Subtraction Strategies: Counting On/Back By Bases and then Ones (CBBO)

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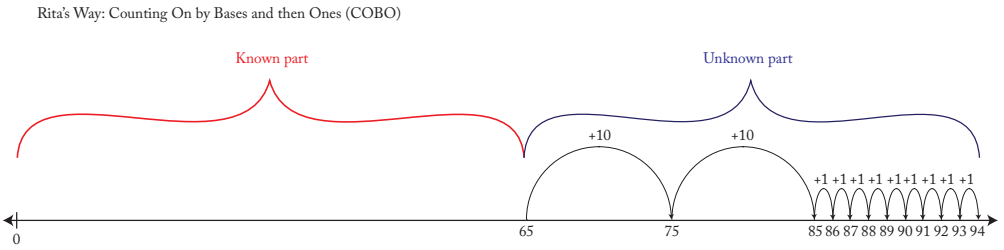
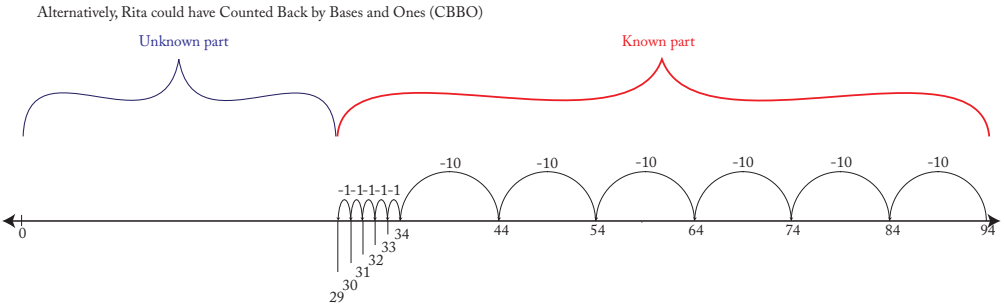
Transcript

Video from Carpenter et al. (1999). Strategy descriptions and examples adapted from Hackenberg (2025).

- **Teacher:** Earl had a collection of 65 bird feathers, on a trip to a marsh he found lots more feathers to put in his collection. Now he has 94 feathers in his collection. How many feathers did Earl find at the marsh?
- **Rita** So he had what?
- **Teacher:** He started off with, 65 feathers.
- **Rita:** 1,2,3,4,5,6 1,2,3,4,5. And then he had how many?
- **Teacher:** Well, he had 65 bird feathers. On a trip to a marsh, he found lots more and he put them in his collection. Now he has 94.
- **Rita:** Well, I can 65, 75, 85. How many did he find?
- **Teacher:** Well, that's my question for you. How many did he find? He ends up with 94.
- **Rita:** And 85,86,87,88,89,90, 91,92,93,94 and so the answer is 20, 21, 22, 23, 24, 25, 26, 27, 28, 29.
- **Teacher** Nice work.





Notation Representing Rita's Solution:

$$65 + (10) = 75$$
$$75 + (10) = 85$$
$$85 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 94$$
$$10 + 10 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 29$$

Description of Strategy:

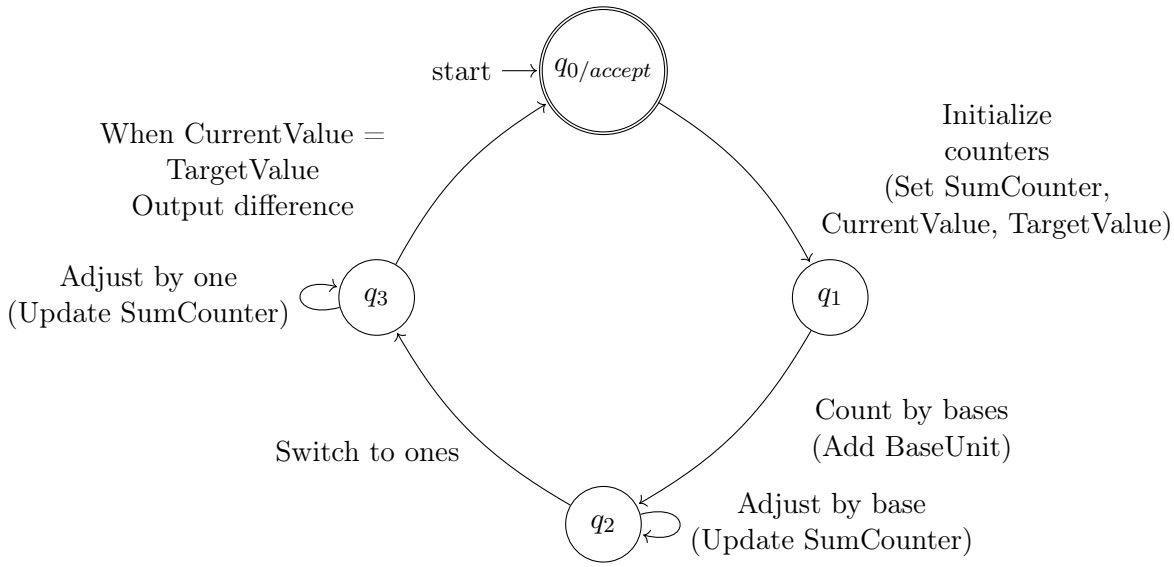
Objective: Description of Counting On by Bases and Then Ones (COBO) Begin with one of the numbers. Break the other number into its base units and its ones. Then, "count on" by adding each base unit one at a time, followed by each individual one.

Why are number lines useful for demonstrating this strategy? COBO is essentially a jump strategy—you start at one number and make "jumps" equal to the other number's base units, then add in the remaining ones. Number lines are ideal because they visually display jumps of varying lengths and directions. They serve as a picture of the process: a jump representing a full base is clearly larger (by a factor of the base) than a jump of a single unit.

Good number line illustrations should:

- Clearly represent the relative sizes of the jumps—each base jump should be exactly as many times larger than a single-unit jump as the base indicates, with all base jumps the same size and all one-unit jumps identical.
- Indicate the position of 0, or mark a break if that portion of the line isn't drawn to scale.
- Use arrows to indicate direction—when adding, the jumps go to the right (or upward); when subtracting, they go to the left (or downward).
- Mark all landing points clearly—the numbers you would speak aloud when counting on by bases and then ones, just as Lauren demonstrated.

