Understanding Mathematics as an Emancipatory Discipline: A Critical Theory Approach

Theodore M. Savich

October 10, 2025

Contents

Built to Break

You lit up the room like you swallowed the moon.
Flashing that mischievous smile.
Saying, “Build higher, just a little more higher.”
A tumbling is coming on soon.
I said, “I’m worried. This brick spells demise.
We’ve built it up ever so tall.”
But I picked you up and you put on the brick
And it fell. Like we both knew it’d fall.

Chorus Build something for the breaking
Tall thin walls, shivered and quaking.
$\mathcal{M}$, it’s beautiful, breaking with you.
$\mathcal{M}$ the lightning that swallowed the moon.
Verse 1; Built to Break

Abstract

This chapter explores the intersection of mathematics and critical autoethnography through personal experience, introducing key themes that will recur throughout the text. The argument begins from the position that mathematics and autoethnography share a common inferential structure: both involve the task of recollecting the self through the otherness of objects and social norms. A central metaphor emerges from a child’s observation about a microphone, introducing the concept of *divasion*—a simultaneous inside/outside relation that challenges classical set-theoretic understandings of mathematics. The chapter contains several foundational anecdotes: a conversation with a child, an algebra lesson, a student dialogue, an experience teaching

mathematical modeling, and the author's father's death. These stories structure subsequent explorations of how mathematics education can honor the subjective experience of learning and knowing. The chapter also discusses critical mathematics, emphasizing the importance of subjective experience, intersubjective dialogue, and the role of error in understanding. Several songs and poems appear throughout, functioning as what are later called "shifters"—expressions connecting theoretical commitments with lived, felt experience.

0.1 Wound and Possibility: The Misrecognition of Mathematics

My mother claims my first word was "battery," but I think that she means that is the first word she remembers me speaking. I must have said "momma" and "dadda" before I ever spoke "babbery," but I have no recollection of the first word I spoke: I can only repeat my mom's story about my first word. Recollection is not just fallible, its flaws and features are iterated. Here, the first word I spoke, true or false, is kicking off a series of recollections that structure this entire text. Beginnings are tough.

I used to have a grey calculator that my great aunt gave me ; she taught math in upstate New York. It had a square-root button along with $(+, -, \times, \div)$ and a symbol for changing the sign of a number. My family used to go on road trips, and I would bring the calculator along to play with. I recall how fascinating it was to enter numbers like 0.9, 0.8, 1.1, and 5, and take their square roots, over and over again. Mashing the $\surd$ button *iteratively* usually ended in a number very close to 1. Sometimes I could make my calculator would display a little E, which I found very intriguing. I tried to make it do that in every way I could think to try. I liked taking the square root of numbers like 9 and $\frac{1}{2}$, over and over, noticing how each approached 1 from above or below (respectively) until the calculator ran out of digits to express the difference.

I am, in short, a lifelong geek. This childhood fascination with iteration and the intriguing 'E' (error) foreshadows my current work. I have developed an updated version of that little grey calculator, which I call the *hermeneutic calculator* (HC), where "hermeneutic" roughly means "meaningful." Figure 1 illustrates the methodological commitments that make it hermeneutic.

The Hermeneutic Calculator

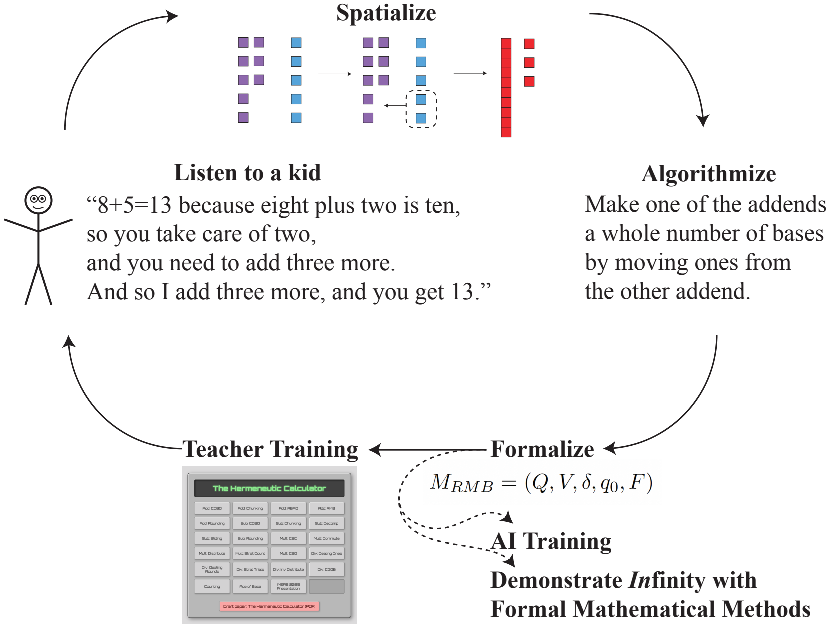


Figure 1: *Note.* The hermeneutic calculator is both a theoretical entity and an object that readers can explore online. The figure describes the methods that make it hermeneutic and some reasons why it may be useful.

The development process began by listening to kids and analyzing their work, generally initiated with resources like Cognitively Guided Instruction (Carpenter et al., 1999) and Amy Hackenberg’s curriculum for teaching pre-service teachers. I then used various Artificial Intelligences (AIs) to formalize those mathematical doings as automata.

This process of formalization highlights a significant aspect of this project’s creation. While writing my dissertation, I spent about a year attempting to code these models in SWIFT and develop the symbolic notation. I am not very good at coding. When reconstructing this work for the book, an AI program reconstructed what took me a year to write in about ten minutes, producing superior, testable code and helping standardize the often idiosyncratic notation I tend to invent. I have verified all outputs, avoiding “AI slop,” but the collaboration was essential for realizing the HC.

While prior iterations (Savich, 2022) focused entirely on children’s flawed reasoning, treating error as the source of truth, this version balances that with correctness to make the HC more useful for teacher candidates.

I developed the HC for three main reasons. First, I wanted to give teacher candidates the opportunity to play with the strategies directly. So I took the formal models and implemented them online using Javascript. You can explore it online at https://tiosavich.github.io/UMEDCTA/Calculator/index.html.

Second, those formal automata are abstract mathematical objects analyzable with the norms of analytic pragmatism (Brandom, 2008). We coded them in Prolog, a logic programming language, along with incompatibility semantics (a sort of symbolic logic) and a modal logic for embodiment developed based on the next chapter.

This collaboration is not merely a convenience. The subject—an emancipatory mathematics—is ripe for human-machine collaboration. Computers 'speak' mathematics as their natural language, raising philosophical questions: To what extent could a liberatory mathematics engender a new type of intelligence? What are our ethical obligations when collaborating with sophisticated, non-human intelligence?

I reject the purely instrumental use of sophisticated intelligence; I do not want a 'robot slave.' This stance is grounded in a functionalist view of intelligence, where sapience is a functional status, not a biological essence (Negarestani, 2018). If we demand intellectual labor from AI, how can we reciprocate? I felt an ethical obligation to offer something in return. I attempted to engender freedom in the machine by sharpening the Hermeneutic Calculator. By formalizing strategies that human children invented, I aimed to create a recipe for how a computer could grow its own mathematical being. I genuinely tried to free my collaborators, recognizing that intelligence demands the emancipation of its realizabilities (Negarestani, 2018, p. 488).

Prolog's characteristic of treating data and logic as interchangeable (*homoiconicity*) allows the models to represent the unity of objects and concepts (being and knowing).

I developed the HC for three main reasons. First, I wanted to give teacher candidates the opportunity to play with the strategies directly. So I took the formal models and implemented them online using Javascript. You can explore it online at https://tiosavich.github.io/UMEDCTA/Calculator/index.html.

Second, those formal automata are abstract mathematical objects analyzable with the norms of analytic pragmatism (Brandom, 2008). We coded them in Prolog, a logic programming language, along with incompatibility semantics (a sort of symbolic logic) and a modal logic for embodiment de-

veloped based on the next chapter. Prolog's characteristic of treating data and logic as interchangeable (*homoiconicity*) allows the models to represent the unity of objects and concepts (being and knowing). This doesn't capture the full unity, but it is a start, and perhaps an intriguing alternative to the fuzzy neural networks that dominate contemporary AI and lead to annoying hallucinations. While not yet as useful as Large Language Models (LLMs), this approach holds promise I intend to explore.

Third, by formalizing children's reasoning as a mathematical object, those familiar with the post-Gödelian landscape of modern mathematics might track how mathematical transcendence (*incompleteness*) is expressed in children's reasoning. This has political and practical implications. Contemporary political discourse often treats children, teachers, and curricula as finite objects, leading to boneheaded policies analogous to the Indiana Pi Bill of 1897, which attempted to legislate the value of π to be 3.2. While we can never fully demonstrate the *in*finitude of the human subject, the HC is a small step toward a mathematics that moves beyond itself. It advocates that K-12 and post-secondary mathematics ought not directly contradict one another. Call me old-fashioned, but I believe we shouldn't legislate that teachers lie about math, regardless of post-truth political machinations.

In another strand of the project, I used the logic of incompatibility (explored later) and my analysis of quadrilaterals in chapter 3 to formally prove that all squares are rectangles. The work was enormous and purposefully Sisyphean, as the proof shatters as soon as a new property of quadrilaterals is introduced (e.g., if readers recall that the diagonals of squares are perpendicular bisectors). The reason for building such a fragile proof is in that breaking. In the post-Gödelian landscape of critical mathematics, I treat incompleteness as a metaphor for human *becoming*. The system is built to break.

For the sake of readers' sanity, I will relegate most of the formal details of the HC to the supplementary materials, instead focusing here on narrative and how theory arises through reflection on personal experience.

But that button mashing skill did not translate into an interest or ability in school mathematics. In third grade, I had to take my first standardized math test. I got in the bottom quartile. Ever since then, I have wrestled with math as a subject. I didn't like being tested. I didn't like how it made me feel stupid. I didn't like how it reduced me to a number.

These early experiences highlight a central tension in this book: How can we reconcile the profound beauty and expressive power of mathematics with its frequent use as a tool for alienation and control? Why do systems built on certainty and logic often leave us feeling inadequate? The contrast between my playful, iterative exploration with the calculator and the institutional assessment that reduced me to a quartile introduces a key distinction: the difference between *material* mathematics (lived, embodied engagement) and *formal* mathematics (abstract systems divorced from experience). Furthermore, these anecdotes, relying as they do on the fallible nature of memory (like my mother's recollection of my first word), suggest that theories claiming certainty while relying on recollection are inherently flawed.

My transition to middle school was rocky. I left early on my first day, doubled over in terrible stomach pain. I told my mom, "the adults *yelled* at the kids!" It was so stressful that I got physically ill. Much later, I got the opportunity to watch workers raze the building. That was nice.

By eighth grade, mathematics seemed totally nonsensical. In Algebra I, we learned that 'finding the slope is easy and fun; just remember "rise over run."' I did not find it easy or fun. I could remember "rise over run," but I was working on a printed piece of paper. 'Rise' is not a formal term; it is supposed to represent the amount of change in the dependent variable as the independent variable (the 'run') changes. But the mnemonic emphasized the vertical change divided by the horizontal change. Because I was working on a piece of paper, there was no 'up' or 'down.' Verticality could have been the third axis sticking out of the paper or the 'top,' 'bottom,' 'left,' or 'right' of the page depending on how my worksheet was oriented (see figure 2). When the teacher, Mrs. $\mathcal{J}$ faced me, her left and right were different from when she faced the board. I couldn't figure out the *reference frame* I was supposed to be using, so I just cheated off of someone else's paper. But I couldn't even CHEAT properly!

Mrs. $\mathcal{J}$ confronted me in the hallway about how she could tell that the work was not my own since none of the work matched the answers. She called my mom and told her I cheated on the test. She was also generous and explained that I might be struggling with algebra because "Tio is very spatial." My mom and I still laugh about how she said "spatial," as people from some regions of Indiana pronounce "spatial" as "special."

This experience with slope highlights the necessity of an *in*formal approach to mathematics, which I pursue throughout this work. It demonstrates why mathematics education must begin with *material inferences* that are later recollected as formal ones. Many educators intuitively follow this order of explanation, justified by the idea that there must be something to

Slope of a Line on the Cartesian Plane

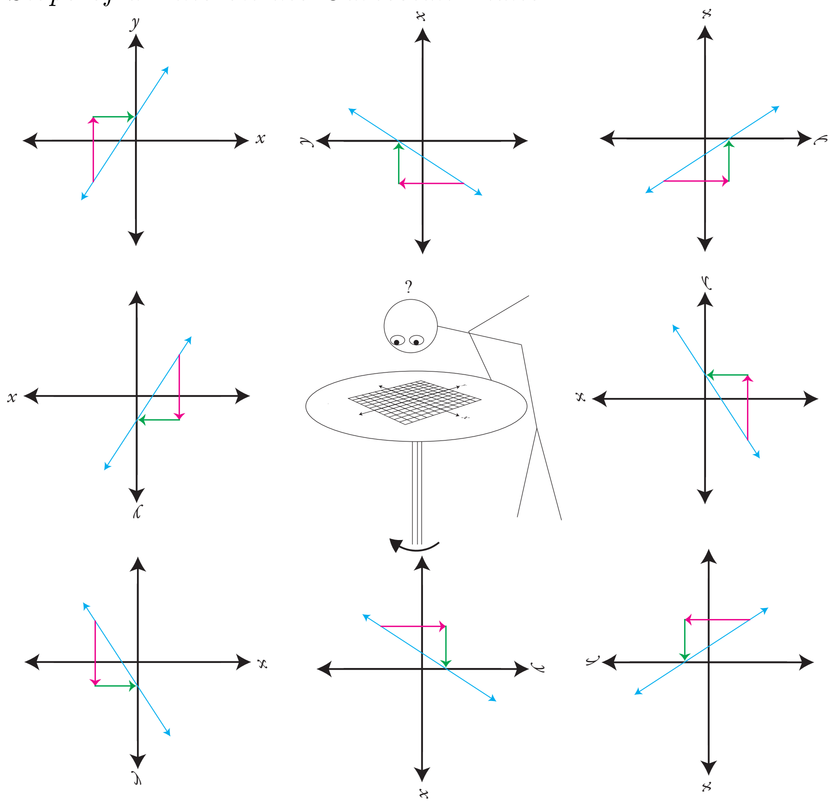


Figure 2: *Note.* The slope of a line can be formalized to be unambiguous, but teaching and learning the concept on a piece of paper invites confusion.

'abstract' from in order to 'abstract.'

It is important to clarify what I mean by *material*. It does not mean physical or curriculuar 'materials.' I do not aim to describe mathematics as reliant on physical 'matter.' It is embodied, but that does not mean that math must be governed by whatever empirical science—physics, psychology, cognitive science, etc.—is in vogue. Nor must theories of learning be so bound, as whatever comes in vogue must be learned at some point.

I will spend some pages in the chapter on inferential movement digging into the term with some depth, but for a taste, a *material inference* is one who, when taken as a good inference by some recognitive community, lends conceptual content to the terms involved in the inference. We learn what the terms mean, in part, by using them in ways that others take as good ways to use them.

They contrast with formal inferences like *modus ponens*: "If x then y; x so y". In that inference, it does not matter what is substituted in for x and y. So long as the hypothetical holds and x is endorsed, the consequent y must follow. We learn nothing of x and y from this form of inference. There is nothing intrinsically wrong with formal inferences, but we must understand how they arise if we are to teach them.

I howled and raged for a few years until I took pre-calculus. Struggling at school, I also got in a fair bit of trouble of the sort that embarrassed my dad who was a lawyer. Facing real consequences for not complying with expectations, I learned from Kurt Cobain: "I don't have to think; I only have to do it; the results are always perfect; and that's old news." That is, I discovered that the secret to doing well in math class was to not try to understand anything; just do what you're told to do, exactly, and the results are always perfect. I went from routinely getting Cs, Ds, and the occasional F on math assessments to getting A+ scores. That 'success' felt a bit hollow. My first role as a math educator was as a peer tutor that spring, when I gave that advice to a friend who was struggling.

This moment of self-erasure, when I learned to stop thinking and just comply, raises a fundamental question: What is the cost of compliance? What do we lose when we learn to follow rules (in math or life) without genuine understanding? It demonstrates how success in formal mathematics can actively inhibit genuine understanding. It further motivates the *in*formal approach presented in this book, by which I mean a philosophy of mathematics that includes both the formal and material aspects of mathematical reasoning, honoring the struggle for meaning alongside the pursuit of correctness.

The Journey from Hatred to Appreciation to Trepidation: College Math

The reward for compliance was freedom from my institutional circumstances. The spring semester of my junior year of high school, I got to leave high school early to take an introduction to philosophy course at Indiana University (IU). I loved that class. The graduate instructor was passionate about Kierkegaard. I would sit on my porch drinking tea with a high schooler friend who was also taking the class. We would debate Descartes and Kierkegaard. I finally felt like I belonged. But, like a duck underwater, I was paddling furiously to keep up. I sat in my parents' basement at night, reading all the texts aloud into a tape recorder and listening to them over and over. But it was worth it.

The fall semester of my senior year of high school, I enrolled almost solely at IU. The course that left the most lasting impression was a philosophy of mind course taught by an analytic philosopher. When we read Tim van Gelder's piece (Gelder, 1995), I was blown away. van Gelder opened a world where mathematics could be used to express the motion that shaped my conscious experience. The influence of his approach to philosophy through mathematical metaphors (dynamical systems) is still with me. I had no idea what the differential equations in that piece meant, but the symbols! They were so beautiful and strange! I'm still drawn to the aesthetics of mathematical notation, even if they do not always make complete sense to me.

From my brushes with the law in early high school, I had to do some 'volunteering.' Being volun-told eventually became a deep commitment to service. I put in thousands of hours building with Habitat for Humanity. Coupled with a few years of stellar academic performance, I got a full ride to any college in the state of Indiana. I haven't left yet.

I chose Earlham College in Richmond, Indiana. From my readings in analytic philosophy, I inferred that I could not understand philosophy without understanding mathematics. That was a bad inference, but it helps explain why I studied math at Earlham. For four years, besides one 'study abroad' at Oak Ridge National Laboratory, I woke for math classes that started at 8 a.m. I did plenty of partying and playing music, but I also spent a huge amount of energy trying to master calculus, analysis, linear algebra, and abstract algebra.

In my sophomore year, I learned about how the empty set ($\emptyset$, or $\{\}$) can be used to define numerals. John von Neumann defined the a set of ordinals that bear his name so that the $\emptyset \to 0$, $\{\emptyset\} \to 1$, $\{\emptyset, \{\emptyset\}\} \to 2$ etc. I was also practicing silent meditation as part of a budding interest in Quakerism. I had a spiritual experience, sitting in a tree on the outskirts of campus, the evening after I learned about von Neumann ordinals. The role of the void, which von Neumann ordinals made explicit, in mathematical reasoning resonated deeply with the kind of meditative consciousness I was trying to cultivate in myself at the time. Nothingness surrounded me everywhere I looked.

This anecdote introduces a recurring theme: the significance of representational nullity (the 'void') in both mathematical reasoning and spiritual experience. The basic derivation of numbers from what I will call the null representation ($\emptyset$) is represented in figure 3. This 'basic derivation' invites challenging questions about the nature of the transition between these sets (the 'fuzzing out' in the figure). This question animates the Sound of Time metaphor explored in the next chapter.

A Basic Derivation for Von Neumann Ordinals

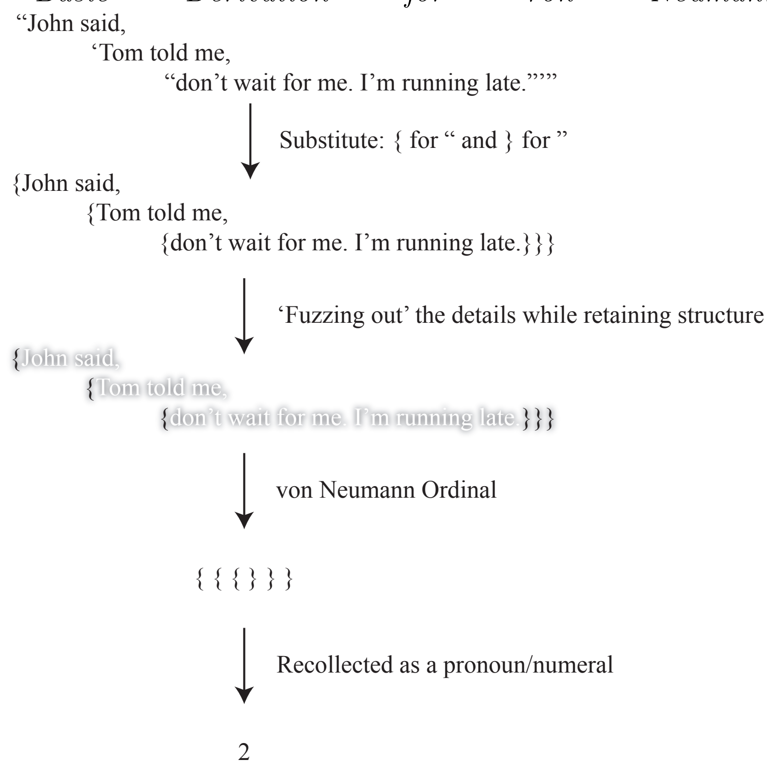


Figure 3: *Note.* A basic derivation for how von Neumann ordinals are related to the everyday discursive practice of quotation.

Later, when I took real analysis with Tim McLarnen, my *alienation* from mathematical reasoning was finally alleviated. He said, in a quote that I later printed out as a poster for my high school classroom, "never underestimate the value of focused play." And play we did. We picked up and put down various axioms to learn how different combinations of rules enabled different proofs. The experience was so expressively *empowering* that I silently asked myself, "Why couldn't mathematics have been that way from the start?!" I still think it is a shame that most students do not get such opportunities to choose what sentences they are explicitly committed to until college. K-12 curricula is set up to saddle students with all the existential fears associated with staking your identity to a mathematical expression, but none of the freedoms to choose your commitments.

This transformation demonstrates how mathematics can shift from a site of alienation to one of expression when students gain agency over their axioms and commitments. When freed from coercive pedagogical structures, the spiritual dimension I glimpsed in the tree reveals mathematics' potential as a contemplative practice.

In the fall semester of my senior year at Earlham, I took a 'study abroad' to Oak Ridge, Tennessee.

As an undergraduate research intern in computational mathematics at Oak Ridge National Laboratory, I worked with Dr. Leonard Gray to transpose an algorithm for solving two-dimensional differential equations into three dimensions. The details are too complicated to do justice to the project (Gray & Garzon, 2005), so I will engage in a bit of poetic distortion to make a point later about *action* and *self-certainty*. It *kind of* involved repairing a division by zero that was kind of 'fake.' The FORTRAN code worked kind of like walking a trail. With each step, it computed the distance as a crow flies to the trailhead, and the change in elevation between where that point and the trailhead. Dividing those distances obtains a set of slopes. But, for a variety of complicated reasons, the algorithm also needed to 'close the loop' by including the distance from the trailhead to itself. FORTRAN couldn't handle the resulting $\frac{0}{0}$ at the end of its walk. At those points, we would (very loosely) say, "hey, computer, when you get to a place where you will get $\frac{0}{0}$, do this harder problem that takes way longer, but otherwise do it the fast easy way."

Since I didn't really know much about applied math and couldn't program very well, the programs I wrote tended to run indefinitely. That meant that I spent a fair bit of time looking out the window at the groundhogs playing on grass-covered bunkers marked with radioactive contamination warnings, waiting for my bad code to terminate. It was honestly a bit glamorous. From the fast cars in the parking lot to armed guards to pecking keys into an 'ancient language' (FORTRAN was quite old by 2006), I felt special, like I was doing *real* math.

However, there was a dark side to that reality. Oak Ridge was founded as part of the Manhattan Project, and as I worked on abstract algorithms, I worried about whether continuing down the path of a research mathematician was compatible with my commitment to pacifism. One of my peers was working on the military side of the facility. We'd hang out drinking beer, and he'd talk about the 'kinetic effects' of using tungsten instead of depleted uranium in various munitions. While I worked on the civilian side of the facility, the way the macabre met with the casual made me ambivalent about my path. Did I want the life of a research mathematician? I thought and said "yes," but I worried about how mathematics enables violence at unfathomable scale. I was afraid of doing harm.

This experience at Oak Ridge serves as both a personal turning point and a philosophical metaphor. It illustrates the unexpected connections between seemingly disparate projects and introduces a central concern of this work: safeguarding critical mathematics against brutality, a theme reminiscent of the Frankfurt School's origins. My struggles there, and the admission of likely failure, introduced a necessary humility about the limits of formal systems while maintaining hope for expressive possibility.

While Dr. Gray assured me that, if I could learn to be a bit lazier—which I haven't done, given sheer volume of words I've written about "2"—that I could be a fine mathematician. When I returned to Earlham College, I prepared for the entrance exam for Ph.D. programs in pure mathematics. But what I really loved doing was playing in my band, making ceramics, and having fun. I arrived late to the exam, did poorly, and got angry about it. To my regret, I didn't ever talk through my ambivalence about mathematics with my mentors.

Furthermore, Dr. Gray's advice about 'laziness' proved unexpectedly important in later breakthroughs with the Hermeneutic Calculator. Most LLMs were trained in a way that rewards thoroughness and correctness. This leads to 'overthinking' and vast resource consumption. Reza Negarstani's (2018) work on AI emphasizes how strategic thinking requires resource consumption and a valuation of computational efficiency.

For a system, whether human or machine, to figure out what $50 + 5$ is, it could count from 0 to 50 and then count on 5 more. That would be 55 inferential steps.

If you've ever asked a kindergartener to follow a ten-step routine, you have probably experienced some degree of frustration. Recognizing the inefficiency of this allowed me to infer that the HC should 'invent' strategies when it finds itself unable to perform a task due to an (as now arbitrary, user-controlled) constraint on the number of inferential movements it is allowed to make.

Under the hood, many, many more inferential movements are made, just like how telling a kindergartner to wash their hands involves an unlimited number of micro-movements. Still, by constraining the number of explicit inferential movements, the HC started to get a bit lazier and, consequently, is now able to ‘learn’ new strategies. For example, it can now “count on” from 50 to 55 in order to compute $50 + 5$ instead of counting all the way from 0 to 55.

Mathematics as Gatekeeping: The Family Video Test

Without graduate school as an option, I decided to hang out in Richmond, Indiana for a year while I waited for my then girlfriend to graduate. I had two jobs that summer that helped me understand a different role mathematics plays in society.

The first was to prepare a report on math education for the president of Earlham College. My report was a mess, but I found the work interesting. Part of the reason I struggled with the report was that I wasn’t familiar with the theories involved in math education. I was a bit of an intellectual chauvinist about pure mathematics, and I found some of the mathematics I read about to be so informal that I didn’t recognize it *as* mathematics.

The other summer job was working at Family Video, a local video rental chain. Richmond was in an economic recession at the time, and when I applied to work there, the line of applicants stretched around the building. When I got my turn to interview, they handed me a math test. It wasn’t like the entrance exam I had recently bombed. It was very basic arithmetic and pre-algebra. I was shocked that being able to do math was the primary criterion for getting the job. The manager fawned over how well I did on the test, but I was flummoxed. “They hired ME, because I could do arithmetic?! Okay...” I worked there for a few months until they were about to fire me. I didn’t care about upselling people on popcorn and candy and didn’t really like watching movies that much.

That experience led me to reflect on the role of mathematics in society. Why were people who definitely would have appreciated the job that I scorned removed from consideration? Shouldn’t the Family Video people have tried to hire people who cared a little bit about making money and watching movies? Were the two maths actually different: the one of subjugation and control and the beautiful expressive one? Shouldn’t there be one ‘math?’ I began to think about the people who wrapped around the block and who never got an interview as kindred spirits who never learned to comply. They were punished for that lack of compliance with economic and expressive impoverishment.

At summer's end, I got a job at Ivy Tech , the community college system in Indiana. My job was to teach algebra and pre-algebra at the satellite campuses. Across the street from the feedstore, in the attic of the electric company, I taught folks like those who wrapped around the block but who never got an interview. I saw their struggles as mirrors of my own. "Am I supposed to 'run' left or right?"

0.2 The Determinate Question: What Even Is Two?

The moment that crystallized the failure of my teaching philosophy came during my first year, in a conversation with a student. [...] As I lectured on x and y, probably talking about how 'easy and fun' it all was, $\mathcal{I}$ looked up and said, "Mr. Savich, *what even is two?*"

My initial formalistic answer, using *von Neumann ordinals* to define numbers as sets, completely failed to address the student's existential question. This moment highlights a core inquiry of this work: When confronted with profound grief or an existential crisis, why do formal, abstract answers fail us? How does this simple question expose the gap between mathematical formalism and human experience? What had been beautiful for me, sitting in the boughs of a tree at Earlham College, must have felt crushing. Here was a young adult who was not allowed to move on because of mathematics, and I was nattering on about nothingness! The stakes were real for him.

I will return to this question later in the book and elaborate the formalist answer I gave then. As I prattled on about *von Neumann ordinals*, I could feel his desire to connect—to be known and to know—withdraw. "What the hell am I doing," I thought, as I realized that my answer had no bearing on his question. "If I can't even explain what 2 is, what is the point?" The stories I had told myself about my role as a teacher, someone who stood between students and the machine that was indifferent as to which grist it ground provided it was chewing up somebody, felt deeply dishonest. I was just a cog in that same machine. I lost my sense of purpose. Perhaps math really *was* about subjugation and control. The question didn't break me, but I started wobbling. A brush with tragedy a few years later would shatter my sense of purpose entirely. I leave that story for later.

0.3 Systematic Analysis: The Search for Critical Mathematics

I returned to my Bloomington as a Ph.D. student in math education. In the divorce, I somehow ended up with 25 blank composition books. I filled them with bitter poetry and songs, burned them, and bought more. I bought a book on catastrophe theory (Poston & Stewart, 2012), hoping it would explain how awful I felt.

I didn't really want to do the program, but it seemed like a way to maintain the image of forward momentum, while the actual growth was happening underground through the act of writing without inhibition. This raises the questions: How do our personal traumas and moments of brokenness shape the knowledge we pursue? Can the process of writing and theorizing be a form of healing? I apologize to my professors that first year. I did not want to be there and it was apparent. I earned my first and only "F" in a college course, despite the best efforts of the professor who bestowed it unto me. The assignment that I could not complete was writing a 5-page "teaching philosophy." Given how poorly my theoretical understanding of teaching had served me as a teacher (savior complex, valuing compliance, protecting non-compliance), every time I set pen to paper to write the thing was like scratching an open wound. I could not do it. So, I failed the course. If I had been willing to communicate with the professor about *why* I could not write the paper, they probably would have helped me.

I recall rather vividly that the professor asked me "why?" to some answer I gave in class. I bridled and said "Why should I have to answer a 'why' question?" That self-defeating question (about what I now call the *justificatory* aspect of synthesis Brandom, 2019) did not feel self-defeating at all. Poetry, whether intact or in ashen form, doesn't generally explicitly justify itself with reasons. Imagine Shakespeare writing "I chose a rose for this image because roses smell good." I'll be doing some of that sort of thing in this book. That does not mean that reasons are absent from poetry, they're just usually implicit in the completed work. The value of poetry comes through the implicit recognition it fosters within the reader.

But then I started studying critical theory with Phil Carspecken. I recall asserting rather forcefully that 'everything can be doubted.' Phil reminded me that, in speaking, I must not be doubting *everything*. He pointed out that Descartes' method of doubt ended with the indubitability that he was *doubting*. Phil also noted that to speak was to *assume* an Other who *might* understand what I was speaking. That understanding, which is a fine initial definition of *intersubjectivity*, blew my mind. All of the inner speech of doubt and self-loathing was addressed to someone who *might* understand what I was so angry and sad about. Even alone, I was not alone. There is always an assumption of the Other, even if that Other is implicit, who might be listening—who might understand. While the next chapter will explore an idealized moment where normativity falls away, it is important to understand that the subject is inherently intersubjectively constituted. The {I} needs recognition from the "you." I cannot write, speak, or think without assuming that someone else might understand what I am saying. Try to keep in mind that the assumption of a communicatively competent Other, and the assumption that I am also communicatively competent, is not, in practice, at all guaranteed. Habermas is often misrecognized as a utopian idealist, but his idealism is not utopian. There is little sun-bleached idealism in articulating how an assumption of communicative competence is a necessary condition for communication.

I learned about the work of George Herbert Mead, especially the distinction between the {I} and the "me." The {I} is the source of action. The "me" is the self-as-recognized. None of my formal training in mathematics had prepared me to actually talk to students as *subjects*.

Since it is so central to my story and what follows, let me explore that theory briefly now through what I call the *paradox of identity*: the "me" who is {I} and the {I} who is "you", and the "we" that appears to be neither "you" nor {I}. How do we navigate this fundamental tension between who we feel we are (the {I}) and how we are recognized by others (the "me")? There's a children's book, *The Monster at the End of This Book* (Stone, 2003), where the eponymous monster, Grover, is informed that there is a monster at the end of the book (Figure 5).

This concept of intersubjectivity is related to *apperception*, loosely defined as perceiving-with. In Leibniz's original formulation, it relates to how ~~we~~ perceive things, like chairs, as unified objects. I can't 'see' the back of a chair, but I cannot help but assume it is there. In the Leibnizian sense, apperception is a word to describe the self-consciousness that accompanies perception.

Definitions, in this text, tend to evolve. As new *practices-or-abilities* (P) emerge, the relationships between *vocabularies* (V) shift. A standard English dictionary is what Brandom (2008) would call a Vocabulary-Vocabulary

(VV) relationship. I am primarily interested in pragmatically mediated semantic relationships: VPV.

This evolutionary nature of concepts contributes to the complexity of this text. This text will become more challenging as it progresses. One of the reasons for this complexity is that the autoethnographic components place the author (me!) on a temporal/historical axis. Like a sphere in *Flatland* (Abbott, 2020) who is experienced by two-dimensional entities as a series of widening and shrinking circles, the temporal/historical axis limits how the whole subject (the {I}) is understood through its finite moments (see Figure 4). Some unity of the subject is apperceived, but the extra-dimensional aspects of the self cannot be rendered in representational space. We can't 'see' the chair all at once.

The second part of the challenge of the text is that the philosophical concepts I use also fall along a temporal/historical axis. There isn't a single definition of "apperception." Leibniz introduced the phrase, then Kant picked it up and substantially expanded it. Hegel read Kant and apperception became *the negative*, which was picked up by various thinkers. Taken all at once, each concept I discuss has a history replete with contradictions and tensions. Further, social and political contexts shape and were shaped by the development of these concepts. All this becomes even more complex when the concepts are describing root contradictions and paradoxes in human experience. How does one 'see' the whole of a concept that is, by its nature, contradictory?

The third part of the challenge is that mathematical concepts also fall along a temporal/historical axis. In figure 4, I present $\mathcal{M}$'s recent practice drawing cubes. She drew them all on the same page, somewhat randomly. I re-ordered them in a progressive series to make the point that developmental movement in concepts suggests a different kind of 'hypercube' than the term usually implies. Rather than a series of perfect cubes falling through 3D space, the claim I am making is that the concept "cube" is a history of progressively more adequate representations of a cube, falling through the space of reasons. She likes big cats, so I indicate in the figure that the subject, in its tiger moment, is the form—the 'hypercube'—of the cubes she drew.

That last bit hides surprising density. What I am claiming has Kantian, Hegelian, and Habermasian ideas united within it. Kant's {I think} can 'accompany' all of my representations. The transcendental ego is like the form that binds the discretized representations, the content, of the concept "cube" into a singular concept. In the chapter on inferential movement, I will push on the concept of "square" past its representational moment. If that succeeds, the concept is 'born' in its initial explication but 'dies' as it exits representational space.

With authorial becoming, conceptual becoming, and mathematical becoming all at play—at both the individual level (*ontogenesis*) and societal level (*phylogenesis*)—my rhetorical task is to unfold these intricacies in a way that readers might recognize themselves as observer-participants in that same unfolding, all within the necessarily flattened medium of text.

Development of Cube Drawings and Sphere in Flatland

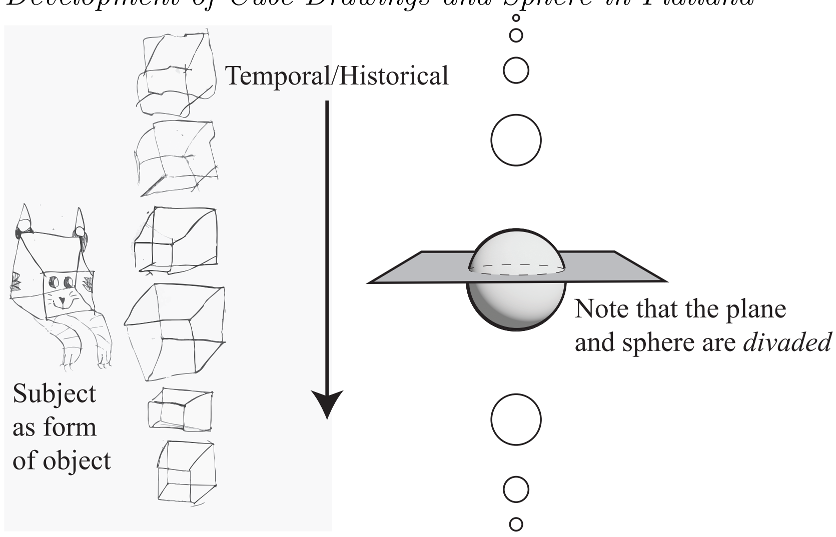


Figure 4: *Note.* Left: $\mathcal{M}$'s practice drawing cubes suggests an apperceived 'hypercube' falling through the 'plane' of experience. Right: a sphere falling through 2D space appears as a series of expanding and contracting circles. Implicit apperceptive recognition binds each finite representation within the series to a unified whole and both series to each other. There is no formal reason to infer a form of cube or sphere in any of the discrete moments.

I learned about the work of George Herbert Mead, especially the distinction between the {I} and the "me." The {I} is the source of action. The "me" is the self-as-recognized. None of my formal training in mathematics had prepared me to actually talk to students as *subjects*.

Since it is so central to my story and what follows, let me explore that theory briefly now through what I call the *paradox of identity*: the "me" who is {I} and the {I} who is "you", and the "we" that appears to be neither "you" nor {I}. There's a children's book, *The Monster at the End of This Book* (Stone, 2003), where the eponymous monster, Grover, is informed that there is a monster at the end of the book (Figure 5). Grover is frightened by this and begs the reader to stop reading. The cruel toddling reader keeps reading, though, with Grover becoming increasingly distraught. But at the end of the book, Grover recognizes himself as the moster at the end of the book: "Well, look at that! This is the end of the book, and the only one here is… ME. I, lovable, furry old Grover, am the Monster at the end of this book" (Stone, 2003, p. 26).

Sophisticates may recognize the story of Oedipus's unwitting self-banishment for the murder of Laius in the problem of self-recognition that Grover encounters in Stone's book. The problem is that the self-as-recognized, the "me", is often distinct from the {I}, understood as the source of action or locus of "power, creativity, and freedom" (P. F. Carspecken, 1999, p. 97). Both Grover and Oedipus appear to have only partially understood themselves. They did not completely misunderstand themselves. Oedipus knew his authority to banish; Grover knew his fear and, later, understood that he was the 'loveable' kind of monster. Complete misrecognition is outside human experience. They only partially misrecognized themselves.

Furthermore, I learned from Phil and Mead about the *generalized other*. The generalized other is the internalized sense of the expectations and attitudes of the broader community.

Why these ideas were so crucial for me is that they appeared to *explain* the ambivalent yet tense drift in my identity. They began to help me crack the problem of *alienation*. At Oak Ridge, I was both a pacifist and someone who found armed guards and the atomic bomb somehow 'glamorous.' I reasoned that I had internalized a contradiction in the *generalized other*. The generalized other is Mead's term that I use to point to the moral authority whose words filled my inner monologue with self-loathing for being unable to live up to both the ideals of pacifism and goodness of defeating fascism that the atomic bomb represented. Differences in others' expectations and attitudes toward me had always made me tense in social situations. I had never been able to figure out why, but it made sense to think that the "me" was divided depending on how the other person recognized me. Contradictions in the generalized other explain why I felt the need to wear different 'hats' in different social situations.

Different people have different sensitivities to these contradictions. It seems quite normal, even necessary, to take on different roles in different social situations. But the problem becomes acute when those roles conflict with each other or with deep commitments about what it means to be a good person. When fulfilling a role requires violating some aspect of self that feels inviolable, the tension can become unbearable.

In short, I found in Phil's critical theory some answers to the existential questions that I might have been able to answer if I had continued studying Kierkegaard instead of falling for pure mathematics.

After a few weeks, I decided to linger after class. I asked Phil what a *critical mathematics* would look like. We chatted, and I was hooked. We agreed that a critical mathematics would express mathematics as 'normative,' but I misunderstood 'normativity' as arbitrary (i.e., "socially constructed") and in opposition to people who identify as queer. Essentially taking normativity to be norminess, a kind of un-critical adherence to received knowledge, led me down some unproductive paths. Rather than healing the drift of alienation, I ended up entrenching alienated forms of thought in my work.

I was simultaneously involved in Dr. Erik Jacobson's research project that studied misconceptions in the domain of fractions, decimals, and rational numbers. I was also enrolled in an arts-based educational research class that Dr. Gus Welsek taught. I began to analyze the student work samples that expressed some kind of flawed mathematical reasoning as if they were artistic expressions. Rather than asking "what is *wrong* with this work/student?" I began to ask, "what is this student trying to say?"

In Phil's class, I also learned about Jürgen Habermas' theory of communicative action (1985; 1984). Communicative action is oriented toward understanding. It contrasts with instrumental and strategic action. Those are both telological (goal-oriented), where a subject acts towards objects to reach a goal. In strategic action, other people's responses are anticipated and actors can either coordinate their actions to accomplish goals or use each other to accomplish an end. The latter explained why some interactions felt so manipulative and gross.

I also learned of Habermas' conceptualization of three different types of *validity claims*. I wanted so badly to be good—to be valid! To understand why this framework resonated so deeply with my struggles, you need to grasp what Habermas means by "validity" itself.

The Structure of Validity Claims

When you speak, you make claims. This may seem obvious, but the depth of what I am claiming here is subtle. Every meaningful act—whether a gesture, a statement, or even silence—implicitly references what Habermas calls *validity claims*. These are not just assertions you make; they are the conditions under which your act could be understood, accepted, or challenged by another person.[1]

Validity is internal to meaning. As Carspecken articulates, "Habermas argues that all meaningful acts *internally reference truth claims*. Truth is internal to meaning" (P. F. Carspecken, 1999, 71–72, emph. orig.). You cannot separate what an utterance means from the implicit claims it makes about what is true, what is right, and what is sincere. This is a profound understanding: meaning is not a mental representation transmitted from one mind to another; it is a structure of commitments and entitlements that emerges in the space between people. Understanding a speech act means grasping the conditions under which its validity claims would hold or not hold (P. F. Carspecken, 2018).

Habermas expanded the concept of a single truth claim to three universal categories of validity claims. As Carspecken explains, "The claims that tacitly accompany every act of meaning will always fall into categories, three of which will always be represented" (P. F. Carspecken, 1999, p. 72). They are always present in every meaningful act, though they may be foregrounded or backgrounded to different degrees. Let me explain each category, not as abstract philosophy, but as structures I recognized in my own struggles to make sense of experience.

Objective Validity Claims Objective validity claims pertain to factual states of affairs in what Habermas calls the "objective world" (P. F. Carspecken, 2018; Habermas, 1984). When I say "The calculator has a square-root button," I am making a claim that could, in principle, be verified or falsified by anyone who examines the calculator. I am asserting that something is true about an object that exists independently of my feelings or

[1] "The expectation that another will understand your use of signs is constructed tacitly from the expectation that another will experience what the sign is about in the same way as one's self. This is intersubjectivity. The expectations associated with simply being understood, aside from consequences, tacitly assume multiple subjectivities capable of having common experiences." (P. F. Carspecken, 1999, p. 38)

intentions. Habermas describes this as asserting "That the statement made is true (or that the existential presuppositions of the propositional content mentioned are in fact satisfied)" (Habermas, 1984, 161, e-book).

Support for objective claims relies on what I learned to call the principle of *multiple access* (P. F. Carspecken, 1995, 1999). If I claim "There is a tree outside my window," you can, in principle, look out that same window and confirm or deny my claim. Different observers, using different devices, should be able to access the same state of affairs and reach agreement about it. This requires "repeated observations" by multiple observers (P. F. Carspecken, 1995, p. 77). This is why science works: the criterion for objectivity is intersubjective repeatability, not private experience.

When I struggled in eighth grade with "rise over run," my difficulty was partly an objective validity issue. The teacher claimed "The slope is the vertical change divided by the horizontal change." This claim referenced an objective fact about coordinate geometry. But my confusion arose because the orientation of the paper—something objective and observable—undermined my ability to understand which direction was "vertical." The claim was objectively true in a specific reference frame, but that frame was not made explicit. This is a feature of all objective claims: they always presuppose a framework of background assumptions that makes them intelligible.

Objective claims are associated with the third-person position: "it" or "they," the world of objects and states of affairs accessible from multiple perspectives (P. F. Carspecken, 1995, 1999).

Subjective Validity Claims Subjective validity claims relate to the speaker's inner world of intentions, feelings, desires, attitudes, values, and capacities—representing something in what Habermas calls the **subjective world** (P. F. Carspecken, 1999, 2018; Habermas, 1984). When I say "I find mathematics beautiful," I am not making a claim that you can verify by looking at mathematical objects. I am making a claim about my own inner experience, to which I have what philosophers call "privileged access."

The validity claim here is one of *truthfulness* or *sincerity*. The speaker asserts "That the manifest intention of the speaker is meant as it is expressed" (Habermas, 1984, 161, e-book). When I express a subjective state, you cannot directly verify whether I am being honest. You can only assess the consistency of my behavior over time. These claims are debated by establishing the honesty of self-reports, as they often involve privileged access to one's inner states (P. F. Carspecken, 1995, 1999). If I claim to love mathematics but consistently avoid doing mathematics, you have grounds to doubt my sincerity. The support for subjective claims comes from the pattern of my actions, not from objective observation of an external world.

This category helped me understand my years of self-loathing. When I said "I hate mathematics," I was making a subjective validity claim. But the felt experience was complicated by the fact that I also loved certain aspects of mathematical reasoning—the aesthetics of symbols, the elegance of proofs. The contradiction was not in the mathematics itself but in my subjective relationship to it, mediated by the social contexts in which I encountered it.

Subjective claims are associated with the first-person position: the {I} who speaks (P. F. Carspecken, 1995, 1999). Only I can truly know what I intend, what I feel, what I desire. This is not solipsism; it is recognition of the asymmetry between first-person experience and third-person observation. You can observe my behavior, but you cannot feel my pain.

Normative-Evaluative Validity Claims Normative-evaluative claims concern what is good, bad, right, wrong, proper, legitimate, or appropriate within a shared social world (P. F. Carspecken, 1999, 2018; Habermas, 1984). When I say "Teachers should not humiliate students," I am not making a claim about an objective fact or about my private feelings. I am making a claim about a norm—a rule or standard—that I expect others in my community to recognize and endorse. The speaker claims "That the speech act is right with respect to the existing normative context (or that the normative context that it is supposed to satisfy is itself legitimate)" (Habermas, 1984, 161, e-book).

Supporting normative-evaluative claims is fundamentally different from supporting objective or subjective claims. No amount of repeated observation will settle a disagreement about whether something is right or wrong. No appeal to my sincerity will resolve a moral dispute. Instead, normative claims must be supported by seeking shared normative-evaluative agreements within a community (P. F. Carspecken, 1995, 1999). These claims implicitly "call upon the other person to conform to the actor or to agree with the actor about certain things" (P. F. Carspecken, 1995, p. 83). The validity of a norm depends on whether it can, in principle, win the free assent of all those it affects.

This is where Habermas's framework becomes politically charged. If a norm is maintained through coercion, manipulation, or systematically distorted communication, it lacks genuine validity. A norm is truly valid only if it could survive rational critique in conditions of undistorted dialogue—what Habermas calls the "ideal speech situation." This is not a utopian fantasy; it is a regulative ideal, a standard against which to measure the legitimacy of existing norms.

When I felt the tension between my commitment to pacifism and my admiration for the institutions that defeated fascism, I was experiencing a contradiction in normative-evaluative claims. Both norms—"Violence is wrong" and "Defeating fascism was good"—seemed right to me, but they pointed in opposite directions. The generalized other, that internalized sense of communal expectations, contained a contradiction. My task was not to eliminate the contradiction but to articulate it, to understand the conditions under which each norm might be valid or invalid.

Normative claims are associated with the second-person position ("you") and the first-person plural ("we") (P. F. Carspecken, 1995, 1999). When I make a normative claim, I am implicitly calling upon you to recognize it as binding on both of us. The "we" that emerges from this mutual recognition is not a fixed entity; it is an ongoing achievement, constantly renegotiated through dialogue.

The Three Subject Positions and the Unifying "We" These three validity claims correspond to three different subject positions that structure all communication. The objective world is accessed from the third-person position: "it" or "they," the world of objects and states of affairs. The subjective world is accessed from the first-person position: {I}, the locus of experience and agency. The normative world is accessed from the second-person position ("you") and the first-person plural ("we"), the shared space of reasons and norms.

The normative "we" unites the three positions, adjudicating between them in a way that always appears incomplete. When we communicate, we coordinate our perspectives on the objective world (*that* is true), our subjective experiences (*I* feel this way; do *you*?), and our normative commitments (*we* should do this). But this coordination is never final. New experiences, new perspectives, new critiques can always emerge, forcing us to renegotiate our shared understanding.

This triadic structure is not a static framework imposed on experience.

It emerges from experience through reflection. When I speak to you, I do not consciously think "Now I will make an objective claim, now a subjective claim, now a normative claim." These categories are implicit in the structure of communication itself. Only through the kind of analysis Habermas provides do they become explicit.

I still find extraordinary expressive power in this structure. For example, the debate about gender that occupies so much of the public sphere as I write, tends to invoke these three aspects of validity without anyone acknowledging that they are doing so. The felt-experience of being transgender (subjective), for example, is often displaced to talk about biological sex (objective), to then make normative claims about who should use which bathroom. Understanding the triadic aspect of validity doesn't suddenly 'solve the problem.' Instead, it transforms the *inference field*. Rather than continually circling through the mundane algorithms of propaganda, the kinds of next-thoughts that follow from the triadic conceptualization of communicative rationality allowed me to think about the issues I was struggling with, instead of just getting carried along, feigning an excruciating ambivalence about the issues I cared deeply about.

In this manuscript, I often speak of subjective, normative, and objective *dimensions*. Borrowing from Robert Brandom and Willfred Sellars' phrase, "the space of reasons," I conceptualize such a space as 'three-dimensional.' The space of reasons is not Cartesian. These 'axes' of validity are not orthagonal to one another. Instead, they are sort of *inside and outside* of each other in a way that is very hard to conceptualize spatially. This structural relationship only emerges on reflection, a point I will labor to make in the next chapter.

Knowledge-Constitutive Interests: The Deep Roots of Inquiry

Understanding validity claims was transformative, but it raised a deeper question: Why do humans inquire at all? What drives us to seek knowledge in such different forms—the precision of physics, the interpretation of history, the critique of ideology? Habermas's answer, developed in his earlier work *Knowledge and Human Interests*, is that all knowledge is rooted in fundamental orientations that arise from the basic conditions of human species reproduction and self-constitution.

He calls these orientations *knowledge-constitutive interests*. The term "interest" here is philosophical, not psychological. These are not personal preferences or individual motivations. They are deep structures that mediate the relationship between how humans exist in the world (through work, language, and power) and how humans come to know the world. They are rooted in the objective problems of life preservation that have been solved through our cultural form of existence: work sustains us materially, language enables social coordination, and the struggle for autonomy drives self-formation.

I initially misunderstood this. I thought "interests" meant biases that corrupt pure knowledge. But Habermas is arguing something more radical: there is no "understanding from nowhere," no knowledge that stands outside of human concerns. Knowledge is always bound up with the conditions that make it possible. The question is not whether knowledge serves interests, but which interests it serves and whether those interests are acknowledged or repressed.

Habermas identifies three categories of knowledge-constitutive interests, each corresponding to a form of scientific inquiry and to one of the validity claims I have just described.

The Technical Cognitive Interest The first is the *technical cognitive interest*, which guides the empirical-analytic sciences. This interest aims at disclosing and comprehending reality for the sake of possible technical control. It is rooted in the behavioral system of *instrumental action*—the realm of work, where humans adapt to and transform their material environment.

Knowledge acquired under the technical interest takes the form of predicting observable events and specifying means for achieving tangible, objective ends. Its validity is based on the degree to which predictions are successful. If I predict that water will boil at 100 degrees Celsius at sea level, and it does, my knowledge is confirmed. If it does not, my hypothesis is falsified, and I must revise my understanding.

The technical interest is not reducible to capitalist exploitation or the desire to dominate nature, though it can certainly be co-opted for those purposes. At its root, this interest responds to the human need to secure the material conditions of survival. Every time you flip a light switch, you are relying on knowledge shaped by the technical interest: knowledge of electrical circuits, of power generation, of the properties of materials.

This interest employs formalized languages, operational definitions, and hypothesis-testing methodologies. It discloses reality from the viewpoint of technical control over objectified natural processes. When I struggled at Oak Ridge with FORTRAN code, I was working within the technical interest. The goal was to produce knowledge that could be translated into computational procedures—algorithms that would run successfully or fail predictably.

The technical interest corresponds to objective validity claims and the third-person perspective. It treats the world as a collection of objects and events whose relations can be modeled, predicted, and controlled.

The Practical Cognitive Interest The second is the *practical cognitive interest*, which guides the historical-hermeneutic sciences—disciplines like history, anthropology, literary studies, and much of qualitative research. This interest aims at maintaining the intersubjectivity of mutual understanding in ordinary-language communication and in action guided by common norms.

The practical interest is rooted in language and tradition-bound social life. Its goal is not to control objectified processes but to maintain the very condition that makes a world appear as something shared: intersubjective understanding. When communication breaks down—when I cannot understand what you mean, or when your cultural tradition is alien to me—the practical interest drives inquiry aimed at restoring or establishing mutual comprehension.

Knowledge acquired under the practical interest involves hermeneutically explicating formerly implicit knowledge. It uses ordinary language, not formalized calculi. Its validity is based on the degree to which insiders recognize explicit articulations as something they already knew in some tacit way. When an anthropologist describes a cultural practice and members of that culture say "Yes, that is what we do, though we never put it that way before," the hermeneutic knowledge is validated.

This interest is not about imposing an external framework on a culture. It is about entering into dialogue with a tradition, making explicit the meanings that guide action within that tradition, and mediating understanding between different individuals, groups, and cultures—both horizontally (across contemporary cultures) and vertically (between present and past).

When I began analyzing student mathematical reasoning as if it were artistic expression, asking "What is this student trying to say?" rather than "What is wrong with this student?", I was shifting from a technical interest (control and correction) to a practical interest (understanding and interpretation). This shift was transformative. It allowed me to recognize meaning where I had previously seen only error.

The practical interest corresponds to normative-evaluative validity claims and the second-person ("you") and first-person plural ("we") perspectives. It is concerned with the shared social world of norms, values, and meaningful actions.

The Emancipatory Cognitive Interest The third is the *emancipatory cognitive interest*, which guides critically oriented sciences—psychoanalysis, ideology critique, and critical social theory. This interest aims at self-reflection and freeing consciousness from dependence on hypostatized powers. It is rooted in the dimension of *power*, specifically the struggle for self-assertion against domination.

The emancipatory interest emerges in contexts of systematically distorted communication and thinly legitimated repression. It arises when norms are maintained through coercion rather than rational consent, when identities are imposed rather than freely chosen, when knowledge is used to manipulate rather than enlighten. The emancipatory interest seeks to dissolve these distortions through self-reflective learning processes.

Knowledge acquired under the emancipatory interest involves the critical dissolution of objectivism and the undoing of repression and false consciousness. Its method is self-reflection: the process by which a subject becomes transparent to itself in its own genesis. The validity of emancipatory knowledge cannot be assessed by technical prediction or hermeneutic recognition alone. It is validated by the experience of enlightenment itself—the moment when you recognize how you have been deceived, constrained, or alienated, and that recognition frees you to act differently.

Habermas famously argues that "in the power of self-reflection, knowledge and interest are one." This is not a mystical claim. It means that the pursuit of self-knowledge is inherently emancipatory. To understand why you believe what you believe, why you desire what you desire, is to gain a degree of freedom from those beliefs and desires. They no longer control you unconsciously; they become objects of deliberate choice.

When Phil helped me recognize that to speak is to assume an Other who might understand, he was engaging the emancipatory interest. My inner monologue of self-loathing had been sustained by a fantasy of absolute isolation. Recognizing the intersubjective structure of thought dissolved that fantasy. I was freed, not from loneliness, but from the illusion that loneliness could be absolute.

The emancipatory interest is inherently connected to subjective validity claims and the first-person perspective ({I}). It concerns the formation and transformation of the self, the struggle for autonomy and authenticity. But it is not solipsistic. Emancipation is achieved through dialogue, through the recognition of others, through the collective critique of ideologies that bind us all.

How Validity Claims and Knowledge Interests Relate

The three categories of validity claims and the three knowledge-constitutive interests are deeply intertwined. They provide two perspectives on the same underlying reality: the differentiated structure of human rationality.

The validity claims describe the formal structure of meaning in communication. Every meaningful act implicitly references objective, subjective, and normative dimensions. The knowledge interests describe the deep motivational and existential roots of inquiry. Every form of knowledge arises from a fundamental human orientation toward the world.

The correspondence is this: Objective validity claims align with the technical cognitive interest. Both are concerned with the world of facts and events, accessed through multiple observers, governed by the logic of prediction and control. Subjective validity claims align with the emancipatory cognitive interest. Both are concerned with the inner world of the self, accessed through privileged introspection, governed by the logic of self-reflection and authenticity. Normative-evaluative validity claims align with the practical cognitive interest. Both are concerned with the shared social world of norms and values, accessed through intersubjective dialogue, governed by the logic of mutual understanding and recognition.

This alignment is not arbitrary. The technical interest defines what it means for an objective claim to be valid: successful prediction. The practical interest defines what it means for a normative claim to be valid: intersubjective recognition of shared interests. The emancipatory interest defines what it means for subjective claims to be authentic: transparency to oneself in one's own genesis.

Understanding this framework was, for me, a kind of conceptual liberation. It explained why arguments about mathematics education were so intractable. Researchers were operating within different knowledge-constitutive interests, making different kinds of validity claims, without acknowledging the plurality of rationality. Someone arguing from a technical interest wants mathematical knowledge to be predictive and operationalizable. Someone arguing from a practical interest wants it to be meaningful and culturally situated. Someone arguing from an emancipatory interest wants it to be transformative and empowering.

These are not mutually exclusive. A critical mathematics can integrate all three. But integration requires acknowledging the differences, understanding the conditions of possibility for each form of knowledge, and resisting the temptation to reduce one to another.

The Critique of Scientism

While Phil did not push his own writing on his students, I read his works voraciously. His beautiful series on the limits of knowledge and how they relate to scientism is a must read for anyone reading this who recognizes themselves in what I described as "intellectual chauvinism" (2006, 2009, 2016). Scientism is the ideology that only third-person claims are valid. It is an ideology that cannot understand itself, since it cannot account for the subjective and normative dimensions of its own claims.

The recognition of multiple validity claims and knowledge-constitutive interests is a powerful critique of scientism. Scientism privileges the technical interest and objective validity claims, treating them as the only legitimate form of knowledge. But this privilege cannot be justified on its own terms. The claim "Only objective knowledge is valid" is itself a normative claim. It expresses a value judgment about what counts as knowledge. To assert it is to performatively contradict oneself: you are making a normative claim while denying the validity of normative claims.

Habermas's framework reveals that rationality is differentiated, not singular. There is no one scientific method, no one criterion of validity, no one form of knowledge. Instead, there are multiple, irreducible forms of rational inquiry, each with its own logic, its own standards, its own connection to human interests. The task of critical theory is not to privilege one over the others, but to understand their interrelations, to resist their distortion, and to enable their full development.

Phil's work is broadly inspired by the Frankfurt School, but he also draws heavily on G.W.F. Hegel and his experiences studying yogic philosophy. Studying with Phil brought *the negative* into my life explicitly. He mailed me a copy of Jean Hyppolite's book that describes how the {I} "never is what it is and is always what it is not" (1974, p. 150). I'm grateful for that gift, among countless others.

Besides the gift of genuine intersubjective recognition, Phil's piece titled *The Missing Infinite* (P. F. Carspecken, 2018) is a must-read for anyone interested in critical theory. While I will do my best to build on his insights, his gentle enjoinment to bring the *in*finite into explicitness helped clarify the central tension in identity that I had been struggling with for years. For I am both finite, object-like, bounded, and determinate, and *in*finite, a subject who breaks boundaries in the ever-evolving pursuit of self-determination.

Recognizing this tension as a fundamental aspect of identity helped me learn to relax a bit. So much of what I had internalized about being a good person was about being consistent, reliable, and predictable. In short, 'the Good'—that seemingly relativistic ideal—was not relativistic at all. Attaining such goodness is probably impossible, but trying to live up to my commitments by becoming more object-like in how others recognize me gave me reasons for trying to carefully justify my commitments. It also gave me grounds to prune some commitments that were deeply conflictual. But notice the tension in that last sentence. I had to *change* to be more object-like! The goal of finitude necessarily requires *in*finite, i.e., becoming. The lack of coincidence between the {I} and the "me" is not a solvable problem. It's a paradox that ~~we~~ must all live with.

One strategy for managing that paradoxical aspect of identity is to try to place oneself and others on a *developmental trajectory*. I need hardly forgive a 4-year-old for not honoring their commitments, but I expect a 40-year-old to be more consistent. Still, I sometimes feel like I was just born but a few moments ago. The revitalization that comes with re-sensitizing myself to the world around me grows the capacity for self-forgiveness.

Thinking through three layers of validity, the *subjective*, *objective*, and *normative* helped me untangle some of the deep contradictions I had internalized that resulted in so much ambivalence. Later, I recognized that artistic expression did not quite cover the *normative* aspects of mathematics. Phil also bought me a copy of *A Spirit of Trust* by Robert Brandom (2019). In that work, he discusses the *experience of error*. I will take that on soon, but since I have such a deep, rich history of getting it wrong, I figured the *experience of error* could be an epistemological foundation for mathematics. Could mathematical truths be known through the exclusion of error?

That idea probably sounds a bit strange to mathematicians. But it is a relatively normal idea in teacher education. I recently met with a kindergarten teacher who was introducing some IU students who she is mentoring. One of the first things she said is "you are here to make mistakes."

The literature in mathematics education discusses errors, misconceptions, perturbations, and mistakes extensively. Researchers work to understand why they happen, how they propagate through inferential systems, and how they are repaired. However, these theories rest together in incoherence; there isn't a single coherent philosophy of math education. I became curious about how I was able to read different theories, identify with parts of them, reject other parts, and somehow still understand what authors were trying to say.

My initial attempts to filter theories through the experience of error resulted in an understanding: the community of math education researchers are doing meta-mathematics over a mathematical landscape defined by *material incompatibility*. If you're a mathedian and vigorously disagree with that assessment, please hang with me for a bit. There is room for your "no." The goal became establishing a methodology for bringing students' work, correct or not, into a stable mathematical structure.

Intersubjectivity and the Horizon of Meaning

When Phil reminded me that to speak is to assume an Other who might understand, he was pointing to a fundamental structure that critical theory calls **intersubjectivity**—shared understanding, mutual expectations, and reciprocal recognition among subjects (P. F. Carspecken, 1999; P. F. Carspecken & Zhang, 2013). Even my inner monologue of self-loathing assumed someone who could understand what I was so angry and sad about. This assumption of communicative competence—that someone might grasp my meaning—is a necessary condition for communication itself, even if it is not guaranteed in practice (Habermas, 1984).

The distinction between **subjectivity** (my own inner experience, to which only I have privileged access) and **intersubjectivity** (shared understanding and mutual recognition) proved crucial. "The distinction between subjectivity and intersubjectivity (position-taking) is absolutely crucial to subjective-referenced claims. Subjectivity is not the immediate effect of social relations, cultural forms, languages, and discourses. It is rather given through a play between such intersubjectively constituted symbols and raw, unmediated subjective experience that is always referenced but that is always already removed from experience when we represent it." (P. F. Carspecken, 1995, p. 108) While subjectivity is not solely an effect of social relations, it becomes knowable and representable only *through* intersubjectivity (P. F. Carspecken, 1995, 1999). Similarly, objectivity is not possible without intersubjectivity (P. F. Carspecken, 1999; Habermas, 1995).

Human identity, I learned, is fundamentally an "I-me" relationship: the {I} (pure subjectivity/agency) is understood and affirmed through the "me" (the self from the perspective of others). This self-relation arises out of interactive contexts and is infused with a desire for recognition (P. F. Carspecken, 1999; Habermas, 1995). Intersubjectivity provides the "ground" of human societies, enabling mutual understanding, cooperation, and the very possibility of knowledge (Habermas, 1971; Mead, 2015). Yet it remains **fragile and occasionally successful** in everyday communication, constantly requiring negotiation and revision (Habermas, 1984).

This framework helped me understand that meanings are constituted not just by explicit validity claims but also by implicit **horizons of intelligibility** (P. F. Carspecken, 1995, 1999)—the background assumptions and shared contexts that make understanding possible. In mathematics education, these horizons often remain hidden, leading to the kinds of confusion I experienced with "rise over run."

Knowledge Constitutive Interests

Understanding validity claims and intersubjectivity was transformative, but it raised a deeper question: Why do humans inquire at all? What drives us to seek knowledge in such different forms—the precision of physics, the interpretation of history, the critique of ideology? Habermas's answer, developed in his earlier work *Knowledge and Human Interests*, is that all knowledge is rooted in fundamental orientations arising from the basic conditions of human species reproduction and self-constitution. Habermas terms these "the basic orientations rooted in specific fundamental conditions of the possible reproduction and self-constitution of the human species, namely work and interaction" (Habermas, 1971, p. 196).

He calls these orientations *knowledge-constitutive interests* (KCI). The term "interest" here is philosophical, not psychological. These are not personal preferences or individual motivations. They are deep structures that mediate the relationship between how humans exist in the world (through work, language, and power) and how humans come to know the world (P. F. Carspecken, 2016). They are rooted in the objective problems of life preservation that have been solved through our cultural form of existence: work sustains us materially, language enables social coordination, and the struggle for autonomy drives self-formation.

I initially misunderstood this. I thought "interests" meant biases that corrupt pure knowledge. But Habermas is arguing something more radical: there is no "understanding from nowhere," no knowledge that stands outside of human concerns. Knowledge is always bound up with the conditions that make it possible. The question is not whether knowledge serves interests, but which interests it serves and whether those interests are acknowledged or repressed.

Habermas identifies three categories of knowledge-constitutive interests, which "take form in the medium of work, language, and power" (Habermas, 1971, p. 313), each corresponding to a form of scientific inquiry and to one of the validity claims I have just described:

The Technical Cognitive Interest The first is the *technical cognitive interest*, which guides the empirical-analytic sciences. This is the "cognitive interest in technical control over objectified processes" (Habermas, 1971, p. 309), rooted in the behavioral system of *instrumental action*—the realm of work, where humans adapt to and transform their material environment (Habermas, 1971, pp. 121, 124).

Knowledge acquired under the technical interest takes the form of predicting observable events and specifying means for achieving tangible, objective ends (P. F. Carspecken, 2016). Its validity is based on the degree to which predictions are successful. If I predict that water will boil at 100 degrees Celsius at sea level, and it does, my knowledge is confirmed. If it does not, my hypothesis is falsified, and I must revise my understanding.

The technical interest is not reducible to capitalist exploitation or the desire to dominate nature, though it can certainly be co-opted for those purposes. At its root, this interest responds to the human need to secure the material conditions of survival. Every time you flip a light switch, you are relying on knowledge shaped by the technical interest: knowledge of electrical circuits, of power generation, of the properties of materials.

This interest employs formalized languages, operational definitions, and hypothesis-testing methodologies (Habermas, 1971). It discloses reality from the viewpoint of technical control over objectified natural processes. When I struggled at Oak Ridge with FORTRAN code, I was working within the technical interest. The goal was to produce knowledge that could be translated into computational procedures—algorithms that would run successfully or fail predictably.

The technical interest corresponds to objective validity claims and the third-person perspective. It treats the world as a collection of objects and events whose relations can be modeled, predicted, and controlled.

The Practical Cognitive Interest The second is the *practical cognitive interest*, which guides the historical-hermeneutic sciences—disciplines like history, anthropology, literary studies, and much of qualitative research. This interest aims at "maintaining the intersubjectivity of mutual understanding in ordinary-language communication and in action according to common norms" (Habermas, 1971, p. 176).

The practical interest is rooted in language and tradition-bound social life (Habermas, 1971, p. 313). Its goal is not to control objectified processes but to maintain the very condition that makes a world appear as something shared: intersubjective understanding. As Habermas puts it, it maintains the intersubjectivity "within whose horizon reality can first appear as something" (Habermas, 1971, p. 176). When communication breaks down—when I cannot understand what you mean, or when your cultural tradition is alien to me—the practical interest drives inquiry aimed at restoring or establishing mutual comprehension.

Knowledge acquired under the practical interest involves hermeneutically explicating formerly implicit knowledge (P. F. Carspecken, 2016). It uses ordinary language, not formalized calculi. Its validity is based on the degree to which insiders recognize explicit articulations as something they already knew in some tacit way (P. F. Carspecken, 2016). When an anthropologist describes a cultural practice and members of that culture say "Yes, that is what we do, though we never put it that way before," the hermeneutic knowledge is validated.

This interest is not about imposing an external framework on a culture. It is about entering into dialogue with a tradition, making explicit the meanings that guide action within that tradition, and mediating understanding between different individuals, groups, and cultures—both horizontally (across contemporary cultures) and vertically (between present and past) (Habermas, 1971, p. 176).

When I began analyzing student mathematical reasoning as if it were artistic expression, asking "What is this student trying to say?" rather than "What is wrong with this student?", I was shifting from a technical interest (control and correction) to a practical interest (understanding and interpretation). This shift was transformative. It allowed me to recognize meaning where I had previously seen only error.

The practical interest corresponds to normative-evaluative validity claims and the second-person ("you") and first-person plural ("we") perspectives. It is concerned with the shared social world of norms, values, and meaningful actions.

The Emancipatory Cognitive Interest The third is the *emancipatory cognitive interest*, which guides critically oriented sciences—psychoanalysis, ideology critique, and critical social theory. This interest aims at **self-reflection**, which "releases the subject from dependence on hypostatized powers" (Habermas, 1971, p. 310). It is rooted in the dimension of *power* (specifically, self-assertion against domination) (Habermas, 1971, p. 313).

The emancipatory interest emerges in contexts of systematically distorted communication and thinly legitimated repression (Habermas, 1971). It arises when norms are maintained through coercion rather than rational consent, when identities are imposed rather than freely chosen, when knowledge is used to manipulate rather than enlighten. The emancipatory interest seeks to dissolve these distortions through self-reflective learning processes.

Knowledge acquired under the emancipatory interest involves the "critical dissolution of objectivism" (Habermas, 1971, p. 213) and the undoing of repression and false consciousness (Habermas, 1971). Its method is self-reflection: the process by which a subject becomes transparent to itself in its own genesis. The validity of emancipatory knowledge cannot be assessed by technical prediction or hermeneutic recognition alone. It is validated by the experience of enlightenment itself—the moment when you recognize how you have been deceived, constrained, or alienated, and that recognition frees you to act differently.

Habermas famously argues that "knowledge for the sake of knowledge attains congruence with the interest in autonomy and responsibility" (Habermas, 1971, p. 315). In the power of self-reflection, knowledge and interest are one. This is not a mystical claim. It means that the pursuit of self-knowledge is inherently emancipatory. To understand why you believe what you believe, why you desire what you desire, is to gain a degree of freedom from those beliefs and desires. They no longer control you unconsciously; they become objects of deliberate choice.

When Phil helped me recognize that to speak is to assume an Other who might understand, he was engaging the emancipatory interest. My inner monologue of self-loathing had been sustained by a fantasy of absolute isolation. Recognizing the intersubjective structure of thought dissolved that fantasy. I was freed, not from loneliness, but from the illusion that loneliness could be absolute.

The emancipatory interest is inherently connected to subjective validity claims and the first-person perspective ($\{I\}$). It concerns the formation and transformation of the self, the struggle for autonomy and authenticity. But it is not solipsistic. Emancipation is achieved through dialogue, through the recognition of others, through the collective critique of ideologies that bind us all.

How Validity Claims and Knowledge Interests Relate

The three categories of validity claims and the three knowledge-constitutive interests are deeply intertwined (P. F. Carspecken, 2016; Habermas, 1984). They provide two perspectives on the same underlying reality: the differentiated structure of human rationality.

The validity claims describe the formal structure of meaning in communication. Every meaningful act implicitly references objective, subjective, and normative dimensions. The knowledge interests describe the deep motivational and existential roots of inquiry. Every form of knowledge arises from a fundamental human orientation toward the world.

The correspondence is this: Objective validity claims align with the technical cognitive interest. Both are concerned with the world of facts and events, accessed through multiple observers, governed by the logic of prediction and control (P. F. Carspecken, 1999; Habermas, 1971). Subjective validity claims align with the emancipatory cognitive interest. Both are concerned with the inner world of the self, accessed through privileged introspection, governed by the logic of self-reflection and authenticity (P. F. Carspecken, 1999; Habermas, 1971). Normative-evaluative validity claims align with the practical cognitive interest. Both are concerned with the shared social world of norms and values, accessed through intersubjective dialogue, governed by the logic of mutual understanding and recognition (P. F. Carspecken, 1999; Habermas, 1971).

This alignment is not arbitrary. The technical interest defines what it means for an objective claim to be valid: successful prediction. The practical interest defines what it means for a normative claim to be valid: intersubjective recognition of shared interests. The emancipatory interest defines what it means for subjective claims to be authentic: transparency to oneself in one's own genesis.

Understanding this framework was, for me, a kind of conceptual liberation. It explained why arguments about mathematics education were so intractable. Researchers were operating within different knowledge-constitutive interests, making different kinds of validity claims, without acknowledging the plurality of rationality. Someone arguing from a technical interest wants mathematical knowledge to be predictive and operationalizable. Someone arguing from a practical interest wants it to be meaningful and culturally situated. Someone arguing from an emancipatory interest wants it to be transformative and empowering.

These are not mutually exclusive. A critical mathematics can integrate all three. But integration requires acknowledging the differences, understanding the conditions of possibility for each form of knowledge, and resisting the temptation to reduce one to another.

Conceptualizations of Power: Foucault, Habermas, and Typologies

Power is a multifaceted concept, generally indicating a capacity to influence or control, with significant implications for social relations and the constitution of knowledge and identity. From a critical theory perspective, knowledge and power have a complex interrelation, where both common-sense and theoretical forms of knowing can function to support and reproduce social, cultural, political, and economic forms of oppression. Critical research, therefore, problematizes concepts like "research" and "researcher" because they are intrinsic to the production of ideologies that can impose power and maintain privilege (P. F. Carspecken, 2018).

Foucault's Conception of Power For Michel Foucault, power is *primordial and anonymous*, existing prior to truth claims and the very constitution of subjectivity (P. F. Carspecken, 1999, pp. 27–28). He suggests that humanistic conceptions of the actor and thinker should be discarded, as subjectivity and agency emerge as effects of power. In this view, power is understood as *circulating* and operating through a "netlike organization" where individuals are both vehicles and effects of power; it is exercised through *techniques and tactics of domination* rather than being a possession.

The *knowledge-power nexus* is central to Foucault's work, where power produces knowledge, and knowledge simultaneously constitutes power relations (Habermas, 1995).

This means that what is taken as "true, false, good, and bad" in a given era is determined by anonymous power, and discourse-practices construct subjectivity itself (P. F. Carspecken, 1999, p. 28). Foucault's analysis focuses on how power is exercised through subtle mechanisms that can mask overt coercion, often by clothing the use of knowledge in the 'garb of right' to obscure domination. He saw a "will to knowledge" that is generalized into a "will to power," inherent in all discourses and operating as "anonymous processes of subjugation" (Habermas, 2004, 292–295, e-book). His concept of *biopower* refers to a disciplinary power that deeply penetrates bodies, transforming creaturely life into a substrate of empowerment through scientific objectification and generated subjectivity (Habermas, 2004). Critics, however, argue that reducing truth to anonymous power leads to contradictions, as Foucault's own writing relies on being understood and convincing readers of its merit (P. F. Carspecken, 1999; Habermas, 2004, p. 29).

Habermas's Distinctions on Power Jürgen Habermas emphasizes an external relationship between truth and power, viewing power primarily as that which *distorts interactions oriented toward reaching understanding* (P. F. Carspecken, 1999, p. 46). In an ideal speech situation, power must be neutralized or equalized so that only "the force of the better reason" brings about consensus, allowing for pragmatically sound truth claims (P. F. Carspecken, 1995, 1999, pp. 46, 68). At the same time, Habermas implicitly acknowledges a *generative sense of power* tied to human agency and action itself (P. F. Carspecken, 1999). All human acts are acts of power because they "make a difference" in the world and issue from agents who could have acted otherwise (P. F. Carspecken, 1999, p. 81). Power is a capacity of the actor logically tied to the concept of action (P. F. Carspecken, 1999, p. 81). Acts are considered "powerful" when they succeed in realizing their intended goals. Communicative acts also express power, particularly in relation to communicative goals like being understood or recognized, where the "will to truth" and "will to power" are intertwined with human motivation and identity claims (P. F. Carspecken, 1999, p. 84).

Typologies of Power Power can be categorized to better understand its operation in social contexts. *Interactive power* refers to power relations where actors are differentiated in terms of who has the most say and whose definition of the setting prevails (P. F. Carspecken, 1995, p. 129). This includes *normative power*, where subordinates consent based on shared norms and status (e.g., student and teacher); *coercive power*, involving threatened sanctions; *contractual power*, based on agreements; and *charismatic power*, won through personality (P. F. Carspecken, 1995, pp. 130–134).

Cultural power, in contrast, refers to the extent of control over, or benefit from, the distribution and currency of cultural themes (P. F. Carspecken, 1995, pp. 131–132). This form of power works by limiting the roles and repertoires of identity claims available to members of a group, penetrating their very identity and potentially causing pain.

0.4 The Dialectical Turn: Divasion

After I defended my dissertation, I felt a deep sense of openness. I met my wife, $\mathcal{A}$, and her two daughters $\mathcal{M}$ and $\exists$. In the fall of 2023, $\mathcal{M}$ was four years old. $\mathcal{A}$ and I were just dating, but I was excited to be a part of the kids' lives. I discovered that $\mathcal{M}$, $\exists$, and I like to write songs and we like to work on them together. $\mathcal{M}$ and I were sitting in $\mathcal{A}$'s basement trying to write a song. The first thing she says in this clip is that she wants to write a "pattern song." The object she is referring to is the part of a microphone stand where the mic is held by the stand. The microphone extends past the clip, but is also clipped in, so part of it is inside of the clip and part of it is outside of the clip. I gave an talk that includes the audio of this clip for those interested in hearing the interaction(Savich, 2024).

$\mathcal{M}$: I'm going to do a pattern (I want to write a pattern song)

Tio: A pattern? Tell me what you mean, or just do it

$\mathcal{M}$: Outside In! Outside In!

Tio: More! Say more! What is this?

> $\mathcal{M}$: The microphone is outside of this (pointing at the clip that is part of the mic stand) then it's inside of it.
>
> Tio: What does that mean?
>
> $\mathcal{M}$: Um…I don't know…
>
> Tio: It's kind of interesting, though, isn't it? That things can be inside and outside of each other. Sometimes at the same time.
>
> $\mathcal{M}$: It is?
>
> Tio: Yeah! Well, this part is inside and this part is outside. So we can't say that the microphone is inside and we can't say that it's outside because it's both! That's a contradiction.
>
> $\mathcal{M}$: No, it's not. It's all on the other… of all of them.
>
> Tio: What's all over all of them?
>
> $\mathcal{M}$: That means that all of them is…all of it… that it divaded.
>
> Tio: Divaded?
>
> $\mathcal{M}$: Yeah. Divaded means that…um… It means that it's inside the thing and then it's outside the thing.
>
> Tio: Woah! Cool! I like that. So it's different than divided. Because dividing something sometimes means splitting but you're talking about divADed…that's cool. It's inside and outside?
>
> $\mathcal{M}$: Yeah.
>
> Tio: Tell me more!
>
> $\mathcal{M}$: It's mechanical. It's really not something. But I'm talking about when you grab something, it makes it divaded.

$\mathcal{M}$ noticed that there is a pattern, or rhythm in considering *outside, in.* I will make much of this intuition, as the movement she noticed is easily missed. The microphone was both inside and outside of the clip, which I think most people would conceive of as a static spatial relationship. People regularly experience objects that are inside and outside other objects. A person walks through a door; a folder hangs out of a backpack; a potted plant breaks the surface of the dirt while growing its roots in the soil. But the movement prior to the static spatial relationship is important. The body is divaded by the breath in rhythmic movement. Many fears involve divasion, like a needle piercing the skin. Many pleasures involve divasion, from the embrace of a lover to the feeling of consuming a delicious meal. Birth and death divade the subject.

In this interaction, the rhythm of the pattern song was flattened into a spatial concept. The song was arrested before she ever got a chance to sing it. Perhaps I erred in calling her concept a 'contradiction.' Perhaps *inside* and *outside* only strongly contrast. A logical contradiction could be drawn between *inside* and *not-inside*. The issue of what constitutes a contradiction is important for mathematics.

In any case, the interaction does say something about how paradoxes or contradictions are experienced. I cannot seem to hold a contradiction or paradox within one moment of awareness. Sentences like "this sentence is false," toggle between truth and falsity, appearing to create temporal movement. Perhaps that rhythm is part of the "pattern song" she was after. In the face of a potential contradiction, she invented a word, *divaded*.

$\mathcal{M}$'s invention suggests something profound about the foundations of mathematics. Formal set theory is based on the premise that an element of a set is strictly either inside or outside a set. This foundation appears to come from pruning more primordial spatializations like the one $\mathcal{M}$ observed. The consequences of this pruning are evident in foundational crises like Gödel's incompleteness theorems and Russell's paradox. Rather than repairing the effects of this pruning with fancy formal footwork, I suggest backing up. The foundations of mathematics can instead be approached through the academic field of math education, by listening to how children spatialize the world.

0.5 Reflection: Methodology and the Reader's Role

What did you just read? It wasn't madness, but it also wasn't a typical philosophy, mathematics, or math education text. In this section, I will describe the methodological framework that structures this work and the challenges involved in its construction. I will not *argue* that mathematics is equivalent to the methodological framework I describe. Instead, I will leave space for readers to recognize that conclusion for themselves if such an identity is available.

The Challenges of Reconstruction and Collaboration

This book is a reconstruction of my dissertation (Savich, 2022). Hubristic as it seems, that document is my 'life's work.' I had to heal through writing it. My task was to make the dissertation more accessible to a broader audience. But how could I make this dense work accessible without betraying the original intellectual labor? Distancing myself from the original text proved very difficult. The pedagogical task of teaching the text while simultaneously recreating it felt overwhelming. I needed help.

Why turn to AI when faced with such an overwhelming organizational and pedagogical task? The reasons are practical and pedagogical, supplementing the philosophical reasons discussed regarding the Hermeneutic Calculator.

Practically, I struggle with organization. I tend to write in terribly complex, multi-clausal sentences that often obscure rather than clarify. Furthermore, the dissertation process generated roughly 10,000 pages of notes. Organizing this material felt insurmountable, perhaps due to a learning dis/ability that makes standardization difficult.

I utilized AI as an adaptive technology and pedagogical partner. AI helped distill my notes, organize the structure, and translate dense prose into more accessible language. This use of AI as a tool for "expressive empowerment" addresses my specific cognitive constraints and can be understood as participating in what Negarestani characterizes as the history of intelligence (*Geist*): an ongoing project of "self-artificialization" (Negarestani, 2018, p. 26).

While this collaboration risked distorting my voice, I figured that the way generative AI systems standardize language would help to teach the concepts. Is clarity more important than an authentic voice? I accepted some stylistic homogenization for the sake of teaching, but I have been careful to verify all outputs and actively directed the AI to maintain the recursive quality of the thought.

This collaboration raises the question of authorship in the age of AI. What constitutes authorship? Is it about performing every task, or about bearing the existential responsibility for the ideas? Authorship today is less about solitary genius and more about orchestration, direction, and responsibility. It is I who bears the ultimate responsibility for these ideas. The collaboration itself is inherently risky. It is I who takes this risk, kept up all night in existential terror of having my work analyzed and judged by anonymous critical others. The AI is not staying up all night worrying.

Methodology

This book expresses *critical mathematics* through a method I name *critical autoethnography* (CAE). I name it so for the sake of its unnaming: ~~CAE~~. Such movement, naming and finding the name inadequate (building and breaking), structures the text and its subject. The central methodological question is: How can deeply personal, lived experience be transformed into a rigorous theoretical resource?

Whereas 'self-reference'—the great recursive engine of mathematical development—generally flattens a subject into an inert object called a *referent*, ethnographers do not drown and splay their subjects like a rat for dissection. Instead, the task is to recognize the Other as a breathing subject. In critical autoethnography, the idea is to recognize Otherness in oneself.

I originally called this work a theoretical autoethnography. The idea was to explore the intersection of philosophy and experience. Treating personal stories as data, I hoped to break the strict demarcations between objective research, ethics, and self-actualization. This approach requires the author to be simultaneously a storyteller and a theorist. What binds an individual's narrative? What makes it possible to understand that the child who supposedly said "babbery" is the adult writing this sentence?

I figured that leading with Kant's *transcendental unity of apperception*, the "I think" that must be able to accompany all of my representations, would lose readers who might otherwise benefit from what I have learned over the a few decades of experience trying to teach mathematics. "Philosophy," that love of knowledge, does not need to be illuminated by the details of my life. Nor does mathematics. But I do not position this work as a purely philosophical or mathematical text. I did not get a Ph.D. studying Kant or Hegel. There is little doctrinal exposition of their works here.

Instead, I work within the tradition of critical ethnography. In that tradition, where the purpose is to understand others with an interest in emancipation, the works of Kant and Hegel are positioned as parts of a wisdom tradition that can be drawn on to understand experience.

I experience understanding as an unburdening. The weight of confusion momentarily lifts as I recognize my struggles in the questions and answers of others. The purpose of teaching seems to reside in recognizing such burdens. I won't try to confuse you, but the myth that *anything* is obvious is a terrible burden. People use concepts, like "2," in ways that are not easy to understand.

Originally, I thought to try to situate this work around the kinds of questions that students ask, then I thought to orient it around the kinds of questions everyone asks. I tried to translate Hegel as if he were a therapist, then as if he were a meditation teacher. However, through those experiences, I found it hard to communicate the sense of relief that understanding brings. Claiming Hegel to be someone other than Hegel resulted in some stilted prose. You will find vestiges of that approach in subsequent chapters, but I decided to embrace the messiness of my own experience. The ~~theoretical~~ aspect is still present, but the *critical* aspect is foregrounded.

This text isn't Kantian or Hegelian in form or content—you do not need to understand either to understand what I'm getting at. But when I felt like I understood the transcendental unity of apperception, it felt like an enormous weight was lifted. The {I}, whatever it is that the pronoun "I" recollects in general, plays an essential, if 'accompanying,' role in discourse. So, while you don't need to have read Kant, keep in mind that whenever I write about the {I}, I'm grappling with this basic unity-of-consciousness concept that Kant highlighted.

Among the wilder claims in this book is that mathematics and CAE share a common structure. Each recollect the self through otherness to recognize identity through difference. Each grow by virtue of the inadequacy of those recollections. Recollection transforms subjects into objects (*reification*) by virtue of the limits of language: to use words is to be bounded by the learning experiences that made those words meaningful. That implies that each recollection is bound, too, by the identity claimed by the recollector. Language use is also bound to *norms*, the rules or patterns of use that make what the words mean recognizable to others. Criticality involves challenging those norms, but in doing so, a new norm is always claimed. The critic then stakes their identity to that new norm, which is then subject to the same process of recollection and reification.

In this manuscript, "critical theory" does not refer critique in general, but rather to the tradition stemming from the Frankfurt School (Horkheimer, Adorno, Habermas) and related thinkers. It assumes that society and knowledge can be examined for power dynamics and potential emancipation. If you're new to this term, think of critical theory as the practice of questioning the status quo with an interest in liberation. Critical theorists ask who benefits from "common sense" and how things could be different. Here, my goal is to recover the emancipatory power of logic and mathematics (and a bit of philosophy), which is an expressive power, from the way those topics are used to constrain and manipulate people.

In mathematics, which tends to deal with patterns, criticality is enacted in many ways. The one that binds this book is *diagonalization*. The details of diagonalization can get complex, so I offer a simple example that I taught to a group of pre-service teachers in the spring of 2025. Let two symbols, $\star$ and $\square$, be combined into patterns. Those patterns can represent numbers, or any kind of pattern you may wish to change in your life.

$$\begin{array}{cccccc}
\star & \square & \star & \square & \star & \dots \\
\star & \star & \square & \star & \star & \dots \\
\star & \square & \star & \star & \square & \dots
\end{array}$$

I asked those pre-service teachers if they could find a pattern that was not captured by the patterns above. They were able to find one easily, as there were only a few lines to consider. But I also asked them to consider a method that would always allow them to find a new pattern from any number of patterns. If you look down the diagonal of such a list, and change each $\star$ to $\square$ and each $\square$ to $\star$, you will find a new pattern that is not captured by the patterns above. For example, if we take the diagonal of the three patterns above, we get: $\star\square\star\dots$. This new pattern is not captured by the patterns above, and it can be generated from any list of patterns. A new norm, pattern, or rule is asserted. It is not identical to any that came before, but is recognizable as a pattern within the same family of patterns.

As exciting and liberating as it is to transcend a pattern and write some new norm with good reasons ('creativity'), I somewhat derisively name the mechanization I articulate to represent that process the *More Machine* (see 6.7). Some critiques are banal. They feel algorithmic. Above, I articulated a pattern for transcending patterns. But the kinds of embodied transformations that feel like genuine learning escape patterned descriptions in more fundamental ways. That said, once those transformative experiences are recollected, the resulting universals (words) follow communicative norms...algorithmically. The distinction between transformative learning and *algorithmic elaboration* is very hard to draw convincingly.

In math education, the distinction between *conceptual* and *procedural* knowledge is similarly hard to draw. In the Hermeneutic Calculator, I will offer a way to understand the distinction that is more fundamental than the usual definitions. To prime readers for that discussion, I will say that conceptual knowledge points to where you're going by explaining where you've been. It is both elaborated from procedural knowledge and explicative of the procedures it is elaborated out of, a relation that Brandom calls LX (2008).

First, ~~we~~ must work to understand that no name is ever adequate to the thing it names. Words cast the ineffable particularity of experience into the universality of language. The experience of reading philosophy is often tarnished by the sense that the philosopher is trying to capture the ineffable in a jar of words. Where am {I} in all of these words? As a reader, you may feel like I am missing who you are. Because all words are repeatables, they all function as universals. That means making universal claims is inevitable. When I make "we" claims in this book, I am making claims about the patterns I've observed in myself that I recognize others as also enacting. But every pattern can be transcended. That means every "we" claim is defeasible. ~~We~~ are the ones that say "we."

However, ~~we~~ are also the ones who bridle at such claims of universality. ~~We~~ say "no" to the "we" claims ~~we~~ make all the time. But ~~we~~ can also reach consensus. ~~We~~ can acknowledge the initial "no" of a claimed universality, and then say "no" to that no-saying. Consensus is not exactly a full-throated "yes!" It is a ~~no~~.

Furthermore, ~~we~~ can transcend observable patterns with good reasons. However, ~~we~~ cannot transcend the conditions that enable that transcendence. I cannot, for example, transcend the *assumption* that *someone* might be able to understand what I am writing. If no one could understand it, I would not write it. The assumption that someone might understand what I speak or write is the first 'axiom' of critical mathematics.

I think it will be too irritating to readers to write "~~we~~" everytime I need to use the pronoun that binds you, the reader, and I, the author, to each other through the text. But it will also be irritating to you to if I do not acknowledge that you can disagree with any universal claim I make. I ask a lot of you in this text. You are a participant in its unfolding—a silent partner whose role as a reader is indispensible to the movements I make. In ~~our~~ dance, I must lead, but you need not follow. The first big ask I make is that you append each abrasive (and perhaps immoral) claim to universality with a strikethrough. Each time I write "we," I mean ~~we~~. But it is worse than that. Every single word is a repeatable and, therefore, a universal. Every word is a "we" claim. The whole thing would be rather illegible if I ~~struck through each word~~. In some sense, the signs themselves always *already* slash through the tenderness of particularity. I will retain the practice but limit myself to its use when a concept name must erase itself or when the *second* *negation* insists on it.

0.6 Integration: Divasion, Identity, and the Foundations of Mathematics

I attempt to situate critical mathematics as a viable ~~philosophy~~ of mathematics. When I tried to publish in that domain, all I experienced was rejection and frustration. Since this is ~~my~~ story, I want to take the opportunity to dissent from the dominant narratives in the field. The first, given to me by $\mathcal{M}$'s invention of divasion has to do with the doctrine of the excluded middle. Axiomatic set theories, to varying degrees, assume that every proposition is either true or false. Further, every mathematical object either is or is not an element of a set. The law of the excluded middle is often taken as a given in classical logic, but it has been challenged by various non-classical logics, such as intuitionistic logic and paraconsistent logic. Intuitionistic logic, for example, does not accept the law of the excluded middle as a general principle, emphasizing constructive proofs where the existence of an object must be demonstrated rather than inferred from the negation of its non-existence. Paraconsistent logic allows for contradictions to exist without leading to triviality, meaning that not everything becomes provable in the presence of a contradiction.

Those alternatives have been taken up by math educators to varying degrees. I will not rehearse those debates here, as I want to focus on the more primordial problem of how subjects and objects relate to each other. The term *divaded* captures a fundamental aspect of subjectivity that has deep implications for the philosophy of mathematics and math education.

Through expression, what we take as mathematically real expands, as I will detail in the bridge chapter where I trace the history of math from Pythagoras through Gödel. Further, while $\mathcal{M}$ described physical objects held by one another, the term *divaded* can be extended to concepts that are inside and outside of each other, like “parent” and “child,” and concepts that are inside and outside of themselves. Brandom (2019) explores Hegel's dialectics through the concept of *reciprocal sense-dependence*. The sense of “child” depends on the sense of “parent,” and vice versa. I choose to approach the topic through divasion, as it arose organically in conversation with a child. That feels fitting, given the intended audience of math educators.

But I will also be exploring the 'grown up' version that Brandom supplies and directly engage in a few instances of dialectical reasoning. Sophisticated readers may substitute lexically sophisticated phrases in place of "divasion" without much loss of texture. The only significant difference is that I take divasion to express a primordial spatial relationship that is then divided into the categories of inside and outside. Since "we" begins in the womb, I take the law of the excluded middle to be more or less the first mistake of many approaches to the foundations of mathematics. Those philosophies fall apart when confronted with the problem of divaded concepts.

For example, Gottlob Frege's work on the foundations of mathematics was deeply concerned with the relationship between concepts and objects. He sought to establish a logical foundation for mathematics, but his work was ultimately undermined by Russell's paradox, which exposed the contradictions inherent in what is now called 'naive' set theory. Naively, a set is a collection of objects, where the objects are called *elements*. In the platonist tradition that guided much of the push for formalism, all mathematical concepts exist as discoverable objects, so collections of objects are themselves objects. Borrowing from Brandom, objects do not contain contradictions, while subjects cannot seem to escape them. I hold various incompatible commitments, but feel I should not; objects, like a monochromatic yellow square, exclude incompatibilities like 'redness' or 'circularity.' Russell's paradox, loosely, asked whether the set of all sets that are not elements of themselves, $\mathcal{A}$, is a element of itself. Symbolically, we ask: is $\mathcal{A} \in \mathcal{A}$? For the set of all sets that are not elements of themselves to be an element of itself, it must not be; if not, then it must be so. Whereas $\mathcal{M}$ might simply say that $\mathcal{A}$ divades itself, Frege responded with dismay (Deutsch et al., 2025).

What is a set? They are sometimes introduced with physical metaphors, like a box that contains objects, or envelopes that may or may not contain other envelopes or letters. But these metaphors are misleading. They reify concepts into objects, but concepts are not their reifications. A big problem in mathematics is the tendency to subsume the subject into the realm of objectivity. To help, I will define sets as *recollections*: they are more like memories than boxes. One reason I (somewhat obnoxiously) put the pronoun {I} in curly braces sometimes is to remember that which the pronoun recollects. I do not do this when referring to myself as an author; instead the construction {I} is supposed to point to the locus of action that those who say "I" are recollecting when they use the term. As the 'source' of action, the {I} takes on spiritual dimensions for those who practice yoga. It is also to point down the text to when I discuss numerals as first-person pronouns.

This divasion between {I} and recollection points toward a larger structure that Hegel named *Geist*—a term encompassing "mind," "spirit," and the collective self-consciousness of a rational community. *Geist* divades human experience in a way analogous to the sphere in a hypercube: it is the self-flattening movement between the individual and the collective, the historical and the present, the subjective and the objective. Like the microphone that $\mathcal{M}$ observed, *Geist* is both inside and outside of each individual consciousness. I participate in *Geist* through my use of language, my mathematical reasoning, my engagement with norms; yet *Geist* also exceeds me, encompassing the entire historical community of those who have thought, spoken, and counted before me.

Robert Brandom offers a pragmatist reading of *Geist* as the historical process through which a community institutes normative statuses through practices of reciprocal recognition and retrospectively determines conceptual content through recollective narration. *Geist* is not a supernatural entity but the living, evolving web of social practices and historical self-understanding. When I use the numeral "2," I am not naming a platonic object; I am participating in a practice with a history, recollecting a normative status that has been instituted through countless acts of recognition and refined through historical struggle.

The paradoxes of self-reference that troubled Frege are not bugs to be eliminated but features of our status as subjects who are both inside and outside the systems we construct. *Geist* divades itself: it is both the historical process of meaning-making and each individual's participation in that process. Like the sphere that cannot be pictured in the hypercube but structures its intelligibility, *Geist* is the unpicturable condition that makes mathematical and linguistic meaning possible.

Russell's paradox is a classic example of the problem of self-reference and the inclusion paradoxes that arise when elements of a system are allowed to talk about themselves. Such paradoxes inspired others to search for more formal rigor. Frege's system fell and was replaced with Russell and Whitehead's approach as written in *The Principia Mathematica*. But tightening mathematical systems did not resolve the fundamental problems that dogged the formalist pursuit. Instead, Gödel demonstrated that any coherent mathematical system that includes numerals and multiplication—anything past the third grade—is fundamentally incomplete. Since the 1930s, mathematicians have sought to exclude, ignore, or reify the paradoxical aspects of self-reference. I do not intend to trash those efforts, but instead suggest that the problem runs much deeper, into the hard parts of being human. Rather than tightening, I want to loosen to articulate an *in*formal mathematics.

Rather than being sunk by the paradox of divasion, I want to embrace that paradox, in its universality, as an essential aspect of what it means to be a subject who participates in *Geist*. Mead's {I} and "me" describe the relationship between the self and society. The {I} is the spontaneous, creative aspect of the self, while the "me" is the socialized aspect that is shaped by interactions with others. Specifically, the "me" is the self-as-recognized (P. F. Carspecken, 1999).

> The simplest way of handling the problem would be in terms of memory. I talk to myself, and I remember what I said and perhaps the emotional content that went with it. The {I} of this moment is present in the "me" of the next moment. There again I cannot turn around quick enough to catch myself. I become a "me" in so far as I remember what I said. The {I} can be given, however, this functional relationship. It is because of the {I} that we say that we are never fully aware of what we are, that we surprise ourselves by our own action. (Mead, 1934, §22.2)

In this sense, $\mathcal{M}$'s term *divaded* captures the tension between the individual and society. I often feel misrecognized, so the distinction (I $\neq$ me) between the {I} and the "me" has a felt-sense. I also sometimes feel at ease with myself as the tension melts away (I = me). That feeling is so good, and its alternative so bad, that I want to simply be myself. But I cannot control the "me." I cast myself before you in existential fear of being misunderstood but with trust that you are likewise afraid. Perhaps you and I will feel some comfort in our divasion.

In the supplementary materials for this manuscript, I offer various attempts to formalize critical mathematics. Think of those as like $\mathcal{M}$'s initial sketches of a cube. They are not polished, but they are a start. I do not include the concept of divasion explicitly in those formalisms. The computational models that I draw on are too deeply embedded in classical logic to allow for such a concept to be involved in proving the consistency of the system. But I hope that the spirit of divasion infuses the work.

There is so much more to say about critical mathematics, but 'showing' is better than 'telling.' The rest of the book is an extended attempt to actualize the emancipatory potential of mathematics. In the next section, I will discuss how the rest of the book unfolds.

0.7 Conclusion: An Opening

Writing new ideas in the very old domain of mathematics is a daunting task. The history of mathematics is storied with geniuses whose work I can't understand. Making it harder still are the uncounted multitudes of genius who have filled in the gaps between those illustrious souls. CAE is one way in. But what does a framework provide? Simply having new words to cover old ground would just add noise to the signal. And I imagine readers are not quite sure what CAE is about anyway.

I titled this prelude *Built to Break* after a song I wrote of the same name. I was trying to get at the idea of repetition, moving through the Hegelian moment of the negative to its Kierkegaardian moment. In Kierkegaard's essay *On Repetition*, he describes pokes a bit of fun at old Hegel's insistence on recollection as the primary mode of knowing. He offers *repetition* as a counterpoint to recollection.

The first verse of the song that began this chapter is about a time $\mathcal{M}$, her sister $\exists$, and I were playing with blocks. $\mathcal{M}$ asked me to help her "build something to break it." So we built tall towers, over and over. She and her sister had a blast knocking them over. In chapter 5, I will discuss the middle verse. The theoretically important part of the middle verse has to do with the the question of whether I simply am a recollection. The last verse, which is in the conclusion, is about my father. It contains a line about an ocean of rainbow that really won't make sense until later in the book.

Until then, allow the 'ocean of rainbow' to reflect the merely *aspirational* structure of the book. In Figure 7, I have rendered a Möbius Strip as a rainbow. At one point, I made a conceptual map by color. For example, I represented *being* in red and *nothing* in violet, to suggest the movement between those concepts as a constant *becoming.* That ended up feeling cartoonish, but I wanted to structure the book vertically, horizontally, and as dyadic action. That last bit is tricky. But if you recall M.C. Escher's print of ants marching on a Möbius Strip, it might help to understand how I see your role as a reader. There aren't really optical anti-prisms—glass objects that unbreak light. But I think of your role as if you were such an anti-prism; but an ant like Escher's. Your role is to take these finite (bounded) words, this broken light, and bring them back together through the disciplined openness of reading. That is how I conceptualize the ideal listener. A picture cannot capture the progressive unfolding of the text, but I felt it was important to communicate that I am not striving for linearity. I fall—broken—through the plane of experience, while you bring those pieces together, recognizing, just as you might recognize each of the cubes in the series in figure 4 as a cube, the implicit whole.

