

Subtraction Strategies: Decomposition

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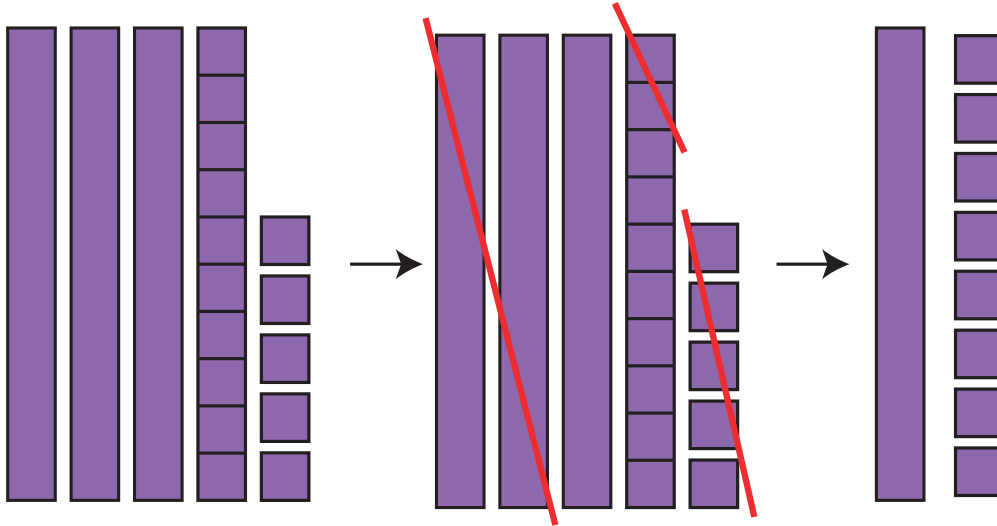
Transcript

Video from Carpenter et al. (1999). Strategy descriptions and examples adapted from Hackenberg (2025)

- **Teacher:** Lucy ordered 45 cupcakes for her birthday. At the party, her guests ate 27 cupcakes, how many cupcakes did she have left? [BACKGROUND]
- **Joel:** This is 10, this is 10, this is 10, this is 10 and this is five. 18.
- **Teacher:** Explain to us what you did there.
- **Joel:** I have, this is 10, this is 10, this is 10, and this is five. So I take away 20 and I take away five. I take away two more. So they enter and then I counted these and those, and so the answer was 18.
- **Teacher:** Nice work

Notation Representing Joel's Solution:

$$\begin{array}{l} 47 - 27 \\ 45 - 20 = 25 \\ 25 - 7 = ? \\ \quad 2 \text{ tens} + 5 \text{ ones} - 7 \text{ ones} \\ 1 \text{ ten} + 1 \text{ ten} + 5 \text{ ones} - 7 \text{ ones} \\ \quad \downarrow \text{DECOMPOSE} \\ 1 \text{ ten} + 10 \text{ ones} + 5 \text{ ones} - 7 \text{ ones} \\ \quad 1 \text{ ten} + 8 \text{ ones} + \underbrace{7 \text{ ones} - 7 \text{ ones}}_{=0} \\ \quad 1 \text{ ten} + 8 \text{ ones} \end{array}$$



Notation Representing Joel's Solution: Imagine representing both numbers by their base units and ones. Begin by subtracting the base components, then subtract the ones. If there aren't enough ones available in the larger number to subtract the ones from the smaller number (while keeping the result positive), break one base unit into its individual ones. Finally, remove only the exact number of ones required to complete the subtraction.

Decomposition

Description of Strategy

- **Objective:** Decompose a base unit from the minuend into ones to have enough ones to subtract the ones in the subtrahend.

Automaton Type

Pushdown Automaton (PDA): Needed to handle the decomposition process and keep track of base units.

