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Departamento de Engenharia de Teleinformática (DETI)
Programa de Pós-Graduação em Engenharia de Teleinformática (PPGETI)

Processamento Estatístico de Sinais - TI 0124

Estimação e Detecção - TIP8417

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Lista de Exercícios No. 3: Teoria da Estimação

1. Assume that $x(0), x(1), \ldots, x(K-1)$ are independent and Gaussian random variables, each one with zero mean and variance σ_x^2. Hence, the sum of their squared terms given as

$$y = \sum_{i=0}^{K-1} [x(i)]^2$$

has a Chi-squared distribution with mean $K\sigma_x^2$ and variance $2K\sigma_x^4$. Design an estimator for the parameters K and σ_x^2 using the ***method of moments***, assuming that we have access to N measurements $y(0), y(1), \ldots, y(N-1)$ of the sum of squared terms.

2. Consider the problem of linear fitting using the ***method of least squares***. Assume that are known N measurements $x(0), x(1), \ldots, x(N-1)$ of the scalar quantity X observed, respectively, in time instantes (or values of the argument) $t(0), t(1), \ldots, t(N-1)$. The task is the to adjust the line

$$x = \alpha_0 + \alpha_1 t$$

to those measurements.

(a) Design the normal equations to this problem using the linear least squares method.

(b) Assume that the sampling interval Δt is constant and it was chosen such that the time instants of the measurements are integers $0, 1, \ldots, N-1$. Solve the equations in this important special case.

3. Consider the sum $z = x_1 + x_2 + \ldots + x_K$, where the scalar x_i are statistically independent and Gaussian, each one with the same zero mean and variance σ_x^2.

(a) Design the ***maximum likelihood estimator*** for the number K of terms in the sum.

(b) Is the estimator unbiased?

4. Consider N measurements of independent observations $x(0), x(1), \ldots, x(N-1)$ of a scalar r.v. X that has a Gaussian distribution of mean μ_x and variance σ_x^2. This time, the mean μ_x is also a r.v. with Gaussian distribution with zero mean and variance σ_μ^2. We assume that both variances σ_x^2 and σ_μ^2 are known and we wish to estimate μ using the ***maximum a posteriori*** (MAP) method; Show that the estimator is given by:

$$\widehat{\mu}_{\text{MAP}} = \frac{\sigma_\mu^2}{\sigma_x^2 + N\sigma_\mu^2} \sum_{i=0}^{N-1} x(i)$$

5. Consider the data

$$x(n) = r^n + v(n), \qquad n = 0, 1, 2, \ldots, N-1$$

where $v(n)$ is a r.v. with normal distrinution with zero mean and variance σ^2. We wish to estimate the parameter r, the exponential factor, which can assume values in the range $r > 0$. Find an estimator by means of the ***maximum likelihood approach***.

6. The data $x(n) = Ar^n + w(n)$ for $n = 0, \ldots, N-1$ are observed. The random variables $w(0), w(1), \ldots, w(N-1)$ are i.i.d. Gaussian random variables with zero mean and variance σ^2. Find the Cramér-Rao bound for A. Does an efficient estimator exist? If so, what is it and what is its variance? For what values of r is it consistent?

7. Suppose, for $i = 1, 2$

$$y_i = x + w_i$$

where x is an unknown constant, and where w_1 and w_2 are statistically independent, zero-mean Gaussian random variables with

$$\begin{aligned} \mathrm{var}(w_1) &= 1 \\ \mathrm{var}(w_2) &= \begin{cases} 1, & x \geq 0 \\ 2, & x < 0. \end{cases} \end{aligned}$$

Calculate the Cramér-Rao bound for unbiased estimators of x based on observation of

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

8. Suppose x is an unknown parameter and we have N observations of the form

$$y_k = \begin{cases} x + w_k, & x \geq 0 \\ 2x + w_k, & x < 0 \end{cases} \quad k = 1, 2, \ldots, N$$

where the w_k are independent and identically distributed Gaussian random variables with zero mean and variance σ^2.

(a) Determine the Cramér-Rao bound on the error variance of unbiased estimates of x.

(b) Does an efficient estimator for x exist? If so, determine $\widehat{x}_{\text{eff}}(y1, y2, ..., yN)$. If not, explain.

(c) Determine $\widehat{x}_{\text{ML}}(y1, y2, ..., yN)$, the maximum likelihood estimate for x based on $y_1, y_2, ..., y_N$.

(d) Is the ML estimator consistent? Explain.