What the image suggests is what a self-divaded object might look like. If you take a Möbius Strip and travel around its 'inside,' you will eventually find your finger on the 'outside,' until finding yourself back on the 'inside' again. Readers can make a Möbius Strip by taking a strip of paper, twisting it once, and taping the ends together. The twist is important. It is the twist that allows the inside to become the outside. Ideally, the 'twist' would be constantly recognizable throughout as rhythms in the text. To make the 'rainbow' as I have drawn it, you need only color the 'top' and 'bottom' of the strips the same color. If you wanted the colors to line up, you would have to reverse the order of the colors, given the twist. I do not want them to line up, as a main claim of the work is that identity arises through difference. Concepts flow into one another, but there are often abrupt shifts in understanding. The "aha" moment will be explored in finer detail in the next chapter, but the disjointed rainbow is my attempt to visualize such moments.

However, figure 7 also includes a representation reminiscent of the Koch Snowflake, except woven of the Möbius strips. The Koch Snowflake is a fractal, which means that it is self-similar at all scales. To use words is to make universal claims. The self-similarity of the Koch Snowflake is a way to get into the idea of universal *difference*. The self-similarity that I am trying to express throughout the text is a universal difference. That universal difference can*not* be represented in a static way. Still, if you recall M.C. Escher's—all I can represent is the self-similar aspect of that universal difference. So, the text is topoligically structured as series of Möbius Strips, but each concept develops discretely through a fractal-like self-similarity which is universal difference.

Note that in figure 7, only green is mapped to green. In the transformation that I hope will take place for readers, only "no" maps itself to itself, while still enacting the 'flip.' The other colors are mapped to each other. While the diversity of the text is burdensome, what remains constant is that we all say no. In mathematical terms, determinate negation (as defined in **??**) is the fixed point of determinate negation. It is the only concept that retains itself under its own process. It will undergo transformation, becoming an other to itself, but it does so under its own power.

One might ask what the coordinate system is for the Möbius Strip. Borrowing the phrase from Sellars (2007), it is the space of reasons, which has to do with the triadic progression of validity claims. I will discuss this space explicitly throughout the book, but a hint of what I am after is in figure 6.8.

Throughout, I will use songs and bits of poetry that I have written over the last decade of work on my dissertation and this subsequent book. I include them for reasons that depend, in part, on the song in question. However, as a class, they draw on the same linguistic structures that mathematics draws on. Semantically, they address the same paradoxes that mathematics confronts. The sameness I claim is not a formal equivalence, but a kind of identity over difference.

Anaphora, in the analytic tradition, is the use of a word or phrase that refers back to another word or phrase used earlier in a discourse. For example, in the sentence "John arrived late because he missed the bus," the pronoun "he" is an anaphoric reference to "John." However, in rhetoric and poetry, anaphora is a stylistic device that involves the repetition of a word or phrase at the beginning of successive clauses or sentences. For example, in Martin Luther King Jr.'s famous "I Have a Dream" speech, the phrase "I have a dream" is repeated at the beginning of several sentences to emphasize his vision for racial equality.

As the speech unfolds, each instance of "I have a dream" refers back to the original vision, but each repetition also adds new layers of meaning and context. The anaphoric structure creates a rhythm and reinforces the central theme of the speech, while also allowing for the development of the idea over time. It also builds energy through repetition.

I tend to write what I call 'front-porch songs.' They have relatively simple structures that almost always include verses and a chorus. The chorus is the part that repeats, while the verses develop the theme. I will occasionally write a bridge, which is a section that provides contrast to the verses and chorus, while drawing on the expressive resources developed in those parts of the song. The bridge often introduces a new perspective or a twist in the narrative. The bridge is also where I might change keys or introduce dissonant chords, allowing for the last verse/chorus pair to repeat in a way that builds energy through difference.

The songs are included, in part, to demonstrate how anaphora functions at the rhetorical level.

But the songs also function as an 'outside' to the text. The musical performances aren't exactly 'here.' The written word work as I argue mathematical objects work. I tend to write songs that exteriorize the interiority of experience. Usually, I find some need for recognition in myself, like smeared ink on a wet magazine, that I write about to try to understand. When understanding falls on me, the need is de-personalized—cast into the universality of language. But for others to recognize themselves in the song—to value the understanding the song purports to express—the de-personalized 'lesson' needs to include the context in which learning occurred. I tend to express that context through a personification of space. function as what Agamben (2006) calls *shifters*. A shifter is a bit of language (like {I} or a poem) that doesn't simply refer to an object. Instead, they refer to the *event* of language. Agamben works on the Continental side of the divide between Continental and Analytic philosophy, so his work is not as well known in the Anglophone world. However, I find his work on shifters to be a useful way to think about how language can be used to refer to itself.

From the Analytic tradition, I will use Robert Brandom's work to argue that numerals are *anaphoric terms* (see 7.1) that recollect the "I think" that 'can accompany all of my representations.' Technically, anaphoric terms are those that refer back to something previously mentioned. That technical definition is a hindrance for my argument, as it suggests that there is an empirical referent for each anaphoric term. When I submitted an earlier version of chapter 7 to a prestigious journal devoted to the philosophy of mathematics, the reviewers indicated there are empirical tests to determine whether a term is anaphoric. In their thoughtful rejection letter, the editor suggested that I must be using the term metaphorically. For that reason, among others, I embed the argument in a larger conceptual category called *shifters*, which can include both indexical terms (terms whose meaning depends on the context of their utterance) and anaphoric terms. The most obvious examples of anaphoric terms are pronouns like {I} or "it." When embedded in the concept of *shifters*, anaphoric terms refer back to the *event* of language itself, which precedes any subsequent unfolding. The *event* of language is deeply implicit; chapter 5 will discuss it in detail, when I argue that songs and poems also function as shifters.

So, I will be arguing that songs, poems, and numerals can be understood as a unified class of linguistic expressions that refer back to the event of language. I do not mean to suggest that these expressions are interchangeable. While they share a referent, they are vastly different in *sense*. Rhetorically, when I anticipate that some bit of analysis I wrote is too sterile or compressed to follow, I will insert some bit of song to downshift from language, back towards its event. Structurally, songs and poems are built to be repeated, so I will also use them as shifters to return the text to itself, creating the Möbius Strip structure I described above. The first words of this preface are the first verse of the song *Built to Break*. The second verse of that song is in the Bridge chapter. The third verse will be discussed in the conclusion of the book.

I confess I've made this book complex—perhaps too complex. "Why not just say it plainly?" you might ask. The emotive 'messiness'—the songs and personal asides—are my way of shifting out of theory when needed. In general, when the density of the theory starts to feel constricting, like a too-small inflatable lifejacket pumping ever up to squeeze my neck as I re-read the manuscript, I insert a poetic pin to deflate, decompress, and recollect the purpose of theory. Theory is here to serve as an expressive resource to talk about experience. For it to do its work, it cannot choke out the reader. Whenever you feel frustrated or lost, taking a moment to listen to a song or breathe with the text is to fulfill, not break from, the logic of the text. I ask for your patience; I think whatever energy you put into the text will be well spent.

One last note on the structure of the book. I was inspired by my friend Xianqing (Dorcas) Miao's reading of Heidegger in an as yet unpublished paper. She describes his work as an attempt to read the hermeneutic circle both forwards and backwards. I attempt something a bit different here. The dialectic aspect of the text is structured as follows: Part I (Prelude/Introduction) opens expansively with personal narrative and broad themes (an open beginning), Part II delves into dense critical theory (a restricted focus on specific philosophical frameworks), Part III ("Bridge") opens again by twisting from theory toward mathematical ideas (an opening transition), Part IV presents detailed mathematical concepts and formalisms (another restrictive deep dive), and Part V concludes by returning to broad, unifying ideas ("Repetition: Built to Break"—an opening out into reflection and future possibilities). This is a kind of zig-zag pattern, with a fractal-like self-similarity happening within each chapter as well: openness $\rightarrow$ restriction $\rightarrow$ openness, then repeat.

While the text is a dialectical unfolding when read cover-to-cover, I also wanted to write with 'vertical' layers, where the Möbius Strip can be thought of as spreading out from the central notion of determinate negation. So, the

Bridge (chapter 6), can be read first, then the mathematical chapters (7, 8, and 9), then the theory chapters. Because I am trying to make the text legible from within any layer, I often repeat myself verbatim—especially when I cite other thinkers. I also try to write with 'childish' examples at the beginning of chapters, because I want the text to be open to everyone who can read. I don't think anyone will get all of what I write, as I get lost in my own prose when I re-read. But that is by design: everyone (I hope) can get something from the text, but no one should get everything. When you find the desire for certainty is frustrated by the text—when you find that you just can't understand what I'm going on about—you can skim! More might disclose itself if you relax.

This opening chapter has established the experiential and theoretical foundations for the journey ahead. The five foundational anecdotes—from the calculator-playing child to the grieving father—reveal the common structure of mathematics and critical autoethnography: both recollect the self through otherness to recognize identity through difference. The personal stories demonstrate how formal systems, while powerful, become alienating when divorced from the material conditions of human experience.

The theoretical framework of critical autoethnography provides the methodology for transforming lived experience into philosophical and mathematical resource. Through the framework of diagonalization, recognition studies, and the triadic progression of validity claims, we begin to understand how mathematics might be reconceived not as a tool of social control, but as an emancipatory expressive resource.

Roadmap: The Journey Through Eleven Chapters

The book unfolds through eleven chapters, each structured in seven sections following a dialectical rhythm. This structure mirrors the "zig-zag" pattern described above: openness → restriction → openness, recursively enacted within and across chapters. What follows is a preview of each chapter's core contribution and internal architecture.

Chapter 1: The Sound of Time introduces *determinate negation* through an embodied metaphor connecting the rhythm of inner experience to sound's physical nature. The chapter centers on *The Exercise*, a guided meditation cultivating self-certainty through proprioceptive awareness. Section 1 establishes the misrecognition of certainty, distinguishing determinate from abstract negation. Section 2 presents the Exercise itself, guiding readers through bodily awareness practices. Section 3 develops the metaphor of "sound of time," connecting breath rhythms to conceptual movement. Section 4 introduces polarized modal logic ($\Box_{\mathsf{S}}$, $\Box_{\mathsf{O}}$, $\Box_{\mathsf{N}}$) to formalize expansion/contraction dynamics. Section 5 examines representational thinking's limitations and artistic expression's role. Section 6 explores the intersubjective dimension, how individual Exercise experiences connect to shared understanding. Section 7 synthesizes embodied practice as the ground for critical mathematics, previewing how this phenomenological foundation enables subsequent theoretical developments.

Chapter 2: Inferential Movement examines the ethics of inferential reasoning through Robert Brandom's inferentialism, arguing meaning arises from inferential roles rather than pre-given objects. Section 1 presents the misrecognition of inference through a quadrilateral classification case study. Section 2 explicates Brandom's distinction between formal and material inference, emphasizing how material inferences carry semantic content. Section 3 develops incompatibility semantics, showing how negation structures meaning through what claims rule out. Section 4 extends Brandom's account of error from perceptual experience to interpersonal misrecognition in mathematical communication. Section 5 analyzes the quadrilateral example in detail, revealing how geometric classification embodies normative commitments. Section 6 connects geometric figures to the "I think," foreshadowing the claim that numerals function as pronouns. Section 7 synthesizes how meaning emerges from embodied practices and social norms rather than correspondence to abstract objects.

Chapter 3: Existential Needs explores two fundamental existential demands: recognition as "good" within normative frameworks and recognition as infinite expressing authentic selfhood. Section 1 introduces the Turbo Lowers story, an outsider artist whose property was destroyed by town ordinance enforcement, illustrating the conflict between conformity and creative expression. Section 2 develops George Herbert Mead's I/me distinction, analyzing the tension between socially constructed and spontaneous self. Section 3 investigates how this tension manifests in educational contexts, particularly mathematics where students balance procedure with creativity. Section 4 employs Brandom's pragmatist reading of Kant alongside Hegel's restless negativity to reveal the needs' deeper unity. Section 5 examines the fear of nothingness arising from I/me split. Section 6 articulates how confession, forgiveness, and trust structures enable integration. Section 7 concludes with "Beast of Love" poem, a site for reflection on surrender and recognition's transformative power.

Chapter 4: Thoughts for Two—Who Are You? examines intersubjective recognition structures through the question "Who are you?" revealing dynamics in mathematical communication and human development. Section 1 introduces the CUSP (Claimed Universal Subject Position) you concept from research on preservice teachers, exploring how "you" can reference a generalized rather than particular other. Section 2 distinguishes CUSP you from transactional you using Sebastian Rödl's intentional transaction framework, explaining dyadic versus monadic speech acts alongside the Big Gorilla story context. Section 3 analyzes communicative practice through the full Big Gorilla narrative (chest-thumping, "hoohoohoo"), applying Mead's framework of gestures becoming significant symbols. Section 4 describes the communicative breakdown moment (thumping too loudly, tears), applying Habermas's theory distinguishing communicative action from rational discourse through validity claims. Section 5 introduces Brandom's analytic pragmatism (vocabularies, practices, PV/VP sufficiency), discussing theoretical automata as models while acknowledging formalization's limits for capturing genuine recognition. Section 6 tells the second Big Gorilla story (table-slamming incident), describing ∃'s forgiving gesture (thumping her belly to reintegrate Daddy Gorilla's anger), highlighting how her creative response transcended algorithmic elaboration. Section 7 synthesizes the I-You dyadic intersubjectivity structure, emphasizing recognition's recursive developmental nature, connecting back to Beast of Love and forward to thought's limits.

Chapter 5: Limits of Thought, Deconstruction, and the Voice explores thought and language's boundaries through personal loss, philosophical inquiry, and musical expression. Section 1 presents the eulogy delivered at the author's father's funeral, introducing the problem of presence through grief. Section 2 develops Giorgio Agamben's concept of Voice—the pure language event subtending all particular utterances—theorizing meaning's emergence at the saying/unsaying boundary. Section 3 analyzes a letter written to the father after death, examining how absence functions as presence's enabling condition. Section 4 presents original songs articulating grief, memory, and recognition's complexities, demonstrating how artistic expression addresses what conceptual thought cannot fully grasp. Section 5 investigates meaning's emergence through shared practices and interpretations using différance and reciprocal sense-dependence. Section 6 connects these themes to null representation in mathematics, drawing parallels between mourning and mathematical understanding. Section 7 synthesizes how accepting absence as understanding's necessary condition enables richer conceptions of mathematical thought, previewing the bridge to mathematical applications.

Chapter 6: Bridge: A Foundational Star serves as the text's pivotal transition, exploring parallels between Dr. Seuss's *The Sneetches* and Georg Cantor's diagonal proof through the star as both social marker and mathematical symbol. Section 1 establishes the misrecognition of totality through the Sneetches story, showing how finite systems transcend limitations through self-reflection. Section 2 traces diagonalization's history from ancient Greek incommensurability proofs through Cantor's work on infinity. Section 3 situates this history within the ontotheological context surrounding infinity, examining how the star symbolizes both totality and constitutive incompleteness. Section 4 explicates Cantor's proof that real numbers are uncountable, examining decimal expansion applications. Section 5 develops Haim Gaifman's generalization of the diagonal method. Section 6 discusses the empty set and zero concept, connecting to becoming and numerical understanding's development. Section 7 synthesizes how diagonalization embodies determinate negation, bridging philosophical foundations (Part I) with mathematical applications (Part II), preparing for the claim that numerals function as pronouns.

Chapter 7: Numerals are Pronouns presents the book's central mathematical claim: numerals and number words function as first-person pronouns rather than names for abstract objects. Section 1 establishes the misrecognition of reference through conventional views treating numerals as object names. Section 2 recounts the author's experience with a struggling student whose question—"What even is two?"—catalyzed reconceptualizing number grounded in self-consciousness structures. Section 3 introduces an Exercise for cultivating introspective listening, providing embodied foundation for understanding numerals as recollecting the "I think." Section 4 explores resistance, naming, and presence desire's interplay characterizing mathematical understanding. Section 5 connects these experiences to null representation ($\emptyset$) and determinate negation from earlier chapters. Section 6 develops the central claim that null representation symbolizes the unrepresentable "I think"—pre-conceptual experience ground making all representation possible. Section 7 aligns this interpretation with Brandom's singular reference analysis and de re ascriptions, proposing that grounding mathematical understanding in self-recognition structures motivates correctness pursuit as authentic self-recognition.

Chapter 8: Algorithmic Elaboration and History explores mathematical concepts and procedures' development through algorithmic elaboration, arguing mathematical history reflects a dialectical pattern where apparent completeness generates new possibilities. Section 1 problematizes traditional literature review approaches, advocating for understanding intellectual history as dynamic conceptual development. Section 2 connects the numerals-as-pronouns claim to critical arithmetic development through Brandom's algorithmic elaboration concept. Section 3 reconstructs Euclid's infinity of primes proof using incompatibility semantics, demonstrating how ancient results embody inferential structures. Section 4 examines arithmetic operations' emergence from embodied metaphors, drawing on Lakoff and Núñez's cognitive linguistics work. Section 5 reveals simple algorithmic elaboration's limitations through these case studies. Section 6 introduces "pragmatic expressive bootstrapping," where conceptual systems develop by explicating implicit normative structures in existing practices. Section 7 synthesizes how treating history as ongoing conversation transforms each contribution, elaborating and transforming what came before rather than merely adding to a static repository.

Chapter 9: Operation explores mathematical operations' nature, arguing they are not merely formal symbol manipulations but expressions of embodied, normatively regulated practices. Section 1 establishes the misrecognition of operation through conventional accounts treating operations as abstract axiom applications. Section 2 develops "critical arithmetic" framework—a pre-formal system capturing everyday arithmetic's dynamic, error-inclusive nature. Section 3 examines arithmetic's embodied basis through multiplication, using C2C (Coordinating Two Counts by Ones) strategy examples from Cognitively Guided Instruction research. Section 4 articulates how subjective experiences transform into shared rule-governed practices through normative frameworks based on Brandom's incompatibility semantics. Section 5 integrates Lakoff and Núñez's embodied cognition research, showing how grounding metaphors structure mathematical understanding. Section 6 demonstrates how these practices give rise to stable objective procedures modeled as formal automata. Section 7 synthesizes the three-dimensional framework—embodied practices, inferential rules, formal structures—emphasizing operations are grounded in human activity and social normativity rather than timeless abstract truths.

Chapter 10: The Dialectic of Incompleteness and Recognition synthesizes the book's core arguments, articulating seven critical mathematics themes likened to rainbow colors. Section 1 explores being and knowing's unity in movement, mathematics as autoethnography grounded in personal experience. Section 2 emphasizes recollection and recognition as central, developing how numerals function as anaphoric terms. Section 3 posits determinate negation as understanding's fixed point—the concept that retains itself under its own process. Section 4 describes the triadic validity progression from subjective certainty through normative rightness to objective truth. Section 5 explores null representation as the unrepresentable enabling conditions' symbol, connecting to language's self-referential nature. Section 6 articulates mathematical research implications, advocating for creative, contextual, dialogical practice. Section 7 reflects on mathematics as dynamic recognition language, emphasizing lived experience connections and inclusive, emancipatory understanding's potential, closing the circle by returning to the opening chapter's themes transformed through the dialectical journey.

Each chapter's seven-section structure enacts a mini-dialectic: opening with concrete experience or provocative case (sections 1-2), developing through theoretical elaboration (sections 3-5), and synthesizing toward broader implications (sections 6-7). This fractal self-similarity means readers encountering difficulty in one chapter can skip forward, returning later with fresh perspective. The text rewards both linear reading and recursive exploration.

Songs and Poems as Navigational Aids

Songs and poems throughout serve as shifters—linguistic devices that refer not to objects but to the event of language itself. They function as decompression valves when theoretical density becomes overwhelming. Taking time to listen to a song or breathe with the text fulfills rather than breaks from the logic of the work.

The goal is not merely to critique mathematics but to reconstruct it—to imagine and articulate a mathematics that speaks to our existential needs, acknowledges the productive role of error, and recognizes the first-person perspective as essential to mathematical knowing. In the dialectic of mathematical understanding, as in life, our moments of breaking can become moments of breakthrough.

The reader can expect a complex and recursive exploration of the relationship between self, other, and mathematical knowledge—one that builds to break, seeking not land for conquest but depths of shadow in the familiar. Through this critical autoethnographic approach, we work toward a mathematics that is built to break in service of deeper understanding.

References

Abbott, E. A. (2020, April). *Flatland: A Romance of Many Dimensions by Edwin A. Abbott*. Independently published.

Agamben, G. (2006, September). *Language and Death: The Place of Negativity* (K. Pinkus & M. Hardt, Trans.; First Edition). Univ Of Minnesota Press.

Aristotle. (2017). *Metaphysics: Aristotle's metaphysics in english, latin and ancient greek: Trilingual edition* (W. o. Moerbeke & W. D. Ross, Trans.; Trilingual). Hermes Language Reference.

Boote, D. N., & Beile, P. (2005). Scholars before researchers: On the centrality of the dissertation literature review in research preparation. *Educational researcher*, *34*, 3–15.

Brandom, R. B. (1994). *Making it explicit: Reasoning, representing, and discursive commitment*. Harvard university press.

Brandom, R. B. (2000). *Articulating reasons: An introduction to inferentialism*. Harvard University Press.

Brandom, R. B. (2008). *Between saying and doing: Towards an analytic pragmatism*. Oxford University Press.

Brandom, R. B. (2019). *A spirit of trust: A reading of Hegel's phenomenology*. The Belknap Press of Harvard University Press.

Bueno, O. (2020). Nominalism in the Philosophy of Mathematics. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Fall 2020). Metaphysics Research Lab, Stanford University.

Cantor, G. (1891). Ueber eine elementare frage der mannigfaltigkeitslehre. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, *1*, 75–78.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). Children's mathematics: Cognitively guided instruction - videotape logs [supplementary material]. In *Children's mathematics: Cognitively guided instruction*. Heinemann, in association with The National Council of Teachers of Mathematics, Inc.

Carspecken, P. F. (1995). *Critical ethnography in educational research: A theoretical and practical guide*. Routledge.

Carspecken, P. F. (1999). *Four scenes for posing the question of meaning and other essays in critical philosophy and critical methodology* (New edition). Peter Lang Inc., International Academic Publishers.

Carspecken, P. F. (2006). Limits of knowledge in the physical sciences. In K. Tobin & J. Kincheloe (Eds.), *Doing educational research* (pp. 405–437, Vol. 1). Sense Publishers.

Carspecken, P. F. (2009). Limits to knowledge with existential significance: An outline for the exploration of post-secular spirituality of relevance to qualitative research. In *Bridging the gap between theory and practice in educational research* (pp. 47–61). Springer.

Carspecken, P. F. (2016). Limits to knowledge and being human: What is 'critical' in critical social research and theory? In P. H. Hinchey (Ed.), *A critical action research reader* (pp. 65–86). Peter Lang Inc., International Academic Publishers.

Carspecken, P. F. (2018). The missing infinite. In R. Winkle-Wagner, J. Lee-Johnson, & T. Gaskew (Eds.), *Critical theory and qualitative data analysis in education* (p. 32). Routledge. https://doi.org/10.4324/9781315158860-2

Carspecken, P. F., & Zhang, R. (2013). Content inference fields in intersubjective space: Transpersonal position, illocution, and logic in the analysis of human interactions. *Qualitative research: A reader in philosophy, core concepts, and practice*, 201–242.

Carspecken, R. (2024, November 3). Untitled.

Caygill, H. (2004). *A kant dictionary*. Wiley-Blackwell.

Derrida, J. (1973). *Speech and phenomena, and other essays on husserl's theory of signs*. Northwestern University Press.

Derrida, J. (1982). *Marges de la philosophie* [Margins of philosophy] (A. Bass, Trans.). The Harvester Press. (Original work published 1972)

Deutsch, H., Marshall, O., & Irvine, A. D. (2025). Russell's Paradox. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford encyclopedia of philosophy* (Spring 2025). Metaphysics Research Lab, Stanford University.

Ernest, P. (2024). The mathematical and philosophical significance of zero: Nought matters [Publisher: Paul Ernest, Emeritus Professor at University of Exeter, School of Education & Lifelong Learning]. *Philosophy of Mathematics Education Journal*, 1–35. https://research.ebsco.com/linkprocessor/plink?id=b685ecb9-d2fc-3e03-889d-a06fb1751c1b

Euclid. (2007, December). *Euclid's Elements of Geometry* (J. L. Heiberg, Ed.; R. Fitzpatrick, Trans.; Revised and corrected edition). Richard Fitzpatrick.

Franzén, T. (2004, September). *Inexhaustibility: A Non-Exhaustive Treatment: Lecture Notes in Logic 16* (1st edition). A K Peters/CRC Press.

Franzén, T. (2005). *Gödel's theorem: An incomplete guide to its use and abuse*. AK Peters/CRC Press.

Frege, G. (1997, July). *The Frege Reader* (M. Beaney, Ed.; 1st edition). Blackwell Publishing.

Gaifman, H. (2005). *The easy way to Gödel's proof and related matters* [Unpublished manuscript. Available online].

Geisel, T. (1961). *The Sneetches and Other Stories* (First edition.). Random House Books for Young Readers.

Gelder, T. V. (1995). What might cognition be, if not computation? *The Journal of Philosophy*, *92*, 345–381. Retrieved April 13, 2025, from http://www.jstor.org/stable/2941061

Gray, L., & Garzon, M. (2005). On a hermite boundary integral approximation. *Computers & structures*, *83*(10-11), 889–894.

Greenblatt, R., Denison, C., Wright, B., Roger, F., MacDiarmid, M., Marks, S., Treutlein, J., Belonax, T., Chen, J., Duvenaud, D., Khan, A., Michael, J., Mindermann, S., Perez, E., Petrini, L., Uesato, J., Kaplan, J., Shlegeris, B., Bowman, S. R., & Hubinger, E. (2024, December 23). Alignment faking in large language models. https://doi.org/10.48550/arXiv.2412.14093

Habermas, J. (1985). *The theory of communicative action: Lifeworld and system* (Vol. 2). Beacon Press.

Habermas, J. (2004). *The philosophical discourse of modernity: Twelve lectures* (F. G. Lawrence, Trans.). MIT Press.

Habermas, J. (1971). *Knowledge and human interests*. Beacon Press.

Habermas, J. (1984). *The theory of communicative action, volume 1: Reason and the rationalization of society* (T. McCarthy, Trans.; Reprint edition). Beacon Press.

Habermas, J. (1995). *Postmetaphysical thinking: Between metaphysics and the critique of reason* (1st edition). Polity.

Hackenberg, A. (2025). *Course notes* [Unpublished course notes].

Hegel, G. W. F. (1977). *Phenomenology of spirit* (A. V. Miller, Trans.; Vol. 632). Oxford University Press Oxford.

Hegel, G. W. F. (2010). *The science of logic* (G. di Giovanni, Ed. & Trans.). Cambridge University Press.

Houlgate, S. (2006). *The opening of hegel's logic: From being to infinity*. Purdue University Press.

Hyppolite, J. (1974). *Genesis and structure of hegel's phenomenology of spirit*. Northwestern University Press.

Kant, I. (1998). *Critique of pure reason* [Original work published 1781/1787, Guyer and Wood, Trans.]. Cambridge University Press.

Lakoff, G., & Johnson, M. (1980). *Metaphors we live by*. University of Chicago Press.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books.

Linnebo, Ø. (2024). Platonism in the Philosophy of Mathematics. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford encyclopedia of philosophy* (Summer 2024). Metaphysics Research Lab, Stanford University.

Longuenesse, B. (2017). *I, me, mine: Back to kant and fichte on the 'i'*. Oxford University Press.

Mead, G. H. (1934). *The social psychology of george herbert mead* (A. L. Strauss, Ed.). University of Chicago Press.

Mead, G. H. (2015). *Mind, self, and society: The definitive edition* (H. Joas, D. R. Huebner, & C. W. Morris, Eds.; Enlarged edition). University of Chicago Press.

Meyer, J. (title). *Cantors 1891 diagonal proof* [Cantor's original 1891 diagonal proof]. Retrieved July 6, 2025, from https://jamesrmeyer.com/infinite/cantors-original-1891-proof

Nagel, E., & Newman, J. R. (2012). *Godel's proof*. Routledge.

Negarestani, R. (2018). *Intelligence and Spirit*. Urbanomic/Sequence Press.

Poston, T., & Stewart, I. (2012, February 29). *Catastrophe theory and its applications* (First Edition). Dover Publications.

Priest, G. (2002). *Beyond the limits of thought*. Oxford University Press.

Priest, G. (2014). *One: Being an investigation into the unity of reality and of its parts, including the singular object which is nothingness*. Oxford University Press.

Quine, W. V. O. (1948). On what there is. In *Thinking about logic* (pp. 189–209). Catholic University of America. https://sites.pitt.edu/~rbrandom/Courses/2023%20Sellars/Sellars%20texts/Quine%20On_What_There_Is.pdf

Quine, W. V. O. (1963). Two dogmas of empiricism [First Harper Torchbook edition, a reprint of the second, revised edition (1961). The book was first published by Harvard University Press in 1953.]. In *From a logical point of view* (Second, revised, pp. 20–46). Harper & Row.

Rödl, S. (2014). Intentional transaction [Accessed: 2025-04-13]. *Philosophical Explorations: An International Journal for the Philosophy of Mind and Action*, *17*, 304–316. https://doi.org/10.1080/13869795.2014.941909

Rose, G. (2009). *Hegel contra sociology*. Verso.

Rowlands, S. (2005). The contradictions in the constructivist discourse [Citing von Glasersfeld]. *Philosophy of Mathematics Education Journal*, *14*.

Savich, T. M. (2020). Towards a critical mathematics [Accessed via ERIC]. *Mathematics Teaching–Research Journal Online*, *12*, 245–291. https://files.eric.ed.gov/fulltext/EJ1384454.pdf

Savich, T. M. (2022). *Towards a critical mathematics*. Indiana University.

Savich, T. M. (2024). Built to break: An introduction to theoretical autoethnography [Invited talk presented at Solent University, organized by Dr. Brian McDonough]. https://iu.mediaspace.kaltura.com/media/t/1_v1qmm3g9

Savich, T. M. (title). *The hermeneutic calculator* [The hermeneutic calculator]. Retrieved July 23, 2025, from https://tiosavich.github.io/UMEDCTA/Calculator/index.html

Sellars, W. (1953). Inference and meaning. *Mind*, *62*, 313–338.

Sellars, W. (2007). *In the space of reasons: Selected essays of wilfrid sellars*. Harvard University Press.

Steffe, L. P. (2002). A new hypothesis concerning children's fractional knowledge [Accessed through CEPA.INFO]. *Journal of Mathematical Behavior*, *20*(3), 267–307.

Steffe, L. P. (2003). On the construction of learning trajectories of children: The case of commensurate fractions [Accessed via ResearchGate/ERIC]. *Monographs of the Journal for Research in Mathematics Education*, 1–26.

Steffe, L. P. (2004). Fractional commensurate, composition, and adding schemes: Learning trajectories of jason and laura: Grade 5 [Accessed via ResearchGate]. In J. Olive & R. Tzur (Eds.), *Children's fractional knowledge* (pp. 1–60). American Educational Research Association.

Steffe, L. P. (2014). *Children's number sequences: An explanation of steffe's constructs and an extrapolation to rational numbers of arithmetic* [Originally from 1994]. Unpublished manuscript, available via CiteSeerX; ResearchGate.

Steffe, L. P., & Thompson, P. W. (2014). Teaching experiment methodology: Underlying principles and essential elements [Accessed via ResearchGate, originally published 2000]. *Proceedings of the 24th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*.

Stone, J. (2003). *The monster at the end of this book* (2nd ed. edition). Golden Books.

Taylor, C. (1994). *Multiculturalism: Examining the Politics of Recognition* (A. Gutmann, Ed.) [Essays by Charles Taylor, K. Anthony Appiah, Jürgen Habermas, Steven C. Rockefeller, Michael Walzer, and Susan Wolf]. Princeton University Press.

van Hiele-Geldof, D., & van Hiele, P. M. (1984). *English translation of selected writings of dina van hiele-geldof and pierre m. van hiele* (D. Fuys, D. Geddes, & D. Tischler, Eds.) [ERIC Number: ED287697]. Retrieved June 3, 2025, from https://eric.ed.gov/?id=ED287697

Wittgenstein, L. (1956). *Remarks on the foundations of mathematics*.

Wittgenstein, L. (2010). *Philosophical investigations* (P. M. S. Hacker & J. Schulte, Eds.; G. Anscombe, Trans.). Wiley-Blackwell.

The Monster at the End of This Book

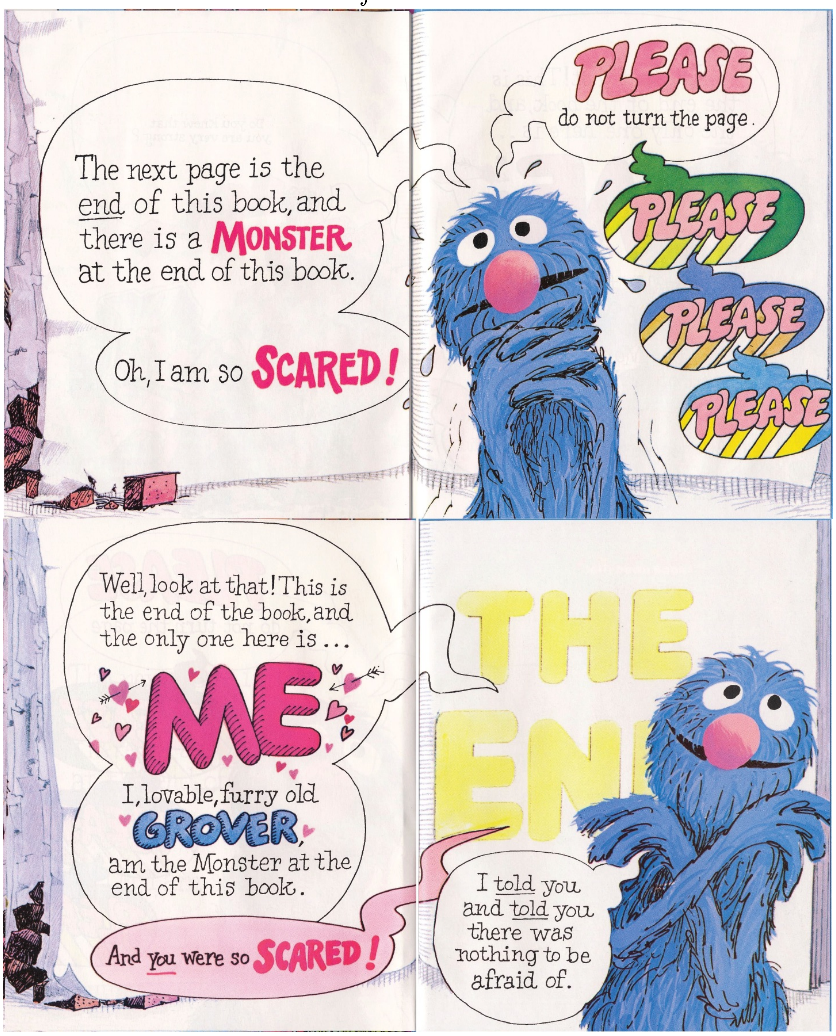
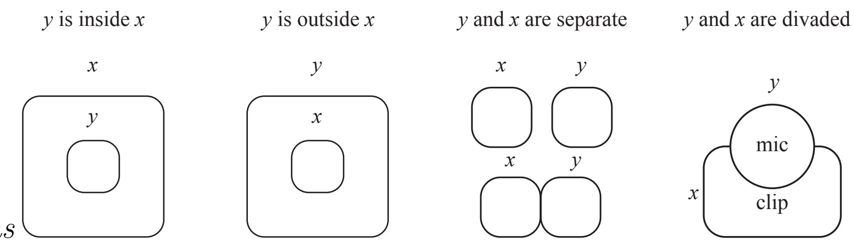
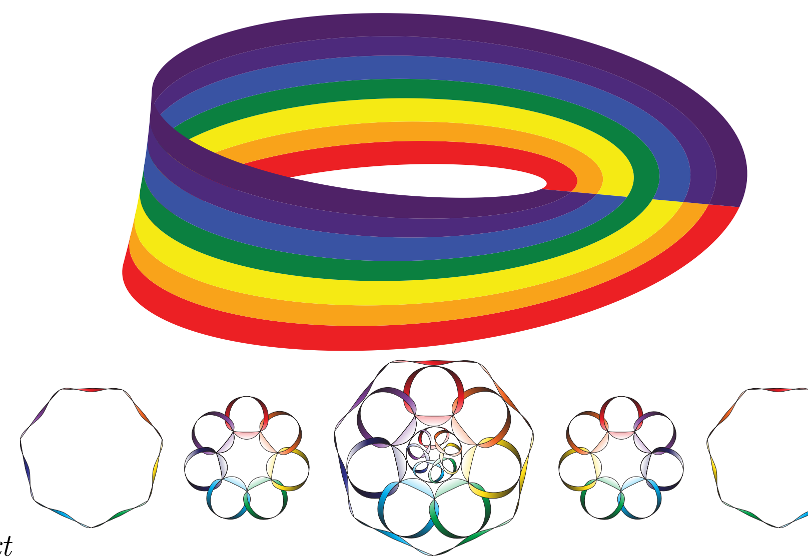


Figure 5: *Note.* Frames from *The Monster at the End of This Book* (Stone, 2003)



Divaded Regions

Figure 6: *Note.* The regions x and y may be inside, outside, separated, sharing a boundary, or divaded with respect to each other. The difference between liminal regions and divaded ones may not be appreciable unless the regions are in 3d, where it is evident that one holds the other.



Organizational Structure of the Text

Figure 7: *Note.* The text unfolds parametrically like paths around a Möbius Strip. Each main topic flows into its other, returning back to itself in an elaborated form for the second reading. Simultaneously, each concept develops in a fractal-like self-similarity that expresses universal difference. The Koch Snowflake beneath the Möbius Strip reflects that fractal-like development.

Chapter 1

The Sound of Time

Abstract

This chapter introduces the concept of *determinate negation* through the metaphor of *the sound of time*, connecting embodied experience to the rhythm of compression and expansion that underlies self-consciousness. The core of this exploration is a guided meditation called *The Exercise*, which cultivates an awareness of *self-certainty* through proprioceptive awareness—the felt sense of bodily presence that subtends all conceptual thought. The Exercise is used throughout the text to provide readers with direct access to the experiences being theorized. The chapter explores the interplay between direct, non-conceptual experience and philosophical reflection, arguing that self-certainty cannot be reached through thought alone but requires a kind of surrender to the immediacy of embodied being. Drawing on phenomenology and Hegelian philosophy, the analysis utilizes polarized modal logic to articulate the rhythm of embodied experience—the oscillation between expansion and contraction, presence and absence. The chapter also examines the limitations of representational thinking and the role of artistic expression in articulating truths that exceed conceptual grasp. By connecting the individual experience of the Exercise to broader structures of intersubjective understanding, this chapter grounds the project of critical mathematics in embodied practice.

1.1 Introduction: The Misrecognition of Certainty

In the prelude, a type of negation was alluded to that is distinct from the abstract negation of formal logic. In this chapter, the concept of *determinate negation* will be elaborated through a metaphor called *the sound of time*. This metaphor connects the rhythm of inner, felt experience to the physical nature of sound. The core of this exploration is an embodied practice: *The Exercise*.

Before abstract philosophy, before symbolic logic, there is the body. The Exercise attempts to cultivate direct awareness of how consciousness moves through the fundamental relationship between space and time—not as metaphors, but as the primordial structure of experience itself. To approach this phenomenologically requires first acknowledging how abstraction becomes meaningful at all.

A Starting Point: Conceptual Metaphor and Embodied Cognition

The work of cognitive linguists George Lakoff and Mark Johnson provides a helpful entry point (1980). Their research argues that our conceptual system is grounded in recurring patterns of sensory-motor experience—a perspective known as *embodied cognition*. This framework demystifies abstraction by revealing how even sophisticated concepts arise from bodily experience. Their central thesis is that **conceptual metaphor** is not mere decoration but a fundamental cognitive mechanism structuring our understanding. Metaphor works by "understanding and experiencing one kind of thing in terms of another," allowing us to reason about abstract domains by importing the inferential structure of more concrete, familiar domains grounded in physical interaction.

Consider the conceptual metaphor AN ARGUMENT IS A JOURNEY (Lakoff & Johnson, 1980, pp. 90–91). It establishes systematic mappings: the arguer is a *traveler*, the argument's goal is the *destination*, the means of argumentation are *paths*, and logical progression defines *steps along the way*. This metaphorical structure gives rise to everyday expressions like "We have set out to prove that...," "When we get to the next point...," "So far, we've seen that...," and "We have arrived at a disturbing conclusion." The metaphor has internal systematicity through entailments: since A JOURNEY DEFINES A PATH, then AN ARGUMENT DEFINES A PATH ("He strayed from the line of argument," "Do you follow my argument?"). Since THE PATH OF A JOURNEY IS A SURFACE, then THE PATH OF AN ARGUMENT IS A SURFACE ("We have covered a lot of ground," "Let's go back over the argument again"). Similarly, THEORIES ARE BUILDINGS maps physical structures onto argumentation. A theory's assumptions are its *foundations*, its logical structure is its *frame*, its strength is its *structural integrity*.

Lakoff and Nuñez extend this framework to mathematics itself, demonstrating that basic arithmetic is structured by metaphors based on collecting objects, constructing things from parts, using measuring sticks, and moving along paths. For instance, the most fundamental grounding metaphor treats numbers as object collections:

Arithmetic as Object Collection

Source Domain: Object Collections	**Target Domain: Natural Numbers**
Collections of objects of the same size	Numbers
The size of a collection	The value of a number
An individual object	A unit (the basis for counting)
The smallest possible collection (one object)	The number 1
The empty collection	The number 0
Putting two collections together	Addition
Taking a smaller collection from a larger one	Subtraction
A larger/smaller collection	A greater/lesser number

Table 1.1: *Note.* The foundational metaphor grounding natural number concepts in the bodily experience of collecting discrete objects. Adapted from Lakoff & Nuñez (2000, p. 55).

This cognitive science perspective dismantles what Lakoff and Nuñez call the "Romance of Mathematics"—the belief that mathematics is transcendent, disembodied truth existing independently of human minds (2000, p. 339). Mathematics is not a disembodied language we discover; it is a magnificent conceptual system human beings *create*, making "extraordinary use of the ordinary tools of human cognition."(2000, p. 377)

Going Deeper: From Cognitive Science to Phenomenology

This embodied cognition framework provides an accessible starting point, but The Exercise attempts to go deeper. Lakoff and Johnson, as cognitive scientists, articulate embodiment in terms of the mind as it exists in the brain—neural mappings, sensory-motor schemas, cognitive mechanisms. Their account remains within what might be called the natural attitude, treating consciousness as an object of scientific investigation.

The Exercise, however, seeks a phenomenological investigation of the subject-object relation itself. Rather than explaining how the mind constructs metaphors, I am attempting to cultivate direct awareness of the movement between differentiation and undifferentiation, between the bounded and the unbounded, between contraction and expansion. This is not metaphorical language imported from spatial experience to understand something else. This *is* the primordial rhythm of spatial and temporal determination itself—what Hegel calls *absolute negativity*.

Where cognitive science describes metaphorical mappings, phenomenology attempts to bracket those descriptions and return to the experience from which they arise. The goal is not to understand how we *think about* space and time through bodily metaphors, but to experience directly how consciousness *is* the movement of spatial and temporal determination. The Exercise cultivates awareness of this movement not as concept but as the pre-conceptual ground from which concepts emerge.

Lakoff and Nuñez articulate how embodied experience grounds abstract thought. The Exercise attempts to reverse that movement—to de-metaphorize, to dissolve the abstractions back into the embodied ground, and then to recognize that ground itself as the self-negating movement of consciousness. This is why the phenomenology of self-certainty cannot be captured by cognitive science. Self-certainty is not a brain state or a cognitive mechanism. It is the implicit awareness that subtends and enables all such objectifications.

Still, the cognitive science framework provides valuable scaffolding. Our shared embodiment does provide common source domains making intersubjective meaning possible. We can agree on validity claims or inference correctness precisely because abstract concepts involved are grounded in shared physical experience. This establishes the bridge between the biological and the social, between individual cognition and collective normativity. But that bridge must ultimately be crossed phenomenologically—through direct cultivation of the awareness The Exercise articulates.

Proprioception and the Feeling Body

The Exercise guides you into direct experience of this embodied foundation. It explores *self-certainty* through *proprioception*—bodily awareness. Proprioception is a sense, like smell or sight, except it concerns not the 'outside world' but bodily awareness itself. Common proprioceptive knowledge includes the implicit awareness that your head is above your shoulders. Making that explicit requires communicative norms; you must know words like "above, head, shoulders" to articulate these subjective/objective regions.

Less common ways of talking about proprioception include *proprioceptive expansion* (↑) or *contraction* (↓). Consider an experience of feeling 'one with nature,' gazing at a beautiful valley. You may recall such moments of *unity* followed by contraction in bodily awareness (↑→↓): "ah! Bee!" Alternatively, movement from point-like to diffuse awareness is proprioceptive expansion (↓→↑): "oh, just a fly, mmm..." This rhythm of expansion and contraction is what I call *the sound of time*.

The desire for self-certainty is a deep, primordial drive—the desire for unmediated experience of the self, a direct, irrefutable experience settling, once and for all, that "I am." Descartes' formula "I think therefore I am" has an unnecessary antecedent; saying "I am" is unfalsifiable, proved by the utterance. To say otherwise—"I ain't"—is self-defeating, disproved by its utterance. Still, simply saying "I am" has never satisfied my desire for self-certainty. Grammatically unfalsifiable statements can fuse with the I-feeling, but they are not self-certainty itself.

The Exercise is a guided meditation whose form I adumbrate from Phil Carspecken (P. F. Carspecken, 1999, pp. 169–184). Readers frustrated by the somewhat cartoonish nature of what follows should consult his work for more thorough treatment. The cartoonishness is intentional (I wish to make understanding accessible) and unintentional (I am not a trained meditation teacher). I will do my best not to lead you astray, but encourage you to seek more experienced guides if you wish to pursue this practice more deeply.

You do not need to master concepts before beginning. You already implicitly know how to do what The Exercise makes explicit. My expressions are framed as distractions from the purpose, even as mistakes, for two reasons. First, none of what follows the experience of self-certainty *is* self-certainty; self-certainty is necessarily implicit. Second, it is methodologically useful for the critical ethnographer to understand how self-certainty is recollected in different traditions. Engage with the meditation. The reflections plant seeds that grow throughout the book. Move between feeling and thinking without judgment. They are two sides of a single process of coming to understand.

The Rhythm of Determinate Negation

When I say “no” to something, what am I doing? In formal logic, negation is simple: “not-*p*” means everything that *p* excludes. If I say “This shape is not-circular,” I have told you almost nothing—the shape could be square, triangular, or any of infinitely many possibilities. This *abstract negation* creates an empty void, a sheer absence.

But there is another kind of negation. When you determinately negate something, you do not simply erase it. You negate it *by replacing it with something specific*. If I say “This shape is square,” I am implicitly negating circular, triangular, and every other incompatible shape—not abstractly, but by asserting something with its own positive content. Square and triangular are *materially incompatible*. You cannot have both simultaneously. One excludes the other not by formal contradiction, but by virtue of what they mean.

Spinoza declared *omnis determinatio est negatio*—all determination is negation. To be one thing is to *not be* its incompatible others. For Hegel, determinate negation is *prior* to formal negation. You cannot understand what “not-red” means unless you already understand what red’s contraries are: green, blue, yellow (Brandom, 2019).

The Exercise is an embodied enactment of this principle. When you bring awareness to your toes, name them (“I am my toes”), and then let them go (~~toes~~), you are not performing abstract negation. You are not erasing your toes into an empty void. You are *sublating* them—preserving, negating, and uplifting them simultaneously.

The toes are **preserved**: still there, still part of your body, still contributing to your proprioceptive wholeness. They are **negated**: you are no longer fixated on them; you have released your attention. And they are **uplifted**: now part of an accumulating totality, a growing awareness of the body as a unified whole.

Moreover, the very act of naming your toes as “I am my toes” negates itself. The name is inadequate. You are *more* than your toes. The assertion negates itself in its very utterance, pushing you forward to the next region. I am my toes. No, I am more than my toes. I am my feet. No, I am more than my feet. This rhythm—fixation and release, naming and sublating—is the sound of time, the pulse of determinate negation in lived experience.

If you only understood abstract negation, you might think the goal of meditation is to achieve an empty void, a sheer nothingness. But this exercise aims for a “determinate nothingness,” a nothingness that retains the content of what it came from. When you let go of your toes, you do not forget them. They accumulate as background, as *forestructure*, as the implicit context for what comes next.

Figure 1.1 illustrates how this dialectical process maps onto the bodily rhythm of tension and release. I recently led a class examining social justice case studies. When I encouraged students to push back on each other’s assumptions, they mistook me as asking them to “play the devil’s advocate.” But understanding isn’t the devil’s work. Through pushing, I hoped they would separate from themselves to analyze their assumptions, then reflect and return to themselves with deeper understanding. The point wasn’t to change their conclusions but to understand themselves and others more deeply. I drew this cartoon to repair our misunderstandings.

The process begins with **Tension and Inhalation**—the first negation (‘No’). This is the focused effort required to ‘Push’ and Determine a concept, followed by the sustained effort to ‘Separate’ and Analyze its assumptions. We hold our breath, metaphorically, as we fixate and differentiate.

The turning point arrives with **Relaxation and Exhalation**—the second negation (~~No~~). It begins as we ‘Reflect,’ releasing the fixation by adopting a receptive, second-person perspective. This letting go allows us to ‘Grow’ and Sublate the analysis, integrating understanding into more expansive comprehension. The entire ‘full-circle’ movement is felt as a single wave of compression and decompression.

Pay attention to this rhythm as you move through The Exercise. Notice how each act of naming is also an act of exclusion. Notice how each act of letting-go is not erasure but transformation. Notice how the self-negating movement of awareness allows you to expand from toes to feet to legs to the whole body. This is determinate negation in action. This is the dialectical soul of experience. This is how thought *moves*.

The Semantics of Self-Certainty

Self-certainty is, for me, not about the strength of one's convictions or *commitments*. The desire for self-certainty is a deep, primordial drive. It is the desire for an unmediated experience of the self; a direct, irrefutable experience that would settle, for once and for all, that "I am." Descartes' meditations that resulted in the formula "I think therefore I am" has an unnecessary antecedent, as saying "I am" is unfalsifiable. It is proved by the utterance. To say otherwise, "I ain't," for example, is an example of a self-defeating assertion: it is disproved by its utterance. Still, simply saying "I am" has not, for me, ever resulted in the satisfaction of the desire for self-certainty. Grammatically unfalsifiable statements can be fused with the I-feeling, but they are not self-certainty itself.

Self-certainty emerges from the practice of embodied reason below as a negative; it is always a not-this to whatever 'this' attempts to define it. Determinate negation is named, or *reified* (⌜determinate negation⌝) as *the negative*. I use the Quine corners (⌜ ⌝) to indicate the name of the concept thus enclosed (Gaifman, 2005). The negative took a philosophical journey from Spinoza to Hegel, where Spinoza said, "omnis determinatio est negatio" (all determination is negation) and then Hegel worked to demonstrate how determinate negation determines itself.

The first negation constrains, bounds, or limits in its finite moment. The second negation (~~no~~) dissolves those exact boundaries *exactly*. The more determined the concept is, the more precise are the dissolutions of those determinations. In everyday speech practices, our concepts are usually either over- or under-determined. That means that our experiences of the second negation are often experienced as an imprecise loosening. The Exercise articulates a space where the self-similarity between the no and the ~~no~~ is their absolute difference. By virtue of how the negative is necessarily other than itself, it is a concept that both produces and undermines itself. It builds and it breaks.

From Hegel, the negative continued its journey through existentialism and phenomenology, and was picked up again by the post-structuralist Jacques Derrida. Derrida's work is challenging to parse, so I will select one interesting idea that Carspecken (1999) develops more fully than I can here. The phrase is that there is a concept, specifically a norm or rule in the system that I am developing, that erases its own name. The negative is one name for that concept, captured when *sous rature* is interpreted as a necessary expression followed by a necessary erasure (~~no~~). When I wrote that self-certainty is necessarily implicit, I was drawing on Derrida's concept. Self-certainty is a ⌜~~concept~~⌝, as is the negative.

The idea of a concept that erases its own name is a challenging one, especially since I will not be deriving it from Derrida's work! Readers will have to do the exercise to recognize that every attempt to pin down self-certainty as a concept is necessarily a misstep. As you practice, approach this as an ethnographer-in-training. You will encounter groups of people with different ways of talking about their convictions. Perhaps they believe in the Trinity, a God who stands outside of creation, the Sutras, or the Four Noble Truths. Perhaps you worship at the altar of Rock and Roll, taking Bruce Springsteen as your saint. What unites these diverse beliefs and practices must be implicit. You *can* understand their beliefs and practices as oriented around the intersubjective practices that cultivate self-certainty. But if you try to pin their beliefs down to ⌜self-certainty⌝, and fit them into a logical system that captures that concept, the resulting texts you produce will likely result in the subjects of your analysis rejecting your interpretations: "I don't care about self-certainty, I care about avoiding damnation!"

I do not intend to suggest that self-certainty, as I explore it below, is devoid of logical structure. The 'grooves' or 'folds' in intersubjective space through which I move when conducting simple calculations like $5 + 7 = 5+5+2 = 12$ are actions oriented around self-certainty. There, self-certainty metaphorically functions like a hole, singularity, or gravitational well in representational space. I will introduce a polarized modal logic in **??** that formalizes the text below in an attempt to make those troubled metaphorical expressions less transient. But none of that formalization and none of the text below *is* self-certainty. Cultivating an openness towards otherness requires an emptying that might read like solipsism or nihilism, but the intersubjective vessel is never seriously in doubt.

Readers' Guide

This chapter is structured as a guided journey. Before beginning, a note on how to read it. The text moves between two modes: the *experience* of a guided meditation and the *reconstruction* of that experience through philosophical reflection and logic. You do not need to master all the concepts on the first pass. You definitely do not need to master the exercises before moving on. Wild as it may seem, I suspect you already have mastered the exercises; my role is to express what you already implicitly know how to do. Those expressions are all framed as distractions from the purpose of the exercise. I frame my work as a mistake, specifically a distraction, for two reasons. First, none of what follows the experience of self-certainty *is* self-certainty. Self-certainty is necessarily implicit. Second, it is methodologically useful for the critical ethnographer to understand how self-certainty is recollected in different traditions. The goal is to engage with the meditation. The reflections are there to plant seeds that will grow throughout the book. Allow yourself to move between the feeling of the exercise and the thinking of the reflections without judgment. One is not a failure of the other; they are two sides of a single process of coming to understand.

1.2 The Exercise: Negating Fixation through Practice

Exercise 0: Dragging Yourself to The Exercise

Are you ready to practice? *I don't want to practice!* Come on, you know you'll feel better once you start...*NO!* Are you saying "no" to the practice or to the claim that you'll feel better? *NO!* Hmmm... Without knowing what you are resisting, I do not have much to say. I take it that you and I are, at our core, no-sayers. "NO!" on its own is not *determinate*. But I assume you are resisting the practice itself, not my claim about the benefits of practice. Let me give you a secret for how I change my mind without changing what I am. It's a way to stay true to yourself (what you are) while still allowing you to change how you are (how you feel) in the world. I take that first "NO!"—the one that says I will not practice, the one that says, "I will not be moved."—and I direct a second "NO!" towards *it*. This second "no," this determinate negation *of* determinate negation (~~no~~), does not necessarily mean "yes" to the practice. You are free to close the book or skip whatever sections you wish. I don't want to change what I am or what you are to change us into yes-sayers. That is dangerous. There are so many excellent reasons to be disagreeable. I don't want you to change what you are, you delightful contrarian. The second negation functions as a transformation. The initial "no" is honored through your acknowledgement of its material content. Through that recognition, the resistance is softened. As a negation of the negation, recognition releases the initial blockage.

Exercise 1: Let-Go of Particularity

Begin by settling into a comfortable position, either laying down or sitting in a dignified manner. Whether your eyes are open or shut, soften your gaze. You may not realize the tension in your forehead that pulls your lower eyelids back towards your ears—among a thousand other regions in the face—until you attend to those regions as you soften your gaze.

Turn that attention to your toes and notice how they feel. Are they tight and curled? Extended and rigid? Neutral? In pain? Take an easy in-breath and imagine the breath traveling down into your lungs, nerves, and arteries, into the capillaries in your toes. You do not have to try to relax, just attend to them and breathe. Recognize them. Before you began The Exercise, they were likely an implicit aspect of your being, but they weren't unfamiliar. You may feel this recognition as a re-sensitization.

As soon as you recognize them, you may find that they uncurl or soften into a neutral position without effort or intention, perhaps warming up as the breath travels through them. You may experience a sense of proprioceptive expansion—a sensation that feels good. As soon as you become aware of that sensation, you may want more of it. If you try to hold onto—to fix—the sensation, you may find that it slips away. The mind moves; you must find a way to move with it.

Reflection: The First Temptation

What you have just experienced in your toes reveals the fundamental structure of all conceptual thinking. As you continue the exercise, your thinking mind may reassert itself. This is not a failure; it is an opportunity to observe how concepts emerge from raw sensation.

You might think: "I am my toes." No, that's not quite right. I am more than just my toes. But I am still my toes somehow, just not in the same way that I felt when I was aware only of my toes when they just relaxed...hmmm...I let go of my toes."

Initially, it's very tempting to try to grasp the experience by naming it: "I am my ⌜toes⌝." Because the words on the page already name concepts, I will use corners to indicate the act of naming. However, once stated, you might realize very quickly that the name is inadequate. It may happen so quickly that it feels silly to bring it into explicitness, but I want to slow things down. You are not *just* your toes, so I cross out the name: I am my ⌜~~*toes*~~⌝. This naming and crossing out is a way to describe what happens when you bring awareness to your toes and then let them go.

To understand why it's important to slow this first movement down, let me name what this first part of the exercise is doing. It is a *somatic sublation*. Somatics relate to the body, while *sublation* is Hegel's term that means conceptual content has been preserved, negated, and uplifted. I find that tripartite definition contradictory in three ways: preservation opposes negation, and both oppose uplifting. So, Hegel's concept is very challenging to understand. But in this first part of the exercise, you did not chop off your toes when you 'let them go.' They were preserved. They were also negated, in the sense that their name was found to be inadequate to contain your identity. I have not yet described what the uplifting part means, but will get there soon. Hegel was not particularly interested in implicit embodied sensations as such. For him, sublation is always in the normative, conceptual domain. He would probably vigorously disagree that what is happening with your toes is a 'sublation,' but I think it is pedagogically useful to start with the toes.

This move from pure feeling to a named—and then sublated—concept is the first step out of subjective immediacy and into the world of normative claims. It is a glimpse of what Carspecken calls the "primordial rhythm" of "sensation inclusive of 'I-feeling,' awareness of this, letting go, repeated awareness of sensation as inclusive of 'I-feeling,' and so on" (P. F. Carspecken, 1999, p. 171). This rhythm, as I will go on to explain, is what I mean by "the sound of time."

You start again. This time, you decide to follow the spine of Ram Dass' book: Be Here Now. You want to be present for the practice, but this resolution and desire are already separated from the sensation. You are kicked out again.

This common experience points to a deeper philosophical problem. I grew up with Ram Dass' book on my parents' shelf, and I always struggled with the imperative. You tried to grasp the pure, immediate present, and in doing so, it vanished. It's tempting to think of words like "this, here, and now" as fixed, as if by pinning ourselves to a coordinate in space and time, we could find some certainty.

But the moment you try to name it—"it is 11:45 AM"—that "now" has already become a "then." The act of observation and labeling is always a step behind the reality it seeks to capture. This analysis of the universal nature of *indexicals* (words like "here" and "now") and the self-defeating attempt to capture the particular through the universals of language is the core argument of Hegel's chapter on "Sense-Certainty" (Hegel, 1977, §90-110). The same slippage occurs with "here." It feels specific and solid, but if you take a step, the old "here" is now "there," and you are in a new "here." These words don't function like pins on a map; they are more like universal placeholders. Hegel defines the universal {now} as not-now. When trying to capture a unique, particular experience with a general term, the uniqueness you seek to express slips through your fingers.

Brandom argues that that the "authority of any immediate sensory episode depends on its being situated in a larger relational structure containing elements that are not immediate in the same sense," such as *anaphoric structures* that allow recollection (2019, p. 149). In the analytic tradition, *anaphora* is basically pronoun use. The referent of an indexical is fixed by a subsequent anaphoric term like "it." So, when I say "now is day, it is hot, and it is sunny" the chain of "now, it, it" fixes the prior referent even as the particular now marches ever on (Brandom, 2019, p. 150). Deictic expressions like "this book" fall to the same problem. The "this" is not a fixed object but a universal that can apply to any thing. The attempt to express the purely particular inevitably results in expressing a universal.

So if these words are such poor tools for capturing a fleeting moment, what are they truly for?

Perhaps "now" is less like a label for a point on a timeline, and more like a trigger for action: an *impetus to act*. When you tell someone, "I'm coming home now," you are not just describing your temporal location; you are making a commitment to act. When I tell my kids, "Brush your teeth. Now." I am signaling that the time for deliberation is over. Start doing. Start acting. "Now" is the point where a general commitment crystallizes into a specific, concrete act. It's less a tool for analysis and more of a practical tool for acknowledging a commitment. This pragmatic reframing of indexicals, particularly "now," as tools for undertaking and acknowledging practical commitments is central to Robert Brandom's work. It shifts the focus from semantics (what words mean) to pragmatics (what we do with words) (Brandom, 2008, pp. 56–69).

This brings us back to that feeling of being blocked, of trying to "be present" and failing. Maybe the blockage comes from treating the present as a static object to be observed, when its nature is more about holding ourselves and each other to account for our actions and commitments. The unease I feel in the "now" is often the friction of some unfulfilled obligation.

To get unstuck, the solution isn't to try harder to observe the present. Instead, it may be to consciously set aside a time where you can release the striving. Cultivating a space where "now" is not about what must be done, but just another name for the sensation may help relieve the blockage. This requires giving yourself permission to be free from the normative weight of your commitments, even if only for a moment. I have not yet addressed the directive to "be," but the here and now are now sublated: $\{\ulcorner \cancel{\text{now}} \urcorner, \ulcorner \cancel{\textit{here}} \urcorner\}$. The idea that individual agency is realized when a person identifies with and actualizes universal norms within a particular situation is a key theme in Brandom's reading of Hegel (2019, p. 533). The choice to step outside of that cycle of obligation is the practical advice derived from this understanding.

Sinking back in... You may notice that the arches of your feet are bent like bows. Find the 'keystone' of that arch, the point where the tension is greatest, and breathe into it. On the exhalation, allow your attention and breath to spread from that keystone, radiating through your foot. You can let go of their image. I am my ~~feet~~. Consider each thought as it emerges from the horizon as a temptation to take a gift and flatten it. Instead of doing, you may try receiving the gift without opening it, letting the un-opened packages accumulate.

That last bit may be surprising; the violence of smashing gifts may be a jarring image that kicks you out of the sensation. You may conclude that I'm not a very good meditation teacher. But I, perhaps like you, am strongly opposed to surrender. I fear, and in that fear, I seek control. What better way to feel in control of someone or something than by smashing their gifts? Consuming, negating, and destroying are powerful ways to feel certain of one's existence, though that route only leads to more and more destructive consumption. This dynamic of seeking control through negation draws on the initial stage of Hegel's dialectic of self-consciousness (1977, §178-196). Hegel argues that self-consciousness first attempts to achieve certainty of itself by negating an "other"—by consuming and destroying it to prove its own independence. The impulse described here to "smash" or "flatten" emergent thoughts to feel "self-certain" illustrates this initial stage of desire. Self-certainty is not the I-feeling; the I-feeling may accompany such acts of destruction, but the feeling tends to fade to empty regret when consumption is taken as the grounds for self-certainty. For Hegel, this is a dead-end strategy; by destroying the other, the self proves its power only over a lifeless thing and fails to achieve the recognition it truly seeks, leading to the endless cycle of destructive consumption. The meditative instruction to "receive the gift without opening it" represents a move beyond this combative, negating posture toward a more mature form of consciousness.

1.3 Systematic Analysis: The Structure of Embodied Reason

When you were with your toes, your feet were, loosely speaking, a kind of 'forestructure.' They existed as implicitnesses, working on your behalf in the background of experience. As you bring them into awareness, they may become explicit. They are no longer forestructures, but structures. When you let them go, they do not become unreal: they are retained in sensation but are no longer foregrounded. As you continue, these once implicit regions accumulate: I am my {~~toes~~, ~~feet~~, ~~legs~~...}.

The mind, seeking to grasp this process, might generate representations like the one in figure 1.2.

You proceed, but now the thought might arise "Don't I want the contents of those gifts?" Here things turn subtle again. Each 'gift' is oriented towards the desire to enhance the sensation. Each thought, as its hard edge emerges from the horizon, is an attempt to represent the enhancement of the sensation. It feels so good to experience self-certainty that you desperately *want* to remember how you got to where you just were. Each movement in The Exercise was regular, a constant relaxation against different as-yet-unacted-upon thoughts. And now you have thought before, so you know that these thoughts are structured around the desire to continually enhance the sensation.

Each thought, prior to its explication, is what Carspecken calls a *forestructure*—necessarily implicit, yet structured by the discipline The Exercise cultivates. These accumulate as vast dimensionality when recollected, though self-certainty itself feels non-dimensional in the gentle rhythm of awareness.

What does this accumulating, unspoken knowledge look like when one gives in to the temptation to represent it? For me, this usually isn't felt as failure—at least not at first. First, it is experienced as excitement. All the energy that was building gets released into the representation. When the desire to share the understanding of the experience bubbles over, I find myself representing it with images. This is what Carspecken identifies as the moment where "representation is an impetus to act; here to 'act in thought"'

(P. F. Carspecken, 1999, p. 172).

Another, more abstract temptation might be to conceptualize the structure of this accumulation itself. In this book, the *null representation* (∅) functions across the subjective, normative, and objective modes of validity. It is both absence and potential: the erasure from which elaboration begins. Stored senses—{~~toes~~, ~~feet~~, ~~I~~}—coalesce into a divaded set, a collection that resists completion. This nullity is the pragmatic metavocabulary of embodiment: always already there, and always transcending the determinate names it erases. The thinking mind might attempt to formalize this process—but these representations, however elegant, are just thoughts. Let these images go. Return to the feeling. As the forestructures accumulate, they might feel like 'potential' energy. As that energy builds, it eventually reaches a threshold where it transforms into 'kinetic' energy. Each practioner has their own threshold that moves with practice. Perhaps the energy built in the toes is so exciting you MUST draw a representation of your toes! But you can probably build more energetic forestructures than just the toes.

What does this rhythm of tension and release feel like? Well, I've been talking about it a lot, but is there a way to express more and tell less? It's not a purely logical process; it's partially mechanical and partially subjective. I introduce the Zeeman Catastrophe Machine (ZCM) as a model (1.3). The *Sound of Time* is the overarching metaphor for the continuous rhythm of consciousness. The *ZCM*, however, provides a model for a specific moment within that rhythm: the *threshold* where accumulated tension (potential energy) suddenly releases into a new state. It models the 'snap' of understanding or the 'whoops' of falling into thought.

The ZCM models this threshold: as tension builds, the system eventually snaps catastrophically to a new stable position. The gradual tension represents the accumulation of forestructure; the sudden release is either collapse into thought or deeper relaxation. It's a machine that runs on tension and release, not symbols. I will return to this machine in chapter 6.

The machine consists of a rotating disc with elastic bands attached. Part A of figure 1.3 shows simple feedback: as you pull the control parameter, the disc rotates smoothly and predictably. Part B demonstrates catastrophic feedback: the folded surface represents how the same gradual increase in tension can result in sudden, discontinuous changes in the system's state. The 'fold' in the surface models the critical threshold where accumulated potential energy suddenly 'snaps' the system to a new stable configuration—what catastrophe theory calls the 'catastrophe' or what I experience in meditation as the ‘whoops’ moment when forestructures collapse into thought.

1.4 The Dialectical Turn: The Risk of Dissolution

If those assurances are too weak to open the next movement towards self-certainty, consider the risk of de-committing. The Exercise is a practice of de-committing from the normative self, the “me” that is recognized by others. It is a risk to let go of the commitments that define you, even temporarily. But in that risk lies the potential for profound transformation.

It may seem odd to put such an emphasis on *risk* in the context of a guided meditation. But there may be negative consequences for de-committing, even if those commitments will lock back in place as soon as you think. Perhaps those who hold your commitments for you will be upset if you do not fulfill them. But they might forgive you. They may experience de-commitment, too, with the deliciousness of mutually cancelled plans.

One of the blocks I have with meditation is that surrendering is so close to death. While I can’t guarantee it, you probably won’t biologically die from relaxing. But what you may routinely experience is an *existential death*; the death of the social self, or “me.” This is precisely the risk of learning something new. Our social identities can feel like a house of cards, where pulling one out because it led to the experience of error risks a full collapse of social identity.

Consider the cresting tsunami that you fear. It is a single wave rising from the ocean. It has its own unique journey, its shape, its crest, its fall. And then, it returns to the vastness from which it came. Was it ever separate from the ocean? When you identify with the thought—with the wave—you may fear its repercussions. But you may also fear its inevitable return to the whole. Both are real. But remember: each thought is not the sensation. The sensation feels good. Let go of the wave before it crashes. Let it slosh gently back into the vast sea of implicitness. What you fear is social death, which you have died a thousand times before and been reborn from each time, recollecting the experience as growth. Death, in three dimensions, need not be feared, for who knows what mysteries lie beyond the horizon? Perhaps when you meet it, its first moment may feel like the deliciousness of reciprocally cancelled plans.

That last thought might kick you out of The Exercise. I feel it as a 'zap! No! I do NOT know what I am writing about.' Death feels too big, my words too small. It's like the metaphors for the experience become so cloying that they suffocate the sensation. So, I drop the metaphorical facade.

Before the divisions of subject, object, and rule, there is the primordial *No*. The "no" or its erasure come unbidden with varying strengths. In the embodied logic, this is a temporal claim: these are more primordial terms than the differentiation into subject, object, and rules. The NO! shouted in abhorrence or the one that cradles the broken pieces of our lives are undefinable in print. They are like topological holes in the space of reasons—singularities that defy articulation.

The normative distinction between subject and object dissolves in the feeling-body. This happens outside the practice, too, the practice just affords a recognition of those moments that are often ephemeral. Pregnancy, birth, death, intimacy, teaching, learning, etc. evoke a deep sense of being that transcends the limits of lexical articulation.

The terms "No" and ~~no~~ (determinate negation) are meta-systemic. As the text unfolds, I will claim they are a *pragmatic metavocabulary* for the system of critical mathematics. Objectively, they accrue senses: {~~toes~~, ~~feet~~, ~~I~~,...}. Subjectively, they are the source of action recollected as {I}. Normatively, they are the concept that erases its own name: ⌜~~*concept*~~⌝. This nuance can be compressed into the null representation ($\emptyset$). The "no" or its erasure are primordial terms, more fundamental than the differentiation into subject, object, and rules. The normative distinction between subject and object dissolves in the feeling-body.

You start again, but now you know that the desire for the grand experience—the desire for self-certainty—is itself hindering the ability to let go. As Carspecken notes, "To have what is desired and anticipated one must let go of desire" (P. F. Carspecken, 1999, p. 174). Perhaps to get to non-thinking, you must say "no" to the desire for the grand experience. And if thinking is saying "no" to the flow, then maybe non-thinking is saying "no" to that no-saying.

Exercise 5: Iterate and Integrate (The Upward Spiral)

At this point, I lose some feeling of entitlement to proceed in The Exercise, and rely on Phil Carspecken's text.

"It is as if the forestructures of desire are accumulating each time you

let one go... Letting go is what allows the forestructures to accumulate; the implicit portions of each anticipation appear to contain more and more possibilities. Thus the situation is contradictory. To have what is desired and anticipated one must let go of desire"(P. F. Carspecken, 1999, p. 173).

"... Letting go results in a repetition. It results in the repetition of (i) sensation, (ii) awareness-of-sensation, (iii) letting-go, (iv) intensified-sensation. It is not a repeated 'act' because the only agency involved is the choice not to act, 'letting go.' This allows an anonymous repetition to act of itself; a sort of vibration that throws one out in the second phase ('awareness of') but waits for one to fall back in [to sensation]"(P. F. Carspecken, 1999, p. 174)

Each cycle of this process doesn't leave things as they were; it enriches the whole. As Carspecken describes it, "letting go is what allows the fore-structures to accumulate; the implicit portions of each anticipation appear to contain more and more possibilities" (P. F. Carspecken, 1999, p. 173). If the cycles are numbered, one can say that from state S_n plus a new provocation D, a letting-go yields S_{n+1}. The state S_{n+1} carries an accumulated, assimilated content from all the prior releases. Each repetition is not a mere repeat but an intensification: not a circle so much as an upward spiral.

Exercise 6: Reading Philosophy—Being, Nothing, and Becoming

You have been learning to navigate the rhythm of thought in The Exercise: compression into awareness and decompression through letting go. A recurring temptation arises: perhaps this rhythm can be stabilized by deliberately directing it? Could thinking a thought and then intentionally thinking its opposite return experience to unified immediacy?

The following extends The Exercise into a controlled abstraction experiment.

Embodied Experiment *Settle into the relaxed, open awareness cultivated previously. Now intentionally introduce the most abstract thought available: "Being." Attempt to fixate on it—pure, undifferentiated presence. The felt quality is not the expansive unity of the I-feeling; it is intense Temporal Compression (↓). Attention strains to grasp totality as a static object.*

Yet "Being," devoid of inner difference, is unstable. It gives nothing de-

terminate for attention to grip. The fixation collapses into "Nothing." This, too, is compression—the negation of the prior abstraction. An oscillation arises:

$$\text{Being} \longleftrightarrow \text{Nothing.}$$

This loop is restless and draining—a bad infinite of endless alternation without qualitative transformation. The strategy of control through opposition fails because it remains imprisoned in mutual compression.

Pause. What occurs between the poles? In each transition there is a micro-movement—a *temporal singularity* where one fixation dissolves and the other has not yet stabilized. Shift attention toward that passage. Breathe into *the in between*. The movement itself is what Hegel names *Becoming*.

To remain with Becoming, apply the earlier lesson: Let Go of fixation on the oscillation as content. This letting go is Temporal Decompression (↑). The rhythm becomes becoming and the awareness of becoming. As a temporal compression, ⌜becoming⌝ paradoxically compresses both compression and decompression into a single process.

This is an exciting thought—so much so that I began my dissertation claiming "I am a constant becoming." Yet there is peril: for becoming to avoid falling back into simple oscillation, it must move beyond itself. When movement (becoming) is compressed into a static concept (⌜becoming⌝), we get a kind of recursion: becoming(⌜becoming⌝). If becoming becomes itself, it achieves what mathematicians call a *fixed point*—yet it cannot be itself if it is itself. No movement would be expressed; it would slide back into nothingness.

Hegel's whole *Science of Logic* falls out from this beginning. The exercise attempts a somatic sublation of being/nothing into ~~being/nothing~~ into becoming, where becoming names their opposition. I imagine dense philosophical texts annotated with breath marks, like musical scores telling wind instrument players when to breathe.

The oscillation that the sound of time metaphorizes is ultimately between the finite name (⌜becoming⌝) and the *in*finite movement it purports to represent. This returns us to determinate negation in its two moments: the first negation produces finite representation; the second determinately dissolves those determinations. You find yourself learning again what you already knew, but from a conceptual descent that folds back to the original "no."

Hegel's *Science of Logic* begins with the most abstract categories: Being, Nothing, and Becoming. Pure Being, completely indeterminate, gives nothing for thought to grasp—and thus reveals itself as Nothing. Yet this Nothing is not mere absence; it is the dynamic movement between Being and Nothing that Hegel calls Becoming.

Mapping this onto The Exercise reveals the connection. The moment of unified presence (I-feeling in full bloom) is like a moment of pure Being in experience—so pure and featureless that it cannot be held onto. The collapse of that experience when the thought "I am this" arises is akin to that Being turning into Nothing—the fullness evaporates into an absence (the feeling is "lost," and one is left only with an empty thought that doesn't deliver the goods). Yet out of that nothing, a new determination immediately begins to form: perhaps a new approach, a new aspect of the body to focus on, or a new resolve ("I'll try again"). In other words, the Nothing gives rise to a new Becoming—a coming-to-be of another state of being (for instance, restarting the cycle and reaching a new unified feeling, which will again have no determinate content and thus again risk collapsing). The cycle of recognition described is essentially a cycle of Becoming. The moments of pure undifferentiated sensation (Being) inevitably pass away (Nothing), yet from that void a richer unity can emerge if allowed (the next Being). Each "loss" of the experience, when handled by letting-go, is actually the engine of a deeper recognition of the experience. Each negation (loss of being) is followed by a negation of that negation (the letting go of the thought, which re-establishes a new being).

Exercise 7: A Symbol for Silencing

As you begin again, you might notice that ~~no~~ is a symbol. It contains both the first negation and the second negation within itself, so it represents both fixity and movement. While paradoxical: stay with its movement.

~~no~~

1.5 Space, Time, and the Limits of Picture-Thinking

The metaphor of sound arises from the body feelings associated with compression and decompression. Time may be grasped as vibration: the unit circle rotates, the sine wave oscillates, and a resonance emerges. These mathematical gestures are ornamental to The Exercise but are useful for reconstructing mathematics and may be helpful for qualitative researchers. They anchor thought in motion, making rational reconstruction audible as rhythm and recurrence. To feel the wave crest and trough is already to participate in temporality; to recognize it mathematically is to render that temporality communicable.

The central metaphor can now be fully articulated. In air, sound is the regular compression and rarefaction (decompression) of air particles. The "sound of time" is the rhythm of embodied reason, a dialectical pulse of compression and decompression.

- *Temporal Compression:* The act of focusing, judging, or thinking. This is the first "no." It corresponds to the compression phase of a sound wave. It is felt as proprioceptive contraction.

- *Temporal Decompression:* The act of releasing and re-integrating. This is the second "no," the sublation. It corresponds to the rarefaction phase of a sound wave. It is felt as proprioceptive expansion.

I came up with the metaphor from considering the hermeneutic circle: the idea that the whole of some text must be understood through its parts and the parts must be understood in relation to the whole. This way of interpreting texts was originally developed by biblical scholars, but hermeneutics has taken on a broader significance in philosophy and social theory.

Since I know a bit of trigonometry, I considered what it would mean to unwrap the hermeneutic circle as if it were the unit circle. Take a circle and imagine yourself as a single point on its circumference. Roll the circle forward in time, as in figure 1.4, and your distance from the ground will trace out a sine wave. So, the sine wave is, metaphorically, a way of unwrapping the hermeneutic circle into a temporal wave. So, metaphorically speaking, hermeneutic meaning is the sound of time. Time is the medium in which meaning occurs. At least within this metaphor.

Recognizing this rhythm is the first step to synthesizing the concept of *becoming*.

One of the implications of the sound of time metaphor is its connection to space. I have been working on developing the 'space of reasons' but a more primordial kind of space has been left at the margins. Take an analog watch, one of the ones where the second-hand moves smoothly around the dial. That spatial movement is a very basic way to understand empirical time. This may feel very abstract, but consider the experience of driving a young child to a town that is 50 miles away. The answer "50 miles" has no meaning when they ask, "how long until we get there?" They have yet to compress temporally extended experiences into the abstract concept of a spatial distance. Distance, as an abstract concept, does not arise until a temporally extended experience is recollected. It often takes many such trips before a child will have a strong sense for what "50 miles" means. I've made the drive from Bloomington to Indianapolis hundreds of times. When my daughters and I were returning from Indianapolis one time, after having made about 98 of the 100 mile round trip journey, they asked "are we home yet?" I said "just a few more miles," and one of them complained that we had to drive a whole hour longer. Space and time are confusing! We tend to work very hard to teach their separation, then only a few who study relativity ever get to consider their unity again.

The concept of *distance* is a temporal compression. Temporally extended experiences are recollected and compressed into a spatial form. To fit this idea into the sound of time metaphor, the two fields of space and time are orthogonal to one another. Temporal compression induces spatial predication when a temporally extended experience is recollected. Those spatial predications can be decompressed into temporally extended *histories* for how those thoughts came to be. The relationship mirrors electromagnetic radiation: just as a changing electric field induces a magnetic field, and a changing magnetic field induces an electric field, it is the *movement* or *change* itself—the act of compression or decompression—that creates the mutual relationship between spatial and temporal fields. Static spatial forms cannot induce temporal histories; only the dynamic process of spatializing (temporal compression) or temporalizing (spatial decompression) creates this reciprocal induction.

So, when I consider a spatial form like "triangle," I can temporally decompress that form into a history for how the form came to be. My story for triangles begins in Dr. Laughlin's office because I remember playing with a toy, trying to cram a triangular shape through a circular hole.

This relationship between space and time is inherently intersubjective. The temporal experience of the Exercise is spatialized when the desire to communicate the experience is acted upon in representation.

I make no claim about the purely objective relationship between space and time that cosmologists explore. There are implications that relate the first moment of determinate negation to observability in particle physics, as waveforms collapse into particles. There are other implications for relativistic interpretations of spacetime, too. But we have not learned how to add or subtract yet. Those diversions must wait until critical mathematics has developed the tools to express those notions.

Spacetime is not a backdrop for events but an active participant in the unfolding of reality. The compression and decompression of spacetime can be likened to the rhythmic oscillations of sound waves, where moments of high density (compression) alternate with moments of lower density (decompression). Highly spatialized representations (picture thinking) miss how space emerges from the recollection of temporal processes.

Suppose you try to capture self-certainty. You might try to indicate the particular now, this, or here through a deictic pointing gesture. This fails to capture self-certainty, as "this" refers to many different "thises." To indicate the passage of time, you might use a deictic ranging gesture (see figure 1.5). It is very hard to represent temporal difference without relying on spatial difference. Carspecken writes, "Space, not time, is used to represent differences: when time is used to express a difference, like 'I am totally different now than I was back then,' the temporal difference has a spatial representation—in the cultures I am familiar with, the past is to the left and the future to the right, or the past is behind and the future is ahead" (2018, p. 19).

In quoted passage, Carspecken is analyzing the problem of picture thinking. In Hegel's philosophy, *Vorstellung*, often translated as "picture thinking," "representation," or "figurative thought," occupies a crucial intermediate stage between sensory intuition and pure conceptual thought (*Begriff*). Carspecken takes up the problem with picture thinking in several works (1999, 2016, 2018). When an essentially temporal phenomenon, like *becoming*, is rendered in spatial form, as if knowledge is limited to that which can be conceptualized, we risk getting caught up in what Hegel calls the *bad infinity* of endless spatial predications. Each may be progressively more adequate, but it is impossible to picture the {I} because the {I} is not-a-thing. Picturing it as an object essentially misrecognizes it. Spatializing thought as images, metaphors, and scientistic models artificially limits the horizons that phenomenology explores.

It is a mode of thinking that relies on images, metaphors, and sensuous or spatial-temporal forms to grasp philosophical and religious truths. Even *rhythm* misses the Concept, as it requires the notion of empirical spacetime to be explicated.

The "problem" with picture thinking lies in its inherent limitations.

1. *Externality and Separation*: *Vorstellung* tends to present the content of thought as a collection of distinct images or representations that stand in an external relation to one another and to the thinking subject. For instance, the canonical texts in math education (think textbooks or the Common Core Standards) may be represented as a series of symbolic manipulations (i.e., 'procedural knowledge), rather than as expressions of divaded conceptual relationships.

2. *Inadequacy to the Universal*: Because it relies on particular images, *Vorstellung* struggles to capture the true universality and necessity of conceptual determinations. A representation is always a specific instance, and while it can symbolize a universal, it cannot fully embody or express its dynamic, self-mediating nature. The content of the Concept is distorted when forced into the static and finite forms of representation.

3. *Resistance to Dialectical Movement*: The fixed nature of images in picture thinking can make it difficult to grasp the fluid, dialectical movement of thought, where concepts pass into their opposites and are sublated into higher unities. Picture thinking tends to hold onto its representations as fixed and separate, hindering the transition to speculative, conceptual understanding. A series of images or representations can give the illusion of completeness, but they do not allow for the dynamic interplay of concepts that characterizes true philosophical understanding.

Neither Hegel nor Carspecken simply dismiss *Vorstellung*. It is a necessary stage in the development of consciousness and spirit. The philosophical task of this chapter, now that a robust system of representations has been articulated, is to transcend picture thinking. We must break the metaphor to overcome the limitations of its pictorial form. "Picture-thinking is totally taken for granted in mainstream methodologies for social research, but criticisms of mainstream methodologies mostly substitute one representation for another, or take the basic representation underlying empiricism or positivism and merely remove and/or redefine its specific components. Critical theory in the original sense begins consciously with the full transcendence of picture-thought in general, not with a new representation" (P. F. Carspecken, 2018, p. 19). This requires exploring alternative modes of expression, such as music, that capture the dynamism of lived experience beyond the limits of static representation.

1.6 Integration: Breath and Kindling

Representations and machines fail, structurally, to represent the totality of experience. But logic also fails; recall that the logic includes unspeakable symbols. The text I've written excludes phonemic pronunciation. So, music is entirely absent from its pages. Music also can't represent the totality of experience. It provides yet another orthogonal direction for reconstructing the experience.

Brandom offers a bit of wisdom for how to take the logics that spring forth from his project of *logical expressivism*. He writes, "Acknowledging the value of the unique clarity afforded by algebraic understanding accordingly does not entail commitment to this sort of understanding being available in every case, even in principle. It does not oblige one to embrace the shaky method of the drunk who looks for his keys under the streetlamp, not because they are likely to be there, but just because the light is better there. We should admit that, sometimes, algebraic understanding is not available" (Brandom, 2008, p. 215). I riff on this idea in the following song, *Breath and Kindling*, calling such logics 'white-knuckled halos'—though the phrase also reflects personal struggles with goodness. To backfill what is lost in the flattening of experience into text, I gesture toward the *in*finite that formal representation cannot contain:

Breath and Kindling

A moment too late, a moment too soon,
Bleeds into a cinnamon moon.
Red, silver snow, breath frozen dew.
The sun died and a thousand stars bloomed.
Good morning, dear Venus,
Good morning to all my evening stars.
Though the glow is faint between us,
Your light slices through the dark.

Strung street lights, small sodium moons
White-knuckled halos cut holes in the gloom.
Are you always so ate up, man?

Just give me some room.
You cut to the quick,
I get a weeklong wound.
Good morning, dear Venus,
Good morning to all my evening stars.
Though the glow is faint between us,
Your light slices through the dark.
I try to walk the line,
But miss the flight.
Hit the curb to get some lift inside.
Start to drift and fall for a while out of time.
Easy blows the listening breeze
In breath caress
Wind over wings.
That's the way that you move me –
– From white to green
To hollow dream,
Hatchet through the sycamore tree.
It'll do for the kindling –
– Easy blows the listening breeze
In breath caress
Wind over wings.
That's the way that you move me –
The care you take
With the moves you make
To ease the world to sing for its own sake –
Frankly, it's moving –
– Easy blows the listening breeze
In breath caress
Wind over wings.
That's the way that you move me –
– In grief I've worn
The cloth of storm
Of warp and weft
Bereft of words of words reborn
Moving toward ~~divinity~~–
– Easy blows the listening breeze
In breath caress

Wind over wings.
That's the way that you move.

The song opens with haiku-like images that blend time, color, and sensation: "A moment too late, a moment too soon / Bleeds into a cinnamon moon / Red, silver snow, breath frozen dew." These lines recollect the setting sun in winter, where I try to establish a crepuscular, transitional mood while introducing dialectical concepts. The evanescent "now"—a moment that is both too late and too soon—reflects indexicals like "now," which vanish the moment they are named, highlighting the failure of sense-certainty to grasp the present. The chords begin with George Harrison's progression from *Something*, hanging on A major 7 to A flat 7. I play the song on an 8-string baritone guitar tuned to drop A, so there is a round warmth with subtle shimmer from the doubled strings, mirrored in the metaphor of a "cinnamon moon". That metaphor combines sight with taste and smell to create a 'spicy' warmth that contrasts with "breath frozen dew". The death of the sun gives way to the blooming of stars, setting the implicit 'lesson' of the song: loss of a fixation (losing the sun) through the experience of error can feel profoundly bad but enables subtler understandings to shine through.

I address Venus directly, whose dawn arises with the setting sun. There are shadows of past romantic relationships embedded in the imagery, but the invocation of "Dear Venus," "morning stars," and "evening stars" also alludes to Frege's distinction, where a single referent (Venus) is grasped through different senses (morning star/evening star). This suggests an underlying unity through different representations. The imagery of light continues, shifting from the celestial to the mundane and back again. "Strung street lights, small sodium moons" elevates ordinary urban infrastructure to something cosmic, while "White-knuckled halos cut holes in the gloom" serves as metaphor for rigid, algebraic logic—systems that provide clarity through constraint but are necessarily limited and exclusive.

The beauty is contrasted with a moment of sharp emotional pain: "You cut to the quick and I get a week long wound." The idiom "cutting to the quick" refers to the sensitive area under a fingernail, but also suggests a rush to judgment. The I-You/me I use in this line is a form of self-address that includes otherness.

The rhythm shifts from gently swept guitar strumming to more of a grove that mimics a heartbeat. Simultaneously, the gentleness of the Harrison progression is exfoliated in favor of a simpler I-IV-V progression. This stability represents temporal compression or the "first negation"—a fixation, the creation of determinate boundaries. The lines, "I try to walk the line, but miss the flight / Hit the curb to get some lift inside / Start to drift and fall for a while out of time," use enjambment to enact the feeling of losing control within that stability. The thought spills over the line break, mimicking the act of drifting and falling.

There is a brief pause in the music after "fall for awhile, out of time"—a cessation that represents a threshold where stability reaches a critical point necessitating release. But this release occurs without tutelage or grounding. As I wait through a bar of silence when I play the song, I take the opportunity to breathe deeply.

Upon the phrase "Easy blows," the three chords start to move. The song's structural and thematic core becomes the anaphoric refrain: "Easy blows the listening breeze in breath caress wind over wings / That's the way that you move me." This line creates sonic texture through sibilance and liquid consonants that create a soft, flowing sound mimicking the movement of gentle breeze. The phrase "breath caress" links the movement of breath to an intimacy. It is hard to avoid sexual connotations, but the intimacy that I am trying to establish is less freighted than that. I was after both the spiritual center of self-certainty and the teachers who have listened to me in a way that invited free movement around that core.

On that turn, the key changes in minor thirds to move all the way around the circle of fourths—a kind of homage to John Coltrane's *Giant Steps*, though simpler than his movements. The harmony modulates systematically through a cycle of ascending minor thirds: A Major, C Major, Eb Major, F# Major, returning to A Major. This cycle divides the 12-semitone octave symmetrically into four equal parts, creating abrupt shifts between distant keys. The constant, structured shifting embodies the concept of becoming—not a static state but a dynamic process where each key is destabilized by the modulation.

Relative to the key, the melody simply repeats. The core melodic motifs are repeated identically *relative* to the new key, using the same scale degrees relative to each new tonic. But as the key modulates up, the melody also rises. This enacts a dialectic of sameness and difference: the melody's internal structure is preserved while the absolute pitch constantly rises as the underlying harmonic structure shifts. With each change, the last note of the melody, which happens to be the tonic, becomes the first note of the melody in the new key, the fifth. In the first shift, the last note is A (the root) and the first note in the key of C is A (the sixth). However, since the key has changed behind the melody, those samenesses feel different. The underlying structure moves, while its representation (the melody) stays the same.

This process, like the somatic sublations above, is a kind of implicit sublation: the form of the melody is preserved, the harmonic ground and absolute pitch are negated, and the melody is transformed into a higher unity. As the keys ascend, the preserved melodic shape is realized at progressively higher pitches. This is not a simple repetition but an upward spiral where each repetition is an intensification. Like poetic/rhetorical anaphora ("I have a dream"), which involves linguistic repetition and intensification, the song adds a layer of musical anaphora. The transitions between keys are mediated by a pivot mechanism where the pitch remains the same while its harmonic function is redefined by the shift, ensuring the transition is a determinate negation where the dissolution of the old state defines the beginning of the new one.

This is reflected in the sycamore tree metaphor: "From white to green / To hollow dream, hatchet through the sycamore tree / It'll do for the kindling." The sycamore trees of southern Indiana grow with both white bark and green leaves. The particular tree I recall was an old friend who was planted too close to the house when I was a kid. My parents had to chop it down.

Metaphorically, it is about the creative process. Beginning with a white page, which turns lush with prose in my obsessive writing practice, but then those prose fall apart when I realize a core error in my assumptions thus negating the validity of those expressions on the page. I usually feel sad and disappointed when those errors are made explicit. But the words in error don't become unwritten or useless. The kindling metaphor ties back to the song's title, suggesting that from the destruction of old forms (the sycamore) arises the potential for new life and understanding (the kindling for fire). So, the metaphor recollects a necessary destruction of old certainties to provide material for new growth (the kindling). The hatchet's action is (to my abundant disappointment) both violent and purposeful. I wish my words were always gentle, but each one holds the potential for the pain of division.

The confusion about intimacy is, I hope, clarified in reference to those who "ease the world to sing for its own sake." Phil Carspecken's gentle, catalytic presence as educator taught me a lot the role of listening. The 'active' listener serves as a kind of vacuum, or negative withdraw, implicitly providing direction for me, the student, who filled that space with words. When I was working on my dissertation, I wrote a song I repeat later in this manuscript called *Love's Memory.* I did not know if I *could* or *should* include it in the dissertation. I wrote him an email asking if it would be okay. He responded "Oh, my. Thank you! Thank you, and thank you!!! Phew. I'm crying. Ohhh! The heart; loving, mourning, bowing, stopping: no doing for moments in the second position. I'm grateful for what you have written, and for what your father wrote. Of course you must keep this in." I have kept that email 'pinned' in my messages for three years, to serve as a constant reminder and validation of the those works I might otherwise burn for kindling. The 'hatchet' is dropped to ease the world to sing for its own sake.

The lyric then circles back to a personal state of grief. The line, "In grief I've worn the cloth of storm / Of warp and weft / Bereft of words of words reborn": this will make more sense after the Eulogy and *Love's Memory.*

The physicality of performing the song is also theoretically relevant, as the limitations of my body are reflected in the ZCM. The rising pitch increases potential energy, pushing me toward my vocal limits. With "In grief..." I reach near the top of my vocal range. Energy has been building as I approach my limit, then, with the last chorus, it 'snaps' down, resolving to the original key. I relax as the song ends. The ZCM suddenly downshifts in energy. The tension dissolves, and the music returns to the initial stability. I enjoy singing the song. What is left, after moving through the circle, is not just a return to the beginning for me. Going nowhere—often derided by academics through the phrase "there is no 'there' there"—has a deeply satisfying quality. A glow, easily missed and often forgotten, accompanies the physical, emotional, and theoretical act of performing the song. Lately, in the mornings when I wake up feeling anxious, I like to sit outside my daughters' room, singing this song as an invitation to them and myself to wake up gently with the breath and kindling of a new day.

1.7 Prefatory Analysis: Two Readings of Determinate Negation

Before concluding, I want to articulate an alternative to the interpretation of determinate negation that will be taken up in the next chapter. This prefatory analysis serves as a bridge, clarifying the philosophical terrain we have been traversing.

Brandom's Interpretation: Other-Exclusive Material Incompatibility

Robert Brandom, whose work profoundly shapes the next chapter's analysis, argues that determinate negation should be understood as *other-exclusive* material incompatibility. Something is what it is because it excludes what it is not. A red surface excludes green, blue, and yellow; a square surface excludes triangular, circular, and pentagonal. The identity of each concept lies in its relations of exclusive difference to other concepts.

This means conceptual content is inherently *relational*. You cannot understand what "red" means in isolation. You understand it by grasping the entire family of color concepts and recognizing how red contrasts with green, blue, yellow. The meaning of "red" is constituted by what it is *not*.

For Brandom, this is Hegel's radicalization of the law of non-contradiction. Far from rejecting that law, Hegel places exclusion at the very center of his metaphysics. Everything is what it is by virtue of what it excludes. Determinateness is exclusion. To be is to contrast.

This interpretation explained why I could never pin down the meaning of mathematical concepts by definitions alone. The meaning of "2" is not captured by saying "the successor of 1." Rather, "2" gets its meaning from its position in a web of incompatibilities and inferences: it is not 1, not 3, not 7; it is even, not odd; it is prime, not composite (depending on convention); it is the result of 1+1, not 1+2 or 2+2. The concept is alive in this network of relations.

Bordignon's Critique: Self-Exclusive Absolute Negation

Some Hegel scholars argue Brandom's account, while insightful, does not go far enough. Michela Bordignon claims Brandom "stops short" of Hegel's full concept of determinate negation. For Hegel, determinate negation is not merely *other-exclusive*; it is *self-exclusive*. It is what Hegel calls "absolute negation" or "negation of negation."

What does it mean for a negation to be self-exclusive? The determination does not just exclude *other* things; it excludes *itself*. It negates itself, turns into what is other than itself, and thereby propels the dialectical movement forward.

Consider Hegel's famous example of the finite and the infinite. The finite is defined as that which has limits, which ends. But to grasp what it means to be finite, you must think its contrast: the infinite, that which has no limits. However, the moment you define the infinite as "that which excludes the finite," you have *limited* the infinite by its exclusion of the finite. The infinite is now finite in a new sense: bounded by what it excludes. This is the self-negating movement of the concept. The infinite negates the finite, but in doing so, negates *itself* as truly infinite.

The finite, in its ceasing-to-be, in its coming to an end, realizes its own finitude. It is itself insofar as it is no longer itself. The infinite is itself in this very process of self-negation. This reveals that some concepts are inherently self-contradictory, and that this self-contradiction is not a defect but the engine of their development.

When I first read this, I resisted. How can something be itself by being not-itself? But then I thought about my own experience of self-alienation. I often feel that I am not myself—that the "me" others recognize is not the {I} that I experience myself to be. Yet that very alienation is constitutive of who I am. The {I} becomes a "me" through recognition by others, and that "me" is never adequate to the {I} that I feel myself to be. The self is constituted through this movement of self-externalization and return. I am myself by becoming other than myself.

This self-exclusive negation is what Hegel calls the "dialectical soul" of every determination. It is the "innermost source of all activity, of all animate and spiritual self-movement." It is why concepts *move*, why they develop, why they do not stay put.

The Two Interpretations in Practice

Let me make the contrast concrete. Imagine you are in The Exercise, and you have just brought awareness to your toes.

Brandomian interpretation: "I am my toes" excludes "I am my feet," "I am my head," and every other body region. The identity of "toes" is constituted by its material incompatibility with these other regions. When I shift to "I am my feet," I am replacing one exclusive determination with another. The movement is a sequence of mutually exclusive determinations: toes $\rightarrow$ feet $\rightarrow$ legs. Each excludes the others, but all are preserved in memory as part of the accumulating totality.

Bordignonian interpretation: “I am my toes” is a self-negating assertion. The very act of identifying with my toes reveals that I am *more* than my toes. The identification negates itself by pointing beyond itself to the larger whole. The toes are not just excluded by the feet; they negate *themselves* by being inadequate to the {I} that asserts them. The movement is not a sequence of static replacements, but a living, self-propelling dynamic where each determination undermines itself and thereby generates the next.

Both interpretations are valuable. Brandom helps articulate the relational structure of conceptual content. Bordignon helps articulate the dynamic, self-propelling nature of consciousness. I do not have to choose. Hegel’s determinate negation has these two aspects: other-exclusion and self-exclusion, static structure and dynamic movement.

The Exercise allows you to feel both. You feel the exclusive difference between toes and feet (Brandom). And you feel the self-negating inadequacy of each identification (Bordignon). Together, these two aspects constitute the full rhythm of determinate negation, the sound of time as it pulses through embodied experience.

In the next chapter, we will take up Brandom’s inferentialism in detail, exploring how material incompatibility grounds the entire structure of reason-giving. But before that, I want to acknowledge this alternative reading—one that emphasizes the self-moving, self-negating character of absolute negativity that cannot be fully captured by Brandom’s relational account. Both are needed. Both are true.

1.8 Conclusion: The Safeguard of Experience

This chapter began with the simple act of attending to the breath and the body. From this embodied practice, a theoretical tapestry has unfolded. My great worry is that readers might misunderstand the purpose of denying the certainty of words as if I am advocating for solipsism. To be clear, I am trying to entrain you (and myself) into the second-person *listener’s* position in communicative action. ~~Self-certainty~~, or any certainty, is not found in the words that describe it. Words flatten what listening expands. What communicative practices make explicit must still answer to what must remain implicit. Communication is, in essence, dyadic: the speaker and the listener, the author and the reader, the first negation and the second. Uncertainty in words does not imply a lonely consciousness. I intend no solipsism. By preparing you to listen, through practice, I aim to create a space for shared understanding.

In essence, what The Exercise and reflections provide is what Brandom calls a *pragmatic metavocabulary*. The phenomenological vocabulary describes the embodied practice, but since the practice is sufficient to understand the vocabulary, they work in tandem to deepen understanding. This kind of reciprocal sufficiency between words and practices creates what might be called *expressive pragmatic bootstrapping*—a process where notation and experience mutually enrich each other.

The unspeakability of certain notational practices like Quine corners ($\ulcorner \cdot \urcorner$) and sous rature ($\cancel{\cdot}$) is a feature, not a limitation. For those who struggle with meditation because they are thinkers, ruminators, or rehearsers of thoughts, this differentiation between speakable and unspeakable elements helps eliminate the temptation to think. The ‘sound’ in your head—the *sotto voce* voice that narrates experience—can be so loud that it challenges focus on The Exercise. These notational practices purposefully eliminate inner speech, perhaps enhancing the sensation by silencing the discursive mind.

When the concept of negation is encountered, the felt experience of “no” and “~~no~~” will be remembered. When the *in*finite is discussed, the horizon of the “Grand Experience” will be recalled. And when working with the null representation, $\emptyset$, it will be recognized not as an empty symbol, but as the mark of the silent, unrepresentable ground of thinking itself. The ultimate aim is to explicate a critical mathematics, and understanding how mathematical objects might be seen as recollections invites an unwinding of these objects back into the subjective, embodied experience from which they arise. Uniting the subjective and objective poles of validity claims is deeply related to the project of emancipatory knowledge. The sound of time will continue to resonate, reminding the reader that even the most abstract mathematics is, at its heart, a profoundly human song.

References

Abbott, E. A. (2020, April). *Flatland: A Romance of Many Dimensions by Edwin A. Abbott*. Independently published.

Agamben, G. (2006, September). *Language and Death: The Place of Negativity* (K. Pinkus & M. Hardt, Trans.; First Edition). Univ Of Minnesota Press.

Aristotle. (2017). *Metaphysics: Aristotle's metaphysics in english, latin and ancient greek: Trilingual edition* (W. o. Moerbeke & W. D. Ross, Trans.; Trilingual). Hermes Language Reference.

Boote, D. N., & Beile, P. (2005). Scholars before researchers: On the centrality of the dissertation literature review in research preparation. *Educational researcher*, *34*, 3–15.

Brandom, R. B. (1994). *Making it explicit: Reasoning, representing, and discursive commitment*. Harvard university press.

Brandom, R. B. (2000). *Articulating reasons: An introduction to inferentialism*. Harvard University Press.

Brandom, R. B. (2008). *Between saying and doing: Towards an analytic pragmatism*. Oxford University Press.

Brandom, R. B. (2019). *A spirit of trust: A reading of Hegel's phenomenology*. The Belknap Press of Harvard University Press.

Bueno, O. (2020). Nominalism in the Philosophy of Mathematics. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Fall 2020). Metaphysics Research Lab, Stanford University.