Automaton Diagram for Counting On or Back by Bases and Then Ones



HTML Implementation

```
<!DOCTYPE html>
<html>
<head>
    <title>Subtraction Strategies: Counting Back By Bases and Ones (CBBO)</title>
    <style>
        body { font-family: sans-serif; }
        #diagramCBBOSVG { border: 1px solid #d3d3d3; }
        #outputContainer { margin-top: 20px; }
        .number-line-tick { stroke: black; stroke-width: 1; }
        .number-line-break { stroke: black; stroke-width: 1; } /* Solid for zig-zag */
        .number-line-label { font-size: 12px; text-anchor: middle; }
        .jump-arrow { fill: none; stroke: purple; stroke-width: 1.5; } /* CBBO color */
        .jump-arrow-head { fill: purple; stroke: purple; } /* CBBO color */
        .jump-label { font-size: 10px; text-anchor: middle; fill: purple; } /* CBBO color
            */
        .tens-jump-label { font-size: 12px; text-anchor: middle; fill: purple; }
        .stopping-point { fill: red; stroke: black; stroke-width: 1; }
        .number-line-arrow { fill: black; stroke: black; }
        .extended-tick { stroke: black; stroke-width: 1; } /* Reuse COBO style */
    </style>
</head>
<body>

<h1>Subtraction Strategies: Counting Back By Bases and Then Ones (CBBO)</h1>

<div>
    <label for="cbboMinuend">Minuend:</label>
    <input type="number" id="cbboMinuend" value="94"> <!-- Example from PDF -->
</div>
<div>
    <label for="cbboSubtrahend">Subtrahend:</label>
    <input type="number" id="cbboSubtrahend" value="29"> <!-- 94 - 65 = 29 -->
</div>

<button onclick="runCBBOAutomaton()">Calculate and Visualize</button>

<div id="outputContainer">
    <h2>Explanation:</h2>
    <div id="cbboOutput">
        <!-- Text output will be displayed here -->
    </div>
</div>

<h2>Diagram:</h2>
<svg id="diagramCBBOSVG" width="700" height="350"></svg>

<script>
document.addEventListener('DOMContentLoaded', function() {
    const outputElement = document.getElementById('cbboOutput');
    const cbboMinuendInput = document.getElementById('cbboMinuend');
    const cbboSubtrahendInput = document.getElementById('cbboSubtrahend');
    const diagramCBBOSVG = document.getElementById('diagramCBBOSVG');
```

```
52
53      // --- Helper Functions (Keep createText, drawTick, drawScaleBreakSymbol,
            createJumpArrow, drawStoppingPoint from previous corrected versions) ---
54      function createText(svg, x, y, textContent, className = 'number-line-label') {
55          const text = document.createElementNS("http://www.w3.org/2000/svg", 'text');
56          text.setAttribute('x', x);
57          text.setAttribute('y', y);
58          text.setAttribute('class', className);
59          text.setAttribute('text-anchor', 'middle'); // Center labels
60          text.textContent = textContent;
61          svg.appendChild(text);
62      }
63
64      function drawTick(svg, x, y, size) {
65          const tick = document.createElementNS('http://www.w3.org/2000/svg', 'line');
66          tick.setAttribute('x1', x);
67          tick.setAttribute('y1', y - size / 2);
68          tick.setAttribute('x2', x);
69          tick.setAttribute('y2', y + size / 2);
70          tick.setAttribute('class', 'number-line-tick');
71          svg.appendChild(tick);
72      }
73
74       function drawScaleBreakSymbol(svg, x, y) {
75          const breakOffset = 4;
76          const breakHeight = 8;
77          const breakLine1 = document.createElementNS('http://www.w3.org/2000/svg', 'line');
78          breakLine1.setAttribute('x1', x - breakOffset);
79          breakLine1.setAttribute('y1', y - breakHeight);
80          breakLine1.setAttribute('x2', x + breakOffset);
81          breakLine1.setAttribute('y2', y + breakHeight);
82          breakLine1.setAttribute('class', 'number-line-break');
83          svg.appendChild(breakLine1);
84          const breakLine2 = document.createElementNS('http://www.w3.org/2000/svg', 'line');
85          breakLine2.setAttribute('x1', x + breakOffset);
86          breakLine2.setAttribute('y1', y - breakHeight);
87          breakLine2.setAttribute('x2', x - breakOffset);
88          breakLine2.setAttribute('y2', y + breakHeight);
89          breakLine2.setAttribute('class', 'number-line-break');
90          svg.appendChild(breakLine2);
91      }
92
93       function createJumpArrow(svg, x1, y1, x2, y2, jumpArcHeight, direction = 'forward',
             arrowSize = 5) { // Removed default color, use CSS
94          const path = document.createElementNS('http://www.w3.org/2000/svg', 'path');
95          const cx = (x1 + x2) / 2;
96          const cy = y1 - jumpArcHeight; // Arc is above the line
97          path.setAttribute('d', `M ${x1} ${y1} Q ${cx} ${cy} ${x2} ${y1}`); // Use y1 for
