} \boldsymbol{w} \qquad = \boldsymbol{\Phi}_x \boldsymbol{w}$$

$$\boldsymbol{u}^p = \boldsymbol{K}^p \left(zI - A + \sum_p B_u^p \boldsymbol{K}^p\right)^{-1} \boldsymbol{w} \qquad = \boldsymbol{\Phi}_u^p \boldsymbol{w}$$

$$\boldsymbol{\Phi}_K^p = \boldsymbol{\Phi}_u^p \boldsymbol{\Phi}_x^{-1}$$

Step 2) System-level parametrisation (time domain)

$$\text{Stability} \iff \begin{matrix} \Phi_{x,n+1} = A\Phi_{x,n} + \sum_p B_u^p \Phi_{u,n}^p \\ \Phi_{x,1} = I_{N_x} \end{matrix}$$

Step 3) System-level constraints

(leading to a finite-horizon optimization problem)

$$\Phi_x, \Phi_u^p \in \left\{\text{FIR w/ } N \text{ factors}\right\}$$

$$\cap \{ \text{ Information as sparsity constraints } \}$$

Step 4) Expectation and chance constraints

(leading to second-order conic constraints)

$$\mathbf{prob}\left[(\text{stuff}) * w_n \leq 1\right] \geq \rho \iff \|B_w^\mathsf{T}(\text{stuff})^\mathsf{T}\|_{\ell_2} \leq 1/Q(\rho)$$

GFNE, problem formulation and system level parametrisation

$$BR^p_\Phi(\Phi_u^{-p}) : \begin{cases} \underset{\Phi_u^p}{\text{minimize}} & \sum_{n=0}^{\infty} \left(\|W_x^p \Phi_{x,n} B_w\|_F^2 + \|W_u^p \Phi_{u,n}^p B_w\|_F^2 \right) \\ \underset{\substack{\forall n \in \mathbb{N} \\ \forall \bar{p} \in \mathcal{P}}}{\text{subject to}} & \Phi_u^p * \Phi_x^{-1} \in \{\text{Stability constraints}\} \cap \{\text{Information constraints}\}, \\ & (G_x \Phi_x + G_u^p \Phi_u) * w_n \preceq 1 \end{cases}$$

Step 1) System-level responses (frequency domain)

$$\begin{aligned} \boldsymbol{x} &= \left(zI - A + \textstyle\sum_p B_u^p \boldsymbol{K}^p\right)^{-1} \boldsymbol{w} &&= \boldsymbol{\Phi}_x \boldsymbol{w} \\ \boldsymbol{u}^p &= \boldsymbol{K}^p \left(zI - A + \textstyle\sum_p B_u^p \boldsymbol{K}^p\right)^{-1} \boldsymbol{w} &&= \boldsymbol{\Phi}_u^p \boldsymbol{w} \\ \boldsymbol{\Phi}_K^p &= \boldsymbol{\Phi}_u^p \boldsymbol{\Phi}_x^{-1} \end{aligned}$$

Step 2) System-level parametrisation (time domain)

$$\text{Stability} \iff \begin{aligned} \Phi_{x,n+1} &= A\Phi_{x,n} + \textstyle\sum_p B_u^p \Phi_{u,n}^p \\ \Phi_{x,1} &= I_{N_x} \end{aligned}$$

Step 3) System-level constraints
(leading to a finite-horizon optimization problem)

$$\Phi_x, \Phi_u^p \in \left\{ \text{FIR w/ } N \text{ factors} \right\} \cap \{ \text{ Information as sparsity constraints } \}$$

Step 4) Expectation and chance constraints
(leading to second-order conic constraints)

$$\mathbf{prob}\left[(\text{stuff}) * w_n \leq 1\right] \geq \rho \iff \|B_w^\mathsf{T} (\text{stuff})^\mathsf{T}\|_{\ell_2} \leq 1/Q(\rho)$$

GFNE, problem formulation and system level parametrisation

$$BR^p_\Phi(\mathbf{\Phi}_u^{-p}) : \begin{cases} \underset{\Phi_u^p}{\text{minimize}} & \displaystyle\sum_{n=0}^{\infty} \left(\|W_x^p \Phi_{x,n} B_w\|_F^2 + \|W_u^p \Phi_{u,n}^p B_w\|_F^2 \right) \\ \underset{\substack{\forall n \in \mathbb{N} \\ \forall \bar{p} \in \mathcal{P}}}{\text{subject to}} & \Phi_{x,n+1} = A\Phi_{x,n} + \sum_{\bar{p}} B_u^{\bar{p}} \Phi_{u,n}^{\bar{p}}, \quad \Phi_{x,1} = I_{N_x} \\ & \Phi_u^p * \Phi_x^{-1} \in \{\text{Information constraints}\}, \\ & (G_x \Phi_x + G_u^p \Phi_u) * w_n \preceq 1 \end{cases}$$