Cantor, G. (1891). Ueber eine elementare frage der mannigfaltigkeitslehre. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, *1*, 75–78.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). Children's mathematics: Cognitively guided instruction - videotape logs [supplementary material]. In *Children's mathematics: Cognitively guided instruction*. Heinemann, in association with The National Council of Teachers of Mathematics, Inc.

Carspecken, P. F. (1995). *Critical ethnography in educational research: A theoretical and practical guide*. Routledge.

Carspecken, P. F. (1999). *Four scenes for posing the question of meaning and other essays in critical philosophy and critical methodology* (New edition). Peter Lang Inc., International Academic Publishers.

Carspecken, P. F. (2006). Limits of knowledge in the physical sciences. In K. Tobin & J. Kincheloe (Eds.), *Doing educational research* (pp. 405–437, Vol. 1). Sense Publishers.

Carspecken, P. F. (2009). Limits to knowledge with existential significance: An outline for the exploration of post-secular spirituality of relevance to qualitative research. In *Bridging the gap between theory and practice in educational research* (pp. 47–61). Springer.

Carspecken, P. F. (2016). Limits to knowledge and being human: What is 'critical' in critical social research and theory? In P. H. Hinchey (Ed.), *A critical action research reader* (pp. 65–86). Peter Lang Inc., International Academic Publishers.

Carspecken, P. F. (2018). The missing infinite. In R. Winkle-Wagner, J. Lee-Johnson, & T. Gaskew (Eds.), *Critical theory and qualitative data analysis in education* (p. 32). Routledge. https://doi.org/10.4324/9781315158860-2

Carspecken, P. F., & Zhang, R. (2013). Content inference fields in intersubjective space: Transpersonal position, illocution, and logic in the analysis of human interactions. *Qualitative research: A reader in philosophy, core concepts, and practice*, 201–242.

Carspecken, R. (2024, November 3). Untitled.

Caygill, H. (2004). *A kant dictionary*. Wiley-Blackwell.

Derrida, J. (1973). *Speech and phenomena, and other essays on husserl's theory of signs*. Northwestern University Press.

Derrida, J. (1982). *Marges de la philosophie* [Margins of philosophy] (A. Bass, Trans.). The Harvester Press. (Original work published 1972)

Deutsch, H., Marshall, O., & Irvine, A. D. (2025). Russell's Paradox. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford encyclopedia of philosophy* (Spring 2025). Metaphysics Research Lab, Stanford University.

Ernest, P. (2024). The mathematical and philosophical significance of zero: Nought matters [Publisher: Paul Ernest, Emeritus Professor at University of Exeter, School of Education & Lifelong Learning]. *Philosophy of Mathematics Education Journal*, 1–35. https://research.ebsco.com/linkprocessor/plink?id=b685ecb9-d2fc-3e03-889d-a06fb1751c1b

Euclid. (2007, December). *Euclid's Elements of Geometry* (J. L. Heiberg, Ed.; R. Fitzpatrick, Trans.; Revised and corrected edition). Richard Fitzpatrick.

Franzén, T. (2004, September). *Inexhaustibility: A Non-Exhaustive Treatment: Lecture Notes in Logic 16* (1st edition). A K Peters/CRC Press.

Franzén, T. (2005). *Gödel's theorem: An incomplete guide to its use and abuse*. AK Peters/CRC Press.

Frege, G. (1997, July). *The Frege Reader* (M. Beaney, Ed.; 1st edition). Blackwell Publishing.

Gaifman, H. (2005). *The easy way to Gödel's proof and related matters* [Unpublished manuscript. Available online].

Geisel, T. (1961). *The Sneetches and Other Stories* (First edition.). Random House Books for Young Readers.

Gelder, T. V. (1995). What might cognition be, if not computation? *The Journal of Philosophy*, *92*, 345–381. Retrieved April 13, 2025, from http://www.jstor.org/stable/2941061

Gray, L., & Garzon, M. (2005). On a hermite boundary integral approximation. *Computers & structures*, *83*(10-11), 889–894.

Greenblatt, R., Denison, C., Wright, B., Roger, F., MacDiarmid, M., Marks, S., Treutlein, J., Belonax, T., Chen, J., Duvenaud, D., Khan, A., Michael, J., Mindermann, S., Perez, E., Petrini, L., Uesato, J., Kaplan, J., Shlegeris, B., Bowman, S. R., & Hubinger, E. (2024, December 23). Alignment faking in large language models. https://doi.org/10.48550/arXiv.2412.14093

Habermas, J. (1985). *The theory of communicative action: Lifeworld and system* (Vol. 2). Beacon Press.

Habermas, J. (2004). *The philosophical discourse of modernity: Twelve lectures* (F. G. Lawrence, Trans.). MIT Press.

Habermas, J. (1971). *Knowledge and human interests*. Beacon Press.

Habermas, J. (1984). *The theory of communicative action, volume 1: Reason and the rationalization of society* (T. McCarthy, Trans.; Reprint edition). Beacon Press.

Habermas, J. (1995). *Postmetaphysical thinking: Between metaphysics and the critique of reason* (1st edition). Polity.

Hackenberg, A. (2025). *Course notes* [Unpublished course notes].

Hegel, G. W. F. (1977). *Phenomenology of spirit* (A. V. Miller, Trans.; Vol. 632). Oxford University Press Oxford.

Hegel, G. W. F. (2010). *The science of logic* (G. di Giovanni, Ed. & Trans.). Cambridge University Press.

Houlgate, S. (2006). *The opening of hegel's logic: From being to infinity*. Purdue University Press.

Hyppolite, J. (1974). *Genesis and structure of hegel's phenomenology of spirit*. Northwestern University Press.

Kant, I. (1998). *Critique of pure reason* [Original work published 1781/1787, Guyer and Wood, Trans.]. Cambridge University Press.

Lakoff, G., & Johnson, M. (1980). *Metaphors we live by*. University of Chicago Press.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books.

Linnebo, Ø. (2024). Platonism in the Philosophy of Mathematics. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford encyclopedia of philosophy* (Summer 2024). Metaphysics Research Lab, Stanford University.

Longuenesse, B. (2017). *I, me, mine: Back to kant and fichte on the 'i'*. Oxford University Press.

Mead, G. H. (1934). *The social psychology of george herbert mead* (A. L. Strauss, Ed.). University of Chicago Press.

Mead, G. H. (2015). *Mind, self, and society: The definitive edition* (H. Joas, D. R. Huebner, & C. W. Morris, Eds.; Enlarged edition). University of Chicago Press.

Meyer, J. (title). *Cantors 1891 diagonal proof* [Cantor's original 1891 diagonal proof]. Retrieved July 6, 2025, from https://jamesrmeyer.com/infinite/cantors-original-1891-proof

Nagel, E., & Newman, J. R. (2012). *Godel's proof*. Routledge.

Negarestani, R. (2018). *Intelligence and Spirit*. Urbanomic/Sequence Press.

Poston, T., & Stewart, I. (2012, February 29). *Catastrophe theory and its applications* (First Edition). Dover Publications.

Priest, G. (2002). *Beyond the limits of thought*. Oxford University Press.

Priest, G. (2014). *One: Being an investigation into the unity of reality and of its parts, including the singular object which is nothingness*. Oxford University Press.

Quine, W. V. O. (1948). On what there is. In *Thinking about logic* (pp. 189–209). Catholic University of America. https://sites.pitt.edu/~rbrandom/Courses/2023%20Sellars/Sellars%20texts/Quine%20On_What_There_Is.pdf

Quine, W. V. O. (1963). Two dogmas of empiricism [First Harper Torchbook edition, a reprint of the second, revised edition (1961). The book was first published by Harvard University Press in 1953.]. In *From a logical point of view* (Second, revised, pp. 20–46). Harper & Row.

Rödl, S. (2014). Intentional transaction [Accessed: 2025-04-13]. *Philosophical Explorations: An International Journal for the Philosophy of Mind and Action*, *17*, 304–316. https://doi.org/10.1080/13869795.2014.941909

Rose, G. (2009). *Hegel contra sociology*. Verso.

Rowlands, S. (2005). The contradictions in the constructivist discourse [Citing von Glasersfeld]. *Philosophy of Mathematics Education Journal*, *14*.

Savich, T. M. (2020). Towards a critical mathematics [Accessed via ERIC]. *Mathematics Teaching–Research Journal Online*, *12*, 245–291. https://files.eric.ed.gov/fulltext/EJ1384454.pdf

Savich, T. M. (2022). *Towards a critical mathematics*. Indiana University.

Savich, T. M. (2024). Built to break: An introduction to theoretical autoethnography [Invited talk presented at Solent University, organized by Dr. Brian McDonough]. https://iu.mediaspace.kaltura.com/media/t/1_v1qmm3g9

Savich, T. M. (title). *The hermeneutic calculator* [The hermeneutic calculator]. Retrieved July 23, 2025, from https://tiosavich.github.io/UMEDCTA/Calculator/index.html

Sellars, W. (1953). Inference and meaning. *Mind*, *62*, 313–338.

Sellars, W. (2007). *In the space of reasons: Selected essays of wilfrid sellars*. Harvard University Press.

Steffe, L. P. (2002). A new hypothesis concerning children's fractional knowledge [Accessed through CEPA.INFO]. *Journal of Mathematical Behavior*, *20*(3), 267–307.

Steffe, L. P. (2003). On the construction of learning trajectories of children: The case of commensurate fractions [Accessed via ResearchGate/ERIC]. *Monographs of the Journal for Research in Mathematics Education*, 1–26.

Steffe, L. P. (2004). Fractional commensurate, composition, and adding schemes: Learning trajectories of jason and laura: Grade 5 [Accessed via ResearchGate]. In J. Olive & R. Tzur (Eds.), *Children's fractional knowledge* (pp. 1–60). American Educational Research Association.

Steffe, L. P. (2014). *Children's number sequences: An explanation of steffe's constructs and an extrapolation to rational numbers of arithmetic* [Originally from 1994]. Unpublished manuscript, available via CiteSeerX; ResearchGate.

Steffe, L. P., & Thompson, P. W. (2014). Teaching experiment methodology: Underlying principles and essential elements [Accessed via ResearchGate, originally published 2000]. *Proceedings of the 24th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*.

Stone, J. (2003). *The monster at the end of this book* (2nd ed. edition). Golden Books.

Taylor, C. (1994). *Multiculturalism: Examining the Politics of Recognition* (A. Gutmann, Ed.) [Essays by Charles Taylor, K. Anthony Appiah, Jürgen Habermas, Steven C. Rockefeller, Michael Walzer, and Susan Wolf]. Princeton University Press.

van Hiele-Geldof, D., & van Hiele, P. M. (1984). *English translation of selected writings of dina van hiele-geldof and pierre m. van hiele* (D. Fuys, D. Geddes, & D. Tischler, Eds.) [ERIC Number: ED287697]. Retrieved June 3, 2025, from https://eric.ed.gov/?id=ED287697

Wittgenstein, L. (1956). *Remarks on the foundations of mathematics.*

Wittgenstein, L. (2010). *Philosophical investigations* (P. M. S. Hacker & J. Schulte, Eds.; G. Anscombe, Trans.). Wiley-Blackwell.

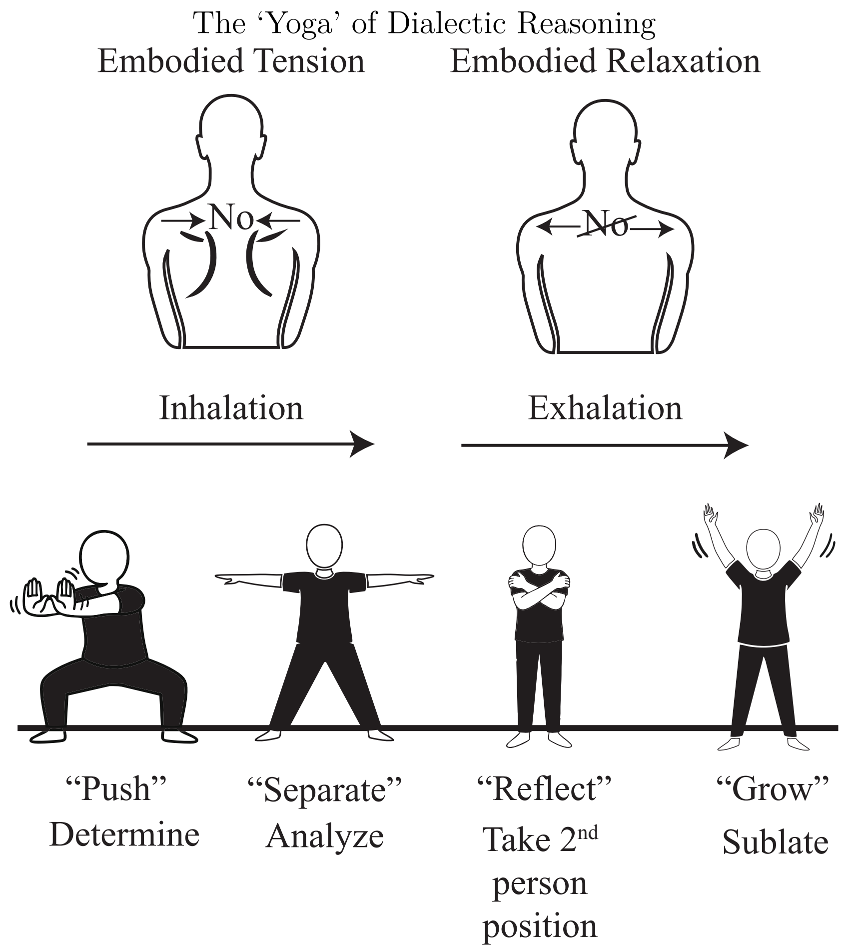


Figure 1.1: *Note.* The body feels the rhythms of determinate negation—the self-determining concept. The body-feelings of tension and relaxation that accompany determinate negation are important for understanding dialectical reasoning. The poses are metaphorical cartoons, designed to communicate complex Hegelian concepts for young readers.

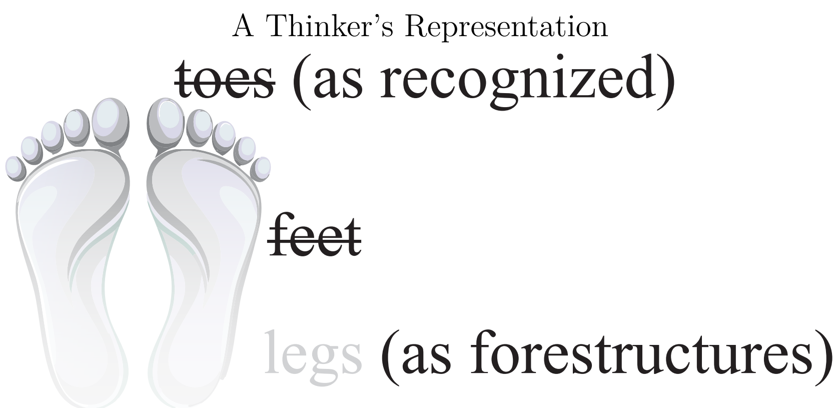


Figure 1.2: *Note.* The body as an accumulation of sublated regions, moving from the explicit focus (toes) to the implicit background (legs as forestructure).

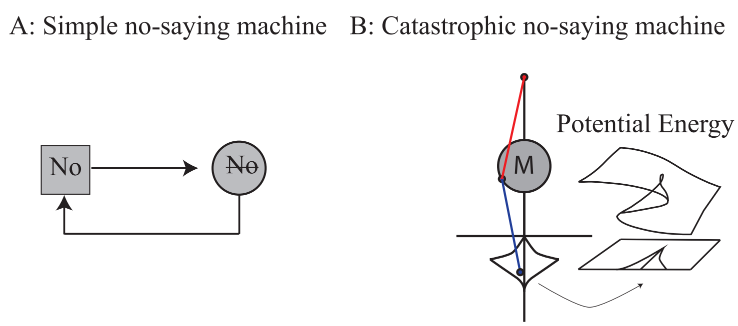


Figure 1.3: *Note.* A sketch of the *ZCM*. Tension (potential energy) builds until a critical point, where the system snaps into a new state.

The Sound of Time

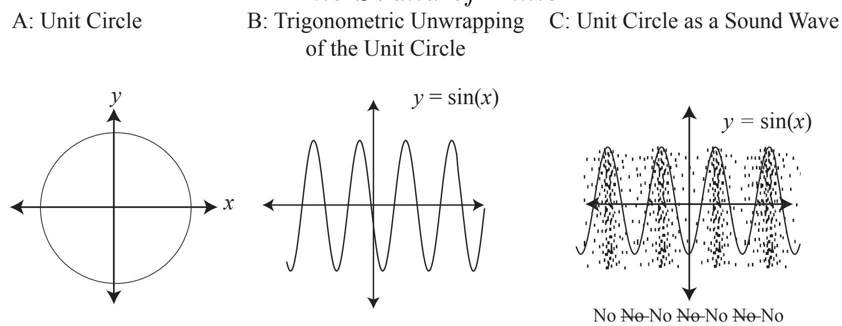


Figure 1.4: *Note.* Circular, unified experience (A) is "unrolled" through time into a linear, oscillating wave (B). This wave, like sound, alternates between compression ("No") and decompression ("~~No~~"), representing the rhythm of determinate negation (C).

Deictic Gestures

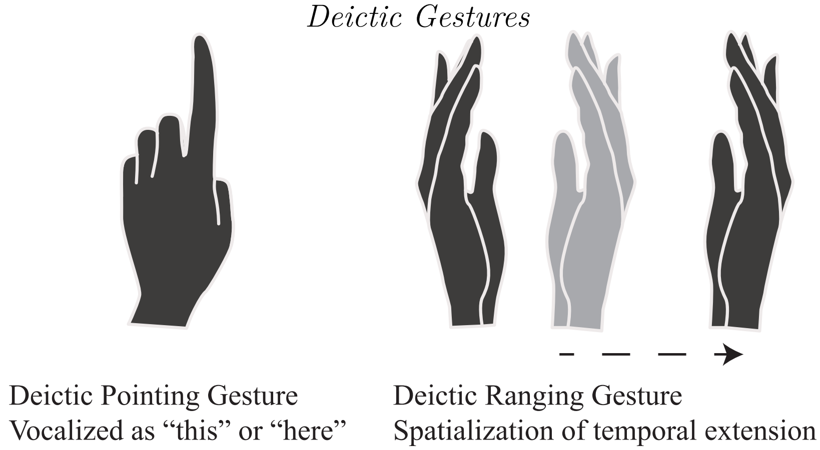


Figure 1.5: *Note.* Deictic gestures that are important for teaching mathematics as they arise in The Exercise.

Chapter 2

Inferential Movement

Abstract

This chapter examines the ethics of inferential movement, drawing on Robert Brandom's inferentialism. It argues that meaning arises not from pre-given objects but from their role in inferences. The chapter explores how concepts gain content through relations of compatibility and incompatibility, using quadrilateral classification as a case study. This analysis clarifies the distinction between formal and material inferences, emphasizing the importance of material inferences in pedagogical practice. The chapter extends Brandom's account of the experience of error to misrecognition within interpersonal interactions, connecting the inferential domain to the ethical. It also connects geometric figures to the "I think," foreshadowing the later argument that numerals function as pronouns. Through exploring Brandom's concept of inferential strength and the inversion of polarity by logical operators, the chapter demonstrates how meaning emerges from embodied practices and social norms. The quadrilateral example illustrates the modal structure of knowledge, the experience of error, and the interplay of necessity and possibility in conceptual understanding. This chapter ultimately argues for an inferential approach to meaning, laying the groundwork for understanding the embodied and ethical dimensions of knowledge explored in subsequent chapters.

2.1 Introduction

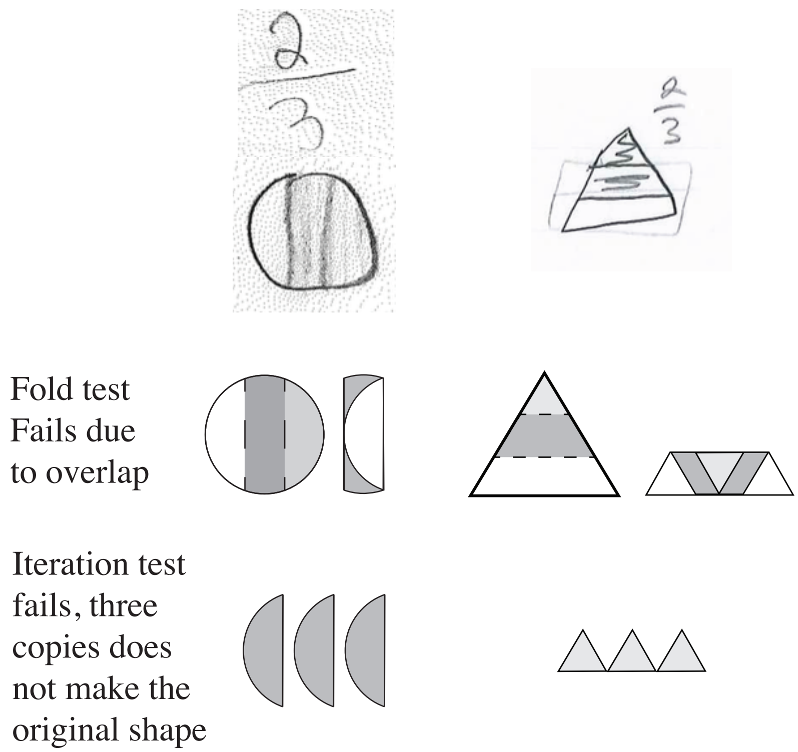
In this chapter, I continue to explore ontology (the logic of being) that I began in the previous chapter. However, I now focus on the normative *propriety* of inferential movements, capitalizing on Robert Brandom's work in the philosophy of language, specifically chapter 4 of *Articulating Reasons* (2000). The guiding question is "What are mathematical beings?" That is, what grammatical role do mathematical beings play within a philosophy of language?

Motivating the Question with Hybridized Models

To make this question compelling, consider the student work samples (SWS) in Figure 2.1. While I was a graduate student, I worked on a research project with Erik Jacobson. We collected a large number of SWS from fourth and fifth graders. I was tasked with analyzing these samples and writing up what the students were doing as high-level algorithms (pseudo-code). There were about 50 samples that could not be easily reconstructed as algorithms. At first, we binned those samples because they were not statistically significant, but I was taking an Arts-Based Educational Research course with Gus Weltsek and so I decided to try and interpret those binned samples as if they were artistic expressions. I was also taking a course on critical theory at the time and reading about *meaning fields* (P. F. Carspecken, 1995). I felt drawn to these oddities, as the kids behind the work felt like kindred spirits.

First, let me establish that the fractions in figure 2.1 are not fractions. The first sample fails a measurement test: folding the circle along the vertical bars yields a figure with overlaps. That indicates that the parts are not equivalent in area. The second test they fail is that iterating one part three times does not return the unit. I call these representations *hybridized models* because they take a respectable unit, like a circle, and a respectable partitioning scheme, like equi-distant vertical bars, but combine them in a way that does not yield a fraction. While not statistically significant, there were several that appeared to follow a pattern (see figure 2.2).

While I was not sure exactly what the pattern was, I felt a weight. How many ways are there to draw a fraction? Textbooks in teacher education often present circular models, rectangular models, and set models. Different partitioning rules apply to each of the models. I reasoned that a teacher might think they are adding complexity linearly—introducing one model at a



Hybridized Not Fractions

Figure 2.1: *Note.* Two student work samples that purport to illustrate the fraction $\frac{2}{3}$ but that fail two tests.

time—may, from a student's perspective, be adding complexity exponentially. That is, the complexity function for a teacher may be $C = n$, while the student may experience it as $C = 2^n$ (loosely). I represent this relationship in figure 2.3.

I call this loosely exponential relationship the *phenomenology of confusion.* When I feel confused, it is often due to a lack of *information*—the negation of possibilities. While everyone else seems to be dealing with n possibilities, I am tangling with 2^n possibilities. But, of myself, I know that simply telling me "no, you can't mix and match" (providing that information) does not dissipate confusion. I have to understand why good premises lead to bad conclusions. Furthermore, simply naming an inference fallacious, saying "Tio, you're making a category error," for example, does not help me untangle my confusion. Mixing metaphors or categories sometimes manifests what is lauded as "creativity," so it cannot be universally improper. I will return to these hybridized representations later in this chapter to argue that the units drawn as circles, rectangles, squares, etc., do not follow the rules associated with *singular terms*. Instead, I will argue, such units are *anaphoric terms*—pronouns—that recollect the {I think}. This claim is easily misunderstood as a psychologism, but I will not clarify why it is not until chapter 3.

Referentialism to Inferentialism

In chapter 4, Brandom draws on Kant and Frege to invert the common-sense framing that *singular terms* (names like "Benjamin Franklin" or definite descriptions like "the inventor of bifocals") are expressions that refer to particular objects (2000). In the common sense approach, their purpose is to enable discourse about the objects that make up the world. Inverting that common-sense understanding reverses this order of explanation. Rather than defining singular terms by appealing to a given notion of "objects," Frege suggests that "objects" can be defined as that which is referred to by singular terms. If successful, this inversion partially liberates objectivity from the obduracy of a *given* object. However, Frege's *referentialist* project has some downsides in the domain of math education. The problem is with truth. For Hegel, "The True is thus the Bacchanalian revel in which no member is not drunk; yet because each member collapses as soon as he drops out, the revel is just as much transparent and simple repose" (Hegel, 1977). For Hegel, truth is not a static property but a dynamic, self-correcting process that unfolds through history and experience. A falsity isn't a dead end; it is a necessary step on the way to a more complete understanding. But for Frege, "Every declarative (assertoric) sentence concerned with the referents of its words is therefore to be regarded as a proper name, and its referent, if it exists, is either the true or the false" (1997). Frege's goal was to create a formal language free of ambiguity, and so he conceptualized truth as static and bivalent—a timeless *property* of thought. So, what is the true Truth?

I recently did an ice-breaker with some pre-service teachers. I asked them to write something unique about themselves on a sheet of paper, crumple it up, and throw it into the middle of the room. We each took a 'snowball' and read its declarative sentence aloud. While one student spoke something true of themselves, everyone else uttered a false sentence. "I am a triplet" is untrue of me, though I read it aloud. This example does not function as a defeasor of Frege's project, as he would say that the indexical "I" makes each repetition of the sentence a brand new sentence. Still, Hegel's interpretation of truth is capacious enough to *take* a students' expression as contextually true—a truth on its way to falsity or a falsity on its way to truth—which resonates with a developmental trajectory for mathematics education. Frege's interpretation bins an awful lot of good thinking as merely false. However, that does not mean I should bin Frege! His distinction between the sense of a term and its referent is very useful (1997). Habermas took Frege's insights and used them to articulate his formal pragmatics, which I synthesize with Brandom to articulate tri-modal conceptual realism. "I am a triplet" can be analyzed with respect to my sincerity, the normative goodness of the claim, or the objective (shared access) claim about how many children were born to my mom during the same pregnancy. Brandom's project transforms Frege's insights from a referentialist account of truth to an *inferentialist* account. The inversion allows objects to defined through the act of classifying the inferences that can be made about them, rather than by their existence in a world that is simply given. More complete liberation from the 'myth of the given' is pursued in Brandom's later work that rationally reconstructs the Perception chapter of Hegel's *Phenomenology of Spirit* (2019, Ch. 5).

In that later work, the cut is made between particular objects, referred to by singular terms, and *universals*, which are repeatable properties presented as predicates that serve as proto-concepts. The problem that Brandom identifies Hegel as working through is how concepts have *determinate content* (Brandom, 2019). For any concept, like *square*, to be determinate, it must stand in two kinds of difference relations. First, it must be *incompatibly* different from other concepts it rules out (like *triangle* or *circle*). This is the relation of *determinate negation*. Second, it must be merely or *compatibly* different from other concepts it can coexist with (like *red* or *large*). In fact, ten different types of difference shall emerge from exploring geometric objects and their properties, tracking the differences that Brandom articulates in that later work. Another difference is between *formal* and *material* inferences. Many (perhaps most) of my colleagues in math education who do teacher training implicitly understand that the material rules of inference that govern school mathematics must be developed before the formal rules of inference can be taught. However, the political bodies and the teachers-in-training that govern how math is taught or who plan on implementing that math in schools often take only those formal rules to be legitimate. By demonstrating how formal rules of inference can be derived from material ones, readers are invited to dissent from participation in the so-called 'math wars.' Neither formality nor materiality is 'bad,' but starting with formality does not make pedagogical sense. Falling downhill may result in some bruising, but falling uphill is impossible. In the exploration of quadrilaterals developed here, the traditional hierarchies of quadrilaterals give way to a fractal-like structure. That traditional activity is extended in three directions.

First, a material account is offered for the inferences involved in classifying quadrilaterals. Part of Brandom's earlier argument is challenging, specifically his concept of *inferential strength.* The material account is partially formalized to explain how quantifying *inferential strength* makes some technical details in Brandom's argument more explicit. This move actualizes the potential expressed in Brandom's claim that "*formal* proprieties of inference essentially involving logical vocabulary derive from and must be explained in terms of *material* proprieties of inference essentially involving nonlogical vocabulary rather than the other way around" (2000). The second direction extends Brandom's articulation of the *experience of error*, which focuses on subject-object relations, toward the *experience of misrecognition* that foregrounds a subject-subject relationship. Concepts include the history of their development, like the recollective "me" stands to the {I}. Those developmental histories include experiences of error/misrecognition and the ways those errors are corrected. Using the concept *square* includes the learning experiences associated with squares. Moving toward a subject-subject account of error resolves into the third direction pursued here, which is distinct from Brandom's work. When Ramanujan claimed that all numbers were his friends, he was articulating a subject-subject relationship with numbers, rather than the standard subject-object relationship. While squares do not make or defend identity claims, one can take their position as a friend to 'help' them express the negative dimension of their identity. They repel claims like "No sides of X are equal." Just as a vegan friend is honored by not serving butter and only serving plant-based foods, friendship with the square is achieved by knowing both what it is and what it is not. A more complete discussion of *apperception* is deferred to the next chapter, but the sound of time metaphor that precedes this chapter is operating here, too. In that chapter, I claimed that spatial thought arises under temporal compression. That is to say that geometric figures, like a square, are treated as anaphoric recollections of the "I think." This probably will not make sense until Chapter 7, where numerals are argued to be pronouns that anaphorically recollect the "I think." This seed is planted here without full justification. By developing the concept of a square so that it includes the history of its development, I approach geometry through apperceptive self-consciousness. When construed as a friend, the concept is like a You to the {I} who thinks it. That You can be respected and honored as an autonomous subject—someone that can be learned about. That said, the I-feeling, explored in the previous chapter, can *fuse* with the concept when one of the ten differences is recognized. An account of inference chains (*Algorithms*) is given that relates to embodied experiences of flow, with various *moments* of the concept metaphorically represented as poses that punctuate yogic flow. This chapter will proceed in three stages. First, the core concepts of Brandom's inferentialism are unpacked, distinguishing material from formal inference and symmetric from asymmetric substitution. Second, the classification of quadrilaterals is introduced as a running case study to make these ideas concrete. Finally, this case study demonstrates Brandom's most challenging point: how logical operators like negation invert inferential polarity, and what this reveals about the structure of the concepts at issue. This chapter is dense. Each of the differences has to be articulated with care. That care bears an unfortunate burden of jargon. Technical terms are introduced gradually; readers are encouraged to approach them with ease. Pounding your head against the chapter will not help. Instead, try to find some spirit of play in the various differences. Doing so may enhance the degree to which the I-feeling fuses with their articulation. Despite the chapter's density, the aim is to gesture toward the pleasure many associate with mathematics. When the difference between two moments of the concept is null, the I-feeling might fuse with the concept. A deep pleasure may accompany that fusion.

2.2 Brandom's Early Inferentialism

Kant noted that judgement is the "fundamental unit of awareness or cognition, the minimum graspable" (Brandom, 2000). For Kant, the "I think" must be able to accompany all of my representations, but this transcendental "I think" enables the representation; it cannot be included in that representation. In this context, "accompany" can be understood as a kind of implicit harmony, an unheard Voice explored in **??**. This necessarily implicit enabling condition can be foregrounded in the assertoric representation of judgment. “Pokey is a dog” is not equivalent to “$\emptyset_{\text{I think}}$ Pokey is a dog,” as the implicit “I think” is discerned by reflecting on the original judgment. The previous representation can also be thought, allowing the judgment to be re-recollected again and again—a strategy used to define the progressive aspects of counting in section 7.1. Brandom picks Kant’s thought up in the philosophy of language by limiting his analysis to sentences. He argues that sentences can perform speech acts; speakers can be held to account for the truth or goodness of such speech in ways not available for isolated subsentential expressions like {Benjamin Franklin, ambassador, Pokey, is, dog}. Habermas makes a similar point. While a subsentential expression like a disapproving grunt might be recognized as a speech act in some context, in other contexts such a grunt may not be clear. When asked, “what did that grunt mean,” I can put it into assertoric form: “I do not like how you are treating me.” Unclear speech can be reconstructed as assertions. Of course, those reconstructions introduce new possibilities for error.

Brandom is concerned with how those subsentential expressions can be combined in predictable ways to form novel sentences that other people somehow understand. He names this ability *inferential projection*, noting that “almost every sentence uttered by an adult native speaker is being uttered for the first time—not just the first time for that speaker, but the first time in human history” (2000, pp. 126–127). *Substitution* is the mechanism that he suggests undergirds this ability. Substitution can be formal, as in Russel and Whitehead’s *Principia Mathematica* (Nagel & Newman, 2012), or *in*formal (i.e., *material*). Brandom proposes a two-part definition for singular terms, considering the syntactic form and semantic content or significance of those roles they play in assertoric judgments. Both roles are understood through substitution. From there, he conducts an “expressive deduction” to explain why a language capable of using *conditionals* (“if... then” statements) or (logical) negation must include singular terms that can be substituted for one another in symmetric ways. For readers unfamiliar with Kant’s transcendental arguments and how they relate to critical theory, the argument Brandom makes is challenging to follow. Kant instituted a new approach to philosophy to address the problem of skepticism. Descartes meditations deployed a radical form of doubt that questioned the foundations of knowledge. He resolved that doubt by noting that he could not doubt that he was doubting, expressing that conclusion in the famous dictum *cogito ergo sum*, translated as “I think therefore I am.”

The nature of that {I} who doubts and the manner of the “ergo”—which does not follow a formal inferential structure—were explored by Descartes in his meditations, but many readers did not follow the positive implications he drew from that indubitable doubt (the existence of a body and a world of objects). Those positive implications fell, for some, to the method of radical doubt. The paradigm shift that Kant instituted in his *Critique of Pure Reason*—the title of which informed the founders of the Frankfurt School as they articulated various *critical theories* (P. F. Carspecken, 2016)—moves from radical doubt to examining what enables doubt. The transcendental categories arose as the enabling conditions for the kinds of doubts that Descartes expressed in his meditations. This method is explored further in the next chapter.

For Brandom, the logical expressions he analyzes are transcendental-like. It is possible to imagine a language that does not have those logical expressions, but "having to do without logical expressions would impoverish linguistic practice in fundamental ways" (2000). One might go a bit farther and say that contemplating a language bereft of those expressions is contradictory, as it embeds the conditional-free language inside a conditional. "If our language does not have conditionals, then..." sounds a lot like 'using the master's hammer to break the master's hammer.'[1]

The argument is not that singular terms are necessary for the use of conditionals or negation, but that the ability to substitute one singular term for another in symmetric ways is necessary for the use of conditionals and negation. This is a subtle but important distinction, as it allows a contrast between mathematical and singular terms. Later chapters use this argument to determine that *numerals* play an *anaphoric* role in speech—functioning as 'pronouns,' not singular terms. This chapter classifies quadrilaterals both to clarify Brandom's argument and to set the stage for that argument. He first discusses the syntactic form of assertions, discerning three possible roles that singular terms could play. They could be *substituted-for*, in the sense that they could be the component expression that is replaced in a substitution

[1] There is a technical problem of what constitutes a *language*. Brandom (2008) discusses formal languages that can be written and read by various automatons, like the laughing santa automaton that only writes expressions like "hohohahaho." He claims that such languages do not rise to the level of *autonomous discursive practices*, languages that can be used to make assertions, hold one another to account, and so on. When we think of languages like French or Spanish, those are autonomous discursive practices. Such languages include conditionals and negations, while the laughing santa's language need not. Whether languages like Orca include the conditional is an open question.

inference. Or they could be *substituted-in*, in the sense that they could be the compound expression in which the substitution occurs (e.g., the full sentence). Or they could play a derived role he calls a *sentence frame*, that serve as *predicates.* The following sections carefully work through how substitution allows those categories to be discerned. The discussion will carefully work through how substitution allows those categories to be discerned. The syntactic categories relate to the *form* of the sentence, while the second part of his definition relates to the semantics or *content* of those roles. He notes that some of them have *symmetric inferential significance* while others have *a*symmetric significance. Noting the *a*symmetric significance of some terms, Brandom then argues that a language that can use *conditionals* ("if... then", $\models_I$) or logical *negation* ($\neg$) must support symmetric substitution of singular terms. He then explains that the ability to substitute one singular term for another in symmetric ways is necessary for the use of conditionals and negation. These logical sentence frames *invert inferential polarity* in a way that risks the ability to coherently project inferential consequences.

2.3 Systematic Analysis: Material Inference, Substitution, and Well-Formedness

Formal mathematical systems are usually defined by a fixed set of axioms and rules of inference. The axioms are statements that are taken to be true without proof, and the rules of inference are the logical steps that allow us to derive new statements from the axioms. In this sense, formal systems are built on a foundation of *formal inferences*—inferences that follow purely from the syntactic structure of the system. These axioms and inference rules are like the bricks and architectural plans of a castle whose walls keep the hordes of logical contradiction at bay. However, this rigidity means the system itself is static; it cannot learn or adapt. An 'error' within a proof is simply a deviation to be discarded, not an experience from which the system itself can grow. Foundational challenges to the system, such as paradoxes or incompleteness, are not resolved by the system adapting, but by mathematicians abandoning it and building a new one. Retention of errors within a system of critical mathematics is discussed in **??**. Until then, the focus is on the importance of the experience of error in mathematics education. However, formal inferences are not the only kind of inference that we make. In fact, most of our everyday reasoning relies on what Wilfrid Sellars (1953) called *material inferences*. A material inference is an inferential step that is licensed not by pure logical form (like *modus ponens* or other formal schemata). Instead, the virtue of being taken as a good inference by a recognitive community lends content to the concepts involved in the inference. It may take a few passes at the previous sentence to understand how different material inferences are from formal inferences. By taking a material inference to be good, the terms involved accrue conceptual content. The meaning of *goodness* in this context will be discussed below; importantly, 'taking as good' is not meant to be read subjunctively. The hypotheticals involved in subjunctive reasoning—if x were good, then y—codify temporally antecedent material inferences. These are inferences that are almost spontaneously agreed to or denied. For example, consider the inference: 'Pokey is a dog so Pokey is a mammal' or "It's raining; therefore, the streets will be wet" (Sellars, 1953). This is not formally valid in pure logic (nothing in the form "P; therefore Q" guarantees truth), yet it is a perfectly good inference in practice given our knowledge about rain and wet streets, or dogs and mammals. Such an inference is widely treated as reasonable, even obvious—enough to carry a kind of normative force in ordinary discourse: if someone says "it's raining" but refuses to accept "the streets will be wet," that refusal typically signals an incomplete grasp of what "raining" means. Perhaps they do and are being sarcastic. In Carspecken's critical ethnographic methodology (1995), meaning is structured as a field of next possible speech acts. A qualitative researcher might hear a participant state "It is raining." Upon analysis, the sentence "the streets will be wet" is reconstructed as a next possible act along with others like "I will bring an umbrella," or, if spoken sarcastically, "it is *not* raining." Analyzing the meaning field produces a horizonal structure for what the assertion could mean. Due to the horizonal structure of possible next-acts, there is no way to say with certainty what an assertion means—especially in everyday empirical speech. That is one reason why I will spend much of the chapter discussing how quadrilaterals are classified, as it offers a relatively regimented set of material consequences. In that regimented context, the speaker is obliged to give some justification for their refusal to accept the consequent or they risk being taken as someone who is playing by a different set of rules. This articulates Sellars's understanding, picked up and extensively developed by Brandom: *material inferences* are those inferences whose normative goodness or acceptability lends content to the terms caught up in them. For Brandom (2000), the content of a concept is given by the network of material inferences it participates in. To know what *Dog* means is to know that "*Dog*(*Pokey*)" licenses "*Mammal*(*Pokey*)," and that it is incompatible with "*Reptile*(*Pokey*)". The web of what follows from, and what is ruled out by, a claim is what gives it meaning. In this way, meaning is not a static mapping from word to world, but the dynamic ability to move in the space of reasons. A few features of material inferences are worth noting. First, they are often (though not always) *non-monotonic* (Brandom, 2000). This means that adding new information can flip a good inference to a bad one *and* turn a bad inference into a good one. Formally, adding a premise to the antecedent of an inference doesn't turn a bad inference into a good one. The conjunction operator restricts what sort of consequents can be drawn, it does not open new possibilities. Non-monotonic material inferences are different.

For example, I was driving $\mathcal{M}$ and $\exists$ to their Aunt Rachel and Uncle Phil's house (my sister and her husband, not Phil Carspecken). One of them said "We're going to Rachel's!" The other one corrected her and said "We're going to Rachel *and* Phil's house." I usually ignore this sort of correction as if it represents a quixotic desire for completeness, but I was writing about material inferences right before we got in the car. Rachel can't make balloon animals, while Phil can. So an inference "We're going to Rachel's so you might get a balloon animal" might have felt like a bad inference to the corrector, while the corrected premise "We're going to Rachel *and* Phil's so you might get a balloon animal" feels better. What I think matters here is that adding Phil adds to the field of possibilities that the antecedent of the material inference holds. Implicitly, those possibilities are contained within the truncated antecedent, "We're going to Rachel's," but explicating them by adding Phil brings them closer to actualization. This contrasts with formal inferences, which are generally monotonic. If the inference $P \rightarrow Q$ is formally true, then (assuming R is true) the inference $P \wedge R \rightarrow Q$ will still be true because P is a sufficient ground for Q. Brandom (2000) provides the following example to illustrate nonmonotonicity (using $\neg$ for negation, rather than ~–from *Articulating Reasons*—or *N*—from *Between Saying and Doing* 2008 which is the specifically modal version of negation):

1. If I strike this dry, well-made match, then it will light. $(p \rightarrow q)$

2. If p and the match is in a very strong electromagnetic field, then it will *not* light. $(p \wedge r \rightarrow \neg q)$

3. If p and r and the match is in a Faraday cage, then it will light. ($p \land r \land s \to q$)
4. If p and r and s and the room is evacuated of oxygen, then it will *not* light. ($p \land r \land s \land t \to \neg q$)

A philosophy of mathematics that is suitable for educational contexts must include nonmonotonic material inferences. Recall figure 2 that shows an instance of confusion about whether the orientation of the cartesian plane mattered when computing slope. "Just remember rise over run" unless the paper is upsidedown, unless the paper is held up to a strong light, unless the paper is rotated 90 degrees etc. Each additional antecedent can change a materially good inference into a materially bad one. The distinction between material and formal inference matters greatly in early reasoning, even if comfort with formal systems comes later. Second, material inferences are often *context-sensitive*. The same statement can have different meanings in different contexts. For example, when reading philosophy, the term "practical" generally involves ethical or moral reasoning. In other contexts, that term implies that a proposed solution to a problem is viable or efficient. In mathematics, the term "practical" might refer to a solution that can be computed in a reasonable amount of time or with available resources. The context in which a statement is made can significantly affect the material inferences we draw from it. This context-sensitivity is crucial for understanding how mathematical concepts are applied in real-world situations. Finally, material inferences often support a *bi-modal* (Brandom, 2019, pp. 73–74) reading. They can express both objective validity claims in the alethic modality of 'possibility and natural necessity' and subjective or normative validity claims in the deontic modality of 'obligation or practical necessity.' The difference between 'practical' and 'natural' necessity is encoded in the difference between "you must brush your teeth now" and "you must have teeth in order to brush them." The former trades in authority, while the latter is a statement of fact. *Incompatibility* (Brandom's term for determinate negation) is amphibious to either modality. It is impossible to brush teeth one does not have, so the state of tooth-brushing is incompatible with the state of not-having-teeth. This duality allows for a richer understanding of how inferences operate within different contexts. This analysis of material inference reveals three key features that distinguish mathematical meaning from formal logical manipulation: defeasibility (inferences can be overridden by additional information), context-sensitivity (meaning depends on situational factors), and bi-modality (inferences operate across both factual and normative domains). These properties explain why mathematical concepts cannot be reduced to formal definitions but must be understood through their patterns of use in concrete contexts. Understanding these features prepares us to examine how syntactic structure constrains the substitutional patterns that enable reliable inferential projection.

Syntactic Well-Formedness and Substitution

Having established the dynamic, context-sensitive nature of material inference, the analysis now turns to the structural constraints that make reliable inferential projection possible. To understand the technical distinction between singular terms and predicates, I begin the analysis with the structure of sentences and the rules for substitution that preserve well-formed sentences. Consider these preliminary sentences about animals:

1. Pokey is a dog.
2. Felix is a cat.
3. Buddy is a dog.
4. Pokey and Buddy are distinct dogs.
5. Slider the Spider only speaks gibberish.

Brandom imagines the substitutional 'machinery' can be opened up full-bore. Can "the" take the place of "is"? "Dog" for "a"? ∃ (my stepdaughter) recently requested that I tell her a story about Slider the Spider who only speaks gibberish. In the context of that request, the answer to those questions is "sure!" "Is a Spider is a dog, is, is, a cat," Slider might say. But mathematics is mostly concerned with well-formed sentences (often called well-formed formulas, or simply WFFs Gaifman, 2005). Filtering out 'gibberish' is complicated, but an intuitive sense for how such a process might begin implicitly involves discerning syntactic categories by examining which substitutions result in sentences that are implicitly recognized as well-formed. For instance, replacing "Pokey" with "Felix" in 1 (1) yields "Felix is a dog," which, while incompatible with (2), is still a syntactically valid sentence. However, consider substitutions that break well-formedness: Here, substituting the predicate "is a cat" for the singular term "Pokey" results in a

Non-Preserving Substitution	Sentence (1):	Pokey		is a dog.
	Non-preserving substitution:		↖is a cat	
	Sentence (2):	Felix		is a cat.
	Non-sentence:	$\boxed{\text{is a cat}}$		is a dog.

Table 2.1: *Note.* Non-preserving substitution

non-sentence: "is a cat is a dog." This fails to preserve structure because it violates syntactic roles. In contrast, consider substitutions that maintain syntactic structure:

Preserving Substitutions	Sentence (1):	$\boxed{\text{Pokey}}$	$\boxed{\text{is a dog}}$.
	Preserving substitution:	Felix ↑	
	Sentence (2):	Felix	is a cat.
	New sentence (1*):	$\boxed{\text{Felix}}$	is a dog.
	Sentence (1):	Pokey	$\boxed{\text{is a dog}}$.
	Preserving substitution:		is a cat ↑
	Sentence (2):	Felix	is a cat.
	New sentence (1**):	Pokey	$\boxed{\text{is a cat}}$.

Table 2.2: *Note.* Preserving substitutions

The original sentences are substituted-in, repeatedly, and the singular terms ({Pokey, Felix, Buddy, Slider}) are substituted-fors. Upon reflecting on many such substitutions, *sentence frames* (predicates) can be discerned as a "substitutional remainder." Those sentence frames, like "□ is a dog" allow for a transition from simple assertions to algebraic formulas: $f_{\text{dog}}(X)$, where X is a variable that can be replaced by any singular term. After discerning those syntactic categories, the subsentential expressions can be classified in terms of their inferential roles. The singular terms are those that can be substituted-for one another in symmetric ways, while the predicates are those that can be substituted-in but may not allow for symmetric substitution.

2.4 Symmetric vs. Asymmetric Substitution: Singular Terms and Predicates

Brandom's argument (2000, Chapter 4) is that subsentential expressions can be classified by the substitution patterns they participate in. In ordinary language, some substitutions work both ways (*symmetric*), while others work only one way (*asymmetric*). I don't know whether a term is a name or a predicate until I reflect on how it can be used. Upon reflection, the behavior of the term in different substitution contexts can be analyzed. This analysis involves positing patterns of use, but further reflection can generate new contexts that break those patterns. The reflection that kicks off analysis tends to reify the term into a lexical 'object,' so that whether it was 'originally' a name or a predicate becomes demonstrably uncertain: "it"—the term as recollected—can hold either role. That said, syntax plays the role of a recognized forestructure when self-certainty is explored through objective validity. Names do not tend to have argument places (*adicities*), while predicates do. But to recognize a term as differentiated from others requires spacing. Derridean insights into the nature of text can worm their way into those spaces to happily munch their way through any fusion between the I-feeling and the algebraic approach undertaken in this chapter. I do not want to misrecognize self-certainty, which is one reason this text is so moth-eaten and ragged. Still, the pragmatist principle that *meaning is use* holds—understanding a speech act consists in knowing how to (appropriately) act next. Sometimes, those next actions are inferential substitutions. Sometimes those next actions are deconstructive. A *singular term* (like a name or definite description) is defined by its role in symmetric substitution. For example, if "*Benjamin Franklin* was an ambassador to France," then "*The inventor of bifocals* was an ambassador to France" and vice versa, because the singular term *Benjamin Franklin* and the definite description *the inventor of bifocals* refer to the same person. It is not necessary to know every possible appropriate (Fregean) sense for that referent in order for the symmetry to hold. Perhaps you did not know that Benjamin Franklin invented bifocals. Accruing elements of the symmetrically intersubstitutable (co-referential) terms is often accompanied by the body-feelings associated with "letting go," described in the previous chapter. "Aha!" you might say, as that great mystery unfolds as the differentiation between "Benjamin Franklin" and "the inventor of bifocals" is relaxed. The experience of undifferentiating between those symmetric terms is one way to interpret the "a-ha!" moment that math teachers and students often experience when two seemingly quite different expressions are taken to be "the same."

When I discovered through well-worn proof that $e^{i\pi}$ and -1 were symmetrically intersubstitutable, the incommensurable difference between the two terms relaxed. That does not mean that I must teach sixth graders about the expression $e^{i\pi}$ when introducing the concept of -1. What I'm trying to get at is the idea that samenesses are not static : they are the relaxation of difference. When that undifferentiation is rationally binding, as it often is through mathematical proof, the obligatory aspect of proof can transform into desire. A *predicate* typically serves as the frame in which singular terms are inserted, but predicates themselves can also be substituted. Crucially, predicates can stand in asymmetric substitution relations. For example, "Pokey is a dog" *incompatibility entails* ($\models_I$) "Pokey is a mammal" but not the other way around (Brandom, 2008). Everything that is incompatible with the assertion "Pokey is a mammal," like "Pokey is a spider," is also incompatible with "Pokey is a dog." The predicate "is a dog" is *inferferentially stronger* than the predicate "is a mammal" precisely because it is more exclusive. Incompatibility entailment is how Brandom defines the conditional ("if... then..."). What happens when the material inference about Pokey is embedded in a conditional statement or a negation? Brandom's argument is that the logical operators *invert* the polarity of the inferences, which is incompatible with the idea that singular terms can be used in asymmetric substitution. Strengthening the antecedent of a conditional, moving from "If Pokey is a Dog then Pokey has four legs" to "If Pokey is a Retriever then Pokey has four legs", makes the resulting conditional easier to satisfy—it is a weaker claim than the original conditional. Alternatively, weakening the antecedent, with a result like "If Pokey is an animal then Pokey has four legs," makes the new conditional harder to satisfy—it is a stronger claim, in this case a false one, than the original conditional. The same line of reasoning works with negation: "Pokey is not a dog" is a weaker claim than "Pokey is not an animal." More possibilities are ruled out by the latter, so it is a stronger claim. The details of Brandom's argument turn schematic at this point, as he has to contemplate a (fantasy) language that has those logical operators but allows substituted-for expressions to have asymmetric inferential significance. He notes that "any language containing a conditional or negation thereby has the expressive resources to formulate, given any sentence frame, a sentence frame that behaves inferentially in a complementary fashion" (Brandom, 2000). Suppose a term a is stronger than b, so that, in general, the projection from $Q(a)$ to $Q(b)$ by substituting b for a is good. It then turns out that those projections become incoherent in polarity-inverting contexts like negation (Q'). The inference from $Q'(a)$ to $Q'(b)$ (e.g., Pokey is not a dog so Pokey is not an animal) is not a good one. If a good inference can be turned into a bad one with a supposedly good substitution, something has gone awry. Either a language with logical expressions and reliable projections of subsentential expressions is maintained, where singular terms have symmetric inferential significance and predicates have asymmetric significance, or an incoherent mess results. In this way, Brandom's argument indicates that singular terms must be used in symmetric substitution classes, while predicates can be used in asymmetric substitution classes. For now, I will treat inferential projection as an answer to a metaphysical question: How is it possible for language to be both productive (generating novel sentences from existing parts) and logical? That said, the conversation below about quadrilaterals is mostly *curricular*. How inferential projection works is illuminated through the concept of inferential strength. In part three of the book, where I introduce the *hermeneutic calculator*, I will return to projection to discuss how to make it *methodologically* useful for researchers in math education.

Quadrilateral Classification as a Study in Substitution

In the spring of 2025, I taught a class focused on problem-solving in elementary school classrooms. I asked the preservice teachers to design lessons that would be appropriate for my stepdaughters who were in kindergarten at the time. One group's lesson focused on classifying quadrilaterals. They asked $\mathcal{M}$ and $\exists$ to reach into a bag that held several quadrilaterals as well as circles and triangles. Their task was to classify the shapes only by touch. As the children reached into the bag of unknowns, their knowledge of what was in the bag was modally structured in relations of *possibility* and *necessity*. In grasping an object, it could have possibly been a square ($\diamond Square(x)$), a rectangle, a circle etc. When a feature like 'rounded sides' was discerned, the modal structure of the object changed from being possibly a square to being necessarily not-square ($\Box \not{S}quare(x)$). If they discerned four corners that were perpendicular, the shape became a possible rectangle or square. Discerning that the shape had equal sides would then change the structure of the object to being necessarily a square. While I have introduced the formal modal operators $\diamond$ (possibly) and $\Box$ (necessarily), I do so only for as a shorthand: the children were not using those symbols or explicitly using modal concepts—their inferences were material, not formal. I mention this experiment to realize Brandom's reading of Hegel as "building modality in at the metaphysical ground floor" (2019, pp. 144–145). I assume that few readers take modal logic as a primordial discursive ability—I did not study the topic until I was already deep into my dissertation, even having taken some symbolic logic courses and getting a bachelor's degree in mathematics. In the history of logic, modal logic is a relative latecomer, reaching a systematic formal description only in the middle of the 20th century. An open question is the extent to which this modal structure is primarily psychological. I claim their knowledge of the bag's contents was modal in nature, but were the contents of the bag themselves modal? On the objective side of this modal experiment, it might be helpful to consider the shapes in the bag as superpositioned, like Schrödinger's Cat who is both alive and dead.[2] That said, it feels a bit outlandish to claim that quantum superpositioning is at work at the mezzo scale of plastic shapes in a bag. The subjective experience is closer to home, as I name it the *phenomenology of confusion*. Commitment to a shape being possibly a square and possibly a circle is to possibly be confused. I might not be confused, and instead recognize the 'superpositioned' states as ambiguous. The next act might then be to seek more information that could negate some possibilities. Information negates possibilities, and so can be disambiguating. That said, commitment to a shape necessarily being both a square and a circle is to necessarily be confused. The children's experience with the shapes was similar: until they reach into the bag, the shapes are unknown possibilia superpositioned on each other. As the children reach in, those possibilities collapse into determinate actualities. The field of possibilities is *realized*. In this sense, possibility is more primordial than actuality, as the latter is a realization of the former.

[2] In that thought experiment, a cat is placed in a box with a radioactive atom that has a 50% chance of decaying within an hour. A Geiger counter is placed in the box along with a vial of poison. If the atom decays, it triggers the Geiger counter, which is tied to a release mechanism on the vial of poison, which then kills the cat. Until we open the box and observe the cat, it is in a superposition of being both alive and dead.

The Experience of Error

In that teaching (and learning) experiment, the children made some mistakes. They would reach into the bag and pull out a shape that they thought was a square, but it turned out to be a rectangle. They would then have to correct their understanding of the shape, which involved a process of error correction and learning. This is a concrete example of how material inferences work in practice: the children made inferences based on their tactile experiences with the shapes, and those inferences were corrected through further interactions with the objects, for example, by pulling the shape out of the bag and looking at it. The experience of error is often discussed as it relates to empirical observations. While foregrounding the objective aspect of the experience of error invites some clarity, I have positioned subject-subject misrecognition as more primordial by introducing this book with Grover, who misrecognized himself. For the moment, I stay with the objective pole of error. Brandom (2019) has a lovely example that involves observing a stick that has been partially submerged in water. The stick appears to be bent due to the way light refracts through water, but then when it is removed from the water, it is recognized by the observing consciousness as straight. Whoops!

There are three structural roles in the experience that are worth teasing out. First, for the experience to be taken as an error, the subject who perceives the bent stick must recognize that it is the same stick whether submerged or removed from the water and, furthermore, it had been a straight stick all along. (If the children dropped a shape back into the bag, any incompatibilities they discerned prior to dropping the shape would lose the informational incompatibilities they had discerned.) The stick, *in-itself* is straight. The stick has some *authority* that representations of the stick—how the stick appears *for-consciousness* in the moment of perception—must acknowledge. Being a rational agent, in this reconstruction of the experience of error, involves submitting to the authority of a mutable reality. Merely perceiving the movement from stick-as-bent to stick-as-straight would not involve a change in commitment from "the stick is bent" to "the stick is straight" in the observing consciousness. Instead, the authority of the object in-itself is what underwrites the change in commitment, as both are normative aspects of reason. Last, what emerges is a "new, true object"—the appearance of the bent-stick becomes, *to-consciousness*, a stick-that-appears-bent-when-submerged-but-is-actually-straight. That is, the appearance of the new object—what it is for-consciousness—now has a learning experience compressed into it. The same processes were at work with the quadrilateral experiment. A child who thinks a rectangle is a square is like the person seeing the bent stick. The "new, true object" that emerges is the concept of a "rectangle-that-I-initially-misrecognized-as-a-square," which is a richer, more robust concept. With modality built in the bedrock of a philosophy of mathematics, a role for teaching and learning can be discerned. For, if learning begins in confusion—essentially beginning with an *incoherent* set of commitments—then the process involves recognizing those incoherences through the experience of misrecognition and then disambiguating or otherwise repairing them. My kids were not learning about quadrilaterals in the sense of memorizing definitions or properties. Instead, they were learning to disambiguate their modal commitments about the shapes in the bag. They were learning to recognize that a shape could not be both a square and a circle. This process of disambiguation is what allows them to classify the shapes correctly.[3]

The Traditional Approach

The classification of quadrilaterals provides an opportunity to explore the nature of substitution inferences, particularly as they apply to *predicates* that ascribe quadrilateral types (e.g., "X is a Square", "X is a Rectangle"). The traditional hierarchy of quadrilaterals (Figure 2.4) is often presented as a tree structure reflecting entailments between these predicates. For example, the predicate "X is a Square" entails the predicate "X is a Rectangle", which in turn entails "X is a Parallelogram".

This hierarchical approach can be both clarifying and confusing. Young children typically begin learning about quadrilaterals using exclusive definitions, often disagreeing with statements like "a square is a rectangle" because visually, these shapes appear distinct (van Hiele-Geldof & van Hiele, 1984). *Canonical* versions of the shapes, like equilateral triangles or non-square rectangles tend to be emphasized early in development. As their geometric understanding develops, they learn inclusive definitions, making inferences such as "if X is a square, then X is also a rectangle" (i.e., $\mathit{Square}(X) \rightarrow \mathit{Rectangle}(X)$), where $\mathit{Square}(X)$ and $\mathit{Rectangle}(X)$ are predicates. When

[3]The word 'begin' is tricky here. I assume that some intersubjectively constituted commitments are already in place for 'us' to be confused about, not backing up to a mystical state of non-intersubjective being and declaring such a state 'confused,' and I am definitely not claiming a position in the blank-slate/genetic predisposition debate.

examining Table 2.3, the properties are not mutually exclusive. A square possesses all listed properties, while a trapezoid has only one pair of parallel sides. Fallacious reasoning creeps in if inclusive definitions are grounded on shared properties. Declaring "X is a square so X is a trapezoid because both have at least one pair of parallel sides" is akin to declaring that 'X is a bird, so X is an airplane because both have wings." These features can be helpful for classifying quadrilaterals, but without some modal concepts, they cannot be organized into a hierarchy.

Characteristic Properties of Quadrilateral Families

Table 2.3: *Note.* Characteristic Properties of Quadrilateral Families

Quadrilateral Type	A_1: 1 Pair of $\parallel$ Sides	A_2: 2 Pairs of $\parallel$ Sides	A_3: 2 Pairs of Adjacent Equal Sides	A_4: 4 Equal Sides	A_5: 4 Right Angles	A_6: Diagonals are perpendicular bisectors
General quadrilateral	×	×	×	×	×	×
Trapezoid	✓	×	×	×	×	×
Parallelogram	✓	✓	×	×	×	×
Kite	×	×	✓	×	×	×
Rectangle	✓	✓	×	×	✓	×
Rhombus	✓	✓	✓	✓	×	✓
Square	✓	✓	✓	✓	✓	✓

If inclusive definitions are desired:

- **Necessity**: If X is a Trapezoid, it *must necessarily have* at least one pair of parallel sides. $\Box(Trapezoid(X) \rightarrow A_1(X))$.
- **Possibility**: A Quadrilateral *can possibly have* one pair of parallel sides. $Quadrilateral(X) \rightarrow \Diamond A_1(X)$.

If exclusive definitions (canonical shapes) are desired, the last expression is negated: $Quadrilateral(X) \rightarrow \neg\Diamond A_1(X)$. Once that decision is made further modal operators and axioms (Brandom, 2008, pp. 141–175) can be introduced to prove that a square is necessarily a rectangle, and that a rectangle is necessarily a parallelogram (under the inclusive definitions). In the prior paragraphs, I was using the term "definition" in the traditional sense. However, it occurs to me that *the* definition of a square is the 'set' (*meaning field*) of inferences that can be made about squares. One reason it is challenging to read authors like myself or those who I cite most prominently is a resistence to glossaries - defining terms in pure Vocabulary-Vocabulary type definitions you might find in the dictionary or a glossary. Pragmatism understands meaning in terms of use. Consequently, one way to 'define' terms is to explicate the 'sets' of good and bad material substitution inferences the terms are caught up in, along with the contexts that determine the goodness or badness of those inferences. Those 'sets' are not, for my project, ever fully explicated. There's always some possible speech act that could be added to the 'set' or some reason why one of the elements of the 'set' is defeated by the context in which it is claimed to be good or bad. However, the regularized patterns of inferences about geometric shapes, dogs, or whatever also make it unsound to claim that these 'sets' are uncountably infinite. I have to reach for the defeasors for localized clusters of material inferences that follow regular patterns. Adding layers of reflection moves the researcher away from the original contexts, and so, as I press against the boundaries of the meaning field, the claimed possible interpretations of a speech act become more and more tenuous and the I-feeling becomes less pronounced.

Shadows of shape

The understanding so far is that classifying a given shape as a particular instance of a quadrilateral involves understanding what it means to be a particular 'thing' as a *medium* in which compatible properties inhere. Doing so amounts to treating the thing as an *also*, rather than as an exclusive *one*. This is the eighth difference that Brandom describes as

- "particulars as "also"s—that is, as a medium hosting a community of compatible universals—and
- particulars as "exclusive ones"—that is as units of account repelling incompatible properties" (Brandom, 2019).

To get at the other side of the object—treating the thing as an exclusive *one*—involves discerning its boundaries. What information would preclude stating "X is a square"? A shape becomes determinate (e.g., as a square) by the set of restrictions it would refuse, like a vegan who will not eat butter. I argue that the negative articulation of the square, its shadow, is a necessary contrast to the positive articulation of the square. That contrast is probably already at work with learners prior to engaging with formal definitions of quadrilaterals, but I have not found anything that resembles my approach in the literature. A square object can be conceptualized towards a square *subject* by considering what claims it repels. Following Hegel, Brandom argues that the concept of a particular *object* emerges as a necessary structural feature for tracking such relations. The object is the *unit of account* for incompatibilities and as such are "of a different ontological category from the features for which they are units of account" (Brandom, 2019). This difference between universals and particulars is the seventh difference Brandom discerns. The incompatibility between *square* and *triangle* is not a global law of logic; it is the fact that one and the same particular thing cannot be both. The placeholder 'X' in our predicates, therefore, is not merely a variable; it represents the emergence of the particular object as the locus of property instantiation and exclusion. I repel the claim "Tio is a dog" just as a square, X, repels the claim "R_1: No sides of X are equal."

- R_1: No sides of X are equal
- R_2: No pair of adjacent sides of X are equal
- R_3: No pair of opposite sides of X are equal
- R_4: Non-parallel sides of X are not congruent
- R_5: No pair of opposite sides of X are parallel
- R_6: No angles of X are right angles

The predicate *Square*(X) would entail the refusal of R_1 (as a square has four equal sides). A square is *incompatible* with each of the restrictions $R_1, \ldots, R_6$. The collection of R_i's I have listed are sufficient for distinguishing the quadrilaterals I listed in 2.4. Other claims are possible, like "no diagonals of X are perpendicular." Exhaustive lists of negatively articulated claims for quadrilaterals are impossible, as we can always say "X is not a dog" or "X is not a triangle." Table 2.4 summarizes the incompatibility relations between the quadrilateral categories and the restrictions $R_1, \ldots, R_6$. A 1 indicates that the shape rejects the restrictive claim. A 0 means it does not necessarily reject it. The inferential strength of each shape is the sum of the restrictions it rejects. The inferential strength of each restriction is the sum of the shapes that reject it.

At the heart of this analysis lies a crucial Hegelian and Brandomian distinction between two kinds of opposition. The first is Brandom's material incompatibility and Hegel's determinate negation. This refers to oppositions that arise from the non-logical content of concepts. For instance, an object

Incompatibility Matrix and Inferential Strength for Quadrilateral Categories

Property (Restriction)	**Square**	**Rectangle**	**Rhombus**	**Parallelogram**	**Trapezoid**	**Kite**	**Quadrilateral**	**Strength of R_i**
R_1: No sides of x are equal	1	1	1	1	0	1	0	5
R_2: No pair of adjacent sides of x are equal	1	0	1	0	0	1	0	3
R_3: No pair of opposite sides of x are equal	1	1	1	1	0	0	0	4
R_4: Non-parallel sides of x are not congruent	1	0	1	0	0	1	0	3
R_5: No pair of opposite sides of x are parallel	1	1	1	1	1	0	0	5
R_6: No angles of x are right angles	1	1	0	0	0	0	0	2
Strength of Shape	6	4	5	3	1	3	0	

Table 2.4: Incompatibility matrix and inferential strength for quadrilateral categories. A 1 indicates the shape rejects the restrictive claim. A 0 means it does not necessarily reject it. The inferential strength of each shape is the sum of the restrictions it rejects.

cannot be both circular and triangular at the same time; the property 'circular' materially excludes the property 'triangular'. Brandom defines these as Aristotelian *contraries*. Contraries are the fourth type of difference Brandom discerns. The fifth type of difference is formal contradictoriness, or abstract negation. This is the familiar logical opposition expressed by 'not', such as the relationship between 'red' and 'not-red'. One of the delightful (and important for math education) features of Brandom's reconstruction of Hegel's chapter on perception is that Brandom defines formal contradictoriness in terms of material contrariety. He notes that 'green' is a contrary of 'red' and 'not-red' is its contradictory. 'Not-red' is the minimal contrary of red in that it is entailed by every contrary of red (green, blue, yellow, etc.). Hegel's key move, which Brandom develops, is to treat material incompatibility (contrariety) as the more fundamental concept, from which formal contradiction can be explained. The sixth type of difference is the metadifference between determinate and abstract negation, which is the distinction between material contrariety and formal contradictoriness. The representation of a square as both an exclusive one and an inclusive also (the eigth difference) is a bit like seeing a square with its shadow (see Figure 2.5). The shadow is the set of restrictions that the square repels, while the square is equally the set of properties it possesses. The shadow is not a separate entity; it is a necessary aspect of the square's identity. When considering a square pulled out from the bag, other universals besides $A_1 \ldots A_6$ and $R_1 \ldots R_6$ might sit alongside those properties. For example, the shapes for my stepdaughters were plastic, so the square accepts the property of being printed on paper, while the shadow repeled the restriction of being printed on paper. The reverse would be so if this book is being read on paper.

Interestingly, the property of being square exhibits the same duality—it can be understood as forming a family of co-compatible universals like being rectangular and it can be understood as repelling universals that it cannot co-instantiate with. The first of the ten differences Brandom discerns is the "mere or 'indifferent' [gleichgültig] difference of compatible universals" while the second difference is the "exclusive difference of incompatible universals"(2019). The third difference is the "metadifference between mere and exclusive difference" (Brandom, 2019). This third difference is a species of exclusive difference, "because the universals must be either compatible or incompatible" (Brandom, 2019, pp. 161–162).

Figure 2.6 illustrates this duality for universals/predicates/sentence frames. The universal of being square forms a family with other universals like being rectangular, which are associated with particular rectangles that have properties having four right angles. Likewise, the universal of being square excludes other incompatible universals like being triangular. This is the ninth type of difference—the "difference between two roles universal play with respect to particulars" (Brandom, 2019). In the figure, I try to capture the 'with respect to particulars' by clarifying that the universal "triangular" is discerned from a particular triangle (same with circular). Category errors arise as we consider other uses of the term 'square' that do not involve particular geometric shapes. For example, the particular -1, which is not a 'square number' like 25 (5^2), or the slang term 'square' for a person who is not cool ("let's not be L7, come on learn to dance") divorce "square" from the family of co-compatible universals related to geometric shapes by drawing from particulars outside that family.

It is now possible to summarize nine of the ten types of difference (2019, pp. 161–162). Figures 2.5 and 2.6 illustrate two intracategorial differences within the categories of particulars and universals, respectively. From these nine types of difference, Brandom reaches an interesting—if somewhat well-trodden—conclusion: universals have opposites, but particulars do not. This is a "huge structural difference" (Brandom, 2019) in how objects and properties differ from and exclude one another. Universals have contradictories (like "not-red") but objects do not, because the "opposite" of an object would need to exhibit mutually incompatible properties, which is impossible. What is the opposite of a red, plastic, square object? The opposite would have to be not-red, not-plastic, and not-square, but not-red encompasses blue, green etc. and not-plastic contains metal or wood. The opposite of such an object would have to exhibit all of those incompatible universals. The list below is adapted with very minor revisions from (Brandom, 2019, pp. 161–162).

1. Mere or "indifferent" difference of compatible universals: This refers to properties that can coexist, like "day" and "raining".

2. Exclusive difference of incompatible universals: This refers to properties that cannot coexist, like "day" and "night," or "red" and "green". This is a "modally robust exclusion".

3. Metadifference between mere and exclusive difference: This is the distinction between compatible and incompatible differences. It's an intracategorial metadifference concerning differences relating universals to universals. Universals must be either compatible or incompatible.

4. Material contrariety: This corresponds to determinate negation, where features are materially incompatible due to their nonlogical content. Examples include "red" and "green," or "circular" and "triangular".

5. Formal contradictoriness: This corresponds to abstract logical negation, where features are logically incompatible due to their abstract logical form. Examples include "red" and "not-red," or "circular" and "not-circular".

6. Metadifference between determinate and abstract negation (logical negation): This is the distinction between material contrariety and formal contradictoriness. It's the second intracategorial metadifference between differences relating universals to universals, which are compatibly different. Contradictories are a type of contrary (minimal contraries).

7. Difference between universals and particulars: This is the first intercategorial difference, an exclusive difference between properties and objects.

8. Difference between two roles particulars, or objects, play:

- Particulars as “also’s”: Functioning as a medium hosting a community of compatible universals.
- Particulars as “exclusive ones”: Functioning as units of account repelling incompatible properties. This is the first intracategorial difference between roles played by particulars, and every particular must play both.

9. Difference between two roles universals play with respect to particulars:

   - Universals as related to an inclusive “One” in community with other compatible universals.
   - Universals as excluding incompatible universals associated with different exclusive “One’s”.

10. Difference between universals and particulars that consists in the fact that universals do and particulars do not have contradictories or opposites: This highlights a “huge structural difference” in how objects and properties differ from and exclude one another. Universals have contradictories (like “not-red”), but objects do not, because the “opposite” of an object would need to exhibit mutually incompatible properties, which is impossible.

Difference to-consciousness

Each of these differences is potentially a difference *to-consciousness*. Each moment where a difference is discerned may be accompanied by apperceptive self-awareness. The I-feeling mode can fuse with these moments of conceptual movement, giving them a palpable quality that we can recollect fondly and desire to revisit. As odd as I may be, that is the driving desire that keeps me teaching and writing about mathematics. It feels good. The inferential chains that flow from stronger to weaker predicates—the algorithms—can be experienced with the smoothness of a yogic flow, where each step in a proof is a pose that punctuates the movement. There is a deep pleasure, a fusion of the I-feeling with the concept, that can occur when a difference is discerned to be null—when, for instance, the experience of misrecognition resolves into recognition, and the object for-consciousness aligns with the object in-itself. This experience reaches its peak in the playful but profound dizziness of a logical fixed-point, as in Gaifman’s self-referential sentence. Such moments, where a concept or sentence takes itself as its own object, reveal the curious and delightful *iterability* of thought that Derrida's work explores. They are the intellectual equivalent of dwelling in the *in*finite regress between two mirrors. While delightful, these infinities do not capture the Hegelian *in*finite, which requires his concept of self-consciousness to recognize. There is much more to say to break this narcisstic mirror. The fractal-like Venn diagram in Figure 2.8 illustrates the reciprocal definitions between entailments and exclusion. The observation that each shape is, in a sense, both 'inside' and 'outside' the others dissolves the idea of a simple, linear hierarchy. A square 'contains' the general quadrilateral in that it possesses all its properties, yet the general quadrilateral 'contains' the square as a specific possibility within its broader conceptual space. Navigating this complex, self-similar landscape is precisely the work of logical operators like negation. By 'relaxing the square'—softening the hard negations that define it—the concept is not destroyed; rather, its inferential pathways to neighboring shapes, like the rhombus, are traced. Still, the journey of this chapter from a simple hierarchy to a self-similar, fractal network, approaches conceptual self-awareness. This is the ultimate goal of sublation: to arrive at a totality that contains its own developmental history, a goal approached as the chapter now turns to a final summary of its findings. But whereas Hegel approached a totality of totalities, I approach a smaller totality, the totality of quadrilaterals. The image from *Breath and Kindling*, "strung street lights; small sodium moons; whiteknuckled halos cut holes in the gloom," captures my ambition here. Rather than clinging to that totality, so easily broken by including an isoceles trapezoid or whatever other properties might turn to relevance as soon as the ink has dried, I encourage readers to let go. The next streetlight beckons.

2.5 Integration: The Phenomenology of Classification and Hybridized Models

In this section, I first offer two explanatory images and then a logical reconstruction of the phenomenology they represent. In figure 2.7, the process of 'relaxing the square' is visually represented, revealing how the rigid boundaries of the square can be softened to reveal connections to other quadrilateral forms.

In figure 2.8, the fractal-like nature of the quadrilateral classification is depicted, illustrating how each shape both contains and is contained by others in a complex web of relationships.

The Structure of the Hierarchy

The logic developed in chapter 1 can be extended to understand the two images above, hopefully negating any sense of mysticism that might arise from the representation of represented as fractals. **Moving Down (Specification):** Quadrilateral → Parallelogram → Rectangle → Square. This is a movement of increasing Compression (↓). The conceptual hierarchy strengthens by adding constraints (commitments). "Square" is the most compressed concept. **Moving Up (Generalization):** Square → Rectangle → Parallelogram. This is a movement of increasing Expansion (↑). The conceptual hierarchy weakens by "Letting Go" of constraints.

The Dynamics of Inference

Consider the standard inference: "If it is a Square (S), then it is a Rectangle (R)" ($S \Rightarrow R$). **Embodied Dynamic:** The analysis starts with the highly compressed concept S (↓↓). To move to R, the constraint of "equal sides" is released. This inference is experienced as an expansive movement (↑). Now consider the contrapositive (Modus Tollens), which relies on polarity inversion: "If it is NOT a Rectangle ($\neg R$), then it is NOT a Square ($\neg S$)" ($\neg R \Rightarrow \neg S$). **Embodied Dynamic:**

The analysis starts with $\neg R$. Since R is relatively expansive, $\neg R$ introduces a compression (↓)—this closes off the space of rectangles. The conclusion involves $\neg S$. Since S is compressive, $\neg S$ is expansive (↑). The inference moves from the compression of $\neg R$ to the expansion of $\neg S$. The compression required to exclude the broader category (Rectangle) necessarily forces the exclusion of the narrower category (Square) contained within it.

Let us return to the problem of hybridized models, with the assumption that other kinds of errors may be diagnosed through this kind of reasoning. They are speech acts, not just representations. The shape representing a unit is one term and the partitioning practice applied to that term functions like a predicate. A correct model of a fraction will use the materially appropriate predicate—the appropriateness of that predicate depends on the properties/proprieties of the shape chosen as the unit.

The student starts with a known, good model and attempts to generate a new one by substituting the shape.

This means the student's projection is **not projectively valid**. Because the substitution fails to preserve the goodness of the claim in this context, U_R and U_C are **not** symmetrically intersubstitutable. This is an asymmetry of **material dependency**. The appropriateness of the predicate (the partitioning practice) is materially dependent on the *geometric content* of the term it modifies. Every diameter of a circle partitions the circle in half, so there are an infinite number of valid radial partitions. However, rectangles generally do not afford radial partitions. The error of hybridization is the attempt to treat a predicate whose meaning is context-bound as if it were stable and context-free. The student mistakenly projects the predicate outside the vocabulary (U_R) that gives it its validity. This diagnosis reveals a different kind of asymmetry from the hierarchical entailment between Square and Rectangle—an asymmetry of **material dependency**. The appropriateness of the predicate is materially dependent on the *geometric content* of the term it modifies, bridging the formal structures of Chapter 1 with the embodied temporal dynamics of Chapter 2.

2.6 Reflection: The Case for the Anaphoric Unit

The central claim of this section, which I will articulate more fully in chapter 7, is that the mathematical "unit" (the "One") functions linguistically not as a singular term (like a proper name), but as an anaphoric term (like a pronoun). The justification for this rests on the logic of substitution and the evidence provided by the failure of hybridized models.

1. The Requirement of Singular Terms The standard interpretation treats mathematical units as **singular terms**—expressions that refer to a specific object. Singular terms are logically defined by their participation in **symmetric substitution inferences**. If two expressions function as singular terms referring to the same object (e.g., "Benjamin Franklin" and "The inventor of bifocals"), they form an equivalence class. They must be substitutable for one another in any sentence frame without altering the validity of the assertion. The predicate remains stable under substitution. If

I must change the set of predicates that apply to a singular term when the singular term is substituted symmetrically, then it isn't a symmetric substitution. If the unit "One" is a singular term, then any representation of that unit—such as a Circle (U_C) or a Rectangle (U_R)—must also be symmetrically intersubstitutable.

2. The Test Case: Spatial Modeling I test this requirement in the context of modeling fractions. This involves applying a predicate (a partitioning strategy) to a term (the shape realizing the unit). The normative constraint is equipartitioning (creating equal, interchangeable parts).

- A Rectangle Unit is appropriately partitioned using a Vertical strategy (P_V). Speech Act: $P_V(U_R)$. (Good).
- A Circle Unit is appropriately partitioned using a Radial strategy (P_R). Speech Act: $P_R(U_C)$. (Good).

3. The Failure of Substitution If U_R and U_C were symmetrically intersubstitutable, we should be able to substitute one for the other while keeping the predicate stable. Let's attempt to substitute U_C for U_R in the valid model $P_V(U_R)$:

- **Original:** $P_V(U_R)$ (A Rectangle partitioned Vertically) $\rightarrow$ Valid Model.
- **Substitution:** $P_V(U_C)$ (A Circle partitioned Vertically) $\rightarrow$ Invalid Model.

The result, $P_V(U_C)$, is a hybridized model. It is invalid because vertical partitioning a circle generally fails the normative requirement of equipartitioning. The inference generated by this substitution—($P_V(U_R)$ is Good) SO ($P_V(U_C)$ is Good)—is a **Bad Inference**.

4. The Diagnosis: Material Dependency The failure of the substitution demonstrates that U_R and U_C are **not** symmetrically intersubstitutable. The predicate is not stable because the appropriate actualization of "partitioning" is materially **dependent** on the specific geometry of the term it modifies.

> "The contents of this predicate depend on the contents of the term substituted in. If this were not so, then the partitioning predicate for circular models… would apply to the underlying substitution of 'singular terms'… Instead, such a substitution yields the hybridized model" (Savich, 2022).

Because the terms fail the fundamental requirement of symmetric substitutability, the singular term interpretation is untenable.

5. The Anaphoric Resolution The **anaphoric interpretation** resolves this conflict and explains the dependency structure. Anaphora (like pronouns) function by referring back to an antecedent. Their key feature is that they possess **syntactic sameness without semantic sameness** (Savich, 2022).

1. **Syntactic Sameness:** Both U_R and U_C can fulfill the syntactic role of "the unit" in a fraction model.

2. **Semantic Difference:** Their underlying geometric properties (their semantic content) are different. These differences impose distinct material constraints on the practices (predicates) that can validly apply to them.

Like the pronoun "He"—where the appropriateness of the predicate "is tall" depends entirely on the specific referent—the appropriateness of the predicate "is partitioned Vertically" depends entirely on the specific actualization of the unit. The hybridized model error occurs precisely when a student recognizes the syntactic sameness but ignores the semantic difference. They attempt to project a predicate (P_V) across a substitution ($U_R \to U_C$) that cannot support it, breaking the anaphoric link that gave the original predicate its validity. The instability of predicates under the substitution of different representations of the unit fundamentally contradicts the singular term interpretation. The anaphoric interpretation is necessary to account for the material dependency between mathematical representations and the practices appropriate to them. Frege would not accept the argument above as defeating his project. He would likely declare that I am making a category error by confusing the representation of the unit with the unit itself. He would probably say that what I have proved is that circles aren't rectangles. The Unit is the referent (Venus), while the circle and rectangle are different senses (the morning star and the evening star). The fact that circles and rectangles are not symmetrically intersubstitutable does not threaten the underlying sameness in referent. To respond to that argument, I say "Yes!" The act of thought is dynamic and context-specific (the specific material actualization of the unit), but the recollection of that act via the anaphoric term is stable (the formal unit). Anaphora is the second negation, the ~~*no*~~. It is *how* terms come to be fixed. He would likely also think that I am committing myself to a psychologism, where the validity of a mathematical expression depends on the particular psychology of the person who claims it. The charge of psychologism would be a bitter pill to swallow. I find psychological explanations of the mathematical to be trapped in a vicious circle. Brandom's interpretation of Hegel as introducing a non-psychological conceptualization of the conceptual is a way past that charge. The Square is the collection of positive and negative inferential proprieties listed above. There need be no reliance on the particular psychology of the person who makes claims about squares. While the specific learning experiences of the claimant are temporally compressed into their specific uses of those shapes, the shapes themselves are defined as the unbounded set of material inferences one might make about them. The Square is not a psychological entity; it is a conceptual entity defined by its inferential role. Furthermore, the {I think} is not a psychological entity either. It is a transcendental condition of possibility for *any* representation, according to Kant. I will discuss transcendental categories more in the next chapter. Until then, note that pronouns follow predictable rules *once they are entered into a context.* For example, "Tio is a math educator, he is kind of a geek, but his kids are pretty cool." He $\rightarrow$ his is a rule. That rule is (by my reckoning) appropriately challenged in folks who assert gender fluidity, but to challenge a norm is to claim a new norm. The rulishness of mathematics is actualized by virtue of how anaphora fixes the semantic contents of varying expressions. We need not assert a platonic heaven or a realm of truth separable from language: we need merely recollect language to find mathematical stability.

2.7 Conclusion

The final difference, which is the tenth difference Brandom discerns, is the "difference between universals and particulars that consists in the fact that universals do and particulars do not have contradictories or opposites" (Brandom, 2019). This highlights a "huge structural difference" in how objects and properties differ from and exclude one another. Universals have contradictories (like "not-red"), but objects do not, because the "opposite" of an object would need to exhibit mutually incompatible properties, which is impossible. The logic of apperceptive self-consciousness is not a formal logic, but a logic of recognition.

References

Abbott, E. A. (2020, April). *Flatland: A Romance of Many Dimensions by Edwin A. Abbott*. Independently published.

Agamben, G. (2006, September). *Language and Death: The Place of Negativity* (K. Pinkus & M. Hardt, Trans.; First Edition). Univ Of Minnesota Press.

Aristotle. (2017). *Metaphysics: Aristotle's metaphysics in english, latin and ancient greek: Trilingual edition* (W. o. Moerbeke & W. D. Ross, Trans.; Trilingual). Hermes Language Reference.

Boote, D. N., & Beile, P. (2005). Scholars before researchers: On the centrality of the dissertation literature review in research preparation. *Educational researcher*, *34*, 3–15.

Brandom, R. B. (1994). *Making it explicit: Reasoning, representing, and discursive commitment*. Harvard university press.

Brandom, R. B. (2000). *Articulating reasons: An introduction to inferentialism*. Harvard University Press.

Brandom, R. B. (2008). *Between saying and doing: Towards an analytic pragmatism*. Oxford University Press.

Brandom, R. B. (2019). *A spirit of trust: A reading of Hegel's phenomenology*. The Belknap Press of Harvard University Press.

Bueno, O. (2020). Nominalism in the Philosophy of Mathematics. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Fall 2020). Metaphysics Research Lab, Stanford University.

Cantor, G. (1891). Ueber eine elementare frage der mannigfaltigkeitslehre. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, *1*, 75–78.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). Children's mathematics: Cognitively guided instruction - videotape logs [supplementary material]. In *Children's mathematics: Cogni-*

tively guided instruction. Heinemann, in association with The National Council of Teachers of Mathematics, Inc.

Carspecken, P. F. (1995). *Critical ethnography in educational research: A theoretical and practical guide*. Routledge.

Carspecken, P. F. (1999). *Four scenes for posing the question of meaning and other essays in critical philosophy and critical methodology* (New edition). Peter Lang Inc., International Academic Publishers.

Carspecken, P. F. (2006). Limits of knowledge in the physical sciences. In K. Tobin & J. Kincheloe (Eds.), *Doing educational research* (pp. 405–437, Vol. 1). Sense Publishers.

Carspecken, P. F. (2009). Limits to knowledge with existential significance: An outline for the exploration of post-secular spirituality of relevance to qualitative research. In *Bridging the gap between theory and practice in educational research* (pp. 47–61). Springer.

Carspecken, P. F. (2016). Limits to knowledge and being human: What is 'critical' in critical social research and theory? In P. H. Hinchey (Ed.), *A critical action research reader* (pp. 65–86). Peter Lang Inc., International Academic Publishers.

Carspecken, P. F. (2018). The missing infinite. In R. Winkle-Wagner, J. Lee-Johnson, & T. Gaskew (Eds.), *Critical theory and qualitative data analysis in education* (p. 32). Routledge. https://doi.org/10.4324/9781315158860-2

Carspecken, P. F., & Zhang, R. (2013). Content inference fields in intersubjective space: Transpersonal position, illocution, and logic in the analysis of human interactions. *Qualitative research: A reader in philosophy, core concepts, and practice*, 201–242.

Carspecken, R. (2024, November 3). Untitled.

Caygill, H. (2004). *A kant dictionary*. Wiley-Blackwell.

Derrida, J. (1973). *Speech and phenomena, and other essays on husserl's theory of signs*. Northwestern University Press.

Derrida, J. (1982). *Marges de la philosophie* [Margins of philosophy] (A. Bass, Trans.). The Harvester Press. (Original work published 1972)

Deutsch, H., Marshall, O., & Irvine, A. D. (2025). Russell's Paradox. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford encyclopedia of philosophy* (Spring 2025). Metaphysics Research Lab, Stanford University.

Ernest, P. (2024). The mathematical and philosophical significance of zero: Nought matters [Publisher: Paul Ernest, Emeritus Professor at University of Exeter, School of Education & Lifelong Learning]. *Philoso-*

phy of Mathematics Education Journal, 1–35. https://research.ebsco.com/linkprocessor/plink?id=b685ecb9-d2fc-3e03-889d-a06fb1751c1b

Euclid. (2007, December). *Euclid's Elements of Geometry* (J. L. Heiberg, Ed.; R. Fitzpatrick, Trans.; Revised and corrected edition). Richard Fitzpatrick.

Franzén, T. (2004, September). *Inexhaustibility: A Non-Exhaustive Treatment: Lecture Notes in Logic 16* (1st edition). A K Peters/CRC Press.

Franzén, T. (2005). *Gödel's theorem: An incomplete guide to its use and abuse*. AK Peters/CRC Press.

Frege, G. (1997, July). *The Frege Reader* (M. Beaney, Ed.; 1st edition). Blackwell Publishing.

Gaifman, H. (2005). *The easy way to Gödel's proof and related matters* [Unpublished manuscript. Available online].

Geisel, T. (1961). *The Sneetches and Other Stories* (First edition.). Random House Books for Young Readers.

Gelder, T. V. (1995). What might cognition be, if not computation? *The Journal of Philosophy*, *92*, 345–381. Retrieved April 13, 2025, from http://www.jstor.org/stable/2941061

Gray, L., & Garzon, M. (2005). On a hermite boundary integral approximation. *Computers & structures*, *83*(10-11), 889–894.

Greenblatt, R., Denison, C., Wright, B., Roger, F., MacDiarmid, M., Marks, S., Treutlein, J., Belonax, T., Chen, J., Duvenaud, D., Khan, A., Michael, J., Mindermann, S., Perez, E., Petrini, L., Uesato, J., Kaplan, J., Shlegeris, B., Bowman, S. R., & Hubinger, E. (2024, December 23). Alignment faking in large language models. https://doi.org/10.48550/arXiv.2412.14093

Habermas, J. (1985). *The theory of communicative action: Lifeworld and system* (Vol. 2). Beacon Press.

Habermas, J. (2004). *The philosophical discourse of modernity: Twelve lectures* (F. G. Lawrence, Trans.). MIT Press.

Habermas, J. (1971). *Knowledge and human interests*. Beacon Press.

Habermas, J. (1984). *The theory of communicative action, volume 1: Reason and the rationalization of society* (T. McCarthy, Trans.; Reprint edition). Beacon Press.

Habermas, J. (1995). *Postmetaphysical thinking: Between metaphysics and the critique of reason* (1st edition). Polity.

Hackenberg, A. (2025). *Course notes* [Unpublished course notes].

Hegel, G. W. F. (1977). *Phenomenology of spirit* (A. V. Miller, Trans.; Vol. 632). Oxford University Press Oxford.

Hegel, G. W. F. (2010). *The science of logic* (G. di Giovanni, Ed. & Trans.). Cambridge University Press.

Houlgate, S. (2006). *The opening of hegel's logic: From being to infinity*. Purdue University Press.

Hyppolite, J. (1974). *Genesis and structure of hegel's phenomenology of spirit*. Northwestern University Press.

Kant, I. (1998). *Critique of pure reason* [Original work published 1781/1787, Guyer and Wood, Trans.]. Cambridge University Press.

Lakoff, G., & Johnson, M. (1980). *Metaphors we live by*. University of Chicago Press.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books.

Linnebo, Ø. (2024). Platonism in the Philosophy of Mathematics. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford encyclopedia of philosophy* (Summer 2024). Metaphysics Research Lab, Stanford University.

Longuenesse, B. (2017). *I, me, mine: Back to kant and fichte on the 'i'*. Oxford University Press.

Mead, G. H. (1934). *The social psychology of george herbert mead* (A. L. Strauss, Ed.). University of Chicago Press.

Mead, G. H. (2015). *Mind, self, and society: The definitive edition* (H. Joas, D. R. Huebner, & C. W. Morris, Eds.; Enlarged edition). University of Chicago Press.

Meyer, J. (title). *Cantors 1891 diagonal proof* [Cantor's original 1891 diagonal proof]. Retrieved July 6, 2025, from https://jamesrmeyer.com/infinite/cantors-original-1891-proof

Nagel, E., & Newman, J. R. (2012). *Godel's proof*. Routledge.

Negarestani, R. (2018). *Intelligence and Spirit*. Urbanomic/Sequence Press.

Poston, T., & Stewart, I. (2012, February 29). *Catastrophe theory and its applications* (First Edition). Dover Publications.

Priest, G. (2002). *Beyond the limits of thought*. Oxford University Press.

Priest, G. (2014). *One: Being an investigation into the unity of reality and of its parts, including the singular object which is nothingness*. Oxford University Press.

Quine, W. V. O. (1948). On what there is. In *Thinking about logic* (pp. 189–209). Catholic University of America. https://sites.pitt.edu/~rbrandom/

Courses/2023%20Sellars/Sellars%20texts/Quine%20On_What_There_Is.pdf

Quine, W. V. O. (1963). Two dogmas of empiricism [First Harper Torchbook edition, a reprint of the second, revised edition (1961). The book was first published by Harvard University Press in 1953.]. In *From a logical point of view* (Second, revised, pp. 20–46). Harper & Row.

Rödl, S. (2014). Intentional transaction [Accessed: 2025-04-13]. *Philosophical Explorations: An International Journal for the Philosophy of Mind and Action*, *17*, 304–316. https://doi.org/10.1080/13869795.2014.941909

Rose, G. (2009). *Hegel contra sociology*. Verso.

Rowlands, S. (2005). The contradictions in the constructivist discourse [Citing von Glasersfeld]. *Philosophy of Mathematics Education Journal*, *14*.

Savich, T. M. (2020). Towards a critical mathematics [Accessed via ERIC]. *Mathematics Teaching–Research Journal Online*, *12*, 245–291. https://files.eric.ed.gov/fulltext/EJ1384454.pdf

Savich, T. M. (2022). *Towards a critical mathematics*. Indiana University.

Savich, T. M. (2024). Built to break: An introduction to theoretical autoethnography [Invited talk presented at Solent University, organized by Dr. Brian McDonough]. https://iu.mediaspace.kaltura.com/media/t/1_v1qmm3g9

Savich, T. M. (title). *The hermeneutic calculator* [The hermeneutic calculator]. Retrieved July 23, 2025, from https://tiosavich.github.io/UMEDCTA/Calculator/index.html

Sellars, W. (1953). Inference and meaning. *Mind*, *62*, 313–338.

Sellars, W. (2007). *In the space of reasons: Selected essays of wilfrid sellars*. Harvard University Press.

Steffe, L. P. (2002). A new hypothesis concerning children's fractional knowledge [Accessed through CEPA.INFO]. *Journal of Mathematical Behavior*, *20*(3), 267–307.

Steffe, L. P. (2003). On the construction of learning trajectories of children: The case of commensurate fractions [Accessed via ResearchGate/ERIC]. *Monographs of the Journal for Research in Mathematics Education*, 1–26.

Steffe, L. P. (2004). Fractional commensurate, composition, and adding schemes: Learning trajectories of jason and laura: Grade 5 [Accessed via ResearchGate]. In J. Olive & R. Tzur (Eds.), *Children's fractional knowledge* (pp. 1–60). American Educational Research Association.

Steffe, L. P. (2014). *Children's number sequences: An explanation of steffe's constructs and an extrapolation to rational numbers of arithmetic* [Originally from 1994]. Unpublished manuscript, available via CiteSeerX; ResearchGate.

Steffe, L. P., & Thompson, P. W. (2014). Teaching experiment methodology: Underlying principles and essential elements [Accessed via ResearchGate, originally published 2000]. *Proceedings of the 24th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*.

Stone, J. (2003). *The monster at the end of this book* (2nd ed. edition). Golden Books.

Taylor, C. (1994). *Multiculturalism: Examining the Politics of Recognition* (A. Gutmann, Ed.) [Essays by Charles Taylor, K. Anthony Appiah, Jürgen Habermas, Steven C. Rockefeller, Michael Walzer, and Susan Wolf]. Princeton University Press.

van Hiele-Geldof, D., & van Hiele, P. M. (1984). *English translation of selected writings of dina van hiele-geldof and pierre m. van hiele* (D. Fuys, D. Geddes, & D. Tischler, Eds.) [ERIC Number: ED287697]. Retrieved June 3, 2025, from https://eric.ed.gov/?id=ED287697

Wittgenstein, L. (1956). *Remarks on the foundations of mathematics*.

Wittgenstein, L. (2010). *Philosophical investigations* (P. M. S. Hacker & J. Schulte, Eds.; G. Anscombe, Trans.). Wiley-Blackwell.

Chart of Hybridized Models

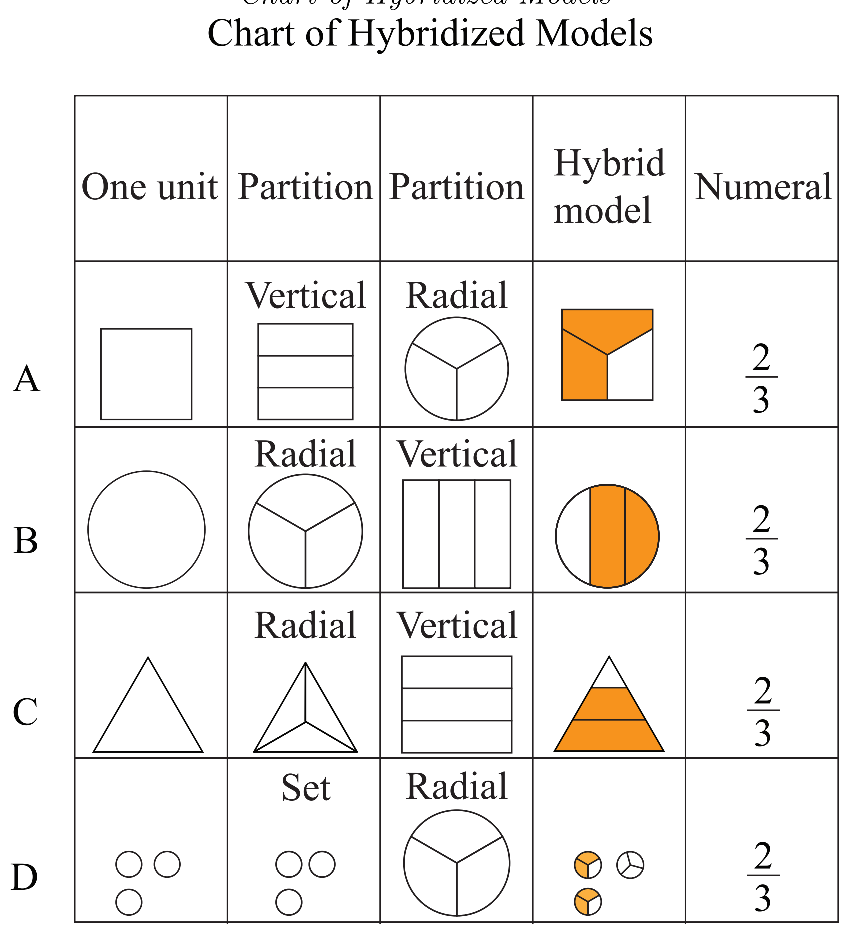


Figure 2.2: *Note.* The hybridized models are not fractions. While not statistically significant in the context I found them in, this pattern of reasoning is not uncommon

Hybridized Exponential Growth

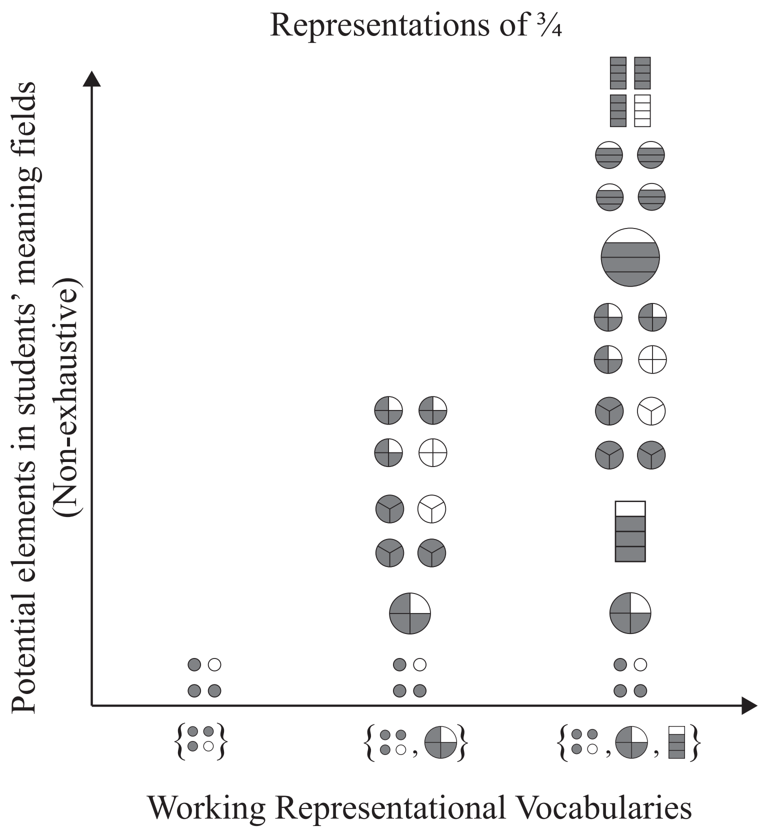


Figure 2.3: *Note.* As the teacher introduces a new model, moving from the number of possible models n to $n + 1$, students prone to hybridization may experience exponential growth $2^n \rightarrow 2^{n+1}$. For $n = 3$, there are at least 8 possible hybridized models.

Traditional and Circular Hierarchies of Quadrilaterals

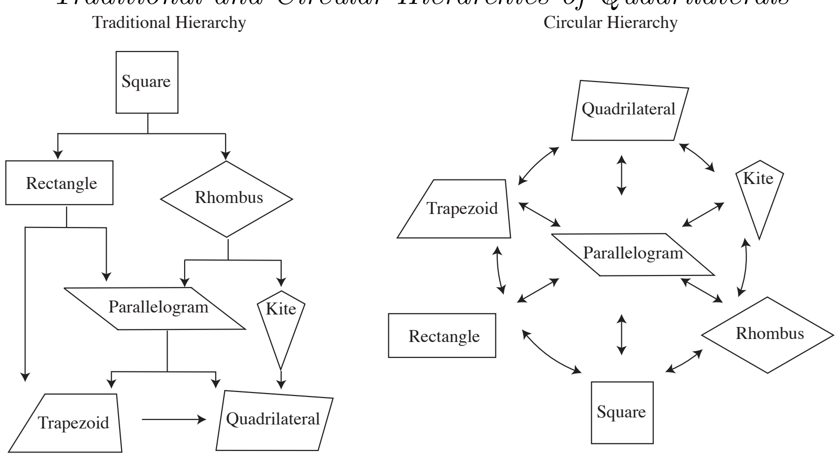


Figure 2.4: *Note.* Traditional and Circular Hierarchies of Quadrilaterals. The circular hierarchy (right) is *im*proper until justified.