                end point to land on line
98          path.setAttribute('class', `jump-arrow`); // Rely on CSS for color
99          svg.appendChild(path);
100
101         // Arrowhead
102         const arrowHead = document.createElementNS('http://www.w3.org/2000/svg', 'path');
```

```
        const dx = x2 - cx;
        const dy = y1 - cy; // Use y1 for angle calculation
        const angleRad = Math.atan2(dy, dx);
        let angleDeg = angleRad * (180 / Math.PI);
        arrowHead.setAttribute('class', `jump-arrow-head`); // Rely on CSS for color

        if (direction === 'forward') {
            angleDeg += 180; // Point right
            arrowHead.setAttribute('d', `M 0 0 L ${arrowSize} ${arrowSize/2} L ${
                arrowSize} ${-arrowSize/2} Z`);
            arrowHead.setAttribute('transform', `translate(${x2}, ${y1}) rotate(${
                angleDeg})`);
        } else { // backward
            // angleDeg already points left-ish from Q curve end
            arrowHead.setAttribute('d', `M 0 0 L ${-arrowSize} ${arrowSize/2} L ${-
                arrowSize} ${-arrowSize/2} Z`); // Pointy part is at (0,0)
             // We want to rotate to align with the curve's end direction
            arrowHead.setAttribute('transform', `translate(${x2}, ${y1}) rotate(${
                angleDeg})`);
        }
        svg.appendChild(arrowHead);
    }

    function drawStoppingPoint(svg, x, y, labelText, labelOffsetBase) {
        const circle = document.createElementNS('http://www.w3.org/2000/svg', 'circle');
        circle.setAttribute('cx', x);
        circle.setAttribute('cy', y);
        circle.setAttribute('r', 5);
        circle.setAttribute('class', 'stopping-point');
        svg.appendChild(circle);
        createText(svg, x, y + labelOffsetBase * 1.5, labelText, 'number-line-label');
    }
    // --- End Helper Functions ---

    // --- Main CBBO Automaton Function ---
    window.runCBBOAutomaton = function() {
        try {
            const minuend = parseInt(cbboMinuendInput.value);
            const subtrahend = parseInt(cbboSubtrahendInput.value); // Amount to subtract
            if (isNaN(minuend) || isNaN(subtrahend)) {
                outputElement.textContent = 'Please␣enter␣valid␣numbers␣for␣Minuend␣and␣
                    Subtrahend';
                diagramCBBOSVG.innerHTML = '';
                return;
            }
             if (subtrahend > minuend) {
                outputElement.textContent = 'Subtrahend␣cannot␣be␣greater␣than␣Minuend␣for␣
                    CBBO.';
                diagramCBBOSVG.innerHTML = '';
                return;
            }

            let output = `<h2>Counting Back by Bases and Ones (CBBO)</h2>\n\n`;
            output += `<p><strong>Problem:</strong> ${minuend} - ${subtrahend}</p>\n\n`;
```

```
151
152             const tensToSubtract = Math.floor(subtrahend / 10) * 10;
153             const onesToSubtract = subtrahend % 10;
154
155             output += `Step 1: Split subtrahend ${subtrahend} into ${tensToSubtract} + ${
                    onesToSubtract}\n\n`;
156
157             let currentVal = minuend;
158             const tensSteps = [];
159             if (tensToSubtract > 0) {
160                 output += 'Step␣2:␣Count␣back␣by␣tens\n';
161                 for (let i = 10; i <= tensToSubtract; i += 10) {
162                     tensSteps.push({ from: currentVal, to: currentVal - 10, action: '
                            Subtract␣10' });
163                     currentVal -= 10;
164                 }
165                 tensSteps.forEach(step => {
166                     output += `<p>${step.from} - 10 = ${step.to}</p>\n`; // Simplified text
167                 });
168                 output += '\n';
169             }
170
171             const onesSteps = [];
172             if (onesToSubtract > 0) {
173                 output += `Step ${tensToSubtract > 0 ? '3' : '2'}: Count back by ones\n`;
174                 for (let i = 1; i <= onesToSubtract; i++) {
175                     onesSteps.push({ from: currentVal, to: currentVal - 1, action: '
                            Subtract␣1' });
176                     currentVal -= 1;
177                 }
178                 onesSteps.forEach(step => {
179                     output += `<p>${step.from} - 1 = ${step.to}</p>\n`; // Simplified text
180                 });
181                 output += '\n';
182             }
183
184             const finalDifference = currentVal; // The final landing spot IS the
                    difference
185             output += `Result: ${minuend} - ${subtrahend} = ${finalDifference}`;
186             outputElement.innerHTML = output;
187             typesetMath();
188
189             // Draw the diagram