STEP 1) SYSTEM-LEVEL RESPONSES (FREQUENCY DOMAIN)

$$\boldsymbol{x} = \left(zI - A + \sum_p B_u^p \boldsymbol{K}^p\right)^{-1} \boldsymbol{w} \qquad = \mathbf{\Phi}_x \boldsymbol{w}$$

$$\boldsymbol{u}^p = \boldsymbol{K}^p \left(zI - A + \sum_p B_u^p \boldsymbol{K}^p\right)^{-1} \boldsymbol{w} \qquad = \mathbf{\Phi}_u^p \boldsymbol{w}$$

$$\mathbf{\Phi}_K^p = \mathbf{\Phi}_u^p \mathbf{\Phi}_x^{-1}$$

STEP 2) SYSTEM-LEVEL PARAMETRISATION (TIME DOMAIN)

$$\text{Stability} \iff \begin{aligned} \Phi_{x,n+1} &= A\Phi_{x,n} + \sum_p B_u^p \Phi_{u,n}^p \\ \Phi_{x,1} &= I_{N_x} \end{aligned}$$

STEP 3) SYSTEM-LEVEL CONSTRAINTS
(leading to a finite-horizon optimization problem)

$$\Phi_x, \Phi_u^p \in \left\{ \text{FIR w/ } N \text{ factors} \right\}$$

$$\cap \, \{ \text{ Information as sparsity constraints } \}$$

STEP 4) EXPECTATION AND CHANCE CONSTRAINTS
(leading to second-order conic constraints)

$$\mathbf{prob}\left[(\text{stuff}) * w_n \leq 1\right] \geq \rho \iff \|B_w^\top (\text{stuff})^\top\|_{\ell_2} \leq 1/Q(\rho)$$

GFNE, problem formulation and system level parametrisation

$$BR^p_{\Phi}(\boldsymbol{\Phi}_u^{-p}) : \begin{cases} \underset{\Phi_u^p}{\text{minimize}} & \displaystyle\sum_{n=0}^{N-1} \left(\|W_x^p \Phi_{x,n} B_w\|_F^2 + \|W_u^p \Phi_{u,n}^p B_w\|_F^2 \right) + \|W_f \Phi_{x,N} B_w\|_F^2 \\ \underset{\forall \bar{p} \in \mathcal{P}}{\underset{\forall n \in \mathbb{N}}{\text{subject to}}} & \Phi_{x,n+1} = A\Phi_{x,n} + \sum_{\bar{p}} B_u^{\bar{p}} \Phi_{u,n}^{\bar{p}}, \quad \Phi_{x,1} = I_{N_x} \\ & \mathtt{Sp}(\Phi_{x,n}) = \mathtt{Sp}(S_n), \quad \mathtt{Sp}(\Phi_{u,n}^p) = \mathtt{Sp}(B_u^{p^\top} S_n), \qquad \text{w/ } S_n = A^{\min(0, \lfloor n - d_a/d_c \rfloor)}, \\ & (G_x \Phi_x + G_u^p \Phi_u) * w_n \preceq 1 \end{cases}$$

Step 1) System-level responses (frequency domain)

$$\boldsymbol{x} = \left(zI - A + \sum_p B_u^p \boldsymbol{K}^p\right)^{-1} \boldsymbol{w} \qquad = \boldsymbol{\Phi}_x \boldsymbol{w}$$

$$\boldsymbol{u}^p = \boldsymbol{K}^p \left(zI - A + \sum_p B_u^p \boldsymbol{K}^p\right)^{-1} \boldsymbol{w} \qquad = \boldsymbol{\Phi}_u^p \boldsymbol{w}$$

$$\boldsymbol{\Phi}_K^p = \boldsymbol{\Phi}_u^p \boldsymbol{\Phi}_x^{-1}$$

Step 2) System-level parametrisation (time domain)

$$\text{Stability} \iff \begin{aligned} \Phi_{x,n+1} &= A\Phi_{x,n} + \sum_p B_u^p \Phi_{u,n}^p \\ \Phi_{x,1} &= I_{N_x} \end{aligned}$$