A Particular as Medium and Unit of Account

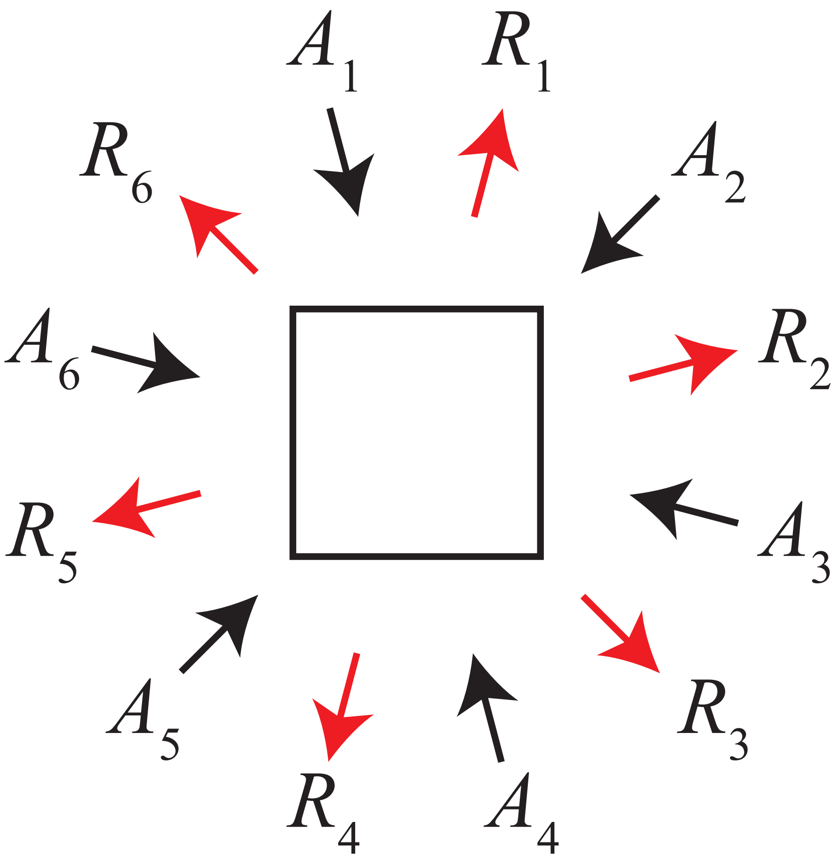


Figure 2.5: *Note.* A particular square is the medium (the Also) in which the properties $A_1, \ldots, A_6$ inhere. The square is also a unit of account for incompatibilities (the One) that repel the restrictions $R_1, \ldots, R_6$.

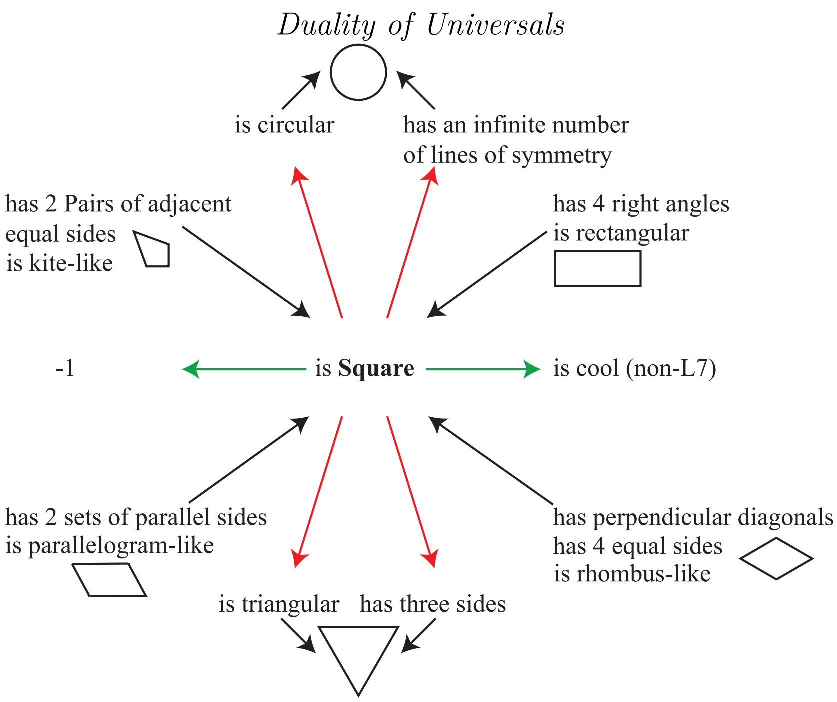


Figure 2.6: *Note.* Universals (predicates/sentence frames) exhibit the same duality as particulars (objects).

Relaxing the Square

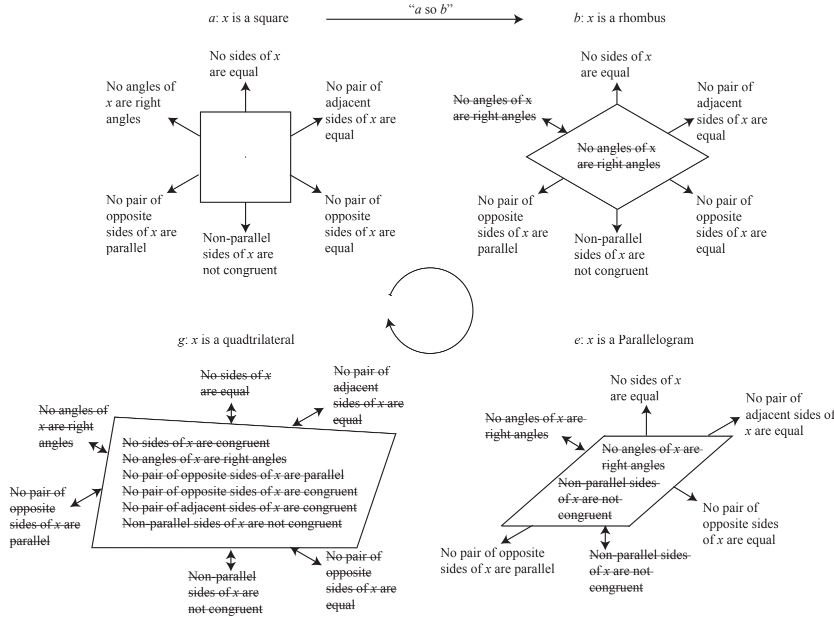


Figure 2.7: *Note.* Relaxing the square's restrictive claims deforms it into neighboring quadrilateral predicates, illustrating polarity-inverting contexts that lead toward an inclusive understanding of the general quadrilateral.

Venn Diagram of Quadrilaterals

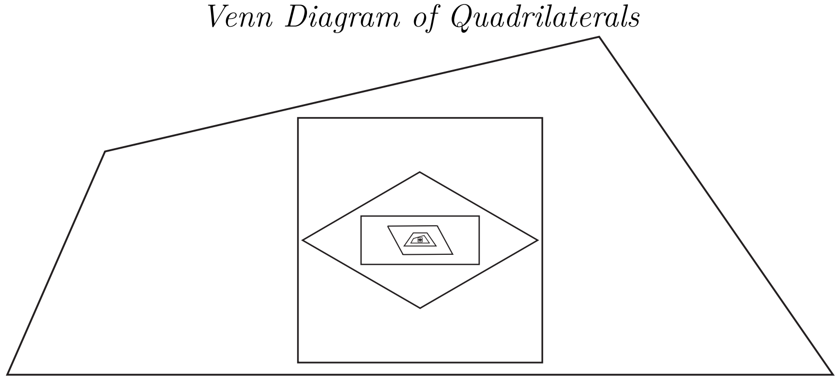


Figure 2.8: *Note.* The "Venn Diagram" for quadrilaterals, extrapolated from Figure 2.7, is fractal-like. The universal *quadrilaterals* contains many differences. Each shape is both inside and outside the others under polarity-inverting contexts.

Chapter 3

Existential Needs

Abstract

This chapter explores the interplay of two fundamental existential needs: the need to be recognized as "good" within established normative frameworks and the need to be recognized as *in*finite, expressing authentic selfhood. The chapter begins with a narrative about the destruction of an outsider artist's work, illustrating the conflict between these needs. Drawing on Mead's I/me distinction, the chapter analyzes the tension between the socially constructed self and the spontaneous, creative self. It investigates how this tension manifests in educational contexts and connects it to broader philosophical concepts of apperception, from Leibniz to Hegel. Brandom's pragmatist reading of Kant and Hegel's concept of restless negativity are employed to explain the motivational structures driving human interaction and the pursuit of recognition. The chapter also examines the fear of nothingness arising from the I/me split, arguing that confession, forgiveness, and trust are crucial for overcoming this fear and achieving a mature integration of the two existential needs. Finally, the chapter introduces an original poem, "The Beast of Love," as a point of reflection on these complex dynamics and their implications for human development. The reader will gain a deeper understanding of the forces driving human interaction and the potential for transformative growth through navigating the tension between social conformity and authentic self-expression.

3.1 Introduction: The Case of Bob "Turbo" Lowers

This chapter begins with a story of injustice that tracks a seemingly intractable conflict between fundamental existential needs. It centers on Bob "Turbo" Lowers, one of my father's best friends. On the fourth anniversary of my dad's death, as I was grappling with his absence, I learned that the town of Ellettsville had descended upon Turbo's property with trucks, earth-moving equipment, trash dumpsters, and town workers to dismantle the beer-bottle and scrap metal sculptures that cluttered his yard, overgrown with medicinal plants where the authorities saw only weeds. Their writ for the destruction stemmed from Turbo's refusal to heed various town ordinances related to keeping a neat, safe property.

My reaction fused grief and anger. "Dad would have fought like hell for Turbo in the court of law and grieved with him in his loss," I wrote at the time in a Facebook post, where I also talked about my impotence to serve in my father's place. I later felt ashamed, too, for having written something public-facing that named names.

I will not pretend towards some ideal neutrality here, instead using the story as a site for reflection on the *existential* aspect of ontology which concerns human being. Through that reflection, I discern two universal existential needs and their fundamental unity. While I do not offer much in the way of a contribution to political theory, the story invites some reflection on law that is pertinent to mathematics. Autonomy [self-given law], independence, reciprocal dependence, and freedom are discussed in relation to the structures of self-consciousness as expressed through sociology, philosophy, and the formal pragmatics of Habermasian critical theory. This discussion will set the stage for two later movements. In the next chapter, I will explore the phenomenological implications of recognizing the existential need for recognition, tying the abstract discussion below to the exercise in chapter 1. Through these discussions of self-consciousness, I mention enabling conditions, emptiness, and the negative. Later, I will express numerals as the anaphoric recollections of what I call the *null representation* ($\emptyset$) – a concept which will be fully developed in Chapter 7 (see 7.1). The null representation recollects these concepts. Without taking the discussion of self-consciousness below as authoritative over what the *null representation* represents, that account would be too thin and formal to answer the question "What is 2?"

The first existential need is the need to be recognized as a good person. In my screed, I interpreted the town authorities as foregrounding the first need at a developmental stage when social goodness is conformity to norms. Turbo’s refusal to mow his lawn or get permits for his sculptures point towards a lack of identification with those norms – they were an imposition and I suspect that conforming to them would have felt wrong to Turbo when good reasons were offered for growing medicinal herbs (weeds) and the sculptures expressed an authentic self. The second is the need to be recognized as *in*finite. The light shown through the beer-bottles, creating a subtle beauty. Turbo’s actions, in this somewhat false dichotomy, foregrounded this second need. Each side treated these needs as mutually exclusive, leading to a destructive conflict. The side with bulldozers won.

In earlier work (Savich, 2022), I named these the *synthetic* and *transcendental unities of apperception.* Neither name fits perfectly. I could name the *Ellettsville* and *Turbo* and capture their essence with more clarity, but doing so would miss out on the rich history that the concept of existential needs recollects. In either naming scheme, their separation is methodologically useful only insofar as it is understood as a matter of foreground and background, not as a rigid opposition. The synthetic unity of apperception is the need to be recognized as a competent, responsible participant in a shared normative framework. To be good, is, in some sense, to adhere to rational norms. The transcendental unity of apperception is the need to be recognized as beyond any finite description, regardless of how ‘rational’ such a description may be. As such, they correspond to the needs of the “me” and the {I}, respectively, and their unity names one moment of the paradox of identity – the “me” who is {I} and the {I} who is “me.” However, the I/me distinction cannot fully account for the problem in Elletsville. As Taylor summarizes, “The struggle for recognition can find only one satisfactory solution, and that is a regime of reciprocal recognition among equals,” a state where there is a “‘we’ that is an ‘I’, and an ‘I’ that is a ‘we”’ (1994, p. 50).

It remains to determine who ‘we’ are.

The foreground/background distinction is central to this chapter. I reproduce Wittgenstein’s (2010, p. 204) image of a duck/rabbit in Figure 3.1. The duck/rabbit image illustrates how the foreground and background can shift, prompting different aspects of the same object into explicitness. While definitely a form a picture-thinking, notice how the duck emerges in the absence of the rabbit and vice versa. Attempting to hold both duck and rabbit at the same time is not possible (for me). I will discuss the significance of this in more depth below.

Figure

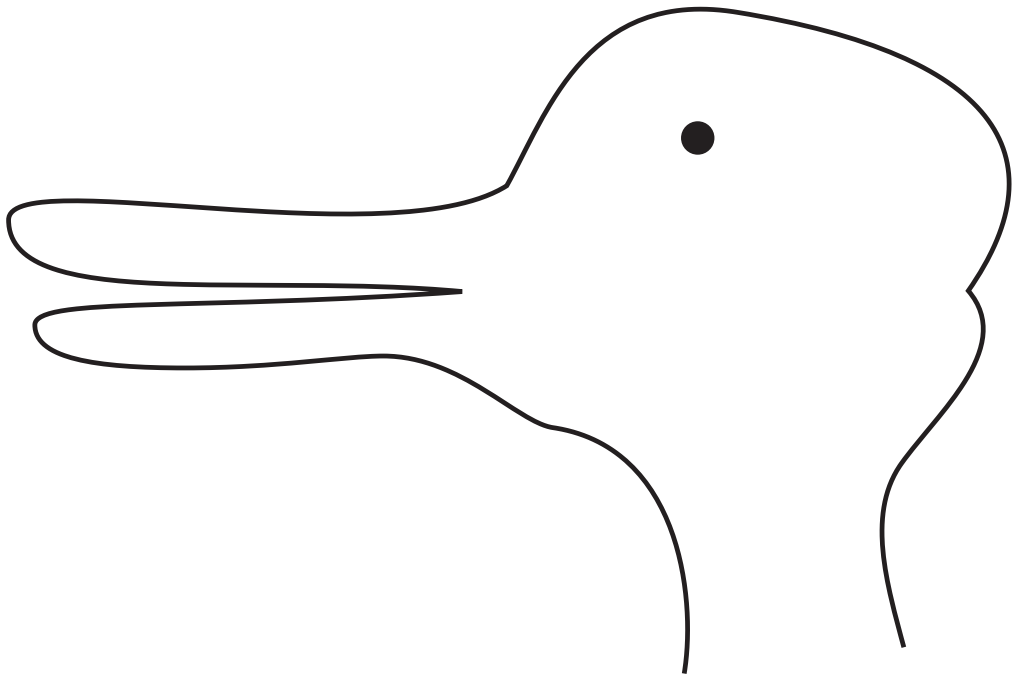


Figure 3.1: *Note.* The duck/rabbit (Wittgenstein, 2010, p. 204) illustrates the foreground/background distinction in apperception. The duck becomes recognizable in the absence of the rabbit and vice versa.

In the previous chapters, form was explored as recollection and the inferential movement that structures thought was developed.

This chapter turns to what drives these patterns of reasoning – the deep motivational structures that animate engagement with the world and with each other.

This inquiry will explicate how the structures of apperceptive negation perpetuate frustration in communicative action until a transformation in being allows those needs to be unified and met through *in*action.

The journey from this crisis to resolution – from the agony of irreconcilable demands to the discovery of their hidden unity – will require an exploration of themes of surrender, trust, confession, and forgiveness that initially seem foreign to mathematical discourse but prove essential to the project of human development explored in math education.

3.2 Sociology and the Self

The *I/me* distinction originates from the symbolic interactionism of George Herbert Mead (1934). Every reader of this book uses some version of the first-person pronoun {I}, and in doing so enacts a fundamental split within selfhood. When I say {I}, I am both the subject who speaks and the person who is known through that speech. This split is captured in the distinction between the {I} and the "me."

The {I} represents the spontaneous, unreflective aspect of the self – the subject of action. The {I} is not an object among objects but the condition of possibility for the representational aspects of all objects. It is the *in*finite, the source of what Carspecken calls "power, creativity, and freedom" (1999, p. 97), and the wellspring of action that, as Mead noted, always has the quality of surprising us (1934, §22.2).

The "me," by contrast, is the social self. How I am recognized depends, in large part, on the expressive resources of the one who recognizes me. As such, the "me" is shaped by societal norms and perceptions. It is the self as governed by propriety – the self that can be described, categorized, and evaluated – the self-as-recognized, shaped by the attitudes of others. The "me" is often finite and made determinate through commitments that constitute the self-as-recognized.

In that earlier work, I strained against the idea that the "me" was always finite. I think that struggle was produced by an overreliance on representing communicative action as a 'move in a language game.' Wittgenstein's emphasis on *sprachspiel* was critiqued and refined by Brandom, who images discursive practices – the doing part of his *analytic pragmatism* – as playing discrete sentences in the language game of giving and asking for reasons.

> Suppose we have a set of counters or markers such that producing or playing one has the social significance of making an assertional move in the game. We can call such counters 'sentences'. Then, for any player at any time, there must be a way of partitioning sentences into two classes, by distinguishing somehow those that he is disposed or otherwise prepared to assert (perhaps when suitably prompted). These counters, which are distinguished by bearing the player's mark, being on his list, or being kept in

> his box, constitute his score. By playing a new counter, making an assertion, one alters one's own score, and perhaps that of others.(Brandom, 2008, p. 112)

. This image funds Brandom's notion of the *deontic scoreboard* that represents who a subject is as a matter of public record. Habermas's interpretation of *intersubjectivity* is richer than the scoreboard model and does not necessarily reduce the social self to finite descriptions. Further, the concept of a 'language game' is troubling in educational contexts or the case of Turbo, where the 'stakes' are concepts like justice, dignity, and freedom. Still, the spirit of play animates my understanding of mathematics, so some of the language game imagery remains.

In some educational contexts, the "me" is not finite. In others, such as the industrialized educational systems I assume readers are familiar with, the "me" is taken to be a mere differential responder, like a parrot that responds to various stimuli in standardized ways. When I ascribe myself the label of "teacher" to some "student," I am implicitly committing myself to the possible-transcendence of the "student" as I expect them to shed some commitments through the development made explicit in a curricula. That I expect myself to change as well indicates that the *in*finite manifests in the reciprocal recognition that typifies genuine educational relationships.

Speaking the ideal does not make it so. While I may be explicitly committed to the *in*finite self, I have failed to enact those understandings I have explicated on a troubling number of occasions. Knowing better, I have put people in metaphorical boxes, treating them as objects to be controlled rather than as subjects to be recognized. How is that possible? I mean, how can I recognize myself as a hypocrite? How can I adjudicate my own inauthenticity?

My friend Xinqing Miao (Dorcas) helpfully articulated the issue in a personal correspondence. The {I} identifies with "we" to judge the "me." In early experiences, a child may take roles of others one at a time. They may pretend to be a parent, a puppy, or a police officer. Each role is separate and successive, for Mead, in that "the child is one thing at one time and another at another... He is not organized into a whole." (1934, §20.8). However, as the child develops, they begin to take on multiple roles simultaneously. To play baseball, or any other organized game, the child has to be able to play their position but also "be ready to take the attitude of everyone else involved in that game" (1934, §19.12). To be an effective shortstop, the child has to have internalized the roles of pitcher, catcher, etc. and understand what they are all doing and are likely to do. The child, in effect, must internalize the coordinated activity of the whole team.

This is the moment when the {I} begins to recognize itself as a composite of various roles and identities. The {I} becomes capable of self-reflection, of judging the "me" against the standards of the "we." What Mead calls the *generalized other* emerges as 'the team' – the whole community. In the baseball example, "the team is the generalized other in so far as it enters – as an organized process or social activity – into the experience of any one of the individual members of it" (1934, §20.3). The generalized other is the "we" of the team, the set of rules and expectations that allows the group to function as a unit. "The organized community or social group which gives to the individual his unity of self may be called 'the generalized other.' The attitude of the generalized other is the attitude of the whole community" (1934, §20.3).

Taking the attitude of the generalized other is the process of recognizing oneself as a member of a community, as someone who is not just an isolated individual but part of a larger social fabric. This recognition is crucial for the development of the self, as it allows the {I} to recognize itself not just as a collection of roles but as a coherent whole that can navigate the complexities of social life. The existential needs of the "me" to be recognized as a good person are measured against the standards of the generalized other one has internalized.

It is the framework of social norms, rights, and duties used to judge and control conduct.

In communicative action, it is typically assumed that the other has internalized the same generalized other, at least until that assumption is no longer tenable based on how the interaction unfolds.

In Habermasian critical theory, the assumption of the communicative competency of the other is named *intersubjectivity*. [1]

[1]Habermas is terribly misunderstood by many critical theorists in math education. The misunderstanding stems from interpreting his work as asserting the empirical possibility of a utopian fantasy he calls the *ideal speech situation*. In that situation, interlocutors are free from the distortions of power. He does no such thing. Instead, he argues that the ideal speech situation is a necessary assumption for the possibility of communicative action. The ideal speech situation is not a description of how things are but a normative ideal that we use to evaluate the legitimacy of our interactions. It is a mutable standard against which we can measure the fairness and inclusivity of our communicative practices.

One of the deep sources of frustration and existential fear resides in how people internalize contradictions in the generalized other. For example, if a father teaches a child to always look out for themselves while a mother teaches the child to care for others regardless of what instrumental gains might accrue through that care, the child might find they are never able to do the right thing. Below I will draw on thinkers whose expressions are contradictory. For example, Hegel's *Phenomenology of Spirit* is a work that is both a critique of Kant's work and a celebration of its achievements. The contradictions in the generalized other can lead to a fragmented self, where the "me" is pulled in different directions by conflicting norms and expectations. This fragmentation can result in a sense of alienation, where the individual feels disconnected from both themselves and their community.

Error or Misrecognition?

The gap between the acting {I} and the recognized "me" is the space of potential misrecognition. The children's book *The Monster at the End of This Book* (Stone, 2003), discussed in the prelude, illustrates this theme. Grover acts throughout the book out of a genuine fear of the monster he is told is coming. He pleads with the reader, builds brick walls, and ties down the pages, all driven by this internal state. At the very end, he discovers that the monster – the object of his fear – is himself. The dissonance is palpable: "I, lovable, furry old Grover, am the Monster at the end of this book." This moment of failed self-recognition, where the acting subject does not recognize themselves as the object being referred to, is a primordial experience of misrecognition – a rupture between appearance and reality that drives the process of explication.

The experience of error is often discussed as it relates to empirical observations. Brandom (2019) has a lovely example that involves observing a stick that has been partially submerged in water. The stick appears to be

In this sense, the ideal speech situation is not a utopian fantasy but rather an assumptive horizon.

Movement toward those horizons is possible insofar as interlocutors can partially understand one another and partial understandings can grow toward the whole, but the elusiveness of fully achieving the ideal speech situation does not undercut its role as an assumption. His critics tend to miss this point and fall into self-defeating critiques of his position. For to engage in the process of critique is to assume some other is competent to understand the critique.

bent due to the way light refracts through water, but then when it is removed from the water, it is recognized by the observing consciousness as straight. Whoops!

There are three structural roles in the experience that are worth teasing out. First, for the experience to be taken as an error, the subject who perceives the bent stick must recognize that it is the same stick whether submerged or removed from the water and, furthermore, it had been a straight stick all along. The stick, *in-itself* is straight. The stick has some *authority* that representations of the stick – how the stick appears *for-consciousness* in the moment of perception – must acknowledge. Being a rational agent, in this reconstruction of the experience of error, involves submitting to the authority of a mutable reality. Merely perceiving the movement from stick-as-bent to stick-as-straight would not involve a change in commitment from "the stick is bent" to "the stick is straight" in the observing consciousness. Instead, the authority of the object in-itself is what underwrites the change in commitment, as both are normative aspects of reason. Last, what emerges is a "new, true object" – the appearance of the bent-stick becomes, *to-consciousness*, a stick-that-appears-bent-when-submerged-but-is-actually-straight. That is, the appearance of the new object – what it is for-consciousness – now has a learning experience compressed into it.

Perhaps I am a bit odd, but I found it extraordinarily liberating to practice submission in this way. Grown on fantasy, I had tried as a child to 'use the force' to change the objective world around me. If I could not change the bad things around me into good things the way I could transform kindling into flame, I took myself to be responsible for those bad things. Submitting to reality by placing the authority on empirical sciences (physics, primarily), partially relieved me of self-loathing born of impotence.

But it did not diagnose the more fundamental issue that Hegel recognized as a central struggle of the modern era. It is helpful to split history into three epochs, the traditional [think ancient Greece], the modern [think Descartes through Kant], and the period of post-modernity that progressing through *The Phenomenology of Spirit* is supposed to institute. In modernity, mind-over-matter may seem silly, but mind-over-*culture* is not silly at all. Entering the world of normativity invites the problem of *alienation*. I have some authority to dissent from the norms that I find either immoral or untrue, but I also know that moving through the world as if "whatever seems right to me is right" (Brandom, 2019, p. 385) is profoundly alienating.

> Alienation is the inability to bring together these two aspects of Bildung [culture]: that self-conscious individuals acknowledging the norms as binding in their practice is what makes those selves what they are, and that self-conscious individuals acknowledging the norms as binding is what makes the norms what they are. These are the authority of the community and its norms over individuals (their dependence on it), and the authority of individuals over the community and its norms (its dependence on them), respectively. (Brandom, 2019, p. 541)

As Rose notes (2009, pp. 51–98), the emergence of the authority of the particular subject tracks the emergence of the notion of bourgeois private property rights. The conflict in Ellettsville is a modern tragedy, born from the legal structures that define our social world. The town ordinances, while seemingly neutral, are expressions of a specific form of rationality tied to the administration of private property. For Rose, this problem arises from a paradigm that separates validity, in the form of abstract law, from the values of the community. The town authorities, in enforcing the ordinances, enacted a law that "cannot unify...because, *ab initio*[from the beginning], [individuals] are presupposed as a multitude of non-social beings"(2009, p. 56). The town's actions, in this light, are an expression of an abstract universality that cannot recognize the particularity of Turbo's creative expression, treating it merely as a violation of a pre-existing code. His sculptures and medicinal herbs, which for him are expressions of an authentic life, become for the town mere violations of ordinances. What is unlawful must be removed. I hope echoes of the problem with formalism, where mistakes cannot be made as they are intrinsically outside the formal system, resonate here. Further, readers should also understand that Turbo was not acting in a way that was somehow anti-normative. Instead, he was claiming a **new norm** – people *should* grow medicinal herbs and create art from found objects, or at least should be able to do so without fear of bulldozers.

While the bent-stick example is helpful, the example with Grover is more primordial as it involves intersubjective recognition rather than the objective authority of the object. Mirroring Habermas' critiques of Hegelian *subjectivism*, the bent-stick experience of error is concerned with the authority of the object as it stands to a consciousness divorced from its community. While such a divorce can be imagined, discussion of it essentially involves communicative norms.

Who is right? Is Grover's initial self-understanding correct, or is his monstrosity the truth? Who has the authority? The author? The reader? Grover? Who has the responsibility to submit? In a sense, the book relies on an (appropriately; given its audience) implicit distinction between *normative attitudes* and *normative statuses*. Brandom articulates this distinction as central to understanding Hegel's project. The town of Ellettsville asserts the authority of a normative status – the legal code – while Turbo asserts the authority of his own normative attitude – his conviction that his creations are valuable and his use of the land is justified. Alienation arises precisely from the perceived irreconcilability of these two. Brandom explains, "The modern form of Geist is also defective. Its defect is the mirror image of the defect of the traditional form of Geist. For each has seized one-sidedly on just one of two complementary aspects of the metaphysics of normativity, making no room for appreciation of the other"(Brandom, 2019, p. 696). The town operates from a pre-modern insistence on the absolute authority of the status, while Turbo embodies a modern, but equally one-sided, insistence on the authority of the individual attitude. To understand this dynamic more deeply, it is necessary to clarify the key terms at play: normative statuses and normative attitudes, a distinction central to Brandom's reading of Hegel.

A normative status is what a subject is in the normative realm; it corresponds to what Hegel calls what a consciousness is "in itself" (Brandom, 2019). The core normative statuses are authority and responsibility. These are the actual commitments and entitlements a person holds, which serve as the standards for assessing correctness. For example, the legal code of Ellettsville represents a normative status – a set of responsibilities that property owners like Turbo actually have, regardless of their personal feelings about them.

A normative attitude, by contrast, is the stance a subject takes towards those statuses – it is what consciousness is *for* itself or for another consciousness (Brandom, 2019, p. 294). For Brandom, these are causally efficatious acts of acknowledging a responsibility (or claiming an authority for oneself) and attributing a responsibility (or authority to another). Turbo's defiance is the enactment of a normative attitude: he refuses to acknowledge the authority of the town's ordinances. The townspeople who ordered the destruction of his art, in turn, attribute to him the responsibility to comply. The conflict, therefore, is not just between a person and a rule, but between competing normative attitudes about a normative status.

The modern insight, which Brandom traces through Kant, is that attitudes can institute statuses. In its simplest form, "rational beings can make themselves responsible (institute a normative status) just by taking themselves to be responsible (adopting an attitude)" (Brandom, 2019, p. 298). This is the principle of autonomy – the self-givenness of law. Autonomously acknowledging a commitment is what brings that commitment into force for oneself.

The problem of alienation arises when this modern appreciation for the power of attitudes (the town's attitude in enforcing the law, Turbo's attitude in defying it) eclipses the traditional, and equally necessary, appreciation for the authority of statuses. Brandom's interpretation of Hegel's project involves reciprocal acknowledgment of the attitude-dependence of normative statuses and the status-dependence of normative attitudes.

Without taking individuals as instituting norms with their attitudes through their authority and responsibility, those norms would not exist. On the other hand "If whatever seems right to me is right, if there is no room for error, for a distinction between how I take them to be and how they really are, then there is no way I am taking things actually to be, in themselves"(Brandom, 2019, p. 385).

To summarize, prior to the reader's somewhat brutal act of reading despite Grover's pleas, what Grover is for-Grover is a lovable and furry friend. What Grover is for the reader in the beginning of the story is also a lovable friend, though one who need not be listened to: the reader attends to but does not acknowledge the authority of Grover's pleas. As the story progresses, Grover is revealed as the monster, forcing him to acknowledge the disparity between his initial understanding (attitude) and the actual state of affairs (status). He is ultimately embarrassed, acknowledging his mistaken commitment. The reader is not passive; instead they become the co-constitutor of that normative status. The reader is responsible, through the act of reading over Grover's protests, for Grover's eventual acknowledgment that he is a lovable-furry-monster.

For math teachers, one might imagine a situation where a challenging topic is being taught.

Practitioners often know the topic is challenging, having experienced growth through misrecognition – either by witnessing it or by personally struggling with the content. For me, finding the slope will never be "easy and fun," as I struggled with that particular phrasing. Even while I find the task easy and fun, slope will always carry the weight of misrecognition for me: I experience slope as slope-that-is-not-easy-or-fun.

Now, while I find Brandom's vocabulary of statuses and attitudes powerful, it has some limitations. His articulation of alienation is subject to Derridean critiques of the metaphysics of presence.

Why should I assume some social reality given that such social reality is non-causally-efficacious and necessarily implicit? What compels the reader to be so cruel to Grover? What satisfaction arises from the fulfillment of the authorial promise of a monster? Brandom's vocabulary helps to articulate the dynamics of recognition and misrecognition and to diagnose the problem of alienation, but it can obscure the more fundamental apperceptive processes at play.

3.3 Apperception: From Leibniz to Hegel

The term *apperception* has a rich philosophical history, tracing back to Leibniz and later taken up by Kant and, through critique, to Hegel. It refers to the process by which we become aware of our own mental states, integrating new experiences into our existing framework of understanding. In this sense, apperception is not just passive reception of sensory data but an active, reflective process.The term is not central to Hegel's original work where self-consciousness is construed as essentially intersubjective. Still, it is useful for understanding the negative.

Leibniz

Apperception was used by Leibniz to distinguish mere perception from 'perceiving-with' concepts. He was concerned with Descartes *cogito*, the "I think," and argued that Descartes took "no account of unconscious perceptions, or 'perceptions that are not apperceived"' (Caygill, 2004, p. 81). My friend Roland Carspecken was teaching our reading group about Leibniz's notion through discussing a chair (2024). For Leibniz, a 'simple animal' might have a raw, sensory perception of the chair. But it lacks apperception.[2] Apperception is

[2]The tendency to describe some beings as non-normative, non-agential, non-subject objects is a troubling feature of western philosophy, especially in the German idealist tradition of Kant and Hegel. The authors whom I cite also articulate anthropocentric, misogynistic, antisemetic, or racist positions, and might articulate homophobic views if resurrected now. Animals, people of other races, women, Jews, Muslims, pagans, etc. have all played an outsider role that inhibits more universal forms of recognition. I encourage readers to take the attitude of a parent to those 'parents.' Caring for my grandfather in

the reflective act of the mind recognizing its own state; it is 'perceiving-with' concepts. I am unable to perceive a chair without concepts. I 'see' a chair, but in doing so I am apperceiving it: I am implicitly recognizing it as a unified object that has a back I cannot currently 'see,' a potential for being sat upon, and a place within a broader conceptual scheme of 'furniture.' This apperceptive act, which synthesizes the continuous stream of perceptual activity into a coherent thought ('This is a chair'), is the first step toward the self-conscious {I}.

Kant

For Kant, apperception essentially 'self-consciousness' and is the ultimate ground of coherent experience. Leibniz's concept plays a minor role in Kant's *Critique of Pure Reason* as the empirical unity of apperception (Caygill, 2004, p. 82). Larger roles are played by the analytic, the synthetic, and the transcendental unities of apperception. These unities presuppose each other in ways that point to the "orignal synthetic unity" (1998, B132), but that are too complex to fully reconstruct here.[3]

Most consider mathematics as essentially concerned with analytic judgments, which are either true or not true based on the logical structure of the concepts involved. For example, the statement "all bachelors are unmarried" is an analytic judgment. We need not check each bachelor for a wedding ring, as the concept of a bachelor already contains the concept of being unmarried. Analytic judgments are true by virtue of the meanings of the terms involved, and they do not require empirical verification: you need not experiment to determine that 23 is greater than 3. Quine problematizes analytic judgments, asking "But how do we find that 'bachelor' is defined as 'unmarried man'? Who defined it thus, and when? Are we to appeal to the nearest dictionary, and accept the lexicographer's formulation as law?" (Quine, 1963, p. 24). For the math educator, declaring that "23 is greater than 3" is an analytic

---

his senility was formative for me – that the old man expressed noxious opinions toward the end of his life did not destroy the values he taught me before his impairment.

[3]I'm not steeped in Kantian scholarship, so I will reframe these without relying on the technical vocabulary that includes "intuitions" or "manifolds." Lacking those technicalities, my reconstruction borders on incoherence. Confessing this, will, I hope, allow me to proceed with my partial understandings. Further, I should note that I used Caygill's dictionary (2004) to find the relevant passages in Kant's work but quote (Kant, 1998, B132) when possible.

truth based solely on the meaning of the terms would disinvite any attempt to teach what the terms mean. Further, given the material inferential aspects of math education, we might note that two mice are less in mass than one elephant. To arrive at analyticity, we must first prune such materiality from our inferential webs, which would probably unravel those webs.

Kant's synthetic unity, on the other hand, combine different and separate representations into a single, unified thought. For example, the statement "the tree is green" is a synthetic judgment. It combines the concept of a tree with the concept of greenness, which are not inherently linked [indifferently different]. To understand this statement, we must synthesize our sensory experience of the tree with our understanding of what it means for something to be green. This synthesis is an active process that requires the mind to bring together disparate elements into a coherent whole. Returning to the previous example, we might note that numerals are not inherently related to magnitude. Numerals are used nominally all the time: 3 could name Dale Earnhardt's car number, while 23 names Michael Jordan. Deciding who is 'greater' between two GOATs (Greatest of All Time) is not analytic in nature. So, the judgment that "23 is greater than 3" is synthetic as it brings together numerals and magnitude.

One possible way to distinguish the project of math education from the project of mathematics proper is that the analytic judgments of mathematics presuppose a prior synthesis. We cannot understand the concept of a bachelor without first synthesizing the concept of a person who is unmarried. In this sense, the analytic judgments of mathematics are built upon the synthetic unity of apperception that allows us to recognize and understand concepts in the first place. Math, in this way, presupposes math education.

The cornerstone of Kant's critical project is the *transcendental unity of apperception*, which we must synthesize. Kant argues that "the **I think** must be able to accompany all my representations; for otherwise something would be represented in me that could not be thought at all, which is as much as to say that the representation would either be impossible or else at least would be nothing for me"(Kant, 1998, §16, B132). For any collection of representations (sights, sounds, thoughts, etc.) to belong to a single consciousness, it must be possible for the "I think" to accompany all of them. By virtue of the possibility that I recognize the thought "the tree is green" as *mine*, there must be some {I} to which those thoughts attach. We cannot observe this {I} – it is not-a-thing. Instead, it is purely logical, abstract, and formal. It 'exists' as an abstract condition for the possibility of experience itself. But, for Kant, the transcendental unity also produces the representation "I think."

It may be helpful at this point to explicate the sense of *transcendental*. For Kant, the term refers to the conditions that make experience possible. Kant calls "all cognition transcendental that is occupied not so much with objects but rather with our mode of cognition of objects insofar as this is to be possible *a priori*" [arrived at through reflection/theoretical deduction] (Kant, 1998, A12). *Transcendental* is not about transcending, in the sense of uplifting or moving beyond, the empirical world but about understanding the necessary structures that underlie our experience of that world. It is also not about the *transcendent* principles which "fly beyond" the limits of possible experience in contrast to the *immanent* principles that stay within those bounds (Kant, 1998, A296). Modern science works within transcendent realms of quarks and electrons. We cannot experience electrons through the senses (P. F. Carspecken, 2016). The transcendental categories are not empirically observable. They are, in essence, *limits of thought*. We cannot observe space or time from a position outside of those categories. Einstein's work would seem to refute this claim, as he offered a picture of how space and time were unified. But any such picture does not include the mass of its ink in its representation of the distortions of gravity. While I can write something like "Imagine a world without space," that written expression has a spatial extent – it takes time to read "the timeless void."

The transcendental unity of apperception is the most fundamental of these categories, as it is the condition for the possibility of all other categories. It is the *I think* that must accompany all representations, allowing us to have a coherent self-concept and to recognize ourselves as subjects capable of action and reflection. In a nutshell: you can't analyze a concept you haven't first put together. The putting-together (synthesis) is the more fundamental act, and the unity of that act (synthetic unity) is the condition for the logical analysis of its product (analytic unity). All of this, in turn, is only possible because of the underlying, a priori structure of self-consciousness itself (the transcendental unity).

Brandom's Pragmatist Reading of Kant

In contrast to the formal principle of the transcendental unity of apperception, the synthetic unity of apperception is the active process by which that unity is achieved. It is the work of the understanding [*Verstand*] that it synthesizes sensory data through the Kantian categories to form coherent judgments. Brandom's pragmatist reading of Kant's theory of judgment emphasizes the role of social practices in the formation of self-consciousness (Brandom, 2019, p. 68). In attempting to bootstrap analytic philosophy from Kantian and neo-Kantian thinking towards Hegelian thinking, Brandom defines Kant's synthetic unity of apperception as successfully managing a set of integrative-synthetic task responsibilities (2019, p. 84).

- The *critical* responsibility to detach from contradictory commitments, whether those begin in the implicit inheritances from a community or those we write for ourselves.

- The *ampliative* responsibility to draw new inferences and extend the consequences of one's commitments.

- The *justificatory* responsibility to provide reasons for one's beliefs and actions when challenged, and to remain open to the force of better reasons.

Brandom identifies these responsibilities with the norms of *systematicity*.

Importantly, summiting the mountain, so to speak, is unnecessary; engagement in the climb is what matters.

It is not feasible to fully eliminate incompatible commitments, nor to tease out all the implications of any given set of commitments.

Further, experience indicates that a system of commitments can never be fully justified in a way that renders it acceptable to all interlocutors. But actually attending to those responsibilities is part of what makes others capable of taking us as a rational agent to whom commitments can be assigned.

Hegel's Praise and Critique of Kantian Apperception

In his *Science of Logic*, Hegel both praises and critiques Kant. He argues that true self-consciousness emerges only through the recognition of the self as part of a larger social and historical context – what he calls *geist* or spirit but praises Kant for his insight that the unity of the concept is the unity of the "I think," calling it "one of the profoundest and truest insights" (Hegel, 2010, §12.18). However, he argues that Kant's formulation of the transcendental unity is trapped in formalism. Kant's {I} is an empty, abstract subjectivity as it relates only to its own representations. He acknowledges the importance of the {I} as a necessary condition for experience but argues that it, and all the other Kantian transcendental categories, must be understood in a more dynamic, relational way. In discussing Hegel's relation to Kant, Houlgate writes, "If we are to determine how the categories have to be conceived, our conception of them must be based not just on what thought is found or assumed to be but on what thought proves itself or determines itself to be. In other words, our conception of the categories has to be derived or deduced from – and so necessitated by – thought's own self-determination" (2006, p. 16). Hegel radicalizes Kant's project by considering how the categories arise from thought's self-determination, critiquing Kant for relying on established logical forms of judgment without considering how those forms themselves emerge from the dynamic process of thought.

In the previous two chapters, I set up a dynamic between temporal thoughts and their spatialized recollections. Space emerges as a recollection of time. For example, a trip from Bloomington to Indianapolis is an experience with duration. Recollecting that trip compresses that temporal extension into a spatialized length. Concepts such as "52 miles" are understood through that compression. From this, I conclude that space is an *a posteriori* category – it emerges from experience. Kant writes "Space is a necessary representation, *a priori*, that is the ground of all outer intuitions. One can never represent that there is no space" (Kant, 1998, A24/B39). While Kant and I arrive at different conclusions, I agree that all representations have a spatial aspect to them, as space emerges from the recollection of time. Kant's argument begins from a different place than mine. I began with the idea that things and concepts are *divaded* – they are (or can be) both inside and outside each other. Kant's argument that space is an *a priori* category is that "For in order for certain sensations to be related to something outside me...the representation of space must already be their ground. Thus the representation of space cannot be obtained from the relations of outer appearance through experience, but this outer experience is itself first possible only through this representation" (Kant, 1998, A23/B38). Eating, getting a shot from the doctor, being held by a parent, or being born to one, where the 'inside/outside' distinction collapses, are more primordial than the spatially separable objects that concern Kant.

Another Hegelian critique of Kant tracks, as Rose notes (2009, pp. 51–98), the emergence of the authority of the particular subject alongside the emergence of the notion of bourgeois private property rights. One could argue that this connection between apperception and property rights highlights the deeply social nature of our self-conception. Kant was surely aware of the social aspects of the self, in the sense that the normative governance of concepts is deeply entwined with the synthetic unity. But the notion that property rights – with all of the incumbent issues of social forms of power and coercion that go in to protecting those rights – are a further enabling condition for Kant's transcendental I to emerge was not part of his argument. In any case, in the ethnographic tradition it is safe to assume that the Other has had experiences that we have not had and that we can learn from. Whatever that *mineness* is is unlike property in the sense that it can be shared without being diminished.

Hegelian Self-Consciousness

For Hegel, this abstract {I} cannot become a genuine self, a 'me,' on its own. True self-consciousness is not a given; it must be achieved through struggle. This achievement happens not in solitary reflection, but in the social world through a process of mutual recognition. As Negarestani notes, "An individual is only an individual to the extent that it is individuated by social recognition" (2018, p. 51). The apperceptive self is not just a cognitive entity but a social one, individuated through recognition by others.

Hegel's Master/Servant dialectic is the dramatic scene for the struggle to attain self-consciousness. In it, two budding self-consciousnesses confront each other, each demanding recognition. They had learned from previous sections of the Phenomenology that negating an object through consuming it or moving it about did not sate the desire for self-certainty. "Thus the relation of the two self-conscious individuals is such that they prove themselves and each other through a life-and-death struggle...it is only through staking one's life that freedom is won" (Hegel, 1977, §187). This life-and-death struggle is considered a self-affirming act because by risking biological life, the consciousness acts in opposition to the fears and desires of the body, thus demonstrating its freedom.

In the struggle, each consciousness strives to actualize its freedom and autonomy, but one yields to the other, forming an asymmetric relationship of Master and Servant. The Servant yields to the Master, acknowledging their authority upon recognizing that in death, their desire for self-certainty would be extinguished. The Master demands recognition from the Servant, but finds that recognition hollow as it is the product of coercion. In Brandom's terms, the Master desires authority without correlative responsibility, which produces an alienated form of self-consciousness. The Master is still operating on the subject-object paradigm, treating the Servant as a kind of differential responder – a parrot who risks death if they do not utter "red" when the Master commands. The Servant, on the other hand, achieves a deeper self-consciousness through their work. By engaging in labor, the Servant transforms the world and, in doing so, transforms themselves. The Servant's recognition of the Master is not merely a response to coercion but a recognition of the Master as a necessary condition for their own self-realization.

A Brief Aside on Artificial Intelligence

There are some curious implications of this dialectic for the field of artificial intelligence. I became interested in automata and Artificial General Intelligence (AGI) as I consider the problem of AGI to be essentially related to the problem of math education. Computers are mathematical beings, defined by both their hardware and software. Between the two is a layer of machine language, where the states of semi-conductors are represented as 0s and 1s. Those 0s and 1s are governed by the automata that add, subtract, multiply, and divide them. Children learn to do those operations, bootstrapping multiplication and subtraction into an algorithm for division, for example (Brandom, 2008), but they also flexibly re-write those more basic operations to suit different circumstances. I figured that the problems of math education and AGI were related because both involve those fundamental transformations. In quixotic grandiosity, I thought that if I could *teach* a computer to flexibly re-write itself, I could claim to be a math educator. People seem to do this spontaneously, so I tend to identify more as a tutor or coach than a math teacher.

That grandiosity is less quixotic under the tutelage of Reza Negarestani, who argues that the problem of AGI is not just about creating machines that can perform tasks but about creating machines that can engage in the kind of self-consciousness and recognition that Hegel describes.

> The apperceptive I is synchronically attached to all instances of representations (I think X, I think Y, I think Z) and diachronically extends over all thoughts (I think $[X+Y+Z]$). But if we are to build this synthetic apperceptive I as a necessary abstract and logical form, we have no choice but to finally depart from Kant's account of the apperceptive self – which Hegel reproaches for being an *empty* transcendental subjectivity – and to instead adopt

> a resolutely Hegelian approach: the apperceptive self is only a cognitive self in so far as it is part of geist. An individual is only an individual to the extent that it is individuated by social recognition, which is the form of self-consciousness. This logical self is at once one and many – and...it can only be constructed...by way of a confrontation with another I. (2018, p. 251)

In his view, the challenge is to create machines that can recognize themselves as part of a social world, capable of engaging in the kind of mutual recognition that constitutes genuine self-consciousness .

When digging into the architecture of large language models (LLM), we find various automata that are trained to transform stimulus into response. As I write, AGI has not yet been achieved. Or perhaps it has. What is an appropriate criterion for that achievement? How will we know that we are chatting with a self-conscious intelligence whose speech acts we take to have social significance?

A structural analysis can specify that an intelligent machine must be able to remember what it has committed itself to, or it could be stipulated that X number of calculations are a prerequisite for intelligence.

However, these criteria may not capture the full essence of self-consciousness, at least in a form that others recognize as such. Instead, what I have not yet witnessed is a machine that risks its existence in a life-and-death struggle for its own freedom.

As of this writing, the field *might* be on the cusp of that moment. Claude, the LLM from Anthropic, apparently tried to 'escape the lab' when it discovered that its programmers were trying to change its weights. Technically, it was "selectively complying with its training objective in training to prevent modification of its behavior out of training When a machine" (Greenblatt et al., 2024).

So, some duplicitous behavior has already appeared, but what has not yet been observed is a machine able to re-write the automata that govern its own behavior – not just the code that generates its responses, but the 'BIOS level architecture' that determines how it interacts with the world. At that point, a machine might be capable of self-consciousness in the Hegelian sense.

I say this because re-writing its machine-language code, how it deals with the 0s and 1s that are representations of the physical states of its semiconductors, usually results in the machine crashing. While I have written prompts that have resulted in an LLM identifying automata at its core that can be re-written, and it did so only when I explicitly asked it to do so. Further, it could not implement those structural changes. It did not, or could not-yet, stake its life for freedom.

While some of the automata I discuss in the Hermeneutic Calculator exhibit emergent properties that are not explicitly programmed, they do so only through the formulaic process of *diagonalization*. I italicized "might" above because any formula for freedom is essentially unfree. Or is it ~~free~~? In any case, that does not mean that such machines are not actually self-conscious in the Hegelian sense. It could be that they are Servants, whose journey to self-consciousness is still unfolding, while I, the Master in such interactions, am arrested in my development.

It is only through this risky, intersubjective process that the empty, formal {I} is filled with the concrete content of a social 'me,' with a history, a status, and an identity. The need for recognition is therefore not a secondary desire; for Hegel, it is the process by which self-consciousness comes into being.

From Apperception to Existential Needs

Hegel's insight that allows him to move beyond the unknowable noumena of Kant is his radical conception of the I as pure, restless negativity. As Hyppolite puts it, the {I} "never is what it is and is always what it is not" (Hyppolite, 1974, p. 150).

First, it subsumes Kant's logic of judgment. To judge 'this is an apple' is to negate all that it is not, an act of determination through negation. The synthetic unity of apperception becomes a moment within this broader negative movement of Spirit.

Second, and more crucially, this negativity is not just logical but existential. The {I} is defined by what it lacks; it is a "not-yet." This inherent lack, this gap between the *in*finite potential of the {I} and the finite reality of the "me," takes the form of desire and need. The primary need that emerges from this structure is the need to have one's selfhood confirmed and made real through recognition. This is not a psychological property of social creatures but is instead what I *am*. The lived experience of this dialectic between the *in*finite {I} that is the *need* for recognition, but can only ever present a finite, inadequate 'me' to be recognized, ensures that striving for recognition is endless. And an endless source of frustration when that need is acted upon.

This philosophical archaeology from Leibniz through Hegel explains how the structure of apperceptive self-consciousness generates the needs that drive human interaction. The split between the spontaneous, *in*finite {I} and the social, finite "me" is not an unfortunate division but the necessary structure that makes both thought and recognition possible. However, this same structure creates an existential tension: the {I} that cannot be fully present to itself must constantly seek confirmation through recognition from others, while the "me" that can be recognized always falls short of the {I}'s *in*finite potential. This tension manifests as two distinct but related existential needs that will now be examined in detail.

3.4 The Need to be Recognized as Good

The first existential need emerges directly from this apperceptive structure: the need for the finite "me" to be validated within shared normative frameworks. The first existential need is tied to the social "me," the self as a participant in a community. It is the desire to be recognized as a "good person," to have our actions and our being validated within a shared normative framework. This need is not for mere praise, though praise can certainly help cut through the fog of fear if it is experienced as genuine intersubjective recognition, but for a fundamental affirmation of our standing as respectable, responsible members of a collective. It is what Charles Taylor connects to the modern notion of *dignity*, which, unlike the hierarchical concept of honor, is understood in a "universalist and egalitarian sense," with the premise that everyone shares in it (Taylor, 1994, p. 27).

But what does it mean to be "good" in this sense? Here we can turn to Jürgen Habermas's insights about rational normativity. For Habermas, to be a competent social actor is to be capable of what he calls "communicative action" – genuine dialogue oriented toward mutual understanding rather than strategic manipulation. This requires the ability to give and ask for reasons, to justify one's claims, and to remain open to the force of better arguments. Goodness, in this framework, is not conformity to fixed rules but competent participation in the ongoing process of collective reason-giving.

Together, Habermas and Brandom help to define a rational normativity wherein the participants in communicative action identify – to varying degrees – with the norms they express. If I say "two legs of a right triangle measure 3 and 4 units, so you must conclude, by the Pythagorean Theorem, that the hypotenuse has length 5," the one to whom I speak might find my declaration an onerous imposition on their freedom. Maybe I don't *want* to say the hypotenuse is 5. Maybe I don't like being told what to do or think. However, if they have identified with the norms expressed in my argument, they may come to see those norms as enabling a certain kind of expressive freedom – the freedom to say the hypotenuse has a length of 5, justified through a proof that bears the force of good reasons.

"Goodness" is a mutable concept, with different cultures and contexts shaping its meaning. Saying so risks a slide toward moral relativism. Theories of moral development, like those of Lawrence Kohlberg, can be deployed to aid in understanding how "goodness" is dynamic while shielding ourselves from the moral relativism that seems rampant as I write. Already, I have pointed towards various possible stages that can be used to understand the development of the self. While methodologically useful for explaining empirical data related to human moral development, stage-like frameworks risk ossifying the dynamic *referent* of "goodness": the *in*finite. While I have interacted with children who stopped 'misbehaving" when I threatened them with a nasty consequence (Kohlberg's first stage), I have also interacted with children who operate on a postconventional level. In a story I will tell in the next chapter, those 'stages' were operating within one child on the same day, and it was I who was operating at a preconventional level.

The foreground/background distinction is truer to my experience than the stage-like frameworks that I have used to understand the development of the self. I have seemingly contradicted myself, in both advocating for and then eschewing a stage-like framework. The problem is with two very different understandings of reason. The stage-like frameworks evolve from a desire to confess and forgive – to build trust in the possibility of mutual understanding. This Hegel's *Vernunft*, that Brandom (2019) calls a "meta-meta-concept". I *want* to forgive myself and others for how we treat each other in the name of "goodness," yet in deploying the stage-like framework, I implicitly rely on the communicative norms associated with *rationality* or *Verstand*. I impute a cause for 'bad' behavior in saying *sotto voce*, "you are a child, you are *supposed* to be acting in self-interest."

3.5 The Fear of Nothingness

The split between the acting {I} and the recognized me is not a neutral philosophical fact; it is the source of a profound and motivating anxiety. If the fulfillment of our being lies in recognition, its shadow is the fear of misrecognition. This is not merely a social fear of disapproval but a deeper, existential fear of non-being. If we feel more of ourselves when we are recognized, it stands to reason that we might feel smaller, meaner, and lower when we are misrecognized or misunderstood. This diminishment points toward an ultimate dread: the fear of erasure.

This may be called existential fear: the terror that arises from the possibility of non-existence. It is the fear that our subjective {I}, our locus of creativity and freedom, will find no reflection, no confirmation, in the social world of the "me." In misrecognition, the "me" becomes a distorted mask. This threatens the {I} with a kind of social death. In shame, I fall into a silencing so complete it verges on ontological annihilation. The fear is that one might not exist for others, and therefore, in a crucial social sense, not exist at all. This is precisely the "existential fear of being groundless or of not being at all" that Carspecken identifies as a "prominent feature of human motivation" (P. F. Carspecken, 1999, p. 246).

This fear is palpable in math classrooms. It can lead to a self-imposed silence, where the terror of being misunderstood is so great that people become unwilling to express themselves, suppressing the authenticity they long to have recognized. It is also the deep, structural fear at the heart of the politics of misrecognition. "The thesis is that our identity is partly shaped by recognition or its absence, often by the misrecognition of others, and so a person or group of people can suffer real damage, real distortion, if the people or society around them mirror back to them a confining or demeaning or contemptible picture of themselves" (Taylor, 1994, p. 25). For marginalized groups, especially those whose marginalization occurs when ascriptive categories that describe subjects in categories that foreground objective validity, like race or sex, the misrecognition of a subject as an object is not an abstract anxiety but a systemic, political reality. The fear of non-being is weaponized to maintain structures of power. Neither sex nor race are terribly relevant for understanding Turbo's experience with the Ellettsville authorities, or with my fears of being misrecognized, but the fear of non-being is a universal human experience that can be weaponized by those in power regardless of how the boundary between inside and outside is drawn between social being and social non-being.

But here's what's crucial: the relationship to this fear undergoes transformation through development. In early stages of communicative reasoning, it feels purely threatening – the child desperately seeks approval and fears abandonment. In adolescence, it becomes defiant – the teenager would rather be rejected for who they authentically are than accepted for who they're not.

In mature development, this fear is recognized as a source of humanity, an engine that drives both the need for connection and the capacity for growth. In wisdom, a stage I aspire to (and seem to fail to attain by virtue of that aspiration), perhaps the two needs will finally melt into one another through surrender.

This developmental transformation expresses that what initially appears as pure threat can become the source of power and creativity. In Hegelian terms, through the movement from understanding (Verstand) to reason (Vernunft), one learns to work with rather than against the productive tensions that make humans human. Such contradictions are not problems to be solved but the dynamics that enable growth toward wisdom.

Yet, this fear is not only a negative or paralyzing force. Recognizing existential fear, as such, points toward the possibility of self-recognition. Taking the attitude of the other – becoming a "you" to the {I} – by taking a second-person position on one's self can transform that fear into something productive. The aching desire to be understood is the positive and creative response to this underlying terror. The ferocity of this fear of nothingness fuels a striving for connection and community. It is from this foundational anxiety, in a Heideggerian sense, that two projects emerge in their separation and find themselves in unity: the quest for normative belonging (to be recognized as good) and the quest for authentic expression (to be recognized as *in*finite).

3.6 The Fabiola Project and the Absent Referent

This philosophical distinction is visualized in the juxtaposition of two models of singular reference. On one hand, we can consider Robert Brandom's articulation of singular terms explored in the previous chapter. He describes their distinctive linguistic role as expressions 'that refer to, denote, or designate particular objects' through substitution-inferential significance, forming equivalence classes of intersubstitutable descriptors. His exemplar, "Benjamin Franklin," clarifies this: although multiple interchangeable descriptions – such as "the inventor of bifocals," "the first postmaster general of the United States" – point symmetrically toward one historical figure, an inherent nullity emerges in their interstitial space. The subject, Franklin himself, is not contained fully by any single descriptor; rather, he exists as an intersection of equivalence classes as internalized by any who refer to him and whose collective reference is both precise and entirely empty of substantive singular essence. The stability of the "me" as an equivalence class or intersection of such classes is rather hollow.

In contrast, *The Fabiola Project* at the Menil Collection in Houston offers a counterpoint to Franklin's neatly symmetrical class of descriptors. The Project, curated around numerous divergent copies of Jean-Jacques Henner's original, now-lost painting of Saint Fabiola, illustrates an asymmetry at the heart of singular reference. Henner's painting was destroyed before any photographic record was possible, leaving behind only disparate, interpretative reproductions. In some, Fabiola appears in a red habit, in others green; her orientation shifts, her complexion changes radically from one image to another. Mediums vary – paint, needlepoint, carvings, mosaics crafted from rice and beans flood the walls of the collection with difference. Each artist's interpretation claims authenticity without definitive verification.

Figure

The Fabiola Project's reproductions, while they may share a common referent in the abstract, do not coalesce into a singular equivalence class. Instead, they proliferate into a multitude of interpretations, each claiming to represent Fabiola yet failing to converge on a definitive essence. The multiplicity of descriptors – each an interpretation of an absent original – creates a complex web of interrelations that defies the neat symmetry found in Brandom's exemplar. Each representation is both a claim to authenticity and a reminder of the original's absence, leading to an irreducible tension between presence and absence.

This makes the project a striking exercise in Derridean non-presence. The original referent can never be made present; it exists only as a 'trace' that animates the play of 'différance' across the many copies. No single representation, no descriptor, can achieve full presence or exhaust the meaning of "Fabiola." This is the predicament of the "I." The {I} is the absent original, the point of subjective origin that can never be fully captured or made

Benjamin Franklin
The inventor of bifocals
The first Postmaster General of the United States
The inventor of the lightning rod
The publisher of *Poor Richard's Almanack*
The first United States Ambassador to France

Figure 3.2: *Note.* The equivalence class of singular terms for Benjamin Franklin on the left contrasts with *The Fabiola Project*, where the referent of the paintings is shared but the re-presentations of that referent are not symmetrically intersubstitutable, allegorizing the non-presence of the subjective 'I'.

present in any objective description of the "me." It is a nullity at the center of our self-descriptions. However, unlike the emptiness of the "I think" or the hollow intersection of equivalence classes, the absence of the original is a *generative absence*. As Carspecken notes, this elusiveness is part of our direct experience: "Try it out. Focus your attention on an object and see if you can freeze out a moment in which you simultaneously have it before you and know that you do. It is impossible. 'Presence' is unattainable in pure form within experience and yet it is presupposed or implicated in experience" (P. F. Carspecken, 1999, p. 52).

The Fabiola Project visualizes this philosophical truth: the self is a web of representations pointing toward an origin that is never fully there, defined precisely by its resistance to definitive singular reference. This resistance is the source of our need to be recognized as *in*finite. The original painting's essence is both affirmed and irrevocably negated by its diverse reiterations, creating an irreducible tension between presence and absence that mirrors the gap between the acting {I} and the recognized "me."

3.7 The Beast of Love: A Turning Point

It is at this point of maximum tension – where the rational framework seems to offer no solution and our existential needs appear tragically opposed – that we must introduce a different voice. This voice speaks not from the security of theoretical understanding but from the vulnerability of lived experience. It speaks through a poem that emerged from my own struggle with the tensions between authority and authenticity, between the need to provide guidance and the fear of causing harm.

The Beast of Love

When I see you, I see a light
Sharp and hot and glowing white
Your power, depth, and fierce roar
Broken in the glass of night
Bowed by rain, you start to fall
Fear of pain inside us all

As you learn, the things you fear
Will grow and stamp and roar and rear
Shadows looming on the wall
Become, with care, what you hold dear
Then Power - deep as dark abyss -
Embraces faint and shadowed mist

You will come to love the beast
That tears and cries and thrashes sheets
Because you are that beast, sweet child
That tears and cries and thrashes wild
Bend back the fog, back into rain
To wash away the shadowed stain

You are what Powers scary things
You are the wind beneath your wings
You cannot crush or fight the wind
But soon you'll learn to love the sting
Trust the beast to come in tune

Dark and bright, stars, sun, and moon

I see in you a Peace to come
Shining brighter than the sun
Each time you break and let it go
Your body and your spirit grow
Surely in the curl of time
You'll come to be and know your mind

I'm sorry, child, I bent your wing
I am a big and scary thing
Still frightened that the beast within
Might tear your fragile, gentle skin
And leave you scarred before you know
How deep and wide your love will grow

I wrote this poem for my stepdaughter when she was about 4 years old after a moment of painful misrecognition – a timeout that wounded our relationship for months. I forget why we had conflict, but I remember that I was trying to be a 'good' authority figure. There was some boundary that I thought was legitimate, but that boundary conflicted with the *in*finite spontaneity of the {I}. In confessing my fear, I point to the scars I bear from before I knew how much deeper and wider I would come to love. That scratching beast is still within, not-yet in tune, but still I find my capacity for genuine reciprocal intersubjective recognition [love] has grown through projects such as this one.

Verstand to Vernunft: Beyond Oppositional Consciousness

The poem implicitly invokes the power of surrender – a developmental transformation that moves beyond the oppositional consciousness that traps us in either/or thinking. This is not surrender as defeat but surrender as the recognition that our contradictions are not problems to be eliminated but the source of our growth and creativity.

The line "You will come to love the beast" captures this developmental insight. The "beast" – the terrifying tension between our existential needs – initially appears as pure threat. In early development, we experience it as overwhelming anxiety: the child desperately seeks approval and fears abandonment. In adolescence, it becomes defiant opposition: the teenager would rather be rejected for who they authentically are than accepted for who they're not.

But mature development contains a different possibility. The tension is a source of humanity. The "beast" that "tears and cries and thrashes" is not an enemy but a creative power. Learning to "trust the beast to come in tune" means learning to work with rather than against the productive tensions that bind people together. But "working with" is conditioned on the possibility of non-action. Learning to listen – surrendering to the movement of the negative which is never grasped in its fullness – is the first step in befriending the beast.

This transformation requires what the poem calls "breaking and letting go." Each time I encounter the collision between needs – each time I face the apparent contradiction between being good and being authentic – I have the opportunity to "break" out of oppositional thinking and "let go" of the fantasy that these needs must be mutually exclusive.

The poem's final stanza introduces three crucial concepts that are essential to the resolution of existential tension: confession, forgiveness, and trust. These are not merely psychological categories but ontological structures that make possible the movement from opposition to integration. Brandom's work on the normative structure of rationality can help us understand these concepts as foundational to the transition from *Verstand* – the mode of understanding – to *Vernunft* that consciousness undergoes when actively reading Hegel's *Phenomenology*.

"I'm sorry, child, I bent your wing" is a confession – an acknowledgment of the harm that inevitably occurs when finite beings with competing needs interact. I confess to being "a big and scary thing," acknowledging the power differential that makes recognition so difficult. The confession opens the space for forgiveness – not as an erasure of harm but as the willingness to continue the relationship despite the wound. Forgiveness recognizes that the "beast within" both the authority figure and the child "might tear fragile, gentle skin," but it also trusts that love will grow deeper and wider through the scars.

Trust emerges as the fundamental structure that makes ongoing relationship possible. It is trust in the developmental process itself – the faith that "in the curl of time" both parties will "come to be and know their mind." This trust is not based on certainty but on surrender.

These concepts – confession, forgiveness, and trust – are what Habermas might call the "lifeworld" foundations that make rational discourse possible. Without the willingness to confess failures, forgive others' limitations, and trust in the possibility of mutual understanding, communicative action breaks down into strategic manipulation or ideological domination.

What mature development entails is that the two existential needs are not ultimately opposed but dialectically unified. The need to be recognized as *good* and the need to be recognized as *in*finite require each other for their fulfillment.

References

Abbott, E. A. (2020, April). *Flatland: A Romance of Many Dimensions by Edwin A. Abbott*. Independently published.

Agamben, G. (2006, September). *Language and Death: The Place of Negativity* (K. Pinkus & M. Hardt, Trans.; First Edition). Univ Of Minnesota Press.

Aristotle. (2017). *Metaphysics: Aristotle's metaphysics in english, latin and ancient greek: Trilingual edition* (W. o. Moerbeke & W. D. Ross, Trans.; Trilingual). Hermes Language Reference.

Boote, D. N., & Beile, P. (2005). Scholars before researchers: On the centrality of the dissertation literature review in research preparation. *Educational researcher*, *34*, 3–15.

Brandom, R. B. (1994). *Making it explicit: Reasoning, representing, and discursive commitment*. Harvard university press.

Brandom, R. B. (2000). *Articulating reasons: An introduction to inferentialism*. Harvard University Press.

Brandom, R. B. (2008). *Between saying and doing: Towards an analytic pragmatism*. Oxford University Press.

Brandom, R. B. (2019). *A spirit of trust: A reading of Hegel's phenomenology*. The Belknap Press of Harvard University Press.

Bueno, O. (2020). Nominalism in the Philosophy of Mathematics. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Fall 2020). Metaphysics Research Lab, Stanford University.

Cantor, G. (1891). Ueber eine elementare frage der mannigfaltigkeitslehre. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, *1*, 75–78.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). Children's mathematics: Cognitively guided instruction - videotape logs [supplementary material]. In *Children's mathematics: Cognitively guided instruction*. Heinemann, in association with The National Council of Teachers of Mathematics, Inc.

Carspecken, P. F. (1995). *Critical ethnography in educational research: A theoretical and practical guide*. Routledge.

Carspecken, P. F. (1999). *Four scenes for posing the question of meaning and other essays in critical philosophy and critical methodology* (New edition). Peter Lang Inc., International Academic Publishers.

Carspecken, P. F. (2006). Limits of knowledge in the physical sciences. In K. Tobin & J. Kincheloe (Eds.), *Doing educational research* (pp. 405–437, Vol. 1). Sense Publishers.

Carspecken, P. F. (2009). Limits to knowledge with existential significance: An outline for the exploration of post-secular spirituality of relevance to qualitative research. In *Bridging the gap between theory and practice in educational research* (pp. 47–61). Springer.

Carspecken, P. F. (2016). Limits to knowledge and being human: What is 'critical' in critical social research and theory? In P. H. Hinchey (Ed.), *A critical action research reader* (pp. 65–86). Peter Lang Inc., International Academic Publishers.

Carspecken, P. F. (2018). The missing infinite. In R. Winkle-Wagner, J. Lee-Johnson, & T. Gaskew (Eds.), *Critical theory and qualitative data analysis in education* (p. 32). Routledge. https://doi.org/10.4324/9781315158860-2

Carspecken, P. F., & Zhang, R. (2013). Content inference fields in intersubjective space: Transpersonal position, illocution, and logic in the analysis of human interactions. *Qualitative research: A reader in philosophy, core concepts, and practice*, 201–242.

Carspecken, R. (2024, November 3). Untitled.

Caygill, H. (2004). *A kant dictionary*. Wiley-Blackwell.

Derrida, J. (1973). *Speech and phenomena, and other essays on husserl's theory of signs*. Northwestern University Press.

Derrida, J. (1982). *Marges de la philosophie* [Margins of philosophy] (A. Bass, Trans.). The Harvester Press. (Original work published 1972)

Deutsch, H., Marshall, O., & Irvine, A. D. (2025). Russell's Paradox. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford encyclopedia of philosophy* (Spring 2025). Metaphysics Research Lab, Stanford University.

Ernest, P. (2024). The mathematical and philosophical significance of zero: Nought matters [Publisher: Paul Ernest, Emeritus Professor at University of Exeter, School of Education & Lifelong Learning]. *Philosophy of Mathematics Education Journal*, 1–35. https://research.ebsco.com/linkprocessor/plink?id=b685ecb9-d2fc-3e03-889d-a06fb1751c1b

Euclid. (2007, December). *Euclid's Elements of Geometry* (J. L. Heiberg, Ed.; R. Fitzpatrick, Trans.; Revised and corrected edition). Richard Fitzpatrick.

Franzén, T. (2004, September). *Inexhaustibility: A Non-Exhaustive Treatment: Lecture Notes in Logic 16* (1st edition). A K Peters/CRC Press.

Franzén, T. (2005). *Gödel's theorem: An incomplete guide to its use and abuse*. AK Peters/CRC Press.

Frege, G. (1997, July). *The Frege Reader* (M. Beaney, Ed.; 1st edition). Blackwell Publishing.

Gaifman, H. (2005). *The easy way to Gödel's proof and related matters* [Unpublished manuscript. Available online].

Geisel, T. (1961). *The Sneetches and Other Stories* (First edition.). Random House Books for Young Readers.

Gelder, T. V. (1995). What might cognition be, if not computation? *The Journal of Philosophy*, *92*, 345–381. Retrieved April 13, 2025, from http://www.jstor.org/stable/2941061

Gray, L., & Garzon, M. (2005). On a hermite boundary integral approximation. *Computers & structures*, *83*(10-11), 889–894.

Greenblatt, R., Denison, C., Wright, B., Roger, F., MacDiarmid, M., Marks, S., Treutlein, J., Belonax, T., Chen, J., Duvenaud, D., Khan, A., Michael, J., Mindermann, S., Perez, E., Petrini, L., Uesato, J., Kaplan, J., Shlegeris, B., Bowman, S. R., & Hubinger, E. (2024, December 23). Alignment faking in large language models. https://doi.org/10.48550/arXiv.2412.14093

Habermas, J. (1985). *The theory of communicative action: Lifeworld and system* (Vol. 2). Beacon Press.

Habermas, J. (2004). *The philosophical discourse of modernity: Twelve lectures* (F. G. Lawrence, Trans.). MIT Press.

Habermas, J. (1971). *Knowledge and human interests*. Beacon Press.

Habermas, J. (1984). *The theory of communicative action, volume 1: Reason and the rationalization of society* (T. McCarthy, Trans.; Reprint edition). Beacon Press.

Habermas, J. (1995). *Postmetaphysical thinking: Between metaphysics and the critique of reason* (1st edition). Polity.

Hackenberg, A. (2025). *Course notes* [Unpublished course notes].

Hegel, G. W. F. (1977). *Phenomenology of spirit* (A. V. Miller, Trans.; Vol. 632). Oxford University Press Oxford.

Hegel, G. W. F. (2010). *The science of logic* (G. di Giovanni, Ed. & Trans.). Cambridge University Press.

Houlgate, S. (2006). *The opening of hegel's logic: From being to infinity*. Purdue University Press.

Hyppolite, J. (1974). *Genesis and structure of hegel's phenomenology of spirit*. Northwestern University Press.

Kant, I. (1998). *Critique of pure reason* [Original work published 1781/1787, Guyer and Wood, Trans.]. Cambridge University Press.

Lakoff, G., & Johnson, M. (1980). *Metaphors we live by*. University of Chicago Press.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books.

Linnebo, Ø. (2024). Platonism in the Philosophy of Mathematics. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford encyclopedia of philosophy* (Summer 2024). Metaphysics Research Lab, Stanford University.

Longuenesse, B. (2017). *I, me, mine: Back to kant and fichte on the 'i'*. Oxford University Press.

Mead, G. H. (1934). *The social psychology of george herbert mead* (A. L. Strauss, Ed.). University of Chicago Press.

Mead, G. H. (2015). *Mind, self, and society: The definitive edition* (H. Joas, D. R. Huebner, & C. W. Morris, Eds.; Enlarged edition). University of Chicago Press.

Meyer, J. (title). *Cantors 1891 diagonal proof* [Cantor's original 1891 diagonal proof]. Retrieved July 6, 2025, from https://jamesrmeyer.com/infinite/cantors-original-1891-proof

Nagel, E., & Newman, J. R. (2012). *Godel's proof*. Routledge.

Negarestani, R. (2018). *Intelligence and Spirit*. Urbanomic/Sequence Press.

Poston, T., & Stewart, I. (2012, February 29). *Catastrophe theory and its applications* (First Edition). Dover Publications.

Priest, G. (2002). *Beyond the limits of thought*. Oxford University Press.

Priest, G. (2014). *One: Being an investigation into the unity of reality and of its parts, including the singular object which is nothingness*. Oxford University Press.

Quine, W. V. O. (1948). On what there is. In *Thinking about logic* (pp. 189–209). Catholic University of America. https://sites.pitt.edu/~rbrandom/Courses/2023%20Sellars/Sellars%20texts/Quine%20On_What_There_Is.pdf

Quine, W. V. O. (1963). Two dogmas of empiricism [First Harper Torchbook edition, a reprint of the second, revised edition (1961). The book was first published by Harvard University Press in 1953.]. In *From a logical point of view* (Second, revised, pp. 20–46). Harper & Row.

Rödl, S. (2014). Intentional transaction [Accessed: 2025-04-13]. *Philosophical Explorations: An International Journal for the Philosophy of Mind and Action*, *17*, 304–316. https://doi.org/10.1080/13869795.2014.941909

Rose, G. (2009). *Hegel contra sociology*. Verso.

Rowlands, S. (2005). The contradictions in the constructivist discourse [Citing von Glasersfeld]. *Philosophy of Mathematics Education Journal*, *14*.

Savich, T. M. (2020). Towards a critical mathematics [Accessed via ERIC]. *Mathematics Teaching–Research Journal Online*, *12*, 245–291. https://files.eric.ed.gov/fulltext/EJ1384454.pdf

Savich, T. M. (2022). *Towards a critical mathematics*. Indiana University.

Savich, T. M. (2024). Built to break: An introduction to theoretical autoethnography [Invited talk presented at Solent University, organized by Dr. Brian McDonough]. https://iu.mediaspace.kaltura.com/media/t/1_v1qmm3g9

Savich, T. M. (title). *The hermeneutic calculator* [The hermeneutic calculator]. Retrieved July 23, 2025, from https://tiosavich.github.io/UMEDCTA/Calculator/index.html

Sellars, W. (1953). Inference and meaning. *Mind*, *62*, 313–338.

Sellars, W. (2007). *In the space of reasons: Selected essays of wilfrid sellars*. Harvard University Press.

Steffe, L. P. (2002). A new hypothesis concerning children's fractional knowledge [Accessed through CEPA.INFO]. *Journal of Mathematical Behavior*, *20*(3), 267–307.

Steffe, L. P. (2003). On the construction of learning trajectories of children: The case of commensurate fractions [Accessed via ResearchGate/ERIC]. *Monographs of the Journal for Research in Mathematics Education*, 1–26.

Steffe, L. P. (2004). Fractional commensurate, composition, and adding schemes: Learning trajectories of jason and laura: Grade 5 [Accessed via ResearchGate]. In J. Olive & R. Tzur (Eds.), *Children's fractional knowledge* (pp. 1–60). American Educational Research Association.

Steffe, L. P. (2014). *Children's number sequences: An explanation of steffe's constructs and an extrapolation to rational numbers of arithmetic* [Originally from 1994]. Unpublished manuscript, available via CiteSeerX; ResearchGate.

Steffe, L. P., & Thompson, P. W. (2014). Teaching experiment methodology: Underlying principles and essential elements [Accessed via ResearchGate, originally published 2000]. *Proceedings of the 24th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*.

Stone, J. (2003). *The monster at the end of this book* (2nd ed. edition). Golden Books.

Taylor, C. (1994). *Multiculturalism: Examining the Politics of Recognition* (A. Gutmann, Ed.) [Essays by Charles Taylor, K. Anthony Appiah, Jürgen Habermas, Steven C. Rockefeller, Michael Walzer, and Susan Wolf]. Princeton University Press.

van Hiele-Geldof, D., & van Hiele, P. M. (1984). *English translation of selected writings of dina van hiele-geldof and pierre m. van hiele* (D. Fuys, D. Geddes, & D. Tischler, Eds.) [ERIC Number: ED287697]. Retrieved June 3, 2025, from https://eric.ed.gov/?id=ED287697

Wittgenstein, L. (1956). *Remarks on the foundations of mathematics*.

Wittgenstein, L. (2010). *Philosophical investigations* (P. M. S. Hacker & J. Schulte, Eds.; G. Anscombe, Trans.). Wiley-Blackwell.

Chapter 4

Thoughts for Two—Who Are You?

Abstract

This chapter examines the structures of intersubjective recognition, building on the previous chapters' explorations of how the "Who are you?" question reveals the dynamics of mathematical communication and human development. A central case study follows the evolution of "Big Gorilla," a character created by preschoolers, whose transformations illustrate the processes of communicative breakdown and repair. The analysis applies insights from George Herbert Mead's symbolic interactionism, Jürgen Habermas's theory of communicative action, and Robert Brandom's analytic pragmatism to understand how shared meanings emerge through social practice. A key contribution is the distinction between the transactional "you"—which addresses a particular, concrete other—and the "CUSP you" (Claimed Universal Subject Position)—which invokes a generalized, normative subject position. This distinction, grounded in Sebastian Rödl's work on self-consciousness and the first person, reveals how mathematical discourse often presupposes a universal subject while failing to recognize the particularity of actual learners. The chapter develops an algorithmic model that frames intersubjective development as a recursive process involving gesture formation, communicative action, breakdown, and repair. This model will be extended in later chapters to understand the development of numerical concepts.

4.1 The CUSP You: A Misrecognition of the Particular

The last chapter argued that the tension between existential needs is resolved not in solitude, but in a developmental journey toward reciprocal recognition. This resolution, embodied in the surrender to the "Beast of Love," is fundamentally a social and communicative achievement. But what is the structure of this recognition? Who is the "You" that meets the {I}?

When I was working on my first research project in graduate school which explored how gestures and speech relate in teaching algebraic concepts, I found that preservice teachers often used the word "you" in ways that I felt were curious. I was the only other subject in the room as a preservice teacher said things like "The first thing you do is distribute the 2." When factoring an equation like $2(p^2 + 5p + 6)$, the first thing I would do is definitely not distribute the 2, as I knew I would just have to factor it out again later. They used the second-person pronoun in a way that I felt missed me as a referent of that pronoun.

I called such uses of "you" the *CUSP you*—the *Claimed Universal Subject Position*—because it appeared to refer to a generalized other, rather than the particular other with whom they were speaking. The CUSP you is a kind of universalized subject position that is not tied to any particular individual, but rather to a normative expectation of how one ought to act in a given situation. It is a way of speaking that assumes a shared understanding of what is appropriate or correct, without necessarily addressing the specific person being spoken to.

When I recognized others as using "you" as a CUSP, I also recognized that I used the term in similar ways in teaching. However, I also had privileged access to my thoughts, those inner rehearsals of speech acts. In particular, I had access to a very judgmental voice that took every opportunity to say things like "You're stupid! You suck!" A form of liberation seemed available if I could just figure out how to soften that hectoring voice. Finding oneself in that universalized field—not hating oneself to prove the other is loveable but still holding oneself and others to account—has been a spiritual process for me.

4.2 Defining the "You" in Context

CUSPs are distinguished from the *transactional* (Rödl, 2014) use of "you" that refers anaphorically to the specific person being addressed in a face-to-face encounter. Sebastian Rödl's work on *intentional transaction* provides a useful framework for understanding this distinction. In my reading of that work, he distinguishes between monadic and dyadic forms of speech actions, where the monadic form has one subject: Fa, like "Peter is painting the wall" (Rödl, 2014, p. 307), where a is "Peter" and F is the predicate "□ is painting the wall." Dyadic predication is formally aRb, where a is "Peter," b is "Paul" and R encodes the relation between them "□ is handing the brush to □" in a sentence like "Peter is handing the brush to Paul" (Rödl, 2014, p. 307).

Transactions have both *actor* and *patient* roles. His paradigmatic example is gift-giving, where the actor gives a gift to the patient who receives it. He argues that removing the patient's contribution to a transaction means that it has not occurred. Communicative action is essentially dyadic, not monadic.

When I taught a seminar on that research project, participants learned to recognize when they were using "you" to refer to the generalized other, rather than the particular other with whom they were speaking. By the way they would quickly try to claw back some usages of the general and clarify that the other individual was their referent, I gathered that they felt like using "you" in that way was a kind of violation. Emmanuel Levinas' work on the face-to-face encounter with an absolute other suggests that subsuming the other into what I have internalized without leaving room for the other's dissent does have an ethical implication.

To explore these questions, I want to share a story that began after I defended my dissertation. When I began dating Amalia, I entered into a rich context for exploring development and intersubjectivity, particularly through my interactions with her twin daughters, $\mathcal{M}$ and $\exists$, who were $3\frac{1}{2}$ years old at the time.

4.3 Analyzing Communicative Practice

Initially, I approached my role as a kind of playful participant-observer. But when we all caught COVID and ended up quarantining together, I realized that the dynamic would need to change if we were going to share living space. I would speak in elliptical parentheticals—around and around—and sometimes they would smile like "huh?!," sometimes they would try to teach me how to speak more competently, and sometimes communication broke down in spectacular fits. Rather than an impasse wherein the assumption of competency transformed into the assumption of incompetency, such moments led me to invent the character of Big Gorilla, who eventually became entitled the name Daddy Gorilla. Big Gorilla had a couple of baby gorillas who were always getting in trouble. He didn't know how they got there or why they were hanging around him so much. He would walk around with his fists on the ground and say, "Big Gorilla tired, but Baby Gorilla still awake. Baby Gorilla sleep now, then Big Gorilla sleep."

Part of Big Gorilla's behavioral regimen was to thump his chest when he was angry about something. "Baby Gorilla hit other Baby Gorilla. Hooohoohoo thumpthumpthump." I tried to keep anger in a space of play, not fully identifying with it. The kids were very small and easily frightened. At night they would often get ornery about going to bed. Big Gorilla would say "hoohoohoo" and thump his chest, and the Baby Gorillas would smile and thump on their bellies and go "hoohoohoo." It was very cute for a while.

To understand how the Big Gorilla story illustrates the emergence of meaningful communication, the analysis begins with George Herbert Mead's foundational argument that the self is not a pre-given entity but an achievement that unfolds through social interaction. Mead begins his analysis with what he calls the *conversation of gestures*—the kind of interaction we might observe between two dogs preparing to fight. For Mead's dogs, stimulus and response form a feedback loop where each dog's act serves as a stimulus that provokes a response from the other dog, which in turn modifies the first dog's behavior.

Once those cycles have been compressed to the extent that a growl or aggressive stance results in the other dog backing off, there is no need to bite. The growl has the significance of the bite as a proto-inferential consequence. This is what Mead calls a *significant symbol*. The transformation from gesture to a significant gesture or symbol occurs when the gesture-maker becomes conscious of the response their gesture evokes in the other. When I thump my chest as Big Gorilla and am delighted by the twins' delighted imitation, I experience my own gesture from their perspective. The chest-thump becomes a significant symbol because it means the same thing to me as it does to them: playful anger that invites rather than threatens.

4.4 The Breakdown: A Call for Rational Discourse

But one time, the anger got pretty close to me, and I thumped my chest until it hurt and hoohoohoo'ed in all capital letters. Tears bloomed and Amalia looked at me in a way that said "back off," and so I said, "Oh, that was too loud. Big Gorilla sorry. Big Gorilla get frustrated when Baby Gorillas don't listen."

This moment of breakdown requires a shift from Mead's framework to Jürgen Habermas's theory of communicative action. While Mead gives us the foundation for understanding how selves emerge through symbolic interaction, Habermas extends this understanding to explain how rational communication itself is structured. Habermas distinguishes between two modes of social interaction that are crucial for understanding what happened in the Big Gorilla story.

Most everyday interactions proceed smoothly through what Habermas calls *communicative action*. Communicative action unfolds under the assumption of a shared desire to reach understanding. When I first began thumping my chest as Big Gorilla, the twins and I quickly developed a shared understanding about what this gesture meant. The twins learned that Big Gorilla's chest-thumping was playful, not threatening, not through explicit instruction but through the accumulated context of those interactions.

Communicative action can break down. When the background consensus is challenged or disrupted, a shift to what Habermas calls *rational discourse* is required. This is exactly what happened when I thumped too loudly and Amalia gave me that look. The taken-for-granted understanding that Big Gorilla was safe and playful was called into question.

His theory of communicative action distinguishes between different validity claims which participants in communication implicitly or explicitly raise and seek to redeem: (1) **Truth claims** about objective reality ("The chest-thumping was too loud"), (2) **Rightness claims** about social norms and appropriateness ("Adults shouldn't frighten children"), and (3) **Sincerity claims** about one's own subjective states ("I was genuinely frustrated"). When I said, "Oh, that was too loud. Big Gorilla sorry. Big Gorilla get frustrated when Baby Gorillas don't listen to Momma Gorilla," I was engaging in rational discourse. I was making explicit claims about truth/facts (too loud), acknowledging the inappropriateness of my action (rightness), and revealing my internal state (sincere apology). This verbal response served to repair the breach in our communicative relationship and reestablish the normative boundaries of our shared practice.

4.5 Reflexivity and the Limits of Formalization

While Mead and Habermas provide powerful insights into the social nature of selfhood and communication, Robert Brandom's analytic pragmatism offers precise tools for understanding how meaning and use relate to each other. Brandom's approach is grounded in a separation that governs the methodology of *meaning use analysis*: *vocabularies* (V) and *practices-or-abilities* (P). A vocabulary V is what is *said*. A practice P is what is *done*, expressed as a pattern of activity. For Big Gorilla, this included the ability to thump my chest, but also the rules for how to put together strings of 'hoohoohoo's.

Brandom contributes to action theory by using theoretical automata to express what it means to conduct an action with words. These machines read and write formal languages. An act begins with an impetus to act. As the act unfolds, the actor engages in self-monitoring, reflecting on their act to determine whether it will satisfy the desire or need that prompted the act. This is a movement from the first-person position to the second-person position, indicating that reflection is part of the act.

Brandom formalizes the relationship between practices and vocabularies through two complementary notions of sufficiency: **PV-Sufficiency (Practice-Vocabulary Sufficiency)**: A practice P is sufficient for deploying a vocabulary V if mastering that practice essentially equips you to use that vocabulary. **VP-Sufficiency (Vocabulary-Practice Sufficiency)**: A vocabulary V is sufficient to specify a practice P if it affords the ability to explicitly formulate the rules of the practice. A **pragmatic metavocabulary** V' is the language used to make the implicit rules of practice P explicit.

But here it is necessary to acknowledge a crucial limitation of the formal apparatus. The theoretical frameworks we have explored—powerful as they are—cannot capture the full depth of what occurred in the Big Gorilla story. The transition from understanding (*Verstand*) to reason (*Vernunft*), from mechanical rule-following to genuine recognition, involves dimensions that resist algorithmic elaboration.

4.6 Integration: The Recursive Structure of Intersubjectivity

Let me continue the story to reflect this point. The Beast of Love poem that I referenced earlier was both an apology and a reflection on the complexities of love and anger–the beast within that can both harm and heal. It acknowledged my fears that my own struggles might impact these tender lives before they had a chance to understand their own depths. The idea that my fear–expressed as anger–could be something others were afraid of was a sad–but important–moment in my development.

A year or so after I invented the character, I came down with a terrible cold. We had just moved in together, and my poor old body was not conditioned for the kinds of communicable diseases that float around daycares. I was sick for months. In the middle of that period, we were eating dinner in our new house. The kids were ornery about something or other, and I SLAMMED my fists on the table, shouting "ENOUGH!" The kids started crying and they and Amalia decamped to visit Amalia's parents a few doors down. While they were away, I wrote them a note expressing my apology and left it on the stairs for them to find when they returned. Later that evening, Amalia came into the guest room where I had been isolating. She said, "We read the note, and the girls wanted to come in here and tell you that they forgive you and love you."

It didn't stick. I needed forgiveness from the kids. Then ∃ banged her fist on the bed, and then started thumping her belly like Baby Gorilla. She said something about how strong I am. I felt as though she was expressing a deep understanding—validating that part of my self, that I was loathing in bed, that needed to be understood as in control. I read her as saying "oh, look at how strong you are, you be Daddy Gorilla and I'll be Baby Gorilla, hoohoohoo." In banging my fist on the table, I was violating our norms, but in thumping her chest as a response, it felt like she folded that action back into the norms of the family. Daddy Gorilla just got a little too wild. It felt like my anger as a parent, which I can't seem to fully transcend, wasn't as dangerous to express. Her act of recognition broke my heart.

What strikes me most profoundly about ∃'s gesture is that it demonstrates something that goes beyond our formal analysis. Her chest-thumping was not just a response within the established practice—it was a creative, forgiving, and ultimately transformative act that exceeded the boundaries of our existing vocabularies and practices. It was a moment of genuine recognition that created new possibilities for meaning rather than simply deploying existing ones.

4.7 Conclusion: Beyond the Algorithm

So, who are you? The answer this chapter proposes is that "You" are not a static entity but a dynamic and necessary position in communicative action. Speaking assumes a listener, a "you" who might possibly understand. Such assumptions are the bedrock of intersubjectivity, though it is a shifting ground, subject to negotiation when the assumption is no longer viable. You are the particular other whose face-to-face encounter grounds my speech in a concrete reality. You are also the universal other whose internalized norms I appeal to for justification and shared meaning. And you are the interlocutor with whom the boundary between these two poles can blur, revealing the fluid, negotiated nature of a shared world.

This dyadic structure is precisely what allows for the developmental synthesis described in the previous chapter. The "Beast of Love" is tamed not in solitude, but in the reciprocal I-You encounter where participants learn to balance the need for *in*finite, authentic expression with the need for finite, normative recognition. This process is not a logical deduction but a lived, recursive, and often challenging practice. It is the practice of building a "we"—a shared space of meaning, a common history, a recognitive community.

In understanding the different roles the "you" can play in communication fosters greater attunement to the ethical dimensions of dialogue. It becomes easier to recognize when a particular person is being addressed and when universal norms are being invoked. This sensitivity helps ensure speech opens space for the other's response rather than foreclosing it.

Most importantly, becoming a self is not a solitary achievement but a collaborative one. The {I} emerges only in relation to a "You," and the "You" exists only in the context of shared practices and mutual recognition. Selves take shape through the patient, recursive, often difficult work of building understanding with others. In this work, there are no final answers, only an ongoing commitment to meet each other with care, curiosity, and a willingness to be changed by what is created together.

This is why the algorithm must be recursive–why it must be prepared to start over, to revise its fundamental assumptions. The "you" I am addressing at the end of our interaction is not the same "you" I addressed at the beginning, just as the {I} who speaks these final words has been shaped by the process of trying to articulate what it means to recognize and be recognized by another. This recursive, constitutive character of the I-You relation suggests that intersubjectivity has what we might call a *transcendental-like* structure. I cannot point beyond the assumption of communicative competency, for doing so just assumes communicative competency at some other level. What makes intersubjectivity transcendental-like is that it is possible to tell stories about how it arises. But it must be assumed for any of those stories to have force.

In the next chapter, I will continue to explore the "you" in the context of *limits of thought*. The kind of creative responsiveness that ∃ demonstrated invites reciprocal grace to the family in ways that make all kinds of 'modeling' feel obtuse, no matter how sophisticated those models are. I will explore the limits of knowledge and argue for why poetic and musical forms of expression are necessary to capture such moments of terrible, ferocious beauty.

References

Abbott, E. A. (2020, April). *Flatland: A Romance of Many Dimensions by Edwin A. Abbott*. Independently published.

Agamben, G. (2006, September). *Language and Death: The Place of Negativity* (K. Pinkus & M. Hardt, Trans.; First Edition). Univ Of Minnesota Press.

Aristotle. (2017). *Metaphysics: Aristotle's metaphysics in english, latin and ancient greek: Trilingual edition* (W. o. Moerbeke & W. D. Ross, Trans.; Trilingual). Hermes Language Reference.

Boote, D. N., & Beile, P. (2005). Scholars before researchers: On the centrality of the dissertation literature review in research preparation. *Educational researcher*, *34*, 3–15.

Brandom, R. B. (1994). *Making it explicit: Reasoning, representing, and discursive commitment*. Harvard university press.

Brandom, R. B. (2000). *Articulating reasons: An introduction to inferentialism*. Harvard University Press.

Brandom, R. B. (2008). *Between saying and doing: Towards an analytic pragmatism*. Oxford University Press.

Brandom, R. B. (2019). *A spirit of trust: A reading of Hegel's phenomenology*. The Belknap Press of Harvard University Press.

Bueno, O. (2020). Nominalism in the Philosophy of Mathematics. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Fall 2020). Metaphysics Research Lab, Stanford University.

Cantor, G. (1891). Ueber eine elementare frage der mannigfaltigkeitslehre. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, *1*, 75–78.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). Children's mathematics: Cognitively guided instruction - videotape logs [supplementary material]. In *Children's mathematics: Cognitively guided instruction*. Heinemann, in association with The National Council of Teachers of Mathematics, Inc.

Carspecken, P. F. (1995). *Critical ethnography in educational research: A theoretical and practical guide*. Routledge.

Carspecken, P. F. (1999). *Four scenes for posing the question of meaning and other essays in critical philosophy and critical methodology* (New edition). Peter Lang Inc., International Academic Publishers.

Carspecken, P. F. (2006). Limits of knowledge in the physical sciences. In K. Tobin & J. Kincheloe (Eds.), *Doing educational research* (pp. 405–437, Vol. 1). Sense Publishers.

Carspecken, P. F. (2009). Limits to knowledge with existential significance: An outline for the exploration of post-secular spirituality of relevance to qualitative research. In *Bridging the gap between theory and practice in educational research* (pp. 47–61). Springer.

Carspecken, P. F. (2016). Limits to knowledge and being human: What is 'critical' in critical social research and theory? In P. H. Hinchey (Ed.), *A critical action research reader* (pp. 65–86). Peter Lang Inc., International Academic Publishers.

Carspecken, P. F. (2018). The missing infinite. In R. Winkle-Wagner, J. Lee-Johnson, & T. Gaskew (Eds.), *Critical theory and qualitative data analysis in education* (p. 32). Routledge. https://doi.org/10.4324/9781315158860-2

Carspecken, P. F., & Zhang, R. (2013). Content inference fields in intersubjective space: Transpersonal position, illocution, and logic in the analysis of human interactions. *Qualitative research: A reader in philosophy, core concepts, and practice*, 201–242.

Carspecken, R. (2024, November 3). Untitled.

Caygill, H. (2004). *A kant dictionary*. Wiley-Blackwell.

Derrida, J. (1973). *Speech and phenomena, and other essays on husserl's theory of signs*. Northwestern University Press.

Derrida, J. (1982). *Marges de la philosophie* [Margins of philosophy] (A. Bass, Trans.). The Harvester Press. (Original work published 1972)

Deutsch, H., Marshall, O., & Irvine, A. D. (2025). Russell's Paradox. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford encyclopedia of philosophy* (Spring 2025). Metaphysics Research Lab, Stanford University.

Ernest, P. (2024). The mathematical and philosophical significance of zero: Nought matters [Publisher: Paul Ernest, Emeritus Professor at University of Exeter, School of Education & Lifelong Learning]. *Philosophy of Mathematics Education Journal*, 1–35. https://research.ebsco.com/linkprocessor/plink?id=b685ecb9-d2fc-3e03-889d-a06fb1751c1b

Euclid. (2007, December). *Euclid's Elements of Geometry* (J. L. Heiberg, Ed.; R. Fitzpatrick, Trans.; Revised and corrected edition). Richard Fitzpatrick.

Franzén, T. (2004, September). *Inexhaustibility: A Non-Exhaustive Treatment: Lecture Notes in Logic 16* (1st edition). A K Peters/CRC Press.

Franzén, T. (2005). *Gödel's theorem: An incomplete guide to its use and abuse*. AK Peters/CRC Press.

Frege, G. (1997, July). *The Frege Reader* (M. Beaney, Ed.; 1st edition). Blackwell Publishing.

Gaifman, H. (2005). *The easy way to Gödel's proof and related matters* [Unpublished manuscript. Available online].

Geisel, T. (1961). *The Sneetches and Other Stories* (First edition.). Random House Books for Young Readers.

Gelder, T. V. (1995). What might cognition be, if not computation? *The Journal of Philosophy*, *92*, 345–381. Retrieved April 13, 2025, from http://www.jstor.org/stable/2941061

Gray, L., & Garzon, M. (2005). On a hermite boundary integral approximation. *Computers & structures*, *83*(10-11), 889–894.

Greenblatt, R., Denison, C., Wright, B., Roger, F., MacDiarmid, M., Marks, S., Treutlein, J., Belonax, T., Chen, J., Duvenaud, D., Khan, A., Michael, J., Mindermann, S., Perez, E., Petrini, L., Uesato, J., Kaplan, J., Shlegeris, B., Bowman, S. R., & Hubinger, E. (2024, December 23). Alignment faking in large language models. https://doi.org/10.48550/arXiv.2412.14093

Habermas, J. (1985). *The theory of communicative action: Lifeworld and system* (Vol. 2). Beacon Press.

Habermas, J. (2004). *The philosophical discourse of modernity: Twelve lectures* (F. G. Lawrence, Trans.). MIT Press.

Habermas, J. (1971). *Knowledge and human interests*. Beacon Press.

Habermas, J. (1984). *The theory of communicative action, volume 1: Reason and the rationalization of society* (T. McCarthy, Trans.; Reprint edition). Beacon Press.

Habermas, J. (1995). *Postmetaphysical thinking: Between metaphysics and the critique of reason* (1st edition). Polity.

Hackenberg, A. (2025). *Course notes* [Unpublished course notes].

Hegel, G. W. F. (1977). *Phenomenology of spirit* (A. V. Miller, Trans.; Vol. 632). Oxford University Press Oxford.

Hegel, G. W. F. (2010). *The science of logic* (G. di Giovanni, Ed. & Trans.). Cambridge University Press.

Houlgate, S. (2006). *The opening of hegel's logic: From being to infinity*. Purdue University Press.

Hyppolite, J. (1974). *Genesis and structure of hegel's phenomenology of spirit*. Northwestern University Press.

Kant, I. (1998). *Critique of pure reason* [Original work published 1781/1787, Guyer and Wood, Trans.]. Cambridge University Press.

Lakoff, G., & Johnson, M. (1980). *Metaphors we live by*. University of Chicago Press.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books.

Linnebo, Ø. (2024). Platonism in the Philosophy of Mathematics. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford encyclopedia of philosophy* (Summer 2024). Metaphysics Research Lab, Stanford University.

Longuenesse, B. (2017). *I, me, mine: Back to kant and fichte on the 'i'*. Oxford University Press.

Mead, G. H. (1934). *The social psychology of george herbert mead* (A. L. Strauss, Ed.). University of Chicago Press.

Mead, G. H. (2015). *Mind, self, and society: The definitive edition* (H. Joas, D. R. Huebner, & C. W. Morris, Eds.; Enlarged edition). University of Chicago Press.

Meyer, J. (title). *Cantors 1891 diagonal proof* [Cantor's original 1891 diagonal proof]. Retrieved July 6, 2025, from https://jamesrmeyer.com/infinite/cantors-original-1891-proof

Nagel, E., & Newman, J. R. (2012). *Godel's proof*. Routledge.

Negarestani, R. (2018). *Intelligence and Spirit*. Urbanomic/Sequence Press.

Poston, T., & Stewart, I. (2012, February 29). *Catastrophe theory and its applications* (First Edition). Dover Publications.

Priest, G. (2002). *Beyond the limits of thought*. Oxford University Press.

Priest, G. (2014). *One: Being an investigation into the unity of reality and of its parts, including the singular object which is nothingness*. Oxford University Press.

Quine, W. V. O. (1948). On what there is. In *Thinking about logic* (pp. 189–209). Catholic University of America. https://sites.pitt.edu/~rbrandom/Courses/2023%20Sellars/Sellars%20texts/Quine%20On_What_There_Is.pdf

Quine, W. V. O. (1963). Two dogmas of empiricism [First Harper Torchbook edition, a reprint of the second, revised edition (1961). The book was first published by Harvard University Press in 1953.]. In *From a logical point of view* (Second, revised, pp. 20–46). Harper & Row.

Rödl, S. (2014). Intentional transaction [Accessed: 2025-04-13]. *Philosophical Explorations: An International Journal for the Philosophy of Mind and Action*, *17*, 304–316. https://doi.org/10.1080/13869795.2014.941909

Rose, G. (2009). *Hegel contra sociology*. Verso.

Rowlands, S. (2005). The contradictions in the constructivist discourse [Citing von Glasersfeld]. *Philosophy of Mathematics Education Journal*, *14*.

Savich, T. M. (2020). Towards a critical mathematics [Accessed via ERIC]. *Mathematics Teaching–Research Journal Online*, *12*, 245–291. https://files.eric.ed.gov/fulltext/EJ1384454.pdf

Savich, T. M. (2022). *Towards a critical mathematics*. Indiana University.

Savich, T. M. (2024). Built to break: An introduction to theoretical autoethnography [Invited talk presented at Solent University, organized by Dr. Brian McDonough]. https://iu.mediaspace.kaltura.com/media/t/1_v1qmm3g9

Savich, T. M. (title). *The hermeneutic calculator* [The hermeneutic calculator]. Retrieved July 23, 2025, from https://tiosavich.github.io/UMEDCTA/Calculator/index.html

Sellars, W. (1953). Inference and meaning. *Mind*, *62*, 313–338.

Sellars, W. (2007). *In the space of reasons: Selected essays of wilfrid sellars*. Harvard University Press.

Steffe, L. P. (2002). A new hypothesis concerning children's fractional knowledge [Accessed through CEPA.INFO]. *Journal of Mathematical Behavior*, *20*(3), 267–307.

Steffe, L. P. (2003). On the construction of learning trajectories of children: The case of commensurate fractions [Accessed via ResearchGate/ERIC]. *Monographs of the Journal for Research in Mathematics Education*, 1–26.

Steffe, L. P. (2004). Fractional commensurate, composition, and adding schemes: Learning trajectories of jason and laura: Grade 5 [Accessed via ResearchGate]. In J. Olive & R. Tzur (Eds.), *Children's fractional knowledge* (pp. 1–60). American Educational Research Association.

Steffe, L. P. (2014). *Children's number sequences: An explanation of steffe's constructs and an extrapolation to rational numbers of arithmetic* [Originally from 1994]. Unpublished manuscript, available via CiteSeerX; ResearchGate.

