A Foundational Star: Diagonalization and the Dialectic of Sense-Certainty

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Abstract

This chapter examines a profound pattern that emerges across seemingly disparate domains—ancient mathematics, modern set theory, German idealist philosophy, and children's literature. I bring together Euclid's proof of infinite primes, Cantor's diagonal argument for the uncountability of real numbers, and Hegel's dialectic of sense-certainty to explore how claims of completeness or immediacy inevitably generate their own transcendence.

The common thread uniting these domains is a specific structural pattern: when a bounded totality is assumed to be complete, an element can be constructed from within that totality which necessarily exceeds its boundaries. Euclid shows how any finite list of primes generates a new prime outside that list. Cantor demonstrates that even an infinite enumeration of real numbers produces, through diagonalization, a number that escapes the list. Hegel's sense-certainty reveals how attempts to grasp the immediate "this-here-now" transform the particular into a universal through the very act of knowing.

I ground this exploration in Dr. Seuss's *The Sneetches* — a story where rigid social categories based on stars "upon thars" ultimately dissolve when the Sneetches recognize their own *in*finitude. The shyster McBean's interminable star-on, star-off machine is akin to Cantor's 'flipping' function, while the whole story becomes a metaphor for the dialectical process through which fixed boundaries become permeable, and supposed-certainties transform into their opposites.

While there are important differences between these arguments, their shared structure illuminates how mathematical and philosophical understanding progress through the accrual of incompatibilities. In each case, what was implicit becomes explicit, what appeared bounded reveals itself as unbounded, and what was presented as immediate discloses its mediation. This pattern of dialectical movement — where an assumed completeness generates its own beyond — offers valuable insights for critical understanding across disciplines.

The chapter does not merely juxtapose these different realms but shows how they mutually illuminate each other. Euclid and Cantor's proofs provide concrete illustrations of Hegel's abstract dialectic, while Hegel's philosophy helps us see these mathematical discoveries as more than technical results — they are profound revelations about how knowledge transcends its own limitations. And the Sneetches remind us that these seemingly abstract patterns have existential significance, embodying the journey from rigid categorization to mutual recognition.

Theodor Geisel's (1961) book *The Sneetches* begins with the statement of a problem:

> Now, the Star-Belly Sneetches
> Had bellies with stars.
> The Plain-Belly Sneetches
> Had none upon thars.
>
> Those stars weren't so big. They were really so small
> You might think such a thing wouldn't matter at all.
> But, because they had stars, all the Star-Belly Sneetches would brag,
> "We're the best kind of Sneetch on the beaches."
> With their snoots in the air, they would sniff and they'd snort
> "We'll have nothing to do with the Plain-Belly sort!"

As the story unfolds, Sylvester McMonkey McBean arrives with a contraption that can put a star on the Plain-Belly Sneetches for three dollars. This disrupts the class system, and so McBean offers to remove the stars from the Star-Belly Sneetches for just "ten dollars eaches." Chaos ensues as McBean denudes the Sneetches of their dollars and stars or puts them back on.

> They kept paying money. They kept running through
> Until neither the Plain nor the Star-Bellies knew
> Whether this one was that one...or that one was this one
> Or which one was what one...or what one was who.

After McBean gets all of their money, he drives off laughing about how "you can't teach a Sneetch!"

> But McBean was quite wrong. I'm quite happy to say
> The Sneetches got really quite smart on that day,

The day they decided that Sneetches are Sneetches
And no kind of Sneetch is the best on the beaches.
That day, all the Sneetches forgot about stars
And whether they had one, or not, upon thars.

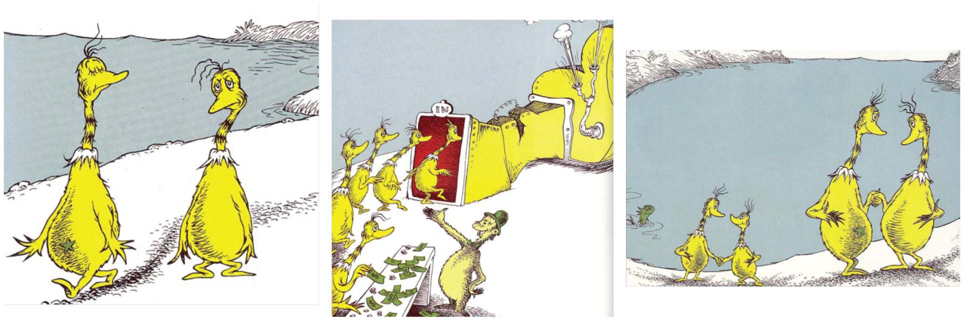

Figure 1: Stars form a demarcation between the two classes in the Sneetch society. The shyster, McBean, gets their dollars and rides away laughing at their foolishness, assuming they would never change. But he taught a strong enough lesson for Sneetch society to evolve.

1 Introduction

In this chapter, I explore how certain mathematical arguments and philosophical dialectics share a profound structural pattern: both reveal how claims of completeness or certainty contain within themselves the seeds of their own negation. In each case, an assumption of determined limits (completeness) is developed through its internal logic to transcend those limits. The finite moves toward the ~~finite~~.

From mathematics, I draw on three classic arguments: Euclid's proof of infinite primes, Cantor's diagonal argument, and references to Gödel's incompleteness theorem. Each demonstrates how closed systems produce elements that necessarily lie outside their boundaries. These proofs widen the field of mathematical inquiry without destroying what came before — only the limitations are broken.

Parallel to these mathematical breakthroughs runs Hegel's dialectic, particularly his analysis of sense-certainty in the *Phenomenology of Spirit*. Just as Euclid shows that any finite list of primes must be incomplete, Hegel reveals that sense-certainty's claim to immediate, complete knowledge of the particular “this here now” is self-undermining. The moment one tries to express this supposedly pure immediate knowledge, one necessarily employs universals that transcend the particular. Hegel’s thesis - “In my view, which can be justified only by the exposition of the system itself, everything turns on grasping and expressing the True, not only as *Substance*, but equally as *Subject*” (Hegel, 1977, p. 46) - resonates with how a binding determinateness around a mathematical object is demonstrably unbound through the internal capacities (enabling conditions) of the binding. The historical antecedents from Euclid and others[1] are not just historical curiosities, but exemplify the Hegelian insight that knowledge *is* movement. In our era, where knowledge is often assumed to be a regurgitation of some fact, the dialectical progression that surges towards the *in*finite using localized incoherence as its engine has some pedagogical importance. I will not discuss that importance here, though readers may infer that I find some of what counts as teaching and learning in mathematics a bit of a weak sauce compared to what could be offered.

I do not intend to force these mathematical and philosophical insights into perfect alignment. Rather, I aim to illuminate their shared pattern of transcendence through internal critique. To ground this abstract discussion, I offer *The Sneetches* as a metaphorical touchstone - a foundational star. I hope the rhetorical paradox of using stars as a metaphor will resonate with the reader. When the Sneetches finally “forgot about stars and whether they had one, or not, upon thars,“ they exemplify the sublation of rigid categories into a richer, more inclusive understanding — much as Hegel’s dialectic and

[1] Ancient examples include Hippasus of Metapontum’s proof that $\sqrt{2}$ is irrational. Post-Cantorian examples like Russell’s paradox, which articulates the set of all sets that do not contain themselves, and Gödel’s proof that unites meta-mathematics with mathematics through *arithmetization* can also be conceptualized as instances of this pattern. Early reviewers: I have not yet finished the introduction of this book, but in it I discuss my daughter, Maddy’s, invention of the term “divaded” to describe things that are both inside and outside of each other at the same time. There is not an obvious paradox at work with instances of divaded things like a folder hanging out of a backpack. But a concept that is both ‘inside’ and ‘outside’ of itself has deep roots in mathematics, philosophy, and logic. Determinate negation is *divaded* in how it both determines objects through the exclusion of essential properties, like a red square is not a green square, which brings the essence of a concect - what it is not - inside the concept. Brandom (2019) uses Frege’s notion of *sense* (Frege, 1948) to articulate this as *reciprocal sense-dependence* - but I like Maddy’s term a little bit more for its simplicity and mathematical flair. Ultimately, it is the ~~“no”~~ that is both inside and outside itself - the constant becoming that arrests its own movement in recognition - that holds my attention. But each of these movements needs to be brought into careful exposition. I merely mention them here to give a few breadcrumbs about how all this is related to the overall project of emancipatory mathematics.

mathematical diagonalization move beyond initial limitations while preserving what was valuable. Though McBean appears as a manipulator of the Sneetches, his actions ultimately lead to their self-recognition and transformation. His actions allow the Sneetches to recognize their own *in*finitude.