Formal Description of the Automaton

We define the PDA as the 7-tuple

$$M = (Q, \Sigma, \Gamma, \delta, q_{0/accept}, Z_0, F)$$

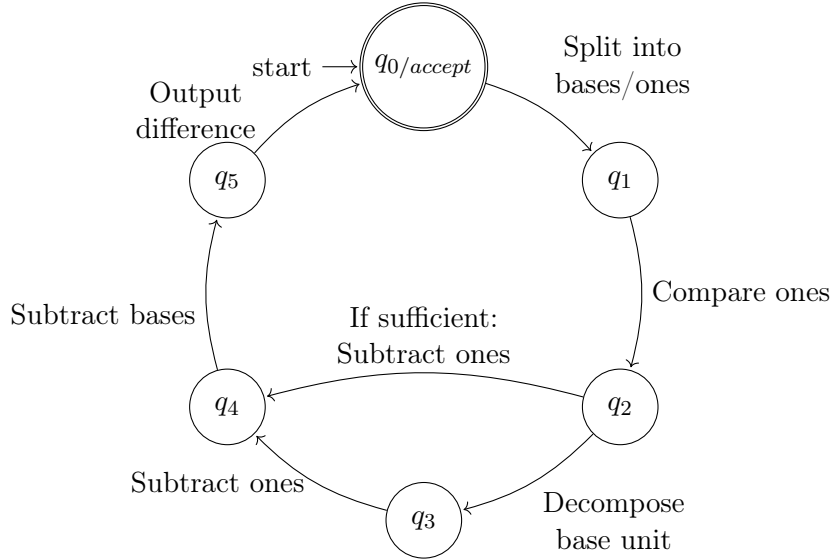
where:

- $Q = \{q_{0/accept}, q_1, q_2, q_3, q_4, q_5\}$ is the set of states.
- $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is the input alphabet.
- $\Gamma = \{Z_0\} \cup \{b \mid b \in \mathbb{N}\}$ is the stack alphabet, where Z_0 is the initial stack symbol and b represents a base unit (e.g., 10 in base-ten).
- $q_{0/accept}$ is the start state, which is also the accept state.
- $F = \{q_{0/accept}\}$ is the set of accepting states.

The transition function δ is defined as:

1. $\delta(q_{0/accept}, "M, S", Z_0) = \{(q_1, Z_0)\}$
(Split the minuend M and subtrahend S into their base and ones components.)
2. $\delta(q_1, \varepsilon, Z_0) = \{(q_2, Z_0)\}$
(Compare the ones in M and S .)
3. $\delta(q_2, \varepsilon, Z_0) = \{(q_3, b Z_0)\}$
(If the ones in M are insufficient, decompose a base unit b into ones.)
4. $\delta(q_2, \varepsilon, Z_0) = \{(q_4, Z_0)\}$
(If the ones in M are sufficient, proceed to subtract ones.)
5. $\delta(q_3, \varepsilon, b) = \{(q_4, b)\}$
(After decomposition, subtract the ones.)
6. $\delta(q_4, \varepsilon, Z_0) = \{(q_5, Z_0)\}$
(Subtract the bases.)
7. $\delta(q_5, \varepsilon, Z_0) = \{(q_{0/accept}, Z_0)\}$
(Output the final difference.)

Automaton Diagram for Decomposition



HTML Implementation

```
1 <!DOCTYPE html>
2 <html xmlns="http://www.w3.org/1999/xhtml" lang="" xml:lang="">
3 <head>
4   <meta charset="utf-8" />
5   <meta name="generator" content="pandoc" />
6   <meta name="viewport" content="width=device-width, initial-scale=1.0, user-scalable=yes" />
7   <meta name="author" content="Theodore M. Savich" />
8   <title>Subtraction Strategies: Decomposition</title>
9   <style>
10    html {
11      color: #1a1a1a;
12      background-color: #fdfdfd;
13    }
14    body {
15      margin: 0 auto;
16      max-width: 36em;
17      padding-left: 50px;
18      padding-right: 50px;
19      padding-top: 50px;
20      padding-bottom: 50px;
21      hyphens: auto;
22      overflow-wrap: break-word;
23      text-rendering: optimizeLegibility;
24      font-kerning: normal;
25    }
26    @media (max-width: 600px) {
27      body {
28        font-size: 0.9em;
29        padding: 12px;
30      }
31      h1 {
32        font-size: 1.8em;
33      }
34    }
35    @media print {
36      html {
37        background-color: white;
38      }
39      body {
40        background-color: transparent;
41        color: black;
42        font-size: 12pt;
43      }
44      p, h2, h3 {
45        orphans: 3;
46        widows: 3;
47      }
48      h2, h3, h4 {
49        page-break-after: avoid;
50      }
51    }
```

```

52  p {
53      margin: 1em 0;
54  }
55  a {
56      color: #1a1a1a;
57  }
58  a:visited {
59      color: #1a1a1a;
60  }
61  img {
62      max-width: 100%;
63  }
64  svg {
65      height: auto;
66      max-width: 100%;
67  }
68  h1, h2, h3, h4, h5, h6 {
69      margin-top: 1.4em;
70  }
71  h5, h6 {
72      font-size: 1em;
73      font-style: italic;
74  }
75  h6 {
76      font-weight: normal;
77  }
78  ol, ul {
79      padding-left: 1.7em;
80      margin-top: 1em;
81  }
82  li > ol, li > ul {
83      margin-top: 0;
84  }
85  blockquote {
86      margin: 1em 0 1em 1.7em;
87      padding-left: 1em;
88      border-left: 2px solid #e6e6e6;
89      color: #606060;
90  }
91  code {
92      font-family: Menlo, Monaco, Consolas, 'Lucida Console', monospace;
93      font-size: 85%;
94      margin: 0;
95      hyphens: manual;
96  }
97  pre {
98      margin: 1em 0;
99      overflow: auto;
100 }
101 pre code {
102     padding: 0;
103     overflow: visible;
104     overflow-wrap: normal;
105 }