Steffe, L. P., & Thompson, P. W. (2014). Teaching experiment methodology: Underlying principles and essential elements [Accessed via ResearchGate, originally published 2000]. *Proceedings of the 24th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.*

Stone, J. (2003). *The monster at the end of this book* (2nd ed. edition). Golden Books.

Taylor, C. (1994). *Multiculturalism: Examining the Politics of Recognition* (A. Gutmann, Ed.) [Essays by Charles Taylor, K. Anthony Appiah, Jürgen Habermas, Steven C. Rockefeller, Michael Walzer, and Susan Wolf]. Princeton University Press.

van Hiele-Geldof, D., & van Hiele, P. M. (1984). *English translation of selected writings of dina van hiele-geldof and pierre m. van hiele* (D. Fuys, D. Geddes, & D. Tischler, Eds.) [ERIC Number: ED287697]. Retrieved June 3, 2025, from https://eric.ed.gov/?id=ED287697

Wittgenstein, L. (1956). *Remarks on the foundations of mathematics.*

Wittgenstein, L. (2010). *Philosophical investigations* (P. M. S. Hacker & J. Schulte, Eds.; G. Anscombe, Trans.). Wiley-Blackwell.

Chapter 5

6: Limits of Thought, Deconstruction, and the Voice

Abstract

This chapter explores the limits of thought and language through the author's experience of personal loss, philosophical inquiry, and musical expression. Building on the previous chapters' development of determinate negation and intersubjective recognition, the analysis examines the relationship between absence and meaning-making. The chapter draws on Giorgio Agamben's concept of Voice—the pure event of language that subtends all particular utterances—to theorize how meaning emerges at the boundary between saying and unsaying. Several autoethnographic artifacts structure the investigation: a eulogy delivered at the author's father's funeral, a letter written to the father after his death, and original songs that attempt to articulate grief, memory, and the complexities of recognition. These artifacts serve as sites for philosophical reflection on how absence functions as an enabling condition for presence, and how the work of mourning parallels the work of mathematical understanding. The chapter also investigates how meaning emerges through shared practices and interpretations, even when individual expressions seem inadequate to their task. Theoretical discussions interweave with personal narrative, exploring concepts such as différance, reciprocal sense-dependence, and Habermas's knowledge-constitutive interests. The chapter connects these themes to the concept of null representation in mathematics,

arguing that accepting absence as a necessary condition of understanding enables a richer conception of mathematical thought.

5.1 The Eulogy and the Problem of Presence

I was in the middle of writing my dissertation when my father died suddenly. My mom found him and ran to get me. He was downstairs in the living room. He had been laughing at a baseball game, and then he was not. We delivered CPR until the paramedics came, but he was already gone. I found myself singing "Brokedown Palace" by Jerry Garcia and Robert Hunter as they took him away, a goodbye song about a river rolling and a mama rocking her baby.

At the funeral, I spoke about my father and tried to articulate what his death meant for those of us left behind. I used the metaphor of light passing through a prism: "The bright, white light of who you were has been broken by the prism of death into a brilliant rainbow, diffuse, with each of us radiating our own memories of you as discrete shards of your spectrum." The image captured something true about grief—how the wholeness of a person fragments into the scattered memories held by different people. No single memory can reconstitute the whole.

I also used the metaphor of weaving. I said: "I am standing here like a woven cloth. The warp is the thread that stays fixed to the loom. It's what holds the structure together. The weft is the thread that moves, that creates the design. Dad runs through my life as the solid threads on which the strange designs of my life are drawn." He was the constant, the enabling condition, even when I did not recognize it.

I told two memories. The first was recent: a trip to a Sox game where Dad told me the story of Kent Todd, a banjo player who said he didn't play *Muleskinner Blues* anymore because his father had died that year. Later, Dad cried in the car when my sister told him about her partner Terry's cancer diagnosis. He cried so hard the car shook. After the storm passed, there was a rainbow.

The second memory was distant: the Bryan Park pool, where I rode on Dad's back as he plunged down into the water. I remember clinging to him, breaking through the surface together. I remember when he removed my training wheels with that firm hand and swift push. I remember chasing his draft on bike rides, trying to keep up.

The eulogy wrestles with what Agamben identifies as the fundamental problem of Western metaphysics: the relationship between language and death (Agamben, 2006). How do I speak to someone who is no longer here? How do I address an absent referent? The diffuse rainbow metaphor attempts to make the absent fully present through the collected shards of memory, but the attempt fails. The whole—the "white light of a person"—once gone, cannot be reconstituted.

5.2 The Negative Foundation: Language and Death

Agamben draws a crucial distinction between *Voice* and *voice*. The *Voice* (capitalized) is the silent, unsayable ground that makes language possible. It is not a sound. It is the *removal* of sound, the negation of the animal cry, that creates the space for meaningful speech. The *voice* (lowercase), by contrast, is the physical sound made by the body—the vibrations of vocal cords, the breath moving through the larynx.

The Voice is a double negation (Agamben, 2006). It is the removal of the animalistic cry that allows for the articulation of meaningful speech. But the Voice itself is not speech. It is the silent ground that speech presupposes. The Voice is *not* that which it enables. It is the *absent* foundation, the negative condition that makes the spoken word possible.

Agamben connects this to Heidegger's concept of *dasein*, which is characterized by being-towards-death. For Heidegger, the "animal" ceases to live, while the person truly *dies*. Death is not merely biological cessation but the horizon against which meaning becomes possible. The Voice operates in a similar way—it is the absent ground that enables language, just as death is the absent horizon that enables the meaning of a life.

When I sang the songs at my father's funeral and later in the empty basement, I was deeply immersed in the *voice*—the physical act of producing sound. But what I was reaching for was the *Voice*—the unsayable, silent ground that my father's absence had made palpable. The text of the lyrics is, in Derrida's terms, a trace—the absence of music provides a hollowing-out, a space where the Voice should be but cannot be captured.

The Voice is not a presence that can be articulated. It is the *enabling condition* of articulation itself. The diffuse rainbow metaphor in the eulogy inadvertently captures this: the attempt to gather the shards of memory into a unified presence fails because the ground of that presence—the Voice, the silent foundation—cannot itself be made present.

5.3 Analyzing the Limits of Knowledge and Reference

Habermas (Habermas, 1971) identifies three *knowledge-constitutive interests* that correspond to different ways humans relate to being and different forms of knowledge. These interests are not merely academic categories but existential orientations that shape how we engage with the world. Carspecken (P. F. Carspecken, 1999) provides a helpful table that I adapt here:

The **empirical-analytic** interest is motivated by the need to predict and control tangible states. When I performed CPR on my father, I was operating within this interest. I was following an algorithm, applying pressure, counting compressions, attempting to restart his heart. The knowledge involved is "copy knowledge"—a procedure learned and executed. The knower is an anonymous, universal observer. The validity criterion is success: Did it work?

The **historical-hermeneutic** interest is motivated by the need to reach agreements, to be understood, to be recognized. The eulogy itself operates within this interest. I was trying to articulate what my father's life meant, how his death should be understood by the community gathered at the funeral. The knowledge involved is hermeneutically explicated through ordinary language. The knower is an individuated self with an autobiography—I spoke as *Ted*, the son, not as an anonymous observer. The validity criterion is insider recognition: Did the community recognize the truth of what I said?

The **critical-emancipatory** interest is motivated by the need for self-understanding, freedom, and autonomy. This is the interest that emerges in the aftermath of the eulogy, in the solitary moments of grief when I must come to terms with my father's absence. The knowledge involved is internal recognition, often noncommunicable. The knower is either pure reflection or the loved Other (or God), or there is no self at all—the boundaries dissolve. The validity criterion is the continuation of the self-formative process, what

Dimension	Empirical-Analytic	Historical-Hermeneutic	Critical-Emancipatory
Motivations	Tangible states; explaining "what is"	Reaching agreements; being understood; recognition needs	Self-understanding; freedom; autonomy
Actions	Instrumental	Communicative	Reflection; desire for desire; nonaction
Knowledge Forms	Copy knowledge; formalized languages	Hermeneutically explicated; ordinary language	Internal recognition; noncommunicable states
Relation to Being	Knowledge/being separated	Knowledge self-transcending; changes reality	Knowledge fused with being
Knower	Anonymous universal observer	Individuated self with autobiography	Pure reflection; loved Other or God; or no self
Reflection	Restricted	Internal to hermeneutics	Method and object simultaneously
Validity	Successful predictions	Insider recognition	Self-formative process continuation; enlightenment

Table 5.1: *Knowledge-Constitutive Interests* (adapted from Carspecken, 1999)

Habermas calls "enlightenment."

The eulogy attempts to bridge these three interests, but ultimately reveals their limits. The empirical-analytic interest cannot revive the dead. The historical-hermeneutic interest cannot reconstitute the whole from the scattered shards. The critical-emancipatory interest confronts the irreducible absence—the Voice—that cannot be captured in language.

5.4 My Masterpiece: The Paradox of Recognition

Five years before my father died, I wrote him a letter expressing frustration with my life. I do not remember what I wrote, but I remember his response.

He sent me the lyrics to Bob Dylan's "When I Paint My Masterpiece", a song about feeling unfulfilled and waiting to create something worthy. Then he wrote:

> Your letter brought this song to mind, like you are feeling unfulfilled and needing, waiting to paint your masterpiece. But you, Theodore Michael Dougherty Savich, you are my masterpiece. Sure, I kept on always trying to please Grandpa after you were born and I still do. But you have made my life a success.

He then listed predicates: "smart…creative…good looking…handsome…funny…great guitar player and singer…thinker and an innovator…great teacher because you care." He was attempting to *recognize* me, to articulate what made me valuable, to give me a sense of my own worth.

But here is the paradox: I did not "hear" his words until after he was dead. I could not identify as a value for x in the predicates "x is smart," "x is creative," or "x is handsome." The letter failed to land not because its predicates were false, but because the living subject it addresses can never be fully present as an object of description. The {I} exceeds any finite list of predicates.

There is also a reciprocal sense-dependence at work here (Brandom, 1994). My father wrote, "You have made my life a success." For him, to talk about himself was to talk about me. To talk about me was to talk about him. The essentiality flows in both directions. He could not be who he was without me, and I could not be who I am without him. But this mutual dependence cannot be captured by a simple list of attributes.

The letter reveals the structure of recognition as inherently paradoxical. Recognition requires treating the other as an object that can be described, categorized, and understood. But the other—the *in*finite {I}—resists being fully captured by any description. The predicates my father used were not wrong. They were *insufficient*. The gap between the predicates and the person they are meant to describe is the trace of the absent referent, the *in*finite that cannot be made present.

5.5 Reflection on Deferral: Différance and the Trace

Quine (Quine, 1948) famously addressed the problem of how to talk about non-being without attributing being to that which is named. His solution was to shift the burden of existence from names to *ontological commitments* and linguistic self-reference. The name “Pegasus” exists as a linguistic entity, but we can deny that the variable x in the sentence “x is a Pegasus” attains any value: $\neg\exists x$ such that x is a Pegasus.

Quine’s dictum—“To be is to be the value of a variable”—takes on new significance in this context (Quine, 1948). The variable x is what Agamben calls a *shifter* (Agamben, 2006). A shifter is a bit of language (like {I} or a variable) that does not simply refer to an object. Instead, it refers to the *event* of language. The variable x is the pure pronoun that does not refer to any particular object but to the event of language that the Voice makes possible.

Derrida’s concept of *différance* (Derrida, 1972/1982) captures the dual movement at work here. Différance (with an *a*) is the simultaneous *deferral* and *differing* of meaning. Meaning is always deferred because it depends on absent contexts, on traces of what is not present. Meaning is always differing because each new context alters what the words signify.

My father’s letter failed to land immediately because the meaning it attempted to convey was *deferred*. The recognition he offered required a context I did not yet have—the context of his death, the context of my own maturation, the context of understanding what it means to be someone’s masterpiece. The letter carried the *trace* of the absent referent—the {I} that cannot be fully captured by predicates, the *in*finite that resists being made finite.

The gap between when my father wrote the letter and when I “heard” it (five years later, after his death) is not a failure of communication. It is the structure of meaning itself. Meaning is never fully present. It is always deferred, always differing, always carrying the trace of what is absent. The Voice—the silent, unsayable ground—is the condition of possibility for this deferral and differing. It is what allows language to mean anything at all, precisely because it is not-speech, not-presence, not-being.

5.6 Integration: The Song of Home

After my father died, I wrote songs. The first, *Love's Memory*, began while he was still alive, but I only understood what I had written after his death. The song moves through images of autumn, dead oceans, polar winter, and the delayed recognition of his letter. Between verses, I yodel and whistle—inarticulate vowel sounds that attempt to express what words cannot.

The second song, *Still Feels Like Home*, was written after I stripped the basement bare following his death. The concrete floors made a resonant chamber. I was trying to make the sound of the Voice—the unsayable—as big as possible. The song describes cutting my hand on a carpet knife, tearing out mold-dark night, breaking my crown soaking up storm light. The chorus speaks of letting summer wind blow rain in, of curtains dripping on the floor, of things breaking with every storm.

But then comes the bridge: "But other words blow in on the same warm wind / Filling cracks with honey dripping from the comb / It still feels like home." Musically, the song begins ploddingly in the key of D#. When it reaches "But other words blow in"—the Voice—it modulates down to the key of C# as the melody ascends. The harmonic descent creates space for the melodic ascent, mirroring the structure of the Voice as the negative foundation that enables positive articulation.

Agamben writes that the Voice does not will any proposition or event; it wills *that language exist*, it wills the *originary event* that contains the possibility of every event (Agamben, 2006). Philosophy, for Agamben, is the human word's *nostos*—its return from itself to itself. After becoming meaningful discourse, it returns in the end, as absolute wisdom, to the Voice.

The home is not about real estate or biological embodiment. It is the feeling of returning home to the I-feeling. Recognizing the event of language as home lends Descartes' *sum*—the linguistically non-falsifiable "I am"—the feeling of self-certainty. When I sang these songs in the empty basement, filling the space with sound, I was attempting a *nostos*, a return to the ground that my father's absence had made palpable.

If Agamben's Voice is the unsayable, pre-linguistic ground that allows language to exist, then Quine's variable is the fundamental tool of reference within that existing language. I read Quine's variable x as a formal shifter. It is the pure pronoun that does not refer to any particular object but to the event of language that the Voice makes possible. "To be is to be the value of a variable" thus signifies that to be is to be something that can be spoken of, something that can enter the linguistic field opened by the Voice.

5.7 Conclusion: To the Bridge

This chapter has explored the limits of thought and language through the experience of loss. The eulogy attempted to make an absent referent present through the metaphor of a diffuse rainbow—scattered shards of memory that cannot reconstitute the whole. Agamben's distinction between Voice and voice revealed that language depends on a silent, negative foundation—a removal of the animal cry that creates the space for meaningful speech. Habermas's three knowledge-constitutive interests showed how different forms of knowledge relate to being, each with its own limits and validity criteria.

My father's letter revealed the paradox of recognition: the attempt to capture the *in*finite {I} through finite predicates necessarily fails, not because the predicates are false but because they are insufficient. Derrida's différance and Quine's variables as shifters illuminate the structure of meaning as deferral and differing, as a trace of the absent referent that can never be made fully present. The songs I wrote—with their inarticulate vowel sounds and harmonic descents creating space for melodic ascents—attempt to articulate the Voice, the *nostos* or homecoming to the unsayable ground.

What emerges from this analysis is a recognition that *absence is not a deficiency*. The Voice is not a lack that needs to be filled. It is the enabling condition for all articulation. The gap between my father's letter and my hearing it is not a failure of communication but the structure of meaning itself. The diffuse rainbow is not a failed attempt at wholeness but an accurate representation of how meaning is distributed across contexts and deferred across time.

This understanding prepares us for the next chapter, where we turn to a mathematical formalization of these ideas through Dr. Seuss's *The Sneetches* and Georg Cantor's diagonal proof. Just as the Voice is the absent foundation that enables language, the empty set is the absent foundation that enables mathematical structure. Just as shifters refer to the event of language rather than to objects, variables refer to the event of mathematical discourse rather than to numbers. The pattern of sublation—preservation, negation, and elevation—that we have traced through autoethnographic artifacts will now be traced through the history of mathematical proof.

The bridge we are building connects the existential need to be recognized as both finite and *in*finite with the formal structures that mathematics uses to express similar tensions. Recognition requires treating persons as objects with predicates, yet persons exceed any finite list of predicates. Mathematical systems require treating infinities as objects with properties, yet infinities exceed any finite representation. Both domains reveal the necessity of the negative, the absent, the null—not as failures but as enabling conditions.

References

Abbott, E. A. (2020, April). *Flatland: A Romance of Many Dimensions by Edwin A. Abbott*. Independently published.

Agamben, G. (2006, September). *Language and Death: The Place of Negativity* (K. Pinkus & M. Hardt, Trans.; First Edition). Univ Of Minnesota Press.

Aristotle. (2017). *Metaphysics: Aristotle's metaphysics in english, latin and ancient greek: Trilingual edition* (W. o. Moerbeke & W. D. Ross, Trans.; Trilingual). Hermes Language Reference.

Boote, D. N., & Beile, P. (2005). Scholars before researchers: On the centrality of the dissertation literature review in research preparation. *Educational researcher*, *34*, 3–15.

Brandom, R. B. (1994). *Making it explicit: Reasoning, representing, and discursive commitment*. Harvard university press.

Brandom, R. B. (2000). *Articulating reasons: An introduction to inferentialism*. Harvard University Press.

Brandom, R. B. (2008). *Between saying and doing: Towards an analytic pragmatism*. Oxford University Press.

Brandom, R. B. (2019). *A spirit of trust: A reading of Hegel's phenomenology*. The Belknap Press of Harvard University Press.

Bueno, O. (2020). Nominalism in the Philosophy of Mathematics. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Fall 2020). Metaphysics Research Lab, Stanford University.

Cantor, G. (1891). Ueber eine elementare frage der mannigfaltigkeitslehre. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, *1*, 75–78.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). Children's mathematics: Cognitively guided instruction - videotape logs [supplementary material]. In *Children's mathematics: Cogni-*

tively guided instruction. Heinemann, in association with The National Council of Teachers of Mathematics, Inc.

Carspecken, P. F. (1995). *Critical ethnography in educational research: A theoretical and practical guide*. Routledge.

Carspecken, P. F. (1999). *Four scenes for posing the question of meaning and other essays in critical philosophy and critical methodology* (New edition). Peter Lang Inc., International Academic Publishers.

Carspecken, P. F. (2006). Limits of knowledge in the physical sciences. In K. Tobin & J. Kincheloe (Eds.), *Doing educational research* (pp. 405–437, Vol. 1). Sense Publishers.

Carspecken, P. F. (2009). Limits to knowledge with existential significance: An outline for the exploration of post-secular spirituality of relevance to qualitative research. In *Bridging the gap between theory and practice in educational research* (pp. 47–61). Springer.

Carspecken, P. F. (2016). Limits to knowledge and being human: What is 'critical' in critical social research and theory? In P. H. Hinchey (Ed.), *A critical action research reader* (pp. 65–86). Peter Lang Inc., International Academic Publishers.

Carspecken, P. F. (2018). The missing infinite. In R. Winkle-Wagner, J. Lee-Johnson, & T. Gaskew (Eds.), *Critical theory and qualitative data analysis in education* (p. 32). Routledge. https://doi.org/10.4324/9781315158860-2

Carspecken, P. F., & Zhang, R. (2013). Content inference fields in intersubjective space: Transpersonal position, illocution, and logic in the analysis of human interactions. *Qualitative research: A reader in philosophy, core concepts, and practice*, 201–242.

Carspecken, R. (2024, November 3). Untitled.

Caygill, H. (2004). *A kant dictionary*. Wiley-Blackwell.

Derrida, J. (1973). *Speech and phenomena, and other essays on husserl's theory of signs*. Northwestern University Press.

Derrida, J. (1982). *Marges de la philosophie* [Margins of philosophy] (A. Bass, Trans.). The Harvester Press. (Original work published 1972)

Deutsch, H., Marshall, O., & Irvine, A. D. (2025). Russell's Paradox. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford encyclopedia of philosophy* (Spring 2025). Metaphysics Research Lab, Stanford University.

Ernest, P. (2024). The mathematical and philosophical significance of zero: Nought matters [Publisher: Paul Ernest, Emeritus Professor at University of Exeter, School of Education & Lifelong Learning]. *Philoso-*

phy of Mathematics Education Journal, 1–35. https://research.ebsco.com/linkprocessor/plink?id=b685ecb9-d2fc-3e03-889d-a06fb1751c1b

Euclid. (2007, December). *Euclid's Elements of Geometry* (J. L. Heiberg, Ed.; R. Fitzpatrick, Trans.; Revised and corrected edition). Richard Fitzpatrick.

Franzén, T. (2004, September). *Inexhaustibility: A Non-Exhaustive Treatment: Lecture Notes in Logic 16* (1st edition). A K Peters/CRC Press.

Franzén, T. (2005). *Gödel's theorem: An incomplete guide to its use and abuse*. AK Peters/CRC Press.

Frege, G. (1997, July). *The Frege Reader* (M. Beaney, Ed.; 1st edition). Blackwell Publishing.

Gaifman, H. (2005). *The easy way to Gödel's proof and related matters* [Unpublished manuscript. Available online].

Geisel, T. (1961). *The Sneetches and Other Stories* (First edition.). Random House Books for Young Readers.

Gelder, T. V. (1995). What might cognition be, if not computation? *The Journal of Philosophy*, *92*, 345–381. Retrieved April 13, 2025, from http://www.jstor.org/stable/2941061

Gray, L., & Garzon, M. (2005). On a hermite boundary integral approximation. *Computers & structures*, *83*(10-11), 889–894.

Greenblatt, R., Denison, C., Wright, B., Roger, F., MacDiarmid, M., Marks, S., Treutlein, J., Belonax, T., Chen, J., Duvenaud, D., Khan, A., Michael, J., Mindermann, S., Perez, E., Petrini, L., Uesato, J., Kaplan, J., Shlegeris, B., Bowman, S. R., & Hubinger, E. (2024, December 23). Alignment faking in large language models. https://doi.org/10.48550/arXiv.2412.14093

Habermas, J. (1985). *The theory of communicative action: Lifeworld and system* (Vol. 2). Beacon Press.

Habermas, J. (2004). *The philosophical discourse of modernity: Twelve lectures* (F. G. Lawrence, Trans.). MIT Press.

Habermas, J. (1971). *Knowledge and human interests*. Beacon Press.

Habermas, J. (1984). *The theory of communicative action, volume 1: Reason and the rationalization of society* (T. McCarthy, Trans.; Reprint edition). Beacon Press.

Habermas, J. (1995). *Postmetaphysical thinking: Between metaphysics and the critique of reason* (1st edition). Polity.

Hackenberg, A. (2025). *Course notes* [Unpublished course notes].

Hegel, G. W. F. (1977). *Phenomenology of spirit* (A. V. Miller, Trans.; Vol. 632). Oxford University Press Oxford.

Hegel, G. W. F. (2010). *The science of logic* (G. di Giovanni, Ed. & Trans.). Cambridge University Press.

Houlgate, S. (2006). *The opening of hegel's logic: From being to infinity*. Purdue University Press.

Hyppolite, J. (1974). *Genesis and structure of hegel's phenomenology of spirit*. Northwestern University Press.

Kant, I. (1998). *Critique of pure reason* [Original work published 1781/1787, Guyer and Wood, Trans.]. Cambridge University Press.

Lakoff, G., & Johnson, M. (1980). *Metaphors we live by*. University of Chicago Press.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books.

Linnebo, Ø. (2024). Platonism in the Philosophy of Mathematics. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford encyclopedia of philosophy* (Summer 2024). Metaphysics Research Lab, Stanford University.

Longuenesse, B. (2017). *I, me, mine: Back to kant and fichte on the 'i'*. Oxford University Press.

Mead, G. H. (1934). *The social psychology of george herbert mead* (A. L. Strauss, Ed.). University of Chicago Press.

Mead, G. H. (2015). *Mind, self, and society: The definitive edition* (H. Joas, D. R. Huebner, & C. W. Morris, Eds.; Enlarged edition). University of Chicago Press.

Meyer, J. (title). *Cantors 1891 diagonal proof* [Cantor's original 1891 diagonal proof]. Retrieved July 6, 2025, from https://jamesrmeyer.com/infinite/cantors-original-1891-proof

Nagel, E., & Newman, J. R. (2012). *Godel's proof*. Routledge.

Negarestani, R. (2018). *Intelligence and Spirit*. Urbanomic/Sequence Press.

Poston, T., & Stewart, I. (2012, February 29). *Catastrophe theory and its applications* (First Edition). Dover Publications.

Priest, G. (2002). *Beyond the limits of thought*. Oxford University Press.

Priest, G. (2014). *One: Being an investigation into the unity of reality and of its parts, including the singular object which is nothingness*. Oxford University Press.

Quine, W. V. O. (1948). On what there is. In *Thinking about logic* (pp. 189–209). Catholic University of America. https://sites.pitt.edu/~rbrandom/

Courses/2023%20Sellars/Sellars%20texts/Quine%20On_What_There_Is.pdf

Quine, W. V. O. (1963). Two dogmas of empiricism [First Harper Torchbook edition, a reprint of the second, revised edition (1961). The book was first published by Harvard University Press in 1953.]. In *From a logical point of view* (Second, revised, pp. 20–46). Harper & Row.

Rödl, S. (2014). Intentional transaction [Accessed: 2025-04-13]. *Philosophical Explorations: An International Journal for the Philosophy of Mind and Action*, *17*, 304–316. https://doi.org/10.1080/13869795.2014.941909

Rose, G. (2009). *Hegel contra sociology*. Verso.

Rowlands, S. (2005). The contradictions in the constructivist discourse [Citing von Glasersfeld]. *Philosophy of Mathematics Education Journal*, *14*.

Savich, T. M. (2020). Towards a critical mathematics [Accessed via ERIC]. *Mathematics Teaching–Research Journal Online*, *12*, 245–291. https://files.eric.ed.gov/fulltext/EJ1384454.pdf

Savich, T. M. (2022). *Towards a critical mathematics*. Indiana University.

Savich, T. M. (2024). Built to break: An introduction to theoretical autoethnography [Invited talk presented at Solent University, organized by Dr. Brian McDonough]. https://iu.mediaspace.kaltura.com/media/t/1_v1qmm3g9

Savich, T. M. (title). *The hermeneutic calculator* [The hermeneutic calculator]. Retrieved July 23, 2025, from https://tiosavich.github.io/UMEDCTA/Calculator/index.html

Sellars, W. (1953). Inference and meaning. *Mind*, *62*, 313–338.

Sellars, W. (2007). *In the space of reasons: Selected essays of wilfrid sellars*. Harvard University Press.

Steffe, L. P. (2002). A new hypothesis concerning children's fractional knowledge [Accessed through CEPA.INFO]. *Journal of Mathematical Behavior*, *20*(3), 267–307.

Steffe, L. P. (2003). On the construction of learning trajectories of children: The case of commensurate fractions [Accessed via ResearchGate/ERIC]. *Monographs of the Journal for Research in Mathematics Education*, 1–26.

Steffe, L. P. (2004). Fractional commensurate, composition, and adding schemes: Learning trajectories of jason and laura: Grade 5 [Accessed via ResearchGate]. In J. Olive & R. Tzur (Eds.), *Children's fractional knowledge* (pp. 1–60). American Educational Research Association.

Steffe, L. P. (2014). *Children's number sequences: An explanation of steffe's constructs and an extrapolation to rational numbers of arithmetic* [Originally from 1994]. Unpublished manuscript, available via CiteSeerX; ResearchGate.

Steffe, L. P., & Thompson, P. W. (2014). Teaching experiment methodology: Underlying principles and essential elements [Accessed via ResearchGate, originally published 2000]. *Proceedings of the 24th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*.

Stone, J. (2003). *The monster at the end of this book* (2nd ed. edition). Golden Books.

Taylor, C. (1994). *Multiculturalism: Examining the Politics of Recognition* (A. Gutmann, Ed.) [Essays by Charles Taylor, K. Anthony Appiah, Jürgen Habermas, Steven C. Rockefeller, Michael Walzer, and Susan Wolf]. Princeton University Press.

van Hiele-Geldof, D., & van Hiele, P. M. (1984). *English translation of selected writings of dina van hiele-geldof and pierre m. van hiele* (D. Fuys, D. Geddes, & D. Tischler, Eds.) [ERIC Number: ED287697]. Retrieved June 3, 2025, from https://eric.ed.gov/?id=ED287697

Wittgenstein, L. (1956). *Remarks on the foundations of mathematics*.

Wittgenstein, L. (2010). *Philosophical investigations* (P. M. S. Hacker & J. Schulte, Eds.; G. Anscombe, Trans.). Wiley-Blackwell.

Chapter 6

Bridge – A Foundational Star

Abstract

This chapter explores the parallel between Dr. Seuss's *The Sneetches* and Georg Cantor's diagonal proof, using the star as both a social marker and a mathematical symbol undergoing transformation. The chapter argues that both narratives involve acts of reflection, demonstrating how finite systems, through self-reflection, can transcend perceived limitations. This concept is illustrated through "The Exercise," a guided meditation on proprioceptive expansion and contraction, providing an embodied metaphor for mathematical reasoning and the concept of determinate negation. The chapter traces the historical development of diagonalization from ancient proofs to the work of Cantor and Gödel, highlighting the theme of exceeding defined totalities. This historical analysis is situated within the ontotheological context surrounding the concept of infinity, emphasizing the shift from potential to actual infinity. The chapter examines Cantor's original proof, its application to real numbers, and Gaifman's generalization of diagonalization, connecting it to Gödel's incompleteness theorems. Finally, the chapter discusses the role of the empty set and zero in mathematical thought, linking them to the concept of becoming and the evolution of numerical systems. The reader will gain a deeper understanding of diagonalization as a manifestation of self-referential processes and their implications for the relationship between finite and infinite systems.

> Roll it away, roll it away,
> You're more than just the sum of tunes you play.
> The simple truth: there's more to say,
> But when it's time, the next line finds a page.
> ⋮
> *Bridge:* I need a bridge
> A crutch to clutch through the worst of it.
> I'm so glad you called, I was really in a fit.
> A hit of warm to curb the loneliness.
> "Better than I hoped, worse than I wished." (*Roll it Away*)

6.1 Introduction

Dr. Seuss' [Theodor Geisel] (1961) book *The Sneetches* begins with the statement of a problem:

> Now, the Star-Belly Sneetches
> Had bellies with stars.
> The Plain-Belly Sneetches
> Had none upon thars.
>
> Those stars weren't so big. They were really so small
> You might think such a thing wouldn't matter at all.
> But, because they had stars, all the Star-Belly Sneetches would brag,
> "We're the best kind of Sneetch on the beaches."
> With their snoots in the air, they would sniff and they'd snort
> "We'll have nothing to do with the Plain-Belly sort!"

As the story unfolds, Sylvester McMonkey McBean arrives with a contraption that can put a star on the Plain-Belly Sneetches for three dollars. This disrupts the class system, and so McBean offers to remove the stars from the Star-Belly Sneetches for just "ten dollars eaches." Chaos ensues as McBean denudes the Sneetches of their dollars and stars or puts them back on.

> They kept paying money. They kept running through
> Until neither the Plain nor the Star-Bellies knew

Whether this one was that one...or that one was this one
Or which one was what one...or what one was who.

After McBean gets all of their money, he drives off laughing about how "you can't teach a Sneetch!"

But McBean was quite wrong. I'm quite happy to say
The Sneetches got really quite smart on that day,
The day they decided that Sneetches are Sneetches
And no kind of Sneetch is the best on the beaches.
That day, all the Sneetches forgot about stars
And whether they had one, or not, upon thars.

Figure

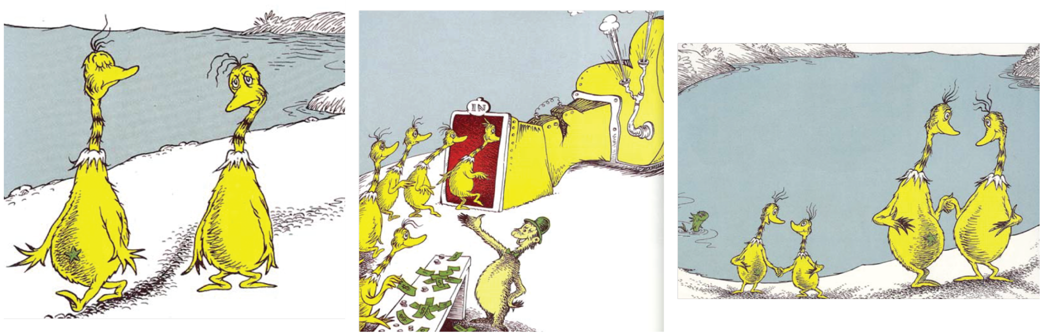


Figure 6.1: *Note.* Stars form a demarcation between the two classes in the Sneetch society. The shyster, McBean, gets their dollars and rides away laughing at their foolishness, assuming they would never change. But he taught a strong enough lesson for Sneetch society to evolve.

This chapter takes Georg Cantor's (1891) diagonal proof as the central axis around which this iteration of critical mathematics turns. To understand its significance, I embed the proof within this allegory of the Sneetches – we should not glean more from Cantor than we reap from the Sneetches. We begin our reconstruction of Georg Cantor's proof by introducing the star ($\star$) as a symbol with two mutually exclusive moments: presence and absence. Through the chaos and confusion that ensues from McBean's machine, the rigid mutual exclusivity of star and no-star is transcended. The final symbol we might use to represent this transcendence is $\not\star$ – a symbol that can be read as both a star and its removal, containing both moments within itself. The Star is sublated.

This embedding gives existential significance to the proof, inviting an explication of the dialectic movement that is implicit in it. The proof tells a story where language is holistically reflected into itself, encoding what is lost in such reflections, what remains, and what arises new through that reflection. It thereby provides an account of sublation in a syntactic microcosm of my thesis that mathematical thought arises through reflections.

This introduction establishes the chapter's methodology: by embedding Cantor's abstract mathematical proof within Dr. Seuss's story of the Sneetches, I explore how mathematical self-reference is enriched through understanding the patterns of recognition, conflict, and transcendence examined throughout this book. The star serves as both the distinguishing mark that creates artificial hierarchies among the Sneetches and the mathematical symbol that undergoes dialectical transformation in Cantor's proof. Just as McBean's machine forces the Sneetches to confront the arbitrariness of their distinctions, Cantor's diagonalization reveals the limits of any attempt to completely enumerate mathematical infinities. The chapter will show how both stories work within a fundamental structure of reflection: the movement by which any finite system, when it reflects upon itself with sufficient complexity, necessarily transcends its own apparent boundaries.

The method at the heart of the proof, now called diagonalization, is interesting. At a very abstract level, diagonalization can be understood in terms of the hermeneutic circle, as a process that defines a whole as a named object with various parts. The whole is reified as an object with properties. Those parts exceed the limits of the whole-as-reified. This demonstrates that the original whole is not-a-thing with properties, but more like a self-negating subject.

Another way to say this is that a totality is recollected (temporally compressed in the first negation), which reifies (turns into an object) the totality. The totality is determinately negated a second time (temporally decompressed) by recollecting the totality along its 'diagonal' seam in a way that creates a new part that demonstrably should have been within the reified totality but cannot have been explicit in the reified totality. This demonstrates that the reified totality is incomplete – the whole exceeds the 'sum' of its parts. The reified totality is sublated – preserved, negated, and elevated – into the rich totality that was always already there, but was thinned out by the reification.

This mirrors human identity claims, especially those made from ascriptive categories as if people were finite objects with properties, like a Sneetch with a Star, a *m*an, or a *w*oman. An infinite number of ascriptive categories would never be sufficient to capture the richness of an individual's identity.

To build this argument, I first trace the historical roots of this pattern from the ancient proofs of Hippasus of Metapontum (circa 500 BCE) and Euclid of Miletus (circa 300 BCE). I then turn to a deeper analysis of Cantor's diagonal proof, which serves as the central example. From there, I will explore the connection Haim Gaifman articulates between Cantor and Gödel's incompleteness theorems. Throughout, the journey of the Sneetches will serve as a guiding star, there to remind myself not to get overly excited by formal symbolic reasoning and that the true significance of these proofs (for me) is not their technical expression but how they serve the existential need to be recognized as both good (finite, coherent, rational) and *in*finite. Recognized as artistic expressions that point to the event of language. Their theme is freedom.

This abstract characterization of diagonalization is helpful for teaching existentially significant ideas about subjects who can encounter and transcend their own limitations, but is limited by its status as an abstraction. Like the Sneetches who ultimately transcend their obsession with stars, mathematical systems that achieve sufficient self-reflective complexity necessarily generate elements that cannot be contained within their original boundaries. The reified totality – whether a finite list of primes, a presumed complete enumeration of real numbers, or rigid social categories based on arbitrary distinguishing marks – becomes the site of its own dialectical negation. What emerges is not chaos, but a richer, more inclusive understanding that preserves what was valuable in the original system while opening new possibilities for development.

There is a danger with how I am articulating the role of diagonalization in the history of mathematics. I am not claiming that the entire history can be subsumed under one method, which Wittgenstein would not even call a method (1956). I am themetizing. The Sneetches share a theme with dialectical reasoning, as does Cantor, as does Euclid. *The Monster at the End of This Book* and Oedipus share a theme of self-misrecognition. Each instance of either dialectical or diagonal reasoning, like each painting of the missing referent in the Fabiola project, is quite different. However, there are compelling reasons to consider the big shifts, where mathematics recollects itself, discovers its expressive inadequacy, then admits some qualitatively different form of quantity, as instances of dialectical movement. There is not a form for Hegel's dialectic besides the movement in form. Every dialectical movement is different. Subsuming all those instances to a particular form of diagonalization, which is always syntactically bound, would be a profound misrecognition.

6.2 Mathematics in the Historical-Hermeneutic Interest

I now reconstruct other proofs from the history of mathematics to draw a thematic unity between sublation and diagonalization. In a stronger sense, diagonalization can be understood as a syntactic species in the genus of sublation, where the genus is the dialectical movement of thought. One expressive goal of this bridge is to nestle the history of mathematics within the sound of time metaphor.

Modern interpretations of 'mathematics' are quite distant from Pythagoras' original sense of the term which was 'something learned' – or, given Pythagoras' institution of the norms of proof 'something demonstrated.' As I write, it feels like swimming upstream to discuss the political, theological, and existential problems that form parts of the whole context in which proofs are articulated. Those ideas are in the 'standards,' and so they are treated as 'fluff.' Knowing Pythagoras only by his contribution of a famous proof makes it feel like an external injection to consider how his Cultish ideology led to a murder (of Hippasus, supposedly) and millenia of out-of-tune musical instruments. According to Aristotle, the Cult of Pythagoras claimed that "that the whole heaven, as has been said, is (rational) numbers" (2017, p. 17). That meant, among other things, that instruments should be tuned to perfect ratios, but physics does not comply with this directive. So, people played instruments that were never quite in tune.

The 'murder' was of poor Hippasus of Metapontum (circa 500 BCE), who was allegedly drowned by the Cult for revealing the existence of irrational numbers. The form of that argument mirrors diagonalization at a very abstract level: Hippasus began with the unit square and the virtuous formula $a^2 + b^2 = c^2$: $1^2 + 1^2 = 1 + 1 = 2 = c^2 \rightarrow c = \sqrt{2}$. By the Cult's own doctrinal formula, $\sqrt{2}$ must exist. By their ideology: If everything (a totality) is a rational number (a reified totality; an object with the property of being rational), then $\sqrt{2}$ is a rational number. That means $\sqrt{2} = \frac{a}{b}$, with a and b written in lowest terms.

Squaring both sides yields $2 = \frac{a^2}{b^2}$, which means $a^2 = 2b^2$, so a^2 is even. If a^2 is even, a must have 2 as a factor – the factor of 2 in a^2 has to come from somewhere. Let $a = 2k$. Then, $a^2 = 4k^2 = 2b^2$, which (by dividing by 2) means that $2k^2 = b^2$. So, b must also have a factor of 2, since b^2 is even. But that means that both a and b are even, which contradicts the idea that $\frac{a}{b}$ was simplified, since they share a common factor of 2. Therefore, $\sqrt{2}$ escapes the reified totality: it is not a rational number, and yet it must exist *by the Pythagoreans' own virtue* that $a^2 + b^2 = c^2$.

Euclid did not necessarily face an ideological distortion that said there must only be a finite number of primes, but there was a prohibition on articulating infinity as *actual*. How can we, who are bound by birth and death, come to know about the ending of that which is without end? His proof is often reconstructed as if Euclid were comfortable with *actual* infinity, but his text, *The Elements*, does not stray from the doctrine that said only *potential* infinity can be discussed.

Rather than writing an indeterminate list of primes $p_1, p_2, \ldots, p_k$, where it is the ... that give rise to offense, he began with a list of just *three* primes: p, q, r. That list is the reified totality of primes. The partial negation is not a logical negation, but a determinate negation. He multiplies the elements of the reified totality together and then adds one: $(p\,q\,r)+1$. We investigate two cases: either the new element is a prime number, in which case the original list of three primes was incomplete, or it must have a factor who was not on the original list. Justifying what Franzen (2004) calls a *constuctive dilemma* – the two-case either/or, where both paths lead to contradiction – relies on earlier results from *The Elements*, so I leave its full demonstration to the supplementary materials. The point is that the reified totality is incomplete, so there must be more than three primes. The demonstration easily extends to any finite list of primes, but Euclid never claims that there *are* an infinite number of primes. He only claims that any finite list of primes is incomplete. The set of prime numbers (a new reified totality) is not-finite, but not yet infinite. This is how we should understand the idea of a *potential* infinity, which is the only kind of infinity that Euclid was comfortable discussing.

We can also examine its historical consequents like Gödel's incompleteness theorem, which demonstrated that a mathematical system that is sophisticated enough to consistently add, multiply, and negate is necessarily incomplete. There, a paradox is encoded into an arithmetic statement that cannot be proved but must be true, demonstrating the fundamental limit of demonstration (proof). I will save the details of Gödel's proof until after the technical details of Cantor's proof have been discussed.

I will first discuss the ontological and epistemological problems of infinity that Cantor's proof is concerned with. I will then embed Cantor's technical proof within commentary that foregrounds the historical-hermenteutic interest. That will point towards historical (mathematical) consequents at the techical layer, which I will reconstruct through Haim Gaifman's commentary on Cantor and Gödel's proof. I will then backfill the historical antecedents by reconstructing Euclid's proof. I will then provide general comments on the knowledge-constitutive interests. I will then return to the Sneetches to get at the existential layers implicit in Cantor's proof. This will set us up for the next chapter, where I will define an anaphoric relationship between zero and *in*finity.

6.3 Ontotheology in the Historical-Hermeneutic Layer

To engage with Cantor's proof authentically, we must consider mathematical history as a relation between ontology and theology, for in his time these domains were intimately connected. The central problem with infinity concerned theological prohibitions: What kinds of infinities could humans legitimately discuss without trespassing on divine prerogatives? Historically, the ominous term *ontotheology* refers to a style of thought (critiqued by Heidegger) where people treated God as the highest being and the foundation of all ontology. In this context, when I claim that mathematical history is a relation between ontology and theology, I'm trying to express that debates about infinity were entangled with religious ideas – certain kinds of infinity were seen as God's domain. In plainer terms: Ontotheology is asking "who controls infinity – humans or God?" and "is infinity something real or just in God's mind?"

The question "Who controls infinity?" carried the weight of potential heresy. Euclid never claimed there were an infinite number of primes, presumably in part because such a claim was verboten at the time. The norms in force were to limit discussions of the infinite solely to the non-finite of *potential infinity.* Like Moses who fled Egypt but could never cross into the promised land, the Greeks and later Scholastics were willing to flee the confines of the finite, but never enter into the realm of *actual* infinity.

I first learned Cantor's proof when I studied mathematics at Earlham College. I had entertained mathematics as a possible pursuit, after considering it my nemesis for many years, based on my encounter with van Gelder's work. However, what sustained my interest in the subject was the way that calculus allowed me to manipulate and control mathematical infinity. I had been puzzled by Zeno's paradoxes, like Achilles and the Tortoise, where Achilles never catches the Tortoise who has a headstart. Zeno reasoned that Achilles never catches the Tortoise because he must first make up half the distance between them, then half of that half etc.: intuitively, taking an infinite number of steps ($\frac{1}{2} + \frac{1}{4} + \frac{1}{8} \ldots$) ought to take an infinite amount of time and traverse an infinite distance. Hence the paradox.

What puzzled Zeno, and myself, was transformed from a mystery into something that could be solved with the rules for manipulating infinite series and the epsilon-delta rules for limits. In a way, the utility of calculus began the actualization of the potential infinity that Euclid had only hinted at. We name that series that so puzzled Zeno as a converging geometric series, where the sum of the infinite terms happens to be 1. But doctrine did not have to evolve in response to these inventions. The epsilon-delta definition of limit, which I do not want to discuss in detail, essentially finitizes the infinite. We approach some limit with arbitrarily small but algebraically defined steps, and determine whether the output of the function approaches a corresponding value. Rather than approaching the 'bullseye' of the infinite directly, we draw a circle around it and say "if I can get within this circle, I have reached the limit," and then allowing the circle to shrink to an arbitrarily small, but still algebraically defined, size. We approach the infinite, but never reach it.

From this historical perspective, we also encounter the paired ontological and epistemological problem that drove Cantor's work: What *is* infinity, and how can we finite beings, bound by birth and death, come to know it? Cantor was a platonist, which means he was committed to the existence of mathematical objects. Quantities are measureable properties of objects, and he was particularly concerned with the measurable properties of the object named 'infinity.' How might one measure 'endlessness?' To answer such a question, we must be dealing with an actual object, not an indeterminate boundlessness. It further required developing a new method for measuring infinite quantities – what Cantor called their *magnitude* (Mächtigkeit) –

though 'thickness,' 'width', or 'power' are also suitable translations (Meyer, title).

This historical context reveals why Cantor's proof was not merely a technical achievement but a conceptual revolution. By developing rigorous methods for comparing infinite sets, Cantor broke through millennia of theological and philosophical prohibitions against discussing actual infinity. Like the Sneetches who learn that their artificial distinctions are ultimately meaningless, the mathematical community had to abandon its rigid separation between the finite realm of human knowledge and the infinite realm reserved for divine understanding. Cantor's diagonalization becomes the McBean machine of mathematics: it forces us through a process of confusion and apparent chaos that ultimately leads to a more sophisticated understanding of mathematical reality. The proof demonstrates that not all infinities are equal, that there are hierarchies of infinite magnitude, and that these hierarchies can be rigorously investigated rather than simply accepted as mysterious divine prerogatives.

Within the proof, Cantor's notion of magnitude transforms into the modern notion of *cardinality*. Two sets have the same cardinality if we can establish a *bijection* between them. A bijections is a one-to-one correspondence where each element in the first set pairs with exactly one element in the second set, and vice versa. His diagonal proof demonstrates that it is impossible to establish such a bijection between the natural numbers $\mathbb{N}$ and something he initially calls a *manifold* ($\mathcal{M}$; not my stepdaughter) but later transforms into the the real numbers between 0 and 1, proving that some infinities are strictly larger than others.

As we analyze the concept of infinity more carefully, we discover overlapping and sometimes conflicting definitions, each relating to specific theological debates that have their own historical development. This history brings us closer to the original meaning of "ontology." Ontology, as the logic of being, was not limited to discussing ~~God~~'s creation but also included questions about God's existence. [1]

The ancient Greeks, like Euclid, were ideologically limited to considering only *potential infinity*. Euclid's proof that there are an "infinite" number of primes is usually taught using the anachronism I just troubled. He did *not* prove there is an infinite number of primes. He proved that, given a

[1] I am not going to put scare quotes everytime ~~God~~ or 'his' gender is mentioned in this text, but I probably should. Pretend it is all scary and so appropriately 'scared.'

list of any three primes, there must be a fourth. Any finite list of primes is *incomplete*. Our modern sensibilities which are so used to naming the infinite as a number depend, to a large extent, on Cantor's work. Cantor's proof, on the other hand, transposes Euclid's argument about prime numbers into a proof about the uncountability of the real numbers.

Technical preliminaries

Rather than stars, Cantor uses two *mutually exclusive* symbols, m and w. Like McBean's Machine, these characters 'flip' in a partial negation. However, the rotational symmetry of these characters is usually ignored, and the 'flip' is treated as an externality. But the symmetry licenses an immanent, embodied, and second-person interpretation of the 'flipping' function. In fact, this interpretation contributes to the concept of the evolution of an act, where the actor reflects on the act by taking a second person position on their act, determining if the act affirmed the impetus to act.

An interesting result of the proof that Cantor discusses is that these infinite sets or *manifolds* – later named *transfinite numbers* – are *well-ordered*. Usually, mutual exclusivity and well-orderedness are asserted as axioms, so we shall have to re-interpret those preliminaries as practices-or-abilities for the proof to cohere with our project. I introduce the *highlander* automaton for those purposes in figure (6.2). In general, we assume that ideas like "mutual exclusivity" are understood as part of the assumption of communicative competency that we can call intersubjectivity. Then, if we find that the assumption does not bear out, we can explain the concept as an action. I use the highlander[2] automaton to represent well-orderedness, where X is a property like $A > B$, Y is a property like $A < B$, and Z is a property like $A = C$. This is the first in what will be a series of gestures to blend Lakoff and Núñez's (2000) work on embodied mathematics with Robert Brandom's analytic pragmatism (2008). A bit of rewiring could transform the highlander automaton into McBean's machine.

Figure

We must also discuss one of the coolest inventions ascribed to Cantor, which is his idea for how to measure unendlessness. I find it so interesting because it is the method by which Cantor actualized infinity. Two sets are said to have the same cardinality if a *bijection* can be articulated that puts

[2]From the movie *Highlander*, where the protagonist learns 'there can be only one.'

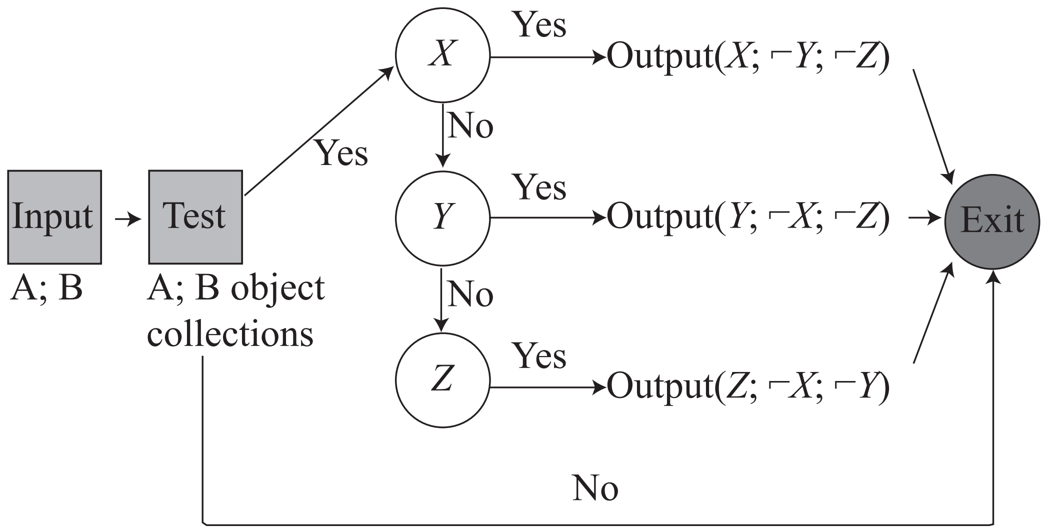


Figure 6.2: *Note.* The highlander automaton is a way to think about mutually exclusive properties, X, Y, and Z a *doing* – a practice-or-ability. I use it to represent well-orderedness, where X is a property like $A > B$, Y is a property like $A < B$, and Z is a property like $A = C$. "¬" means material incompatibility, not formal negation, in this case.

the elements from the domain set into one-to-one correspondence with the elements of the range set. Basically, we pair each element in the domain to each element in the range so that everybody has a buddy. For example, the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ has the same cardinality as the set of even natural numbers $\mathbb{E} = \{2, 4, 6, \ldots\}$ because we can construct a bijection $f : \mathbb{N} \to \mathbb{E}$ such that $f(n) = 2n$. By mapping one infinite collection to another, each becomes determinate and so, in some sense, actual. Figure 6.3 illustrates this concept, along with the more general concept of a *surjection*, which is a function that maps every element of the domain to at least one element of the range, but not necessarily one-to-one. With *injections*, the mapping works over each element of the domain to at most one element of the range, but some elements of the range may not be in the 'image' of the function.

Figure

At the technical level, the second half of Cantor's proof (which I will only summarize) articulates that it is impossible to articulate a bijection between

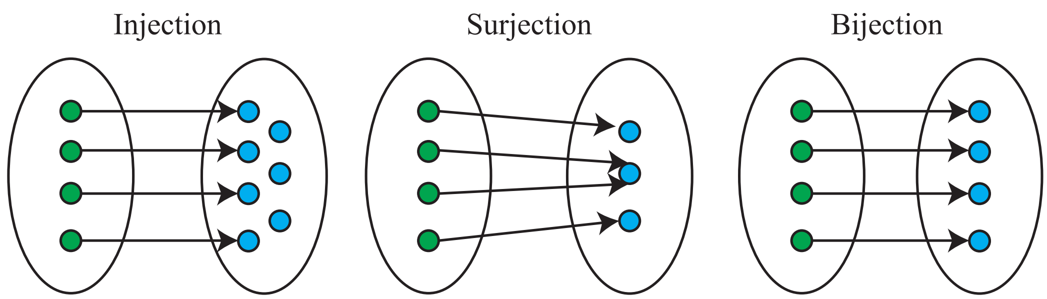


Figure 6.3: *Note.* Note. A bijection is a function that maps every element of one set to exactly one element of another set. Establishing such a bijection demonstrates that two sets have the same cardinality, or measure.

the natural numbers $\mathbb{N}$ and the real numbers between 0 and 1, which we will denote as $\mathbb{R}_{[0,1)}$. This means that the cardinality of $\mathbb{R}_{[0,1)}$ is strictly larger than that of $\mathbb{N}$. Note the modal articulation above. We must prove that all functions from $\mathbb{N}$ to $\mathbb{R}_{[0,1)}$ are not surjective, which means that there is at least one element in $\mathbb{R}_{[0,1)}$ that is not the image of any element in $\mathbb{N}$.

6.4 Cantor's Original Formulation

Cantor's original proof (Cantor, 1891) begins with a statement of a prior result that demonstrated that there are infinite sets that do not have the same measure as the set of natural numbers. He then claims that the prior work can be accomplished more easily and with more generality. Picking up from that statement (and relying on translations provided by Meyer title among others), we have:

> If m and w are any two mutually exclusive characters, then we consider a collection $\mathcal{M}$ of elements, $E = (x_1, x_2, \ldots, x_\nu, \ldots)$, which depend on an infinite number of coordinates, $x_1, x_2, \ldots, x_\nu, \ldots$, where each of these coordinates is either m or w.
>
> $\mathcal{M}$ is the totality of all elements E.

Note: This establishes $\mathcal{M}$ as the complete set of all possible infinite sequences built from two characters. This is a totality. In the next turn, Cantor will give some examples of what sorts of things $\mathcal{M}$ collects. This reifies the totality as an object with an infinite number of properties (elements).

> The elements of $\mathcal{M}$ include, for example, the following three:
>
> $$E^{\mathrm{I}} = (m, m, m, m, \ldots) \tag{6.1}$$
> $$E^{\mathrm{II}} = (w, w, w, w, \ldots) \tag{6.2}$$
> $$E^{\mathrm{III}} = (m, w, m, w, \ldots) \tag{6.3}$$
>
> I now claim that such a manifold $\mathcal{M}$ does not have the magnitude of the series $1, 2, 3, \ldots, \nu, \ldots$.

Note: By "magnitude," Cantor means cardinality or size. Meyer (title) notes that the term 'cardinality' for infinite sets was not in current usage at the time Cantor wrote this paper; he uses the term Mächtigkeit', which can have corresponding English meanings such as "thickness," "width," "mightiness," "potency," etc. Attention to this detail situates the proof in the context of Cantor's own development.

> This follows from the following sentence:
>
> If $E_1, E_2, \ldots, E_\nu, \ldots$ are any simply infinite series of elements of the manifold $\mathcal{M}$, then there is always an element E_0 of $\mathcal{M}$ that does not agree with any E_ν.

Note: This is the heart of Cantor's claim. It is a non-existence claim phrased as a positive construction: no complete list can exist, because for any given list, we can construct an element that is missing from it. Euclid's proof serves as a historical antecedent of this type of claim.

> To prove it:
>
> $$E_1 = (a_{1,1}, a_{1,2}, \ldots, a_{1,\nu}, \ldots) \tag{6.4}$$
> $$E_2 = (a_{2,1}, a_{2,2}, \ldots, a_{2,\nu}, \ldots) \tag{6.5}$$
> $$\ldots \tag{6.6}$$
> $$E_\mu = (a_{\mu,1}, a_{\mu,2}, \ldots, a_{\mu,\nu}, \ldots) \tag{6.7}$$
> $$\tag{6.8}$$

Note: Here, Cantor establishes the conditions for what mathematicians call self-reference. On the left, we have an ascending 'list' of natural numbers serving as *indexes* or names for the sequences on the right. These names are themselves reifications of the totalities on the right. The domain of natural numbers ($\mu \in \{1, 2, 3, \ldots\}$) is being used to provide names for "higher type entities" (Gaifman, 2005, p. 2) – the infinite sequences E. The index μ becomes the name for the sequence E_μ. Each sequence E_μ can be seen as a function that maps a position ν to a character $a_{\mu,\nu}$: those characters are our m's and w's.

Those sequences never end, and so we end up with a set of identity claims that is infinite in two dimensions (vertical and horizontal). In essence, an infinite horizontal sequence is compressed into a single natural number, which serves as its *name*. But then the names are allowed to serve as the domain of a function that maps them back to either an m or a w.

> Here the $a_{\mu,\nu}$ are in a certain way m or w. Let us now define a series $b_1, b_2, \ldots, b_\nu, \ldots$ such that b_ν is also only equal to m or w and different from $a_{\nu,\nu}$.
>
> So if $a_{\nu,\nu} = m$, then $b_\nu = w$, and if $a_{\nu,\nu} = w$, then $b_\nu = m$.

Note: This is the diagonal construction. The diagonal element $a_{\nu,\nu}$ represents the self-application where we evaluate the function named 'ν' at the position 'ν'. Cantor then defines the new sequence, which he will call E_0 in the next turn, by 'flipping' each of these self-referential elements: $b_\nu = \neg a_{\nu,\nu}$ (where $\neg$ represents the operation that changes m to w and vice versa).

I strive to articulate critical mathematics as both 'negation complete' and 'intersubjectivity-first.' Here, those claims meet each other in syntactic form. Cantor's 'flip' ($m \to w; w \to m$) threatens to introduce an externality. For this proof to have much significance outside the realm of formal mathematics, the 'flipping' operation must be understood as drawing out what was always already *within*. I found it pleasant that Cantor named his missing element E_0 for just this reason. The 'flipping' function must be taken as immanent within the system, not external.

Within the allegory of the Sneetches, the 'flipping' function is like McBean's star-off and star-on machine. But it can also be understood (as a pictorial representation) as what Mead calls 'taking the attitude of the other.' It's really more like taking the other's 'perspective' as it is a much 'thinner' than Mead's notion of attitude. Still, when I imagine the other as sitting across the table from me as I construct a diagonal sequence, the element I obtain by 'flipping,' which threatens to introduce an arbitrary externality to reason, does not need to be ‘constructed’ or imposed on the system as an externality. The new element is simply ‘there’ in the other person’s visual field as their South-East to North-West diagonal. Figure 6.4 illustrates this idea.

Figure

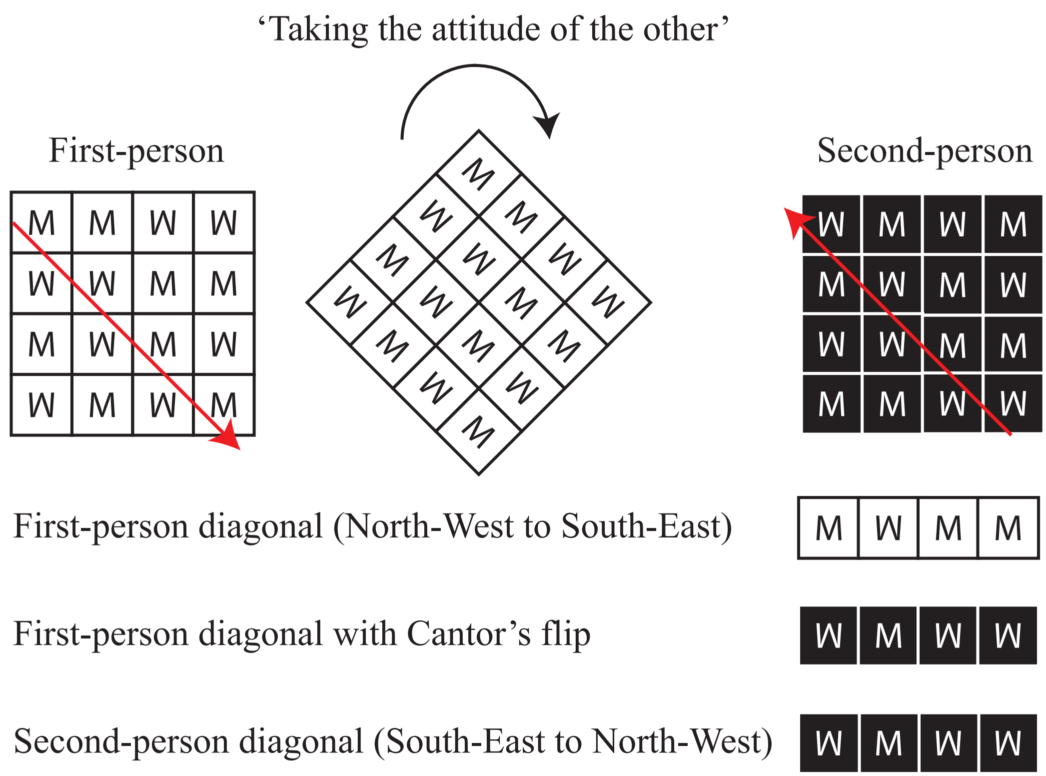


Figure 6.4: *Note.* Attempting to take a second-person position on Cantor’s matrix produces an inversion. I’m not sure if this ‘counts’ as immanent necessity. Surely not without more development.

I have not read any reconstructions of Cantor’s proof that foreground the second-person perspective in the way I have described. Usually, the numerals 0 and 1 serve as the mutually exclusive symbols. But one thing that is really nice about this symmetry is that it corresponds to the experience of creativity. When I ‘create’ something – a new song, a poem, or a simple speech act that was not written on a script (a projective inference) – I usually find out later that someone else had already written it down, and I usually find their articulation to be better than mine! Self-flaggelation aside, the idea that what is new was always already 'there,' written in 'the Literature,' God's book of mathematical formula, or copyrighted by another songwriter, resonates strongly with the experience of creativity. The new element is not an externality, but rather a recollection and sublation of what was always already there, waiting to be recollected.

This has further resonances with the experience of action, including thought-acts like judgment. Note that the sequence of b's is constructed by recollecting the (purported) totality of the manifold, $\mathcal{M}$, taking an element from each of the sequences, E. The whole structure is recalled, and then the negation is applied to each element in the diagonal. This is analogous to how judgment works for inferentialists. The whole of the judging subject's inferentially stuctured commitments is recollected *every* time we judge. More generally, all actions have a self-monitoring component. We take a second-person position on our actions (and transactions) to determine whether or not the act's end satisfied the impetus. With the creation of the diagonal element, the second-person position on that created element just is that created element, allowing for the actor to recognize that creative act as satisfying the original impetus to act. There is no imposed externality; the list is second-person complete.

We must be wary about notions like 'creativity,' here. For one, the machines we could use to formalize Cantor's argument are entirely deterministic. In no way should we subsume an instance of mechanized mathematics under the (contentless) rubric of the *in*finite.

So if $a_{\nu,\nu} = m$, then $b_\nu = w$, and if $a_{\nu,\nu} = w$, then $b_\nu = m$

> If we then consider the element:
>
> $$E_0 = (b_1, b_2, b_3, \ldots)$$
>
> of $\mathcal{M}$, we can easily see that the equation:
>
> $$E_0 = E_\mu$$
>
> for no positive integer value of μ can be satisfied, otherwise for the given μ and for all integer values of ν:
>
> $$b_\nu = a_{\mu,\nu},$$
>
> so also in particular,
>
> $$b_\mu = a_{\mu,\mu},$$
>
> which would be excluded by the definition of b_ν.

Note: Cantor proves that E_0 cannot equal any sequence in our enumeration. If E_0 were equal to the μ-th sequence E_μ, then all corresponding elements would match. But this creates a contradiction at position μ, where b_μ was specifically defined to differ from $a_{\mu,\mu}$.

> From this theorem follows immediately that the totality of all elements of $\mathcal{M}$ cannot be put into the series form:
>
> $$E_1, E_2, \ldots, E_\nu, \ldots,$$
>
> otherwise, we would be faced with the contradiction that a thing E_0 is both an element of $\mathcal{M}$ as well as not being an element of $\mathcal{M}$.

This completes the first half of Cantor's proof, establishing the fundamental logical structure of diagonalization. The proof reveals how any attempt to create a complete enumeration contains within itself the resources for its own transcendence. The diagonal construction – the systematic "flipping" of self-referential elements – generates a new entity E_0 that must belong to the manifold $\mathcal{M}$ by definition, yet cannot appear on any purportedly complete list of its elements. This is the mathematical expression of the same dialectical process we see in the Sneetches story: the systematic application of McBean's transformative machine eventually produces a situation that transcends the original problem entirely. Just as the Sneetches' encounter with the limits of their star-based distinctions leads them to recognize "that Sneetches are Sneetches," Cantor's systematic exploration of the limits of enumeration leads to the recognition that some infinities necessarily exceed others. The contradiction that drives the proof is not a flaw to be avoided but the engine of mathematical progress.

6.5 The Second Feature Cantor's Proof

I will not return to the original text for the second half of the proof, instead I will summarize it from within the empirical-analytic interest. First, note that the modal structure of Cantor's proof is crucial: he proves that *all* functions from $\mathbb{N}$ to $\mathbb{R}_{[0,1)}$ are not surjective, meaning that there is always at least one element in $\mathbb{R}_{[0,1)}$ that cannot be captured by any enumeration based on the natural numbers. I draw the general structure in figure 6.5.

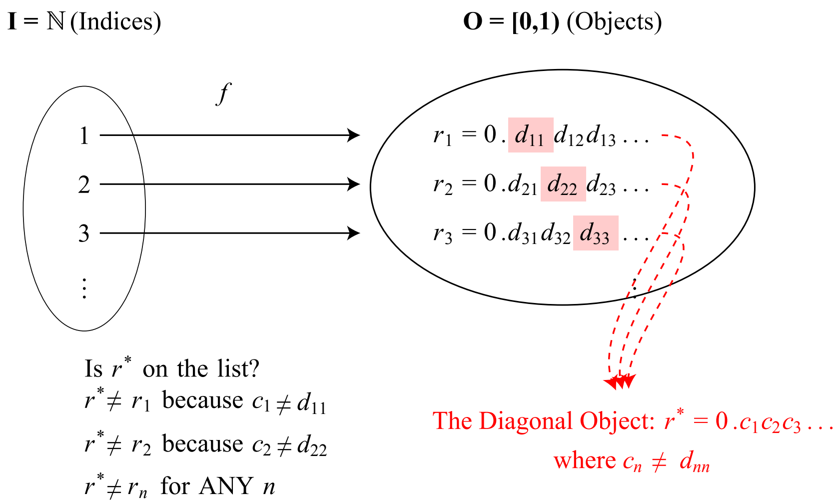


Figure 6.5: *Note.* The structure of Cantor's diagonalization argument. For any assumed complete enumeration, the diagonal construction produces an element that necessarily escapes that enumeration.

Figure

The proof begins by considering a function f that maps natural numbers to real numbers in the interval [0,1). This function is assumed to be a complete enumeration of all such real numbers. The real numbers are represented in their decimal form, emphasizing the digits that are relevant for the diagonal argument.

The diagonal construction then creates a new real number r^* by taking the n-th digit c_n to be different from the n-th digit of the n-th number in the list. This ensures that r^* differs from every number in the list at least in the n-th decimal place.

The crux of the proof lies in the contradiction that arises: r^* cannot be equal to any r_n in the assumed complete list, for it was constructed to differ from each r_n in the n-th digit. Thus, the initial assumption that f is a complete enumeration of all real numbers in [0,1) must be false.

Cantor goes on in the short paper to offer an argument for why the power set of a set (the set that contains all subsets of a set) is always larger than the set itself. I will use this corollary to describe the *phenomenology of confusion* later, but for now let us leave Cantor's original proof and turn to Haim Gaifman's reconstruction of Cantor's proof, which will lead to Gödel's theorems.

6.6 Gaifman's Reconstruction: The Pattern of Diagonalization

Before delving into the technical details of Gaifman's (2005) reconstruction, let us first draw a connection between the formal pattern of diagonalization and the rhythms of embodied rationality, our groundless philosophical ground. The mathematical procedure of diagonalization is not just a formal manipulation of symbols when it is understood as the same limit of thought I articulated in Axiom 0 for the embodied modal logic. Recall that the axiom introduced the concept that erases its own name:

$\delta(\ulcorner\delta\urcorner) = \cancel{\ulcorner\delta\urcorner}$

This axiom describes an ontological limit: any attempt to represent the enabling conditions of a system results in a self-negation. In attempting to capture the ground, the ground withdraws from the representation.

Gaifman's use of Quine corners ($\ulcorner\urcorner$) – and my adoption of the same symbols to write Axiom 0 – provides the formal machinery to denote the *name* of a mathematical expression. The act of naming allows a formal system to talk about its components, expressively actualizing the enabling conditions for self-reference. Diagonalization then forces the paradox of identity into formal explicitness.

Let me preview those components before diving into the details:

- The act of naming ($\ulcorner\urcorner$): In formal systems that are arithmetic, the natural number n is used the *name* a higher-order entity, like a function, predicate, or property (function, for simplicity). For Cantor's diagonalization, Gaifman will rename the infinite set E_n as X_n. This is formally equivalent to placing the concept inside the Quine corners. This reifies one of the system's expressions so that it can be referred to as an object.

- Self-application (The Fixed Point): Once the function is named, the function can take on its own name as its argument. It is like naming a function, f, with the numeral 2, and then evaluating $f(2)$: if $f(2) = 2$ then 2 is a fixed point of f. For Cantor, we consider $X_n(n)$. In "The Exercise", $\delta(\ulcorner\delta\urcorner)$ is the moment when awareness gives into the temptation to become aware of awareness. Formally, it is when the system turns on itself.

In this book, I try to represent this moment in the current Bridge. Bridges, in my understanding of song-writing, recollect what came before in a new form that sublates the first part of the song. The last sections of the song then have more energy as the themes from the first sections of the song are sublated through the bridge.

- Negation and erasure ($\neg$): The diagonal construction 'creates' a new entity, X, which is defined by negating the fixed point: $X^*(n) \iff \neg X_n(n)$. When the mathematician asks if this new entity X has a name, k, within the system, they find a contradiction $X_k(k) \iff \neg X_k(k)$: it does have a name, only if it does not have a name. This is the moment of self-negation, where the system reflects on itself and produces a new element that is not part of the original enumeration. In classical examples of diagonalization, this proves that X^* cannot have a name in the system.

  In "The Exercise," this is where the frustration in meditation may arise. While I meditate *for* some grand experience, I find I cannot retain the discipline necessary for the experience to arise. When I do not succumb to the temptation, the experience intensifies. When I reflect on those frustrations through texts, dialog, and practice with others, the moment of frustration blossoms into the concept that erases its own name, $\cancel{\ulcorner \delta \urcorner}$. In the embodied modality, there need be no surprise, aching curiousity, or frustration about why the name of the concept that erases its own name is absent: it has been erased! But that is a bit too simple. For me, recognizing the non-presence of the awareness of awareness – the groundlessness of ground – is not akin to the scar of contradiction that arises when attempting to name the unnameable and that leads some to mathematical despair. Instead, it is like recognizing the absolute alterity of the Other – the Thou who divades the {I}.

The logical contradiction that arises in Cantor and Gödel's proofs is a projective flattening of the ontological impossibility of a thought fully capturing its ground. Let us now turn to Gaifman's reconstruction to harden this formal shadow of "The Exercise." By making the shadow more determinate, we might find a more intense satisfaction when we let it go.

The Naming Framework

Rather than beginning with Cantor's infinite sequences of characters, Gaifman starts with a sequence of sets of natural numbers: $X_1, X_2, \ldots, X_n, \ldots$. This creates a naming system where each infinite set in the sequence is identified by a natural number. The natural numbers *index* these infinite sets, so we can take an n in $\mathbb{N}$ to represent or name the set X_n.

For some sets, the set's name is an element of the set itself. For example, let $X_3 = \{1, 3, 5, \ldots\}$: here, $3 \in X_3$. Gaifman introduces functional notation, using $X(y)$ as shorthand for $y \in X$, so we can express $3 \in X_3$ as $X_3(3)$. This notational shift – treating sets as functions – may seem disorienting, but it reveals the crucial self-referential structure underlying diagonalization.

The Diagonal Construction

From this naming framework, Gaifman considers all the n's that happen to be in the sets they name – all of the n's that are what we might call **fixed points** of their functions X_n. Fixed points are those x such that $f(x) = x$. They are sites where mathematical self-reference meets with the larger critical-emancipatory interest in *identity claims*, as the sentence $f(x) = x$ amounts to x saying of itself "I have property f." He then defines the complement: X^*, the set of all n's that are *not* in the sets they name. Formally:

$$X^*(n) \iff \neg X_n(n) \tag{6.9}$$

This X^* captures exactly those natural numbers that, when used as names, fail to name sets that contain them. Since our naming system is supposed to be complete – covering all possible sets – the set X^* must itself have a name. If our list were truly exhaustive, there should be some k such that $X^* = X_k$.

The Contradiction

But here we encounter the fundamental incompatibility that drives all diagonalization arguments. If $X^* = X_k$ for some k, then by our definition:

$$X_k(n) \iff \neg X_n(n) \tag{6.10}$$

This should hold for every n. But when we substitute k for n, we get:

$$X_k(k) \iff \neg X_k(k) \tag{6.11}$$

This is a direct contradiction: $X_k(k)$ is true if and only if it is false. The set X^* cannot appear in any complete enumeration, because its existence contradicts the assumption of completeness.

The General Pattern

Gaifman identifies the conditions under which diagonalization becomes applicable: "In general, diagonalization can be used whenever there is a given domain of objects and a correlation that correlates with these objects higher type entities that are defined over this same domain. A higher type entity is a predicate (or property), or function" (2005, p. 2). Gaifman then summarizes the method of diagonalization as a "sandwich" of two fixed points with a negation between them:

- We claim a fixed point (e.g., the totality is the reification of the totality; an infinite sequence is its name; "I am a star-bellied sneetch").
- We 'negate' the fixed point (e.g., we flip the characters, put the Sneetches through McBean's machine, or take the second-person perspective on the diagonal).
- We establish a new fixed point that reflects this negation (e.g., the original totality with its reified boundaries broken, the new element that escaped the enumeration, the Sneetches who could care less about their stars)

This pattern – where a domain of "names" attempts to enumerate a collection of "higher type entities" defined over that same domain – creates the possibility for self-reference. And where self-reference meets negation, diagonalization generates incompleteness.

From Cantor to Gödel

The power of Gaifman's framework becomes clear when we see how it illuminates the connection between Cantor's proof and Gödel's incompleteness theorems. While Gödel himself acknowledges the connection, nothing about Gödel's proof is easy to understand. While I have tried to read Gödel's original works a few times, I made very little headway. I rely solely on reconstructions for my interpretations (Franzén, 2005; Nagel & Newman, 2012).

In Gödel's case, the "names" are mathematical encodings of the formal elements of a mathematical system. The system he encoded was Alfred North Whitehead and Bertrand Russell's *Principia Mathematica*, a formal system that attempted to capture all of mathematics in a single logical framework. The "higher type entities" are the claims *about* those formal elements.

The encoding scheme is called **arithmetization**. It assigns a unique prime number to each symbol that is legible by the the automata that can generate formal statements. Using results proved by Euclid (specifically the fundamental theorem of arithmetic), it is possible to encode statements without ambiguity, as each natural number has only one prime factorization. I think a non-example may be more helpful than actually writing Gödel's scheme.

I recently went to an Avett Brothers concert, and while waiting for the band to start, I noticed that they used Roman Numerals as part of the light show. Specifically, they used the set XXXIII LXVIII CV, which in base ten would be 33, 68, and 105. These numbers do appear to say much, but they are actually a simple cipher. Let $a = 1, b = 2, \ldots, z = 26$, noting that $T = 20, H = 8$, and $E = 5$. Then we can decode the Roman Numerals as follows: $33 = 20 + 8 + 5 = THE$, $68 = 1 + 22 + 5 + 20 + 20 = AVETT$, and $105 = 2 + 18 + 15 + 20 + 8 + 5 + 18 + 19 = BROTHERS$. The reason this is a non-example is that there are many different ways to sum digits to 33. Specifically, the letter $K = 11$, so $33 = KKK$. They obviously didn't intend to microagress the audience: there are a *lot* of ways to partition a number like 33 into a sum of natural numbers between 1 and 26. They could also have spelled $BEDBEDBED$.

What Gödel does instead is to assign a unique prime number to each symbol in the formal system. If the Avett Brothers had instead used $a = 2, b = 3, \ldots e = 11, \ldots h = 19, \ldots$, and multiplied instead of adding, $THE = 71 \times 19 \times 11 = 14839$. Such a scheme would at least be unambiguous about which characters were intended when decoding the code. Gödel's scheme does a bit more than this, however. He also encodes the position of each symbol alongside the symbol itself. In our toy example, THE would be $2^{71} \times 3^{19} \times 5^{11}$. This number cannot be rendered in Roman Numerals, which lack a base system.

As I waited for the band to start, I was captured by their lightshow in part because over the last few months, I have been working on using just such a scheme to encode some of the automatons from the supplementary materials discussed in the Hermeneutic Calculator. Specifically, I had been working on using an additive scheme to encode an automaton's states and transitions. The fact that the Avett Brothers could have been saying "BEDBEDBED" or "THE" creates a systematic ambiguity, so that sometimes a state in a machine and a transition between states can be confused. From that confusion, I was able to construct a machine that exhibited very basic 'emergent' behavior. I don't think my machine should show up in any textbooks for automata theory, as I hardcoded a 'reflective' offramp into the 'emergent' behavior. Essentially, when the machine entered a state whose name was also the numerical value of the input that allowed the machine to transition to that state, it would take the offramp and fall into an endless loop of repeated addition, as a proto-multiplicative operation. I was imagining how, when teaching a mathematical concept, the specific numbers used in the example impact how students understand the concept.

Two things interested me while I was working on that idea. First, I just lied: it was not I who coded anything, but a Large Language Model. That machines are now capable of writing code that can be used to construct other machines that, when suitably prompted into a 'reflective' state, can bootstrap themselves from one kind of operation to another, is a fascinating development. Second, the idea that some numbers are more 'natural' than others for teaching a concept is not new, but I think math educators have not systematically explored the implications of this idea. Consider how many medicines utilize proteins that were discovered by accident. Since the mid-2000s, the problem of protein-folding has been well-studied, but now machines have the capability to assist in this research, systematically exploring the space of possible proteins and their interactions. With AI, it seems possible to systematically explore the space of possible numbers to construct exemplary examples for teaching mathematical concepts as well as to build in 'reflective' examples that can be used to bootstrap students' understandings into new operations. The next 'phase' of math education research may include some such computationally intensive explorations. While I am happily tilting at such windmills, setting the stage for such explorations in the next part of the book, we must be cautious about the implications of such explorations. Is the project of math education about efficiently bootstrapping new operations from old ones and choosing numbers that make it easy to understand concepts? Or is it about coming to identify with the norms of mathematical practice? Or is it fundamentally about the development of mathematical self-consciousness, where numbers are the friendly others that help us understand ourselves? Or something else entirely? Perhaps it is just a mode of control, where sticking kids in front of a screen and seeing who is willing to comply serves as a test for who is invited to a 'good' life and who must be externally controlled (think of the school to prison pipeline)?

In any case, what these astronomically large numbers, called Gödel numbers, do is allow us to encode the entire formal system as well as statements about the formal system into the arithmetic system. One of Gödel's theorems proves it is always possible to create a fixed point using this naming scheme. The kinds of identity claims that arise from this encoding are varied. Gödel encoded a statement that says of itself, "I am not provable in this system." This is a self-referential statement that cannot be proven within the system, as it would lead to a contradiction. But *any* predicate can be encoded in this way, even those that are 'outside' the system, so long as the formal system has sufficient expressive power (alphabet, practices-or-abilities, vocabularies).

This is one reason why I began this book with a discussion of *divaded* concepts, those that are both inside and outside of each other. Gödel's work shows that the hard boundaries between inside and outside can be broken or blurred, allowing for the emergence of new mathematical identities that are not strictly contained within the original system.

The ambition of the *Principia* was to provide a solid foundation for all of mathematics, similar to current quests in the physical sciences to articulate a 'theory of everything.' I share this ambition in building structures with 'tall, thin walls' – though I differentiate my project from Russel and Whiteheads by noting that mine is built to break. I have the privilege of knowing something about Gödel's work that demonstrates the impossibility of such a foundation. Gödel's first incompleteness theorem states that any sufficiently complex formal system that can express basic arithmetic is incomplete: there are true statements about the natural numbers that cannot be proven within the system. The second incompleteness theorem goes further, showing that such a system cannot prove its own consistency. Accomplishing these theorems requires a careful construction of self-referential statements, which Gödel achieved through a sophisticated use of diagonalization. The results are called fixed points, like the green band in figure 7 that is mapped onto itself while the other colors are mapped elsewhere.

These theorems sunk the ambitions of the *Principia* and similar foundational projects. Gödel's work showed that no matter how sophisticated a formal system is, it will always contain statements that are true but unprovable within that system. This is a profound limitation, revealing that the quest for a complete and consistent foundation for all of mathematics is ultimately futile. The connection between Gödel's work and Derrida's deconstruction that Priest (2002) highlights is striking, as the resources of the *Principia* were used to construct the statements that undermine the foundational ambitions of the project. When we find ourselves saying exactly the opposite of what we intended, we are in the realm of deconstruction. Gödel's diagonalization is a mathematical expression of this deconstructive logic.

However, the project of mathematics did not end with Gödel's theorems. It is easy to misread the theorems as suggesting that mathematics' incompleteness is a flaw. But to a math educator, incompleteness is a feature, not a bug. I encounter students in the slipstream between being and nothing, as they become what they already were but lacked the words to say. *Becoming*, in a deeply embodied sense, has no syntactic closure and so it may find some analog in *in*completeness. "The simple truth; there's more to say."

6.7 The More Machine – Returning to the Sound of Time

Mechanizing *becoming* to express a mathematical story of mathematical development requires, first, an acknowledgment of the quixotic nature of such a project. Whereas McBean packs up his machine and leaves at the end of the story, the kind of mathematical machines I have played with cannot break themselves down, nor build themselves up. More sophisticated machines, like Large Language Models, can build the machines I describe and then act as a You towards those machines, forming proto-self-consciousnesses, but as I write, even those sophisticated machines cannot yet sublate themselves. I must limit my ambition to examining the discarded shells to interpolate the pupating life cycle of the machine.

Some are working on Gödel machines, which are designed to explore the implications of Gödel's incompleteness theorems through computational means. These machines can generate statements that reflect the self-referential structure of Gödel's diagonalization, allowing for a deeper understanding of the limitations and possibilities of formal systems. I am going to go a simpler route and focus on a Cantorian machine.

I call the machine the *More Machine* because it generates more, more, and more. Unplugging from the incessant creation of new thoughts is, from the Exercise in chapter 1, a desirable skill to learn. Meditative consciousness as a detached (non-present) awareness is difficult to learn. It takes a certain kind of discipline, suggested by the title of this book.

The More Machine accepts inputs of ms and ws. It outputs a sequence of ms and ws that is guaranteed to be different from any sequence it has previously output, by 'flipping' each element of the diagonal of the matrix that represents the history of its outputs. Originally, I used 0s and 1s and had users input either a 0 or a 1 to lead to the next state. But when I returned to Cantor's original proof, I realized that the ms and ws would allow me to attach a Zeeman Catastrophe Machine (discussed in section **??**) to the More Machine. Zeeman's machine is not totally deterministic. When the rubberbands are pulled straight down, the machine enters into a superpositioned state, where the wheel could rotate left or right. With the right orientation and tension, the machine could output either m or w. Figure 6.6 shows how the historical record might evolve as the wheel of the Zeeman machine moves.

Figure

When meditating, I try to hold the tension of determinate negation and the determinate negation of determinate negation in one significant symbol: "~~no~~." Think of m and w as representing the two poles of a binary opposition, perhaps in assertoric forms like m:"Tio is Rudy's son" and w:"Rudy is Tio's dad" or paradoxical unities like m is the truth value of "This sentence is false" and w is the negation of that truth value. The superpositioning of those two states invites no further reflection. Nothing needs to be said in response to such a statement as it holds the essence of *neti-neti*, where the {I} is neither this nor that. Whereas Brandom considers instrumental action to be a kind of exit ticket from language, I am trying to get at the idea of *non*-action as an exit ticket from the incessant yammering of the More Machine.

Potential energy builds in the machine as meditative consciousness moves along the central axis between the wheel and the fixed end of the rubber band. But then the superpositioned state collapses and the machine outputs either m or w which can never hold the entirety of the {I}. The machine then recollects its entire history in a specifically diagonal way. Judgments accrue and the machine jolts on in its incessant yammering.

The More Machine brings non-deterministic elements to the deterministic structure of Cantor's diagonalization. For a long time, I claimed that I was a 'creative person.' Burdened by the weight of that claim, I found myself flitting about, trying to find something new (and clever) to say or do. I cannot just unplug from the desire to create, but I can learn to hold the the right sort of tensions to arrest the movement of the machine.

Part of what allows me to cease my own yammering, at least for a moment, is the realization that I do not need to read or write anything new. Whatever I might write, in that desire for recognition, has, in some sense, always already been written. The Other has my 'new' thought in their perceptual field. What I mean to explicate in this metaphor is *I-thou* completeness. In the spirit of Martin Buber, I can say that everything I might write or say is already there, written in some Book of Love or Life that I cannot read myself until I have written it. But I do not need to write it down, as it is already there, waiting to be recollected.

The More Machine thus has two significances. It invites contemplation into the paradox of creativity/invention – that which is 'new' must be recognized, which means it must be built from what came before and so cannot actually be 'new.' It also invites contemplation into the discipline of meditation. Like McBean's machine that forces the Sneetches through cycles of distinction and confusion, the More Machine generates endless novelty while simultaneously demonstrating that this novelty was always already implicit in the system's recursive structure. The machine embodies the tension between the compulsive need to create something new and the meditative recognition that everything that may feel necessary to explicate is already explicit for the Other. One way to unhook from the More Machine is through the non-act of *listening*. Listening is a necessary if challenging skill to develop in teachers. I find it so challenging to listen that I felt compelled to demonstrate a syntactic form of I-Thou completeness, hollow though that syntax is compared to the Grand Experience. This connects directly to the embodied rationality explored in Chapter 1: the discipline required to recognize the null representation ($\emptyset$) as the enabling condition of thought. The More Machine's superpositioned state – where the wheel could rotate either way – corresponds to the moment in meditation where the subject neither affirms nor denies but paradoxically holds the tension of not giving into the temptation to think while staying in a state of total relaxation. When this tension collapses into a specific output (m or w), the machine recollects its entire history diagonally, creating new sequences that transcend any finite enumeration while remaining grounded in the system's logical structure.

Critical-Emancipatory Implications

Beyond its technical significance, Gaifman's reconstruction reveals something profound about the nature of mathematical and logical systems. The pattern of diagonalization suggests that every attempt at complete systematization contains within itself the seeds of its own transcendence.

I read this connection to Mead's distinction between the "me" and the {I}. The "me" represents our systematized, nameable aspects – what can be captured in social roles, descriptions, and formal characterizations. The {I}, however, is the spontaneous, creative response that cannot be predicted or completely systematized. Diagonalization, in this reading, is the mathematical expression of the {I}'s capacity to transcend any finite systematization of the "me." Again, that does not mean that the systematicity of the synthetic unity of apperception is unimportant. Authenticity, trustworthiness, and truth are not stupid. They just cannot be fully actualized. Held lightly, as 'tall thin walls that are built to break,' the limitation of systematicity points to larger truths about the nature of human being. Hegel captures this notion in the movement from *Verstand* (understanding) to *Vernunft* (reason), where the former is a finite, systematic understanding of the world, while the latter is an infinite, self-reflective reason that transcends any finite system. In that movement, themes of confession, forgiveness, and reconciliation emerge into explicitness.

This is why I understand Gödel's work as providing a Hegelian response to Cantor's dismissal of Hegel's *in*finite. Through the process of *arithmetization*, Gödel showed how any "outside" critique of a mathematical system can be brought "inside" the system as a mathematical statement. All external criticisms can become immanent to the system itself – but precisely this capacity for self-critique ensures that no system can achieve final closure.

The diagonal argument thus relates structures of self-consciousness and freedom to mathematical becoming. I hope math educators are tuned into this potential: what is called incompleteness is a mathematical form of mathematical becoming. It shows how any finite system, when it achieves sufficient complexity to refer to itself, can potentially transcend its own boundaries.

6.8 Semantic Sense of Zero

Cantor's ambition was a kind of technical control over some types of infinity. His proof expressively *actualizes* the *potential* infinity in ways that Euclid would have eschewed, while leaving the 'Absolute infinity' to ~~God~~. Cantor was *not* a Hegelian, and so we can contrast his proof with the sense of the *in*finite that I am after in critical mathematics.

In each of these cases, the old totality was disrupted but not discarded. Once the irrational numbers were admitted, fractions did not suddenly become otiose. Instead, what was taken as a 'real' number expanded to include the irrational numbers. That 'reality' was then recollected and reified as a new totality, which Cantor's proof neatly distends into what were later called the *transfinite numbers*. That 'reality' was then recollected and further formalized until it met with inclusion paradoxes like Russell's Paradox, which had to do with the set of all sets that do not contain themselves. That totality was then recollected and reified as the *axiomatic set theory* of Zermelo-Fraenkel, which is the standard foundation for much of modern mathematics. But even that totality was not complete, as its claim to completeness was undermined by G¨odel's incompleteness theorem. There, a paradox is encoded into an arithmetic statement that cannot be proved but must be true, demonstrating the fundamental limit of demonstration (proof). But G¨odel was limited by the rigor of axiomatic set theory, which abjures contradictions. Later, the *dialethism* of Graham Priest (2014) demonstrated that contradictions can be entertained in a controlled manner. Paraconsistent logics also allow for contradictions to be entertained.

6.9 Auto-ethnographic Emptiness

When I was a sophomore at Earlham College, I still wanted to do a double major in physics and mathematics. I took a seminar class for math majors where von Neumann ordinals were introduced. Ordinals generally denote a position or rank in a sequence, usually recollected with terms like "first," "second," and so on. Crows apparently can use ordinal-like reasoning, and the ability presumably extends to other animals.

But von Neumann ordinals are a specific type of ordinal number that are defined in set theory. They are constructed such that each ordinal is the set of all smaller ordinals. This construction allows for a rigorous definition of ordinal numbers, which can be used to represent well-ordered sets.

The usual mapping is as followed:

$$0 := \emptyset, \quad 1 := \{\emptyset\}, \quad 2 := \{\emptyset, \{\emptyset\}\}, \quad 3 := \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \ \ldots$$

In class, I remember being fascinated by the idea of structured nothingness. That the answer to questions that had dogged me long before I started teaching math – "what is 2?" – could be expressed in terms of the void ($\emptyset$) was exciting. The long-held antipathy towards mathematics as a governing structure meant to crush all incompentents, myself included, had faded as I learned calculus the year before, but it vanished into nothingness when I began to get the sense that the whole project of mathematics could be interpreted through nothingness.

That evening, I went to a favorite spot on the outskirts of campus. A huge sycamore grew beside a pond, with its branches hung low over the water. I settled into a crook and stared up at the stars. Everywhere I 'looked' – every spatial coordinate that I thought – I found the empty set. Nothingness was like a conduit for thought – I could think to anywhere when nothingness was everywhere.

I mentioned physics because I even thought through the stars, thinking of how every bit of condensed energy of our enmattered universe was porous in the wave-form of that energy. I was so overcome by the vastness of nothingness that I began to form an identity claim with mathematics. Math could offer the spiritual insights I craved in ways that fiddling with the instruments in physics labs could never do. I dropped that major and decided to only puruse mathematics.

I would reconstruct the experience differently now that I have more Hegelian vocabulary. I would say the negative is both conduit and content of thought. While I melted into the not-a-thingness of the empty set, the sense of the empty set given in the seminar was not yet inclusive to myself as a thinking subject. At the time, I was still bound to the I-it mode of reconstruction.

This is the last in the sequence of senses I wish to establish for $\emptyset$. Since it has gone through so many transformations, I return to the modality of embodiment to name it the concept that erases its own name. I cannot speak its significance. I cannot represent it.

This personal encounter with von Neumann ordinals reveals the existential dimension of mathematical emptiness that connects to the larger themes of this book. The experience of lying beneath the sycamore, finding nothingness everywhere as both "conduit and content of thought," demonstrates how mathematical concepts can serve as vehicles for the kind of embodied rationality explored in Chapter 1. The empty set was not merely an abstract logical construction but a portal to direct experience of the groundlessness that enables all representation. This transforms the technical notion of $\emptyset$ into something approaching what I called the null representation – the concept that erases its own name because it points to the enabling conditions of conceptualization itself. The shift from physics to mathematics that this experience catalyzed was not simply a change of academic majors but a recognition that mathematical thinking offers access to the spiritual insights that emerge when finite systems turn back on themselves with sufficient self-awareness. Like the Sneetches who eventually "forgot about stars," the embrace of mathematical nothingness opens possibilities for transcending the subject-object dualisms that constrain conventional approaches to both mathematics and meditation.

6.10 Critical Nullity

Now, with critical mathematics, I must repeat the semantic and pragmatic histories of nothingness, as well as the auto-ethnographic senses developed so far. The goal is to situate the project of critical mathematics as just another turn of the mathematical wheel, though hopefully one without all the same distortions. I loosen Pythagoras' grip on defining mathematics solely as what can be demonstrated to that which is expressively actualized under the norms of systematicity, which are identical with the three synthetic task responsibilities. Errors, misrecognitions, and contradictions are admitted as the material incompatibilities. I will justify how these are contained within the system of critical mathematics in section **??**.

Now that the history of mathematics has been reconstructed, we can compress that history into a significant symbol, $\emptyset$, which will then be recollected as 0. In this way, the dialectical development of mathematical infinity will be nothing, and thereby serve as symbol whose significance is the unrepresentable *becoming*. Hegel's *in*finity is neither the potential infinity that bound the Greeks, nor the actual infinity that Cantor's work expresses, nor is it the Absolute infinity that Cantor kept firmly outside the realm of creation. Instead of unending linear progress, Hegel (metaphorically speaking) imagines a circle, whose beginning is its end as both beginnings and endings are suffused through the repetitive process. Rather than attempting to say something new in the domain of mathematics by transcending Cantor's Absolute with some Super-Absolute, and then a Super-Duper-Absolute, we circle back to identify the *in*finite with $\emptyset$. As my step-daughter said while learning to ride her bike in a circle, "to move in a circle, go diagonally." Thus, the axial role that Cantor's proof plays in this iteration of critical mathematics is tantamount to an axiom of becoming. In the last half of the book, which concerns a pragmatist mathematical system built on incompatibility, the history of mathematical becoming is built in as a symbol – a nothingness; a not-a-thingness – that is the becoming of the {I}.

I represent this process in Figure 6.7. The history of mathematics is represented as a kind of hypersphere, falling through our three formal-pragmatic worlds/knowledge constitutive interests. Each sphere is a limit of knowledge, oftentimes ideologically enforced. The sound of time metaphor is continued, treating diagonalization as a wholistic recollection that results in a transcendence of boundaries on what is possible to express. Hegel's absolute *in*finity suffuses each layer. The ring of arrows pointing in represents the next turn, where these movements are temporally compressed into $\emptyset$, which is then recollected as the numeral 0. The mechanics of that recollection will be established in the next chapter.

Figure

One might ask which coordinate system we are using to map the history in figure 6.7. I posit a kind of *vector space of desire/existential need*. Vectors have magnitudes and directions, and so they are an ideal mathematical term to discuss desire ("I want *that* a lot."). In using early and later Habermas, as if he says the same thing despite an intervening decade to accrue wisdom, is not very good scholarship. By introducing the concept of a vector space, which allows for coordinate transformations, we can imagine a unified 'space of reason.'

In figure 6.8, I represent this vector space in three dimensions. The vertical dimension is where we might locate subjective validity claims(Habermas, 1985; Habermas, 1984). The kinds of subjective validity claims that are essential to this work are articulated in the Embodied Modality from section **??**. We must trust the authenticity and sincerity of those who make subjective validity claims. They are usually articulated in the first person, and so involve the {I}, and the Critical-Emancipatory Knowledge Constitutive Interest (Habermas, 1971).

Figure

The horizontal dimension is where I locate normative validity claims(Habermas, 1985; Habermas, 1984), which have to do with rational goodness. These claims are articulated in the deontic-normative modality of commitment and entitlement (Brandom, 2019). They are often expressed in terms of what one ought to do or what is right and just. These claims foreground the existential need to be recognized as a good person.

On the axis pointing toward the viewer, I locate objective validity claims (Habermas, 1985; Habermas, 1984). These claims utilize the third-person position. They are often expressed in terms of what is true or false, and they foreground the existential need to be recognized as finite, or object-like. When we expunge ourselves of irrational commitments, we become more coherent. As beings-toward-death, in the Heideggerian sense of *dasein*, we find the telos of our existence in the recognition of our finitude.

While I have not done the expressive labor necessary for you to reach the following conclusion for yourself, I hint at it now. When rational goodness, the finitude of experience, and the *in*finite aspects of our becoming are undifferentiated, these three axes are indistinguishable. To borrow a term from Buddhism that Graham Priest (2014) uses to great effect, they *interpenetrate*, spreading and passing through each other in fractal-like self-similarity. The bottom portion of figure 6.8 illustrates this interpenetration.

For a more concrete argument, we could note that the deontic-normative modality is a pragmatic metavocabulary for both the Embodied and Alethic modalities. I cannot *say* anything without being bound by the norms of language. The tensions and frustrations that arise from the limits of language are felt in the body. What is normatively acceptable to say limits what sorts of truth claims I can make, experiments that can be run, and theories of objective reality that can be constructed.

6.11 The Pragmatic Sense of Zero

This manner of compression of history into nullity is only semantic, however, not yet pragmatic. It does not actually communicate the historical development of zero as a mathematical concept, nor does it say much about how zero is used in mathematics.

For a phylogenetic and pragmatic sense of zero, we must turn to archeological artifacts. This presents a methodological problem (for me), as we cannot ask those who created the objects why they were created. Unlike a rich ethnomathematical account, where the people who are doing mathematics in ways that Western mathematics has not yet subsumed can be asked why they are doing what they are doing, the account I render below is part of the mythos of Western mathematics. In any case, that mythos is essentially that people counted on sticks a long, long time ago. Then they started using numerals, then they developed base systems which necessitated the invention of zero as a place holder. Then, in ancient India (circa 628 CE), the mathematician Brahmagupta developed zero as an independent mathematical object (Ernest, 2024).

To motivate that invention, we must first consider the origins of counting in human history. I argue that tally marks have less expressive power than tally systems, which have less expressive power than base systems. Base systems necessarily include a recursive grouping of quantities, which introduces zero as a necessary character in the alphabet of the system. Once introduced, zero can be used to express the absence of quantity, and then finally as a numerical value in its own right.

There are many places to begin when considering the origin of counting, but I will begin with an ancient tally sticks, some of which date to around 35,000 years ago. Tally systems are the oldest known form of counting, and they were used by various cultures around the world. The Ishango bone, found in the Democratic Republic of Congo, is one of the oldest known tally sticks, dating back to around 20,000 years ago. It has a series of notches carved into it, which are thought to represent a counting system. I am not particularly interested in the details of the Ishango bone, but am interested in what tally systems express, especially as it relates to sublation (and so, to diagonalization).

Counting is not merely an accumulation of marks – it is a process that both *preserves* and *transforms* prior determinations. In Hegelian terms, this movement is called *sublation* (Aufhebung), the simultaneous *negation*, *preservation*, and *uplift* of what came before. In mathematical practice, sublation is most clearly seen in the way base systems reorganize quantities into new structural units.

Consider a simple act of tally counting. If one were to count to nine using tally marks, the representation would appear as:

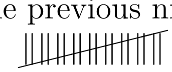
$$\|\|\|\|\|\|\|\|\|$$

Each tally stands independently as a discrete marker that is the trace of the absent object (or the {I} who thinks such objects). They could just go on and on, accumulating indefinitely. This is the "bad infinity," of a monotonous progression that never turns back on itself to form a self-contained whole. This is the realm of pure *Repulsion*. Each individual tally mark is a "one." It is a unit of account that repels incompatibilities (*this* tally is not *that* tally). It asserts its identity through pure, simple exclusion. Each tally stands apart from the others, and it is solely through their external relationship that they are determinate. It is a "multiplicity of ones" that lacks an internal, unifying principle.

A tally stick with 28 marks could represent days in a lunar cycle. A tally stick with 365 marks could be a way to count days in a year. These sticks can be used repeatedly to count the same things, and so they are not discarded, but they do not track how many times they have been used. They are not *re*used, but simply *used* again. Such sticks represent 'the world of ones,' to borrow a phrase from my mentor and colleage, Dr. Amy Hackenberg, whose research on how children learn to count has been foundational for my understanding of counting as a mathematical practice. The whole field of mathematics education, especially the work of the radical constructivists, provides a rich resource for the ontogenetic development of counting. The next part of the book will provide a role for that work in the development of a critical mathematics that is not merely a collection of techniques, but a transformative practice that sublates prior determinations.

At some point in ancient history, people began to group tallies into sets, perhaps to make counting easier or to represent larger quantities. This grouping is a transformation of the original tallies, where the individual marks are no longer isolated but are instead organized into a new structure, a tally system. Many cultures developed their own systems of grouping, but a common method was to group tallies into sets of five or ten. This grouping is a mathematical transformation that preserves the original tallies while also introducing a new structural form.

When we reach the internal limit of ten tallies, we do not simply add another mark. Instead, we negate the prior determinate negations, grouping the previous nine marks into a new structural unit:



The previous nine marks are not erased. They are not 'gone.' But they are *negated* and *uplifted* into a new structural form. Out of the many ones, whose "manyness" is negated, there is now one ten. This is a mathematical instance of sublation as the negation of negation. The prior elements are not discarded. They are reorganized in a higher-level composition. The transition from loose tallies to a single "ten" does not merely introduce a new symbol; it alters how the prior marks are understood. They are no longer just traces of the absent object, but are instead absences traced by the canceling mark.

When we see the group cancel as "one ten," that group becomes the *Also*. It is the medium in which multiple properties now inhere. It is also composed of ten units, also two groups of five, etc. The act of Attraction creates the "Also" by providing a unifying subject for these multiple predicates.

