Subtraction Strategies: Decomposition

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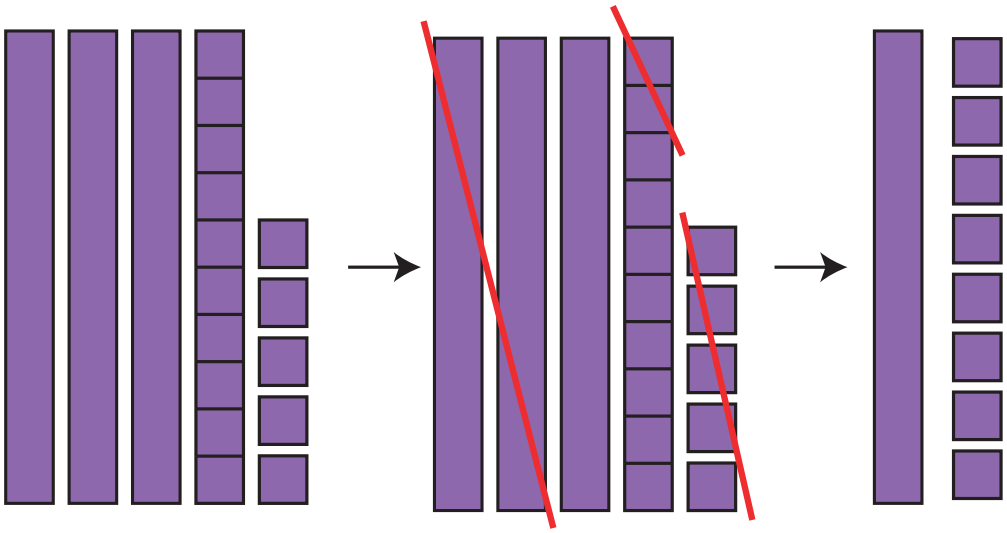
Transcript

Video from Carpenter et al. (1999). Strategy descriptions and examples adapted from Hackenberg (2025)

- **Teacher:** Lucy ordered 45 cupcakes for her birthday. At the party, her guests ate 27 cupcakes, how many cupcakes did she have left? [BACKGROUND]
- **Joel:** This is 10, this is 10, this is 10, this is 10 and this is five.18.
- **Teacher:** Explain to us what you did there.
- **Joel:** I have, this is 10, this is 10, this is 10, and this is five. So I take away 20 and I take away five. I take away two more. So they enter and then I counted these and those, and so the answer was 18.
- **Teacher:** Nice work

Notation Representing Joel's Solution:

$$
\begin{aligned}
&47 - 27 \\
&45 - 20 = 25 \\
&\ 25 - 7 = ? \\
&\qquad 2 \text{ tens } + 5 \text{ ones } - 7 \text{ ones} \\
&1 \text{ ten } + 1 \text{ ten } + 5 \text{ ones } - 7 \text{ ones} \\
&\qquad \downarrow \text{DECOMPOSE} \\
&1 \text{ ten } + 10 \text{ ones } + 5 \text{ ones } - 7 \text{ ones} \\
&\qquad 1 \text{ ten } + 8 \text{ ones } + \underbrace{7 \text{ ones } - 7 \text{ ones}}_{=0} \\
&\qquad 1 \text{ ten } + 8 \text{ ones}
\end{aligned}
$$



Notation Representing Joel's Solution: Imagine representing both numbers by their base units and ones. Begin by subtracting the base components, then subtract the ones. If there aren't enough ones available in the larger number to subtract the ones from the smaller number (while keeping the result positive), break one base unit into its individual ones. Finally, remove only the exact number of ones required to complete the subtraction.

Decomposition

Description of Strategy

- **Objective:** Decompose a base unit from the minuend into ones to have enough ones to subtract the ones in the subtrahend.

Automaton Type

Pushdown Automaton (PDA): Needed to handle the decomposition process and keep track of base units.