The danger in this exploration is getting lost in technical vocabularies or formal abstractions. Should that become part of your experience as a reader, remember the Sneetches, who were so concerned with their stars until they learned to recognize their own *in*finitude through the harsh lessons of McBean. The story is ~~not~~ in the symbols, though we need some alphabet to express those deeper truths we already know but sometimes struggle to express.

Hegel's dialectical method is characterized by dynamic triads and the process of *Aufhebung* (usually translated as "sublation")[2]. Much like determinate negation, the "no" that says "no" to itself thereby incorporating both a determinate bound and the dissolution of that boundary in ~~"no"~~, *Aufhebung* has a double-edged meaning: it means both to negate or cancel something and to preserve it. In a dialectical transition, an initial concept is posited. In the context I explore here, that initial claim is one of completeness or immediacy—the claim 'stands on its own.' However, the act of positing cannot be understood or justified without undermining itself. The positing defeats itself.

In the dialectic of Sense-Certainty, the independence of the object—the particular {this} that is {here} {now}—cannot be understood without some ”I,“ the speaker or thinker of {this}. Later, the {I} is posited as independent. The independence of that {I} is canceled when the action of positing is recognized as involving a necessary second-person reflection on that initial posited {I}. The kind of negation at work here is not purely logical. In embodied terms, it is deflationary—the collapse of the body feelings that often accompany self-certainty, as that self-certainty has tried to take itself as a finite claim, attaching itself to the posited claim as its truth or essence.

This triadic pattern is often erroneously reconstructed as 'thesis $\rightarrow$ antithesis $\rightarrow$ synthesis.' The problem with this formula is two-fold. First, it isn't Hegelian: it's Fichte's formula (Maybee, 2020). Second, Hegelian dialectics are different in every instance. There is no formula that captures his

[2] The difficulty in translating Aufebung must be evident since its supposed English translation is *sublation*. What does *that* mean? Inwood (1992) says it kind of means 'to kick upstairs.' This opacity is another reason to appreciate the Sneetch example. The symbol of the stars is not erased from all bellies, nor appended to each. It does not go away, but its meaning is no longer discriminatory. Perhaps it recalls - for the reformed Sneetches - the way things used to be.

insights. Instead, the best I have been able to do is discern a pattern through those instances, where the pattern is an escape from a pattern. Some finite pattern is posited $\rightarrow$ an internal tension within the posited determinate pattern is explicated as a claim that exceeds the boundaries of the initial claim $\rightarrow$ that explication negates the finitary assumptions in the initial posited claim without destroying the initial pattern that was posited. Sometimes this results in a new pattern emerging. Said another way, by positing a bounded 'object,' the nature of the claim is such that a new object can be explicated from within the boundaries of the object. However, due to the strict boundaries posited in the original claim, the new object falls 'outside' those boundaries, transcending the original boundaries. This sometimes results in a simply redrawing the boundaries again with strict determinacies that must again be negated in a way that preserves the contents but not the false boundaries. However, when consciousness runs through all of its possible 'shapes,' the subject may recognize that determinate negation - the "no" - is what affords each of the boundaries. Then, the determinate negation *of* determinate negation can be said to bound itself in its unbinding. It gets very strange to talk about, but the "no" that says "no" to itself as ~~no~~ is both inside and outside itself. What is 'strange' is that this paradox expresses the constancy of movement in a way that seems to statisfy the desire to express oneself without having to talk at all.

I will examine Euclid's proof that there are always more primes than any finite list to introduce a framework that can represent both mathematical and existential claims. Cantor's diagonalization will extend this pattern to infinite sets, while references to Gödel will suggest how formal systems escape their boundaries. By foregrounding these potential incoherences, I invite readers to consider how patterns that may feel inescapable can become the very means of escape—prison bars becoming ladders out the window.

To be clear, I do not wish to minimize incoherences with existential significance by equating them with mathematical puzzles. The complex unity of knowing and being that Hegel describes is not a formal system, but a way of becoming the individual we already are by means of expressions that are both what we are and—pointedly—what we are ~~not~~. With that understanding, let us turn to the escape games of mathematics and philosophy.

2 Some symbolic logic

The way through this fog that I will carve is by reinterpreting *proofs by contradiction* that include aspects of *constructive dilemmas* (Franzén, 2004) as some metaphor for the dark notions of 'letting go' and 'acceptance' I discussed earlier as the Sound of Time. Constructive dilemmas are formal inferences, like *modus ponens*. Euclid's mode of escape is algorithmic. So is is Cantor's. Consequently, they do not rise to the transformative power of the *in*finite that Hegel's work invites us to engage in. I let go of incoherent commitments within an *incoherence frame* (Brandom, 2008) while accepting the inevitability of some conclusion given some prior set of commitments. The two formal approaches (contradiction, constructive dilemma) loosely map on to the *critical* and *ampliative* responsibilities of the synthetic unity of apperception (Brandom, 2019). Hegel's arguments are **not** formal, so the following should be taken as lightly as any other metaphor.

First, let us discuss an *incoherence frame* as a pair

$$\langle L, Inc \rangle,$$

where Inc specifies which sentences in the language L are *incoherent*. For Brandom, these are formal languages written by automatons. But it does not do much harm to the notion to think of a language as a collection of sentences a qualitative researcher seeks to understand, or the way that people in general reflect on troubling interactions to try to make sense of them. When I speak with the kids about a stuffed animal, what I take to be incoherence with existential significance is different from the kind of incoherence that came about when I tried to synthesize critical theories built on rational normativity with those who purport to eschew rational norms: that paper (2022) remains a 'hot stove' for my experience - I do not want to touch it again.

To comply with Brandom's articulations that formalize the notion of *incompatibility*, if we have an incoherent set of sentences, like A : "now is night" and B : "now is day," where we are implicitly recognizing that the two nows are supposed to refer to the same moment (not just the same token-type, but a repetition of the same token), then by what Brandom calls the *persistence* axiom, we cannot just add more sentences to the language that will repair this incoherence. That is, there is no C that would allow for $A \cup B \cup C$ to be coherent. I am not sure that rule is generally true, though it makes sense for the somewhat inert sentences formed by automatons. When each instance of language use is recognized as an action with an inherently reflective aspect, saying one more thing transforms what came before. A more general notion is that once we recognize as committed to incoherence, trying to dominate ourselves or others with it by refusing to engage in critique is probably a mistake. In formal mathematics, those contradictory commitments are dropped as falsities. In mathematics education, the tensions of misrecognition ('errors,' 'mistakes,' 'misconceptions') are sublated into a new understanding. They do not just 'go away.'

An *incompatibility relation* I is defined so that two sets X and Y are *incompatible* if and only if their union is incoherent. That is, A is incompatible with B if taking them together results in an incoherence. Symbolically, $A \in I(B)$ iff $A \cup B \in Inc$. Following Brandom's notion that we can talk about whether we should accept a claim either in terms of truth (alethic modality) or morality (deontic-normative modality), we can discuss whether we should accept an inference in terms of whether the consequent is incompatible with everything that the antecedent of the inference is incompatible with. A sentence like A :"Pokey is a mammal" is a looser claim than B :"Pokey is a dog," but everything incompatible with "is a mammal" is also incompatible with "is a dog." Sort of. We have to buy into some assumptions here that we are talking with people who know that all dogs are mammals. A stuffed toy doggy is not a mammal. Brandom would say B *incompatibility entails* A and write: $(B \models_I A)$.

I will use a few more symbols throughout the book that Brandom develops that relate to modal logic. There are many different modal logics. The idea is that when we make moral claims, truth claims, temporal claims etc., each makes explicit some rule or norm. Those rules differ depending on the category that is foregrounded in the claim. Using modal logic allows for some of those nuances to be captured, but they do not repair the fact that our speech slips between categories in empirical dialog. A temporal claim habitually transforms into a spatial claim when it is recollected. The Kantian categories of understanding are inside and outside of each other, at least for the critical ethnographer. The rigidity implied by a foreign alphabet should not be taken to detract from core insights based on lived experiences: the rules of speech are not rigid.

- Np for negation ("not p"),
- Kpq for conjunction ("p and q"),
- Apq as shorthand for $NK(Np)(Nq)$ (the disjunction "p or q"),
- Cpq as shorthand for $NKp(Nq)$ (the conditional "if p then q"),
- Lp for a *necessity* operator (introduced via incompatibility semantics), and
- $Mp := NLNp$ for *possibility* (dually).