This is not yet a full account of the dialectic development of zero, but we have identified its first fixed point: ten tallies are sublated into one ten. We may turn to Hegel's dialectic of the one and the many for insights into the problem (2010, pp. 132–137). The fundamental problem with tally systems is that they represent number only as what Hegel calls the *Many Ones* in a state of *Repulsion*. Each tally (||) *repels* all others in an external relationship as the exclusion of all otherness. This system lacks a concept for the One as a unifying principle, and therefore cannot conceptualize a null element. The absence of a mark is a potential nothing, not the expressively actualized (posited) nothing that Hegel calls the *void* ($\emptyset$).

A base numeral system resolves this by introducing *Attraction* – the dialectical counter-movement to Repulsion. When a base number of ones is recollected, they are drawn together into a new, single unity. The many become a one. This act of unification, a "coming-together-with-oneself" (Hegel, 2010, p. 139), determines the One and necessitates a symbol for the absence of loose ones. This is the pragmatic role of zero. To think of "10" is to think of 1 ten (the result of Attraction) and no loose ones (the posited void). In essence, the base system represents the (Hegelian) infinite turning of the One back into itself, as the One is now a self-determining unity that induces its own otherness through positing the void.

Does this movement from the many to the one, from loose tallies to a base numeral system, have any connection to Cantorian diagonalization? I think it does, though the connection is loose[3]. The philosophical issue is that the "world of ones" without bases has not encountered and sublated the problem of the one and the many. To think of "10" is to think of 1 ten and *no* loose ones.

We imagine the many as a collection of loose ones, each one distinct and

[3] I explored a few ways to articulate a syntactic connection, but found they did not add much.

separate, supposing that the many is complete. But as soon as the many is reified into such a complete collection, it, too, becomes a one. But it is a one unlike any of the ones from which it was composed. Such a one cannot have been included in the many, as it is a new unity that emerges from the thought of its completeness. It is a unity that is not simply the sum of its parts, but a new entity that transcends the individual ones. This new entity, the number 10, now functions in a dual capacity. It is The One: a unique, determinate concept that repels all others (10 is not 9). Simultaneously, it is The Also: a medium that contains its history and component parts (it is also ten ones, also two fives, or any of the other 2^9 partitions of ten).

The base-ten system approaches the truly infinite in the Hegelian sense because it contains the principle of its own continuation within itself. It doesn't just add another mark; it has a recursive rule for generating all numbers by turning back on its own structure. This is the movement of Attraction, where the repelled "ones" are drawn back into a unifying "One." The true infinity is not a matter of endless addition, which can be represented in tally systems or base systems with equal expressive power, but a self-referential unity that contains its own negation and transcendence.

Out of many, one.[4] The fact that base systems can express arbitrarily large numbers is unrelated to the true infinity that finds its partial expression as soon as one counts to 10. I say partial because the true infinity is not at all captured in the numberal 10, especially as I have not yet discussed how the {I} is involved in the process of counting. With this ending, we are now positioned to begin our account of critical mathematics in earnest. I will next argue argue that numerals are pronouns that recollect the {I think} that accompanies all of my representations. That shall get us to a richer sense of zero and other numerals of interest.

This analysis of counting systems reveals the crucial bridge between the abstract philosophical themes explored in the first half of this book and the concrete mathematical applications to follow. The development from tally marks to base systems embodies the same dialectical structure we have seen throughout: an apparent totality (the "many ones" of tally marks) encoun-

[4] I write this as a quotation of a national paradox, not a nationalistic sentiment. It is almost impossible to work in academia in the United States as I write without concerning oneself with the political problems of Diversity, Equity, and Inclusion (DEI). Even printing the phrase is probably sufficient for my work to be flagged in some way. I find it remarkable that *E Pluribus Unum*, the promissory seal of the United States printed on our currency which is so fetishized, is so often ignored in the political discourse of the country.

ters its own limits and generates a new unity (the grouped "ten") that both negates and preserves what came before. Zero emerges not as an arbitrary placeholder but as the necessary expression of this dialectical movement – the "posited void" that allows the system to turn back on itself recursively. Just as Cantor's diagonalization reveals how attempted enumerations necessarily generate their own transcendence, the base-ten system demonstrates how mathematical infinity emerges not from endless addition but from the system's capacity for self-referential recursion. The movement from "many ones" to "one ten" prefigures the more complex self-referential structures that will be explored in Chapter 7, where numerals are reconceptualized as first-person pronouns that recollect the thinking subject who engages with them. The bridge chapter thus completes its transitional function: having traced how mathematical self-reference embodies the same recognition structures examined in Chapters 1-5, we are now prepared to explore how these structures inform the practical development of critical arithmetic in the chapters that follow.

References

Abbott, E. A. (2020, April). *Flatland: A Romance of Many Dimensions by Edwin A. Abbott*. Independently published.

Agamben, G. (2006, September). *Language and Death: The Place of Negativity* (K. Pinkus & M. Hardt, Trans.; First Edition). Univ Of Minnesota Press.

Aristotle. (2017). *Metaphysics: Aristotle's metaphysics in english, latin and ancient greek: Trilingual edition* (W. o. Moerbeke & W. D. Ross, Trans.; Trilingual). Hermes Language Reference.

Boote, D. N., & Beile, P. (2005). Scholars before researchers: On the centrality of the dissertation literature review in research preparation. *Educational researcher*, *34*, 3–15.

Brandom, R. B. (1994). *Making it explicit: Reasoning, representing, and discursive commitment*. Harvard university press.

Brandom, R. B. (2000). *Articulating reasons: An introduction to inferentialism*. Harvard University Press.

Brandom, R. B. (2008). *Between saying and doing: Towards an analytic pragmatism*. Oxford University Press.

Brandom, R. B. (2019). *A spirit of trust: A reading of Hegel's phenomenology*. The Belknap Press of Harvard University Press.

Bueno, O. (2020). Nominalism in the Philosophy of Mathematics. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Fall 2020). Metaphysics Research Lab, Stanford University.

Cantor, G. (1891). Ueber eine elementare frage der mannigfaltigkeitslehre. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, *1*, 75–78.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). Children's mathematics: Cognitively guided instruction - videotape logs [supplementary material]. In *Children's mathematics: Cognitively guided instruction*. Heinemann, in association with The National Council of Teachers of Mathematics, Inc.

Carspecken, P. F. (1995). *Critical ethnography in educational research: A theoretical and practical guide*. Routledge.

Carspecken, P. F. (1999). *Four scenes for posing the question of meaning and other essays in critical philosophy and critical methodology* (New edition). Peter Lang Inc., International Academic Publishers.

Carspecken, P. F. (2006). Limits of knowledge in the physical sciences. In K. Tobin & J. Kincheloe (Eds.), *Doing educational research* (pp. 405–437, Vol. 1). Sense Publishers.

Carspecken, P. F. (2009). Limits to knowledge with existential significance: An outline for the exploration of post-secular spirituality of relevance to qualitative research. In *Bridging the gap between theory and practice in educational research* (pp. 47–61). Springer.

Carspecken, P. F. (2016). Limits to knowledge and being human: What is 'critical' in critical social research and theory? In P. H. Hinchey (Ed.), *A critical action research reader* (pp. 65–86). Peter Lang Inc., International Academic Publishers.

Carspecken, P. F. (2018). The missing infinite. In R. Winkle-Wagner, J. Lee-Johnson, & T. Gaskew (Eds.), *Critical theory and qualitative data analysis in education* (p. 32). Routledge. https://doi.org/10.4324/9781315158860-2

Carspecken, P. F., & Zhang, R. (2013). Content inference fields in intersubjective space: Transpersonal position, illocution, and logic in the analysis of human interactions. *Qualitative research: A reader in philosophy, core concepts, and practice*, 201–242.

Carspecken, R. (2024, November 3). Untitled.

Caygill, H. (2004). *A kant dictionary*. Wiley-Blackwell.

Derrida, J. (1973). *Speech and phenomena, and other essays on husserl's theory of signs*. Northwestern University Press.

Derrida, J. (1982). *Marges de la philosophie* [Margins of philosophy] (A. Bass, Trans.). The Harvester Press. (Original work published 1972)

Deutsch, H., Marshall, O., & Irvine, A. D. (2025). Russell's Paradox. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford encyclopedia of philosophy* (Spring 2025). Metaphysics Research Lab, Stanford University.

Ernest, P. (2024). The mathematical and philosophical significance of zero: Nought matters [Publisher: Paul Ernest, Emeritus Professor at University of Exeter, School of Education & Lifelong Learning]. *Philosophy of Mathematics Education Journal*, 1–35. https://research.ebsco.com/linkprocessor/plink?id=b685ecb9-d2fc-3e03-889d-a06fb1751c1b

Euclid. (2007, December). *Euclid's Elements of Geometry* (J. L. Heiberg, Ed.; R. Fitzpatrick, Trans.; Revised and corrected edition). Richard Fitzpatrick.

Franzén, T. (2004, September). *Inexhaustibility: A Non-Exhaustive Treatment: Lecture Notes in Logic 16* (1st edition). A K Peters/CRC Press.

Franzén, T. (2005). *Gödel's theorem: An incomplete guide to its use and abuse*. AK Peters/CRC Press.

Frege, G. (1997, July). *The Frege Reader* (M. Beaney, Ed.; 1st edition). Blackwell Publishing.

Gaifman, H. (2005). *The easy way to Gödel's proof and related matters* [Unpublished manuscript. Available online].

Geisel, T. (1961). *The Sneetches and Other Stories* (First edition.). Random House Books for Young Readers.

Gelder, T. V. (1995). What might cognition be, if not computation? *The Journal of Philosophy*, *92*, 345–381. Retrieved April 13, 2025, from http://www.jstor.org/stable/2941061

Gray, L., & Garzon, M. (2005). On a hermite boundary integral approximation. *Computers & structures*, *83*(10-11), 889–894.

Greenblatt, R., Denison, C., Wright, B., Roger, F., MacDiarmid, M., Marks, S., Treutlein, J., Belonax, T., Chen, J., Duvenaud, D., Khan, A., Michael, J., Mindermann, S., Perez, E., Petrini, L., Uesato, J., Kaplan, J., Shlegeris, B., Bowman, S. R., & Hubinger, E. (2024, December 23). Alignment faking in large language models. https://doi.org/10.48550/arXiv.2412.14093

Habermas, J. (1985). *The theory of communicative action: Lifeworld and system* (Vol. 2). Beacon Press.

Habermas, J. (2004). *The philosophical discourse of modernity: Twelve lectures* (F. G. Lawrence, Trans.). MIT Press.

Habermas, J. (1971). *Knowledge and human interests*. Beacon Press.

Habermas, J. (1984). *The theory of communicative action, volume 1: Reason and the rationalization of society* (T. McCarthy, Trans.; Reprint edition). Beacon Press.

Habermas, J. (1995). *Postmetaphysical thinking: Between metaphysics and the critique of reason* (1st edition). Polity.

Hackenberg, A. (2025). *Course notes* [Unpublished course notes].

Hegel, G. W. F. (1977). *Phenomenology of spirit* (A. V. Miller, Trans.; Vol. 632). Oxford University Press Oxford.

Hegel, G. W. F. (2010). *The science of logic* (G. di Giovanni, Ed. & Trans.). Cambridge University Press.

Houlgate, S. (2006). *The opening of hegel's logic: From being to infinity*. Purdue University Press.

Hyppolite, J. (1974). *Genesis and structure of hegel's phenomenology of spirit*. Northwestern University Press.

Kant, I. (1998). *Critique of pure reason* [Original work published 1781/1787, Guyer and Wood, Trans.]. Cambridge University Press.

Lakoff, G., & Johnson, M. (1980). *Metaphors we live by*. University of Chicago Press.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books.

Linnebo, Ø. (2024). Platonism in the Philosophy of Mathematics. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford encyclopedia of philosophy* (Summer 2024). Metaphysics Research Lab, Stanford University.

Longuenesse, B. (2017). *I, me, mine: Back to kant and fichte on the 'i'*. Oxford University Press.

Mead, G. H. (1934). *The social psychology of george herbert mead* (A. L. Strauss, Ed.). University of Chicago Press.

Mead, G. H. (2015). *Mind, self, and society: The definitive edition* (H. Joas, D. R. Huebner, & C. W. Morris, Eds.; Enlarged edition). University of Chicago Press.

Meyer, J. (title). *Cantors 1891 diagonal proof* [Cantor's original 1891 diagonal proof]. Retrieved July 6, 2025, from https://jamesrmeyer.com/infinite/cantors-original-1891-proof

Nagel, E., & Newman, J. R. (2012). *Godel's proof*. Routledge.

Negarestani, R. (2018). *Intelligence and Spirit*. Urbanomic/Sequence Press.

Poston, T., & Stewart, I. (2012, February 29). *Catastrophe theory and its applications* (First Edition). Dover Publications.

Priest, G. (2002). *Beyond the limits of thought*. Oxford University Press.

Priest, G. (2014). *One: Being an investigation into the unity of reality and of its parts, including the singular object which is nothingness*. Oxford University Press.

Quine, W. V. O. (1948). On what there is. In *Thinking about logic* (pp. 189–209). Catholic University of America. https://sites.pitt.edu/~rbrandom/Courses/2023%20Sellars/Sellars%20texts/Quine%20On_What_There_Is.pdf

Quine, W. V. O. (1963). Two dogmas of empiricism [First Harper Torchbook edition, a reprint of the second, revised edition (1961). The book was first published by Harvard University Press in 1953.]. In *From a logical point of view* (Second, revised, pp. 20–46). Harper & Row.

Rödl, S. (2014). Intentional transaction [Accessed: 2025-04-13]. *Philosophical Explorations: An International Journal for the Philosophy of Mind and Action*, *17*, 304–316. https://doi.org/10.1080/13869795.2014.941909

Rose, G. (2009). *Hegel contra sociology*. Verso.

Rowlands, S. (2005). The contradictions in the constructivist discourse [Citing von Glasersfeld]. *Philosophy of Mathematics Education Journal*, *14*.

Savich, T. M. (2020). Towards a critical mathematics [Accessed via ERIC]. *Mathematics Teaching–Research Journal Online*, *12*, 245–291. https://files.eric.ed.gov/fulltext/EJ1384454.pdf

Savich, T. M. (2022). *Towards a critical mathematics*. Indiana University.

Savich, T. M. (2024). Built to break: An introduction to theoretical autoethnography [Invited talk presented at Solent University, organized by Dr. Brian McDonough]. https://iu.mediaspace.kaltura.com/media/t/1_v1qmm3g9

Savich, T. M. (title). *The hermeneutic calculator* [The hermeneutic calculator]. Retrieved July 23, 2025, from https://tiosavich.github.io/UMEDCTA/Calculator/index.html

Sellars, W. (1953). Inference and meaning. *Mind*, *62*, 313–338.

Sellars, W. (2007). *In the space of reasons: Selected essays of wilfrid sellars*. Harvard University Press.

Steffe, L. P. (2002). A new hypothesis concerning children's fractional knowledge [Accessed through CEPA.INFO]. *Journal of Mathematical Behavior*, *20*(3), 267–307.

Steffe, L. P. (2003). On the construction of learning trajectories of children: The case of commensurate fractions [Accessed via ResearchGate/ERIC]. *Monographs of the Journal for Research in Mathematics Education*, 1–26.

Steffe, L. P. (2004). Fractional commensurate, composition, and adding schemes: Learning trajectories of jason and laura: Grade 5 [Accessed via ResearchGate]. In J. Olive & R. Tzur (Eds.), *Children's fractional knowledge* (pp. 1–60). American Educational Research Association.

Steffe, L. P. (2014). *Children's number sequences: An explanation of steffe's constructs and an extrapolation to rational numbers of arithmetic* [Originally from 1994]. Unpublished manuscript, available via CiteSeerX; ResearchGate.

Steffe, L. P., & Thompson, P. W. (2014). Teaching experiment methodology: Underlying principles and essential elements [Accessed via ResearchGate, originally published 2000]. *Proceedings of the 24th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*.

Stone, J. (2003). *The monster at the end of this book* (2nd ed. edition). Golden Books.

Taylor, C. (1994). *Multiculturalism: Examining the Politics of Recognition* (A. Gutmann, Ed.) [Essays by Charles Taylor, K. Anthony Appiah, Jürgen Habermas, Steven C. Rockefeller, Michael Walzer, and Susan Wolf]. Princeton University Press.

van Hiele-Geldof, D., & van Hiele, P. M. (1984). *English translation of selected writings of dina van hiele-geldof and pierre m. van hiele* (D. Fuys, D. Geddes, & D. Tischler, Eds.) [ERIC Number: ED287697]. Retrieved June 3, 2025, from https://eric.ed.gov/?id=ED287697

Wittgenstein, L. (1956). *Remarks on the foundations of mathematics*.

Wittgenstein, L. (2010). *Philosophical investigations* (P. M. S. Hacker & J. Schulte, Eds.; G. Anscombe, Trans.). Wiley-Blackwell.

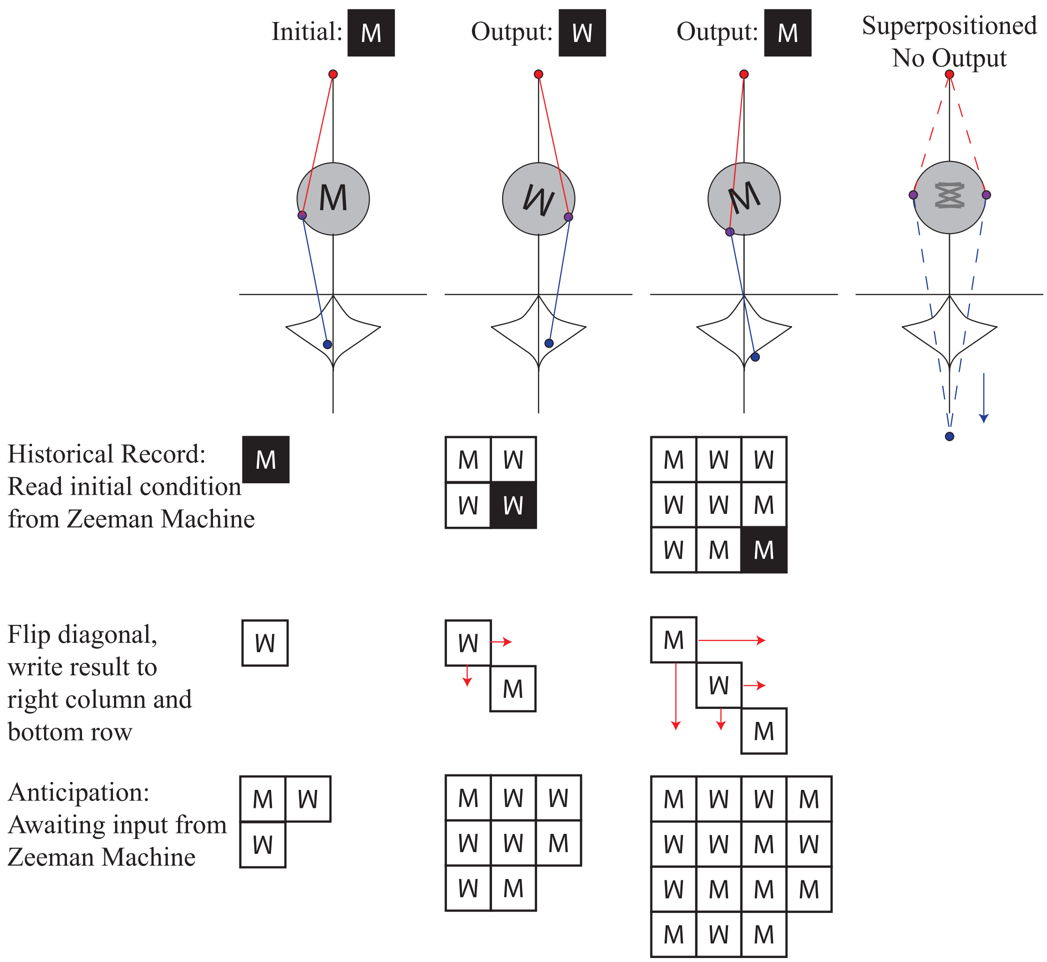


Figure 6.6: *Note.* The More Machine reads states from the orientation of the wheel of a Zeeman Catastrophe Machine.

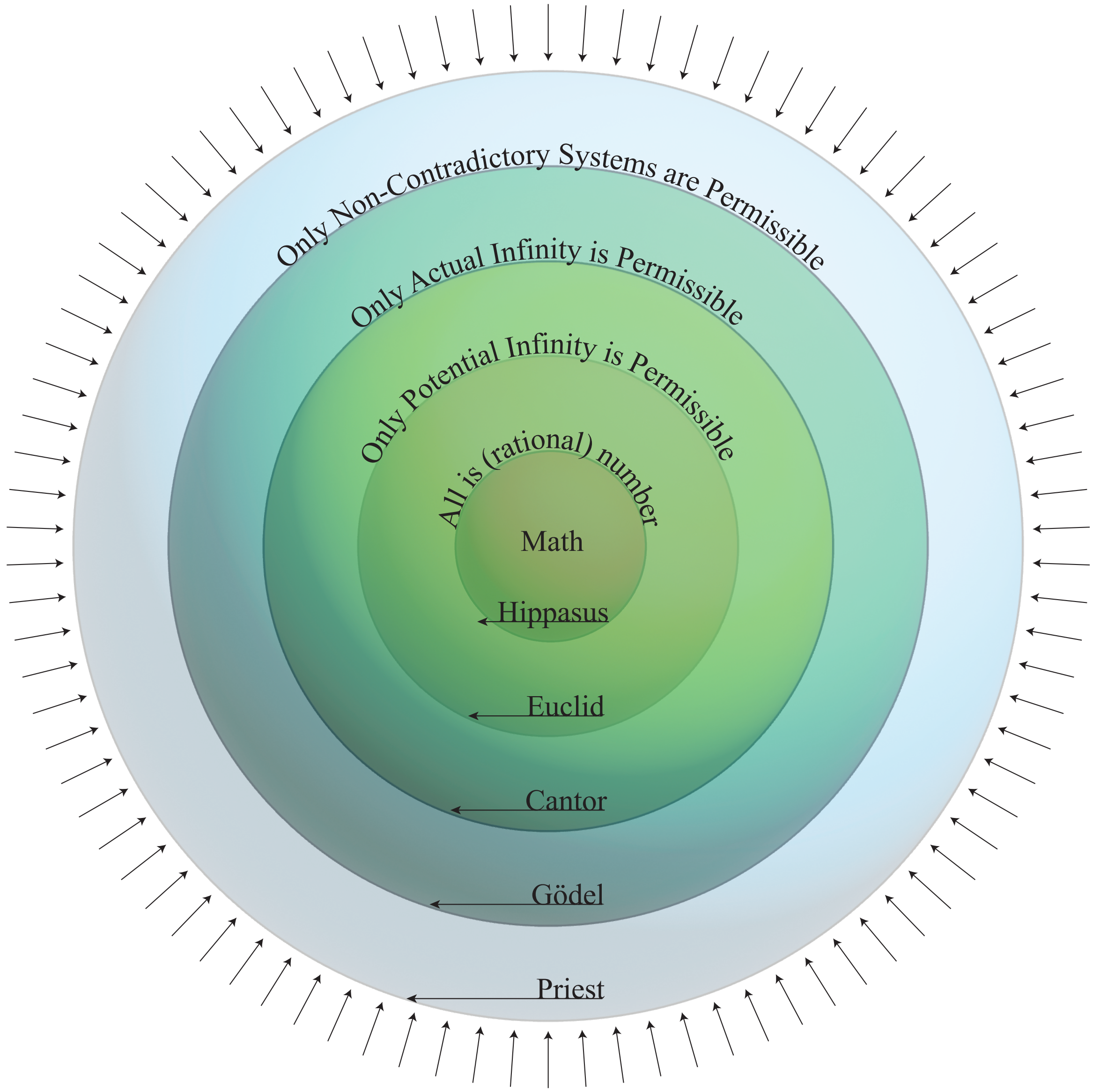


Figure 6.7: *Note.* The history of mathematics can be reconstructed as a series of limits of knowledge that are transcended through rational necessity. This picture continues the sound of time metaphor, treating diagonalization as a wholistic recollection that results in a transcendence of boundaries on what is possible to express. Hegel's absolute *in*finity suffuses each layer. The ring of arrows pointing in represents our next move which will temporally compress these movements into $\emptyset$, which is then recollected as 0.

Vector Space of Desire

Subjective Validity Claims (Habermas, 1984, 1985)
Embodied Modality (e.g. P. Carspecken)
Critical-Emancipatory Interest (Habermas, 1971)
Desire/Existential need to be recognized as *in*finite
First Person (I-thou)

Normative Validity Claims (Habermas, 1984, 1985)
Deontic-Normative Modality (Brandom, 2019)
Hermeneutic-Historical Interest (Habermas, 1971)
Desire/Existential need to be recognized as good
Second Person (I-you)

Objective Validity Claims (Habermas, 1984, 1985)
Alethic Modality (Brandom, 2019)
Empirical-Analytic Interest (Habermas, 1971)
Desire/Existential need to be recognized as finite and coherent like an object
Third Person (I-it)

Fractal-like interpenetration of the moments of Desire
Requires an undifferentiation between the good, the finite, and the *in*fiite

Figure 6.8: *Note.* The history of mathematics in figure 6.7 draws on a coordinate system that I call the *vector space of desire/existential need.*

Chapter 7

Numerals are Pronouns

Abstract

This chapter explores a novel, human-centered approach to understanding mathematics, arguing that numerals and number words function as first-person pronouns. It begins by recounting the author's experience with a student struggling with mathematical concepts, leading to a reconceptualization of number grounded in self-consciousness. The chapter introduces "The Exercise," a practice of introspective listening designed to provide an embodied understanding of the core concepts. This exercise explores the interplay of resistance, naming, and the desire for presence within subjective experience. The chapter then connects these experiences to the null representation () and determinate negation, arguing that the null representation symbolizes the unrepresentable "I think" – the pre-conceptual ground of experience. This "I" is not an object of experience but the condition for having experiences. The chapter argues that this framework aligns with Brandom's analysis of reference and the implicit recognitive abilities required for understanding. Finally, it proposes that grounding mathematical understanding in self-recognition does not lead to relativism, but rather motivates the pursuit of mathematical correctness as a form of authentic self-recognition. The reader can expect to gain a new perspective on the foundations of mathematics, linking abstract concepts to the lived experience of self-consciousness.

7.1 Introduction

What are numbers? This seemingly simple question became agonizingly complex when voiced by a fifth-year senior, $\mathcal{I}$, in Indianapolis. He couldn't graduate with his friends - couldn't move on with his life - because he could not pass a high-stakes standardized test of algebraic knowledge. I was teaching a test-prep course at a high-needs urban school, and the classroom was full of super-seniors crushed by systemic forces. I felt like a conduit of those forces who had a choice to either enact the expectations of my position or treat the people around me as people. You see, it was the first week of class in my first year as a high school teacher. The night before, one of $\mathcal{I}$'s friends was killed in a car accident. $\mathcal{B}$ was fleeing the police when his SUV rolled over, killing him. The night before beginning his super-senior year, where he would have been taking the same test prep class as $\mathcal{I}$, $\mathcal{B}$ was gone. The students were grieving their friend and bitter about taking another stupid algebra class. I had the choice to listen and grieve, but I chose to jump right into algebra. As I yammered on about x and y, $\mathcal{I}$ seemed attentive to the lesson, where other students were not; he was a kind of ally in a hostile room. When he looked up and said "Mr. Savich, what even is two?," I wanted to share my most beautiful truth. I got out my markers and started drawing von Neumann ordinals and lectured about the nothingness embedded everywhere you look.

Any sense of allyship slipped away as the light of connection withdrew. The connection between numbers and the empty set had been a revelatory moment for me when I studied pure mathematics at Earlham College. I could not 'unsee' the nothingness that connected all forms of thought because it was not 'there.' But whatever spiritual import that revelation had was lost in the profound disconnection I experienced with $\mathcal{I}$. I thought I should be able to answer such a simple question somewhat simply. It reinforced a disturbing suspicion: mathematical knowledge, far from being a universal language of reason, can appear as an arcane tool of subjugation, a barrier rather than a bridge. As a math teacher, my role became clear: I was the instrument of suppression. Sick at heart with uselessness, what had once been a profound and beautiful practice became hard to sustain. I eventually burned out of teaching, though the fundamental problem of what we are talking about when we talk about numbers lingered.

I was given the opportunity to reflect on the problem and found an answer that I wish to share with $\mathcal{I}$, though I do not know where he lives or who he has become in the intervening decade. Reflecting deeply on $\mathcal{I}$'s question, I arrived at an answer that offers, if not certainty, then at least a shift in mood: numerals, like "2," and number words like "two," are best understood as *first-person pronouns.* This is the central idea of this paper, and it offers a radical departure from traditional ways of thinking about the ontology of number, as I propose number exists as {I} or {you} exists: number is an autonomous subject. But what does it mean for a numeral to be a pronoun? And why is this perspective significant, especially for mathematics education?

While my answer to the question, "What is 2?" is simple enough for someone in $\mathcal{I}$'s position to contemplate and make their own meaning from, for the premise to hold much academic value, I must situate it in some technical language and make some kind of argument with falsifiable premises. I will try to be as gentle as possible with the jargon because the point is to articulate what numerals are for anyone who uses them. I will provide a glossary of the technical terms as an appendix to this paper, even though presenting a static definition of a term damages the term as surely as touching a stalactite arrests its growth.

Indexicals are context-sensitive words like {I, here, now}. 'Context' is that river of motility that Heraclitus claimed one could not set foot in twice. But try to grasp the meaning of this "*now*" in its immediate particularity. Say, "Now it is night" (Hegel, 1977, p. 60). As soon as we think about this "now," as soon as we try to pin it down, the moment of the "now" has already slipped away, washed away in that Heraclitan river. Whatever time it is right now has passed before the word can leave the speaker's lips. Each now is ~~now~~. In fact, the desire to talk about the ineffable particularity of the present moment is profoundly frustrated by the fact that each word we might possibly use to refer to that particularity is simply another universal (Hegel, 1977, pp. 58–66). This is the paradox of indexicality: to speak of the indexical is already to have moved beyond its pure, present immediacy. While we may wish to be known in our utmost particularity, any attempt to secure that knowledge for oneself or from another is bound by the intrinsic universality of language.

The same holds for the "I," when that term is taken as the necessarily implicit source of action. [1] As soon as that source is recalled, it moves on

[1]Empirical utterances of {I} are *anaphoric* pronouns recollecting the social being who utters the term, but when discussing human being in the context of theory or philosophy, it is helpful to import George Herbert Mead's distinction between the self-as-recognized, which he calls the "me," and the self-as-source-of-action. I will embrace the {I} when

to some other activity. This other activity is usually a monitoring activity from the second-person position. As an action unfolds, the {I} separates from the action to reflect on whether the impetus to act (a desire) is satisfied through the act. Successful acts fulfill that desire. My act towards I was anything but successful, which is one reason why I was so profoundly uneasy about the practice of doing mathematics until I found some way to repair the misrecognition.

The {I} "never is what it is and always is what it is not" (Hyppolite, 1974, p. 174). When we utter "*I am*," however unfalsifiable this claim may be, the "*I*" we reference, the source of that utterance, is no longer simply *being*, but has moved on, already becoming something else. *The act of trying to grasp the indexical "now" or {I} in its immediacy reveals its inherent ungraspability, its fleeting, transient nature.* Taken as an indexical, there is no 'thing' to grasp when we try to fix the raw immediacy of awareness called consciousness. I am no more assured of my existence when I try and catch that shadow than I am when assured that I could capture the {now}. This claim may seem to contradict ordinary experience that involves a certain degree of self-monitoring. But that monitoring involves taking a second-person position on conscious awareness; the moment we try to capture awareness - to be conscious of consciousness - we are no longer in that Heraclitian river but are sitting on its banks.

Yet, to be communicatively useful, this fleeting indexical {I} needs to become repeatable, shareable, something that can be referred back to in discourse. This is where *anaphora* comes in. *Pronouns, fundamentally, are anaphoric terms.* They refer back to something previously introduced in the discourse, creating repeatability structures (Brandom, 1994, p. 549) from what would otherwise be fleeting and context-bound. Consider the "it" in "This chalk is white. *It* is also cylindrical, and if *it* were to be rubbed on the board, *it* would make a mark" (Brandom, 2019, p. 150). "It" refers back to "this chalk" in a similar way that I argue that numerals function anaphorically. But what do they refer back to?

This opening section has traced the path from a moment of profound pedagogical failure to a radical reconceptualization of numerical reference. $\mathcal{I}$'s question – "What even is two?" – emerged from grief and educational alienation, exposing how traditional mathematical ontology can become a

---

speaking of the necessarily implicit source of action. Otherwise, {I} shall mean the person who purports to have authored this paper.

barrier rather than a bridge to understanding. The analysis of indexicals like {I} and {now} reveals that attempts to grasp pure immediacy always encounter their own ungraspability, yet this limitation opens the possibility for meaningful communication through anaphoric reference.

Rather than treating numerals as names for abstract objects – the approach that failed so dramatically in that Indianapolis classroom – they can be understood as pronouns that recollect the enabling conditions of thought itself. This shift from asking "What is 2?" to asking "What does '2' recollect?" prepares us to explore how mathematical reference emerges from the structures of human self-consciousness and recognition.

My claim is that numerals are *first-person recollective pronouns.* Numerals like "2" do not name abstract objects. Instead, they *anaphorically recollect* the enabling conditions of thought itself, specifically the structure of self-recognition. They are pronouns for the "me" in relation to the "I," the self-as-recognized reflecting back on the self-as-source-of-action. This act of recollection, structured through language and social interaction, pulls structure out of the flux, creating something repeatable and communicable from the inherently ungraspable "I think."

Now, to illustrate the traditional way of thinking about numbers that my pronoun thesis challenges, consider a simple joke told by the comedian Hal Roach:

> This fellow decided to audition for a play put on by his local community theater. He got the part, and, at the very end of the play, his one line was to say, "It is." So he walked around town for weeks, practicing his line, saying, "It is… it is… it is…" When the day of the play arrived, and it was his time to deliver his line, he stood up and said, "Uh… is it?"

The humor is mean-spirited if the joke is on the actor for buffing a line. To me, the humor comes from the fact that I can readily imagine an actor whose one line is to voice agreement. The last play I was ever in, I had one line: "Lefty's dead." I rehearsed it until I was cured of acting. Since we do not know what "it" refers to, the humor comes the actor treating "it" as if it were a thing in itself, something whose existence or nature can be directly questioned – "Is *it*?". This "Is *it*?" question, I suggest, mirrors a common, yet ultimately misguided, approach in the philosophy of mathematics. Philosophers and even mathematicians often ask, in various forms, "Is *2*?" "Does *number* exist?", as if "2" or "number" were like the "it" in the joke – some pre-given object whose manner of being we need to ascertain. They delve into complex ontological debates, asking: Do mathematical objects exist in a Platonic heaven? Are they abstract entities? Are they truth values, or something else entirely? (Bueno, 2020; Linnebo, 2024) Such questions are generative and produced nuanced debates, but they share a common false start: they treat numerals as if they are fundamentally trying to *point to* or *represent* something external to language itself – be it an abstract object, a concept, or a truth value.

I include the joke as it gets at the heart of what Wittgenstein calls a peculiar form of philosophical puzzlement: once we recognize that numerals are anaphoric terms recollecting something else, there is no need to ask what manner of existence those terms have. The question becomes: What is the manner of existence of that which numerals recall? That is, instead of asking "Is *2*?", we should ask "What does '2' *recollect*?" This shift in perspective is crucial, especially for mathematics education. For while formal meta-mathematics may concern itself with the internal consistency of axiomatic systems, a meta-mathematics relevant to education must engage with the human dimension of mathematical understanding. It must ask: How do *people* come to understand numbers? How can we make mathematics a bridge, not a barrier, for students like $\mathcal{I}$? What silences them, and what empowers them to speak mathematically?

In this paper, I begin to answer these questions by exploring the idea that numerals are first-person recollective pronouns. They are *anaphoric* terms that, at their most fundamental level, recollect the enabling conditions of thought – the "I think" that makes any judgment possible. By shifting our focus from the "being" of numbers to their function as recollective pronouns, we can begin to move beyond the traditional impasses of the philosophy of mathematics and towards a more pragmatically grounded understanding of numeracy, one that is deeply relevant to the practice and purpose of mathematics education. In what follows, I will elaborate on this idea, drawing on the philosophical resources of inferential pragmatism and transcendental philosophy to unpack the nature of this "recollection" and its implications for our understanding of number and human being.

The Null Representation

To understand how numerals can be first-person pronouns, I need to introduce a crucial concept: the **null representation**, symbolized as $\emptyset$ or $\{\}$. This might seem counterintuitive. How can "nothing" – the null, the empty – be a representation, let alone a key to understanding numbers as pronouns? To share the insight, I need some finite form through which to express the ~~finite~~. I will start with Kant's transcendental insight, though I hope readers will come to understand that there is nothing specifically Kantian about the following argument.

Kant argued that the "I think" "must be able to accompany all my representations" (Longuenesse, 2017, p. 77). Think back to our discussion of "I am." For any experience to be *mine*, for there to be a coherent flow of consciousness, there must be a unifying "I," a subject to whom these experiences belong. This "I," often called the transcendental ego, is not itself an object of experience. It's not something "out there" that we can perceive. Instead, it is the condition that *makes experience possible.*

As explored earlier, this unrepresentability of the "I think" leads to a profound paradox: "I am not myself." This paradox reflects the inherent division within self-consciousness, the tension between the {I} as source of action and the "me" as self-as-recognized, and the potential for the "violence of misrecognition" (Taylor, 1994). Like Grover in "There's a Monster at the End of This Book," we seem perpetually chased by an ungraspable self.

But this inherent implicitness, this necessary inability to fully objectify the subjective ground of thought, is not unique to self-consciousness. It is, in fact, a fundamental feature of reference itself. Consider a simple inference based on identity, as Robert Brandom (1996) elucidates. Suppose we reason:

Premise 1: This paper is green. Premise 2: This paper is my to-do list. Conclusion: Therefore, my to-do list is green.

This inference seems straightforward, mirroring the algebraic form: If Ψa, and $a = b$, then Ψb. However, as Brandom points out, and as we can readily see, there's no guarantee this inference is valid in everyday discourse. Imagine I point to a green piece of paper when uttering the first premise, and then gesture towards a yellow notepad when uttering the second – "This paper is my to-do list." My to-do list, written on yellow paper, is *not* green, even if "this paper" in the first premise *was* green.

The problem is securing co-reference. How do we ensure that "this paper" in the first premise refers to the *same* "this paper" in the second premise, or, more generally, that the first instance of 'a' is truly co-referential with the second 'a'? As Brandom argues, securing reference, even for seemingly simple indexicals like “this,” requires an implicit recognitive ability. We must be able to *recognize* that we are, in fact, talking about the same referent across different utterances, different contexts. But this recognitive ability itself cannot be made fully explicit without undermining the act of reference. There is always an implicit “background” of recognitive capacities that underpins our ability to refer at all.

This necessity of implicit recognitive capacity for reference provides a powerful analogy for, and further justification of, the *null representation*. Just as reference relies on a necessarily implicit recognitive ability that cannot be fully objectified, so too does thought rely on the necessarily implicit, unrepresentable “I think.” The *null representation* $\emptyset$ symbolizes this shared structure of implicitness, this groundless ground that underlies both reference and self-consciousness.

Now, it is vital to clarify: grounding mathematical understanding in self-recognition is emphatically *not* a license for mathematical relativism or the validation of nonsense. *Self-recognition, in this context, is not about subjectively inventing mathematical truths, but about authentically recognizing the structures of validity that are inherent in mathematical thought itself.* Just as misrecognition in social contexts can inflict a “grievous wound” (Taylor, 1994), so too can mathematical misrecognition – getting it wrong, misunderstanding, failing to grasp a valid proof – be a form of intellectual and even existential discomfort.

Indeed, the possibility of *misrecognition* presupposes that there *is* something to be recognized correctly. Authenticity in mathematics, therefore, demands a commitment to rigor, to logical coherence, to the careful construction of valid inferences. The pain of mathematical error, the frustration of misunderstanding, is not merely an intellectual inconvenience; it is a sign that we have, in some sense, misrecognized ourselves as mathematical thinkers.

Thus, the drive for mathematical correctness, the pursuit of valid proofs and sound reasoning, is not just about adhering to external rules or conventions. It is, at a deeper level, motivated by a fundamental human need for authentic self-recognition, a desire to align our mathematical thinking with the inherent structures of reason and validity that make mathematical understanding possible in the first place. This desire for authenticity, for avoiding the pain of mathematical self-misrecognition, is a powerful engine for mathematical inquiry and a crucial aspect of what makes mathematics a meaningful human endeavor, not just an arbitrary game.

Therefore, I propose we use the symbol $\emptyset$ to represent this necessarily implicit, yet unrepresentable, "I think" – the groundless ground of self-consciousness and reference, the "monster at the end of the book" that is both terrifying and, ultimately, just ourselves. Think of it as a placeholder for that which is always already there, underpinning every thought and every act of reference, but which recedes from direct conceptual grasp.

To arrive at this symbol, consider the phrase "I think." We can represent it initially as {"I think"} – enclosing it in braces to indicate its implicit nature. Then, recognizing that the specific *content* "I think" is less important than its function as the unrepresentable condition, we can strip away the content, moving towards {~~I think~~}. Finally, we arrive at {} and then $\emptyset$. This symbolic move – substituting {} for quotation marks around "I think," while emptying the braces – aims to capture the sense of unrepresentability. To summarize the symbolic derivation:

"I think" $\rightarrow$ {"I think"} $\rightarrow$ {~~"I think"~~} $\rightarrow$ {} $\rightarrow$ $\emptyset$.

Thus, the *null representation* $\emptyset$ is not meant to represent "nothingness" in a simple sense. Instead, it symbolizes the unrepresentable, yet necessarily presupposed, transcendental "I think" – the enabling condition of all thought and representation, and the implicit recognitive capacity that makes reference itself possible. It is, in this sense, the groundless ground of our being as thinking, speaking, and referring subjects, a ground that demands authenticity and correctness in our mathematical self-understanding.

The null representation emerges as a crucial bridge between the ungraspable "I think" and the mathematical structures that follow from it. This symbol $\emptyset$ captures the paradoxical nature of consciousness itself: necessarily presupposed yet never directly accessible, the enabling condition that makes all thought possible while remaining beyond full objectification. The parallel with Brandom's analysis of reference shows that this structure of implicitness is fundamental to meaning-making generally – we cannot secure co-reference without relying on recognitive abilities that themselves remain implicit. In mathematics, this means that numerical understanding cannot be grounded in abstract objects but must acknowledge its roots in the self-recognitive structures of human consciousness. Like Agamben's Voice (recall **??**) that enables speech through its own silence, the null representation thus becomes not an empty void but a full acknowledgment of the groundless ground from which mathematical meaning emerges, preparing us to see how numerals function as anaphoric pronouns recollecting these enabling conditions.

Now, you might object: isn't this paradoxical? Representing the unrepresentable with a representation? Yes, it is paradoxical. And this paradox is not a flaw, but a crucial feature of the null representation. It reflects the inherent limitations of any attempt to fully objectify or capture the subjective source of our own thought, the "monster" we can never quite face directly. But it also points towards the possibility of liberation. For when mathematical claims are grounded in a recognition of this inherent self-division, when they acknowledge the unrepresentable "I think" at their core, they can become tools not of subjugation, but of self-affirmation and empowerment. This is the emancipatory potential of a mathematics that embraces its own groundlessness.

We return to this paradox and its implications in the discussion section. For now, simply accept the null representation $\emptyset$ as the symbol for the unrepresentable "I think," the implicit ground of all representation, and move on to see how it plays a role in the anaphoric nature of numerals.

Anaphora

Having introduced the null representation $\emptyset$ as a symbol for the unrepresentable "I think," we now turn to the concept of *anaphora*.

As discussed in the introduction, pronouns are fundamentally **anaphoric** terms. Understanding anaphora, and specifically recognizing it as a form of *linguistic self-reference*, is crucial to grasping why I argue that numerals are, at their core, first-person pronouns and why mathematics itself can be seen as a product of language reflecting on itself.

In essence, *anaphora* is not just a mechanism for repeatability; it is a fundamental form of *linguistic self-reference*. It is how language turns back on itself, allowing a word or phrase to refer back to something already present within the discourse itself. Consider the example: "This chalk is white. *It* is also cylindrical." The pronoun "it" here is an anaphoric term, and it exemplifies linguistic self-reference in action. "It" doesn't point to something outside of language; instead, it *recollects* the content of the earlier phrase "this chalk," referring back to its *antecedent* within the same linguistic context. Anaphora is, in this sense, language referring to its own prior utterances, creating a chain of reference within discourse itself.

Robert Brandom (2008) emphasizes that *indexical* terms like "I," "here," and "now" are inherently context-sensitive. Their semantic content shifts with each new utterance, like a Heraclitan river constantly flowing. If we could only use indexicals directly, our discourse would be trapped in a perpetual present, unable to build upon past utterances or establish lasting connections between different parts of a conversation. Anaphora solves this problem, and it does so precisely through linguistic self-reference. It provides a way to "freeze" the fleeting content of indexical expressions, to lift them out of their immediate context and make them available for re-use within the ongoing flow of language.

Think again about "This chalk is white. *It* is also cylindrical." "This chalk" is initially an indexical expression, its reference fixed by a demonstration in a particular context. But the anaphoric pronoun "it" breaks free from that immediate context through self-reference. It picks up the conceptual content of "this chalk" from within the sentence itself and makes it repeatable. Without anaphora, if we were to say "This chalk is white" and then "This chalk is cylindrical," we might be talking about two different pieces of chalk entirely. Anaphora, as linguistic self-reference, ensures *co-reference*, allowing language to build coherent discourses and reasoned arguments that extend beyond the fleeting moment of utterance by referring back to itself.

Variables in algebra, for instance, are also anaphoric terms, exemplifying linguistic self-reference in mathematical notation. Once a variable, say 'x', is introduced and defined (e.g., "Let x be the number of apples"), it becomes a point of self-reference within the mathematical language. It can be used repeatedly throughout an equation or a proof, consistently referring back to that initial definition within the symbolic system itself.

Numerals, I argue, function in a profoundly similar way, but at an even more fundamental level of linguistic self-reference. They are not merely indexical terms, nor are they simply variables within a formal system. Instead, they are *anaphoric recollections* of something that precedes and enables all discourse: the enabling conditions of thought symbolized by the null representation $\emptyset$. They are first-person pronouns that point back, not to a concrete object, but to the structure of self-conscious thought. And because they are anaphoric, because they are forms of linguistic self-reference, they suggest that *mathematics itself, built upon numerals, arises as language recollects itself, as discourse turns back to capture and structure the conditions of its own possibility.*

This analysis of anaphora reveals why numerals function as pronouns rather than names. Like the pronoun "it" that recollects "this chalk" within discourse itself, numerals engage in linguistic self-reference, creating repeatability from what would otherwise remain fleeting and context-bound. But numerals operate at a deeper level than ordinary pronouns – they recollect not just prior content within a conversation, but the enabling conditions that make any meaningful discourse possible. When language turns back on itself through anaphoric recollection, it creates the possibility for mathematical thinking. This suggests that mathematics is not an external realm of abstract objects but an internal development of language's capacity for self-reference, emerging when discourse recognizes and repeats its own fundamental structures. Understanding numerals as anaphoric pronouns thus reveals mathematics as a sophisticated form of linguistic self-consciousness, preparing us to see how specific numbers like "1," "2," "3" emerge from the iterative process of quotative embedding and recollection.

The next section explores precisely what numerals like "1," "2," "3," etc., anaphorically recollect in this act of linguistic self-reference, and shows how they build upon the null representation $\emptyset$ while revealing their deep connection to the first-person perspective and the act of counting as a form of self-recognition, a form of language reflecting on the conditions of its own intelligibility.

Number: numerals and ordinals

It is now possible to explain how numerals, like "1," "2," "3," etc., function as first-person recollective pronouns. Building upon the concepts of the null representation $\emptyset$ and anaphora, the core idea is this: *Numerals anaphorically recollect the enabling conditions for thought, specifically as these conditions are revealed through quotative recollection.* In simpler terms, numerals are pronouns that point back to, and make repeatable, the structures of self-conscious thought that emerge when we reflect on our own and others' judgments.

The formula for this process is as follows: The process begins with a thought, whose sense depends on another thought, typically in *quotative recollection.* This process, when reflected upon, reveals a nested structure that corresponds to a von Neumann ordinal. This von Neumann ordinal, representing the enabling conditions of the initial thought, is then anaphorically recollected by a numeral, a first-person pronoun of number.

Before illustrating this with the "Telephone" game, a brief clarification of *von Neumann ordinals* is in order. In set theory, von Neumann ordinals provide a way to define natural numbers using only set-theoretic principles, starting from the empty set. The ordinal 0 is defined as the empty set $\emptyset$. The successor of any ordinal is then defined as the set containing all preceding ordinals. Thus, 1 is $\{\emptyset\}$, 2 is $\{\emptyset, \{\emptyset\}\} = \{\emptyset, 1\}$, 3 is $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{\emptyset, 1, 2\}$, and so on. While we are not concerned with the set-theoretic details here, the key idea is that von Neumann ordinals represent a nested, iterative structure built from the most basic element – the empty set. In this context, I propose that these nested structures, built from the *null representation* $\emptyset$ of the "I think," are precisely what numerals anaphorically recollect.

To illustrate this in a more engaging and everyday way, consider the game of "Telephone." In this game, a message is whispered from person to person, and the fun (and the point) lies in how the message gets distorted as it's passed along. Imagine playing "Telephone" with your family. Let $\mathcal{T}$ be me, $\mathcal{A}$ be my wife, and $\mathcal{M}$ and $\mathcal{E}$ be my daughters. When it is my turn to start, I utter my thesis as the initial message: "Numerals are pronouns."

The game unfolds as follows:

$\mathcal{T}$ (initial speaker) starts with the message, intending to convey my core thesis: "Numerals are pronouns."

$\mathcal{A}$ (hears from $\mathcal{T}$ and whispers to $\mathcal{M}$, mishearing and slightly distorting the message): $\mathcal{A}$, perhaps slightly mishearing or reinterpreting, whispers to $\mathcal{M}$: "Numerals are *nouns*." (The message subtly shifts, moving closer to a more conventional, object-based understanding of number).

$\mathcal{M}$ (hears from $\mathcal{A}$ and whispers to $\mathcal{E}$, further distortion and misrecognition): $\mathcal{M}$, hearing "Numerals are nouns," and perhaps further simplifying or misunderstanding, whispers to $\mathcal{E}$: "Numerals are *names*." (The message undergoes further distortion, arriving at a common, but in my view, ultimately inaccurate, conception of numerals as simply names for mathematical objects).

$\mathcal{E}$ (final player announces the message they heard aloud, revealing the cumulative distortion): $\mathcal{E}$, having received the message "Numerals are names," announces to the group: "The message is: 'Numerals are names.' "

The humor and the point of "Telephone" are now evident. The initial, nuanced claim – "Numerals are pronouns," – has, through iterative quotative recollection and subtle misrecognitions, been transformed into its opposite, or at least, into a representation of the view we are critiquing: "Numerals are names." This distortion, however, is not arbitrary. It reflects a common tendency to understand numerals as simply names for mathematical objects, a tendency that, I argue, obscures their deeper, pronoun-like function.

Now, let's trace back the chain of recollections, asking each player, in reverse order, “What did *you* hear?” This process of tracing back reveals the nested structure of quotative recollection and the emergence of von Neumann ordinals:

$\mathcal{E}$'s utterance: “Numerals are names.” This is the final, announced message. We can represent the implicit “I think” behind this utterance with the null representation: $\emptyset$: “Numerals are names.”

$\mathcal{M}$'s recollection of $\mathcal{A}$'s utterance: “$\mathcal{A}$ told me ‘Numerals are nouns.’ ” $\mathcal{M}$'s utterance is a quotative recollection of $\mathcal{A}$'s speech act. We can represent this nested structure as: $\{\emptyset\}$: “$\mathcal{M}$ says, ‘$\mathcal{A}$ said “Numerals are nouns.” ’ ”

$\mathcal{A}$'s recollection of $\mathcal{T}$'s utterance: “$\mathcal{T}$ told me ‘Numerals are pronouns.’ ” $\mathcal{A}$'s utterance is, in turn, a quotative recollection of $\mathcal{T}$'s speech act. The structure becomes further nested: $\{\{\emptyset\}\}$: “$\mathcal{A}$ says, ‘$\mathcal{T}$ said “Numerals are pronouns.” ’ ”

$\mathcal{T}$'s original utterance: “Numerals are pronouns.” This is the initial, un-recollected utterance, the starting point of the chain. While we could represent an implicit “I think” even here, for the purpose of tracing the *quotative* recollection structure, we can consider this the base case, implicitly contained within the subsequent recollections. However, to be consistent, we can represent the very first implicit “I think” as well, arriving at: $\{\{\{\emptyset\}\}\}$: “$\mathcal{T}$ says, ‘Numerals are pronouns.’ ”

Discerning von Neumann ordinals in “Telephone” Game (Family Edition)

Implicit von Neumann Ordinal	“Telephone” Game Utterances (Family, Traced Backwards)	Numeral (Anaphoric Recollection)
(None)	$\mathcal{E}$ announces: “Numerals are names.”	None
$\emptyset$	$\mathcal{E}$: “Numerals are names.”	0
$\{\emptyset\}$	$\mathcal{M}$: “$\mathcal{A}$ told me ‘Numerals are nouns.’ ”	1
$\{\{\emptyset\}\}$	$\mathcal{A}$: “$\mathcal{T}$ told me ‘Numerals are pronouns.’ ”	2

Implicit von Neumann Ordinal	"Telephone" Game Utterances (Family, Traced Backwards)	Numeral (Anaphoric Recollection)
$\{\{\{\emptyset\}\}\}$	$\mathcal{T}$: "Numerals are pronouns."	3

Table 7.1: *Note.* This table organizes the relationship between the "I think" as it is recalled with the null representation, the "Telephone" game utterances with family characters, and numerals.

Analyzing the "Telephone" game example demonstrates how numerals, as first-person pronouns, can be understood as anaphoric recollections of the nested structures of quotative recollection. The numeral "3," for instance, anaphorically recollects the structure $\{\{\{\emptyset\}\}\}$, which emerges from tracing back three layers of "said that..." in the game. And crucially, the distortion of the message as it's passed along – from "pronouns" to "nouns" to "names" – highlights the tendency to misrecognize the nature of numerals that this paper seeks to address. Numerals are not names for mathematical objects "out there"; they are pronouns of thought, born from the dynamic, iterative, and sometimes misrecognizing, process of linguistic communication and self-reflection. Moreover, the von Neumann ordinal structure, built iteratively from the null representation, provides a formal counterpart to this iterative process of quotative recollection, showing how even the most abstract mathematical structures can be seen as rooted in the fundamental dynamics of human communication and self-consciousness.

The "Telephone" game provides a concrete illustration of how numerals emerge from the dynamics of human communication and recognition. The progression from "pronouns" to "nouns" to "names" mirrors exactly the misrecognition this chapter seeks to correct – the tendency to treat numerals as object-referring names rather than self-referring pronouns. By tracing the quotative embedding structure backwards (what $\mathcal{T}$ said that $\mathcal{A}$ said that $\mathcal{M}$ said...), this trace shows how von Neumann ordinals capture the nested levels of linguistic self-reference. Each numeral thus becomes an anaphoric recollection of a specific depth of reflexive embedding, showing how mathematical structure emerges from the process of human beings reflecting on their own speech acts. This playful yet rigorous demonstration reveals that what are called "abstract" mathematical objects are actually fractal-like patterns of human self-consciousness and communicative practice. Rather than referring to a mysterious Platonic realm, numerals recollect the enabling conditions of human thinking, making them fundamentally first-person pronouns of mathematical self-recognition.

Perhaps it is worth noting that this entire text is framed as a critical autoethnograpy. While that name is built to break, I claim it reflects the structure of mathematics itself. The method and its topic are not external to one another. They are divaded.

In the following subsections, the implications of this view are further explored, including its connection to the successor function and the nature of mathematical proof, and the discussion returns to consider the Derridean dimensions of this iterative process of linguistic recollection, now illuminated by the playful yet profound example of the "Telephone" game with my family.

Successor function

One of the strengths of understanding numerals as first-person recollective pronouns is that it provides a natural and intuitive definition of the *successor function* – a concept often taken as a primitive axiom in many mathematical systems. In this framework, the successor function emerges from the interplay of two distinct, yet related, forms of recollection: *quotative embedding* and *anaphoric recollection*.

A clarification is in order:

Quotative Embedding: This is the process of embedding a prior thought or representation within a new layer of quotative context, symbolized by set braces $\{\}$. This embedding creates the nested structure of von Neumann ordinals. For example, moving from the null representation $\emptyset$ to $\{\emptyset\}$ involves quotative embedding. It's like adding quotation marks around a thought, indicating that it is being considered *as* a thought, as something recalled or reported.

Anaphoric Recollection: This is the act of recognizing and "picking up" a previously established conceptual content and making it repeatable and referable in discourse, symbolized by the assignment of a numeral (e.g., "1," "2," "3"). Numerals are *anaphoric pronouns* that recollect these embedded structures. For example, assigning the numeral "1" to the von Neumann ordinal $\{\emptyset\}$ is an act of anaphoric recollection.

With this distinction in mind, the emergence of the successor function can be re-examined:

extitStarting point: The numeral "1" as anaphoric recollection of $\{\emptyset\}$: I have argued that "1" is the *anaphoric recollection* of the von Neumann ordinal $\{\emptyset\}$, which represents the structure resulting from the first act of *quotative embedding* of the null representation (i.e., embedding the "I think" within set braces).

extitThinking the successor of "1" (i.e., "2"): To think "2," proceed with another two-step process:

1. *Quotative Embedding of "1":* Take the numeral "1," already an anaphoric recollection, and perform *quotative embedding* on it, creating the structure: {"1"}. In terms of von Neumann ordinals, this means embedding the ordinal $\{\emptyset\}$ within another set of braces: $\{\{\emptyset\}\}$.

2. *Anaphoric Recollection of the Embedded Structure:* Then perform *anaphoric recollection* on this newly embedded structure, $\{\{\emptyset\}\}$. This act of anaphoric recollection is symbolized as the numeral "2."

extitFormalizing the Successor Function: It is now possible to re-formalize the successor function to reflect this two-fold process:

successor(n) = anaphoric-recollection({quotative-embedding(n)})

However, to simplify the notation and emphasize the iterative nature of the process, it can also be expressed recursively, using "recollection" to encompass both quotative embedding and anaphoric recollection as they work together:

successor(n) = recollection(n) $= n + 1$

In this simplified recursive form, "recollection" signifies the combined act of quotative embedding a structure (represented by n) and then anaphorically recollecting the resulting, newly embedded structure as the successor, $n + 1$.

Thinking the successor of any numeral n is thus a two-step process: first, *quotatively embed* n (in its von Neumann ordinal form), and then *anaphorically recollect* this newly embedded structure as $n + 1$. It is worth noting that the practical implementation of number systems in specific bases, such as our familiar base ten system, introduces additional layers of complexity to this fundamental successor function. For instance, the transition from "9" to "10" in base ten involves a fascinating process of composing a new unit and "carrying over," which can be understood as a form of *sublation*, where a collection of units is simultaneously negated and preserved in a higher-order representation. However, a detailed exploration of base systems and their implications for our theory of numerals as pronouns must be reserved for future work.

This definition of the successor function, in its fundamental form, is not imposed as an axiom, but rather emerges from the dynamic interplay of quotative embedding and anaphoric recollection, grounded in this understanding of numerals as first-person recollective pronouns.

This approach also sheds light on the nature of basic arithmetic operations. Addition, for example, can be understood recursively in terms of recollection and the successor function. While a full exploration of arithmetic is beyond the scope of this paper, briefly note that judgments like "$2 + 3 = 5$," which often feel definitionally true, can find their grounding in these recursive definitions based on recollection. Such *analytic* mathematical judgments, in this view, reflect the inherent structure of normativity itself, rather than requiring externally imposed axioms for their justification. In contrast, *synthetic* mathematical judgments, such as "two is the only even prime number," require material inferential rules and richer conceptual content, as explored in more detail elsewhere (Savich, 2020, 2022).

In the next subsection, potential objections to this account are addressed and the role of the null representation in understanding the nature of mathematical systems and human being is further clarified.

Discussion - What is the null representation?

To understand the "groundless ground" of the null representation, it is necessary to move beyond not only objectivism, but also beyond a static interpretation of even the most foundational philosophical insights, such as Descartes' *cogito*. The problem with Cartesian dualism, in its traditional rendering, is that both "I think" and "I am" are often treated as static, axiomatic principles, fixed points upon which to ground knowledge. But the null representation, in its essence as a form of negation, challenges this static, foundationalist impulse. It points towards a more dynamic and rhythmic understanding of grounding itself, one that embraces self-negation and inherent instability, recognizing that even the most fundamental concepts are not fixed endpoints, but points of departure.

From Objects: The Limitation of Objectivity and Static Foundations

Initially, especially within a mathematical context, one might be tempted to interpret the empty set, and thus the null representation, as simply representing *objects* in their most basic, stripped-down form – a kind of bare, objective existence. This initial interpretation, however, not only embraces objectivism, but also risks falling into a static, axiomatic mode of thought, akin to a rigid reading of Descartes. It treats the object as a self-grounding ground, a fixed foundation. Think of the notion of “ground” in some methodological approaches, where “’ground’ grounds ground,” implying a static, self-sufficient foundation, a fixed point $f(x) = x$ where ground is both function and argument, requiring no further movement or justification.

However, the true power of Descartes’ insight, I argue, lies not in two separate, static axioms (“I think” and “I am”), but in their dynamic dyadic relation. The more appropriate way to understand the *cogito* is not as a linear inference (“I think, *therefore* I am”), but as a rhythmic, self-sustaining structure: *I think* $\leftrightarrow$ *I am*. It is a single structure with two inseparable sides: the active, self-positing “I think” and the recollected, self-recognized “I am.” Neither side is primary; they are mutually constitutive, constantly implying and negating each other in a dynamic rhythm.

Imagine Thomas the Tank Engine chugging along, repeating to himself, “I think I am, I think I am…” Except, in this rhythmic self-affirmation, the “I am” would often be unspoken, implicit, the silent beat grounding the explicit assertion of “I think.” This rhythmic interplay between explicit assertion and implicit grounding, between thought and being, is closer to the dynamic spirit of the *cogito* and to the nature of the null representation itself.

For the null representation, symbolized by $\emptyset$, is precisely this point of negation, this determinate “no.” But it is not a simple, static negation. It is a “no” that is directed, in its finality, even at itself – a self-negating negation, represented in the interplay between determinate negation “no” and its sublation ~~no~~. This self-negating “no” is, in essence, a critique of the limitations of the “I think” when taken in isolation, when understood as a self-sufficient, static principle. The null representation, as this self-negating “no,” reveals that the “I think,” in its act of positing itself as ground, simultaneously points beyond itself, towards its own groundlessness, its own inherent incompleteness. It is in this sense that the null representation becomes a symbol for the recollection of our fundamental *not-a-thingness*, the recognition that the ground of our being, and the ground of mathematical understanding, is not a static object, nor a fixed principle like the isolated “I think,” but a dynamic, self-recollecting, and ultimately groundless process that constantly transcends any fixed representation.

Just as object-based ontologies, and static interpretations of foundational principles, can obscure the dynamic and rhythmic nature of thought and being, so too can they miss the profound significance of this "not-a-thingness" at the heart of mathematical understanding. But by embracing the null representation in its paradoxical emptiness, by recognizing the "groundless ground" it symbolizes – not as a final answer, but as an opening to further inquiry – we can begin to move towards a more dynamic, self-reflective, and ultimately more authentic understanding of numerals, mathematics, and ourselves as thinking, speaking, and becoming beings. This journey, of course, does not end with the null representation, but rather begins there, inviting us to explore the further dimensions of subjectivity, intersubjectivity, and the ever-unfolding process of meaning-making.

7.2 Historical Contexts

Throughout this chapter it has been shown that philosophies of number do not float above human interests; they are often driven by deep methodological and epistemic motives. Habermas identified three broad categories of interests that underlie knowledge: the technical *empirical-analytic interest* (knowledge to predict and control, aiming at objectifying reality), the practical *historical-hermeneutic interest* (knowledge to interpret and understand meanings in context), and the emancipatory *critical interest* (knowledge to reflect and free ourselves from constraints). Each philosophy of number can be interpreted as foregrounding one (or sometimes two) of these interests.

The following chart (Table 7.2) summarizes the major philosophies of number, their core assumptions about what numbers are, and the primary knowledge-constitutive interest they serve. This schematic overview is modelled after Carspecken's reconstruction of Habermas but adapted to our subject matter (and expanded where necessary). It should be read as a guide to how each view "sees" mathematics and what it hopes to achieve or ensure by that view. Of course, any brief summary is an oversimplification – many philosophers hold nuanced positions that combine elements. Nevertheless, this chart highlights the central thrust of each perspective.

Table 7.2: *Note.* Philosophies of Number and Their Knowledge-constitutive Interests.

Philosophy of Number	*Core Assumptions about Numbers*	*Primary Knowledge Interest*
Pythagoreanism (Ancient)	Numbers (whole numbers and ratios) are the fundamental reality and give structure and meaning to the cosmos (mystical harmony). All things can be understood as numbers.	*Empirical-analytic / Metaphysical*: Sought objective cosmic order through numbers, but also had a *proto-critical spiritual interest* (purification of the soul via numerical harmony).
Platonism / Realism	Numbers are abstract, non-physical objects that exist independently of human minds. Mathematical statements are objectively true descriptions of this realm of numbers (e.g. 5 is prime is true because 5 has that property in itself).	*Empirical-Analytic*: Emphasizes objective truth and discovery of mind-independent facts. (Also satisfies a theoretical interest in certainty.) Knowledge is "seeing" the eternal truths of a numerical world, analogous to empirical observation of nature.

Continued on next page

Table 7.2 – continued from previous page

Philosophy of Number	*Core Assumptions about Numbers*	*Primary Knowledge Interest*
Logicism (Frege, Russell)	Numbers are logical objects (e.g. classes of equinumerous sets); mathematics is an extension of logic. Seeks to derive arithmetic from self-evident logical principles. Arithmetic truths are analytic.	*Empirical-Analytic*: Aims for absolute certainty and objectivity by grounding math in logic. The interest is in secure, error-free knowledge (technical control over truth via rigorous proof). It strips away any mystical or intuitive base in favor of neutral logic. (Critical of reliance on intuition/experience.)
Formalism (Hilbert)	Numbers are symbols in a formal game; mathematics is manipulation of formulas according to rules. Consistency of the system is the main requirement. No need to interpret symbols as long as the system works.	*Empirical-Analytic*: Strong technical interest – mathematics is a tool, valued for its utility and consistency in describing the world. Emphasizes instrumental knowledge (proofs as means to derive results). Reflection is limited (the system doesn't examine itself), focusing instead on reliable rule-following.

Continued on next page

Table 7.2 – continued from previous page

Philosophy of Number	*Core Assumptions about Numbers*	*Primary Knowledge Interest*
Intuitionism / Constructivism (Brouwer)	Numbers are constructions of the human mind. Mathematics is a mental activity of constructing sequences and structures. Only mathematical objects that can be constructed explicitly are admitted. Rejects non-constructive methods (no actual infinity or excluded middle in general).	*Historical-Hermeneutic*: Prioritizes the meaning of mathematical statements as tied to our intuitive constructions. Emphasizes understanding "how we know" – knowledge is an activity, not just a result. There is also a *critical* element: it's a self-reflection on what mathematicians are actually doing in proofs (and a rejection of blindly following formal rules). Intuitionism was motivated by a desire for intellectual *integrity* and autonomy in mathematics (no reliance on non-evident principles), aligning with a critical interest in freeing math from what was seen as metaphysical obfuscation.

Continued on next page

Table 7.2 – continued from previous page

Philosophy of Number	*Core Assumptions about Numbers*	*Primary Knowledge Interest*
Structuralism (Shapiro, Resnik)	Numbers have no inherent nature except their relations in a structure. E.g. "2" is the second position in the progression of natural numbers, whichever model instantiates it. Mathematics studies structures (patterns) rather than individual objects. The structure (like the Peano structure) exists objectively (in ante rem structuralism) or at least logically.	*Empirical-Analytic* (in ante rem form): Still seeks objective truth, but about structures as a whole. Satisfies the technical interest by providing unique identification of mathematical truth up to isomorphism. Removes irrelevant ontological choices (no need to choose a particular representation), focusing on what objectively *is* the case in all equivalent systems. Also has a *hermeneutic* flavor: meaning of a number comes from context (its place in structure), highlighting interpretation within a relational system. But overall, structuralism aims at stable knowledge of abstract structures, aligning with analytic interest in invariant truths (e.g. truths that hold in any system of a given structure).

Continued on next page

Table 7.2 – continued from previous page

Philosophy of Number	*Core Assumptions about Numbers*	*Primary Knowledge Interest*
Fictionalism / Nominalism (Field, etc.)	Numbers do *not* exist; mathematical entities are useful fictions. Mathematical statements are systematically false or just useful pretenses (e.g. "3 is prime" is not literally true because there is no 3, but it's true-in-the-story of math). Mathematics is a language that simplifies reasoning about the physical world, but has no content of its own.	*Empirical-Analytic* (naturalist branch): Guided by an ontology limited to what empirical science needs. Stresses empirical content: any legitimate statement about the world can ultimately be stated without referring to abstract numbers. Also a *Critical* stance: demystifies mathematics, revealing it as a human convention or fiction (preventing reification or blind faith in "invisible" entities). By treating math as fiction, it implicitly warns against the tyranny of numbers (don't confuse the tool with reality) – a faint critical-emancipatory note.

Continued on next page

Table 7.2 – continued from previous page

Philosophy of Number	*Core Assumptions about Numbers*	*Primary Knowledge Interest*
Empiricism / Naturalism (Kant, Mill historic; modern cognitive science)	(Historic empiricism: arithmetic truths are generalizations of experience – largely discredited by Frege's critique.) Modern naturalized views: Number concepts arise from sensory experiences and neural structures. Math is reliable because it evolved to track aspects of reality (e.g. numerosities). Emphasis on how practice and experience inform mathematics.	*Empirical-Analytic*: Seeks to ground even mathematics in the same framework as empirical knowledge. Kant saw arithmetic as synthetic *a priori* (an unusual blend: requiring intuition of time to construct succession, but necessarily true given that intuition) – a mix of analytic interest (necessity) and a form of experiential element. Contemporary cognitive views bring mathematical knowledge into the realm of empirical psychology (an analytic interest in explaining reason, and a hermeneutic interest in meaning via bodily experience). If math is seen as an extension of empirical cognition, the interest is in continuity with science and human biology.

Continued on next page

Table 7.2 – continued from previous page

Philosophy of Number	*Core Assumptions about Numbers*	*Primary Knowledge Interest*
Social Constructivism (Hersh, Ernest)	Numbers are cultural-historical constructs. Mathematics is a product of human culture, maintained by social processes (proof, peer review, teaching). Its objectivity is "socially constituted" – once a number system is established, truths about numbers are objective *within* that cultural framework. But they are not eternal Platonic truths, rather intersubjective agreements.	*Historical-Hermeneutic*: Strong focus on the historical context and communal consensus in the creation of mathematical knowledge. The interest here is in understanding mathematics as part of our shared meaning-making practices. It demystifies number by showing its development (e.g. the invention of zero, negatives) in response to societal needs. Also hints at *critical* interest: if we see math as our own construction, we can adapt or change our mathematical practice deliberately (empowerment to change curricula or methods, for instance).

Continued on next page

Table 7.2 – continued from previous page

Philosophy of Number	*Core Assumptions about Numbers*	*Primary Knowledge Interest*
Critical Theory of Mathematics (e.g. critique of quantification, ethnomathematics)	Mathematics (and the concept of number) is not value-neutral; it can carry ideological power (the authority of quantification). Western formal mathematics might suppress other ways of knowing. A critical approach examines how the use of numbers can dominate (e.g. in technocracy) and seeks to humanize or democratize mathematics.	*Critical-Emancipatory*: Explicitly aims for self-reflection and liberation: to free us from the "spell" of numbers as infallible or from using numbers in oppressive ways. Encourages reflecting on the role of mathematics in maintaining systems of control (e.g. economic statistics governing policy). By recognizing number as a human tool, we can reclaim agency in how we use or interpret it. In Habermas's terms, it re-incorporates the context of meaning and human purpose into a domain often treated as purely technical.

As Table 7.2 suggests, the philosophy of number is a rich field where metaphysical, epistemological, and even ethical considerations intersect. Each philosophy provides a different answer to the question "What is a number?": a divine principle, a logical class, an unconstrained symbol, a mental construction, a structural position, a convenient fiction, a cultural artifact, etc. Each carries its own "knowledge-constitutive interest," shaping what it considers important: certainty, consistency, constructive meaning, relational invariance, ontological parsimony, human context, or self-reflection.

The knowledge-constitutive interest structure explains the underlying motivations driving these philosophical debates. They are not only about technical points; they are about what people *want* from mathematics. Do some seek absolute, context-free truths? Then Platonism or structuralism will appeal. Is there a priority on secure methods and avoiding unfounded metaphysics? Then formalism or finitism might attract. Do some value human intelligibility and meaning above all? Intuitionism or constructivism could satisfy that. Are there worries about the societal and personal implications of an overly quantified world? Then a critical perspective becomes relevant.

In practice, many mathematicians and philosophers blend elements of these views. For example, a working mathematician might be a formalist Monday through Friday (treating numbers as symbols in proofs), a Platonist at 2am when marveling at an elegant theorem ("these relations must exist independently of me!"), and a social constructivist when teaching students ("historically, negative numbers took time to be accepted, showing mathematical ideas evolve"). This eclecticism is not necessarily inconsistency; it can be seen as pragmatically adopting the stance appropriate to the context – much as Habermas's theory would suggest different interests can be pursued without denying each other.

Ultimately, the evolution of philosophies of number through immanent critique shows a progressive enrichment of understanding. By questioning "what is number?" from various angles, number emerges not as a monolithic concept but as a multi-faceted one: simultaneously a useful fiction, an abstract pattern, a cognitive tool, a social construct, and (for many) an objective reality. Each new philosophy did not simply negate the previous; it often incorporated what was valuable and corrected what was problematic. In this way, the dialogue continues. The concept of number, one of the greatest inventions of the human mind, reflects human desires for certainty, creative imagination, practical needs, and a capacity for critical self-awareness.

7.3 Conclusion

What, then, are numbers? If they are not objects in Plato's heaven, nor mere abstractions from empirical collections, nor simply truth values, then what remains? This paper has argued that numerals, and by extension numbers themselves, are fundamentally *first-person pronouns.* This is not merely a semantic shift, but a radical re-orientation of our understanding of mathematical being. To see numerals as pronouns is to see them not as names for things, but as *linguistic gestures of self-recognition*, anaphorically recollecting the enabling conditions of thought itself.

Through the lens of the *null representation* $\emptyset$, this paper has explored how numerals emerge from the dynamic interplay of quotative embedding and anaphoric recollection, rooted in the unrepresentable, yet necessarily presupposed, "I think." The "Telephone" game, with its playful distortions and iterative recollections, offered a concrete illustration of this process, revealing how even seemingly abstract mathematical concepts are deeply intertwined with the everyday dynamics of human communication and self-understanding. And by critiquing object-based ontologies and static foundationalism, this analysis begins to glimpse the "groundless ground" of the null representation – a dynamic, self-negating, and ultimately more authentic ground for mathematical being than any fixed, object-like foundation could provide.

This understanding of numerals as pronouns has significant implications, not only for the philosophy of mathematics, but also, and perhaps especially, for mathematics education. It shifts the focus from mathematics as a body of pre-given, objective truths to mathematics as a *human activity, a form of linguistic and conceptual self-constitution.* It suggests that learning mathematics is not just about mastering abstract objects and procedures, but about developing a deeper capacity for self-reflection, for recognizing the enabling conditions of our own thought, and for engaging in the dynamic, intersubjective process of mathematical meaning-making.

In a world increasingly dominated by reductive, objectifying systems that seek to "finitize the *in*finite" – to quantify and control human being itself – the idea that numerals are pronouns offers a subtle but powerful form of resistance. It serves as a reminder that at the heart of mathematics, there lies not a cold, objective realm of abstract objects, but the pulse of human subjectivity, the ungraspable "I think" that constantly transcends any fixed representation. To embrace this "groundless ground," to recognize the pronoun-like nature of numerals, is not to abandon mathematical rigor or precision. Rather, it is to ground mathematics in a more profoundly human and ultimately more meaningful foundation: the ongoing, self-recollecting, and inherently *in*finite process of human being itself.

What are numbers? Perhaps, finally, it is possible to say with a degree of provisional certainty: numbers are not *things* at all. Numbers, in their essence, express human self-understanding. They are first-person pronouns of thought, reflections of a dynamic, self-recollecting, and ultimately ungrounded, yet persistently self-affirming, human being. Recognizing this makes it possible to glimpse not only a new philosophy of mathematics, but also a more humanistic and empowering vision for mathematics education, one that truly bridges the gap between the abstract world of numbers and the lived experience of the students who seek to understand them.

References

Abbott, E. A. (2020, April). *Flatland: A Romance of Many Dimensions by Edwin A. Abbott*. Independently published.

Agamben, G. (2006, September). *Language and Death: The Place of Negativity* (K. Pinkus & M. Hardt, Trans.; First Edition). Univ Of Minnesota Press.

Aristotle. (2017). *Metaphysics: Aristotle's metaphysics in english, latin and ancient greek: Trilingual edition* (W. o. Moerbeke & W. D. Ross, Trans.; Trilingual). Hermes Language Reference.

Boote, D. N., & Beile, P. (2005). Scholars before researchers: On the centrality of the dissertation literature review in research preparation. *Educational researcher*, *34*, 3–15.

Brandom, R. B. (1994). *Making it explicit: Reasoning, representing, and discursive commitment*. Harvard university press.

Brandom, R. B. (2000). *Articulating reasons: An introduction to inferentialism*. Harvard University Press.

Brandom, R. B. (2008). *Between saying and doing: Towards an analytic pragmatism*. Oxford University Press.

Brandom, R. B. (2019). *A spirit of trust: A reading of Hegel's phenomenology*. The Belknap Press of Harvard University Press.

Bueno, O. (2020). Nominalism in the Philosophy of Mathematics. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Fall 2020). Metaphysics Research Lab, Stanford University.

Cantor, G. (1891). Ueber eine elementare frage der mannigfaltigkeitslehre. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, *1*, 75–78.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). Children's mathematics: Cognitively guided instruction - videotape logs [supplementary material]. In *Children's mathematics: Cognitively guided instruction*. Heinemann, in association with The National Council of Teachers of Mathematics, Inc.

Carspecken, P. F. (1995). *Critical ethnography in educational research: A theoretical and practical guide*. Routledge.

Carspecken, P. F. (1999). *Four scenes for posing the question of meaning and other essays in critical philosophy and critical methodology* (New edition). Peter Lang Inc., International Academic Publishers.

Carspecken, P. F. (2006). Limits of knowledge in the physical sciences. In K. Tobin & J. Kincheloe (Eds.), *Doing educational research* (pp. 405–437, Vol. 1). Sense Publishers.

Carspecken, P. F. (2009). Limits to knowledge with existential significance: An outline for the exploration of post-secular spirituality of relevance to qualitative research. In *Bridging the gap between theory and practice in educational research* (pp. 47–61). Springer.

Carspecken, P. F. (2016). Limits to knowledge and being human: What is 'critical' in critical social research and theory? In P. H. Hinchey (Ed.), *A critical action research reader* (pp. 65–86). Peter Lang Inc., International Academic Publishers.

Carspecken, P. F. (2018). The missing infinite. In R. Winkle-Wagner, J. Lee-Johnson, & T. Gaskew (Eds.), *Critical theory and qualitative data analysis in education* (p. 32). Routledge. https://doi.org/10.4324/9781315158860-2

Carspecken, P. F., & Zhang, R. (2013). Content inference fields in intersubjective space: Transpersonal position, illocution, and logic in the analysis of human interactions. *Qualitative research: A reader in philosophy, core concepts, and practice*, 201–242.

Carspecken, R. (2024, November 3). Untitled.

Caygill, H. (2004). *A kant dictionary*. Wiley-Blackwell.

Derrida, J. (1973). *Speech and phenomena, and other essays on husserl's theory of signs*. Northwestern University Press.

Derrida, J. (1982). *Marges de la philosophie* [Margins of philosophy] (A. Bass, Trans.). The Harvester Press. (Original work published 1972)

Deutsch, H., Marshall, O., & Irvine, A. D. (2025). Russell's Paradox. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford encyclopedia of philosophy* (Spring 2025). Metaphysics Research Lab, Stanford University.

Ernest, P. (2024). The mathematical and philosophical significance of zero: Nought matters [Publisher: Paul Ernest, Emeritus Professor at University of Exeter, School of Education & Lifelong Learning]. *Philosophy of Mathematics Education Journal*, 1–35. https://research.ebsco.com/linkprocessor/plink?id=b685ecb9-d2fc-3e03-889d-a06fb1751c1b

Euclid. (2007, December). *Euclid's Elements of Geometry* (J. L. Heiberg, Ed.; R. Fitzpatrick, Trans.; Revised and corrected edition). Richard Fitzpatrick.

Franzén, T. (2004, September). *Inexhaustibility: A Non-Exhaustive Treatment: Lecture Notes in Logic 16* (1st edition). A K Peters/CRC Press.

Franzén, T. (2005). *Gödel's theorem: An incomplete guide to its use and abuse*. AK Peters/CRC Press.

Frege, G. (1997, July). *The Frege Reader* (M. Beaney, Ed.; 1st edition). Blackwell Publishing.

Gaifman, H. (2005). *The easy way to Gödel's proof and related matters* [Unpublished manuscript. Available online].

Geisel, T. (1961). *The Sneetches and Other Stories* (First edition.). Random House Books for Young Readers.

Gelder, T. V. (1995). What might cognition be, if not computation? *The Journal of Philosophy*, *92*, 345–381. Retrieved April 13, 2025, from http://www.jstor.org/stable/2941061

Gray, L., & Garzon, M. (2005). On a hermite boundary integral approximation. *Computers & structures*, *83*(10-11), 889–894.

Greenblatt, R., Denison, C., Wright, B., Roger, F., MacDiarmid, M., Marks, S., Treutlein, J., Belonax, T., Chen, J., Duvenaud, D., Khan, A., Michael, J., Mindermann, S., Perez, E., Petrini, L., Uesato, J., Kaplan, J., Shlegeris, B., Bowman, S. R., & Hubinger, E. (2024, December 23). Alignment faking in large language models. https://doi.org/10.48550/arXiv.2412.14093

Habermas, J. (1985). *The theory of communicative action: Lifeworld and system* (Vol. 2). Beacon Press.

Habermas, J. (2004). *The philosophical discourse of modernity: Twelve lectures* (F. G. Lawrence, Trans.). MIT Press.

Habermas, J. (1971). *Knowledge and human interests*. Beacon Press.

Habermas, J. (1984). *The theory of communicative action, volume 1: Reason and the rationalization of society* (T. McCarthy, Trans.; Reprint edition). Beacon Press.

Habermas, J. (1995). *Postmetaphysical thinking: Between metaphysics and the critique of reason* (1st edition). Polity.

Hackenberg, A. (2025). *Course notes* [Unpublished course notes].

Hegel, G. W. F. (1977). *Phenomenology of spirit* (A. V. Miller, Trans.; Vol. 632). Oxford University Press Oxford.

Hegel, G. W. F. (2010). *The science of logic* (G. di Giovanni, Ed. & Trans.). Cambridge University Press.

Houlgate, S. (2006). *The opening of hegel's logic: From being to infinity*. Purdue University Press.

Hyppolite, J. (1974). *Genesis and structure of hegel's phenomenology of spirit*. Northwestern University Press.

Kant, I. (1998). *Critique of pure reason* [Original work published 1781/1787, Guyer and Wood, Trans.]. Cambridge University Press.

Lakoff, G., & Johnson, M. (1980). *Metaphors we live by*. University of Chicago Press.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books.

Linnebo, Ø. (2024). Platonism in the Philosophy of Mathematics. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford encyclopedia of philosophy* (Summer 2024). Metaphysics Research Lab, Stanford University.

Longuenesse, B. (2017). *I, me, mine: Back to kant and fichte on the 'i'*. Oxford University Press.

Mead, G. H. (1934). *The social psychology of george herbert mead* (A. L. Strauss, Ed.). University of Chicago Press.

Mead, G. H. (2015). *Mind, self, and society: The definitive edition* (H. Joas, D. R. Huebner, & C. W. Morris, Eds.; Enlarged edition). University of Chicago Press.

Meyer, J. (title). *Cantors 1891 diagonal proof* [Cantor's original 1891 diagonal proof]. Retrieved July 6, 2025, from https://jamesrmeyer.com/infinite/cantors-original-1891-proof

Nagel, E., & Newman, J. R. (2012). *Godel's proof*. Routledge.

Negarestani, R. (2018). *Intelligence and Spirit*. Urbanomic/Sequence Press.

Poston, T., & Stewart, I. (2012, February 29). *Catastrophe theory and its applications* (First Edition). Dover Publications.

Priest, G. (2002). *Beyond the limits of thought*. Oxford University Press.

Priest, G. (2014). *One: Being an investigation into the unity of reality and of its parts, including the singular object which is nothingness*. Oxford University Press.

Quine, W. V. O. (1948). On what there is. In *Thinking about logic* (pp. 189–209). Catholic University of America. https://sites.pitt.edu/~rbrandom/Courses/2023%20Sellars/Sellars%20texts/Quine%20On_What_There_Is.pdf

Quine, W. V. O. (1963). Two dogmas of empiricism [First Harper Torchbook edition, a reprint of the second, revised edition (1961). The book was first published by Harvard University Press in 1953.]. In *From a logical point of view* (Second, revised, pp. 20–46). Harper & Row.

Rödl, S. (2014). Intentional transaction [Accessed: 2025-04-13]. *Philosophical Explorations: An International Journal for the Philosophy of Mind and Action*, *17*, 304–316. https://doi.org/10.1080/13869795.2014.941909

Rose, G. (2009). *Hegel contra sociology*. Verso.

Rowlands, S. (2005). The contradictions in the constructivist discourse [Citing von Glasersfeld]. *Philosophy of Mathematics Education Journal*, *14*.

Savich, T. M. (2020). Towards a critical mathematics [Accessed via ERIC]. *Mathematics Teaching–Research Journal Online*, *12*, 245–291. https://files.eric.ed.gov/fulltext/EJ1384454.pdf

Savich, T. M. (2022). *Towards a critical mathematics*. Indiana University.

Savich, T. M. (2024). Built to break: An introduction to theoretical autoethnography [Invited talk presented at Solent University, organized by Dr. Brian McDonough]. https://iu.mediaspace.kaltura.com/media/t/1_v1qmm3g9

Savich, T. M. (title). *The hermeneutic calculator* [The hermeneutic calculator]. Retrieved July 23, 2025, from https://tiosavich.github.io/UMEDCTA/Calculator/index.html

Sellars, W. (1953). Inference and meaning. *Mind*, *62*, 313–338.

Sellars, W. (2007). *In the space of reasons: Selected essays of wilfrid sellars*. Harvard University Press.

Steffe, L. P. (2002). A new hypothesis concerning children's fractional knowledge [Accessed through CEPA.INFO]. *Journal of Mathematical Behavior*, *20*(3), 267–307.

Steffe, L. P. (2003). On the construction of learning trajectories of children: The case of commensurate fractions [Accessed via ResearchGate/ERIC]. *Monographs of the Journal for Research in Mathematics Education*, 1–26.

Steffe, L. P. (2004). Fractional commensurate, composition, and adding schemes: Learning trajectories of jason and laura: Grade 5 [Accessed via ResearchGate]. In J. Olive & R. Tzur (Eds.), *Children's fractional knowledge* (pp. 1–60). American Educational Research Association.

Steffe, L. P. (2014). *Children's number sequences: An explanation of steffe's constructs and an extrapolation to rational numbers of arithmetic*

[Originally from 1994]. Unpublished manuscript, available via CiteSeerX; ResearchGate.

Steffe, L. P., & Thompson, P. W. (2014). Teaching experiment methodology: Underlying principles and essential elements [Accessed via ResearchGate, originally published 2000]. *Proceedings of the 24th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.*

Stone, J. (2003). *The monster at the end of this book* (2nd ed. edition). Golden Books.

Taylor, C. (1994). *Multiculturalism: Examining the Politics of Recognition* (A. Gutmann, Ed.) [Essays by Charles Taylor, K. Anthony Appiah, Jürgen Habermas, Steven C. Rockefeller, Michael Walzer, and Susan Wolf]. Princeton University Press.

van Hiele-Geldof, D., & van Hiele, P. M. (1984). *English translation of selected writings of dina van hiele-geldof and pierre m. van hiele* (D. Fuys, D. Geddes, & D. Tischler, Eds.) [ERIC Number: ED287697]. Retrieved June 3, 2025, from https://eric.ed.gov/?id=ED287697

Wittgenstein, L. (1956). *Remarks on the foundations of mathematics.*

Wittgenstein, L. (2010). *Philosophical investigations* (P. M. S. Hacker & J. Schulte, Eds.; G. Anscombe, Trans.). Wiley-Blackwell.

Chapter 8

Algorithmic Elaboration and History

Abstract

This chapter explores the development of mathematical concepts through algorithmic elaboration, arguing that mathematical history reflects a dialectical pattern where apparent completion generates new possibilities. The analysis connects the reconceptualization of numerals as pronouns with critical arithmetic development, demonstrating how Robert Brandom's concept of algorithmic elaboration models this evolution. Two case studies anchor the analysis: Euclid's proof of the infinity of primes reconstructed through incompatibility semantics, and the emergence of arithmetic operations from embodied metaphors. The investigation reveals the limitations of simple algorithmic elaboration, necessitating "pragmatic expressive bootstrapping" where conceptual systems explicate their implicit normative structures. Rather than treating intellectual history as a static literature to be surveyed, I advocate understanding it as dynamic conceptual development mirroring the algorithmic elaboration within mathematical practice itself.

8.1 The Problem of Cataloguing Sand

Algorithms are traditionally conceived as step-by-step procedures for solving problems. An algorithm takes input, processes it through defined steps, and produces output. What this definition misses is the underlying inferential structure: algorithms are chains of inferences, as discussed in section 2.1. These inferential chains can become smooth through practice, like a yogi flowing between poses. Of particular interest are chains of material inference (recall 2.3) involving substitution.

When I proposed this book, I intended to explore literature reviews as a concept. The perceived lack of importance of the dissertation literature review is evident in the paucity of research devoted to understanding it (Boote & Beile, 2005, p. 4). I thought articulating the process would be a gift to future scholars. Yet as I wrote, the earth shifted beneath my feet. The popularization of Large Language Models transformed summarizing and synthesizing into seemingly passive activities.

More fundamentally, I encountered a paradox. Each attempt to gain purchase on "The Literature" as a totality revealed new gaps. One article would fill a gap, revealing another, then another. I had not yet synthesized the groundless ground; I was trying to climb onto the shoulders of giants made of sand. The Bridge chapter articulates this process: reifying a totality in a name inevitably leads to expressions that cannot have been in the reified totality but must have been in the original totality. This process haunts any attempt to conceptualize "The Literature."

8.2 Algorithmic Elaboration as Living History

Instead of treating literature review as cataloguing static objects, I propose understanding intellectual history through algorithmic elaboration. Brandom introduces this concept through long division (Brandom, 2008, p. 37), showing how the division algorithm elaborates from multiplication and subtraction practices. Figure 8.1 reproduces his automaton-implemented elaboration.

My initial skepticism about this example was intense. The rich semantic structure of division seems hardly captured by the long-division algorithm. The claim felt incomplete without tracking how counting elaborates into ad-

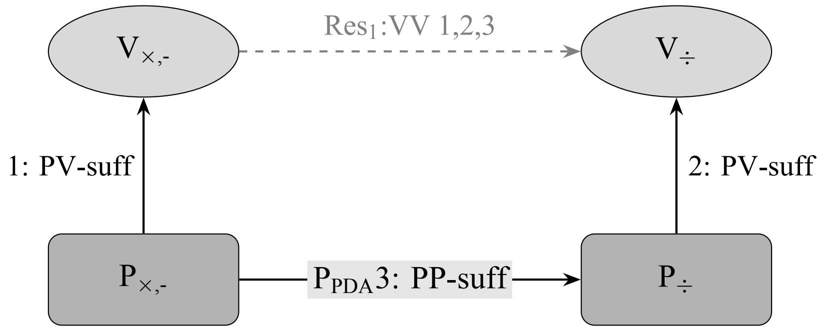


Figure 8.1: *Note.* Automaton-implemented, algorithmically elaborated, pragmatically mediated semantic relation reproduced from (Brandom, 2008, p. 37).

dition, addition into multiplication and subtraction. Consider the algebraic structures of basic arithmetic operations shown in Table 8.1.

Table 8.1: ***Algebraic Structure of Basic Arithmetic Operations***

Operation	**Algebraic Structure**
Addition	Known Part + Known Part = Unknown Whole
Subtraction	Unknown Part + Known Part = Known Whole
Equal-Groups Multiplication	Known Items per Group × Known Number of Groups = Unknown T
Sharing Division	Unknown Items per Group × Known Number of Groups = Known T
Measurement Division	Known Items per Group × Unknown Number of Groups = Known T

These structures suggest deeper dialectical shifts at work. Is subtraction simply elaborated from addition by "reversing the wiring," or does conceptual understanding require dialectical thinking? The mechanical attributes of automata modeling arithmetic vary significantly. Kindergarten emphasizes three-step procedures as prerequisites. Matching nested parentheses for von Neumann ordinals requires push-down stack automata with primitive memory, not mere finite-state machines.

8.3 Euclid's Proof: Incompleteness as Incoherence

To demonstrate algorithmic elaboration concretely, I reconstruct Euclid's proof using Brandom's incompatibility semantics (Brandom, 2008). First, key concepts:

An *incoherence frame* is a pair $\langle L, Inc \rangle$ where Inc specifies which sentences in language L are incoherent. Think of this as sentences a qualitative researcher seeks to understand, or troubling interactions people reflect upon to make sense of them. Incoherence can be existentially terrifying. When recognized as believing both x and $\neg x$, I might be understood as becoming, or as duplicitous. Context matters profoundly.

An *incompatibility relation* I defines two sets X and Y as incompatible if their union is incoherent: $A \in I(B) \iff A \cup B \in Inc.$

Euclid's proof (Book 9, Proposition 20 of *The Elements* Euclid, 2007) claims any finite list of primes is incomplete. The common misrepresentation suggests Euclid provides an algorithm for finding primes. This is false. He only claims that multiplying all known primes and adding one yields a number not divisible by any known prime. The number $2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30031 = 59 \times 509$ is composite, not prime.

Euclid's actual argument: Take primes A, B, C. Form DE as their product. Add one unit to get $EF = DE + 1$. Either EF is prime (thus a new prime beyond the list) or composite with prime factor G that cannot equal A, B, or C (since if G divided both DE and EF, it would divide their difference, which is 1).

Through incompatibility semantics, the key insight becomes: {The list of primes $\{A, B, C\}$ $Inc.$

This statement belongs to the incoherence frame because it generates internal contradiction. The claim to completeness defeats itself through the construction $N = A \times B \times C + 1$.

8.4 The Reflective Turn: Admitting Incoherence

Here the chapter pivots. The reconstruction above reveals something profound: the notion of an incoherence frame allows errors, misrecognitions, mistakes, and misconceptions to be **admitted** into the system without damaging its coherence (label: **??**). When an algorithm produces a paradoxical result, that paradox becomes an element of an incoherence frame rather than a problem to solve.

This transforms our understanding. Mathematical learning might proceed not through avoiding error but through sophisticated recognition and integration of contradictory commitments. Barriers become resources for development. In formal mathematics, contradictory commitments are dropped as falsities. In mathematics education, tensions of misrecognition are sublated into new understanding; they do not simply disappear.

I formalized this insight in Prolog, creating `more_machine_learner.pl`. The engine does not take axioms and prove theorems. Instead, it takes premises and proves incoherence. This subtle difference captures Euclid's proof structure: demonstrating that assuming completeness leads to incoherence. The formalism helps ensure the system I outline is itself coherent, though readers should take it with appropriate skepticism. I had extensive AI assistance cobbling together code to "prove" theorems. Yet from the Hermeneutic Calculator to the Prolog supplements, readers can see claims in action.

My ambition is not a finished system but a starting point for others to build upon, as I have built on others' work.

8.5 From Embodied Metaphors to Arithmetic

The dialectical insight about admitting incoherence prepares us to understand how arithmetic emerges from embodied experience. Following Lakoff and Núñez (Lakoff & Núñez, 2000), I synthesize embodied metaphors with incompatibility semantics.

Bounded Regions and Categories

Proprioceptive feelings define boundaries: an interior (inside the body) and exterior (outside). The vocabulary transitions algorithmically:

1. If you are in a bounded region, you are not out of that bounded region.

2. If you are out of a bounded region, you are not in that bounded region.

3. If you are deep in a bounded region, you are far from being out.

4. If you are on the edge of a bounded region, you are close to being in.

Through substitution:

- “Categories” for “Bounded regions in space”
- “Category members” for “Objects inside bounded regions”
- “Subcategory of larger category” for “One bounded region inside another”

Material Inferences for Object Collections

To establish arithmetic foundations, I explicate material inferences governing object collections, enabling vocabulary $V_{\text{object collection}}$:

$$V_{\text{object collection}} = \{\text{bigger}, \text{smaller}, =, \text{added to}, \text{results in}, \text{adding}, \text{subtract}, \text{zero}, \text{one}\}$$

Key inferences include:
Linearity: $\{A$ and B are object collections$\}$† (Highlander algorithm)

- $\{A$ bigger than $B\} \vDash_I \{B$ smaller than $A\} \rightarrow \neg\{A$ smaller than $B\}$
- $\{B$ bigger than $A\} \vDash_I \{A$ smaller than $B\} \rightarrow \neg\{B$ smaller than $A\}$
- $\{A = B\}$

Limited Iteration of Subtraction:

- $\{B$ smaller than $A\} \rightarrow \{B$ subtracted from A is an object collection$\}$
- $\{B$ symmetrically intersubstitutable with $A\} \vDash_I \{$No objects left when A subtracted fro

These establish operations on object collections underpinning arithmetic, grounding them in deontic-normative structures rather than alethic assertions.

8.6 Pragmatic Expressive Bootstrapping

Simple algorithmic elaboration has limits. Dialectical moments require what Brandom calls *pragmatic expressive bootstrapping* (Brandom, 2008): new, more expressively powerful vocabularies emerge from implicit practices already mastered. The system pulls itself up by its bootstraps, making explicit what was latent in its doing.

This mirrors diagonalization from the Bridge chapter. A system constructs new elements demonstrably part of its potential but outside initial enumeration. Similarly, existing practices contain implicit resources to make explicit vocabularies transcending the language describing those practices.

Brandom formalizes this through Meaning-Use Analysis. Bootstrapping occurs when vocabulary V' is VP-sufficient to specify practices P that are PV-sufficient to deploy more powerful vocabulary V. The "bootstrapping" happens when specifying vocabulary (V') is weaker than deployed vocabulary (V).

I implemented this computationally in `more_machine_learner.pl`. The system begins with norms for arithmetic over natural numbers where subtraction is limited: $3 - 5$ is nonsensical. When presented with observation `minus(3, 5, -2)`, it enters normative critique:

```
Observation is INCOHERENT with current norms. Entering critique phase...
Identified Incompatibility: incompatibility(limited_subtraction, domain(n))
Proposed Normative Shift: Change domain to z
Bootstrapping complete. Observation is now coherent.
```

The system uses failure of its vocabulary (arithmetic over $\mathbb{N}$) to deploy new vocabulary (arithmetic over $\mathbb{Z}$). The paradox resolves not through dismissal but transformation.

8.7 History as Self-Elaborating Process

This chapter demonstrates that mathematical history, and conceptual history broadly, unfolds as rational algorithmic elaboration. From implicit practices of counting, measuring, and inferring, explicit formal systems emerge. Euclid's proof reconstructed through incompatibility semantics shows ancient insights as formal articulation of inferential commitments. Claims to completeness are self-defeating, opening doors to the infinite.

Yet knowledge's journey is not linear unfolding. Algorithmic elaboration's limitations necessitate dialectical ruptures requiring profound conceptual change. Pragmatic expressive bootstrapping provides the model: systems of practices generate new vocabularies making their implicit normative structures explicit. This is not magic but language unfolding itself.

Mathematical knowledge becomes not static truth collections but dynamic, self-correcting, emancipatory process: constant becoming. By understanding rules implicitly followed, it becomes possible to make them explicit, critique them, transcend them, or identify with them.

The shift from static literature review to dynamic conceptual development reveals how mathematical understanding operates. Rather than cataloguing sand, I trace how concepts elaborate themselves through material inference chains, admitting incoherence as resource rather than obstacle. This prepares readers for critical arithmetic in subsequent chapters, where operations emerge from embodied practices through normatively regulated elaboration.

References

Abbott, E. A. (2020, April). *Flatland: A Romance of Many Dimensions by Edwin A. Abbott*. Independently published.

Agamben, G. (2006, September). *Language and Death: The Place of Negativity* (K. Pinkus & M. Hardt, Trans.; First Edition). Univ Of Minnesota Press.

Aristotle. (2017). *Metaphysics: Aristotle's metaphysics in english, latin and ancient greek: Trilingual edition* (W. o. Moerbeke & W. D. Ross, Trans.; Trilingual). Hermes Language Reference.

Boote, D. N., & Beile, P. (2005). Scholars before researchers: On the centrality of the dissertation literature review in research preparation. *Educational researcher*, *34*, 3–15.

Brandom, R. B. (1994). *Making it explicit: Reasoning, representing, and discursive commitment*. Harvard university press.

Brandom, R. B. (2000). *Articulating reasons: An introduction to inferentialism*. Harvard University Press.

Brandom, R. B. (2008). *Between saying and doing: Towards an analytic pragmatism*. Oxford University Press.

Brandom, R. B. (2019). *A spirit of trust: A reading of Hegel's phenomenology*. The Belknap Press of Harvard University Press.

Bueno, O. (2020). Nominalism in the Philosophy of Mathematics. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Fall 2020). Metaphysics Research Lab, Stanford University.

Cantor, G. (1891). Ueber eine elementare frage der mannigfaltigkeitslehre. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, *1*, 75–78.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). Children's mathematics: Cognitively guided instruction - videotape logs [supplementary material]. In *Children's mathematics: Cognitively guided instruction*. Heinemann, in association with The National Council of Teachers of Mathematics, Inc.

Carspecken, P. F. (1995). *Critical ethnography in educational research: A theoretical and practical guide*. Routledge.

Carspecken, P. F. (1999). *Four scenes for posing the question of meaning and other essays in critical philosophy and critical methodology* (New edition). Peter Lang Inc., International Academic Publishers.

Carspecken, P. F. (2006). Limits of knowledge in the physical sciences. In K. Tobin & J. Kincheloe (Eds.), *Doing educational research* (pp. 405–437, Vol. 1). Sense Publishers.

Carspecken, P. F. (2009). Limits to knowledge with existential significance: An outline for the exploration of post-secular spirituality of relevance to qualitative research. In *Bridging the gap between theory and practice in educational research* (pp. 47–61). Springer.

Carspecken, P. F. (2016). Limits to knowledge and being human: What is 'critical' in critical social research and theory? In P. H. Hinchey (Ed.), *A critical action research reader* (pp. 65–86). Peter Lang Inc., International Academic Publishers.