Formal Description of the Automaton

We define the PDA as the 7-tuple

$$M = (Q, \Sigma, \Gamma, \delta, q_{0/accept}, Z_0, F)$$

where:

- $Q = \{q_{0/accept}, q_1, q_2, q_3, q_4, q_5\}$ is the set of states.
- $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is the input alphabet.
- $\Gamma = \{Z_0\} \cup \{b \mid b \in \mathbb{N}\}$ is the stack alphabet, where Z_0 is the initial stack symbol and b represents a base unit (e.g., 10 in base-ten).
- $q_{0/accept}$ is the start state, which is also the accept state.
- $F = \{q_{0/accept}\}$ is the set of accepting states.

The transition function δ is defined as:

1. $\delta(q_{0/accept}, \text{“}M, S\text{”}, Z_0) = \{(q_1, Z_0)\}$
   (Split the minuend M and subtrahend S into their base and ones components.)

2. $\delta(q_1, \varepsilon, Z_0) = \{(q_2, Z_0)\}$
   (Compare the ones in M and S.)

3. $\delta(q_2, \varepsilon, Z_0) = \{(q_3, b\,Z_0)\}$
   (If the ones in M are insufficient, decompose a base unit b into ones.)

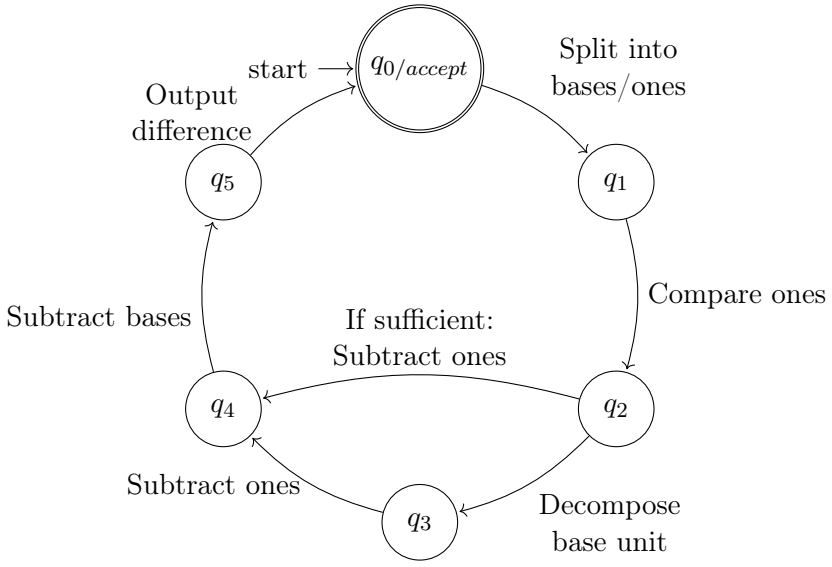
4. $\delta(q_2, \varepsilon, Z_0) = \{(q_4, Z_0)\}$
   (If the ones in M are sufficient, proceed to subtract ones.)

5. $\delta(q_3, \varepsilon, b) = \{(q_4, b)\}$
   (After decomposition, subtract the ones.)

6. $\delta(q_4, \varepsilon, Z_0) = \{(q_5, Z_0)\}$
   (Subtract the bases.)

7. $\delta(q_5, \varepsilon, Z_0) = \{(q_{0/accept}, Z_0)\}$
   (Output the final difference.)