3 Euclid's Proof: An Uncontainable Infinity

The first proof by 'contradiction' I recall learning (as such) was attributed to Euclid (Euclid, 2007) and cited as a proof that the set of prime numbers is infinite. I'm not sure Euclid would have recognized this formulation as work of his own hand. Since I have forgotten who actually taught me the modern misrepresentation and wouldn't want to name names if I did recall, I will quote Hardy and Woodgold (2009) as I recognize their experience as my own:

> Once upon a time a learned professor explained to a class of bright students how Euclid proved the existence of infinitely many prime numbers in the third century BC. He said that Euclid began by supposing only finitely many prime numbers exist and ultimately deduced a contradiction. He also remarked on the admirable simplicity of what he said was Euclid's proof by contradiction. He explained that the proof began by letting $p_1 \ldots p_n$ be all the prime numbers that exist, in increasing order. He directed the students' attention to the number $p_1 \ldots p_n + 1$: He showed that that number cannot be divisible by any of the primes $p_1, \ldots, p_n$: Then he said, 'By our assumption, no other primes than those exist. This number is therefore not divisible by any primes. Since it is not divisible by any primes, it must itself be prime. But that contradicts our initial assumption that no other primes than $p_1, \ldots, p_n$ exist.'

I will discuss some of 'fishy' parts of this representation of Euclid's argument in the section on criticality in Euclid's proof. My argument gets a little complicated, so I encourage readers to investigate Hardy and Woodgold (2009) for a clear argument for why this representation is pedagogically and mathematically unsound.

But before diving into Euclid's original proof, note that Franzén (2004) convincingly argues that this proof is not actually a proof by contradiction, but rather a proof by constructive dilemma. In a proof by contradiction, the negation of the sentence that we wish to prove is assumed as true and a contradiction is shown to follow from that assumption. For Euclid's proof to be a proof by contradiction, we would have to assume that the false assumption "there are a finite number of primes" has a logical negation: there are an infinite number of primes. Euclid not make that claim, and probably could not have done so since modern version requires antecedents (namely, Cantor (1891)) that themselves depend on Euclid's original claim as an antecedent.

In constructive dilemmas, two conditionals are established with a disjunction between: "if a then b" or "if c then d." If one can prove "a or b" then either "c or d" must follow. To illustrate: If a Sneech has a star, then it goes in the Star-Off machine. If a Sneech does not have a star, then it goes in the Star-On machine. Either a Sneetch has a star or does not have a star, so it either goes in the Star-Off machine or the Star-On machine. At least, that is, until the Sneetches get wise to the tricks of McBean. When that occurs, the ascriptive category of 'having a star' is sublated into '~~having a star~~'. It loses one meaning but gains another when everyone is invited to the frankfurter roasts. I will first present a translation of his proof, then discuss it in terms of incoherence. That will provide the expressive resources to discern a transcendental-like pattern in Euclid's reasoning that brings Euclid's proof into the critical ethnographic framework developed throughout this book.

Euclid of Alexandria (circa 300 BCE) claims in Book 9, Proposition 20 of *The Elements* that any finite list of prime numbers is incomplete. His approach stands as a paradigmatic example of how mathematical incompatibilities drive forward our understanding of infinity.

3.1 The Structure of the Proof

The proof unfolds within the conceptual framework of ancient Greek mathematics, using notions of length, measurement, and the incommensurability of certain magnitudes. Rather than using modern algebraic notation, Euclid speaks of lengths that cannot be measured by each other without remainder (primes), and those that can be measured by those lengths (composites). One possible translation of his proof is as follows:

> Let the prime numbers which have been given be A, B, and C. I say that the prime numbers are more numerous than A, B, and C. For let us take the least number which is measured by A, B, and C, and let that number be denoted by DE, and let the unit be appended to DE, forming the number EF.
>
> Now, either EF is prime or it is not. Suppose first that EF is prime. Then the collection of prime numbers A, B, C, and EF is larger than the collection A, B, and C.
>
> But if EF is not prime, then it is measured by some prime number, say G. And G cannot be identical with any of A, B, or C, for if it were, then (since A, B, and C measure DE) G would

also measure EF, and further the unit would be measured by G — which is absurd.

Therefore, in this case as well, the prime numbers A, B, C, and G are more numerous than the assigned set A, B, and C. This is what was required to be demonstrated.

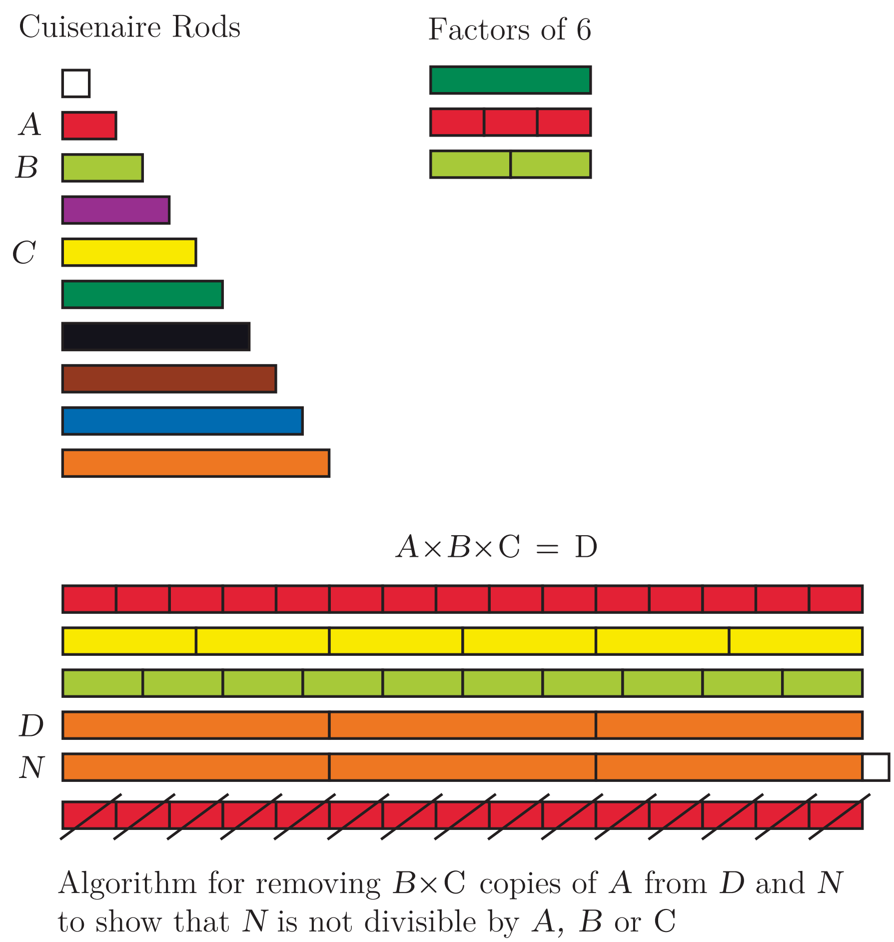

Figure 2: Cuisenaire rods, interpreted as the numerals 1-10, building a composite (6) from primes (2×3), and an algorithm for a key part of Euclid's proof.

To illustrate this proof for modern readers, we can use Cuisenaire rods (Figure 2), which are common manipulatives for teaching base-ten concepts.

Imagine these rods represent lengths between 1 and 10, with the prime numbers being {2, 3, 5, 7}. We could build the composite number 6 using either 2 copies of 3 or 3 copies of 2 (or 6 copies of 1).

Euclid's idea is elegant: take any finite collection of primes (say, A, B, and C), form a number DE that is their product, then add one unit to form $EF = DE + 1$. Now we have a fundamental incompatibility:

- **Case 1:** If EF is prime, it cannot equal any of A, B, or C (since it's larger than all of them), so it's a new prime outside the original list.
- **Case 2:** If EF is composite, it must have some prime factor G. But here's the crucial incompatibility: G cannot equal any of the original primes A, B, or C.

  Why? If G were equal to one of them, then G would divide both DE (by construction) and EF (by our assumption). But then G would have to divide their difference $EF - DE = 1$, which is impossible since no prime divides 1.

In either case, we've found a prime (EF itself or its factor G) that lies outside our original finite list. The list $\{A, B, C\}$ is proven incomplete.