```

```

106 .sourceCode {
107     background-color: transparent;
108     overflow: visible;
109 }
110 hr {
111     background-color: #1a1a1a;
112     border: none;
113     height: 1px;
114     margin: 1em 0;
115 }
116 table {
117     margin: 1em 0;
118     border-collapse: collapse;
119     width: 100%;
120     overflow-x: auto;
121     display: block;
122     font-variant-numeric: lining-nums tabular-nums;
123 }
124 table caption {
125     margin-bottom: 0.75em;
126 }
127 tbody {
128     margin-top: 0.5em;
129     border-top: 1px solid #1a1a1a;
130     border-bottom: 1px solid #1a1a1a;
131 }
132 th {
133     border-top: 1px solid #1a1a1a;
134     padding: 0.25em 0.5em 0.25em 0.5em;
135 }
136 td {
137     padding: 0.125em 0.5em 0.25em 0.5em;
138 }
139 header {
140     margin-bottom: 4em;
141     text-align: center;
142 }
143 #TOC li {
144     list-style: none;
145 }
146 #TOC ul {
147     padding-left: 1.3em;
148 }
149 #TOC > ul {
150     padding-left: 0;
151 }
152 #TOC a:not(:hover) {
153     text-decoration: none;
154 }
155 code{white-space: pre-wrap;}
156 span.smallcaps{font-variant: small-caps;}
157 div.columns{display: flex; gap: min(4vw, 1.5em);}
158 div.column{flex: auto; overflow-x: auto;}
159 div.hanging-indent{margin-left: 1.5em; text-indent: -1.5em;}