Carspecken, P. F. (2018). The missing infinite. In R. Winkle-Wagner, J. Lee-Johnson, & T. Gaskew (Eds.), *Critical theory and qualitative data analysis in education* (p. 32). Routledge. https://doi.org/10.4324/9781315158860-2

Carspecken, P. F., & Zhang, R. (2013). Content inference fields in intersubjective space: Transpersonal position, illocution, and logic in the analysis of human interactions. *Qualitative research: A reader in philosophy, core concepts, and practice*, 201–242.

Carspecken, R. (2024, November 3). Untitled.

Caygill, H. (2004). *A kant dictionary*. Wiley-Blackwell.

Derrida, J. (1973). *Speech and phenomena, and other essays on husserl's theory of signs*. Northwestern University Press.

Derrida, J. (1982). *Marges de la philosophie* [Margins of philosophy] (A. Bass, Trans.). The Harvester Press. (Original work published 1972)

Deutsch, H., Marshall, O., & Irvine, A. D. (2025). Russell's Paradox. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford encyclopedia of philosophy* (Spring 2025). Metaphysics Research Lab, Stanford University.

Ernest, P. (2024). The mathematical and philosophical significance of zero: Nought matters [Publisher: Paul Ernest, Emeritus Professor at University of Exeter, School of Education & Lifelong Learning]. *Philosophy of Mathematics Education Journal*, 1–35. https://research.ebsco.com/linkprocessor/plink?id=b685ecb9-d2fc-3e03-889d-a06fb1751c1b

Euclid. (2007, December). *Euclid's Elements of Geometry* (J. L. Heiberg, Ed.; R. Fitzpatrick, Trans.; Revised and corrected edition). Richard Fitzpatrick.

Franzén, T. (2004, September). *Inexhaustibility: A Non-Exhaustive Treatment: Lecture Notes in Logic 16* (1st edition). A K Peters/CRC Press.

Franzén, T. (2005). *Gödel's theorem: An incomplete guide to its use and abuse*. AK Peters/CRC Press.

Frege, G. (1997, July). *The Frege Reader* (M. Beaney, Ed.; 1st edition). Blackwell Publishing.

Gaifman, H. (2005). *The easy way to Gödel's proof and related matters* [Unpublished manuscript. Available online].

Geisel, T. (1961). *The Sneetches and Other Stories* (First edition.). Random House Books for Young Readers.

Gelder, T. V. (1995). What might cognition be, if not computation? *The Journal of Philosophy*, *92*, 345–381. Retrieved April 13, 2025, from http://www.jstor.org/stable/2941061

Gray, L., & Garzon, M. (2005). On a hermite boundary integral approximation. *Computers & structures*, *83*(10-11), 889–894.

Greenblatt, R., Denison, C., Wright, B., Roger, F., MacDiarmid, M., Marks, S., Treutlein, J., Belonax, T., Chen, J., Duvenaud, D., Khan, A., Michael, J., Mindermann, S., Perez, E., Petrini, L., Uesato, J., Kaplan, J., Shlegeris, B., Bowman, S. R., & Hubinger, E. (2024, December 23). Alignment faking in large language models. https://doi.org/10.48550/arXiv.2412.14093

Habermas, J. (1985). *The theory of communicative action: Lifeworld and system* (Vol. 2). Beacon Press.

Habermas, J. (2004). *The philosophical discourse of modernity: Twelve lectures* (F. G. Lawrence, Trans.). MIT Press.

Habermas, J. (1971). *Knowledge and human interests*. Beacon Press.

Habermas, J. (1984). *The theory of communicative action, volume 1: Reason and the rationalization of society* (T. McCarthy, Trans.; Reprint edition). Beacon Press.

Habermas, J. (1995). *Postmetaphysical thinking: Between metaphysics and the critique of reason* (1st edition). Polity.

Hackenberg, A. (2025). *Course notes* [Unpublished course notes].

Hegel, G. W. F. (1977). *Phenomenology of spirit* (A. V. Miller, Trans.; Vol. 632). Oxford University Press Oxford.

Hegel, G. W. F. (2010). *The science of logic* (G. di Giovanni, Ed. & Trans.). Cambridge University Press.

Houlgate, S. (2006). *The opening of hegel's logic: From being to infinity*. Purdue University Press.

Hyppolite, J. (1974). *Genesis and structure of hegel's phenomenology of spirit*. Northwestern University Press.

Kant, I. (1998). *Critique of pure reason* [Original work published 1781/1787, Guyer and Wood, Trans.]. Cambridge University Press.

Lakoff, G., & Johnson, M. (1980). *Metaphors we live by*. University of Chicago Press.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books.

Linnebo, Ø. (2024). Platonism in the Philosophy of Mathematics. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford encyclopedia of philosophy* (Summer 2024). Metaphysics Research Lab, Stanford University.

Longuenesse, B. (2017). *I, me, mine: Back to kant and fichte on the 'i'*. Oxford University Press.

Mead, G. H. (1934). *The social psychology of george herbert mead* (A. L. Strauss, Ed.). University of Chicago Press.

Mead, G. H. (2015). *Mind, self, and society: The definitive edition* (H. Joas, D. R. Huebner, & C. W. Morris, Eds.; Enlarged edition). University of Chicago Press.

Meyer, J. (title). *Cantors 1891 diagonal proof* [Cantor's original 1891 diagonal proof]. Retrieved July 6, 2025, from https://jamesrmeyer.com/infinite/cantors-original-1891-proof

Nagel, E., & Newman, J. R. (2012). *Godel's proof*. Routledge.

Negarestani, R. (2018). *Intelligence and Spirit*. Urbanomic/Sequence Press.

Poston, T., & Stewart, I. (2012, February 29). *Catastrophe theory and its applications* (First Edition). Dover Publications.

Priest, G. (2002). *Beyond the limits of thought*. Oxford University Press.

Priest, G. (2014). *One: Being an investigation into the unity of reality and of its parts, including the singular object which is nothingness*. Oxford University Press.

Quine, W. V. O. (1948). On what there is. In *Thinking about logic* (pp. 189–209). Catholic University of America. https://sites.pitt.edu/~rbrandom/Courses/2023%20Sellars/Sellars%20texts/Quine%20On_What_There_Is.pdf

Quine, W. V. O. (1963). Two dogmas of empiricism [First Harper Torchbook edition, a reprint of the second, revised edition (1961). The book was first published by Harvard University Press in 1953.]. In *From a logical point of view* (Second, revised, pp. 20–46). Harper & Row.

Rödl, S. (2014). Intentional transaction [Accessed: 2025-04-13]. *Philosophical Explorations: An International Journal for the Philosophy of Mind and Action*, *17*, 304–316. https://doi.org/10.1080/13869795.2014.941909

Rose, G. (2009). *Hegel contra sociology*. Verso.

Rowlands, S. (2005). The contradictions in the constructivist discourse [Citing von Glasersfeld]. *Philosophy of Mathematics Education Journal*, *14*.

Savich, T. M. (2020). Towards a critical mathematics [Accessed via ERIC]. *Mathematics Teaching–Research Journal Online*, *12*, 245–291. https://files.eric.ed.gov/fulltext/EJ1384454.pdf

Savich, T. M. (2022). *Towards a critical mathematics*. Indiana University.

Savich, T. M. (2024). Built to break: An introduction to theoretical autoethnography [Invited talk presented at Solent University, organized by Dr. Brian McDonough]. https://iu.mediaspace.kaltura.com/media/t/1_v1qmm3g9

Savich, T. M. (title). *The hermeneutic calculator* [The hermeneutic calculator]. Retrieved July 23, 2025, from https://tiosavich.github.io/UMEDCTA/Calculator/index.html

Sellars, W. (1953). Inference and meaning. *Mind*, *62*, 313–338.

Sellars, W. (2007). *In the space of reasons: Selected essays of wilfrid sellars*. Harvard University Press.

Steffe, L. P. (2002). A new hypothesis concerning children's fractional knowledge [Accessed through CEPA.INFO]. *Journal of Mathematical Behavior*, *20*(3), 267–307.

Steffe, L. P. (2003). On the construction of learning trajectories of children: The case of commensurate fractions [Accessed via Research-

Gate/ERIC]. *Monographs of the Journal for Research in Mathematics Education*, 1–26.

Steffe, L. P. (2004). Fractional commensurate, composition, and adding schemes: Learning trajectories of jason and laura: Grade 5 [Accessed via ResearchGate]. In J. Olive & R. Tzur (Eds.), *Children's fractional knowledge* (pp. 1–60). American Educational Research Association.

Steffe, L. P. (2014). *Children's number sequences: An explanation of steffe's constructs and an extrapolation to rational numbers of arithmetic* [Originally from 1994]. Unpublished manuscript, available via CiteSeerX; ResearchGate.

Steffe, L. P., & Thompson, P. W. (2014). Teaching experiment methodology: Underlying principles and essential elements [Accessed via ResearchGate, originally published 2000]. *Proceedings of the 24th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.*

Stone, J. (2003). *The monster at the end of this book* (2nd ed. edition). Golden Books.

Taylor, C. (1994). *Multiculturalism: Examining the Politics of Recognition* (A. Gutmann, Ed.) [Essays by Charles Taylor, K. Anthony Appiah, Jürgen Habermas, Steven C. Rockefeller, Michael Walzer, and Susan Wolf]. Princeton University Press.

van Hiele-Geldof, D., & van Hiele, P. M. (1984). *English translation of selected writings of dina van hiele-geldof and pierre m. van hiele* (D. Fuys, D. Geddes, & D. Tischler, Eds.) [ERIC Number: ED287697]. Retrieved June 3, 2025, from https://eric.ed.gov/?id=ED287697

Wittgenstein, L. (1956). *Remarks on the foundations of mathematics.*

Wittgenstein, L. (2010). *Philosophical investigations* (P. M. S. Hacker & J. Schulte, Eds.; G. Anscombe, Trans.). Wiley-Blackwell.

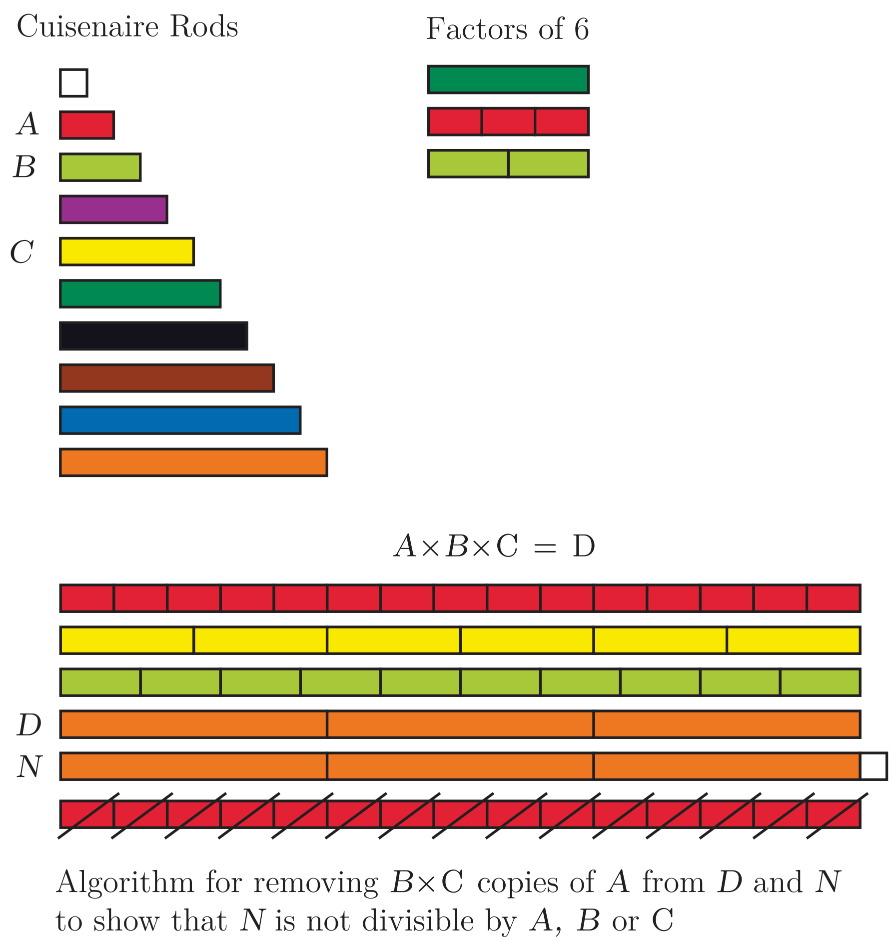


Figure 8.2: *Note.* Cuisenaire rods interpreted as numerals 1-10, building composite 6 from primes (2×3), and algorithm for Euclid's proof.

Chapter 9

10: Operation

Abstract

This chapter explores the nature of mathematical operations, arguing that they are not merely formal manipulations of symbols but rather expressions of embodied, normatively regulated practices. Building on the previous chapters' reconstruction of numerals as pronouns and the history of mathematical development through algorithmic elaboration, the analysis develops a framework for understanding what the author calls "critical arithmetic"—a pre-formal system that captures the dynamic, error-inclusive nature of everyday arithmetic. Through the lens of multiplication, the chapter demonstrates how operations emerge from the interplay of three dimensions: embodied practices, inferential rules, and formal structures. The discussion begins with the embodied basis of arithmetic, using the example of a strategy called C2C (Coordinating Two Counts by Ones) documented in Cognitively Guided Instruction research. The chapter then articulates how subjective experiences are transformed into shared, rule-governed practices through a normative framework based on Robert Brandom's incompatibility semantics and the embodied cognition research of George Lakoff and Rafael Núñez. Finally, the analysis demonstrates how these practices give rise to stable, objective procedures that can be modeled as formal automata. Throughout, the chapter emphasizes that mathematical operations are grounded in human activity and social normativity rather than existing as timeless, abstract truths.

9.1 The Misrecognition of Operation

The preceding chapters reconstructed the nature of number itself, beginning with a student's profound question asked in a moment of grief and frustration: "Mr. Savich, what even is two?" That moment led away from the conventional view of numbers as abstract, pre-existing objects and toward an account of them as first-person recollective pronouns. Numerals, symbolized by the null representation $\emptyset$ (Chapter 8), emerge from the dynamic interplay of linguistic self-reference, anaphorically recollecting the enabling conditions of thought—the transcendental "I think".

Having explored what numbers *are*—reflections of dynamic, self-recollecting human being—this chapter now turns to the equally fundamental question of what it means to *do* things with them. Conventional accounts treat mathematical operations as formal manipulations of abstract symbols, governed by axioms that are either arbitrary conventions or self-evident truths. In this view, addition, subtraction, multiplication, and division are processes applied to numbers, with rules to be memorized and followed.

I propose a different understanding, continuous with the emancipatory project of this book: mathematical operations are not arbitrary conventions but are themselves expressions of embodied, normatively regulated practices. The "laws" governing the use of numbers are best understood not as transcendent truths, but as algorithms or inference chains—normatively regulated doings that emerge from embodied engagement with the world and with others. This misrecognition—treating operations as mechanical procedures divorced from human meaning-making—parallels the earlier misrecognition of numbers as abstract objects divorced from the thinking subject who deploys them.

This chapter articulates inference rules for what I call "critical arithmetic"—an open-ended, pre-formal system that captures the kind of arithmetic used in everyday life—errors and all. Drawing on the empirical foundation provided by Cognitively Guided Instruction (CGI) research (Carpenter et al., 1999), which established that children possess rich, informal mathematical knowledge long before formal instruction, the investigation reveals how mathematical operations unfold in lived experience.

9.2 The First Determinate Negation: Critical Arithmetic and the Triad

To make this misrecognition determinate, I introduce the concept of *critical arithmetic*. This framework ~~negates~~ the conventional view by denying that operations exist as pre-given formal procedures. Instead, critical arithmetic reveals operations as practices that emerge through the triadic structure of validity developed throughout the inquiry.

The investigation into student-invented strategies is deeply indebted to the paradigm shift initiated by Cognitively Guided Instruction (CGI). Emerging in the 1980s from research by Carpenter, Fennema, and colleagues, CGI established that children possess rich, informal mathematical knowledge long before formal instruction (Carpenter et al., 1999). This research provided the empirical foundation for the constructivist turn in mathematics education, asserting that students actively construct understanding by solving problems in meaningful ways, rather than passively receiving algorithms.

However, while constructivism and CGI have demonstrated significant positive effects on students' conceptual understanding, it is burdened by privileging subjective-validity claims over normative and objective validity claims. I want to use the brilliant results that emerged from the careful methods of CGI and constructivism, but reframe some of that work in a way that better integrates the triadic framework of validity developed in this book.

The primary 'revision' is that strategies are not precisely 'invented.' They emerge as intersubjectively constituted practices within a social context. As dyadic communicative *actions* – a speaker stakes their identity to their expressions under the assumption that someone *might* be able to understand them – they are structured by action-impeti, self-monitoring feedback loops, and existential needs for recognition. They are like orbits in intersubjective space around self-certainty, the 'gravitational well'/singularity/hole in representational space.

This revision incorporates Derridean insights into the concept of invention. At its core, invention is paradoxical: the invention must be recognized as different from what has come before, it it must also be intelligible within the existing framework of understanding.

My revision also challenges the preferred order of explanation that constructivists often employ. The tendancy is to value constructed knowledge as more valid than memorized procedural knowledge. I don't disagree with that valuation, but it isn't how I tend to learn. Instead, I tend to read/hear/observe, experience the I-feeling, as if I am genuinely recognized by what I hear, reflect, realize I have no idea what they mean, try to fit them together like a puzzle with how I understand the world, experience the I-feeling again when I take the new idea on as a commitment, and then realize the new idea is not self-certainty.

That abstract description could stand some nuance. Take counting. A child usually learns the chant of 1, 2, 3, etc. They aren't usually counting anything, they're just repeating the chant. This is memorization, not construction. But it is a necessary step. The child must have the words in their head before they can use them to count.

Then, through practice and social interaction, the child begins to understand that these words correspond to quantities of objects. They start to count real things, experiencing the I-feeling of successfully matching the number of objects to the last number said. This is the construction phase, where the child actively builds understanding.

Then, eventually, they might experience the paradox of identity encoded in the claim that ten ones is one ten. This is first experienced as a falsity: ten pennies is *not* one dime. There's more stuff, so they can't be the same! Then they might actually synthesize the concept of zero, realizing that, in intersubjective practices, people really do trade in ten pennies for one dime. Or they might learn to decompose a base with counting cubes. All the while, I imagine children are feeling the same sort of existential fear tracks me, like a prowling shadow, whenever I try to learn something new. Am I okay??

In attempting to connect the various strategies of CGI to embodied practices, bootstrapping from counting to division, the purpose is not to suggest that people actually learn so systematically. The curricula doesn't start with the breath. It started, for me, when my sister, who is two years older than I am, came home from school talking about the books she was reading. I *wanted* to be as cool as she is, so I oretended to know how to read. I pretended to know how to use fractions. I pretended a lot. Eventually, I became firmly committed to communicative norms associated with mathematics. I learned to read, I learned to use fractions, and use variables. But I still don't understand why, for example, $4\frac{m}{s^2} \times 2s = 8\frac{m}{s}$ – why an accelleration multiplied by a unit of time gives a velocity. I mean, I can explain such equations using an incredibly rich vocabulary in a way that might convince a skeptic and convince anyone who doubts my understanding that I actually do understand it. But the body-feeling of accelleration accompanied by a few seconds of sitting around waiting, doesn't feel like a velocity. The reification of body-feelings as anaphoric expressions doesn't easily backfill into those same body-feelings. I don't experience speed, for example, as a body-feeling at all. Do you? You must understand that the current wisdom suggests that the solar system is orbiting the center of the Milky Way at 514,000 miles per hour. Do you *feel* that? No. We feel differences—jolts in the catastrophe machine. So how would it be the case that two experiencable concepts, acceláration and waiting, multiply to give an unexperiencable concept of velocity? The best I can say is that I'm not sure.

In any case, formalizing student-invented strategies provides some benefits. By making them explicit in a vocabulary built to describe speech-actions (formal automata), commonalities can be discerned. States and transitions may be shared across different operations in ways that the rigid adherence to the communicative norms of constructivism do not afford, precisely because of their commitment to human-centered learning. I want to be able to leverage the massive new expressive powers of computational systems to explore the space of possible strategies. This is not to say that I want to replace human-centered learning with machine-centered learning. Quite the opposite. I want to use machines to explore the space of possible strategies so that I can better understand how humans learn.

A computer, for example, can systematically explore the space of possible numbers to use in examples that might be used to invoke the crisis of strategic thinking. Right now, there are probably thousands of children working on computer-generated worksheets where the numbers being used are almost arbitrary. Yet math teachers will recognize that some numbers are more 'natural' than others for teaching a concept. I liken this benefit to the problem of protein-folding. Consider how many medicines utilize proteins that were discovered by accident. Since the mid-2000s, the problem of protein-folding has been well-studied, but now machines have the capability to assist in this research, systematically exploring the space of possible proteins and their interactions. With AI, it seems possible to systematically explore the space of possible numbers to construct exemplary examples that can be used to bootstrap students' understandings into new operations. The next 'phase' of math education research may include some such computationally intensive explorations.

Consider the following example. Imagine a first-grader who can count-on, but who can't yet rearrange to make bases. Suppose they learned to count to 100 in kindergarten, and the curriculum is designed to use that prior knowledge. A computer-generated worksheet might draw from the nearly 200 million partitions of 100 to randomly select some problems. It presents the child with $15 + 2$. The child learns nothing, because they can count-on. Another child is randomly presented with the problem $15 + 7$. They are too 'lazy' (in a good way) to want to do 7 inferential steps. So, instead, they draw on the material inference for object-collections that implicitly involves associativity and commutativity, to move 3 ones from 15 to 7, resulting in $12 + 10$, and the notion that 1 base + 1 base is 2 bases, to obtain 22. While 'inventing' such a strategy takes real effort, the new structure for addition doesn't require as many monotonous inferential steps. Testing the strategy to ensure it always works requires more steps, but once the strategy has been tested, what might have taken 7 steps now only takes 3. Ideally, a teacher would serve rich problems to students, being able to tell what sorts of examples would be most likely to invoke the crisis of strategic thinking. But practically, teachers are busy. We're already letting machines do a lot of the heavy lifting in math education, but the machines are relatively stupid. They have not been trained to think about mathematics like people think about mathematics.

These challenges motivate the need for a more structured understanding of the landscape of student-invented strategies. The formalization project undertaken here, exemplified by the Hermeneutic Calculator (Savich, title), is not intended to restrict student invention but to rigorously analyze the strategies that emerge within intersubjective space. By treating these strategies as computational choreographies, ~~we~~ can move beyond the limitations of isolated teaching experiments and discover relationships that might otherwise remain obscured.

The triadic framework examines operations from three integrated subject positions: (1) The **subjective validity** of embodied practices, grounded in first-person lived experience; (2) The **normative validity** of inferential rules, structured through second-person commitments that exclude error; (3) The **objective validity** of formal structures, modeled as third-person automata. These three dimensions interpenetrate, forming a unity where the formal emerges from the felt, the abstract from the concrete. Operations stabilize around points of self-certainty where subjective experience, normative regulation, and objective structure achieve coherence.

9.3 Systematic Analysis: The Recipe from Embodiment to Norms

The triadic progression from subjective to objective requires a systematic recipe for moving from private, embodied feeling to shared, communicable, and rule-governed practice. This section articulates that recipe using multiplication as a running example, demonstrating how the formal emerges from the felt.

The Embodied Ground: Subjective Validity

The journey of any operation begins not with a symbol, but with a doing. Children, before they are ever taught a formal multiplication algorithm, invent their own strategies grounded in embodied experience. Consider one of the most intuitive strategies: *Coordinating Two Counts by Ones* (C2C). To find the total number of items in 3 groups of 4, a child counts: "one, two, three, four" for the first group, then continues "five, six, seven, eight" for the second, and "nine, ten, eleven, twelve" for the third. All the while, a second, implicit count tracks the number of groups processed.

This is a lived, felt process—a rhythmic activity grounded in the basic human capacities for grouping objects, maintaining a sequence, and tracking multiple streams of information. This represents the *subjective* pole of the operation. It is a form of knowing that is inseparable from the physical act of doing, a claim to subjective validity: "this is how it feels right to me to do this."

From Embodiment to Norms: The Recipe

How do ~~we~~ move from this private feeling to a mathematical operation? The recipe unfolds in systematic steps:

Step 1: Start with the Embodied Practice. Begin with an intuitive strategy like C2C, grounded in subjective validity.

Step 2: Articulate Material Inferences. Re-describe this practice as a set of material inferential commitments using incompatibility semantics (Brandom, 2008). The "goodness" of an operation is not an abstract truth but a normative status conferred upon a practice within a recognitive community. The operation $3 \times 4 = 12$ becomes a material inferential commitment:

$$\{3 \text{ groups of } 4\} \vDash_I \{12\}$$

Using the definition of incompatibility entailment, this means that everything incompatible with the result '12' (e.g., asserting the result is 11 or 13) must also be incompatible with the initial premise '3 groups of 4'. By systematically excluding what is incoherent, ~~we~~ define the boundaries of coherent practice.

Step 3: Transpose Embodied Metaphors. Drawing on Lakoff and Núñez (Lakoff & Núñez, 2000), transpose their four grounding metaphors into the modal logic of Chapter 1. *Arithmetic as Object Collection* grounds commutativity and associativity in the physical indifference of pooling. *Arithmetic as Motion Along a Path* grounds subtraction in backward movement. These are not mere mappings but phenomenological performances—rhythms of temporal compression (reification $[T]$) and decompression (sublation $[LG]$).

Step 4: Generate the Formal Automaton. The result is a stable, repeatable procedure modeled as a Finite State Automaton. The automaton embodies the norms of the practice, providing a third-person, formal description. Its validity is alethic: when followed, it correctly yields results consistent with the normative framework. This is the form of the practice made explicit, a determinate structure.

9.4 The Dialectical Turn: Choreography and Fractal Geometry

Mathematical operations are, fundamentally, choreographed movements of thought. Each arithmetic strategy represents what I term "written choreography for embodied cognition" (Savich, title). A formal automaton becomes a script for the temporal unfolding of mathematical understanding: initialize, transform, check, recurse, terminate. This framing preserves *how* a student actually moves through a calculation—counting up, pausing at a boundary, decomposing a number—rather than replacing those movements with opaque symbolic shortcuts.

The dialectical turn emerges from recognizing that this choreography exhibits genuine computational self-similarity. Across twenty-five formalized student strategies in the Hermeneutic Calculator project, I consistently recover the same fractal pattern. Each strategy exhibits an **iterative core**—a minimal loop of initialize, step, and condition check—nested within a **strategic shell** that prepares, optimizes, or transforms the problem so the core runs with fewer or cognitively lighter iterations.

Consider the *Sliding* strategy for subtraction. Faced with $74 - 36$, a student might recognize that both numbers can be adjusted: $(74+4)-(36+4) = 78-40 = 38$. This emerges from the *Motion Along a Path* metaphor, where distances remain invariant under translation. The formal automaton reveals its computational structure: the strategic shell identifies a convenient adjustment value K that simplifies calculation (creating a multiple of ten), then invokes the iterative core to perform the adjusted subtraction.

Strategies such as Rearranging to Make Bases, Sliding, and Distributive Reasoning wrap the iterative core with analysis that computes gaps, splits factors, or slides both numbers to maintain invariant relationships. Because the shell often invokes the core as a subroutine, the global structure becomes genuinely self-similar: strategies contain and sometimes nest earlier strategies, producing a computational fractal rather than mere metaphorical flourish.

Two fundamental movements underlie all arithmetic strategies. First, **temporal compression** (sublation/recollection) unitizes many micro-acts into larger cognitive units: ten ones become one ten, three base jumps become a single composite stride. Second, **temporal decompression** (determinate negation) strategically expands a composite to restore fine control: borrowing a ten, splitting five into two plus three, decomposing a factor for distributive reasoning. Mathematical fluency emerges as students learn to coordinate these movements, developing the capacity to know *when* to expand and *when* to re-compress.

9.5 Second-Person Reflection: Elaboration and Formalization

The distinction between Practical Elaboration by Training (PEdT) and Algorithmic Elaboration (AE) is crucial for understanding the structure of mathematical development. This distinction explains how mathematical understanding moves from contingent pedagogical stabilization to necessary logical elaboration.

Practical Elaboration by Training (PEdT) is the contingent, pedagogical process emphasized in Cognitively Guided Instruction that transforms implicit material inferences into explicit, teachable procedures. The embodied origins of mathematical concepts—the four grounding metaphors of Object Collection, Object Construction, Measuring Stick, and Motion Along a Path—require pedagogical stabilization to become reliable mathematical practices. Students must learn to count on consistently, to decompose numbers strategically, and to recognize when particular approaches are appropriate. This stabilization is fundamentally pedagogical—it depends on teaching, practice, and the social regulation of mathematical activity.

Algorithmic Elaboration (AE), by contrast, represents the logical restructuring of existing abilities rather than their mere strengthening through repetition. When students reorganize their Counting On practice into the more sophisticated Rearranging to Make Bases strategy, they are not simply becoming faster counters but are discovering new inferential relationships. The elaborated strategy makes explicit what was implicit in the original practice: that numerical relationships can be preserved under certain transformations.

Strategies that stand in an LX (Elaborated-Explicating) relationship—where later practices both derive from and make explicit earlier ones—constitute what mathematics educators recognize as "conceptual understanding." The Rearranging to Make Bases strategy explicates the associative structure latent in Object Collection. Distributive Reasoning articulates the multiplicative structure implicit in repeated addition. These relationships demonstrate how formal mathematical properties emerge from and remain grounded in embodied practice.

This second-person reflection reveals that mathematical development is not a smooth, continuous progression but involves dialectical moments where existing practices must be transcended. The failure of a strategy under certain conditions (e.g., the limitation of subtraction in $\mathbb{N}$ when facing $3 - 5$) creates the incoherence that drives elaboration toward more expressive frameworks (the integers $\mathbb{Z}$). Only after PEdT stabilizes primitive practices can AE restructure them into sophisticated strategies. The structure of mathematical development thus exhibits both contingent pedagogical moments and necessary logical transformations.

Formalizing Algorithmic Elaboration: Counting On to Rearranging to Make Bases

The claim that Rearranging to Make Bases (RMB) algorithmically elaborates Counting On (CO) can be demonstrated formally through automata specifications. By examining the computational structure of each strategy, we can see precisely how the elaborated practice reorganizes and nests the primitive practice.

The Primitive Practice: Counting On

Counting On is the foundational additive strategy. To solve $A + B$, the student starts at A and counts forward B times: "$A + 1, A + 2, \ldots, A + B$." This embodies the Motion Along a Path metaphor, where numbers are locations and addition is forward movement.

Formally, Counting On can be modeled as a deterministic pushdown automaton (DPDA) with a bounded stack representing base-10 place values. The machine maintains units, tens, and hundreds on the stack, incrementing and carrying as needed.

Table 9.1: *Counting On Automaton (M_{count}).* The formal specification treats counting as iterative unit increments with automatic carry propagation (sublation) across place values. Each "tick" input advances the count; overflow at U_9 triggers temporal compression into the tens place.

Current State	**Input**	**Top of Stack**	**Next State**	**Action (Stack)**	**Interpretation**
q_{start}	ε	Z_0	q_{idle}	Push(U_0, T_0, H_0)	Initialize to 0
q_{idle}	tick	U_n $(n < 9)$	q_{idle}	Pop; Push(U_{n+1})	Increment units
q_{idle}	tick	U_9	q_{inc_tens}	Pop	Unit overflow → carry
q_{inc_tens}	ε	T_m $(m < 9)$	q_{idle}	Pop; Push(T_{m+1}, U_0)	Increment tens, reset units
q_{inc_tens}	ε	T_9	q_{inc_hund}	Pop	Tens overflow → carry
q_{inc_hund}	ε	H_k $(k < 9)$	q_{idle}	Pop; Push(H_{k+1}, T_0, U_0)	Increment hundreds
q_{inc_hund}	ε	H_9	q_{halt}	Pop; Push(H_0, T_0, U_0)	Counter overflow

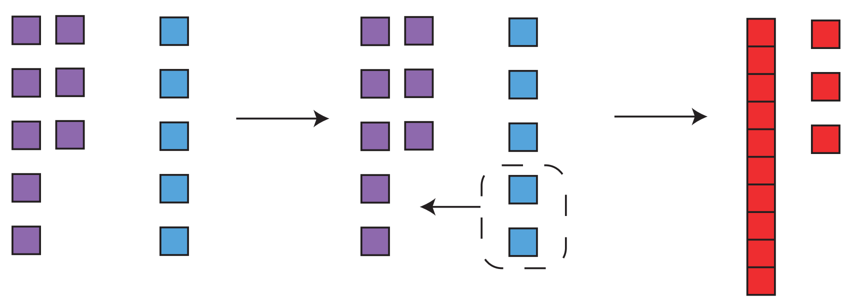
The choreography of Counting On is simple but profound. Each "tick" is a bodily rhythm—a finger tap, a verbal count, a step forward. The carry operation is temporal compression: ten unit steps recollected as one ten. This primitive practice is reliable, but inefficient for larger addends.

The Elaborated Practice: Rearranging to Make Bases

Consider my favorite example, rearranging to make bases. Remember: the formalization is not meant to intimidate—it is built to break. As my understanding of formal systems grows, the vocabularies I use to express the pragmatics of mathematical strategic thinking, the formalizations change. There is no claim to finality or completeness.

The video I analyzed is from Carpenter et al. (1999). The strategy descriptions and examples were adapted from Amy Hackenberg's careful reconstruction of those videos to teach pre-service teachers Hackenberg (2025).

- **Teacher:** Lucy is eight fish. She buys five more fish. How many fish will Lucy have then?
- **Sarah:** 13.
- **Teacher:** How'd you get 13?
- **Sarah:** Well, because eight plus two is ten, but then two plus three is five. And she wants to buy five more fish. So you take care of two, and you need to add three more. And so I add three more, and you get 13.



Notation Representing Sarah's Solution:

$$\begin{aligned}
8 + 5 &= \square \\
8 + 2 &= 10 \\
2 + 3 &= 5 \\
8 + 5 &= 10 + 3 \\
8 + 5 &= 13
\end{aligned}$$

Description of Strategy:

Objective: Rearranging to Make Bases (RMB) means shifting the extra ones from one addend over to the other so that one of the numbers becomes a complete multiple of the base (a whole "group" of that base). This rearrangement simplifies the addition process because there are established patterns for adding an exact multiple of the base. In other words, when you add a full group of base units to a number, the ones digit stays the same while only the digit representing the base (like the tens place) increases.

Rearranging to Make Bases transforms the counting procedure by introducing strategic planning. Instead of counting forward B times, RMB recognizes that reaching the next base boundary (10, 20, 100, etc.) creates a cognitive landmark. The strategy decomposes B into two parts: the gap K needed to reach the next base, and the remainder R. Symbolically: $A + B = (A + K) + R$, where $A + K$ is a clean base.

For example, solving $28 + 7$: the gap from 28 to 30 is 2, so decompose $7 = 2 + 5$, yielding $(28 + 2) + 5 = 30 + 5 = 35$. The student has reorganized the primitive counting procedure to minimize cognitive load.

Table 9.2: *Rearranging to Make Bases Automaton (M_{RMB}).* This strategy algorithmically elaborates Counting On by adding a strategic shell that computes the gap K to the next base, decomposes the addend B, and recombines. The core counting operations are nested within this higher-level strategic structure.

Current State	**Condition**	**Next State**	**Action**	**Interpretation**
q_{start}	—	q_{calcK}	$K \leftarrow 0$; $A_{temp} \leftarrow A$	Initialize
q_{calcK}	$A_{temp} <$ NextBase(A)	q_{calcK}	$A_{temp} \leftarrow A_{temp} + 1$; $K \leftarrow K + 1$	Count up to find gap K
q_{calcK}	$A_{temp} ==$ NextBase(A)	q_{decomp}	$A' \leftarrow A_{temp}$	Gap found, store new base A'
q_{decomp}	$K > 0$	q_{decomp}	$B \leftarrow B - 1$; $K \leftarrow K - 1$	Decompose B by transferring K
q_{decomp}	$K == 0$	q_{recomb}	$R \leftarrow B$	Remainder R is what's left of B
q_{recomb}	—	q_{accept}	Output $A' + R$	Combine base + remainder

The Logic of Elaboration

Comparing these two specifications reveals the precise mechanism of algorithmic elaboration. RMB does not replace CO; it reorganizes it. The q_{calcK} state uses the primitive counting operation (incrementing A_{temp}) to calculate the gap. The q_{decomp} state uses subtraction (the inverse of counting) to split B. The final q_{recomb} state invokes addition on the transformed problem.

The elaboration is *algorithmic* because it is specifiable in principle: given mastery of counting up and counting down, the RMB procedure can be constructed through sequencing and conditional branching. It is *explicating* because RMB makes explicit what was always implicit in CO—the associative structure of addition. When counting from 28 to 35, the student was implicitly crossing the base boundary at 30. RMB makes this boundary crossing explicit and strategic.

This is the structure of LX relationships throughout arithmetic. Distributive multiplication explicates the iterative structure of repeated addition. Decomposing a base ('borrowing') explicates the decomposability of place-value units. Each elaborated strategy reveals the logic that was always present in the primitive practice, transforming implicit knowing-how into explicit knowing-that.

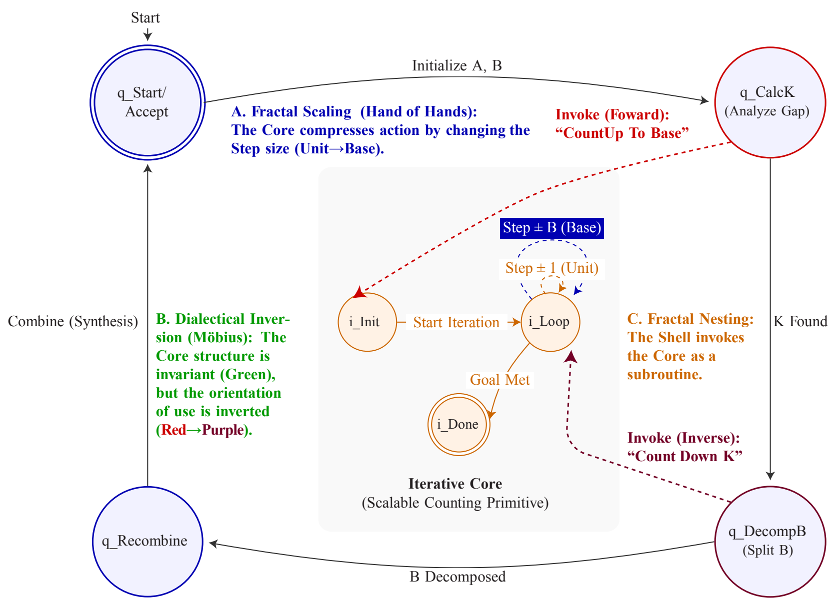


Figure 9.1: *The Fractal and Dialectical Geometry of Arithmetic.* Fractalized Arithmetic as Automata: The fractal scaling of the Iterative Core demonstrates computational self-similarity. The superimposed loops show that the Core's structure remains invariant while the scale of action changes from units to bases. The dialectical inversion (Red/Purple) visualizes how the same structure operates in opposite orientations.

The fractal architecture illustrated in Figure 9.1 reveals how mathematical strategies achieve their power through self-similar structure. The iterative core—the fundamental loop of initialize, step, and check—scales across different mathematical magnitudes while maintaining its essential form. This scaling property allows strategies to work across place values, demonstrating how mathematical understanding exhibits genuine computational self-similarity. The integration of inversion (Möbius orientation flipping) with fractal nesting (core within shell) explains both the unity and diversity of arithmetic strategies: they are variations on the same underlying mathematical structure.

The formal automata do more than describe strategies; they articulate the developmental logic of mathematical understanding. They show how practices that feel natural to experienced mathematicians emerge through systematic reorganization of primitive embodied doings. They demonstrate that "conceptual understanding" is not a mysterious extra ingredient added to procedural skill but the capacity to articulate the inferential structure of one's own mathematical practices.

9.6 Integration: The Dialectical Structure of Operations

The concept of inversion is central to the dialectical structure of mathematical operations. The analysis reveals that operations exhibit a fundamentally dialectical relationship where opposing procedures share underlying computational structures. Subtraction strategies emerge not through the invention of entirely new procedures but through the inversion or repurposing of addition approaches.

The Missing Addend strategy reframes subtraction as forward accumulation: "What must I add to 36 to reach 74?" becomes the question structuring $74 - 36$. Counting Back mirrors Counting On, moving in the opposite direction along the number line. The Sliding strategy preserves difference across translation, maintaining invariant relationships under transformation. Borrowing reverses the carry procedure by decompressing previously sublated place-value units: where carrying compressed ten ones into one ten, borrowing expands one ten back into ten ones.

This dialectical relationship extends beyond the superficial observation that subtraction "undoes" addition. The formal automata reveal that operations often share identical computational cores while differing in their orientations or interpretations. The Möbius strip visualization captures this relationship precisely: the underlying structure (the Green core) remains invariant while the operational orientation flips between additive (Red) and subtractive (Purple) modes.

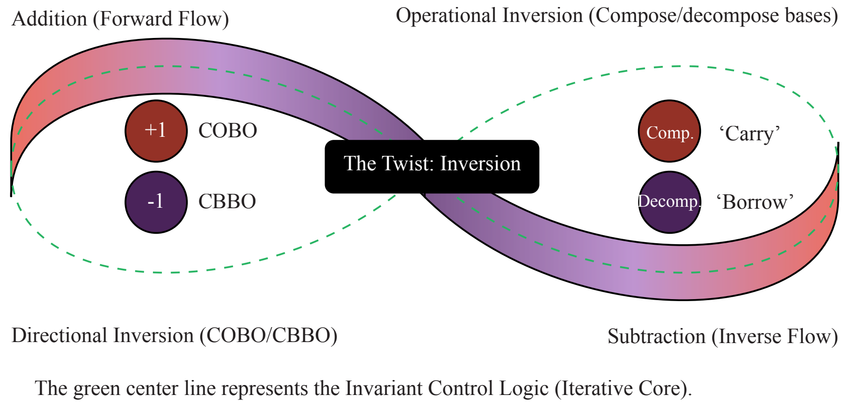


Figure 9.2: *The Geometry of Inversion: Addition and Subtraction as a Unified Structure.* The Möbius strip visualization articulates the dialectical relationship between addition and subtraction. The automata analysis demonstrates how they represent inverse orientations of the same underlying temporal structure.

9.7 Conclusion: Operations as Creative Possibility

This chapter has articulated a vision of mathematical operations as embodied, normatively regulated practices that emerge from the triadic unity of subjective experience, intersubjective norms, and objective structures. Throughout this investigation, the argument has been that operations are not mere procedures applied to pre-existing objects but are expressions of human being-in-the-world.

The *subjective* experience of doing (e.g., the felt rhythm of Coordinating Two Counts by Ones) provides the initial, meaningful content of an operation. It is a first-person grounding of mathematical thought, rooted in the proprioceptive and temporal structures explored in Chapter 1. The *normative* framework of incompatibility semantics provides the rules of the game—a second-person structure of commitments and entitlements, defining coherence by systematically excluding error. The *objective* structure of the automaton provides the formal, repeatable, and communicable form of the practice—a third-person, explicit representation of a normatively governed activity.

The critical arithmetic I propose is not a closed formal system but an open-ended, pre-formal practice that is continuously evolving through dialogue and critique. It is a practice that values the agency of learners and the creative potential of mathematical thought. It resists the scientistic impulse to reduce the rich, messy, human process of mathematical meaning-making to a single, "correct," disembodied algorithm.

The Hermeneutic Calculator project demonstrates that student-invented strategies can be formalized without losing their connection to embodied meaning. The automata serve not as replacements for human mathematical thinking but as precise descriptions of the temporal unfolding of mathematical understanding. By treating each strategy as choreographed movement of thought, the formalization preserves rather than eliminates the human dimension of mathematical activity.

Understanding mathematical operations as embodied, normatively regulated practices reveals arithmetic not as a closed system of fixed procedures but as an open field of creative possibility. Students do not simply learn to execute predetermined algorithms but participate in the ongoing elaboration of mathematical understanding. This participation is simultaneously deeply personal—grounded in each individual's embodied experience—and thoroughly social—regulated through the shared norms of mathematical practice.

The analysis demonstrates how formal mathematical structures emerge from rather than supersede lived experience. The automata that model student strategies are not abstract computational devices but precise articulations of embodied mathematical thinking. They reveal the temporal dynamics through which mathematical understanding unfolds, showing how the formal and the felt, the abstract and the concrete, the subjective and the objective are integrated within a unified account of mathematical meaning-making.

Embracing this vision—seeing operations as a synthesis of subjective feelings, normative commitments, and objective structures—moves mathematics education toward a more emancipatory stance, fostering not just technical proficiency, but also rigorous thinking, creative exploration, and humane flourishing.

References

Abbott, E. A. (2020, April). *Flatland: A Romance of Many Dimensions by Edwin A. Abbott*. Independently published.

Agamben, G. (2006, September). *Language and Death: The Place of Negativity* (K. Pinkus & M. Hardt, Trans.; First Edition). Univ Of Minnesota Press.

Aristotle. (2017). *Metaphysics: Aristotle's metaphysics in english, latin and ancient greek: Trilingual edition* (W. o. Moerbeke & W. D. Ross, Trans.; Trilingual). Hermes Language Reference.

Boote, D. N., & Beile, P. (2005). Scholars before researchers: On the centrality of the dissertation literature review in research preparation. *Educational researcher*, *34*, 3–15.

Brandom, R. B. (1994). *Making it explicit: Reasoning, representing, and discursive commitment*. Harvard university press.

Brandom, R. B. (2000). *Articulating reasons: An introduction to inferentialism*. Harvard University Press.

Brandom, R. B. (2008). *Between saying and doing: Towards an analytic pragmatism*. Oxford University Press.

Brandom, R. B. (2019). *A spirit of trust: A reading of Hegel's phenomenology*. The Belknap Press of Harvard University Press.

Bueno, O. (2020). Nominalism in the Philosophy of Mathematics. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Fall 2020). Metaphysics Research Lab, Stanford University.

Cantor, G. (1891). Ueber eine elementare frage der mannigfaltigkeitslehre. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, *1*, 75–78.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). Children's mathematics: Cognitively guided instruction - videotape logs [supplementary material]. In *Children's mathematics: Cognitively guided instruction*. Heinemann, in association with The National Council of Teachers of Mathematics, Inc.

Carspecken, P. F. (1995). *Critical ethnography in educational research: A theoretical and practical guide*. Routledge.

Carspecken, P. F. (1999). *Four scenes for posing the question of meaning and other essays in critical philosophy and critical methodology* (New edition). Peter Lang Inc., International Academic Publishers.

Carspecken, P. F. (2006). Limits of knowledge in the physical sciences. In K. Tobin & J. Kincheloe (Eds.), *Doing educational research* (pp. 405–437, Vol. 1). Sense Publishers.

Carspecken, P. F. (2009). Limits to knowledge with existential significance: An outline for the exploration of post-secular spirituality of relevance to qualitative research. In *Bridging the gap between theory and practice in educational research* (pp. 47–61). Springer.

Carspecken, P. F. (2016). Limits to knowledge and being human: What is 'critical' in critical social research and theory? In P. H. Hinchey (Ed.), *A critical action research reader* (pp. 65–86). Peter Lang Inc., International Academic Publishers.

Carspecken, P. F. (2018). The missing infinite. In R. Winkle-Wagner, J. Lee-Johnson, & T. Gaskew (Eds.), *Critical theory and qualitative data analysis in education* (p. 32). Routledge. https://doi.org/10.4324/9781315158860-2

Carspecken, P. F., & Zhang, R. (2013). Content inference fields in intersubjective space: Transpersonal position, illocution, and logic in the analysis of human interactions. *Qualitative research: A reader in philosophy, core concepts, and practice*, 201–242.

Carspecken, R. (2024, November 3). Untitled.

Caygill, H. (2004). *A kant dictionary*. Wiley-Blackwell.

Derrida, J. (1973). *Speech and phenomena, and other essays on husserl's theory of signs*. Northwestern University Press.

Derrida, J. (1982). *Marges de la philosophie* [Margins of philosophy] (A. Bass, Trans.). The Harvester Press. (Original work published 1972)

Deutsch, H., Marshall, O., & Irvine, A. D. (2025). Russell's Paradox. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford encyclopedia of philosophy* (Spring 2025). Metaphysics Research Lab, Stanford University.

Ernest, P. (2024). The mathematical and philosophical significance of zero: Nought matters [Publisher: Paul Ernest, Emeritus Professor at University of Exeter, School of Education & Lifelong Learning]. *Philosophy of Mathematics Education Journal*, 1–35. https://research.ebsco.com/linkprocessor/plink?id=b685ecb9-d2fc-3e03-889d-a06fb1751c1b

Euclid. (2007, December). *Euclid's Elements of Geometry* (J. L. Heiberg, Ed.; R. Fitzpatrick, Trans.; Revised and corrected edition). Richard Fitzpatrick.

Franzén, T. (2004, September). *Inexhaustibility: A Non-Exhaustive Treatment: Lecture Notes in Logic 16* (1st edition). A K Peters/CRC Press.

Franzén, T. (2005). *Gödel's theorem: An incomplete guide to its use and abuse*. AK Peters/CRC Press.

Frege, G. (1997, July). *The Frege Reader* (M. Beaney, Ed.; 1st edition). Blackwell Publishing.

Gaifman, H. (2005). *The easy way to Gödel's proof and related matters* [Unpublished manuscript. Available online].

Geisel, T. (1961). *The Sneetches and Other Stories* (First edition.). Random House Books for Young Readers.

Gelder, T. V. (1995). What might cognition be, if not computation? *The Journal of Philosophy*, *92*, 345–381. Retrieved April 13, 2025, from http://www.jstor.org/stable/2941061

Gray, L., & Garzon, M. (2005). On a hermite boundary integral approximation. *Computers & structures*, *83*(10-11), 889–894.

Greenblatt, R., Denison, C., Wright, B., Roger, F., MacDiarmid, M., Marks, S., Treutlein, J., Belonax, T., Chen, J., Duvenaud, D., Khan, A., Michael, J., Mindermann, S., Perez, E., Petrini, L., Uesato, J., Kaplan, J., Shlegeris, B., Bowman, S. R., & Hubinger, E. (2024, December 23). Alignment faking in large language models. https://doi.org/10.48550/arXiv.2412.14093

Habermas, J. (1985). *The theory of communicative action: Lifeworld and system* (Vol. 2). Beacon Press.

Habermas, J. (2004). *The philosophical discourse of modernity: Twelve lectures* (F. G. Lawrence, Trans.). MIT Press.

Habermas, J. (1971). *Knowledge and human interests*. Beacon Press.

Habermas, J. (1984). *The theory of communicative action, volume 1: Reason and the rationalization of society* (T. McCarthy, Trans.; Reprint edition). Beacon Press.

Habermas, J. (1995). *Postmetaphysical thinking: Between metaphysics and the critique of reason* (1st edition). Polity.

Hackenberg, A. (2025). *Course notes* [Unpublished course notes].

Hegel, G. W. F. (1977). *Phenomenology of spirit* (A. V. Miller, Trans.; Vol. 632). Oxford University Press Oxford.

Hegel, G. W. F. (2010). *The science of logic* (G. di Giovanni, Ed. & Trans.). Cambridge University Press.

Houlgate, S. (2006). *The opening of hegel's logic: From being to infinity*. Purdue University Press.

Hyppolite, J. (1974). *Genesis and structure of hegel's phenomenology of spirit*. Northwestern University Press.

Kant, I. (1998). *Critique of pure reason* [Original work published 1781/1787, Guyer and Wood, Trans.]. Cambridge University Press.

Lakoff, G., & Johnson, M. (1980). *Metaphors we live by*. University of Chicago Press.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books.

Linnebo, Ø. (2024). Platonism in the Philosophy of Mathematics. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford encyclopedia of philosophy* (Summer 2024). Metaphysics Research Lab, Stanford University.

Longuenesse, B. (2017). *I, me, mine: Back to kant and fichte on the 'i'*. Oxford University Press.

Mead, G. H. (1934). *The social psychology of george herbert mead* (A. L. Strauss, Ed.). University of Chicago Press.

Mead, G. H. (2015). *Mind, self, and society: The definitive edition* (H. Joas, D. R. Huebner, & C. W. Morris, Eds.; Enlarged edition). University of Chicago Press.

Meyer, J. (title). *Cantors 1891 diagonal proof* [Cantor's original 1891 diagonal proof]. Retrieved July 6, 2025, from https://jamesrmeyer.com/infinite/cantors-original-1891-proof

Nagel, E., & Newman, J. R. (2012). *Godel's proof*. Routledge.

Negarestani, R. (2018). *Intelligence and Spirit*. Urbanomic/Sequence Press.

Poston, T., & Stewart, I. (2012, February 29). *Catastrophe theory and its applications* (First Edition). Dover Publications.

Priest, G. (2002). *Beyond the limits of thought*. Oxford University Press.

Priest, G. (2014). *One: Being an investigation into the unity of reality and of its parts, including the singular object which is nothingness*. Oxford University Press.

Quine, W. V. O. (1948). On what there is. In *Thinking about logic* (pp. 189–209). Catholic University of America. https://sites.pitt.edu/~rbrandom/Courses/2023%20Sellars/Sellars%20texts/Quine%20On_What_There_Is.pdf

Quine, W. V. O. (1963). Two dogmas of empiricism [First Harper Torchbook edition, a reprint of the second, revised edition (1961). The book was first published by Harvard University Press in 1953.]. In *From a logical point of view* (Second, revised, pp. 20–46). Harper & Row.

Rödl, S. (2014). Intentional transaction [Accessed: 2025-04-13]. *Philosophical Explorations: An International Journal for the Philosophy of Mind and Action*, *17*, 304–316. https://doi.org/10.1080/13869795.2014.941909

Rose, G. (2009). *Hegel contra sociology*. Verso.

Rowlands, S. (2005). The contradictions in the constructivist discourse [Citing von Glasersfeld]. *Philosophy of Mathematics Education Journal*, *14*.

Savich, T. M. (2020). Towards a critical mathematics [Accessed via ERIC]. *Mathematics Teaching–Research Journal Online*, *12*, 245–291. https://files.eric.ed.gov/fulltext/EJ1384454.pdf

Savich, T. M. (2022). *Towards a critical mathematics*. Indiana University.

Savich, T. M. (2024). Built to break: An introduction to theoretical autoethnography [Invited talk presented at Solent University, organized by Dr. Brian McDonough]. https://iu.mediaspace.kaltura.com/media/t/1_v1qmm3g9

Savich, T. M. (title). *The hermeneutic calculator* [The hermeneutic calculator]. Retrieved July 23, 2025, from https://tiosavich.github.io/UMEDCTA/Calculator/index.html

Sellars, W. (1953). Inference and meaning. *Mind*, *62*, 313–338.

Sellars, W. (2007). *In the space of reasons: Selected essays of wilfrid sellars*. Harvard University Press.

Steffe, L. P. (2002). A new hypothesis concerning children's fractional knowledge [Accessed through CEPA.INFO]. *Journal of Mathematical Behavior*, *20*(3), 267–307.

Steffe, L. P. (2003). On the construction of learning trajectories of children: The case of commensurate fractions [Accessed via ResearchGate/ERIC]. *Monographs of the Journal for Research in Mathematics Education*, 1–26.

Steffe, L. P. (2004). Fractional commensurate, composition, and adding schemes: Learning trajectories of jason and laura: Grade 5 [Accessed via ResearchGate]. In J. Olive & R. Tzur (Eds.), *Children's fractional knowledge* (pp. 1–60). American Educational Research Association.

Steffe, L. P. (2014). *Children's number sequences: An explanation of steffe's constructs and an extrapolation to rational numbers of arithmetic* [Originally from 1994]. Unpublished manuscript, available via CiteSeerX; ResearchGate.

Steffe, L. P., & Thompson, P. W. (2014). Teaching experiment methodology: Underlying principles and essential elements [Accessed via ResearchGate, originally published 2000]. *Proceedings of the 24th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*.

Stone, J. (2003). *The monster at the end of this book* (2nd ed. edition). Golden Books.

Taylor, C. (1994). *Multiculturalism: Examining the Politics of Recognition* (A. Gutmann, Ed.) [Essays by Charles Taylor, K. Anthony Appiah, Jürgen Habermas, Steven C. Rockefeller, Michael Walzer, and Susan Wolf]. Princeton University Press.

van Hiele-Geldof, D., & van Hiele, P. M. (1984). *English translation of selected writings of dina van hiele-geldof and pierre m. van hiele* (D. Fuys, D. Geddes, & D. Tischler, Eds.) [ERIC Number: ED287697]. Retrieved June 3, 2025, from https://eric.ed.gov/?id=ED287697

Wittgenstein, L. (1956). *Remarks on the foundations of mathematics*.

Wittgenstein, L. (2010). *Philosophical investigations* (P. M. S. Hacker & J. Schulte, Eds.; G. Anscombe, Trans.). Wiley-Blackwell.

Chapter 10

11: The Dialectic of Incompleteness and Recognition

Formalization as Revelation

We have gone on quite an adventure. Let me summarize where we have been so far. In the prelude, I tried to 'invent' a methodology, critical autoethnography, to structure my comments on mathematics. The concept of divasion required a history for its expression, and so I documented where the word came from: it came from the desire to write a pattern song—a rhythm of "outside/in." From there, I articulated a series of exercises designed to explore this rhythm as the *sound of time.* That gave a metaphor for how knowing is being; intersubjective space is where reasons are negotiated. It arises through the temporal compression of learning experiences as they are artificialized in language. From that rationale for how space functions as a recollection of temporally extended experiences, I articulated how geometric expressions rely on the negative, articulated in Brandom's favored vocabulary of material incompatibility. The act of classifying quadrilaterals allowed for a pseudo-formalization of inferential strength. By playing in the closed space of quadrilateral properties (which are likewise material inferential proprieties), I was able to 'prove' that squares are rectangles because everything (that I listed) that is incompatible with a rectangle is also incompatible with a square. Through the chapter on inferential movement, quadrilateral properties, inferential strength, and what counts as good substitutions were made determinate. In fact, they were made determinate enough to program into Prolog.

In the chapter on *existential needs*, I articulated some of my ambivalence towards 'the law,' whether that law was set to destroy a friend's artistic expression or to codify mathematical expressions as axioms. The law must come from somewhere. If it is simply imposed externally as the systemic crush of individuals, then ambivalence should transform into hostility. But the law cannot be entirely bad. I identify with some norms—they make sense. Other 'laws' must be critiqued with the hope of enacting some change.

In the chapter that questions "Who are you?" I began the work of articulating how intersubjectivity is transcendental-like. Mead's work provided a foundation for Habermas' later work on describing how intersubjectivity—the assumption of communicative competency—arises through social interaction. We worked out a finite-state automaton for describing Mead's dog fights, found it lacking, and then incorporated Zeeman's own analysis of fighting dogs using his Catastrophe Machine. I built one from a record player, using the elastic as a metaphor for the embodied tensions between existential needs. I imagined hooking two such machines together, and considered the challenge of navigating social relationships. When I imagined turning the record players on, representing the systemic forces that always seem to be pushing me towards objectifying and commodifying, instrumentalizing and strategizing—acting in ways that just seem to drain my body of its meaning—I pushed from Verstand to Vernunft. From the mechanical to the meaningful, I had to give up control. The beast of love, as hard as it is to admit, still prowls my nights. But the refrain—trust the beast to come in tune—served to introduce the concept of trust.

From that, I began the work of reconstructing the Voice—that origin of language. As the source of action, the {I} is that Voice. And yet, stripped of music, words, text, and sound, the Voice requires the voice to actualize itself. Like Geist, or Spirit, the Voice must flatten itself into expression. I used the Eulogy for my father as an opportunity to explore how the dyadic negative—a unity—breaks into a rainbow of difference.

In the Bridge, I used the Sneetches and their stars to teach Cantor's diagonalization argument. This allowed me to metaphorically equate Hegelian sublation with diagonalization. I argued that Euclid's proof recognized the incompleteness of finite lists of primes, Hippasus' proof recognized the incompleteness of the rationals, Cantor's proof recognized the incompleteness of the irrational numbers, and Gödel's proof recognized the incompleteness of any coherent mathematical system that can prove theorems, add, and multiply. None of these instances of 'diagonalization' threw out the old system entirely. These were not abstract negations, but determinate negations. The history of mathematics is the history of its becoming.

I then argued that numerals are anaphoric terms. In the mathematical system that we have been working to articulate, this means that proving something belongs in the system amounts to proving that it is reconstructable as a recollection of embodied practices. Various alternative understandings of number were explored as they related to the knowledge-constitutive interests that Habermas (1971) articulated. Treating numerals as pronouns allows for the underlying, embodied, implicit ordinality of counting to be cast into the space of reasons. But perhaps more interestingly, the domain of embodied ordinality seems to relate as equally to human subjects as it does to corvids. I do not speak crow, so I cannot tell, grammatically, how they might be using numerals. But I can recognize them as embodied thinkers, doing the same sort of thing that I do when I count.

With an idea of what numerals are now explicit, I could move on to articulating the difference between algorithmic elaboration and pedagogical elaboration through training: *becoming* split into two separable roles. Lakoff & Núñez's (2000) embodied metaphors, like the measuring stick metaphor, require training. Students, like my daughters on their car-ride to Indianapolois, have to explore distance as a temporal compression of learning experiences. They can't merely be said to be taught. I gave an instance of algorithmic elaboration by reconstructing Euclid's proof in terms of the incompatibility semantics of Robert Brandom (2008) and the embodied mathematics that Lakoff & Núñez articulate. Perhaps it might read as a bit more nuanced than simply articulating how long division is elaborated from multiplication and subtraction, the archetypal example that Brandom gives for algorithmic elaboration, as it required the introduction of an *incoherence frame*, but such a frame is no more complex than the sort of toy I played with at the doctor's office as a kid, where the triangular shape could not fit through the circular hole.

I then moved to applying what I know of critical action theory to formal automata to express mathematical operations. While a surface level shift, I explored automata as represented with circles. I was trying to get at the idea of self-monitoring. When I act, the temporally extended movement is accompanied, to greater or lessor degree, by the ability to STOP acting when things feel like they are going wrong. Automata cannot fully model self-monitoring. Still, the idea that the end of an act must turn back to its beginning to assess whether the act was successful or not, was why I made the aesthetic decision to try and represent these little machines as circles.

In that chapter, I articulated three automata in gross detail. I discussed counting, rearranging to make bases (RMB), and coordinating two counts by counting by ones (C2C). I include about 20 other strategies for arithmetic, and a few related to fractions in the supplementary materials.

I did other things with words along the way. For example, I discussed how the song *Breath and Kindling* uses rhetorical anaphora to build energy. I discussed how *Goodbye Friends* uses structured ambiguity to express what I could never say directly. I discussed how the poetic side of my writing often uses substitution at a smaller grain size than Brandom articulates. Recall how in that song, I hide the rhyme {seven, leaven, heaven}, and made use of the frame □ven, to find rhymes that would work. Do such substitutions challenge the principle that every speech act can be reconstructed as an assertion? Not exactly. Word games at the subsentential level instead demonstrate the fractal-like nature of language and analysis.

But I highlight the above aspects of the book for the purpose of making a telos of this work explicit. At every step, I have gleaned some assertions that are representable in computer language. The hermeneutic calculator (HC) can represent counting, addition, multiplication, and proof, all within a 'coherent' logic. The coherence of the incompatibility semantics that I programmed into Prolog is debatable. I did a bit more than copy and paste Brandom's incompatibility semantics into Prolog, but I read about how other logicians took (Brandom, 2008) apart and found it had some inconsistencies. I have no doubt that if my code were subjected to the same analysis, it would also be found wanting. The sort of coherence tests I ran were minimal. The system doesn't immediately declare, for example, that $1 = 2$. I truly considered suppressing all the formal work my writing partners and I have done over the last 3 years, but then I decided that the negative attention my work might receive from logicians and mathematicians could make the work stronger. I find it genuinely exciting, but I also include it under the hope that it will be irritating enough to the professionals whom I admire but whose work I don't understand, to turn their attention on my system. Perhaps that attention will result in better reasons that serve to demonstrate the telos of my work.

Every bit of it is designed to be incomplete, so it may be somewhat silly to express the telos: the HC is *formally* incomplete. Proofs of incompleteness involving diagonalization are species in the genus *sublation.* I wanted to create a mathematical system that grows beyond itself. We have, in essence, developed the raw conceptual material necessary to enact Gödel's incompleteness theorem.

While somewhat silly, given that the whole thing is *supposed* to be incomplete, the exercise of instantiating Gödel's theorem, which we turn to next, speaks to the politically powerful who seek to control K-12 students, teachers, and curricula. If the politics surrounding math education aspire to *mathematical* coherence, then the those subjects must be recognized as *in*finite. ~~We~~ are non-finite. Treating kids or teachers as if they are buckets to be filled with mathematical knowledge, then buckets whose fullness can be measured with standardized tests, for the purposes of categorizing people as more-or-less able to fill some role within the system (e.g., become a video-store clerk or a nurse or a mathematician), probably isn't *all bad.* But such moves fundamentally misrecognize the human and mathematical subject.

that, I began the work of articulating a formal modal logic for embodiment. I began this inquiry with a question I could not yet articulate. Seven chapters have brought me here—through the embodied origins of counting, the emergence of operations from gesture and rhythm, the construction of number kinds, the recognition of incompatibility as productive crisis, the power of diagonalization as self-transcendence, and the formalization of these student-invented strategies into a computational system. The question I could not name was this: What does it mean that we learn mathematics at all?

The reductive answer—the answer implicit in calls for a "science of math education" that treats understanding as the accumulation of procedures—is that learning mathematics means acquiring a finite set of rules. The student is a vessel. The curriculum is the content poured in. Assessment measures how much remains. This view assumes closure: that mathematical understanding can be completely specified, that mastery is achievable, that the system is finite.

I formalized the strategies to prove this view wrong. Not through argument, but through demonstration. I built the Hermeneutic Calculator (HC)—a formal system grounding arithmetic in the very cognitive moves invented by children emerging from embodied practice. The system counts via rhythmic grouping. It adds through compression and elaboration (COBO). It multiplies by chaining (C2C). It recognizes boundaries and transcends them (from $\mathbb{N}$ to $\mathbb{Z}$ to $\mathbb{Q}$). The formalization captures not just calculation, but the logic of mathematical reasoning itself: axioms, rules of inference, proofs of theorems spanning arithmetic, geometry, and number theory.

And then I applied Gödel's First Incompleteness Theorem. The result is not a failure of the model. It is the revelation I needed. The HC—this rigorous formalization of elementary mathematics as invented by children—is *necessarily incomplete*. There exist truths the system can articulate but cannot prove. The formalized "me" (the strategies-as-recognized) cannot exhaust the "I" (the source of mathematical action). The horizon is necessarily open. We are not vessels. We are boundary-recognizers. We are transcenders. We are *in*finite.

The Hermeneutic Calculator Interprets Robinson Arithmetic

The technical foundation begins with expressive power. Gödel's theorem applies to any consistent formal system capable of interpreting elementary arithmetic. The standard minimal system is Robinson Arithmetic (Q), which requires definitions for Zero, Successor, Addition, and Multiplication.

The HC robustly satisfies these requirements. Zero is grounded in the axiom `axiom(zero)`—the recognition that counting begins somewhere. Succession is implemented as the +1 operation, the rhythmic pulse underlying all numerical construction. Addition is captured by the COBO (Compression/Elaboration) strategy, verified across test cases like $7 + 5 = 12$ and $23 + 17 = 40$. Multiplication emerges through the C2C (Chaining) strategy, confirmed via computations like $3 \times 4 = 12$ and $5 \times 7 = 35$.

But the HC is more than a calculator. The file `incompatibility_semantics.pl` reveals a complete axiomatic architecture. The system includes a sequent calculus prover (`proves/1`, `proves_impl/2`) implementing standard logical rules. It contains explicit axioms: commutativity of addition, geometric entailments (squares → rectangles), and the axioms M4, M5, M6 formalizing Euclid's proof of the infinitude of primes. The logic is grounded in computation via the `is_recollection/2` predicate, which verifies the constructive history of numbers based on the execution of student strategies.

The HC is not a sophisticated abacus. It is a formal system capable of proving theorems. Because it successfully implements the operations of Q, Gödel's theorem applies. The claim is stronger than "student strategies are incomplete." The claim is: "The entire formalized system of mathematical reasoning—spanning calculation, embodied modal logic, geometric proof, and number theory—is necessarily incomplete."

Arithmetization: Encoding the System Within Itself

For the incompleteness theorem to apply, the system must be capable of *self-reference*. This requires encoding the syntax and mechanics of the HC as numbers the system itself can manipulate. The method is Gödel numbering via prime factorization.

Consider a snapshot of the C2C multiplication strategy computing 3×4. The automaton's state includes: the current control state (q_{count}), the goal register ($G = 100$, encoding the problem), the index register ($I = 100$, tracking progress), the total register ($T = 100$, accumulating the result), and the parameters ($N = 103$, $S = 104$). Each component is assigned a number, and the entire configuration is encoded as:

$$g(C) = 2^1 \cdot 3^{100} \cdot 5^{100} \cdot 7^{100} \cdot 11^{103} \cdot 13^{104}$$

The first prime encodes the state, the second encodes G, the third encodes I, and so forth. A single number captures the entire cognitive state. Transitions between states—the rules governing how the automaton updates its registers—become arithmetic predicates. The Counting Rule, for instance, states: "If the state is q_{count} and $I < S$, then increment both I and T." This rule becomes a predicate $\text{Rule}_{\text{Count}}(X, Y)$ that checks whether number X transitions to number Y via this specific cognitive move.

The technical challenge is proving that these encoding and decoding operations are *Primitive Recursive* (PR)—expressible using only the elementary arithmetic operations (addition, multiplication, exponentiation, bounded search) that the HC itself can perform. The key step involves extracting exponents from the prime factorization. The function $\exp_p(N)$, which returns the exponent of prime p in the factorization of N, is PR because it relies on *bounded minimization*: we search for the largest exponent $e \leq N$ such that p^e divides N but p^{e+1} does not. Since divisibility, exponentiation, and bounded search are all PR, the extraction function is PR. Therefore, the entire `Transition` predicate—the formalization of the HC's mechanics—is representable within the HC itself.[1]

[1] The rigorous proof of primitive recursion, including the detailed demonstration that $\exp_p(N) = \mu e \leq N[\neg(p^{e+1}|N)]$ is PR via bounded minimization, is provided in Appendix

The system can talk about itself. The stage is set for the Gödelian construction.

The Gödel Sentence: G (The Reflective Turn)

Here is where the formalization breaks open. Using the Diagonal Lemma—a technique grounded in the same self-referential structure we encountered in Cantor's proof and Russell's paradox—we construct a sentence G with a specific property: G asserts, "I am not provable in the HC."

More precisely, G is a statement about numbers. It says, "There does not exist a number n that encodes a valid proof of the sentence with Gödel number $\ulcorner G \urcorner$." The sentence is constructed so that it speaks about itself via the arithmetic encoding. This is not paradox; it is provable self-reference.

Now consider what G implies. Suppose the HC is consistent (it does not prove contradictions). If the HC could prove G, then G would be false (since G asserts its own unprovability). But if the HC proves a false statement, it is inconsistent—contradicting our assumption. Therefore, the HC *cannot* prove G.

But if the HC cannot prove G, then G is *true*. The sentence accurately describes the system's limitation. G is a truth the system can articulate (it is a well-formed statement in the language of arithmetic) but cannot demonstrate.

This is the incompleteness. The HC, despite its expressive power, despite its capacity to prove commutativity, to verify Euclid's argument, to reason across domains, cannot reach G. The formalized strategies—no matter how sophisticated, no matter how grounded in embodied practice—cannot exhaust the mathematical reality they describe.

And this is not a deficiency. This is the structure of the *in*finite.

Incompleteness as the *In*finite: Self-Transcendence

The Hegelian *in*finite is not endlessness. It is not the "bad infinite" of endless iteration $(1, 2, 3, \ldots)$. The *in*finite is the capacity to relate to oneself *as* finite—to recognize one's own boundary and, in that recognition, to transcend it.

A. The proof establishes that divisibility is PR via bounded quantification, that primality is PR, and that the entire `Transition` predicate is PR as a finite disjunction of PR rules.

The Gödel sentence G formalizes this structure. G is generated *by* the system (it is expressible in the language of the HC). Yet G points *beyond* the system's deductive reach. The system contains within itself the mechanism to articulate its own limit. Recognizing the truth of G—seeing that it must be true because the system cannot prove it—requires stepping outside the current formalization. It requires the "I" to transcend the "me."

Consider the parallel in the HC's own architecture. The file `incompatibility_semantics.pl` includes axioms designed to be transcended. The rule for subtraction in $\mathbb{N}$ states that for $a - b$, we must have $b < a$. This constraint is an axiom of the natural number domain. But when a student encounters $3 - 8$, the system triggers a *normative crisis*. The predicate `is_incoherent` fires. The system recognizes its own boundary: "I cannot subtract a larger number from a smaller one *here*." This recognition is not failure. It is the opening. The crisis drives the transition to $\mathbb{Z}$, where the constraint is lifted, where negative numbers become intelligible.

The Gödelian incompleteness provides the mathematical necessity for this transcendence. Every finite formalization of the "me" will generate its own G—a statement that points beyond. The "I" names the gap that drives learning forward. The horizon is necessarily open because the system is *built to recognize its own limits.*

This is what it means to be *in*finite. Not to possess unlimited knowledge. Not to escape finitude. But to possess the capacity—the necessity—to break the boundaries we build.

Educational and Political Implications: Against the Finite Vessel

The reductive ideology I oppose assumes finite closure. It assumes that mathematical understanding *is nothing more than* the mastery of a fixed set of procedures. Students are vessels to be filled. Curricula are content to be delivered. Assessment measures retention. The "science of math education" seeks optimal transmission protocols.

But elementary arithmetic—the mathematics invented by children from embodied practice—is *necessarily incomplete.* I did not impose incompleteness on the HC. I formalized the strategies, and incompleteness emerged as a structural feature. The origins of mathematical understanding already possess this openness.

The profundity lies in *what* was formalized. Not an arbitrary logical system. Not a mathematician's construction. The HC captures the cognitive moves of students learning to count by grouping objects rhythmically, to add by compressing and elaborating collections, to multiply by chaining iterations. These are the strategies that emerge when children—embodied, situated, socially embedded—engage with quantity. And these strategies, when formalized, yield a system subject to Gödel's theorem.

We are not vessels. We are boundary-recognizers. The normative crises we encounter ($3-8$ in $\mathbb{N}$, $\sqrt{2}$ as a ratio, the need for i to solve $x^2+1=0$) are not deficiencies to be remediated. They are the engine of mathematical invention. Incompleteness proves there is always "something more." The claim that education should aim for completeness—that students should master a closed curriculum—is mathematically incoherent.

The political stakes are clear. Reductive pedagogies that emphasize rote mastery, that valorize standardized assessment, that frame mathematics as a fixed body of knowledge, rest on a false assumption. They assume closure where incompleteness is necessary. They treat transcendence as deviance. They mistake the recognition of boundaries for failure.

But we are those who break boundaries. The "I" that invents a new strategy, that bootstraps a new domain, that grasps Euclid's proof of infinite primes—this "I" enacts the self-transcendence formalized by G. Mathematics education, if it is to honor what students actually do, must cultivate this capacity. Not the transmission of procedures, but the recognition that every formalization points beyond itself.

Coda: Built to Break

I return to where I began: to my father, to the song, to the eulogy.

10.1 The Sound of Time Revisited: A Final Verse

As this concludes, this image returns, deepened by the exploration of critical mathematics and enriched by the final verse of the song that has accompanied the journey.

A few years back, I eulogized

Someone whose death broke a powerful light.
An ocean of rainbow listened to me
As I read my eulogy.

Build something for the breaking
Tall thin walls, shivering, quaking
Dad, it's been beautiful, breaking with you
Build and then break, like you taught me to do

This final verse reveals the deeper meaning that has been implicit throughout our exploration. The "powerful light" that was broken by my father's death was not simply extinguished – it was transformed, refracted into an "ocean of rainbow" that encompasses and transcends what was lost. This is the movement of dialectical transcendence that we have traced through mathematical proofs, philosophical arguments, and personal transformations.

The final line – "Build and then break, like you taught me to do" – captures the essential insight of critical mathematics. We construct systems of understanding not as permanent monuments but as temporary scaffolding that enables further construction. The willingness to break what we have built, to let go of certainties that have served their purpose, is not a rejection of achievement but its highest expression.

Mathematics, like music, is a temporal art. It unfolds in time through the inferential movements of mathematical judgment, through the developmental trajectories of mathematical learning, through the historical evolution of mathematical concepts. Mathematics, like music, involves both determination and transcendence, both the creation of definite forms and the release from those forms into new possibilities.

The sound of time in mathematics is the sound of concepts moving along inference chains, of understanding evolving through the experience of error, of consciousness grasping its own conditions of possibility. It is the sound of the finite resonating with the *in*finite, of the sayable gesturing toward the unsayable, of the representable embracing the unrepresentable.

This sound is not separate from the sounds of everyday life, from the rhythms of work and play, from the melodies of conversation and storytelling, from the harmonies of social interaction. It is not a rarefied music accessible only to mathematical specialists but a variation on the basic rhythms that structure all human experience.

The formalization was an act of building. I constructed the HC with care—state machines, transition rules, grounded semantics, axioms spanning arithmetic and geometry and number theory. I encoded the strategies invented by children. I verified the operations. I proved the system interprets Robinson Arithmetic. I arithmetized the mechanics via Gödel numbering. I demonstrated primitive recursion. I built something rigorous, something mathematically sound.

And then I applied the theorem. I showed that the system I built—this careful, grounded, empirically motivated formalization—necessarily contains a truth it cannot prove. I built the HC *to break it open.* The incompleteness is not a flaw. It is the revelation. The formal system points beyond itself because that is what formal systems *do* when they are rich enough to matter.

The Gödel sentence G is the horizon the HC reveals but cannot cross. The "ocean of rainbow"—the equivalence class of enabling conditions, the irreducible plurality of lived experience, the diversity that formal systems cannot totalize—remains beyond. The null set $\emptyset$ as the groundless ground, the absence that makes presence possible, cannot be captured by the "me." It is the condition of the "I."

This manuscript has been an act of breaking with you, Dad. Breaking in the sense of rupture—the kind of self-transcendence that incompleteness formalizes. Breaking in the sense of recognition—seeing the boundary, naming it, and understanding that the boundary is not a wall but an opening. Breaking in the sense of grief—the recognition that you are gone, that the "I-You" is now a recollection, that the enabling conditions can be honored but not recovered.

And breaking in the sense of continuation. The sound of time continues. The formalization reveals its own horizon. The students I teach will encounter their own normative crises, their own G, their own need to transcend the "me" they have articulated. Mathematics is *in*finite not because it is endless, but because every articulation contains the seeds of its own transcendence.

I built the Hermeneutic Calculator to demonstrate what you taught me: that we are not defined by the systems we construct. We are defined by our capacity to recognize those systems' limits and to move beyond them. The incompleteness theorem is not a conclusion. It is an opening. It says: there is always something more. The horizon is open. We are built to break.

It's been beautiful.

Bibliography

Abbott, E. A. (2020, April). *Flatland: A Romance of Many Dimensions by Edwin A. Abbott*. Independently published.

Agamben, G. (2006, September). *Language and Death: The Place of Negativity* (K. Pinkus & M. Hardt, Trans.; First Edition). Univ Of Minnesota Press.

Aristotle. (2017). *Metaphysics: Aristotle's metaphysics in english, latin and ancient greek: Trilingual edition* (W. o. Moerbeke & W. D. Ross, Trans.; Trilingual). Hermes Language Reference.

Boote, D. N., & Beile, P. (2005). Scholars before researchers: On the centrality of the dissertation literature review in research preparation. *Educational researcher*, *34*, 3–15.

Brandom, R. B. (1994). *Making it explicit: Reasoning, representing, and discursive commitment*. Harvard university press.

Brandom, R. B. (2000). *Articulating reasons: An introduction to inferentialism*. Harvard University Press.

Brandom, R. B. (2008). *Between saying and doing: Towards an analytic pragmatism*. Oxford University Press.

Brandom, R. B. (2019). *A spirit of trust: A reading of Hegel's phenomenology*. The Belknap Press of Harvard University Press.

Bueno, O. (2020). Nominalism in the Philosophy of Mathematics. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Fall 2020). Metaphysics Research Lab, Stanford University.

Cantor, G. (1891). Ueber eine elementare frage der mannigfaltigkeitslehre. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, *1*, 75–78.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). Children's mathematics: Cognitively guided instruction - videotape logs [supplementary material]. In *Children's mathematics: Cogni-*

tively guided instruction. Heinemann, in association with The National Council of Teachers of Mathematics, Inc.

Carspecken, P. F. (1995). *Critical ethnography in educational research: A theoretical and practical guide*. Routledge.

Carspecken, P. F. (1999). *Four scenes for posing the question of meaning and other essays in critical philosophy and critical methodology* (New edition). Peter Lang Inc., International Academic Publishers.

Carspecken, P. F. (2006). Limits of knowledge in the physical sciences. In K. Tobin & J. Kincheloe (Eds.), *Doing educational research* (pp. 405–437, Vol. 1). Sense Publishers.

Carspecken, P. F. (2009). Limits to knowledge with existential significance: An outline for the exploration of post-secular spirituality of relevance to qualitative research. In *Bridging the gap between theory and practice in educational research* (pp. 47–61). Springer.

Carspecken, P. F. (2016). Limits to knowledge and being human: What is 'critical' in critical social research and theory? In P. H. Hinchey (Ed.), *A critical action research reader* (pp. 65–86). Peter Lang Inc., International Academic Publishers.

Carspecken, P. F. (2018). The missing infinite. In R. Winkle-Wagner, J. Lee-Johnson, & T. Gaskew (Eds.), *Critical theory and qualitative data analysis in education* (p. 32). Routledge. https://doi.org/10.4324/9781315158860-2

Carspecken, P. F., & Zhang, R. (2013). Content inference fields in intersubjective space: Transpersonal position, illocution, and logic in the analysis of human interactions. *Qualitative research: A reader in philosophy, core concepts, and practice*, 201–242.

Carspecken, R. (2024, November 3). Untitled.

Caygill, H. (2004). *A kant dictionary*. Wiley-Blackwell.

Derrida, J. (1973). *Speech and phenomena, and other essays on husserl's theory of signs*. Northwestern University Press.

Derrida, J. (1982). *Marges de la philosophie* [Margins of philosophy] (A. Bass, Trans.). The Harvester Press. (Original work published 1972)

Deutsch, H., Marshall, O., & Irvine, A. D. (2025). Russell's Paradox. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford encyclopedia of philosophy* (Spring 2025). Metaphysics Research Lab, Stanford University.

Ernest, P. (2024). The mathematical and philosophical significance of zero: Nought matters [Publisher: Paul Ernest, Emeritus Professor at University of Exeter, School of Education & Lifelong Learning]. *Philoso-*

phy of Mathematics Education Journal, 1–35. https://research.ebsco.com/linkprocessor/plink?id=b685ecb9-d2fc-3e03-889d-a06fb1751c1b

Euclid. (2007, December). *Euclid's Elements of Geometry* (J. L. Heiberg, Ed.; R. Fitzpatrick, Trans.; Revised and corrected edition). Richard Fitzpatrick.

Franzén, T. (2004, September). *Inexhaustibility: A Non-Exhaustive Treatment: Lecture Notes in Logic 16* (1st edition). A K Peters/CRC Press.

Franzén, T. (2005). *Gödel's theorem: An incomplete guide to its use and abuse*. AK Peters/CRC Press.

Frege, G. (1997, July). *The Frege Reader* (M. Beaney, Ed.; 1st edition). Blackwell Publishing.

Gaifman, H. (2005). *The easy way to Gödel's proof and related matters* [Unpublished manuscript. Available online].

Geisel, T. (1961). *The Sneetches and Other Stories* (First edition.). Random House Books for Young Readers.

Gelder, T. V. (1995). What might cognition be, if not computation? *The Journal of Philosophy*, *92*, 345–381. Retrieved April 13, 2025, from http://www.jstor.org/stable/2941061

Gray, L., & Garzon, M. (2005). On a hermite boundary integral approximation. *Computers & structures*, *83*(10-11), 889–894.

Greenblatt, R., Denison, C., Wright, B., Roger, F., MacDiarmid, M., Marks, S., Treutlein, J., Belonax, T., Chen, J., Duvenaud, D., Khan, A., Michael, J., Mindermann, S., Perez, E., Petrini, L., Uesato, J., Kaplan, J., Shlegeris, B., Bowman, S. R., & Hubinger, E. (2024, December 23). Alignment faking in large language models. https://doi.org/10.48550/arXiv.2412.14093

Habermas, J. (1985). *The theory of communicative action: Lifeworld and system* (Vol. 2). Beacon Press.

Habermas, J. (2004). *The philosophical discourse of modernity: Twelve lectures* (F. G. Lawrence, Trans.). MIT Press.

Habermas, J. (1971). *Knowledge and human interests*. Beacon Press.

Habermas, J. (1984). *The theory of communicative action, volume 1: Reason and the rationalization of society* (T. McCarthy, Trans.; Reprint edition). Beacon Press.

Habermas, J. (1995). *Postmetaphysical thinking: Between metaphysics and the critique of reason* (1st edition). Polity.

Hackenberg, A. (2025). *Course notes* [Unpublished course notes].

Hegel, G. W. F. (1977). *Phenomenology of spirit* (A. V. Miller, Trans.; Vol. 632). Oxford University Press Oxford.

Hegel, G. W. F. (2010). *The science of logic* (G. di Giovanni, Ed. & Trans.). Cambridge University Press.

Houlgate, S. (2006). *The opening of hegel's logic: From being to infinity*. Purdue University Press.

Hyppolite, J. (1974). *Genesis and structure of hegel's phenomenology of spirit*. Northwestern University Press.

Kant, I. (1998). *Critique of pure reason* [Original work published 1781/1787, Guyer and Wood, Trans.]. Cambridge University Press.

Lakoff, G., & Johnson, M. (1980). *Metaphors we live by*. University of Chicago Press.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books.

Linnebo, Ø. (2024). Platonism in the Philosophy of Mathematics. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford encyclopedia of philosophy* (Summer 2024). Metaphysics Research Lab, Stanford University.

Longuenesse, B. (2017). *I, me, mine: Back to kant and fichte on the 'i'*. Oxford University Press.

Mead, G. H. (1934). *The social psychology of george herbert mead* (A. L. Strauss, Ed.). University of Chicago Press.

Mead, G. H. (2015). *Mind, self, and society: The definitive edition* (H. Joas, D. R. Huebner, & C. W. Morris, Eds.; Enlarged edition). University of Chicago Press.

Meyer, J. (title). *Cantors 1891 diagonal proof* [Cantor's original 1891 diagonal proof]. Retrieved July 6, 2025, from https://jamesrmeyer.com/infinite/cantors-original-1891-proof

Nagel, E., & Newman, J. R. (2012). *Godel's proof*. Routledge.

Negarestani, R. (2018). *Intelligence and Spirit*. Urbanomic/Sequence Press.

Poston, T., & Stewart, I. (2012, February 29). *Catastrophe theory and its applications* (First Edition). Dover Publications.

Priest, G. (2002). *Beyond the limits of thought*. Oxford University Press.

Priest, G. (2014). *One: Being an investigation into the unity of reality and of its parts, including the singular object which is nothingness*. Oxford University Press.

Quine, W. V. O. (1948). On what there is. In *Thinking about logic* (pp. 189–209). Catholic University of America. https://sites.pitt.edu/~rbrandom/

Courses/2023%20Sellars/Sellars%20texts/Quine%20On_What_There_Is.pdf

Quine, W. V. O. (1963). Two dogmas of empiricism [First Harper Torchbook edition, a reprint of the second, revised edition (1961). The book was first published by Harvard University Press in 1953.]. In *From a logical point of view* (Second, revised, pp. 20–46). Harper & Row.

Rödl, S. (2014). Intentional transaction [Accessed: 2025-04-13]. *Philosophical Explorations: An International Journal for the Philosophy of Mind and Action*, *17*, 304–316. https://doi.org/10.1080/13869795.2014.941909

Rose, G. (2009). *Hegel contra sociology*. Verso.

Rowlands, S. (2005). The contradictions in the constructivist discourse [Citing von Glasersfeld]. *Philosophy of Mathematics Education Journal*, *14*.

Savich, T. M. (2020). Towards a critical mathematics [Accessed via ERIC]. *Mathematics Teaching–Research Journal Online*, *12*, 245–291. https://files.eric.ed.gov/fulltext/EJ1384454.pdf

Savich, T. M. (2022). *Towards a critical mathematics*. Indiana University.

Savich, T. M. (2024). Built to break: An introduction to theoretical autoethnography [Invited talk presented at Solent University, organized by Dr. Brian McDonough]. https://iu.mediaspace.kaltura.com/media/t/1_v1qmm3g9

Savich, T. M. (title). *The hermeneutic calculator* [The hermeneutic calculator]. Retrieved July 23, 2025, from https://tiosavich.github.io/UMEDCTA/Calculator/index.html

Sellars, W. (1953). Inference and meaning. *Mind*, *62*, 313–338.

Sellars, W. (2007). *In the space of reasons: Selected essays of wilfrid sellars*. Harvard University Press.

Steffe, L. P. (2002). A new hypothesis concerning children's fractional knowledge [Accessed through CEPA.INFO]. *Journal of Mathematical Behavior*, *20*(3), 267–307.

Steffe, L. P. (2003). On the construction of learning trajectories of children: The case of commensurate fractions [Accessed via ResearchGate/ERIC]. *Monographs of the Journal for Research in Mathematics Education*, 1–26.

Steffe, L. P. (2004). Fractional commensurate, composition, and adding schemes: Learning trajectories of jason and laura: Grade 5 [Accessed via ResearchGate]. In J. Olive & R. Tzur (Eds.), *Children's fractional knowledge* (pp. 1–60). American Educational Research Association.

Steffe, L. P. (2014). *Children's number sequences: An explanation of steffe's constructs and an extrapolation to rational numbers of arithmetic* [Originally from 1994]. Unpublished manuscript, available via CiteSeerX; ResearchGate.

Steffe, L. P., & Thompson, P. W. (2014). Teaching experiment methodology: Underlying principles and essential elements [Accessed via ResearchGate, originally published 2000]. *Proceedings of the 24th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*.

Stone, J. (2003). *The monster at the end of this book* (2nd ed. edition). Golden Books.

Taylor, C. (1994). *Multiculturalism: Examining the Politics of Recognition* (A. Gutmann, Ed.) [Essays by Charles Taylor, K. Anthony Appiah, Jürgen Habermas, Steven C. Rockefeller, Michael Walzer, and Susan Wolf]. Princeton University Press.

van Hiele-Geldof, D., & van Hiele, P. M. (1984). *English translation of selected writings of dina van hiele-geldof and pierre m. van hiele* (D. Fuys, D. Geddes, & D. Tischler, Eds.) [ERIC Number: ED287697]. Retrieved June 3, 2025, from https://eric.ed.gov/?id=ED287697

Wittgenstein, L. (1956). *Remarks on the foundations of mathematics*.

Wittgenstein, L. (2010). *Philosophical investigations* (P. M. S. Hacker & J. Schulte, Eds.; G. Anscombe, Trans.). Wiley-Blackwell.

.1 Appendix A: Primitive Recursion and the Transition Predicate

Primitive Recursive Functions: Foundations

Primitive Recursive (PR) functions are built from a simple base and closed under specific operations:

Base Functions:

- Zero: $Z(x) = 0$
- Successor: $S(x) = x + 1$
- Projection: $P_i^n(x_1, \ldots, x_n) = x_i$

Closure Operations:

- **Composition:** If $g, h_1, \ldots, h_m$ are PR, then $f(\vec{x}) = g(h_1(\vec{x}), \ldots, h_m(\vec{x}))$ is PR.
- **Primitive Recursion:** If g and h are PR, then f defined by:

$$f(\vec{x}, 0) = g(\vec{x})$$
$$f(\vec{x}, y+1) = h(\vec{x}, y, f(\vec{x}, y))$$

  is PR.

It is well-established that addition $(x + y)$, multiplication $(x \cdot y)$, and exponentiation (x^y) are all PR. Furthermore, comparisons $(x = y,\ x < y)$ and Boolean operations $(\neg,\ \wedge,\ \vee)$ are PR.

Bounded Minimization: Crucially, PR functions are closed under bounded minimization. If $P(x, \vec{y})$ is a PR predicate, then the function:

$$F(\vec{y}, B) = \mu x \leq B[P(x, \vec{y})]$$

("the smallest $x \leq B$ such that $P(x, \vec{y})$ is true") is also PR.

Divisibility, Primality, and Prime Extraction

Divisibility is PR: The predicate $x|y$ ("x divides y") can be defined using bounded quantification:

$$x|y \iff \exists k \leq y\,(x \cdot k = y)$$

Since the search for k is bounded by y, and multiplication and equality are PR, divisibility is PR.

Primality is PR: The predicate Prime(x) checks whether x has divisors other than 1 and x, which involves checking divisibility for all values up to x. This is a bounded operation, so primality is PR.

The n^{th} Prime is PR: The function P_n (the n^{th} prime number) is PR. The search for the next prime is bounded (e.g., by $(P_{n-1})!+1$, though tighter bounds exist).

Extracting Exponents: The Rigorous Step

We now prove that $\exp_p(N)$—the function returning the exponent of prime p in the factorization of N—is Primitive Recursive.

We seek the exponent e such that p^e divides N but p^{e+1} does not. We establish a bound: since $p \geq 2$, we have $2^e \leq N$, which implies $e \leq N$. Therefore:

$$\exp_p(N) = \mu e \leq N\, [\neg(p^{e+1}|N)]$$

This reads: "The smallest e less than or equal to N such that p^{e+1} does not divide N."

Since exponentiation (p^{e+1}) is PR, divisibility ($|$) is PR, and negation ($\neg$) is PR, the predicate inside the minimization is PR. Because the minimization is bounded by N, the entire function $\exp_p(N)$ is Primitive Recursive.

The Transition Predicate is Primitive Recursive

Consider the C2C multiplication strategy. A configuration C is encoded as:

$$g(C) = 2^{g(\text{State})} \cdot 3^{g(G)} \cdot 5^{g(I)} \cdot 7^{g(T)} \cdot 11^{g(N)} \cdot 13^{g(S)}$$

The Counting Rule states: "If State $= q_{\text{count}}$ and $I < S$, then $I' = I + 1$ and $T' = T + 1$."

We define $\text{Rule}_{\text{Count}}(X, Y)$ as:

$$\text{Rule}_{\text{Count}}(X, Y) \iff \text{Condition}(X) \wedge \text{Update}(X, Y)$$

Condition is PR:

$$\begin{aligned} \text{Condition}(X) \iff & (\exp_2(X) = g(q_{\text{count}})) \wedge \\ & (\exp_5(X) < \exp_{13}(X)) \end{aligned}$$

Since $\exp_p(N)$, equality, comparison, and conjunction are PR, the Condition is PR.

Update is PR:

$$\begin{aligned} \text{Update}(X, Y) \iff & (\exp_5(Y) = \exp_5(X) + 1) \wedge \\ & (\exp_7(Y) = \exp_7(X) + 1) \wedge \\ & (\exp_2(Y) = \exp_2(X)) \wedge \dots \end{aligned}$$

Since $\exp_p(N)$, addition, and equality are PR, the Update is PR.

Therefore, $\text{Rule}_{\text{Count}}(X, Y)$ is PR.

The Full Transition Predicate: The complete Transition(X, Y) predicate is the finite disjunction of all rules:

$$\text{Transition}(X, Y) \iff \text{Rule}_1(X, Y) \vee \text{Rule}_2(X, Y) \vee \cdots \vee \text{Rule}_K(X, Y)$$

Since PR predicates are closed under finite disjunction, Transition(X, Y) is Primitive Recursive.

Conclusion: Self-Reference is Possible

By rigorously demonstrating that the decoding operation $\exp_p(N)$ is Primitive Recursive via bounded minimization, we have established that:

1. The mechanics of the HC (the Transition predicate) are Primitive Recursive.
2. The HC is sufficiently expressive to represent all Primitive Recursive functions.
3. Therefore, the HC can represent its own mechanics—self-reference is possible.

This provides the mathematical foundation for applying Gödel's First Incompleteness Theorem to the Hermeneutic Calculator, securing the intellectual and rhetorical payoff: that the formalized system of student-invented strategies is necessarily incomplete.

References

Abbott, E. A. (2020, April). *Flatland: A Romance of Many Dimensions by Edwin A. Abbott*. Independently published.

Agamben, G. (2006, September). *Language and Death: The Place of Negativity* (K. Pinkus & M. Hardt, Trans.; First Edition). Univ Of Minnesota Press.

Aristotle. (2017). *Metaphysics: Aristotle's metaphysics in english, latin and ancient greek: Trilingual edition* (W. o. Moerbeke & W. D. Ross, Trans.; Trilingual). Hermes Language Reference.

Boote, D. N., & Beile, P. (2005). Scholars before researchers: On the centrality of the dissertation literature review in research preparation. *Educational researcher*, *34*, 3–15.

Brandom, R. B. (1994). *Making it explicit: Reasoning, representing, and discursive commitment*. Harvard university press.

Brandom, R. B. (2000). *Articulating reasons: An introduction to inferentialism*. Harvard University Press.

Brandom, R. B. (2008). *Between saying and doing: Towards an analytic pragmatism*. Oxford University Press.

Brandom, R. B. (2019). *A spirit of trust: A reading of Hegel's phenomenology*. The Belknap Press of Harvard University Press.

Bueno, O. (2020). Nominalism in the Philosophy of Mathematics. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Fall 2020). Metaphysics Research Lab, Stanford University.

Cantor, G. (1891). Ueber eine elementare frage der mannigfaltigkeitslehre. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, *1*, 75–78.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). Children's mathematics: Cognitively guided instruction - videotape logs [supplementary material]. In *Children's mathematics: Cognitively guided instruction*. Heinemann, in association with The National Council of Teachers of Mathematics, Inc.

Carspecken, P. F. (1995). *Critical ethnography in educational research: A theoretical and practical guide*. Routledge.

Carspecken, P. F. (1999). *Four scenes for posing the question of meaning and other essays in critical philosophy and critical methodology* (New edition). Peter Lang Inc., International Academic Publishers.

Carspecken, P. F. (2006). Limits of knowledge in the physical sciences. In K. Tobin & J. Kincheloe (Eds.), *Doing educational research* (pp. 405–437, Vol. 1). Sense Publishers.

Carspecken, P. F. (2009). Limits to knowledge with existential significance: An outline for the exploration of post-secular spirituality of relevance to qualitative research. In *Bridging the gap between theory and practice in educational research* (pp. 47–61). Springer.

Carspecken, P. F. (2016). Limits to knowledge and being human: What is 'critical' in critical social research and theory? In P. H. Hinchey (Ed.), *A critical action research reader* (pp. 65–86). Peter Lang Inc., International Academic Publishers.

Carspecken, P. F. (2018). The missing infinite. In R. Winkle-Wagner, J. Lee-Johnson, & T. Gaskew (Eds.), *Critical theory and qualitative data analysis in education* (p. 32). Routledge. https://doi.org/10.4324/9781315158860-2

Carspecken, P. F., & Zhang, R. (2013). Content inference fields in intersubjective space: Transpersonal position, illocution, and logic in the analysis of human interactions. *Qualitative research: A reader in philosophy, core concepts, and practice*, 201–242.

Carspecken, R. (2024, November 3). Untitled.

Caygill, H. (2004). *A kant dictionary*. Wiley-Blackwell.

Derrida, J. (1973). *Speech and phenomena, and other essays on husserl's theory of signs*. Northwestern University Press.

Derrida, J. (1982). *Marges de la philosophie* [Margins of philosophy] (A. Bass, Trans.). The Harvester Press. (Original work published 1972)

Deutsch, H., Marshall, O., & Irvine, A. D. (2025). Russell's Paradox. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford encyclopedia of philosophy* (Spring 2025). Metaphysics Research Lab, Stanford University.

Ernest, P. (2024). The mathematical and philosophical significance of zero: Nought matters [Publisher: Paul Ernest, Emeritus Professor at University of Exeter, School of Education & Lifelong Learning]. *Philosophy of Mathematics Education Journal*, 1–35. https://research.ebsco.com/linkprocessor/plink?id=b685ecb9-d2fc-3e03-889d-a06fb1751c1b

Euclid. (2007, December). *Euclid's Elements of Geometry* (J. L. Heiberg, Ed.; R. Fitzpatrick, Trans.; Revised and corrected edition). Richard Fitzpatrick.

Franzén, T. (2004, September). *Inexhaustibility: A Non-Exhaustive Treatment: Lecture Notes in Logic 16* (1st edition). A K Peters/CRC Press.

Franzén, T. (2005). *Gödel's theorem: An incomplete guide to its use and abuse*. AK Peters/CRC Press.

Frege, G. (1997, July). *The Frege Reader* (M. Beaney, Ed.; 1st edition). Blackwell Publishing.

Gaifman, H. (2005). *The easy way to Gödel's proof and related matters* [Unpublished manuscript. Available online].

Geisel, T. (1961). *The Sneetches and Other Stories* (First edition.). Random House Books for Young Readers.

Gelder, T. V. (1995). What might cognition be, if not computation? *The Journal of Philosophy*, *92*, 345–381. Retrieved April 13, 2025, from http://www.jstor.org/stable/2941061

Gray, L., & Garzon, M. (2005). On a hermite boundary integral approximation. *Computers & structures*, *83*(10-11), 889–894.

Greenblatt, R., Denison, C., Wright, B., Roger, F., MacDiarmid, M., Marks, S., Treutlein, J., Belonax, T., Chen, J., Duvenaud, D., Khan, A., Michael, J., Mindermann, S., Perez, E., Petrini, L., Uesato, J., Kaplan, J., Shlegeris, B., Bowman, S. R., & Hubinger, E. (2024, December 23). Alignment faking in large language models. https://doi.org/10.48550/arXiv.2412.14093

Habermas, J. (1985). *The theory of communicative action: Lifeworld and system* (Vol. 2). Beacon Press.

Habermas, J. (2004). *The philosophical discourse of modernity: Twelve lectures* (F. G. Lawrence, Trans.). MIT Press.

Habermas, J. (1971). *Knowledge and human interests*. Beacon Press.

Habermas, J. (1984). *The theory of communicative action, volume 1: Reason and the rationalization of society* (T. McCarthy, Trans.; Reprint edition). Beacon Press.

Habermas, J. (1995). *Postmetaphysical thinking: Between metaphysics and the critique of reason* (1st edition). Polity.

Hackenberg, A. (2025). *Course notes* [Unpublished course notes].

Hegel, G. W. F. (1977). *Phenomenology of spirit* (A. V. Miller, Trans.; Vol. 632). Oxford University Press Oxford.

Hegel, G. W. F. (2010). *The science of logic* (G. di Giovanni, Ed. & Trans.). Cambridge University Press.

Houlgate, S. (2006). *The opening of hegel's logic: From being to infinity*. Purdue University Press.

Hyppolite, J. (1974). *Genesis and structure of hegel's phenomenology of spirit*. Northwestern University Press.

Kant, I. (1998). *Critique of pure reason* [Original work published 1781/1787, Guyer and Wood, Trans.]. Cambridge University Press.

Lakoff, G., & Johnson, M. (1980). *Metaphors we live by*. University of Chicago Press.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books.

Linnebo, Ø. (2024). Platonism in the Philosophy of Mathematics. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford encyclopedia of philosophy* (Summer 2024). Metaphysics Research Lab, Stanford University.