Step 3) System-level constraints

(leading to a finite-horizon optimization problem)

$$\Phi_x, \Phi_u^p \in \left\{ \text{FIR w/ } N \text{ factors} \right\}$$

$$\cap \{ \text{ Information as sparsity constraints } \}$$

Step 4) Expectation and chance constraints

(leading to second-order conic constraints)

$$\textbf{prob}\left[(\text{stuff}) * w_n \leq 1 \right] \geq \rho \iff \|B_w^\top (\text{stuff})^\top\|_{\ell_2} \leq 1/Q(\rho)$$

GFNE, problem formulation and system level parametrisation

$$BR^p_\Phi(\boldsymbol{\Phi}_u^{-p}) : \begin{cases} \underset{\Phi_u^p}{\text{minimize}} & \sum_{n=0}^{N-1} \left(\|W_x^p \Phi_{x,n} B_w\|_F^2 + \|W_u^p \Phi_{u,n}^p B_w\|_F^2 \right) + \|W_f \Phi_{x,N} B_w\|_F^2 \\ \underset{\substack{\forall n \in \mathbb{N} \\ \forall \bar{p} \in \mathcal{P}}}{\text{subject to}} & \Phi_{x,n+1} = A\Phi_{x,n} + \sum_{\bar{p}} B_u^{\bar{p}} \Phi_{u,n}^{\bar{p}}, \quad \Phi_{x,1} = I_{N_x} \\ & \mathtt{Sp}(\Phi_{x,n}) = \mathtt{Sp}(S_n), \quad \mathtt{Sp}(\Phi_{u,n}^p) = \mathtt{Sp}(B_u^{p^\top} S_n), \qquad \text{w/ } S_n = A^{\min(0, \lfloor n - d_a/d_c \rfloor)}, \\ & (G_{x,i}\Phi_x B_w + G_{u,i}^p \Phi_u B_w)^\top \preceq_{K_2} 1/Q(\rho), \qquad (\forall i) \end{cases}$$

Step 1) System-level responses (frequency domain)

$$\boldsymbol{x} = \left(zI - A + \sum_p B_u^p \boldsymbol{K}^p\right)^{-1} \boldsymbol{w} \qquad = \boldsymbol{\Phi}_x \boldsymbol{w}$$

$$\boldsymbol{u}^p = \boldsymbol{K}^p \left(zI - A + \sum_p B_u^p \boldsymbol{K}^p\right)^{-1} \boldsymbol{w} \qquad = \boldsymbol{\Phi}_u^p \boldsymbol{w}$$

$$\boldsymbol{\Phi}_K^p = \boldsymbol{\Phi}_u^p \boldsymbol{\Phi}_x^{-1}$$

Step 2) System-level parametrisation (time domain)

$$\text{Stability} \iff \begin{array}{l} \Phi_{x,n+1} = A\Phi_{x,n} + \sum_p B_u^p \Phi_{u,n}^p \\ \Phi_{x,1} = I_{N_x} \end{array}$$

Step 3) System-level constraints
(leading to a finite-horizon optimization problem)

$$\Phi_x, \Phi_u^p \in \left\{\text{FIR w/ } N \text{ factors}\right\} \cap \{ \text{ Information as sparsity constraints } \}$$

Step 4) Expectation and chance constraints
(leading to second-order conic constraints)

$$\textbf{prob}\left[(\text{stuff}) * w_n \leq 1\right] \geq \rho \iff \|B_w^\top (\text{stuff})^\top\|_{\ell_2} \leq 1/Q(\rho)$$

GFNE, problem formulation and system level parametrisation

$$BR^p_\Phi(\boldsymbol{\Phi}_u^{-p}) : \begin{cases} \underset{\Phi_u^p}{\text{minimize}} & \sum_{n=0}^{N-1}\left(\|W_x^p\Phi_{x,n}B_w\|_F^2 + \|W_u^p\Phi_{u,n}^p B_w\|_F^2\right) + \|W_f\Phi_{x,N}B_w\|_F^2 \\ \underset{\forall \bar{p} \in \mathcal{P}}{\underset{\forall n \in \mathbb{N}}{\text{subject to}}} & \Phi_{x,n+1} = A\Phi_{x,n} + \sum_{\bar{p}} B_u^{\bar{p}}\Phi_{u,n}^{\bar{p}}, \quad \Phi_{x,1} = I_{N_x} \\ & \mathtt{Sp}(\Phi_{x,n}) = \mathtt{Sp}(S_n), \quad \mathtt{Sp}(\Phi_{u,n}^p) = \mathtt{Sp}(B_u^{p^\top} S_n), \qquad \text{w/ } S_n = A^{\min(0,\lfloor n-d_a/d_c \rfloor)}, \\ & (G_{x,i}\Phi_x B_w + G_{u,i}^p \Phi_u B_w)^\top \preceq_{K_2} 1/Q(\rho), \qquad (\forall i) \end{cases}$$