```

```

160  /* The extra [class] is a hack that increases specificity enough to
161     override a similar rule in reveal.js */
162  ul.task-list[class]{list-style: none;}
163  ul.task-list li input[type="checkbox"] {
164      font-size: inherit;
165      width: 0.8em;
166      margin: 0 0.8em 0.2em -1.6em;
167      vertical-align: middle;
168  }
169  </style>
170  <script
171  src="https://cdn.jsdelivr.net/npm/mathjax@3/es5/tex-ctml-full.js"
172  type="text/javascript"></script>
173  </head>
174  <body>
175  <header id="title-block-header">
176  <h1 class="title">Subtraction Strategies: Decomposition</h1>
177  <p class="author">Theodore M. Savich</p>
178  </header>
179  <h2 class="unnumbered" id="decomposition">Decomposition</h2>
180  <h3 class="unnumbered" id="description-of-strategy">Description of
181  Strategy</h3>
182  <ul>
183  <li><p><strong>Objective:</strong> Decompose a base unit from the
184  minuend into ones to have enough ones to subtract the ones in the
185  subtrahend.</p></li>
186  </ul>
187  <h3 class="unnumbered" id="automaton-type">Automaton Type</h3>
188  <p><strong>Pushdown Automaton (PDA)</strong>: Needed to handle the
189  borrowing (decomposition) process and keep track of base units.</p>
190  <h3 class="unnumbered" id="formal-description-of-the-automaton">Formal
191  Description of the Automaton</h3>
192  <p>We define the PDA as the 7-tuple <span class="math\_display">\[M =
193  (Q, \, \, \Sigma, \, \, \Gamma, \, \, \delta, \, \, q_{0/accept}, \, \, Z_0, \, \, F)\]</span>
194  where:</p>
195  <ul>
196  <li><p><span class="math\_inline">\(Q = \{q_{0/accept}, \, \, q_1, \, \, q_2, \, \,
197  q_3, \, \, q_4, \, \, q_5\})\)</span> is the set of states.</p></li>
198  <li><p><span class="math\_inline">\(\Sigma =
199  \{0,1,2,3,4,5,6,7,8,9\}\)</span> is the input alphabet.</p></li>
200  <li><p><span class="math\_inline">\(\Gamma = \{Z_0\} \cup \{b \mid b \in
201  \mathbb{N}\}\)</span> is the stack alphabet, where <span
202  class="math\_inline">\(Z_0\)</span> is the initial stack symbol and <span
203  class="math\_inline">\(b\)</span> represents a base unit (e.g., 10 in
204  base-ten).</p></li>
205  <li><p><span class="math\_inline">\(q_{0/accept}\)</span> is the start
206  state, which is also the accept state.</p></li>
207  <li><p><span class="math\_inline">\(F = \{q_{0/accept}\}\)</span> is the
208  set of accepting states.</p></li>
209  </ul>
210  <p>The transition function <span class="math\_inline">\(\delta\)</span>
211  is defined as:</p>
212  <ol>
213  <li><p><span class="math\_inline">\(\delta(q_{0/accept}, \, \,

```

```

214 \text{'M,S\text{&\#39;\&\#39;},\, Z_0) = \{(q_1,\, Z_0)\}\}</span><br />
215 (Split the minuend \,(M\)</span> and subtrahend
216 \,(S\)</span> into their base and ones
217 components.)</p></li>
218 <li><p><span class="math_{\text{inline}}">\,(\delta(q_1,\, \text{\varepsilon},\, Z_0) =
219 \{(q_2,\, Z_0)\}\}</span><br />
220 (Compare the ones in \,(M\)</span> and \,(S\)</span>.)</p></li>
221 <li><p><span class="math_{\text{inline}}">\,(\delta(q_2,\, \text{\varepsilon},\, Z_0) =
222 \{(q_3,\, b,Z_0)\}\}</span><br />
223 (If the ones in \,(M\)</span> are insufficient,
224 decompose a base unit \,(b\)</span> into
225 ones.)</p></li>
226 <li><p><span class="math_{\text{inline}}">\,(\delta(q_2,\, \text{\varepsilon},\, Z_0) =
227 \{(q_4,\, Z_0)\}\}</span><br />
228 (If the ones in \,(M\)</span> are sufficient,
229 proceed to subtract ones.)</p></li>
230 <li><p><span class="math_{\text{inline}}">\,(\delta(q_3,\, \text{\varepsilon},\, b) =
231 \{(q_4,\, b)\}\}</span><br />
232 (After decomposition, subtract the ones.)</p></li>
233 <li><p><span class="math_{\text{inline}}">\,(\delta(q_4,\, \text{\varepsilon},\, Z_0) =
234 \{(q_5,\, Z_0)\}\}</span><br />
235 (Subtract the bases.)</p></li>
236 <li><p><span class="math_{\text{inline}}">\,(\delta(q_5,\, \text{\varepsilon},\, Z_0) =
237 \{(q_{\text{0/accept}},\, Z_0)\}\}</span><br />
238 (Output the final difference.)</p></li>
239 </ol>
240 </div>
241 <h3 class="unnumbered"
242 id="automaton-diagram-for-decomposition">Automaton Diagram for
243 Decomposition</h3>
244 <div style="text-align:center;">
245 
246 </div>
247 </body>
248 </html>

```

References

- Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). Children's mathematics: Cognitively guided instruction – videotape logs [supplementary material]. In *Children's mathematics: Cognitively guided instruction*. Heinemann, in association with The National Council of Teachers of Mathematics, Inc.
- Hackenberg, A. (2025). *Course notes* [Unpublished course notes].