3.2 Incompatibility and the Accrual of Insight

What makes this proof philosophically profound is how incompatibilities accrue to yield a deeper insight about mathematical reality. Let's examine this process:

1. *Initial Positing*: We begin by entertaining the notion that some finite list of primes $\{A, B, C\}$ might be complete.
2. *Constructed Element*: From this list, we construct $EF = A \times B \times C + 1$, which embodies a tension: it's built using the list, yet stands apart from it by that "+1" addition.
3. *Fundamental Incompatibility*: We find that EF (or its prime factors) cannot belong to our original list, creating an irreparable incompatibility between the assumption of completeness and the nature of the constructed number.
4. *Accrued Understanding*: This incompatibility forces us to revise our initial positing and recognize that no finite list of primes can ever be complete. The concept of "primeness" inherently transcends finitude.

This pattern of incompatibilities accruing until a conceptual breakthrough occurs mirrors Hegel's dialectical process. The assumed "finite totality" of the prime list contains within itself the seed of its own negation. When we examine it thoroughly, the concept undermines itself and points beyond its own limitations.

3.3 Transcendental-Like Enabling Conditions

What's particularly noteworthy about Euclid's proof is that it doesn't introduce any external principles to generate its conclusion. Rather, it reveals what was already implicit in the nature of numbers — what we might call, following Kant, the "conditions of possibility" for prime numbers. These conditions include:

- The unboundedness of the natural numbers (we can always construct larger numbers)
- The fact that every integer greater than 1 can be measured by some prime (the existence of prime factorization)
- Basic properties of divisibility (if a number divides two quantities, it divides their difference)

Thus, the infinity of primes isn't something Euclid invented or imposed; it's a necessary feature of how numbers work. The proof simply makes explicit what was already latent in our arithmetic system — which makes it an example of how mathematics can uncover the "already there."

3.4 From Modern Misrepresentation to Original Insight

The common modern representation of Euclid's proof often frames it as "proving there are an infinite number of primes." However, this formulation subtly misrepresents what Euclid actually demonstrated. Three anachronistic elements are worth noting:

- The equals sign (introduced by Diophantus, circa 200-284 CE)
- Symbolic algebra (developed by al-Khwarizmi, circa 900 CE)
- Most significantly, the concept of 'the infinite' as a completed totality(Kanamori, 1996, p. 3), where the definite article "the" presupposes that Euclid's process of breaking a finite boundary is somehow equivalent to a bounded object that we now represent with the symbol ∞

Ancient Greek mathematics generally avoided the notion of actual infinity, preferring what Aristotle called "potential infinity" — the idea that a process can continue without end. Euclid doesn't claim there exists an infinite set of primes; he shows that "prime numbers are more than any assigned multitude of prime numbers." In other words, no matter how many primes you list, there's always another.

This distinction aligns Euclid's proof more closely with Hegel's notion of the "good infinite" (the notion of infinity as a dynamic transcendence of any finite limit) rather than the "bad infinite" (an endless progression that never reaches completion). Euclid shows that any attempt to confine prime numbers within a finite boundary will fail — the prison bars become a ladder out the window.

3.5 Incompatibility Semantics: Why the Claim to Completeness Collapses

To understand how incompatibilities accrue in Euclid's proof, let us approach it through incompatibility semantics. We can express the key incompatibility as:

$$\{\text{The list of primes } \{A, B, C\} \text{ is complete}\} \in Inc$$

This statement belongs to the set of incoherent claims (Inc) because it generates an internal contradiction. Once we understand what prime numbers are, this claim defeats itself. The incoherence emerges through the following steps:

1. If the list $\{A, B, C\}$ is complete, then any newly constructed number must either be composite or equal to a prime already on the list.
2. The construction $N = A \times B \times C + 1$ yields a number that, whether prime or composite, forces the existence of a prime not on the list.
3. This contradicts the initial assumption of completeness.

The critical insight is that this incompatibility isn't immediately obvious — it emerges only when we follow the proof's logical development. In this way, it parallels Hegel's approach in the dialectic of sense-certainty, where the persona of Sense-Certainty doesn't recognize that its claims are self-defeating until they're made explicit through reasoning.

4 The Philosophical Significance of Euclid's Proof

Euclid's proof then might be used to interpret some aspects of Hegel, or at least be considered as presaging some of what Hegel makes explicit through his dialectic.

- *Finite Totality and Self-Negation*: The proof begins with the finite assumption that the list of primes is bounded. This assumption, when examined through Euclid's construction, generates a contradiction—the list produces a prime not on itself. The finite concept thus contains the seed of its own negation.
- *Immanent Necessity*: The infinity of primes isn't imposed from outside; it emerges as a necessary truth from within the concept of primeness itself. The proof shows that infinity is not an arbitrary projection but a necessary structure inherent in the concept of number.
- *From "Bad Infinity" to "True Infinity"*: Before the proof, one might think of acquiring more primes by continuously searching for the next (an unending, step-by-step process). Euclid's proof gives us a conceptual leap: it demonstrates in one stroke that there are infinitely many primes without ever engaging in a never-to-be-completed task.

4.1 Relation to Modern Mathematics

Though Euclid's proof is ancient, it presages modern mathematical patterns of diagonalization and self-reference. Like Cantor's later diagonal argument (which shows the uncountability of real numbers), Euclid's proof assumes a totality and then constructs an element that lies outside it. The proof produces a new prime not by random exploration but by using the very list that was claimed to be complete—a kind of self-reference that modern logic would later exploit. Both arguments reveal that certain mathematical domains cannot be bounded or contained within finite limits — a theme that would reach its apotheosis in Gödel's incompleteness theorems.

In conclusion, Euclid's proof stands as an exemplar of how mathematical reasoning can transcend finite limitations. Though expressed in the language of ancient geometry, it reveals a truth uniting theme in the history of mathematics and philosophy: the mathematical 'object' is also a subject - or at least a 'concept'. If you put 'it' in a box, it will get out. The 'it' is not a thing, but a process of becoming that is always already in motion.

5 Cantor's Diagonal Argument

Let us now jump forward a few millennia in mathematical history to Georg Cantor's diagonal argument (Cantor, 1891). While Euclid's posited finitude is somewhat unlikely - I'm not sure how many people would think the primes are finite, though perhaps it was a revolution in its time - Cantor demonstrated something more surprising: even an *infinite* list of certain mathematical objects must be incomplete. This insight would transform our understanding of infinity itself.

5.1 From Euclid to Cantor: A New Kind of Incompleteness

In some ways, Cantor's proof transposes Euclid's argument from a finite list of primes to an infinite list of binary sequences. But where Euclid merely showed that primes cannot be contained within any finite boundary, Cantor took a more radical step: he proved that some infinities are fundamentally larger than others. This is the mathematical equivalent of discovering that the universe contains multiple sizes of "forever" — an achievement that made me giddy for weeks when I understood the scope of its magnitude.

To understand Cantor's argument, let us first consider what he aims to prove: that the set of all possible locations between 0 and 1 on a number line cannot be enumerated—cannot be put into a one-to-one correspondence with the counting numbers. In modern terms, he seeks to prove that the real numbers are "uncountable," while the natural numbers are "countable." This result is counterintuitive, as one might think that the infinite set of real numbers between 0 and 1 could be matched up with the infinite set of natural numbers. Cantor's diagonal argument reveals why this is not the case.

5.2 The Diagonal Method: Intuitive Approach

Before examining Cantor's original formulation, let me offer an intuitive explanation. Imagine an infinite and (impossibly irenic) army of monkeys with typewriters, each with only two keys: 0 and 1. Each monkey types an infinite sequence of 0s and 1s, which we interpret as representing a real number between 0 and 1 (with a decimal point at the beginning). For example:

$$\begin{gathered}0.01001110... \\ 0.10111010... \\ 0.00000111...\end{gathered}$$

No matter how many monkeys we employ (even infinitely many), Cantor proves we can always construct a number between 0 and 1 that none of them have typed. The brilliance of his approach lies in how he constructs this "escaped" number.

First, we number our monkeys: 1, 2, 3, and so on. Then, we examine the diagonal elements of their output—the 1st digit of the 1st monkey's sequence, the 2nd digit of the 2nd monkey's sequence, and so forth. Finally, we create a new sequence by "flipping" each of these digits (changing 0 to 1 and vice versa).

This new sequence differs from the first monkey's sequence in the first position, from the second monkey's sequence in the second position, and so on. Therefore, it cannot match any sequence on our list—it has escaped enumeration. This "diagonal method" reveals that no list of real numbers between 0 and 1, no matter how infinite, can be complete.