$K^\star$ is a vGFNE of $\mathcal{G}_\infty^{\mathrm{LQ}}$ [dynamic game]

$$\Updownarrow$$

$\Phi_u^\star$ is a vGNE of $\mathcal{G}_\Phi^{\mathrm{LQ}}$ [static game]

Algorithm: vGFNE-Seeking via FB-Splitting

for $k = 0, 1, 2, \ldots$ **do**
- **for** *each player* $p \in \mathcal{P}$ **do**
  - $\boldsymbol{K}_k^p \coloneqq \boldsymbol{\Phi}_{u,k}^p \boldsymbol{\Phi}_{x,k}^{-1}$;
  - $\boldsymbol{\Phi}_{u,+}^p \coloneqq \boldsymbol{\Phi}_{u,k}^p - \eta \nabla L_\Phi^p(\boldsymbol{\Phi}_{u,k}^p \mid \boldsymbol{\Phi}_{u,k}^{-p})$;
- **for** *coordinator* **do**
  - $\boldsymbol{\Phi}_{u,k+1} \coloneqq \mathrm{proj}_{\mathcal{U}_{\boldsymbol{\Phi}},\mathcal{G}}(\boldsymbol{\Phi}_{u,+})$;

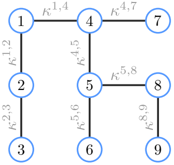
GFNE, partially-observed dynamic games

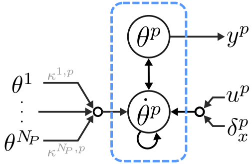
What about **partially-observed** (i.e., output-feedback) Nash equilibrium problems?

$$\begin{cases} x_{t+1} = Ax_t + \sum_p B_u^p u_t^p + w_t \\ y_t = Cx_t + v_t \\ u_t^p = K^p * y_t \end{cases} \Rightarrow \begin{cases} \boldsymbol{x} = \boldsymbol{\Phi}_{xx}\boldsymbol{w} + \boldsymbol{\Phi}_{xy}\boldsymbol{v} \\ \boldsymbol{u}^p = \boldsymbol{\Phi}_{ux}^p\boldsymbol{w} + \boldsymbol{\Phi}_{uy}^p\boldsymbol{v} \\ \boldsymbol{K}^p = \boldsymbol{\Phi}_{uy}^p - \boldsymbol{\Phi}_{ux}^p\boldsymbol{\Phi}_{xy}^{-1}\boldsymbol{\Phi}_{xy} \end{cases}$$

⇝ Similar procedure (with much more involved formulas) so I will spare you the long details!

Example, power-grid with capacity constraints



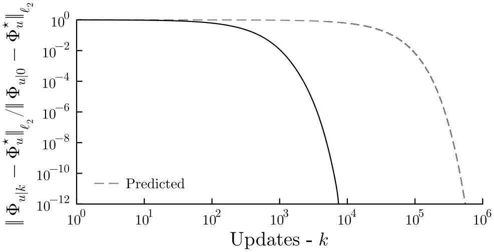


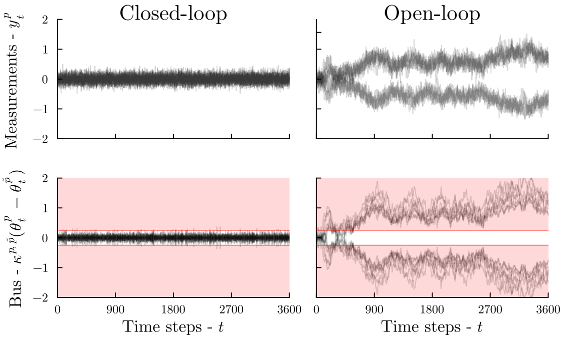
Continuous-time dynamics (each node):

$$m^p\ddot{\theta}^p + d^p\dot{\theta}^p = -\sum_{\tilde{p}\in\mathcal{P}} \kappa^{p,\tilde{p}}(\theta^p - \theta^{\tilde{p}}) + u^p + \delta_x^p$$

Link capacity constraint (each node):

$$-5 \le \kappa^{p,\tilde{p}}(\theta^p - \theta^{\tilde{p}}) \le 5, \quad \forall t \in \mathbb{N},\ \tilde{p} \in \mathcal{P}$$





Convergence to vGFNE (albeit slow) while ensuring stability and constraint satisfaction

I am always looking for opportunities to collaborate– so hit me up!

[tiominho.github.io]

Thank you!

Questions?

May 26, 2025