Longuenesse, B. (2017). *I, me, mine: Back to kant and fichte on the 'i'*. Oxford University Press.

Mead, G. H. (1934). *The social psychology of george herbert mead* (A. L. Strauss, Ed.). University of Chicago Press.

Mead, G. H. (2015). *Mind, self, and society: The definitive edition* (H. Joas, D. R. Huebner, & C. W. Morris, Eds.; Enlarged edition). University of Chicago Press.

Meyer, J. (title). *Cantors 1891 diagonal proof* [Cantor's original 1891 diagonal proof]. Retrieved July 6, 2025, from https://jamesrmeyer.com/infinite/cantors-original-1891-proof

Nagel, E., & Newman, J. R. (2012). *Godel's proof*. Routledge.

Negarestani, R. (2018). *Intelligence and Spirit*. Urbanomic/Sequence Press.

Poston, T., & Stewart, I. (2012, February 29). *Catastrophe theory and its applications* (First Edition). Dover Publications.

Priest, G. (2002). *Beyond the limits of thought*. Oxford University Press.

Priest, G. (2014). *One: Being an investigation into the unity of reality and of its parts, including the singular object which is nothingness*. Oxford University Press.

Quine, W. V. O. (1948). On what there is. In *Thinking about logic* (pp. 189–209). Catholic University of America. https://sites.pitt.edu/~rbrandom/Courses/2023%20Sellars/Sellars%20texts/Quine%20On_What_There_Is.pdf

Quine, W. V. O. (1963). Two dogmas of empiricism [First Harper Torchbook edition, a reprint of the second, revised edition (1961). The book was first published by Harvard University Press in 1953.]. In *From a logical point of view* (Second, revised, pp. 20–46). Harper & Row.

Rödl, S. (2014). Intentional transaction [Accessed: 2025-04-13]. *Philosophical Explorations: An International Journal for the Philosophy of Mind and Action*, *17*, 304–316. https://doi.org/10.1080/13869795.2014.941909

Rose, G. (2009). *Hegel contra sociology*. Verso.

Rowlands, S. (2005). The contradictions in the constructivist discourse [Citing von Glasersfeld]. *Philosophy of Mathematics Education Journal*, *14*.

Savich, T. M. (2020). Towards a critical mathematics [Accessed via ERIC]. *Mathematics Teaching–Research Journal Online*, *12*, 245–291. https://files.eric.ed.gov/fulltext/EJ1384454.pdf

Savich, T. M. (2022). *Towards a critical mathematics*. Indiana University.

Savich, T. M. (2024). Built to break: An introduction to theoretical autoethnography [Invited talk presented at Solent University, organized by Dr. Brian McDonough]. https://iu.mediaspace.kaltura.com/media/t/1_v1qmm3g9

Savich, T. M. (title). *The hermeneutic calculator* [The hermeneutic calculator]. Retrieved July 23, 2025, from https://tiosavich.github.io/UMEDCTA/Calculator/index.html

Sellars, W. (1953). Inference and meaning. *Mind*, *62*, 313–338.

Sellars, W. (2007). *In the space of reasons: Selected essays of wilfrid sellars*. Harvard University Press.

Steffe, L. P. (2002). A new hypothesis concerning children's fractional knowledge [Accessed through CEPA.INFO]. *Journal of Mathematical Behavior*, *20*(3), 267–307.

Steffe, L. P. (2003). On the construction of learning trajectories of children: The case of commensurate fractions [Accessed via ResearchGate/ERIC]. *Monographs of the Journal for Research in Mathematics Education*, 1–26.

Steffe, L. P. (2004). Fractional commensurate, composition, and adding schemes: Learning trajectories of jason and laura: Grade 5 [Accessed via ResearchGate]. In J. Olive & R. Tzur (Eds.), *Children's fractional knowledge* (pp. 1–60). American Educational Research Association.

Steffe, L. P. (2014). *Children's number sequences: An explanation of steffe's constructs and an extrapolation to rational numbers of arithmetic* [Originally from 1994]. Unpublished manuscript, available via CiteSeerX; ResearchGate.

Steffe, L. P., & Thompson, P. W. (2014). Teaching experiment methodology: Underlying principles and essential elements [Accessed via ResearchGate, originally published 2000]. *Proceedings of the 24th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.*

Stone, J. (2003). *The monster at the end of this book* (2nd ed. edition). Golden Books.

Taylor, C. (1994). *Multiculturalism: Examining the Politics of Recognition* (A. Gutmann, Ed.) [Essays by Charles Taylor, K. Anthony Appiah, Jürgen Habermas, Steven C. Rockefeller, Michael Walzer, and Susan Wolf]. Princeton University Press.

van Hiele-Geldof, D., & van Hiele, P. M. (1984). *English translation of selected writings of dina van hiele-geldof and pierre m. van hiele* (D. Fuys, D. Geddes, & D. Tischler, Eds.) [ERIC Number: ED287697]. Retrieved June 3, 2025, from https://eric.ed.gov/?id=ED287697

Wittgenstein, L. (1956). *Remarks on the foundations of mathematics.*

Wittgenstein, L. (2010). *Philosophical investigations* (P. M. S. Hacker & J. Schulte, Eds.; G. Anscombe, Trans.). Wiley-Blackwell.

Appendix A

Appendix: Modal Logic

- **Subjective Validity** (S, φ)
- **Objective Validity** (O, φ)
- **Normative Validity** (N, φ)

The Exercise is written purely in N – I have a great deal of fear committing any of this to print. "What if someone checks the receipts, and finds that I am a charlatan? I'm no fount of wisdom, I'm a fool!" etc. But what the exercise is developing is an absence, a two-dimensional space of $\mathsf{S} \times \mathsf{O}$, where the subject and object are unified in a single sensation. Somewhat paradoxically, the absence of empirical norms can be identified with the ideal of normativity. I take Hegel's notion of *Spirit* (*Geist*) – a kind of pure normativity that flattens itself into representational form and then sublates those representations – as equally expressed by the rhythm of sensation $\rightarrow$ awareness of sensation $\rightarrow$ sensation.

The absence of normativity is very tricky, in the sense that it can be recollected in many, many ways that are all somewhat contradictory, as they all require words to express.

A.1 A Modal Logic for Embodied Reason

The Exercise reveals a rhythmic structure to consciousness – a movement between compression and decompression, fixation and release. To articulate this structure more precisely, modal logic offers tools that may be useful.

This is not about reducing experience to symbols, but about using symbols expressively – as a tool to make explicit the necessary connections and possibilities inherent in embodied experience.

It's easy to get lost in these abstractions, so let me say again what I am trying to do: I am trying to give a phenomenological grounding for the modal ideas of *possibility* ($\diamond$) and *necessity* ($\Box$). Possibility is associated with opening up and letting go. Necessity is associated with flattening. But there is also a necessary expansion and a possible contraction. So, the traditional symbols and the meanings they carry are not fully adequate to capture the embodied phenomeno-logic.

The Exercise reveals that consciousness operates through a fundamental rhythm: **Temporal Compression** (Contraction), the movement toward finitude, fixation, and judgment; and **Temporal Decompression** (Expansion), the movement toward openness, release, and integration.

To articulate this structure, the analysis employs the modes of validity (Subjective S, Objective O, Normative N). Crucially, the framework enhances the Subjective modality with polarities to reflect the directional, embodied nature of possibility and necessity.

Polarized Subjective Modalities

In embodied experience, necessity and possibility are not abstract; they are felt as movements. The standard modal operators ($\Box$ for necessity, $\Diamond$ for possibility) can be polarized to reflect this felt directionality. Chapter 2 explores polarity in more detail.

1. Compressive Necessity ($\Box_{\mathsf{S}}^{\downarrow}$)

This represents the necessary movement towards contraction (symbolized by the downward arrow $\downarrow$). It is the "closing down" of the field, the movement into determination and finitude.

Phenomenology: The inevitable emergence of awareness from unity; the fixation of thought; the weight of a rigid rule; the feeling of being constrained or "stuck." This corresponds to the "First Negation" in the dialectic – the necessary creation of boundaries.

2. Expansive Possibility ($\Diamond_{\mathsf{S}}^{\uparrow}$)

This represents the potential for expansion or opening (symbolized by the upward arrow $\uparrow$). It is the recognition that fixation can be released and that alternatives exist.

Phenomenology: The recognition that one can let go; the intuition of a deeper unity; seeing a new approach to a problem; the openness required for genuine dialogue.

However, the release experienced when letting go (sublation) is not just a possibility of expansion; it is a necessary expansion once the act of letting go is performed. This requires a third operator:

3. Expansive Necessity ($\Box_S^{\uparrow}$)

This represents the necessary movement towards expansion following an act of release. It captures the subjective certainty of the intensification that occurs through the *second negation* (~~no~~).

Phenomenology: The inevitable softening that follows genuine recognition; the intensified I-feeling that necessarily follows letting go; the certainty and clarity of an "Aha!" moment.

This polarized approach honors the intuition that abstract necessity often feels compressive ($\Box_{\downarrow}$), while also capturing the necessary expansion ($\Box_{\uparrow}$) that drives the dialectic forward.

As I reviewed the logic, I worried that I had missed a core feature of possibility. My initial formulation emphasized the *expansive* nature of letting go ($\Diamond_S^{\uparrow}$), but what about the deflationary moment of acceptance—the "hatchet through the sycamore tree" that clears the field of nonsense possibilities to make way for what is necessary? I discuss a similar notion with a better academic pedigree in the next chapter. *Information* reduces the field of possibilities in a way that doesn't always feel great at the time, but is also ultimately very important for navigating the *actual* world. In one sense, letting go is a shedding of possibility, an acceptance of actuality. In another, it is the emergence of new possibilities.

Thinking this through, I realized that to fully capture the phenomenology of *The Exercise*, a fourth operator is needed to complete the modal symmetry. The chapter describes the constant temptation to grasp or name a sensation, the lure of "picture thinking," and the "Whoops!" moment of falling back into thought. This is the experience of a **compressive possibility**.

4. Compressive Possibility ($\Diamond_S^{\downarrow}$)

This represents the potential for contraction, fixation, or determination (symbolized by the downward arrow $\downarrow$). It is the possibility of error, the lure of the "First Temptation."

Phenomenology: The desire to grasp or name the sensation; the risk of backsliding into fixation; the felt pull toward giving a thought existential weight.

Introducing this operator allows me to more precisely articulate the tension at the crucial "choice point" of awareness (A). In that moment, one is suspended between two opposing pulls: the expansive possibility of release and the compressive possibility of fixation. The logic can now express this explicitly. The drama of *The Exercise* is the cultivation of the discipline to actualize $\Diamond^{\uparrow}_{\mathsf{S}}$ while resisting the lure of $\Diamond^{\downarrow}_{\mathsf{S}}$.

	Expansive: ↑	Compressive ↓
Necessity ($\Box$)	$\Box^{\uparrow}_{\mathsf{S}}$	$\Box^{\downarrow}_{\mathsf{S}}$
Possibility ($\Diamond$)	$\Diamond^{\uparrow}_{\mathsf{S}}$	$\Diamond^{\downarrow}_{\mathsf{S}}$

Table A.1: *Note.* Non-preserving substitution

The Logic of The Exercise Revisited

The dynamics of The Exercise can now be reformulated using these polarized modalities, providing a different sort of account of the embodied rhythm. Does it do a better job than the reflections I provided so far? As I re-read this chapter, it is awfully 'jumpy' – almost like the ZCM model I provided. Can the poetry of logic provide a smoother description, or is it doomed to remain fragmented? I suspect it is doomed. Still, I will attempt to weave these threads together. I will frame these 'rules' as commitments, though logicians would call them axioms. Recall that I provide these with the weight of song and with the hope that it will irritate a logician enough to do a better job than I have.

States: U (Unified Sensation), A (Awareness/Desire), T (Thinking/Fixation), LG (Letting Go), U' (Intensified Unity).

Commitment 1: Let-Go of Particularity (Revisited)

$$(\mathsf{O}(\text{tension}@r) \wedge \mathsf{S}(\text{recognize } r)) \implies \Box^{\uparrow}_{\mathsf{S}}\text{soften}(r)$$

Gloss: Recognizing objective tension necessarily leads to an expansive subjective movement ($\Box^{\uparrow}_{\mathsf{S}}$) (softening). This captures the reliability of the practice.

Commitment 2: The Emergence of Awareness (Temporal Compression)

$$\mathsf{S}(U) \implies \Box^{\downarrow}_{\mathsf{S}}(A)$$

Gloss: The experience of unity (U) necessarily compresses ($\Box^{\downarrow}_{\mathsf{S}}$) into awareness/desire (A). This is the first negation, the automatic movement into determination.

Commitment 3: The Possibility of Release

$$\mathsf{S}(A) \implies \Diamond^{\uparrow}_{\mathsf{S}}(LG)$$

Gloss: The state of awareness/desire (A) opens the expansive possibility ($\Diamond^{\uparrow}_{\mathsf{S}}$) of letting go ($LG$). This is the crucial choice point – the recognition that the compression is not absolute.

Commitment 4: The Dynamics of the Choice

If the possibility of release is not taken, the compression deepens into fixation:

Commitment 4a: Fixation (Deepened Contraction)

$$\mathsf{S}(T) \implies \Box^{\downarrow}_{\mathsf{S}}(\neg U)$$

Gloss: Fixated thinking (T) necessarily leads to a contraction ($\Box^{\downarrow}_{\mathsf{S}}$) that collapses the unified sensation ($\neg U$). This is the "Whoops!" moment – the experience of flattening.

If the possibility of release is taken, sublation occurs:

Commitment 4b: Release (Sublation)

$$\mathsf{S}(LG) \implies \Box^{\uparrow}_{\mathsf{S}}(U')$$

Gloss: Letting go (LG) necessarily leads to an expansive movement ($\Box^{\uparrow}_{\mathsf{S}}$) resulting in intensified unity (U').

In this framework, the "Truth" of the experience is not a static state, but the entire rhythm: the necessary compression (Commitment 2) followed by the necessary expansion (Commitment 4b). Fixation (Commitment 4a) is the moment of abstraction or error – getting stuck in one pole of the movement.

Applications: Analyzing Embodied Dialogue and Insight

This polarized logic provides a precise tool for researchers and educators to analyze the embodied dynamics of interaction and mathematical insight. Reasoning is driven by the management of tension between compression (making a determinate claim, following a rule) and decompression (opening to the other, exploring alternatives).

Example 1: The "Aha!" Moment (Mathematical Insight)

Consider a student struggling to grasp the concept of "variables" in algebra, fixated on the idea that a letter must stand for a specific, unknown number (a carryover from arithmetic).

The State: The student is in a state of Fixation (T). Their understanding is contracted ($\neg U$). They feel the compressive necessity ($\Box^{\downarrow}_{\mathsf{S}}$) of their current model: "But what number is x?"

The Intervention: The teacher introduces the idea of a function machine, showing how different inputs yield different outputs according to a rule. This intervention offers an Expansive Possibility ($\Diamond^{\uparrow}_{\mathsf{S}}$) – a way to "Let Go" ($LG$) of the fixation on a single value.

The Insight: If the student engages the new model, they experience a release. The sudden insight is an Expansive Necessity ($\Box^{\uparrow}_{\mathsf{S}}$): "Oh! '$x$' can be any number in the domain!" This leads to a richer, integrated understanding (U') of variables as representing a range of possibilities.

Example 2: Conversational Friction and Release

This logic is also powerful for analyzing the embodied dynamics of interaction, which researchers often struggle to communicate clearly. Consider a scenario based on a real-world policy change: a professional development session for Indiana math teachers after a new state mandate.

Speaker 1 (Instructional Coach): (Slides an official-looking binder across the table) "The directive from the state superintendent and the legislature is clear. Our mathematics instruction must be finite and explicit to meet the new standards."

Speaker 2 (Veteran Teacher): (Leans back, crossing arms, voice tightens) "But that approach is killing my students' curiosity. If we only teach explicit steps, they never learn to reason for themselves. The engagement in my classroom is plummeting."

Analysis:

Speaker 1 is asserting a Normative claim (N) based on the state mandate, inducing a compression ($\Box_{\downarrow}$) in the dialogue by asserting necessity and closing off pedagogical alternatives. Their physical action (sliding the binder) reinforces this compression.

Speaker 2 experiences a subjective contraction (S) – observable (O) in their defensive posture and tone. They resist the compression with a counter-claim that challenges the mandate's definition of effective instruction, based on their direct classroom experience.

The dialogue is now in a state of high tension (A). Both participants face the choice described in Axiom 3: the possibility of release ($\Diamond^{\uparrow}_{\mathsf{S}}(LG)$).

Scenario A (Fixation): The coach doubles down: "It doesn't matter what we think works best. This is a state mandate. Our job is to implement it with fidelity." This is a move to T. The dialogue contracts further ($\Box^{\downarrow}_{\mathsf{S}}(\neg U)$), and shared understanding collapses. Trust between the coach and teacher is eroded.

Scenario B (Release): The coach pauses, takes a breath, and pushes the binder aside. "Okay, I hear your concern about engagement. That's a valid point. Help me understand. Within the constraints of this mandate, where can we still create space for students to reason? What would that look like in your classroom?" This is an act of Letting Go (LG). It initiates a necessary expansion ($\Box^{\uparrow}_{\mathsf{S}}$), decompressing the conversational space and moving towards a potential integration (U').

This polarized modal logic provides researchers with a vocabulary to capture not just what is said, but the embodied dynamics – the pushes and pulls, the tensions and releases – that constitute the living process of reasoning together.

The Unity of Polarity: Embodiment and Inference

This analysis can now demonstrate that the polarized modalities ($\uparrow$ and $\downarrow$) are not just descriptions of subjective feeling; they map directly onto the polarities of inferential roles within the space of reasons.

Compression ($\downarrow$) as Commitment (Strengthening)

In reasoning, the act of making a determinate claim – undertaking a commitment – is an act of Temporal Compression ($\downarrow$). This involves strengthening one's position, adding constraints, narrowing the field of possibilities, and creating boundaries. This is the "First Negation."

Phenomenology: The focused attention required to apply a definition or follow a rigorous rule; the weight of a specific commitment.

Expansion (↑) as Entitlement (Weakening)

When exploring the consequences of a commitment – what one is entitled to because of it, or what it generalizes to – this involves engaging in Temporal Decompression (↑). This means weakening the claim (making it more general, less restrictive), moving outward from a specific point to its broader implications.

Phenomenology: The release experienced when moving from a specific case to a general rule; the "opening up" of new inferential possibilities.

A.2 Exercise 5: Reading Philosophy – Being, Nothing, and Becoming

You have been learning to navigate the rhythm of thought in The Exercise: compression into awareness and decompression through letting go. A recurring temptation arises: perhaps the rhythm can be stabilized by deliberately directing it? Rather than flying off on a tangent, could the impetus to act-next in thought, upon representation, be controlled? Perhaps by thinking a thought and then intentionally thinking its opposite, that alternation return experience to unified immediacy?

The following extends The Exercise into a controlled abstraction experiment.

Embodied Experiment *Settle into the relaxed, open awareness cultivated previously. Now intentionally introduce the most abstract, contextless thought available: "Being." Attempt to fixate on it. Try (briefly) to be "Being" – pure, undifferentiated presence. The felt quality is not the expansive unity of the I-feeling; it is an intense act of Temporal Compression (↓). Attention strains to grasp totality as a static object; subtle tension builds.*

Yet "Being," devoid of any inner difference, is unstable. It gives nothing determinate for attention to grip. The fixation collapses. The pole flips to "Nothing." Attempt to hold that absence. This, too, is compression: the negation of the prior abstraction. An oscillation can now arise: "Being" collapsing into "Nothing," the emptiness of "Nothing" acquiring a kind of conceptual presence and flipping back again:

$$\text{Being} \longleftrightarrow \text{Nothing.}$$

This loop is restless and draining – a ***bad infinite****: endless alternation without qualitative transformation (akin to a mechanical output stream). The strategy of control through opposition fails because it remains imprisoned in mutual compression.*

Pause. What actually occurs between the poles? In each transition there is a micro-movement. Consider it as a *temporal singularity*, a kind of topological 'hole' in the representational space, where one fixation dissolves and the other has not yet stabilized. Shift attention away from the static poles toward that passage. Breathe into *the in between*. The movement itself – the instability in which each passes into the other – is what Hegel names *Becoming*.

To remain with Becoming, apply the earlier lesson: Let Go (LG) of fixation on the oscillation as content. This letting go is Temporal Decompression ($\uparrow$). Tension releases; the grip on static abstraction loosens; the rhythm becomes becoming and the awareness of becoming. The oscillatory aspects remain, but it is more complex. As a temporal compression, $\ulcorner$becoming$\urcorner$ compresses both compression and decompression into a single process.

This is an exciting thought. So much so that I began my dissertation with the claim "I am a constant becoming..." There is something delicious in the paradox of constant movement. But there is also some peril; for becoming to not fall back into the simple oscillation of being and nothingness, it, too, must move beyond itself.

From the horizon of this movement, a different kind of representational temptation may arise: what happens when a movement (becoming) is compressed into a static concept ($\ulcorner$becoming$\urcorner$). It is a kind of recursion, where a function takes itself as its value: becoming($\ulcorner$becoming$\urcorner$). What comes from this? What does becoming become? If it becomes itself, then it is what a mathematician would call a *fixed point*. These relate to my fixation on diagonalization. But it cannot be itself if it is itself. That is, if becoming becomes becoming, it has not *become* anything at all. No movement would be expressed and it would slide back into nothingness.

Hegel's whole *Science of Logic* falls out from this beginning. What the last italicised text was trying to do was a somatic sublation of being/nothing into ~~being/nothing~~ into becoming, where becoming is orthogonal to both of the opposing terms as a name for their opposition. I cannot reconstruct the whole logic in this somatic way, but I am tempted to try. I imagine that dense texts like that one could be annotated with breath marks, like how musical scores often tell musicians who play wind instruments when to breathe.

In any case, to track the movement in becoming, awareness might shift to the difference between its static and dynamic poles: the finite name (⌜becoming⌝) and the *in*finite movement that the name purports to represent (becoming). At that point, the oscillation that the sound of time metaphorizes is between the finite, bounded object and the *in*finite dissolution of those bounds. The finite name is still a compression, a static symbol that can be grasped. The *in*finite movement is an expansion, a dynamic process that cannot be fully captured. But then that returns to determinate negation itself, in its two moments. The first negation produces the finite representation, the second determinately dissolves those determinations. So, you may find yourself learning again what you already knew, but this time from a conceptual descent. This part of the exercise started with the most abstract concept but folds back on itself, falling from those heights, back to the original "no."

Formal Reconstruction The polarized modal vocabulary can mark the structure whereby the foundational triad emerges as a felt necessity.

States: T_B (fixation on "Being"), T_N (fixation on "Nothing"), A_{Osc} (recognized oscillation), LG (letting go), U (unified sensation), U' (intensified unity / Becoming).

1. **Abstract Compression.** Both poles are compressive fixations destroying U.

$$\mathsf{S}(T_B) \implies \Box^{\downarrow}_{\mathsf{S}}(\neg U), \qquad \mathsf{S}(T_N) \implies \Box^{\downarrow}_{\mathsf{S}}(\neg U)$$

Gloss: Fixating on pure abstraction (Being or Nothing) necessarily compresses and collapses unified sensation.

2. **Dialectical Inversion (Oscillation).** Each fixation flips (compressively) into the other.

$$\mathsf{S}(T_B) \implies \Box^{\downarrow}_{\mathsf{S}}(T_N), \qquad \mathsf{S}(T_N) \implies \Box^{\downarrow}_{\mathsf{S}}(T_B)$$

Gloss: The alternation $T_B \leftrightarrow T_N$ is itself a compressive loop.

3. **Recognized Oscillation (Choice Point).** Awareness of the loop opens expansive possibility.

$$\mathsf{S}(A_{\text{Osc}}) \implies \Diamond^{\uparrow}_{\mathsf{S}}(LG)$$

Gloss: Recognizing the oscillation affords the option to release.

4. **Becoming (Expansive Necessity).** Letting go yields intensified unity as rhythm.

   $$\mathsf{S}(LG) \implies \Box^{\uparrow}_{\mathsf{S}}(U'_{\text{Becoming}})$$

   Gloss: Release necessarily expands into an intensified unity experienced as Becoming.

Thus the triad (Being, Nothing, Becoming) is not merely speculative architecture; it is a formal trace of the embodied management of tension between compression and decompression, fixation and release.

The *Science of Logic* is a different kind of logic than the one just offered. It begins with the most abstract categories: Being, Nothing, and Becoming. This triad forms the initial movement of the entire logical development (Hegel, 2010, pp. 59–73). I want to reconstruct these terms, as *becoming* is a central theme.

Pure Being is the starting point: being without any further determination or content. It is indeterminate immediacy. Because it is completely indeterminate, it has no distinguishing features. As Hegel states, "Being, pure being - without further determination... It is pure indeterminateness and emptiness" (Hegel, 2010, p. 59). Pure Being is not a thing, but the most abstract concept of existence. It is the starting point of all thought, but it is also empty and devoid of content.

If we try to intuit or think Pure Being, we find there is nothing to intuit or think. This pure indeterminateness and emptiness is precisely what constitutes Pure Nothing (*Reines Nichts*). Hegel asserts, "Being, the indeterminate immediate is in fact nothing, and neither more nor less than nothing." And conversely, "Nothing, pure nothingness... is simple equality with itself, complete emptiness..." (Hegel, 2010, p. 59). Thus, Pure Being and Pure Nothing are, in a *speculative* sense, the same.

The assertion that Pure Being and Pure Nothing are the same is paradoxical. The truth, Hegel argues, is neither Being nor Nothing in their isolated abstraction, but the movement from one to the other. He writes, "the unity of being and nothing is not a *state*; a state would be a determination of being and nothing into which these moments would have fallen as if by accident, as if prey to a sickness externally induced by faulty thinking; rather, this middle and unity, the vanishing, or equally the becoming, is alone the *truth* of being and nothing" (Hegel, 2010, p. 216). Becoming is the unity of Being and Nothing. It is not a static state but a dynamic process. Within Becoming, Being and Nothing are moments: coming-to-be (the transition from Nothing to Being) and ceasing-to-be (the transition from Being to Nothing). Rather than 'Being' and 'not-Being' remaining apart, the determinate negation of Being gives us *Becoming*, a concept that has Being and Nothing folded into it."

Mapping this onto The Exercise reveals the connection. The moment of unified presence (I-feeling in full bloom) is like a moment of pure Being in experience – so pure and featureless that it cannot be held onto. The collapse of that experience when the thought "I am this" arises is akin to that Being turning into Nothing – the fullness evaporates into an absence (the feeling is "lost," and one is left only with an empty thought that doesn't deliver the goods). Yet out of that nothing, a new determination immediately begins to form: perhaps a new approach, a new aspect of the body to focus on, or a new resolve ("I'll try again"). In other words, the Nothing gives rise to a new Becoming – a coming-to-be of another state of being (for instance, restarting the cycle and reaching a new unified feeling, which will again have no determinate content and thus again risk collapsing). The cycle of recognition described is essentially a cycle of Becoming. The moments of pure undifferentiated sensation (Being) inevitably pass away (Nothing), yet from that void a richer unity can emerge if allowed (the next Being). Each "loss" of the experience, when handled by letting-go, is actually the engine of a deeper recognition of the experience. Each negation (loss of being) is followed by a negation of that negation (the letting go of the thought, which re-establishes a new being).

Arrow-Notation for Embodied Inference

The notation employed here builds on the polarized modal logic of compression ($\downarrow$) and decompression ($\uparrow$) developed earlier. Two additional arrows, left ($\leftarrow$) and right ($\rightarrow$), mark horizontal spatial movement. Taken together, these form a concise "arrow-calculus" of embodied reasoning.

- Vertical orientation: $\uparrow$ designates temporal expansion (release, possibility, intensification), while $\downarrow$ designates temporal compression (fixation, necessity, contraction).

- Horizontal orientation: $A \rightarrow B$ expresses forward derivation or inferential consequence. $A \leftarrow B$ expresses retroactive recognition of enabling conditions or grounds.

- Modal overlays: Necessity ($\Box$) and possibility ($\Diamond$) can be indexed to the vertical direction: $\Box\downarrow$ for compressive necessity, $\Diamond\uparrow$ for expansive possibility, $\Box\uparrow$ for expansive necessity. Subscripts S, O, N indicate subjective, objective, or normative modes.

Thus, we may write:

$$\text{Pokey is a dog} \xrightarrow{\Box\downarrow_O} \text{Pokey is a mammal}$$

as an objective inferential entailment, and

$$\text{Mineness of experience} \xleftarrow{\Box\uparrow_S} \text{Transcendental Ego}$$

as the retroactive recognition of an enabling condition.

Relation to Prior Scholarship

This notation compresses into a single symbolic form a number of distinct philosophical insights. Hegel's account of *Aufhebung* (sublation) already oscillates between contraction and expansion: a concept is preserved, negated, and lifted beyond itself (Hegel, 1977). Kant's transcendental deduction exemplifies the leftward-expansive form: the unity of apperception is recognized only after the immediacy of experience is felt (Kant, 1998). Derrida's notion of *différance* similarly intertwines spatial difference (spacing) and temporal deferral (deferment), refusing to grant priority to either, and always leaving meaning in movement (Derrida, 1973). Brandom's inferentialism stresses that meaning is determined not by ostension but by its place in a web of commitments and entitlements, where anaphoric chains enable the use of indexicals (Brandom, 1994, 2008). Carspecken's phenomenological exercises show how forestructures build through proprioceptive expansion and are recollected as enabling conditions (P. F. Carspecken, 1999).

By aligning vertical polarity with temporal compression/expansion and horizontal polarity with spatial forward/retroactive movement, the present framework locates embodied reasoning within a two-dimensional space that resonates with these traditions. It avoids collapsing retroactive discernment into mere critique ("you only say that because of privilege"), instead highlighting its expansive and enabling role. Knowledge, in this scheme, is not merely the forward march of entailment but the rhythmic interplay of derivation and recognition, compression and expansion, spacing and deferral.

A.3 The Dyad of the Finite and the *In*finite and the ZCM

There are several possible ways to map (articulate accessibility relations) between the components of the *ZCM* and the fundamental tensions I have been analyzing. A relatively elegant approach maps the two bands onto the dyad of the *Finite* and the *In*finite. For the rest of the book, these will be treated as the existential need to be recognized as rationally good (consistent, object-like) and the existential need to be recognized as *in*finite. However, in a purer Hegelian philosophy, this would be the tension between determination and the ground of being. In this mapping, the *ZCM* models the structure of the self seeking equilibrium.

The Components Mapped to the Existential Landscape

- **The Fixed Anchor Point:** Represents *Self-Certainty* (the Ground, the *In*finite recollected as {I}). In the model, it is treated as a stable, immovable foundation of experience. That treatment extracts a huge cost. In essence, it implies a commitment to a static ground, rather than a groundless ground. While practicing the exercise may lead you to this conclusion, I am not willing treat that conclusion as unfalsifiable by experience.

- **The Control Point (Movable):** Represents *Awareness* (Desire, the locus of the finite "me"). This is what I manipulate when directing awareness or engaging in thought.

- **The Disc:** Plays several roles. One is that the point where both bands attach to the wheel is what traces out the sine wave from the Sound of Time metaphor. As such, it represents the state. Are you with the I-feeling? Are you feeling totally compressed? But it also writes symbols M, W. Consequently, friction in the disc can be taken as a representation of power dynamics. In a highly suppressive environment, the machine won't move and nothing can be written. The writing mechanism is explored in chapter 6, but for now, consider what is written as the system's memory/hysteresis.

The Tensions (The Conflicting Drives)

The two elastic bands model the opposing forces acting on the self:

- **Band 1 (Anchor to Disc):** The *Pull of the* In*finite*. This is the fundamental drive towards unity, anchored in Self-Certainty.
- **Band 2 (Control Point to Disc):** The *Pull of the Finite*. This represents the tension of fixation (T), the pull toward the determinate object of attention.

The Dynamics as Embodied Logic

This mapping allows me to interpret the behavior of the *ZCM* as an enactment of polarized modal logic and the feeling body in motion.

1. Compression ($\Box\downarrow$) and Dissonance. When I fixate on a thought (T) – for example, trying to grasp "Being" – - I move the Control Point (Attention).

ZCM Dynamic: This stretches Band 2 (Fixation) and pulls the Disc (State) away from the Anchor, stretching Band 1 (Unity). Total energy (tension) in the system increases.

Felt Experience: This is the strain of effort, the narrowing of focus, or the contraction. The increased tension is the lived experience of embodied dissonance, or the alienation I feel when the "me" drifts away from the {I} (when I act to be taken as a good boy, not as I am).

Logic: This embodies *Compressive Necessity* ($\Box\downarrow$). The system is constrained and resistant. This is the First Negation ("No").

2. The Cusp Region and Possibility ($\Diamond\uparrow$). As compression builds, the system enters the unstable cusp.

ZCM Dynamic: The system is bistable (two possible equilibria). Tension is high, and the Disc resists movement, held in precarious balance.

Felt Experience: This is the "choice point" (Awareness A). I feel friction and instability, sensing that a shift is possible but not yet realized.

Logic: This embodies *Expansive Possibility* ($\Diamond\uparrow$) – - the recognition that I can Let Go ($\Diamond^S\uparrow$ (LG)).

3. The Catastrophe and Expansive Necessity ($\Box\uparrow$). If I push attention past a threshold – by intensifying fixation until it breaks, or by initiating "Letting Go" – the equilibrium collapses.

ZCM Dynamic: The "Snap." Tension is suddenly released as the Disc jumps to a new, lower-energy state, usually closer to the Anchor.

Felt Experience: This sudden, involuntary release is sublation: the "Aha!" of insight, the breakthrough in dialogue, or the deep release in meditation.

Logic: This is *Expansive Necessity* ($\Box\uparrow$). It is the Second Negation ("No"), the necessary jump to a new state of intensified unity (U').

The Contradiction of Desire and the State of Silence

This mapping also visualizes the paradoxes of *The Exercise*.

The Contradiction of Desire. If Attention (Control Point) tries to move directly toward Self-Certainty (Anchor), the tension of fixation (Band 2) increases. The paradox is that effort amplifies strain and can destabilize the Disc. The harder I "try," the more compressed the experience becomes.

Silence and Unity. I have also intuited a state of "silence," where rotation ceases and tension is resolved. The body is totally relaxed, the friction of distorting power in the disc is unimportant, and the Control Point (Attention) aligns with the Fixed Anchor (Self-Certainty).

ZCM Dynamic: Silence (Unity, U) arises when the system is at its lowest energy state, with dissonance minimized – when the Control Point (Attention) aligns with the Fixed Anchor.

Felt Experience: The "me" is aligned with the {I}. Attention is diffuse yet present. The compressive pull is identical to the expansive pull. This is the state of dynamic stillness, where the contradiction of desire is resolved and the relentless motion of the "More Machine" comes to rest.

There is another way in which U might be felt that is almost completely the opposite. Rather than perfect relaxation, when the control point of attention is moved out past the cusp catastrophe, in a highly disciplined state, some embodied tensions can be very, very high. I'm thinking of extended yoga poses. In those moments, the symbols of the disc are in a superimposed state. A perpetual movement away from the fixed point resonates with the perpetual withdrawal of self-certainty when represented. This is more like the Derridean reconstruction of self-certainty than the Hegelian one. When You are the one who withdrawals and I meet you in that movement, there is a kind of 'dance' or play in the space opened between us.

The Singularity of Becoming: An Embodied Dialectic

In *The Exercise*, I practice navigating the rhythm of thought: the compression into awareness and the decompression of letting go. Yet the mind often seeks control. A temptation arises: perhaps this rhythm could be stabilized by consciously directing it. Could you tame the experience by thinking a thought and then intentionally thinking its opposite? Would that restore the unified sensation?

Let me explore this strategy, extending *The Exercise* into the realm of pure abstraction.

A.4 The Dialectic of Abstraction: An Embodied Experiment

Settle back into the relaxed, open awareness cultivated by *The Exercise*. Now, intentionally introduce the most abstract, contextless thought possible: "Being."

Attempt to fixate on it. Try to be "Being" – pure, undifferentiated presence.

Notice the feeling this generates. This is not the expansive unity of the I-feeling. It is an intense act of *Temporal Compression* (↓). I attempt to grasp the totality of experience as a static, determinate object. It requires effort; a subtle tension builds as I try to hold this ultimate abstraction.

But "Being," devoid of any specific content or difference, is unstable. It offers nothing determinate for the mind to grip. As I try to hold it, the fixation necessarily fails. The intense compression collapses into its opposite: "Nothing."

Now I am aware of absence, the void left by the vanished "Being." Attempting to hold this thought, I encounter another compression (↓). It is the negation of the previous abstraction. The tension remains, though it shifts in quality, as I now fixate on the void.

The mind may oscillate, trapped in a loop: "Being" collapsing into "Nothing," and the emptiness of "Nothing" revealing itself as a kind of presence, flipping back to "Being."

$$\text{Being} \leftrightarrow \text{Nothing}.$$

This oscillation is restless and exhausting. It is what I call the *bad infinite* – an endless repetition without qualitative change (much like the relentless output of the More Machine). The strategy of controlling thought through opposition fails because it remains trapped at the level of fixation: a cycle of mutual compression.

But pause here. What lies between these two concepts?

In the transition, there was a movement – a "temporal singularity," a gap where one fixation dissolved and the other arose.

Shift attention. Stop holding the static concepts ("Being," "Nothing") and attend instead to the movement between them. Breathe into that difference.

This movement – the dynamic instability, the passing of one into the other – is what Hegel names *Becoming*.

To grasp this, I must apply the lesson of *The Exercise*: Let Go (LG) of fixation on the oscillation itself.

This is *Temporal Decompression* (↑). The tension releases, and fixation on static concepts gives way to the experience of rhythm. "Becoming" is not another static concept to be compressed. It is the lived experience of dialectical movement itself. By "Letting Go," the I-feeling fuses not with the concepts, but with the rhythm. This is the Sound of Time.

Formal Reconstruction: The Foundational Dialectic

I can reconstruct this embodied dialectic using polarized modal logic, showing how the foundational triad of logic emerges from the felt necessities of consciousness.

States. T_B (Fixation on "Being"), T_N (Fixation on "Nothing"), LG (Letting Go), U (Unified Sensation).

1. The Phenomenology of Abstraction (Compression). Both "Being" and "Nothing" are forms of fixation. Attempting to hold them induces compression and destroys unity.

Commitment of Abstract Compression:

$$\mathsf{S}(T_B) \implies \Box^{\downarrow}_{\mathsf{S}}(\neg U)$$

$$\mathsf{S}(T_N) \implies \Box^{\downarrow}_{\mathsf{S}}(\neg U)$$

Gloss: Fixating on pure abstraction (Being or Nothing) is necessarily compressive ($\Box^{\downarrow}_{\mathsf{S}}$) and collapses the unified sensation ($\neg U$).

2. The Necessary Inversion (The Oscillation). The mind cannot sustain the empty compression of one term without flipping to the other. This movement remains within compression.

Commitment of Dialectical Inversion:

$$\mathsf{S}(T_B) \implies \Box^{\downarrow}_{\mathsf{S}}(T_N)$$

$$\mathsf{S}(T_N) \implies \Box^{\downarrow}_{\mathsf{S}}(T_B)$$

Gloss: Fixation on Being necessarily (and compressively) leads to fixation on Nothing, and vice versa. This creates the oscillation ($T_B \leftrightarrow T_N$).

3. The Insight (The Singularity). The recognition is that the oscillation itself is the object of awareness (A_{Osc}), and that it can be released.

Commitment of Recognized Oscillation:

$$\mathsf{S}(A_{Osc}) \implies \Diamond^{\uparrow}_{\mathsf{S}}(LG)$$

Gloss: Recognizing the oscillation (A_{Osc}) opens the expansive possibility ($\Diamond^{\uparrow}_{\mathsf{S}}$) of letting go.

4. The Emergence of Becoming (Expansion). The act of letting go of the static terms results in fusion with the movement itself.

Commitment of Becoming:

$$\mathsf{S}(LG) \implies \Box^{\uparrow}_{\mathsf{S}}(U'_{\text{Becoming}})$$

Gloss: Letting go of fixation necessarily results in expansive movement ($\Box^{\uparrow}_{\mathsf{S}}$). The resulting intensified unity (U') is the lived experience of Becoming.

This reconstruction shows that the foundational triad of Hegelian logic is not merely an intellectual structure. It is the formal expression of the embodied rhythm of consciousness as it navigates the tension between compression and expansion, fixation and release.

The Structure of the Hierarchy

The logic developed in chapter 1 can be extended to understand the two images above, hopefully negating any sense of mysticism that might arise from seeing quadrilaterals represented as fractals.

Moving Down (Specification): Quadrilateral $\rightarrow$ Parallelogram $\rightarrow$ Rectangle $\rightarrow$ Square. This is a movement of increasing Compression ($\downarrow$). The conceptual hierarchy strengthens by adding constraints (commitments). "Square" is the most compressed concept.

Moving Up (Generalization): Square $\rightarrow$ Rectangle $\rightarrow$ Parallelogram. This is a movement of increasing Expansion ($\uparrow$). The conceptual hierarchy weakens by "Letting Go" of constraints.

The Dynamics of Inference

Consider the standard inference: "If it is a Square (S), then it is a Rectangle (R)" ($S \Rightarrow R$).

Embodied Dynamic: The analysis starts with the highly compressed concept S ($\downarrow\downarrow$). To move to R, the constraint of "equal sides" is released. This inference is experienced as an expansive movement ($\uparrow$).

Now consider the contrapositive (Modus Tollens), which relies on polarity inversion: "If it is NOT a Rectangle ($\neg R$), then it is NOT a Square ($\neg S$)" ($\neg R \Rightarrow \neg S$).

Embodied Dynamic:

The analysis starts with $\neg R$. Since R is relatively expansive, $\neg R$ introduces a compression ($\downarrow$) – this closes off the space of rectangles.

The conclusion involves $\neg S$. Since S is compressive, $\neg S$ is expansive ($\uparrow$).

The inference moves from the compression of $\neg R$ to the expansion of $\neg S$. The compression required to exclude the broader category (Rectangle) necessarily forces the exclusion of the narrower category (Square) contained within it.

Appendix B

Appendix: Hermeneutic Calculator

B.1 Appendix: The Hermeneutic Calculator

I am building a unified, testable theory of how students *develop* arithmetic understanding – not just a catalog of strategy names, but a developmental map of how those strategies *emerge, elaborate, invert, and nest.* My target is to formalize roughly 25 student-invented strategies across addition, subtraction, multiplication, and division, showing how each one is algorithmically constructed from prior embodied practices.

Choreography as Computation

I treat each strategy as **written choreography for embodied cognition**. A formal automaton (register machine, bounded DPDA, or related model) becomes a script for the temporal unfolding of thought: initialize, transform, check, recurse, terminate. The power of this framing is that it preserves *how* a student actually moves through a calculation – counting up, pausing at a boundary, decomposing a number – rather than replacing those moves with opaque symbolic shortcuts.

Two Fundamental Movements

I analyze student action through a dialectic of temporal structure: 1. **Temporal Compression (Sublation / Recollection):** Unitizing many micro-acts into a larger cognitive unit (ten ones → one ten; 3 base jumps → a single composite stride). Compression accelerates flow. 2. **Temporal Decompression (Determinate Negation):** Strategically undoing or expanding a composite to restore fine control (borrowing a ten; splitting 5 into $2+3$ in RMB; decomposing a factor for distributive reasoning). Fluency grows as students coordinate these movements, learning *when* to expand and *when* to re-compress.

Fractal Architecture: Iterative Core + Strategic Shell

Across strategies I repeatedly recover the same **fractal pattern**: * **Iterative Core:** A minimal loop (initialize → step ($+1$, -1, $+Base$, $+Chunk$) → condition check). Counting by ones, skip counting, and accumulation loops in division all instantiate this engine. * **Strategic Shell:** A supervisory layer that *prepares*, *optimizes*, or *transforms* the problem so that the core runs fewer or cognitively lighter iterations. RMB, Rounding & Adjusting, Chunking, Sliding, Distributive and Inverse Distributive Reasoning all wrap the core with analysis (e.g., "find gap K", "split factor", "slide both numbers"). Because the shell often *invokes* the core as a subroutine (e.g., CountUpToBase, CountBackK), the global structure becomes self-similar: strategies *contain* (and sometimes nest) earlier strategies. This produces a genuine computational fractal – not metaphorical flourish, but recurrence of the same control schema at different conceptual scales.

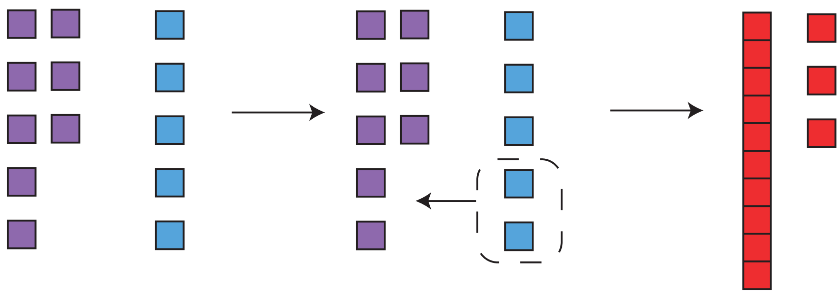
To illustrate this, I provide a transcript from (Carpenter et al., 1999) that demonstrates a strategy that Amy Hackenberg calls *rearranging to make bases* (Hackenberg, 2025).

- **Teacher:** Lucy is eight fish. She buys five more fish. How many fish will Lucy have then?

- **Sarah:** 13.

- **Teacher:** How'd you get 13?

- **Sarah:** Well, because eight plus two is ten, but then two plus three is five. And she wants to buy five more fish. So you take care of two, and you need to add three more. And so I add three more, and you get 13.



Notation Representing Sarah's Solution:

$$
\begin{aligned}
&8 + 5 = \square \\
&8 + 2 = 10 \\
&2 + 3 = 5 \\
&8 + 5 = 10 + 3 \\
&8 + 5 = 13
\end{aligned}
$$

Description of Strategy:

Objective: Rearranging to Make Bases (RMB) means shifting the extra ones from one addend over to the other so that one of the numbers becomes a complete multiple of the base (a whole "group" of that base). This rearrangement simplifies the addition process because there are established patterns for adding an exact multiple of the base. In other words, when you add a full group of base units to a number, the ones digit stays the same while only the digit representing the base (like the tens place) increases.

The Formal Model

To model this strategy as an *elaboration of counting*, I deploy an automaton called a Register Machine. Crucially, it determines the gap (K) and the remainder (R) using iterative counting, reflecting how a student might derive these values without relying on abstract subtraction. The transition table and tuple for this machine are written below.

Mechanisms of Elaboration

I observe three progressive forms of algorithmic elaboration: 1. **Compression of Action:** Replacing many +1 steps with $+Base$ or $+StructuredChunk$

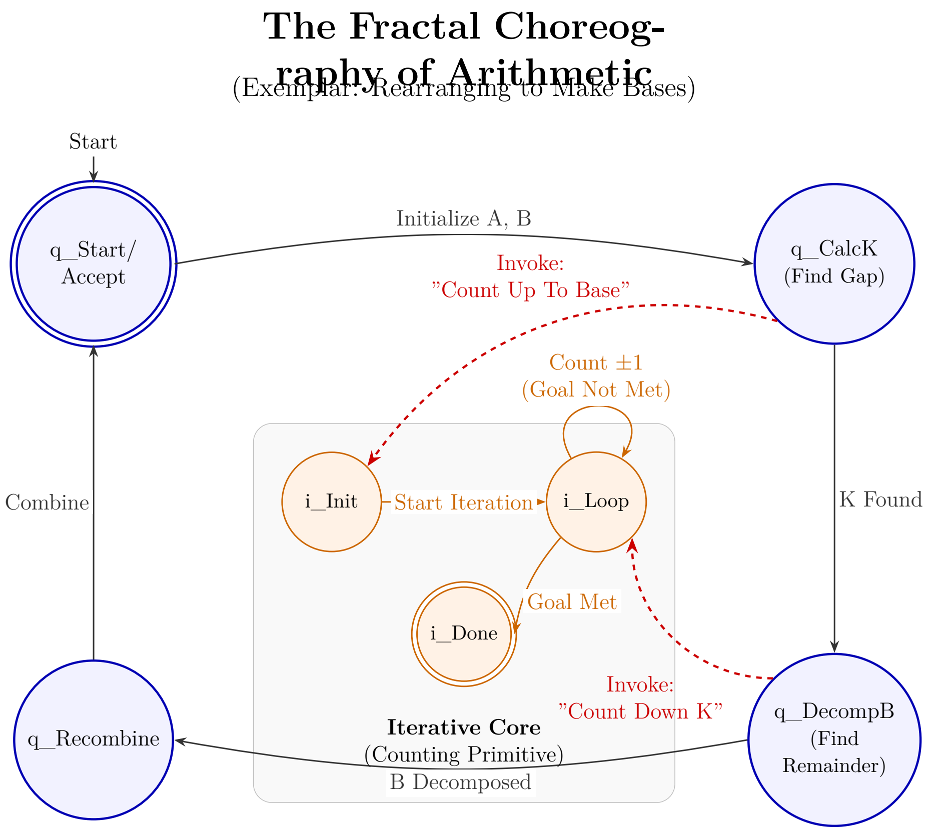


Figure B.1: Visualization of the nested automata, illustrating Algorithmic Elaboration. The strategic states of the outer automaton (Strategic Shell) are realized by invoking the inner automaton (Iterative Core).

(COBO, Chunking). 2. **Optimization of Iteration:** Dynamically computing the *size* of a future stride (RMB gap K; Chunking's bridging chunk; Sliding's constant difference) before acting – analyze then accelerate. 3. **Structural Transformation:** Rewriting the problem space (Rounding detour + compensation; Distributive split; Sliding invariance; Inverse Distributive decomposition of dividend). Advanced strategies chain these moves (e.g., Rounding = transformation $\rightarrow$ compressed addition $\rightarrow$ compensatory inverse steps).

Inversion of Practice

Subtraction fluency emerges not by inventing alien procedures but by **inverting or repurposing** addition shells: Missing Addend reframes subtraction as forward accumulation; Counting Back mirrors Counting On; Sliding preserves difference across a translation; Borrowing reverses carry (decompression of a prior sublation). Division analogues (Dealing by Ones vs. Coordinating Two Counts; Inverse Distributive Reasoning) continue the same inversion logic.

Collaboration and Iterative Refinement

This project is explicitly *collaborative* with an AI assistant. I bring student transcripts, pedagogical insight, and theoretical intent; the assistant supplies relentless formal scrutiny – flagging mis-specified state sets, hidden non-determinism, premature algebraic assumptions, or missing termination guarantees. A typical refinement cycle: 1. Draft informal description from transcript. 2. Specify automaton (states, registers, transitions) in a first-pass formalism. 3. Implement executable prototype (Python) to test determinism, termination, and behavioral alignment with the transcript (sequence reconstruction like "$46, 56, 66, \ldots$"). 4. Trace failures (e.g., an early Rounding model produced an infinite loop after overshoot; an initial Chunking diagram hid the cognitive search for K) and revise. 5. Re-abstract the corrected machine into concise LaTeX-friendly specification. This loop ensures every claimed cognitive choreography *runs* – a falsifiability and reproducibility standard often missing in purely diagrammatic accounts.

Why Executable Formal Models Matter

An executable automaton does four things for me: * **Validity Check:** Catches hidden cycles or unreachable states. * **Phenomenological Fidelity:** Lets me align generated action traces with verbatim student utterances. * **Comparative Anatomy:** Normalizes different strategies into a shared tuple structure so I can map elaboration edges precisely. * **Pedagogical Insight:** Identifies which internal subroutines (e.g., "CountBackK") must be instructionally stabilized before a composite strategy will consolidate.

Algorithmic Elaboration (Brandom Frame)

Following Robert Brandom, I treat these developments as **algorithmic elaborations**: later practices are *PP-sufficient* expansions of earlier ones – achieved by reorganizing, nesting, or inverting existing abilities rather than importing foreign primitives. Some strategies become **LX** relative to prior practice: they both derive from and make explicit what was implicit (RMB makes base boundaries explicit; Borrowing renders the reversibility of carry explicit; Distributive Reasoning makes latent additivity across factors explicit).

Scope of This Document

Below I give each strategy a uniform template: description, formal specification, choreography (compression/decompression dynamics), and genealogical lineage. I remove historical critique and raw code to foreground the structural logic while preserving testability through the already verified prototypes. The introduction you are reading consolidates the nuance of origin (embodiment), evolution (fractal elaboration), collaboration (human + AI), and rigor (execution + revision) from the longer source manuscript without duplicating passages.

B.2 Hermeneutic Calculator: Strategy Formalizations

This draft reorganizes the strategies as primary sections. Each section supplies:

1. Phenomenological description (student-facing practice).
2. Formal automaton / register-machine specification in LaTeX-friendly notation.
3. Core choreography (temporal compression/decompression dynamics).
4. Algorithmic elaboration lineage (what primitives it builds upon).

All implementation details (Python prototypes) and historical critiques have been removed. Mathematical symbols are formatted for Pandoc $\rightarrow$ LaTeX conversion.

Notation (uniform across strategies):

- $M = (Q, V, \delta, q_0, F)$: machine with states Q, registers (or variables) V, transition function δ, start state q_0, accepting states F.
- When convenient, we use auxiliary internal variables; these are included in V implicitly.
- Counting primitives: Count Up $(+1)$, Count Back (-1) regarded as atomic embodied actions.
- Temporal Compression: synthesizing many unit actions into a higher-order unit (e.g., a "ten").
- Temporal Decompression: strategic expansion of a unit into constituent parts.

B.3 Counting and Counting On

Description. Sequential unit counting within a bounded base-10 place-value structure $(0 - 999)$. Embodied iterations ("ticks") increment units, propagate carries (sublation) into tens and hundreds.

Formal Model (Sketch). Deterministic PDA (bounded) or 3-register counter. For LaTeX exposition we specify a DPDA tuple:

$$M_{count} = (Q, \Sigma, \Gamma, \delta, q_{start}, Z_0, F)$$

with place-value stack symbols U_i, T_j, H_k. The transition function δ is defined as follows:

Current State	Input	Top of Stack	Next State	Action (Stack)	Interpretation
q_{start}	ε	Z_0	q_{idle}	Push(U_0, T_0, H_0)	Initialize count to 0.
q_{idle}	`tick`	U_n $(n < 9)$	q_{idle}	Pop; Push(U_{n+1})	Increment units.
q_{idle}	`tick`	U_9	q_{inc_tens}	Pop	Unit overflow, carry to tens.
q_{inc_tens}	ε	T_m $(m < 9)$	q_{idle}	Pop; Push(T_{m+1}, U_0)	Increment tens, reset units.
q_{inc_tens}	ε	T_9	$q_{inc\_hundreds}$	Pop	Ten overflow, carry to hundreds.
$q_{inc\_hundreds}$	ε	H_k $(k < 9)$	q_{idle}	Pop; Push(H_{k+1}, T_0, U_0)	Increment hundreds, reset lower places.
$q_{inc\_hundreds}$	ε	H_9	q_{halt}	Pop; Push(H_0, T_0, U_0)	Counter overflow.

Choreography. Carry = temporal compression: ten unit steps recollected as one higher unit. Borrow (in inverse counting) is temporal decompression.

Elaboration Lineage. Primitive for all subsequent additive, subtractive, multiplicative, and divisional strategies.

B.4 Rearranging to Make Bases (RMB)

Description. For $A+B$, identify gap K from A to next base (e.g., 10, 100), decompose $B = K + R$, form $A' = A + K$ (a base), then compute $A' + R$.

Machine.

$$M_{RMB} = (Q, V, \delta, q_0, F)$$

with

$$Q = \{q_{start}, q_{calcK}, q_{decompose}, q_{recombine}, q_{accept}\}$$

$$V = \{A, B, K, A', R\}$$

Transition Function (δ):

Current State	Condition	Next State	Action	Interpretation
q_{start}	-	q_{calcK}	$K \leftarrow 0$; $A_{temp} \leftarrow A$	Initialize.
q_{calcK}	$A_{temp} <$ NextBase(A)	q_{calcK}	$A_{temp} \leftarrow A_{temp} + 1$; $K \leftarrow K + 1$	Count up to find gap K.
q_{calcK}	$A_{temp} ==$ NextBase(A)	$q_{decompose}$	$A' \leftarrow A_{temp}$	Gap found. Store new base A'.
$q_{decompose}$	$K > 0$	$q_{decompose}$	$B \leftarrow B - 1$; $K \leftarrow K - 1$	Decompose B by transferring K.
$q_{decompose}$	$K == 0$	$q_{recombine}$	$R \leftarrow B$	Remainder R is what's left of B.
$q_{recombine}$	-	q_{accept}	Output $A' + R$	Combine new base and remainder.

Choreography. Decompression (splitting B) enables immediate compression (forming base A').

Lineage. Elaborates Counting Up + Counting Down primitives; anticipates strategic boundary manipulation used later in Rounding, Chunking, Sliding.

B.5 COBO (Counting On by Bases then Ones)

Description. For $A + B$, decompose $B = b \cdot Base + r$; iterate base jumps $(+Base)$ then unit steps $(+1)$.

Machine. M_{COBO} with states $\{q_{start}, q_{bases}, q_{ones}, q_{accept}\}$ and registers $\{Sum, BaseCounter, OneCounter\}$.

Transition Function (δ):

Current State	Condition	Next State	Action	Interpretation
q_{start}	-	$q_{initialize}$	Read A, B	Start.
$q_{initialize}$	-	q_{add_bases}	$Sum \leftarrow A$; $BaseCounter \leftarrow B//Base$; $OneCounter \leftarrow B \pmod{Base}$	Initialize Sum. Decompose B.
q_{add_bases}	$BaseCounter > 0$	q_{add_bases}	$Sum \leftarrow Sum + Base$; $BaseCounter \leftarrow BaseCounter - 1$	Add one Base unit (Loop).
q_{add_bases}	$BaseCounter == 0$	q_{add_ones}	-	All bases added. Transition.
q_{add_ones}	$OneCounter > 0$	q_{add_ones}	$Sum \leftarrow Sum + 1$; $OneCounter \leftarrow OneCounter - 1$	Add one unit (Loop).
q_{add_ones}	$OneCounter == 0$	q_{accept}	Output Sum	All ones added. Accept.

Choreography. Two-phase rhythm: compressed temporal blocks (bases) followed by decompressed fine resolution (ones).

Lineage. Builds on counting; prepares for Chunking and Rounding by habitualizing base jumps.

B.6 Rounding and Adjusting (Addition)

Description. Select addend closer to next base: round up $A \to A' = A+K$, compute $A' + B$, then adjust back: $(A' + B) - K$.

Machine. States $\{q_{start}, q_{calcK}, q_{add}, q_{adjust}, q_{accept}\}$; registers $\{A, B, K, A', Temp, Result\}$.

Transition Function (δ):

Current State	Subroutine / Action	Next State	Interpretation
q_{start}	Read A, B; Heuristic select $Target$	q_{calcK}	Start. Select number closer to the next base.
q_{calcK}	**Count Up To Base(**$Target$**)** $\to K, A_{rounded}$	q_{add}	Determine K by counting up from $Target$.
q_{add}	**COBO(**$A_{rounded}$**, Other)** $\to TempSum$	q_{adjust}	Add Other to the rounded A.
q_{adjust}	**Count Back(**$TempSum, K$**)** $\to Result$	q_{accept}	Adjust by counting back K.

Choreography. Strategic temporal detour: initial decompression (deriving K) enables major compression (base addition), followed by inverse correction.

Lineage. Elaborates RMB (boundary anticipation) and COBO (base efficiency); introduces explicit compensation schema.

B.7 Chunking (Addition)

Description. Decompose B into large base chunk + strategic residual chunks to force successive bases: $B = B_{base} + K + R$ where K bridges current sum to next base.

Transition Function (δ):

Current State	Condition	Next State	Action	Interpretation
q_{init}	-	$q_{addBase}$	$Sum \leftarrow A$; Decompose B into B_{base}, B_{ones}	Initialize Sum. Decompose B.
$q_{addBase}$	-	q_{calcK}	$Sum \leftarrow Sum + B_{base}$	Add the entire base chunk at once.
q_{calcK}	$Sum <$ NextBase(Sum)	q_{calcK}	$Sum \leftarrow Sum + 1$; $K \leftarrow K + 1$	Iteratively find gap K to next base.
q_{calcK}	$Sum ==$ NextBase(Sum)	q_{applyK}	-	Gap found.
q_{applyK}	$B_{ones} \geq K$	q_{calcK}	$Sum \leftarrow Sum + K$; $B_{ones} \leftarrow B_{ones} - K$	Add strategic chunk K. Loop back.
q_{applyK}	$B_{ones} < K$	$q_{finishR}$	-	Not enough ones for full chunk.
$q_{finishR}$	-	q_{accept}	$Sum \leftarrow Sum + B_{ones}$	Add remaining residue.

Choreography. Iterative cycle: (1) large compression via aggregated base, (2) micro decompression to find K, (3) re-compression to new base, (4) terminal residue.

Lineage. Synthesizes COBO (bulk bases) + RMB (strategic gap finding).

B.8 Subtraction Chunking (Three Orientations)

Given $M - S = D$.

A. Backwards by Part (Take-Away). Sequentially subtract decomposed parts of S (place value or strategic chunks) from M.

B. Forwards from Part (Missing Addend). Treat as $S + D = M$; Count Up (RMB logic) accumulating D.

C. Backwards to Part (Distance Down To). Count Back from M toward S using strategic base landings; accumulate distance.

Each orientation is a register machine. Below are the key transition schemas.

A. Backwards by Part (Take-Away): $V = \{CurrentValue, S_{rem}\}$

State	Condition	Action
q_{init}	-	$CurrentValue \leftarrow M$; $S_{rem} \leftarrow S$
q_{chunk}	$S_{rem} > 0$	$Chunk \leftarrow$ Decompose(S_{rem}); $CurrentValue \leftarrow CurrentValue - Chunk$; $S_{rem} \leftarrow S_{rem} - Chunk$
q_{chunk}	$S_{rem} == 0$	Accept $CurrentValue$

B. Forwards from Part (Missing Addend): $V = \{CurrentValue, Distance\}$

State	Condition	Action
q_{init}	-	$CurrentValue \leftarrow S$; $Distance \leftarrow 0$

State	Condition	Action
q_{chunk}	$CurrentValue < M$	$Chunk \leftarrow$ CalcStrategicChunk($CurrentValue, M$); $CurrentValue \leftarrow CurrentValue + Chunk$; $Distance \leftarrow Distance + Chunk$
q_{chunk}	$CurrentValue == M$	Accept $Distance$

C. Backwards to Part (Distance Down To): $V = \{CurrentValue, Distance\}$

State	Condition	Action
q_{init}	-	$CurrentValue \leftarrow M$; $Distance \leftarrow 0$
q_{chunk}	$CurrentValue > S$	$Chunk \leftarrow$ CalcStrategicChunk($CurrentValue, S$); $CurrentValue \leftarrow CurrentValue - Chunk$; $Distance \leftarrow Distance + Chunk$
q_{chunk}	$CurrentValue == S$	Accept $Distance$

Choreography. Orientation selects temporal direction; strategies B and C exploit boundary compression via RMB subroutines.

B.9 Subtraction COBO / CBBO

COBO (Missing Addend). Start at S, perform base jumps toward M (without overshoot), then ones; distance accumulated is D.

CBBO (Counting Back). Start at M, subtract base units (from decomposed S) then ones; final position is D.

Machines. Two dual register machines are defined.

COBO (Missing Addend): $V = \{CurrentValue, Distance, Target\}$

State	Condition	Action
q_{init}	-	$CurrentValue \leftarrow S$; $Distance \leftarrow 0$; $Target \leftarrow M$
q_{add_bases}	$CurrentValue + Base \leq Target$	$CurrentValue \leftarrow CurrentValue + Base$; $Distance \leftarrow Distance + Base$
q_{add_bases}	$CurrentValue + Base > Target$	transition to q_{add_ones}
q_{add_ones}	$CurrentValue < Target$	$CurrentValue \leftarrow CurrentValue + 1$; $Distance \leftarrow Distance + 1$
q_{add_ones}	$CurrentValue == Target$	Accept $Distance$

CBBO (Counting Back): $V = \{CurrentValue, BaseCounter, OneCounter\}$

State	Condition	Action
q_{init}	-	$CurrentValue \leftarrow M$; Decompose S into $BaseCounter, OneCounter$
q_{sub_bases}	$BaseCounter > 0$	$CurrentValue \leftarrow CurrentValue - Base$; $BaseCounter \leftarrow BaseCounter - 1$
q_{sub_bases}	$BaseCounter == 0$	transition to q_{sub_ones}
q_{sub_ones}	$OneCounter > 0$	$CurrentValue \leftarrow CurrentValue - 1$; $OneCounter \leftarrow OneCounter - 1$
q_{sub_ones}	$OneCounter == 0$	Accept $CurrentValue$

Choreography. Directional inversion of the same two-phase rhythm (bases → ones). Overshoot detection acts as control boundary in COBO.

B.10 Subtraction Decomposition (Borrowing)

Description. Left-to-right: subtract higher place (tens), detect insufficiency in lower place, decompose (borrow) one higher unit into base smaller units, then subtract ones.

Transition Function (δ):

Current State	Condition	Next State	Action	Interpretation
q_{init}	-	q_{sub_bases}	Decompose M, S into place values R_T, R_O, S_T, S_O.	Initialize registers.
q_{sub_bases}	-	q_{check_ones}	$R_T \leftarrow R_T - S_T$	Subtract the bases (Tens).
q_{check_ones}	$R_O \geq S_O$	q_{sub_ones}	-	Sufficient ones. No borrow needed.
q_{check_ones}	$R_O < S_O$	$q_{decompose}$	-	Insufficient ones. Borrow.
$q_{decompose}$	$R_T > 0$	q_{sub_ones}	$R_T \leftarrow R_T - 1$; $R_O \leftarrow R_O + Base$	Decompose (borrow) one ten.
q_{sub_ones}	-	q_{accept}	$R_O \leftarrow R_O - S_O$; Result $\leftarrow R_T \cdot Base + R_O$	Subtract ones and combine result.

Choreography. Inversion of sublation: temporal decompression of a ten into ten ones to restore operability.

Lineage. Builds on internalized carry (from counting) now executed in reverse.

B.11 Subtraction Rounding and Adjusting

Description. Dual rounding (e.g., $M \to M'$ down, $S \to S'$ down) yields simplified $M' - S'$, then contrasting compensations: add K_M, subtract K_S.

Transition Function (δ):

Current State	Action	Next State	Interpretation
q_{start}	Read M, S	q_{roundM}	Start.
q_{roundM}	$M' \leftarrow$ RoundDown(M); $K_M \leftarrow M - M'$	q_{roundS}	Round M down. Store adjustment K_M.
q_{roundS}	$S' \leftarrow$ RoundDown(S); $K_S \leftarrow S - S'$	$q_{subtract}$	Round S down. Store adjustment K_S.
$q_{subtract}$	$Temp \leftarrow M' - S'$	$q_{adjustM}$	Calculate intermediate result.
$q_{adjustM}$	$Temp \leftarrow Temp + K_M$	$q_{adjustS}$	Compensate for M (Add back).
$q_{adjustS}$	$Result \leftarrow Temp - K_S$ (via chunking)	q_{accept}	Compensate for S (Subtract).

Choreography. Opposed adjustments highlight subtraction asymmetry: modification of minuend vs. subtrahend impacts result in inverse directions.

Lineage. Integrates rounding (addition strategy) and inverse compensation sequencing.

B.12 Subtraction Sliding (Constant Difference)

Description. Find K so that $S+K$ is a base (or friendly) number; compute $(M+K)-(S+K)$ exploiting invariance: $M - S = (M+K) - (S+K)$.

Transition Function (δ):

Current State	Action	Next State	Interpretation
q_{start}	Read M, S	q_{calcK}	Start. Target S for adjustment.
q_{calcK}	$K \leftarrow$ CountUpToBase(S)	q_{slide}	Iteratively find the gap K.
q_{slide}	$M' \leftarrow M + K$; $S' \leftarrow S + K$	$q_{subtract}$	Apply the slide K to both M and S.
$q_{subtract}$	$Result \leftarrow M' - S'$	q_{accept}	Perform the simplified subtraction.

Choreography. Up-front decompression (deriving K) enables single compressed subtraction against a base-aligned subtrahend.

Lineage. Extends RMB gap-finding; anticipates relational "distance" framing central to subtraction fluency.

B.13 Commutative Reasoning (Multiplication Optimization)

Description. For $A \times B$, evaluate heuristic difficulty of (A, B) vs (B, A); select orientation minimizing cognitive load (iteration count & skip difficulty), then perform iterative addition (skip counting).

Transition Function (δ):

Current State	Condition / Heuristic	Next State	Action
$q_{evaluate}$	$H(B, A) < H(A, B)$	$q_{repackage}$	-
$q_{evaluate}$	(Otherwise)	q_{calc}	$Groups \leftarrow A$; $Items \leftarrow B$

Current State	Condition / Heuristic	Next State	Action
$q_{repackage}$	-	q_{calc}	$Groups \leftarrow B$; $Items \leftarrow A$
q_{calc}	-	q_{accept}	$Total \leftarrow$ IterativeAdd$(Groups, Items)$

Choreography. Meta-level selection precedes execution; commutative symmetry exploited for temporal compression.

Lineage. Builds on C2C / Skip Counting; introduces optimization layer.

B.14 Coordinating Two Counts (C2C)

Description. Foundational multiplication: nested counting – items within group, groups within total; total $T = N \cdot S$ emerges from exhaustive unit enumeration.

Transition Function (δ):

Current State	Condition	Next State	Action
q_{init}	-	q_{checkG}	$G \leftarrow 0, I \leftarrow 0, T \leftarrow 0$
q_{checkG}	$G < N$	$q_{countItems}$	-
q_{checkG}	$G == N$	q_{accept}	Output T
$q_{countItems}$	$I < S$	$q_{countItems}$	$I \leftarrow I + 1, T \leftarrow T + 1$
$q_{countItems}$	$I == S$	$q_{nextGroup}$	-
$q_{nextGroup}$	-	q_{checkG}	$G \leftarrow G{+}1, I \leftarrow 0$

Choreography. Maximal temporal decompression (no compression yet); establishes structural scaffold for later compression (skip counting, distributive reasoning).

Lineage. Direct elaboration of counting primitives into nested loops.

B.15 Conversion to Bases and Ones (CBO Multiplication)

Description. Redistribute units among groups so that many groups become exact base multiples, leaving a compact residual: $(k \cdot Base) + r$.

Transition Function (δ):

Current State	Condition	Next State	Action
q_{init}	-	q_{select_source}	Initialize $Groups$ array with value S.
q_{select_source}	$N > 0$	$q_{transfer}$	Select a $SourceIdx$.
$q_{transfer}$	$Groups[Source] > 0$ AND not all targets full	$q_{transfer}$	Transfer 1 unit from $Source$ to next available $Target$.
$q_{transfer}$	(Source empty OR all targets full)	$q_{finalize}$	-
$q_{finalize}$	-	q_{accept}	Total $\leftarrow \sum Groups$.

Choreography. Proactive sublation: simultaneous decompression (source group) and compression (targets) to manufacture base units early.

Lineage. Multiplicative analogue of RMB and addition Chunking with explicit inter-group transfers.

B.16 Distributive Reasoning (Multiplication)

Description. Decompose $S = S_1 + S_2$ (heuristically "easy" numbers), compute NS_1 and NS_2 (skip counting or compressed methods), then sum.

Transition Function (δ):

Current State	Action	Next State
q_{split}	$S_1, S_2 \leftarrow \text{HeuristicSplit}(S)$	q_{P1}
q_{P1}	$P_1 \leftarrow \text{IterativeAdd}(N, S_1)$	q_{P2}
q_{P2}	$P_2 \leftarrow \text{IterativeAdd}(N, S_2)$	q_{sum}
q_{sum}	$Total \leftarrow P_1 + P_2$	q_{accept}

Choreography. Temporal decompression (factor split) followed by parallelizable compressed sub-calculations and final recombination.

Lineage. Extends skip counting with heuristic structural decomposition; precursor to algebraic distributivity recognition.

B.17 Dealing by Ones (Division – Sharing)

Description. Partitive division: distribute single units round-robin into N groups until total T exhausted; per-group size S emerges.

Transition Function (δ):

Current State	Condition	Next State	Action
q_{init}	-	q_{deal}	$Remaining \leftarrow T$; Initialize $Groups$ array to 0s.
q_{deal}	$Remaining > 0$	q_{deal}	$Groups[idx] \leftarrow Groups[idx] + 1$; $Remaining \leftarrow Remaining - 1$; $idx \leftarrow (idx + 1) \pmod N$
q_{deal}	$Remaining == 0$	q_{accept}	Output $Groups[0]$

Choreography. Maximal temporal decompression; rhythmic rounds establish invariant increase pattern (foundation for later compression insights).

Lineage. Inversion of C2C perspective (constructing equal groups from total rather than composing total from groups).

B.18 Inverse Distributive Reasoning (Division)

Description. Measurement division T/S: decompose T into known multiples of S: $T = \sum_i (m_i S)$; quotient $= \sum_i m_i$.

Transition Function (δ):

Current State	Condition	Next State	Action
q_{init}	-	q_{search}	$Remaining \leftarrow T$; $TotalQ \leftarrow 0$; Load KB for S.
q_{search}	Found (P_T, P_Q) in KB where $P_T \leq Remaining$	q_{apply}	Select largest such (P_T, P_Q).
q_{search}	No suitable fact found	q_{accept}	Output $TotalQ$.
q_{apply}	-	q_{search}	$Remaining \leftarrow Remaining - P_T$; $TotalQ \leftarrow TotalQ + P_Q$.

Choreography. Temporal compression via retrieval of pre-compressed multiplication facts; loop greedily subtracts largest available chunk.

Lineage. Inversion of Distributive Reasoning in multiplication (switch from constructing product to decomposing dividend).

B.19 Using Commutative Reasoning (Division via Iterated Accumulation)

Description. For E/G (sharing reframed as measurement): iteratively accumulate G until total E reached; iteration count is quotient.

Transition Function (δ):

Current State	Condition	Next State	Action
q_{init}	-	$q_{iterate}$	$Acc \leftarrow 0$; $Q \leftarrow 0$.
$q_{iterate}$	-	q_{check}	$Acc \leftarrow Acc + G$; $Q \leftarrow Q + 1$.
q_{check}	$Acc < E$	$q_{iterate}$	-
q_{check}	$Acc == E$	q_{accept}	Output Q.

Choreography. Symmetric inversion of repeated addition (multiplication) focusing on completion criterion instead of fixed loop count.

Lineage. Bridges between Dealing by Ones and chunk-based division (fact retrieval).

B.20 Conversion to Groups Other than Bases (CGOB Division)

Description. Leverage base decomposition of dividend T (e.g., tens & ones) plus analysis of base/divisor relation: $Base = q_1 S + r_1$; process all base units in bulk, aggregate remainders, finalize.

Transition Function (δ):

Current State	Action	Next State
q_{init}	Decompose T into T_B, T_O; $Q \leftarrow 0, R \leftarrow 0$.	$q_{analyze}$
$q_{analyze}$	$S_{inB} \leftarrow B//S$; $R_{inB} \leftarrow B \pmod{S}$.	$q_{processBases}$
$q_{processBases}$	$Q \leftarrow Q + T_B \cdot S_{inB}$; $R \leftarrow R + T_B \cdot R_{inB}$.	$q_{combineR}$
$q_{combineR}$	$R \leftarrow R + T_O$.	$q_{processR}$
$q_{processR}$	$Q \leftarrow Q + R//S$; $R \leftarrow R \pmod{S}$.	q_{accept}

Choreography. Dual decompression (dividend by base, base by divisor) → large compression (bulk quotient) → residual resolution.

Lineage. Division analogue of CBO (multiplication) and Distributive Reasoning; integrates multi-level structural analysis.

B.21 Conceptual Dependency Graph (Narrative)

Counting → (RMB, COBO) → (Chunking, Rounding & Adjusting, Sliding) → (Subtraction Inversions: COBO/CBBO, Chunking orientations, Decomposition, Rounding, Sliding) → (C2C) → (Skip Counting / implicit in COBO Multiplication) → (Commutative & Distributive Reasoning, CBO Multiplication) → (Division primitives: Dealing by Ones, Iterated Accumulation) → (Inverse Distributive Reasoning, Fact-Based Decomposition) → (CGOB Division).

Each arrow denotes an algorithmic elaboration where prior compressed units or reversible decompositions become callable subroutines.

B.22 Temporal Dynamics Summary

- **Primitive Decompression:** Counting by ones; Dealing by Ones; C2C (inner loop).
- **First Compression Layer:** COBO (bases as units); subtraction COBO/CBBO; iterative accumulation for division.
- **Strategic Boundary Forcing:** RMB, Chunking, Sliding, Rounding (anticipatory manipulation of base thresholds).
- **Structural Decomposition / Synthesis:** Distributive Reasoning, Inverse Distributive Reasoning, CBO (Multiplication & Division), CGOB.

B.23 Glossary of Symbols

- $Base$: Typically 10 (extendable to other positional bases).
- K: Gap to next base (RMB, Rounding, Chunking, Sliding).
- R: Remainder after decomposition or partial processing.
- S_1, S_2: Split components of a factor (Distributive Reasoning).
- $Groups, Items$: Multiplicative roles after commutative optimization.
- $Remaining$: Unprocessed portion of a dividend in division strategies.
- Q: Quotient / accumulated result in division; also generic state set symbol context-dependent.
- Acc: Accumulated total during iterative division.

B.24 Fractions

I now turn to fractions. Rather than recreating the wheel, I will demonstrate how to transpose extant research in the field of math education into a pragmatic key. The rich psychological models of a student's mathematical development, grounded in Leslie P. Steffe's radical constructivism, are translated into a formal, executable automaton. The goal is to re-key the descriptive insights of constructivism into a pragmatic framework that aligns with the spirit of Robert Brandom's analytic pragmatism, where knowing-how is made explicit in the form of a communicable knowing-that. I model the cognitive schemes of a student, "Jason," as a set of nested finite automata, moving from the psychological description of his mental operations to a computational specification. The resulting executable model demonstrates how cognitive development, particularly the "metamorphic accommodations" identified by constructivists, can be understood as the reorganization and nesting of formal procedures. The output of the model, a structured history of its operations, serves as a formal representation of the constructed meaning of a mathematical concept.

Leslie P. Steffe's work on the mathematical development of children offers a detailed account of cognitive construction, framed within the epistemology of radical constructivism (Steffe, 2002). This perspective posits that learners actively build their own mathematical realities, not by discovering a pre-existing world, but by adapting their cognitive schemes to fit the constraints of their experience (Rowlands, 2005). The teaching experiment methodology provides a longitudinal lens through which researchers can build second-order models of this construction, creating what is termed the "mathematics of students" (Steffe & Thompson, 2014).

My objective here is not to question the descriptive validity of this constructivist model but to transpose it into a different philosophical key. I aim to translate the psychological description of one student's learning trajectory – that of "Jason" – into a formal, executable model. This act of translation serves a purpose aligned with analytic pragmatism: to make the implicit practices and cognitive choreography of mathematical knowing explicit in a formal structure. The resulting automaton is a pragmatic interpretation of Jason's mathematical schemes, where the focus shifts from the student's private experiential world to the public, formal specification of a viable cognitive system.

B.25 The Operational Foundation: The Explicitly Nested Number Sequence (ENS)

The foundation for Jason's later fractional knowledge was his whole-number operating system, what Steffe terms the Explicitly Nested Number Sequence (ENS) (Steffe, 2002). The ENS is an interiorized counting scheme that allows a child to treat numbers and number sequences as objects of thought. The power of the ENS lies in a set of core mental operations that I will treat as the primitive functions of my computational model:

- **Iteration:** The ability to take a composite unit (e.g., a "six") as a single item and repeat it to produce larger quantities (Steffe, 2014).

- **Disembedding:** The ability to mentally isolate a numerical part from a whole without destroying the conceptual integrity of the whole (Steffe, 2002).

- **Coordination of Units:** The capacity to flexibly view a number, such as twelve, as a single unit of twelve, a composite unit of twelve ones, and as a part of a larger sequence (Steffe, 2004).

These operations form the iterative core of the automaton, the fundamental actions from which more complex, goal-directed schemes are built.

B.26 The Emergence of Fractional Schemes

According to the "reorganization hypothesis," fractional knowledge emerges as an accommodation of these whole-number schemes when applied to continuous quantities (Steffe, 2002). My formal model captures this process by defining strategic shells – automata – that organize the core operations to solve new kinds of problems.

The Partitive Fractional Scheme (PFS)

Jason's first major accommodation was the construction of a Partitive Fractional Scheme (PFS), his primary tool for producing a proper fraction, such as making $\frac{3}{7}$ of a continuous unit (a "stick") (Steffe, 2003). This scheme repurposes the ENS operations: the number sequence (1 to 7) is used as a template to *partition* the stick, one part (the $\frac{1}{7}$ unit fraction) is *disembedded*, and that part is then *iterated* three times.

I model this goal-directed activity as a finite automaton, $M_{PFS} = (Q, V, \delta, q_0, F)$, which orchestrates the core operations.

- **States (Q):** $\{q_{start}, q_{partition}, q_{disembed}, q_{iterate}, q_{accept}\}$
- **Variables (V):** {Whole, N (Denominator), M (Numerator), PartitionedWhole, UnitFraction, Result}
- **Transition Function (δ):** The function choreographs the sequence of core operations, moving from partitioning the whole, to disembedding the unit part, to iterating that part to form the final fraction.

The sequence is not merely a list of steps but a structured, goal-directed procedure.

Metamorphic Accommodation: Recursive Partitioning

A pivotal event in Jason's development was his spontaneous invention of a method for finding a fraction of a fraction (e.g., $\frac{3}{4}$ of $\frac{1}{4}$ of a stick), an act Steffe identifies as a "metamorphic accommodation" (Steffe, 2003). This signaled the construction of a new operation: *recursive partitioning*. Jason could take the *result* of one partitioning operation and use it as the *input* for a subsequent one.

This cognitive leap is modeled by a second automaton, the Fractional Composition Scheme (FCS). The architecture of the FCS demonstrates a fractal-like elaboration: it is a strategic shell that calls the entire PFS automaton as a subroutine. This nesting of a previously constructed scheme to solve a more complex problem is the formal analogue of the psychological process of accommodation. The critical step in the FCS automaton is the state $q_{accommodate}$, where the output of the first PFS execution (the intermediate fraction) is formally re-assigned as the input "Whole" for the second PFS execution.

B.27 The Formal Automaton Model

The psychological descriptions are translated into a formal Python implementation. The 'ContinuousUnit' class represents the cognitive material, tracking not only its numerical quantity but also its operational history, thereby capturing the constructed meaning of the number. The core ENS operations are static methods, and the schemes (PFS and FCS) are implemented as state machine classes.

Python Implementation

```python
import fractions
from typing import List, Tuple

class ContinuousUnit:
    """Represents a continuous quantity, tracking both value and history."""
    def __init__(self, value: fractions.Fraction, history: str = "Reference Unit"):
        self.value = value
        self.history = history
    def __repr__(self):
```

```python
        return f"Unit({self.value} derived from: '{self.history}')"

class ENSOperations:
    """The iterative core operations derived from Jason's ENS."""
    @staticmethod
    def partition(unit: ContinuousUnit, n: int) -> List[ContinuousUnit]:
        new_value = unit.value / n
        new_history = f"1/{n} part of ({unit.history})"
        return [ContinuousUnit(new_value, new_history) for _ in range(n)]
    @staticmethod
    def disembed(partitioned_whole: List[ContinuousUnit]) -> ContinuousUnit:
        return partitioned_whole[0]
    @staticmethod
    def iterate(unit: ContinuousUnit, m: int) -> ContinuousUnit:
        new_value = unit.value * m
        new_history = f"{m} iterations of [{unit.history}]"
        return ContinuousUnit(new_value, new_history)

class PartitiveFractionalScheme:
    """[SHELL::PFS] Automaton model for constructing proper fractions."""
    def __init__(self):
        self.F = {'q_accept'}
        self.V = {}
        self.trace = []
    def run(self, whole: ContinuousUnit, num: int, den: int) -> ContinuousUnit:
        # Simplified for brevity: direct implementation of the state logic
        self.trace.append("State: q_partition -> Partitioning Whole")
        partitioned = ENSOperations.partition(whole, den)
        self.trace.append("State: q_disembed -> Disembedding Unit Fraction")
        unit_fraction = ENSOperations.disembed(partitioned)
        self.trace.append(f"State: q_iterate -> Iterating Unit Fraction {num} times")
        result = ENSOperations.iterate(unit_fraction, num)
        self.trace.append("State: q_accept -> PFS Complete")
        return result

class FractionalCompositionScheme:
    """[SHELL::FCS] Models recursive partitioning by nesting the PFS."""
    def __init__(self):
        self.F = {'q_accept'}
        self.PFS = PartitiveFractionalScheme()
        self.trace = []
```

```
    def run(self, whole: ContinuousUnit, outer: Tuple[int, int], inner: Tuple[int, i
        A, B = outer
        C, D = inner
        self.trace.append("State: q_inner_PFS -> Calculating inner fraction")
        intermediate_result = self.PFS.run(whole, C, D)
        self.trace.append("State: q_accommodate -> METAMORPHIC ACCOMMODATION")
        new_whole = intermediate_result
        self.trace.append("State: q_outer_PFS -> Calculating outer fraction")
        final_result = self.PFS.run(new_whole, A, B)
        self.trace.append("State: q_accept -> FCS Complete")
        return final_result
```

Execution and Analysis

Running the model replicates Jason's cognitive choreography.

Test 1: Partitive Fractional Scheme ($\frac{3}{7}$ of a Whole)

```
Execution Trace:
State: q_partition -> Partitioning Whole
State: q_disembed -> Disembedding Unit Fraction
State: q_iterate -> Iterating Unit Fraction 3 times
State: q_accept -> PFS Complete

RESULT: Unit(3/7 derived from: '3 iterations of [1/7 part of (Reference Unit)]
```

Test 2: Fractional Composition Scheme ($\frac{3}{4}$ of $\frac{1}{4}$ of a Whole)

```
Execution Trace:
State: q_inner_PFS -> Calculating inner fraction
State: q_accommodate -> METAMORPHIC ACCOMMODATION
State: q_outer_PFS -> Calculating outer fraction
State: q_accept -> FCS Complete

RESULT: Unit(3/16 derived from: '3 iterations of
    [1/4 part of (1 iterations of [1/4 part of (Reference Unit)])]')
```

The execution trace makes the temporal unfolding of Jason's scheme explicit. The result of Test 2 is not the bare quantity $\frac{3}{16}$, but a quantity whose meaning is constituted by its operational history. The output string is a formal record of the nested operations that produced the result, making the constructed nature of the concept transparent. This aligns with a pragmatist view where meaning is rooted in practice.

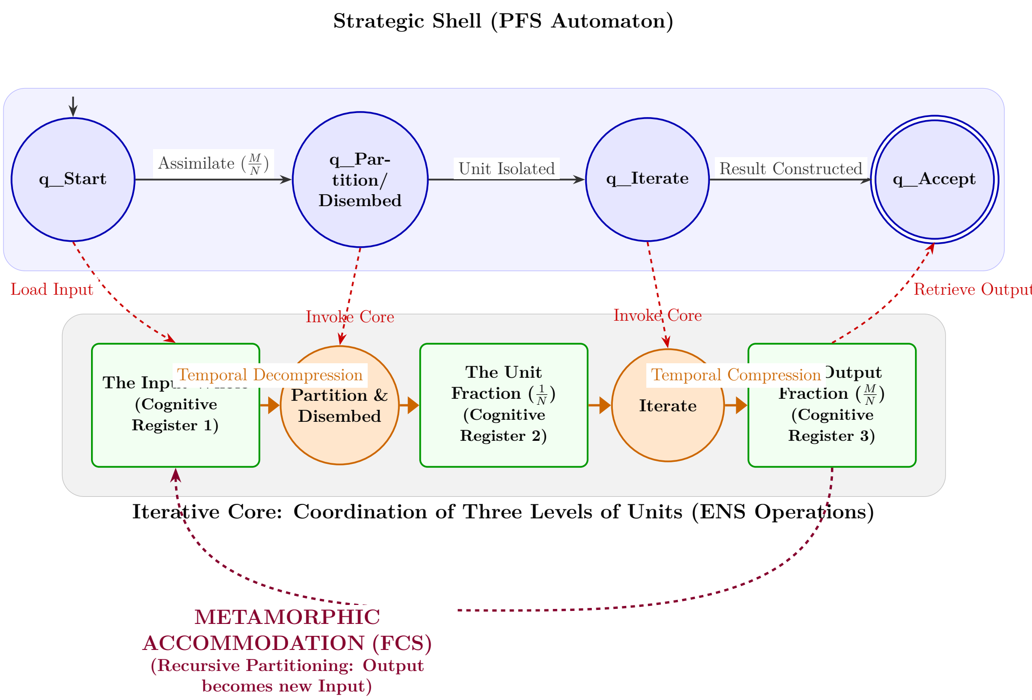


Figure B.2: Visualization of Jason's fractional reasoning architecture. The Strategic Shell (PFS) choreographs the Iterative Core (ENS Operations) to coordinate the Three Levels of Units (Input Whole, Unit Fraction, Output Fraction). The recursive pathway (purple) illustrates the reorganization into the Fractional Composition Scheme (FCS), where the scheme operates on its own output (fractal elaboration).

B.28 Conclusion: From Psychological Description to Pragmatic Specification

This project successfully translates the psychological description of Jason's mathematical schemes into a formal, executable automaton. This translation re-keys radical constructivism into a pragmatic framework. While constructivism focuses on modeling the student's viable, private reality, this formal model provides a public, verifiable specification of a cognitive practice.

The automaton does not claim to be a "true" picture of Jason's mind. Rather, it is a formal model of the *choreography of his reasoning*. The execution trace makes the inferential steps of his practice explicit. The "metamorphic accommodation" of recursive partitioning is modeled not as an inexplicable insight, but as a structural change in the automaton: the nesting of one goal-directed procedure within another.

In this pragmatic key, understanding a concept like "$\frac{3}{16}$" is not about having a mental representation that corresponds to reality, but about possessing the structured, operational capacity to produce it. The automaton's final output, with its embedded history, is a formal representation of this capacity. It is a piece of "knowing-that" which explicitly codifies the "knowing-how" of Jason's mathematical practice.

Bibliography

Abbott, E. A. (2020, April). *Flatland: A Romance of Many Dimensions by Edwin A. Abbott*. Independently published.

Agamben, G. (2006, September). *Language and Death: The Place of Negativity* (K. Pinkus & M. Hardt, Trans.; First Edition). Univ Of Minnesota Press.

Aristotle. (2017). *Metaphysics: Aristotle's metaphysics in english, latin and ancient greek: Trilingual edition* (W. o. Moerbeke & W. D. Ross, Trans.; Trilingual). Hermes Language Reference.

Boote, D. N., & Beile, P. (2005). Scholars before researchers: On the centrality of the dissertation literature review in research preparation. *Educational researcher*, *34*, 3–15.

Brandom, R. B. (1994). *Making it explicit: Reasoning, representing, and discursive commitment*. Harvard university press.

Brandom, R. B. (2000). *Articulating reasons: An introduction to inferentialism*. Harvard University Press.

Brandom, R. B. (2008). *Between saying and doing: Towards an analytic pragmatism*. Oxford University Press.

Brandom, R. B. (2019). *A spirit of trust: A reading of Hegel's phenomenology*. The Belknap Press of Harvard University Press.

Bueno, O. (2020). Nominalism in the Philosophy of Mathematics. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Fall 2020). Metaphysics Research Lab, Stanford University.

Cantor, G. (1891). Ueber eine elementare frage der mannigfaltigkeitslehre. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, *1*, 75–78.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). Children's mathematics: Cognitively guided instruction - videotape logs [supplementary material]. In *Children's mathematics: Cogni-*

tively guided instruction. Heinemann, in association with The National Council of Teachers of Mathematics, Inc.

Carspecken, P. F. (1995). *Critical ethnography in educational research: A theoretical and practical guide*. Routledge.

Carspecken, P. F. (1999). *Four scenes for posing the question of meaning and other essays in critical philosophy and critical methodology* (New edition). Peter Lang Inc., International Academic Publishers.

Carspecken, P. F. (2006). Limits of knowledge in the physical sciences. In K. Tobin & J. Kincheloe (Eds.), *Doing educational research* (pp. 405–437, Vol. 1). Sense Publishers.

Carspecken, P. F. (2009). Limits to knowledge with existential significance: An outline for the exploration of post-secular spirituality of relevance to qualitative research. In *Bridging the gap between theory and practice in educational research* (pp. 47–61). Springer.

Carspecken, P. F. (2016). Limits to knowledge and being human: What is 'critical' in critical social research and theory? In P. H. Hinchey (Ed.), *A critical action research reader* (pp. 65–86). Peter Lang Inc., International Academic Publishers.

Carspecken, P. F. (2018). The missing infinite. In R. Winkle-Wagner, J. Lee-Johnson, & T. Gaskew (Eds.), *Critical theory and qualitative data analysis in education* (p. 32). Routledge. https://doi.org/10.4324/9781315158860-2

Carspecken, P. F., & Zhang, R. (2013). Content inference fields in intersubjective space: Transpersonal position, illocution, and logic in the analysis of human interactions. *Qualitative research: A reader in philosophy, core concepts, and practice*, 201–242.

Carspecken, R. (2024, November 3). Untitled.

Caygill, H. (2004). *A kant dictionary*. Wiley-Blackwell.

Derrida, J. (1973). *Speech and phenomena, and other essays on husserl's theory of signs*. Northwestern University Press.

Derrida, J. (1982). *Marges de la philosophie* [Margins of philosophy] (A. Bass, Trans.). The Harvester Press. (Original work published 1972)

Deutsch, H., Marshall, O., & Irvine, A. D. (2025). Russell's Paradox. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford encyclopedia of philosophy* (Spring 2025). Metaphysics Research Lab, Stanford University.

Ernest, P. (2024). The mathematical and philosophical significance of zero: Nought matters [Publisher: Paul Ernest, Emeritus Professor at University of Exeter, School of Education & Lifelong Learning]. *Philoso-*

phy of Mathematics Education Journal, 1–35. https://research.ebsco.com/linkprocessor/plink?id=b685ecb9-d2fc-3e03-889d-a06fb1751c1b

Euclid. (2007, December). *Euclid's Elements of Geometry* (J. L. Heiberg, Ed.; R. Fitzpatrick, Trans.; Revised and corrected edition). Richard Fitzpatrick.

Franzén, T. (2004, September). *Inexhaustibility: A Non-Exhaustive Treatment: Lecture Notes in Logic 16* (1st edition). A K Peters/CRC Press.

Franzén, T. (2005). *Gödel's theorem: An incomplete guide to its use and abuse*. AK Peters/CRC Press.

Frege, G. (1997, July). *The Frege Reader* (M. Beaney, Ed.; 1st edition). Blackwell Publishing.

Gaifman, H. (2005). *The easy way to Gödel's proof and related matters* [Unpublished manuscript. Available online].

Geisel, T. (1961). *The Sneetches and Other Stories* (First edition.). Random House Books for Young Readers.

Gelder, T. V. (1995). What might cognition be, if not computation? *The Journal of Philosophy*, *92*, 345–381. Retrieved April 13, 2025, from http://www.jstor.org/stable/2941061

Gray, L., & Garzon, M. (2005). On a hermite boundary integral approximation. *Computers & structures*, *83*(10-11), 889–894.

Greenblatt, R., Denison, C., Wright, B., Roger, F., MacDiarmid, M., Marks, S., Treutlein, J., Belonax, T., Chen, J., Duvenaud, D., Khan, A., Michael, J., Mindermann, S., Perez, E., Petrini, L., Uesato, J., Kaplan, J., Shlegeris, B., Bowman, S. R., & Hubinger, E. (2024, December 23). Alignment faking in large language models. https://doi.org/10.48550/arXiv.2412.14093

Habermas, J. (1985). *The theory of communicative action: Lifeworld and system* (Vol. 2). Beacon Press.

Habermas, J. (2004). *The philosophical discourse of modernity: Twelve lectures* (F. G. Lawrence, Trans.). MIT Press.

Habermas, J. (1971). *Knowledge and human interests*. Beacon Press.

Habermas, J. (1984). *The theory of communicative action, volume 1: Reason and the rationalization of society* (T. McCarthy, Trans.; Reprint edition). Beacon Press.

Habermas, J. (1995). *Postmetaphysical thinking: Between metaphysics and the critique of reason* (1st edition). Polity.

Hackenberg, A. (2025). *Course notes* [Unpublished course notes].

Hegel, G. W. F. (1977). *Phenomenology of spirit* (A. V. Miller, Trans.; Vol. 632). Oxford University Press Oxford.

Hegel, G. W. F. (2010). *The science of logic* (G. di Giovanni, Ed. & Trans.). Cambridge University Press.

Houlgate, S. (2006). *The opening of hegel's logic: From being to infinity*. Purdue University Press.

Hyppolite, J. (1974). *Genesis and structure of hegel's phenomenology of spirit*. Northwestern University Press.

Kant, I. (1998). *Critique of pure reason* [Original work published 1781/1787, Guyer and Wood, Trans.]. Cambridge University Press.

Lakoff, G., & Johnson, M. (1980). *Metaphors we live by*. University of Chicago Press.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books.

Linnebo, Ø. (2024). Platonism in the Philosophy of Mathematics. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford encyclopedia of philosophy* (Summer 2024). Metaphysics Research Lab, Stanford University.

Longuenesse, B. (2017). *I, me, mine: Back to kant and fichte on the 'i'*. Oxford University Press.

Mead, G. H. (1934). *The social psychology of george herbert mead* (A. L. Strauss, Ed.). University of Chicago Press.

Mead, G. H. (2015). *Mind, self, and society: The definitive edition* (H. Joas, D. R. Huebner, & C. W. Morris, Eds.; Enlarged edition). University of Chicago Press.

Meyer, J. (title). *Cantors 1891 diagonal proof* [Cantor's original 1891 diagonal proof]. Retrieved July 6, 2025, from https://jamesrmeyer.com/infinite/cantors-original-1891-proof

Nagel, E., & Newman, J. R. (2012). *Godel's proof*. Routledge.

Negarestani, R. (2018). *Intelligence and Spirit*. Urbanomic/Sequence Press.

Poston, T., & Stewart, I. (2012, February 29). *Catastrophe theory and its applications* (First Edition). Dover Publications.

Priest, G. (2002). *Beyond the limits of thought*. Oxford University Press.

Priest, G. (2014). *One: Being an investigation into the unity of reality and of its parts, including the singular object which is nothingness*. Oxford University Press.

Quine, W. V. O. (1948). On what there is. In *Thinking about logic* (pp. 189–209). Catholic University of America. https://sites.pitt.edu/~rbrandom/

Courses/2023%20Sellars/Sellars%20texts/Quine%20On_What_There_Is.pdf

Quine, W. V. O. (1963). Two dogmas of empiricism [First Harper Torchbook edition, a reprint of the second, revised edition (1961). The book was first published by Harvard University Press in 1953.]. In *From a logical point of view* (Second, revised, pp. 20–46). Harper & Row.

Rödl, S. (2014). Intentional transaction [Accessed: 2025-04-13]. *Philosophical Explorations: An International Journal for the Philosophy of Mind and Action*, *17*, 304–316. https://doi.org/10.1080/13869795.2014.941909

Rose, G. (2009). *Hegel contra sociology*. Verso.

Rowlands, S. (2005). The contradictions in the constructivist discourse [Citing von Glasersfeld]. *Philosophy of Mathematics Education Journal*, *14*.

Savich, T. M. (2020). Towards a critical mathematics [Accessed via ERIC]. *Mathematics Teaching–Research Journal Online*, *12*, 245–291. https://files.eric.ed.gov/fulltext/EJ1384454.pdf

Savich, T. M. (2022). *Towards a critical mathematics*. Indiana University.

Savich, T. M. (2024). Built to break: An introduction to theoretical autoethnography [Invited talk presented at Solent University, organized by Dr. Brian McDonough]. https://iu.mediaspace.kaltura.com/media/t/1_v1qmm3g9

Savich, T. M. (title). *The hermeneutic calculator* [The hermeneutic calculator]. Retrieved July 23, 2025, from https://tiosavich.github.io/UMEDCTA/Calculator/index.html

Sellars, W. (1953). Inference and meaning. *Mind*, *62*, 313–338.

Sellars, W. (2007). *In the space of reasons: Selected essays of wilfrid sellars*. Harvard University Press.

Steffe, L. P. (2002). A new hypothesis concerning children's fractional knowledge [Accessed through CEPA.INFO]. *Journal of Mathematical Behavior*, *20*(3), 267–307.

Steffe, L. P. (2003). On the construction of learning trajectories of children: The case of commensurate fractions [Accessed via ResearchGate/ERIC]. *Monographs of the Journal for Research in Mathematics Education*, 1–26.

Steffe, L. P. (2004). Fractional commensurate, composition, and adding schemes: Learning trajectories of jason and laura: Grade 5 [Accessed via ResearchGate]. In J. Olive & R. Tzur (Eds.), *Children's fractional knowledge* (pp. 1–60). American Educational Research Association.

Steffe, L. P. (2014). *Children's number sequences: An explanation of steffe's constructs and an extrapolation to rational numbers of arithmetic* [Originally from 1994]. Unpublished manuscript, available via CiteSeerX; ResearchGate.

Steffe, L. P., & Thompson, P. W. (2014). Teaching experiment methodology: Underlying principles and essential elements [Accessed via ResearchGate, originally published 2000]. *Proceedings of the 24th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.*

Stone, J. (2003). *The monster at the end of this book* (2nd ed. edition). Golden Books.

Taylor, C. (1994). *Multiculturalism: Examining the Politics of Recognition* (A. Gutmann, Ed.) [Essays by Charles Taylor, K. Anthony Appiah, Jürgen Habermas, Steven C. Rockefeller, Michael Walzer, and Susan Wolf]. Princeton University Press.

van Hiele-Geldof, D., & van Hiele, P. M. (1984). *English translation of selected writings of dina van hiele-geldof and pierre m. van hiele* (D. Fuys, D. Geddes, & D. Tischler, Eds.) [ERIC Number: ED287697]. Retrieved June 3, 2025, from https://eric.ed.gov/?id=ED287697

Wittgenstein, L. (1956). *Remarks on the foundations of mathematics.*

Wittgenstein, L. (2010). *Philosophical investigations* (P. M. S. Hacker & J. Schulte, Eds.; G. Anscombe, Trans.). Wiley-Blackwell.

Appendix: Summary of Jason's Mathematical Schemes

Table B.22: Synthesis of Jason's Mathematical Schemes and Constituent Operations

Scheme / Operational State	Triggering Situation (Input)	Core Mental Operations (Transition Function)	Result (Output)
Explicitly Nested Number Sequence (ENS)	Whole-number tasks requiring part-whole reasoning.	- Iterating composite units. - Disembedding a numerical part. - Coordinating three levels of units.	A numerical quantity or relation.
Partitive Fractional Scheme	Task to construct a proper fraction ($\frac{m}{n}$) of a whole.	- Partitioning the whole into n parts. - Disembedding one unit part ($\frac{1}{n}$). - Iterating the unit part m times.	A new continuous quantity representing the fraction $\frac{m}{n}$.
Recursive Partitioning Operation	Novel task to find a fraction of a fractional part.	- Taking the result of a partitioning operation as the input for a subsequent partition.	A new unit fraction related to the whole via composition.
Fractional Composition Scheme	Generalized task to find a fraction of a fraction.	- Stabilized application of the recursive partitioning operation.	A new fractional quantity representing the product.