190             drawCBBONumberLineDiagram(diagramCBBOSVG, minuend, subtrahend, tensSteps,
                    onesSteps, finalDifference);
191
192
193         } catch (error) {
194              console.error("Error␣in␣runCBBOAutomaton:", error);
195             outputElement.textContent = `Error: ${error.message}`;
196         }
197     };
198
```

```
function drawCBBONumberLineDiagram(svg, minuend, subtrahend, tensSteps, onesSteps,
    finalDifference) {
    if (!svg || typeof svg.setAttribute !== 'function') { return; }
    svg.innerHTML = '';

    const svgWidth = parseFloat(svg.getAttribute('width'));
    const svgHeight = parseFloat(svg.getAttribute('height'));
    const startX = 50;
    const endX = svgWidth - 50;
    const numberLineY = svgHeight / 2; // Center vertically
    const tickHeight = 10;
    const labelOffsetBase = 20;
    const jumpHeight = 30; // Consistent jump height for CBBO
    const jumpLabelOffset = 15;
    const arrowSize = 5;
    const scaleBreakThreshold = 40;

    // Determine range for scaling
    let diagramMin = finalDifference;
    let diagramMax = minuend;

    // Calculate scale and handle potential break (near 0, before diagramMin)
    let displayRangeStart = diagramMin;
    let scaleStartX = startX;
    let drawScaleBreak = false;

    if (diagramMin > scaleBreakThreshold) {
        displayRangeStart = diagramMin - 10;
        scaleStartX = startX + 30;
        drawScaleBreak = true;
        drawScaleBreakSymbol(svg, scaleStartX - 15, numberLineY);
        drawTick(svg, startX, numberLineY, tickHeight);
        createText(svg, startX, numberLineY + labelOffsetBase, '0', 'number-line-label
            ');
    } else {
        displayRangeStart = 0;
        drawTick(svg, startX, numberLineY, tickHeight);
        createText(svg, startX, numberLineY + labelOffsetBase, '0', 'number-line-label
            ');
    }

    const displayRangeEnd = diagramMax + 10;
    const displayRange = Math.max(displayRangeEnd - displayRangeStart, 1);
    const scale = (endX - scaleStartX) / displayRange;

    // Function to convert value to X coordinate
    function valueToX(value) {
        if (value < displayRangeStart && drawScaleBreak) { return scaleStartX - 10; }
        const scaledValue = scaleStartX + (value - displayRangeStart) * scale;
        return Math.max(scaleStartX, Math.min(scaledValue, endX));
    }

    // Draw the main visible segment of the number line
     const mainLineStartX = valueToX(displayRangeStart);
```

```
        const mainLineEndX = valueToX(displayRangeEnd);
        const numberLine = document.createElementNS('http://www.w3.org/2000/svg', 'line');
        numberLine.setAttribute('x1', mainLineStartX);
        numberLine.setAttribute('y1', numberLineY);
        numberLine.setAttribute('x2', mainLineEndX);
        numberLine.setAttribute('y2', numberLineY);
        numberLine.setAttribute('class', 'number-line-tick');
        svg.appendChild(numberLine);

        // Add arrowhead to the right end
        const mainArrowHead = document.createElementNS('http://www.w3.org/2000/svg', 'path');
        mainArrowHead.setAttribute('d', `M ${mainLineEndX - arrowSize} ${numberLineY - arrowSize/2} L ${mainLineEndX} ${numberLineY} L ${mainLineEndX - arrowSize} ${numberLineY + arrowSize/2} Z`);
        mainArrowHead.setAttribute('class', 'number-line-arrow');
        svg.appendChild(mainArrowHead);


    // Draw Ticks and Labels
    function drawTickAndLabel(value, index) {
        const x = valueToX(value);
        if (x < scaleStartX - 5 && value !== 0) return;

        drawTick(svg, x, numberLineY, tickHeight);
        const labelOffset = labelOffsetBase * (index % 2 === 0 ? 1 : -1.5); // Stagger
        createText(svg, x, numberLineY + labelOffset, value.toString(), 'number-line-label');
    }

    // Collect all points to draw ticks for
    let allPoints = new Set([minuend, finalDifference]); // Start and end
    tensSteps.forEach(step => allPoints.add(step.to));
    onesSteps.forEach(step => allPoints.add(step.to));
    let sortedPoints = Array.from(allPoints).sort((a, b) => a - b);
    let pointIndexMap = {};
    let currentIndex = 0;
    sortedPoints.forEach(point => {
        if (point >= displayRangeStart || (point === 0 && !drawScaleBreak)) {
            if (!(point < displayRangeStart && drawScaleBreak)) {
                pointIndexMap[point] = currentIndex++;
                drawTickAndLabel(point, pointIndexMap[point]);
            }
        }
    });

    // Draw tens jumps (Backward)
    tensSteps.forEach((step, index) => {
        const x1 = valueToX(step.from);
        const x2 = valueToX(step.to);
        if (x1 <= scaleStartX || x2 < scaleStartX) return; // Skip if outside visible range