Automaton Diagram for Decomposition



HTML Implementation

```
<!DOCTYPE html>
<html xmlns="http://www.w3.org/1999/xhtml" lang="" xml:lang="">
<head>
  <meta charset="utf-8" />
  <meta name="generator" content="pandoc" />
  <meta name="viewport" content="width=device-width, initial-scale=1.0, user-scalable=yes" />
  <meta name="author" content="Theodore M. Savich" />
  <title>Subtraction Strategies: Decomposition</title>
  <style>
    html {
      color: #1a1a1a;
      background-color: #fdfdfd;
    }
    body {
      margin: 0 auto;
      max-width: 36em;
      padding-left: 50px;
      padding-right: 50px;
      padding-top: 50px;
      padding-bottom: 50px;
      hyphens: auto;
      overflow-wrap: break-word;
      text-rendering: optimizeLegibility;
      font-kerning: normal;
    }
    @media (max-width: 600px) {
      body {
        font-size: 0.9em;
        padding: 12px;
      }
      h1 {
        font-size: 1.8em;
      }
    }
    @media print {
      html {
        background-color: white;
      }
      body {
        background-color: transparent;
        color: black;
        font-size: 12pt;
      }
      p, h2, h3 {
        orphans: 3;
        widows: 3;
      }
      h2, h3, h4 {
        page-break-after: avoid;
      }
    }
```

```
    p {
      margin: 1em 0;
    }
    a {
      color: #1a1a1a;
    }
    a:visited {
      color: #1a1a1a;
    }
    img {
      max-width: 100%;
    }
    svg {
      height: auto;
      max-width: 100%;
    }
    h1, h2, h3, h4, h5, h6 {
      margin-top: 1.4em;
    }
    h5, h6 {
      font-size: 1em;
      font-style: italic;
    }
    h6 {
      font-weight: normal;
    }
    ol, ul {
      padding-left: 1.7em;
      margin-top: 1em;
    }
    li > ol, li > ul {
      margin-top: 0;
    }
    blockquote {
      margin: 1em 0 1em 1.7em;
      padding-left: 1em;
      border-left: 2px solid #e6e6e6;
      color: #606060;
    }
    code {
      font-family: Menlo, Monaco, Consolas, 'Lucida␣Console', monospace;
      font-size: 85%;
      margin: 0;
      hyphens: manual;
    }
    pre {
      margin: 1em 0;
      overflow: auto;
    }
    pre code {
      padding: 0;
      overflow: visible;
      overflow-wrap: normal;
    }
```

```
    .sourceCode {
     background-color: transparent;
     overflow: visible;
    }
    hr {
      background-color: #1a1a1a;
      border: none;
      height: 1px;
      margin: 1em 0;
    }
    table {
      margin: 1em 0;
      border-collapse: collapse;
      width: 100%;
      overflow-x: auto;
      display: block;
      font-variant-numeric: lining-nums tabular-nums;
    }
    table caption {
      margin-bottom: 0.75em;
    }
    tbody {
      margin-top: 0.5em;
      border-top: 1px solid #1a1a1a;
      border-bottom: 1px solid #1a1a1a;
    }
    th {
      border-top: 1px solid #1a1a1a;
      padding: 0.25em 0.5em 0.25em 0.5em;
    }
    td {
      padding: 0.125em 0.5em 0.25em 0.5em;
    }
    header {
      margin-bottom: 4em;
      text-align: center;
    }
    #TOC li {
      list-style: none;
    }
    #TOC ul {
      padding-left: 1.3em;
    }
    #TOC > ul {
      padding-left: 0;
    }
    #TOC a:not(:hover) {
      text-decoration: none;
    }
    code{white-space: pre-wrap;}
    span.smallcaps{font-variant: small-caps;}
    div.columns{display: flex; gap: min(4vw, 1.5em);}
    div.column{flex: auto; overflow-x: auto;}
    div.hanging-indent{margin-left: 1.5em; text-indent: -1.5em;}
```

```
    /* The extra [class] is a hack that increases specificity enough to
       override a similar rule in reveal.js */
    ul.task-list[class]{list-style: none;}
    ul.task-list li input[type="checkbox"] {
      font-size: inherit;
      width: 0.8em;
      margin: 0 0.8em 0.2em -1.6em;
      vertical-align: middle;
    }
  </style>
  <script
  src="https://cdn.jsdelivr.net/npm/mathjax@3/es5/tex-chtml-full.js"