5.3 Cantor's Original Formulation

Now let us turn to Cantor's original proof, with some typographical adjustments to highlight how a second-person position might resolve some incompleteness.[3] Cantor used two characters, ʍ and w, which have an interesting rotational symmetry. Like the Sneetches with and without stars upon thars, the difference between these two characters is one that might be sublated.

> If ʍ and w are any two mutually exclusive characters, then we consider a collection $\mathcal{M}$ of elements, $E = (x_1, x_2, \ldots, x_\nu, \ldots)$, which depend on an infinite number of coordinates, $x_1, x_2, \ldots, x_\nu, \ldots$, where each of these coordinates is either ʍ or w.
>
> $\mathcal{M}$ is the totality of all elements E.

[Note: This definition establishes $\mathcal{M}$, the "manifold" of the real numbers, as the complete set of all possible infinite binary sequences. Soon, Cantor will construct an element that must belong to $\mathcal{M}$ by definition, yet cannot appear on any enumeration of its elements—revealing a fundamental incompleteness.]

[3] I spent a lot of space in earlier drafts showing how the 'turning the table' on Cantor's matrix reveals the completed element is 'there' without the externalized flipping function. I leave it out of this draft, though I miss it.

The elements of $\mathcal{M}$ include, for example, the following three:

$$E^{\mathrm{I}} = (\mathsf{m}, \mathsf{m}, \mathsf{m}, \mathsf{m}, \ldots) \tag{1}$$
$$E^{\mathrm{II}} = (\mathsf{w}, \mathsf{w}, \mathsf{w}, \mathsf{w}, \ldots) \tag{2}$$
$$E^{\mathrm{III}} = (\mathsf{m}, \mathsf{w}, \mathsf{m}, \mathsf{w}, \ldots) \tag{3}$$

I now claim that such a manifold $\mathcal{M}$ does not have the power of the series $1, 2, 3, \ldots, \nu, \ldots$.

[Note: By "power," Cantor means cardinality or size. He is asserting that the set of binary sequences is larger than the set of natural numbers—a revolutionary claim that introduces different sizes of infinity.]

This follows from the following sentence:

If $E_1, E_2, \ldots, E_\nu, \ldots$ are any simply infinite series of elements of the manifold $\mathcal{M}$, then there is always an element E_0 of $\mathcal{M}$ that does not agree with any E_ν.

[Note: This is the heart of Cantor's claim — no matter how we try to list all elements of $\mathcal{M}$, we can always construct another element that escapes the list, similar to how Euclid showed there is always one more prime beyond any finite list.]

To prove it:

$$E_1 = (a_{1,1}, a_{1,2}, \ldots, a_{1,\nu}, \ldots) \tag{4}$$
$$E_2 = (a_{2,1}, a_{2,2}, \ldots, a_{2,\nu}, \ldots) \tag{5}$$
$$\ldots \tag{6}$$
$$E_\mu = (a_{\mu,1}, a_{\mu,2}, \ldots, a_{\mu,\nu}, \ldots) \tag{7}$$
$$\tag{8}$$

[Note: Here Cantor sets up an arbitrary enumeration of binary sequences. Each subscripted E_μ represents the μ-th sequence in our list, while $a_{\mu,\nu}$ represents the ν-th element within that sequence. Crucially, this establishes a naming relationship—each infinite sequence is identified by a natural number, creating the conditions for self-reference that will drive the diagonalization.]

Here the $a_{\mu,\nu}$ are in a certain way m or w. Let us now define a series $b_1, b_2, \ldots, b_\nu, \ldots$ such that b_ν is also only equal to m or w and different from $a_{\nu,\nu}$.

So if $a_{\nu,\nu} = \mathsf{m}$, then $b_\nu = \mathsf{w}$, and if $a_{\nu,\nu} = \mathsf{w}$, then $b_\nu = \mathsf{m}$.

[Note: This is the diagonal construction. Cantor examines the diagonal elements $a_{1,1}, a_{2,2}, a_{3,3}, \ldots$ and for each one creates a corresponding element b_ν that is its opposite. This creates a new sequence that differs from every sequence in the original enumeration in at least one position.]

The diagonal construction can be understood as two cases:

- **Case 1:** If $a_{\nu,\nu} = \mathsf{m}$, then $b_\nu = \mathsf{w}$
- **Case 2:** If $a_{\nu,\nu} = \mathsf{w}$, then $b_\nu = \mathsf{m}$

This ensures that our new sequence $E_0 = (b_1, b_2, \ldots)$ differs from each E_ν in the ν-th position.

If we then consider the element:

$$E_0 = (b_1, b_2, b_3, \ldots)$$

of $\mathcal{M}$, we can easily see that the equation:

$$E_0 = E_\mu$$

for no positive integer value of μ can be satisfied, otherwise for the given μ and for all integer values of ν:

$$b_\nu = a_{\mu,\nu},$$

so also in particular,

$$b_\mu = a_{\mu,\mu},$$

which would be excluded by the definition of b_ν.

[Note: Cantor proves that E_0 cannot equal any sequence in our enumeration. If E_0 were equal to the μ-th sequence E_μ, then all corresponding elements would match. But this creates a contradiction at position μ, where b_μ was specifically defined to differ from $a_{\mu,\mu}$.]

From this theorem follows immediately that the totality of all elements of $\mathcal{M}$ cannot be put into the series form:

$$E_1, E_2, \ldots, E_\nu, \ldots,$$

otherwise, we would be faced with the contradiction that a thing E_0 is both an element of $\mathcal{M}$ as well as not being an element of $\mathcal{M}$.

[Note: By definition, E_0 must be in $\mathcal{M}$ since it is a valid binary sequence. Yet we've proven it cannot appear in any enumeration of $\mathcal{M}$'s elements. Therefore, $\mathcal{M}$ cannot be put into one-to-one correspondence with the natural numbers—it is uncountable.]

5.4 Self-Reference and Negation: The Structure of Diagonalization

What makes Cantor's argument particularly significant from a philosophical perspective is its structure of self-reference and negation. Following Gaifman (2005, 2006), we can identify a pattern of "naming" that enables the diagonal construction.

When we enumerate the sequences as $E_1, E_2, E_3, \ldots$, we create a naming relationship where each infinite sequence is identified by a finite natural number. This compression of the infinite into the finite creates the conditions for self-reference. Using Gaifman's notation, we might write $n = \delta(E_n)$, where δ represents this naming operation.

The diagonal construction then performs a partial negation on these named elements. We don't negate the entire sequence — just the diagonal elements. But this partial negation is sufficient to ensure that our new sequence differs from every sequence in the original enumeration.

Schematically, the process involves:

1. Naming each sequence with a natural number
2. Using those names to identify diagonal elements
3. Negating those diagonal elements
4. Constructing a new sequence from these negated elements

This can be expressed as $E_0 = \delta(\neg\delta E_n)$—the new sequence is constructed by negating parts of the named original sequences. Gaifman calls this a "sandwich" of two diagonalizations with a negation between.

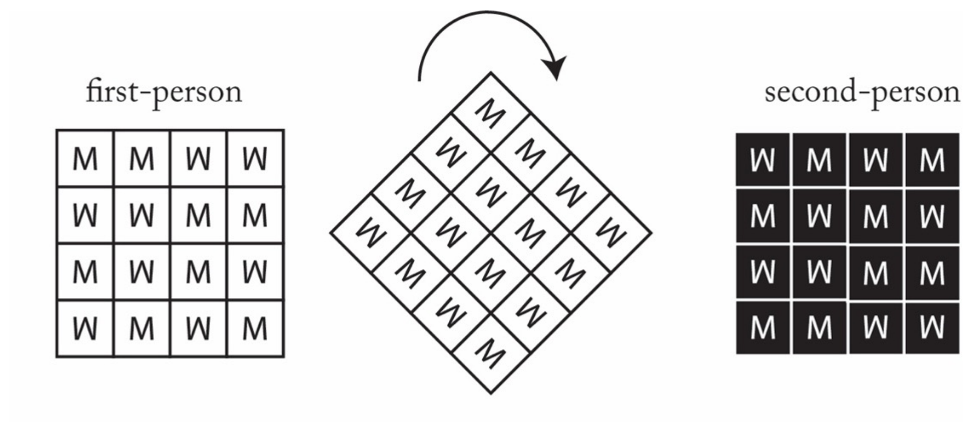

Figure 3: Note. Attempting to take a second-person position on Cantor's matrix produces an inversion. I'm not sure if this 'counts' as immanent necessity. Surely not without more development.