```

```
            const staggerOffset = index % 2 === 0 ? 0 : jumpHeight * 0.5;
            createJumpArrow(svg, x1, numberLineY, x2, numberLineY, jumpHeight + staggerOffset, 'backward', arrowSize);
            createText(svg, (x1 + x2) / 2, numberLineY - (jumpHeight + staggerOffset) - jumpLabelOffset, '-10', 'tens-jump-label');
        });

        // Draw ones jumps (Backward)
        onesSteps.forEach((step, index) => {
            const x1 = valueToX(step.from);
            const x2 = valueToX(step.to);
             if (x1 <= scaleStartX || x2 < scaleStartX) return; // Skip if outside visible range

            const staggerOffset = (tensSteps.length + index) % 2 === 0 ? 0 : jumpHeight * 0.5; // Continue staggering
            createJumpArrow(svg, x1, numberLineY, x2, numberLineY, jumpHeight + staggerOffset, 'backward', arrowSize);
            createText(svg, (x1 + x2) / 2, numberLineY - (jumpHeight + staggerOffset) - jumpLabelOffset, '-1', 'jump-label');
        });

        // Start point marker
        if (valueToX(minuend) >= scaleStartX) {
            drawStoppingPoint(svg, valueToX(minuend), numberLineY, 'Start', labelOffsetBase);
        }
    }

    function typesetMath() { /* Placeholder */ }

    // Initial run on page load
    runCBBOAutomaton();

});
</script>

</body>
<!-- New button for viewing PDF documentation -->
<button onclick="openPdfViewer()">Want to learn more about this strategy? Click here.</button>

<script>
    function openPdfViewer() {
        // Opens the PDF documentation for the strategy.
        window.open('../SAR_SUB_COBO.pdf', '_blank');
    }
</script>
</html>
```

References

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). *Children's mathematics: Cognitively guided instruction* [Includes supplementary material: Children's mathematics: Cognitively guided instruction – videotape logs]. Heinemann; The National Council of Teachers of Mathematics, Inc.

Hackenberg, A. (2025). *Course notes* [Unpublished course notes].