  type="text/javascript"></script>
</head>
<body>
<header id="title-block-header">
<h1 class="title">Subtraction Strategies: Decomposition</h1>
<p class="author">Theodore M. Savich</p>
</header>
<h2 class="unnumbered" id="decomposition">Decomposition</h2>
<h3 class="unnumbered" id="description-of-strategy">Description of
Strategy</h3>
<ul>
<li><p><strong>Objective:</strong> Decompose a base unit from the
minuend into ones to have enough ones to subtract the ones in the
subtrahend.</p></li>
</ul>
<h3 class="unnumbered" id="automaton-type">Automaton Type</h3>
<p><strong>Pushdown Automaton (PDA)</strong>: Needed to handle the
borrowing (decomposition) process and keep track of base units.</p>
<h3 class="unnumbered" id="formal-description-of-the-automaton">Formal
Description of the Automaton</h3>
<p>We define the PDA as the 7-tuple <span class="math␣display">\[M =
(Q,\,\Sigma,\,\Gamma,\,\delta,\,q_{0/accept},\,Z_0,\,F)\]</span>
where:</p>
<ul>
<li><p><span class="math␣inline">\(Q = \{q_{0/accept},\, q_1,\, q_2,\,
q_3,\, q_4,\, q_5\}\)</span> is the set of states.</p></li>
<li><p><span class="math␣inline">\(\Sigma =
\{0,1,2,3,4,5,6,7,8,9\}\)</span> is the input alphabet.</p></li>
<li><p><span class="math␣inline">\(\Gamma = \{Z_0\} \cup \{b \mid b \in
\mathbb{N}\}\)</span> is the stack alphabet, where <span
class="math␣inline">\(Z_0\)</span> is the initial stack symbol and <span
class="math␣inline">\(b\)</span> represents a base unit (e.g., 10 in
base-ten).</p></li>
<li><p><span class="math␣inline">\(q_{0/accept}\)</span> is the start
state, which is also the accept state.</p></li>
<li><p><span class="math␣inline">\(F = \{q_{0/accept}\}\)</span> is the
set of accepting states.</p></li>
</ul>
<p>The transition function <span class="math␣inline">\(\delta\)</span>
is defined as:</p>
<ol>
<li><p><span class="math␣inline">\(\delta(q_{0/accept},\,
```

```
\text{‘‘}M,S\text{&#39;&#39;},\, Z_0) = \{(q_1,\, Z_0)\}\)</span><br />
(Split the minuend <span class="math␣inline">\(M\)</span> and subtrahend
<span class="math␣inline">\(S\)</span> into their base and ones
components.)</p></li>
<li><p><span class="math␣inline">\(\delta(q_1,\, \varepsilon,\, Z_0) =
\{(q_2,\, Z_0)\}\)</span><br />
(Compare the ones in <span class="math␣inline">\(M\)</span> and <span
class="math␣inline">\(S\)</span>.)</p></li>
<li><p><span class="math␣inline">\(\delta(q_2,\, \varepsilon,\, Z_0) =
\{(q_3,\, b\,Z_0)\}\)</span><br />
(If the ones in <span class="math␣inline">\(M\)</span> are insufficient,
decompose a base unit <span class="math␣inline">\(b\)</span> into
ones.)</p></li>
<li><p><span class="math␣inline">\(\delta(q_2,\, \varepsilon,\, Z_0) =
\{(q_4,\, Z_0)\}\)</span><br />
(If the ones in <span class="math␣inline">\(M\)</span> are sufficient,
proceed to subtract ones.)</p></li>
<li><p><span class="math␣inline">\(\delta(q_3,\, \varepsilon,\, b) =
\{(q_4,\, b)\}\)</span><br />
(After decomposition, subtract the ones.)</p></li>
<li><p><span class="math␣inline">\(\delta(q_4,\, \varepsilon,\, Z_0) =
\{(q_5,\, Z_0)\}\)</span><br />
(Subtract the bases.)</p></li>
<li><p><span class="math␣inline">\(\delta(q_5,\, \varepsilon,\, Z_0) =
\{(q_{0/accept},\, Z_0)\}\)</span><br />
(Output the final difference.)</p></li>
</ol>
<h3 class="unnumbered"
id="automaton-diagram-for-decomposition">Automaton Diagram for
Decomposition</h3>
<div style="text-align:␣center;">
  <img src="../images/SAR_SUB_DECOMPOSITION.svg" alt="Diagram␣description">
</div>
</body>
</html>
```

References

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). Children's mathematics: Cognitively guided instruction – videotape logs [supplementary material]. In *Children's mathematics: Cognitively guided instruction*. Heinemann, in association with The National Council of Teachers of Mathematics, Inc.

Hackenberg, A. (2025). *Course notes* [Unpublished course notes].