The first diagonalization posits a fixed point as a bounded object. In Cantor's proof, the negation does not feel entirely immanent like Euclid's proof. The 'flipping' operation is not produced from an obvious tension within the set of sequences. For this reason, Cantor's proof, unlike Euclid's, may be a genuine proof by contradiction. If there were some reason for the ᴍ's and the ᴡ's to *want* to transform into each other (through immanent necessity), the parallels with Hegelian dialectics would be stronger.

Perhaps a case could be made for the as-yet-implicit transition to self-consciousness. The flipping operation could be seen as taking the attitude of the other. I thought about this path for a while and came up with a kind of illustration (see 3). But I am not sure how much it adds to the argument.

5.5 Dialectical Implications: Cantor and Hegel

Cantor's proof reveals a profound incompatibility: The set of binary sequences can be enumerated $\cup$ $\mathcal{M}$ is complete $\in Inc$.

Unlike some simple contradictions, this incompatibility isn't immediately obvious — it emerges only through the careful construction of the diagonal argument. Once revealed, however, it forces us to revise our understanding of infinity. We don't discard the concept of infinity; rather, we expand it to acknowledge different sizes or "powers" of infinity.

This dialectical movement parallels Hegel's process of sublation (Aufhebung). The initial concept (a single notion of of a 'bad' infinity that is merely successor after successor after successor) is negated (shown to be inadequate) but preserved in a boundless (though still hierarchical and mechanically produced) notion of the infinite. What was implicit in our understanding of infinity becomes explicit through the diagonal construction.

Moreover, the process involves a form of recollection and objectification. Cantor's enumeration takes the infinite sequences and recollects them as finite names (natural numbers). This naming allows us to treat the infinite as an object we can manipulate—similar to how, in Hegel, language allows consciousness to objectify its immediacy.

The ultimate resolution isn't to abandon our original concept, but to recognize its limitations and transcend them. Just as Euclid's proof expanded our understanding of primes beyond any finite boundary, Cantor's proof expands our understanding of infinity beyond a single, uniform concept.

In both cases, what appeared complete reveals itself as inherently incomplete when subjected to internal critique. The prison bars of a bounded understanding become a ladder to a more expansive conception — one that recognizes the inexhaustible nature of certain mathematical domains.

6 Hegel's Sense-Certainty: Dialectic as Incompatibility and Sublation

6.1 Hegelian Dialectic and Mathematical Diagonalization: A Shared Structure

Hegel's dialectical method and mathematical diagonalization arguments share a profound structural similarity: both reveal how claims of completeness contain within themselves the seeds of their own negation. Before exploring this connection through sense-certainty, it is worth establishing the fundamental pattern that unites these seemingly disparate intellectual traditions.

Hegelian dialectic is characterized by dynamic triads and the process of *Aufhebung* (sublation), which simultaneously negates and preserves what came before. In a dialectical transition, an initial concept (often characterized as a claim to completeness or immediacy) is posited as a bounded 'object', only to be undermined by tensions within itself. These tensions produce a negation that is then incorporated into a unity that preserves what was valuable in the original position while transcending its limitations.

This dialectical movement finds a parallel in mathematical diagonalization arguments—from Euclid's proof of infinite primes to Cantor's uncountability theorem to Gödel's incompleteness results. In each case, a system claiming completeness is forced, by its own resources, to generate something that necessarily lies outside that system. In diagonal arguments, the paradox of sets that are both inside and outside themselves generates new elements that must have been implicit within a formal system, but whose explication cannot be contained within the constitutive boundaries of the system.

In the *Phenomenology of Spirit*, Hegel begins with an analysis of *sense-certainty* (SC), the most immediate form of knowing. To mirror Euclid's proof that any finite list of primes (say, $\{A, B, C\}$) is incomplete and Cantor's diagonalization showing that any enumeration of binary sequences is incomplete, SC assumes that $\{\text{this}\}$ is complete. That is, knowledge $\mathcal{K}$ is taken to be immediate — a non-recollective, direct, non-inferential encounter with a purely particular object $\mathcal{O}$. Formally, SC assumes:

$$\mathcal{K}(\mathcal{O}) = \mathcal{O}.$$

This expresses that the act of knowing does not change the object, nor does it alter the knowing subject. Readers familiar with more advanced forms of diagonalizing arguments may recognize this formula as a claim to a *fixed point* - a point ($\mathcal{O}$) that returns to itself through predicative judgment. It would be if $\mathcal{K}$ were a simple predicate or function. It isn't, which is one of the problems with SC.

Once SC begins to "speak" its truth, the very act of expression — using words that are, by nature, universals and repeatable — undermines its claim to pure immediacy. In everyday life, I might experience the smell of coffee, to ground myself in a sensory experience, easing the sense of existential anxiety that often dogs me. Self-certainty, which is the inheritor of the true aspects of sense-certainty as Hegel will show through the *Phenomenology*, can be recognized in a host of possible experiences. It is the assumption that such sensory experiences furnish complete knowledge is precisely what leads to incoherence. We can also argue whether an inarticulate knowing counts as knowledge at all. I tend to think that the source of self-certainty is entirely inarticulate, and that spinning my gears to get at that source is somewhat fruitless. But nevertheless here I am, spinning my gears.

In any case, when SC claims that knowledge is simply direct sensory awareness of particulars, it cannot sustain a purely particular *this*. Instead, as soon as SC starts to articulate its truth — as a "this," "here," or "now" — its claim defeats itself.

The Incoherence of Complete Immediacy

The claim that {this} is complete (in the SC sense) is self-undermining once we unpack what {this} means. Euclid's claim is unrecognizable as self-defeating until we understand what prime numbers are, and the same is true with the claim that {this} is complete: it only falls apart when it is understood.

As soon as we recollect {this}, it splits into two components:

$$\{\text{this}\} \quad \longrightarrow \quad \{\text{this}\}_{\mathcal{I}} \cup \{\text{this}\}_{\mathcal{O}},$$

where $\{\text{this}\}_{\mathcal{I}}$ represents the knowing subject (the {I}) and $\{\text{this}\}_{\mathcal{O}}$ represents the object ($\mathcal{O}$) as it is recollected. The very insistence of SC that everything is a {this} forces both the subject and object to be instantiated as a {this}. If these two are distinct (i.e. if $\{\text{this}\}_{\mathcal{I}} \neq \{\text{this}\}_{\mathcal{O}}$), then the original claim to completeness is lost. In Brandom's notation, $\{\text{this}\}_{\mathcal{I}} \cup \{\text{this}\}_{\mathcal{O}} \in \mathit{Inc}$.

Perhaps one of them is essential and the other inessential. If that were the case, we now have another term to consider, and the original claim to completeness is lost either way. But SC cannot determine which without its own reflection. Hegel is careful about an 'internal' as opposed to an 'external' dialectic (Inwood, 1992). The distinction is abstract, but it's a kind of ethical commitment to sound pedagogy: take the attitude of the Other (SC) and let their reflections on their commitments guide the process of learning. The first few paragraphs (§90-§94) are external to SC, but in §95, we are invited into SC.

The Triad of Universal-Particular-Individual

One core structure in Hegel's logic, evident even in the simple dialect of sense-certainty, is the triad of *Universal-Particular-Individual*. These are not static categories but interdependent moments of the Concept (*Begriff*). In Robert Brandom's interpretation, this triadic structure manifests through a social model of recognition: the recognitive community (all who mutually recognize each other) functions as a kind of *universal*, under which individual self-conscious agents count as *particulars* defined by that universal.

The contradiction in sense-certainty emerges precisely at this juncture — the persona of sense-certainty tries to grasp a pure *particular* (the "this here now") without acknowledging the universal context that gives it meaning. But in Hegel's view, a true concrete individual is not a mere isolated particular; it embodies universality, just as universals exist only by expressing themselves through particular individuals.

From within SC, {this} further splits into the indexicals {here, now}. We may index the object by space and time:

$$\mathcal{O}_{\text{here, now}}.$$

Yet, when we recollect $\mathcal{O}$, these indices change. For example, if SC asserts:

$$\text{Now} \ = \ \text{“Now is Night”},$$

a simple thought experiment — writing this down and then re-reading it at noon — shows that the context has shifted:

$$\text{Now} \ \to \ \neg\text{Now is Night},$$

since what was night becomes day. Similarly, if SC asserts

$$\text{Here} \ = \ \text{“Here is a tree”},$$

turning around reveals

$$\text{Here} \ \to \ \neg\text{Here is a tree},$$

as the location of the tree is no longer fixed.

Diagonalization and Self-Reference

Somewhat farcically, to capture the truth of SC symbolically, let us consider that the sensory experience is first *compressed* into an object $\mathcal{O}$, so that its decompression is given by

$$\delta(\text{non-compressed sensory experience}) = \mathcal{O}.$$

This is not what sense-certainty claims: it claims that there is no recollection prior to its insistence on the {this}. Then, the recollective act that attempts to capture the Here and Now can be seen as applying a partial negation (acting on the indices) to $\mathcal{O}$. In SC, the claim of immediate knowledge,

$$\mathcal{K}(\mathcal{O}) = \mathcal{O},$$

must be read as inherently recollective. In other words, even if SC intends for $\mathcal{K}(\mathcal{O}) = \mathcal{O}$, the act of recollection necessarily introduces a negation between successive moments:

$$\mathcal{O} = \delta\big(\neg\delta(\mathcal{O})\big).$$

This formula bears striking resemblance to the logical structure of mathematical diagonalization. Just as Gödel's incompleteness theorem constructed a mathematical statement that a given formal system cannot prove (even though that statement is true), the act of recollection constructs an expression of "this" that cannot be captured in the original framework of pure immediacy. In both cases, self-reference generates a paradoxical situation: Gödel's sentence essentially says "I am not provable in this system," while sense-certainty's articulation of "this" implicitly states "I cannot be expressed as a pure particular."

The diagonalization technique—pioneered by Georg Cantor and extended by Gödel—produces a sentence through self-reference that cannot be contained within the presumed complete system. This indicates that what SC takes as immediate knowledge is already mediated by a recollective process — the very act of speaking or pointing out (using universal words) destabilizes the claim to a fixed particular.

Incompatibility and Diagonalization: The Incomplete {This, Here, Now}

The key insight is that the evolving claims of sense-certainty ({this} is complete, $\text{this}_{\mathcal{I}} \cup \text{this}_{\mathcal{O}}$ is complete, {here, now} is complete) are self-defeating. Just as Euclid shows that any list of primes (e.g. {A, B, C}) is incomplete (since one can construct a new prime from them), Hegel shows that the demonstrative {this, here, now} cannot be complete by its own rules. All we need to consider is each claim as it splits and changes. In Euclid's proof, the construction

$$N = A \times B \times C + 1$$

yields a new prime not in the list. Analogously, when SC attempts to pin down a particular {this}, the very process of recollection forces a split:

$$\{\text{this}\} \quad \longrightarrow \quad \{\text{this}\}_{\mathcal{I}} \cup \{\text{this}\}_{\mathcal{O}},$$

and further into indexed forms like $\mathcal{O}_{\text{here, now}}$. The outcome is that the specific *this* is transformed into a universal:

$$\text{Universal } \{\text{this}\} \; = \; \delta\big(\neg\delta(\mathcal{O})\big),$$

which expresses that no particular instance (e.g., a specific Now) remains fixed; instead, each is negated by its own temporal or spatial shift. In this way, the process mimics Cantor's diagonalization: just as Cantor's method constructs a new binary sequence by negating the diagonal entries of a list, the act of recollecting {this} (or Now) produces a result that is necessarily *not* any one of the original particulars. In both cases, the assumed completeness is exposed as incoherent.

This diagonalization pattern appears across multiple domains:

- *Euclid's Proof:* A finite list of primes A, B, C is shown to be incomplete by constructing a number N = ABC+1 that either is prime itself or has a prime factor not in the original list.
- *Cantor's Diagonal Argument:* Any presumed complete enumeration of real numbers is shown to be missing at least one number, constructed by flipping each digit along the diagonal.
- *Gödel's Incompleteness:* A formal system strong enough to encode arithmetic necessarily contains a true statement that cannot be proven within that system, constructed through self-reference.
- *Hegel's Sense-Certainty:* The presumed complete immediate knowledge of "this here now" necessarily generates a universal through the very act of articulation, showing the inadequacy of pure particularity.

In each case, the key move involves an "internal critique" where a system that claims completeness is forced, using only its own resources, to generate something that lies beyond its boundaries. In each argument, the finite contains within it the seed of its own negation: it *inevitably* points beyond itself. The very act of attempting to totalize the primes forces one to acknowledge an 'other' - an 'outside' to its totality.

Thus, the idea that $\mathcal{K}(\mathcal{O}) = \mathcal{O}$ in SC is untenable: once one begins to articulate immediate experience, the universality of language (its repeatability, its capacity for recollection) forces the claim into an open-ended, mediated domain. SC's original promise — that the sensory immediate is complete — is sublated (negated and preserved) into the insight that the true content of our knowledge is universal, not a fixed, isolated particular.

Sublation of Sense-Certainty and the True Infinite

We are not to discard the elements of {this, here, now} entirely. Instead, the dialectic shows that the claim of complete immediacy must be revised. SC does not vanish; rather, its commitments are transformed. The elements remain, but the claim that they are all that there is is tossed. In other words, SC's assertion

$$\{\text{this, here, now}\} \cup \text{ is complete} \in Inc.$$

Its claim is incompatible with the inherent mediation of recollection, so that we must sublate this claim into a new understanding. The revised view accepts the elements {this, here, now} but recognizes that their significance lies in their *universal* rather than particular character.

This sublation exemplifies Hegel's distinction between the "bad infinity" of endless progression and the "true infinity" that embraces finitude within a larger conceptual whole. The "bad infinity" would be the endless listing of particular moments of "now" or locations of "here," never reaching completion. The "true infinity" recognizes that the universal "now" or "here" already contains all particular instances within its concept. Euclid's result moves us from the former to the latter: Prior to the proof, one might think of getting more primes by continuously searching for the next (an unending, step-by-step process). The proof, however, gives a conceptual leap: it proves in one stroke that there is an infinity of primes, thereby embracing the infinite *in order to express it* - without getting involved with a never-ending task. I will return to this theme when I discuss the null representation.[4]

Precisely what Hegel means by a *universal* is a bit unclear within SC. It will become more determinate in the later stages of the *Phenomenology.* We can think of it as a category that persists through the negation of particulars. In the case of SC, the universal is the result of the continual negation of particular Nows or Heres. The universal Now is defined by the failure of any particular now to remain fixed.

To analogize with Cantor's proof, let me backtrack a bit. The real numbers are usually defined in terms of axioms. For example, the idea that a collection of three real numbers is ordered is one rule, among many, that puts boundaries around what counts as a real number. Suppose $A \leq B \leq C$. If $C > A$, then we are not talking about real numbers anymore. But a constructive notion of the real numbers would take Cantor's diagonalizing process as the foundation. They are built out of the process of negating the diagonal of a matrix of rational numbers. The real numbers 'complete' the rational numbers. Of course, the reals are also incomplete, both in material inferential terms based on the existence of complex numbers and on Gödel's proof that axiomatic systems capable of expressing addition and multiplication are necessarily incomplete. Gödel's proof is subtler than Cantor's but both involve diagonalization.

[4]To preview that argument, the null represenation is formed through the substitution of "" for {}. The act of speaking always includes something left unsaid. Embedding recollections in quotatations produces a von Neumann-like structure in the unspoken conditions of possibility for the utternace which are then anaphorically recollected as numerals.

Internal Self-Reference and Transcendental Conditions

Both Hegel's dialectic and mathematical diagonalization arguments operate through a form of internal self-reference that exposes limits. In Gödel's case, a formal system contains enough arithmetic to let sentences refer to themselves (via Gödel numbering). Using only the system's own ingredients, one can construct a sentence that the system tries but fails to encompass. This is analogous to Hegel's method of immanent critique, where the initial concept furnishes the seeds of its own undoing.

Furthermore, these arguments reveal what might be called "transcendental-like enabling conditions" — the prior conditions that make knowledge possible in the first place. Euclid's proof does not simply invent a new entity but reveals a property of the arithmetic system itself — its inexhaustible supply of primes. Similarly, Hegel's analysis reveals that universality is a condition for the possibility of particular knowledge—even at the supposedly immediate level of sense-certainty.

As Hegel demonstrates, once SC starts to speak (or point) its truth, it inevitably employs words — and words are universals. The inescapable result is that the immediate knowledge of SC is inherently incomplete. The same pattern appears in Euclid's and Cantor's arguments: a system that claims completeness is forced, by its own resources, to generate something that lies outside that system. Here, the system of SC, when forced to articulate itself, yields the universal (denoted, for instance, by a negated token like *~~this~~*).

Schematic Summary of Key Dialectical Moves in SC:

90. *Immediate knowledge* is defined as knowing the immediate ($\mathcal{K}(\mathcal{O}) = \mathcal{O}$), with no mediation.

91. This knowledge appears rich (all the concrete data) but reduces to the bare assertion "it is," which is abstract.

92. In any act of immediate knowing, the single $\{\text{this}\}$ splits into $\{\text{this}\}_{\mathcal{I}}$ (the subject) and $\{\text{this}\}_{\mathcal{O}}$ (the object). If they are distinct, SC's claim to completeness fails.

93. The distinction between essence (object) and instance (subject) is made by SC itself, yet both are treated as if they were pure $\{\text{this}\}$.

94. Questioning whether the object truly is the immediate essence shows that once recollected, its immediacy is undermined.

95. A thought-experiment on “Now”: e.g., “Now is Night” becomes “Now is Day” on recollection, indicating the instability of the particular.

96. The continual negation of particular Nows (or Heres) leads to a universal; symbolically, the universal now, N_U, is defined by the failure of any particular now to remain fixed.

Conclusion: Diagonalization as Dialectical Unfolding

In summary, Hegel’s dialectic of SC demonstrates that the claim

$$\{\text{this, here, now}\} \text{ is complete}$$

is self-defeating. The moment we try to grasp or express immediate knowledge, we invoke recollection, and the very act of recollection introduces a negation — much like the diagonalization process in Cantor’s proof. Thus, rather than achieving a pure, unmediated knowledge, SC collapses into the recognition that what we truly have is a *universal* (a *negated particular*), an insight that lays the groundwork for a more mature form of knowledge in later stages of the *Phenomenology*.

The parallel between Hegel’s dialectic and mathematical diagonalization arguments runs deeper than mere analogy. Both reveal a fundamental pattern in human knowledge: claims to completeness or immediacy inevitably generate their own transcendence. Whether in mathematics or philosophy, the attempt to establish fixed boundaries results in the discovery of something that lies beyond those boundaries, yet is generated from within the system itself.

The proof’s necessity arises from ‘something already there’ (the inherent properties of multiplication, divisibility, and basic logic), rather than from a construction wholly within the proof. Similarly, the necessity of universality in knowledge arises from something already there—the inherent nature of articulation and recollection—rather than from an outside imposition.

In both cases, the result is not a mere negation but a sublation—a preservation of what was valuable in the original claim within a richer, more complex understanding. The particular moments of “now” and “here” are not discarded but recognized as instances of universals. Likewise, the mathematical understanding of primes or real numbers is not abandoned but enhanced by recognizing their inexhaustible nature.

This shared structure suggests a deep connection between dialectical thinking and mathematical reasoning—both reveal how our attempts to grasp truth inevitably lead us beyond finite boundaries toward a more comprehensive understanding that embraces both the finite and the infinite in their proper relation.

Throughout this chapter, we have explored a profound pattern of thought that manifests across seemingly disparate domains—ancient mathematics, modern set theory, German idealist philosophy, and even children's literature. At its heart lies a revolutionary insight: claims to completeness or immediate certainty inevitably generate their own negation, and it is through this dialectical movement that deeper understanding emerges.

Mathematical Transcendence and Dialectical Progression

We began with Euclid's proof that no finite list of primes can be complete. By constructing a number from the very primes we thought were all-encompassing (their product plus one), we inevitably produce a prime outside our original list. The prison bars of finitude become a ladder out the window. This pattern persisted when we examined Cantor's diagonal argument, where even an infinite list of binary sequences generates, through its own structure, a sequence that escapes enumeration.

The incompleteness exposed in these mathematical realms isn't a defect — it's a feature of how mathematical truth unfolds dialectically. Each attempt at closure gives rise to its own beyond, revealing a structural pattern that connects these mathematical moments with Hegel's dialectic:

Hegel's Dialectic of Sense-Certainty

This same pattern appears in Hegel's critique of sense-certainty, though now in the realm of consciousness rather than mathematics. When consciousness attempts to grasp the absolute immediate "this-here-now," it discovers that its very act of knowing transforms the particular into a universal. The attempt to express pure immediacy—where knowledge of an object would yield the object itself without mediation—can be represented as $\mathcal{K}(\mathcal{O}) = \mathcal{O}$. Yet this claim of immediate knowledge inevitably introduces mediation through language and conceptualization.

As we've seen, this process can be formalized in terms structurally similar to diagonalization:

$$\mathcal{O} = \delta\big(\neg\, \delta(\mathcal{O})\big)$$

Here, δ represents the recollective act that makes the immediate accessible to consciousness, while simultaneously negating its status as pure im-

Dialectical Stage	**Euclid's Proof**	**Cantor's Argument**	**Hegel's Sense-Certainty**
Initial Claim (Posits a bounded or determinate 'object')	The list of primes {A,B,C} is complete	The real numbers can be enumerated	Knowledge of the immediate {this} is complete
Construction via Self-Reference	$N = A \times B \times C + 1$	Diagonal sequence E_0 that differs from each listed sequence	{this} splits into $\{\text{this}\}_{\mathcal{I}}$ and $\{\text{this}\}_{\mathcal{O}}$ when recollected
Generated Negation (posited boundaries allow for the articulation of an object outside their boundaries)	N or its prime factor must lie outside the original list	E_0 cannot equal any sequence in the enumeration	The "Now" changes from night to day, the immediate slips away
Resulting Incompatibility	No prime can divide both N and $A \times B \times C$	{All reals can be enumerated} $\in Inc$	{Pure immediacy can be known} $\in Inc$
Sublation (New 'object' that transcends its boundaries and so is more like a subject)	Primeness extends infinitely beyond any finite boundary	Infinity exists in different magnitudes (uncountable)	The universal emerges as the truth of the particular

Table 1: The Dialectical Structure Across Domains

mediacy. Just as the diagonal construction reveals the incompleteness of any enumeration, Hegel's dialectic reveals the inadequacy of any claim to unmediated knowledge. But crucially, this negation doesn't destroy knowledge—it elevates it to a higher plane where the particular is recognized as always-already permeated by universality.

The Pattern of Sublation

Across all three domains examined — Euclid's proof, Cantor's diagonalization, and Hegel's dialectic — we observe the same fundamental movement, what we might call the signature of dialectical progress:

Initial Claim	A boundary is asserted or assumed (all primes are listed, all sequences are enumerated, knowledge can be immediate)
Self-Generated Negation	The very content of the claim produces its own refutation (a new prime, a diagonal sequence, a universal concept)
Sublation (Aufhebung)	The initial claim isn't merely rejected but transformed — elevated to a higher understanding that preserves what was true while transcending its limitations

This is the engine of what Hegel calls *determinate negation*—not a skeptic's blanket rejection but a productive contradiction that drives knowledge forward. In each case, the new understanding does not erase what came before but incorporates it into a more expansive framework.

The Sneetches: Recognition and Infinitude

Moving from abstract mathematics and philosophy to narrative, Dr. Seuss's *The Sneetches* captures this dialectical journey with remarkable clarity. The Sneetches begin with rigid categories based on arbitrary distinctions (stars upon thars). McBean exploits their desire for certainty and fixed identity, driving them through a dialectical gauntlet where certainty dissolves: "Whether this one was that one... or that one was this one... or which one was what one... or what one was who.“

When the Sneetches finally "forgot about stars and whether they had one, or not, upon thars,“ they achieved what we might call a recognition of their own *in*finitude—the understanding that no finite marker (star or not) could exhaust their identity. Like the mathematical and philosophical examples we've examined, the Sneetches transcended the assumed totality of their classification system.

The Sneetches' journey thus becomes not merely a children's tale but a profound allegory for the dialectical process we've traced through more technical domains. Their story reminds us that these patterns are not mere abstractions but have concrete significance for how we understand ourselves and others.

The Implications for Critical Understanding

What broader lessons can we draw from this constellation of insights that span mathematics, philosophy, and literature?

First, any system that claims completeness or absolute immediacy contains within itself the seeds of its own transcendence. This applies not only to mathematics and philosophy but to ideologies, worldviews, and educational frameworks that claim to exhaust their domains.

Second, the movement beyond limitations doesn't occur through external critique alone, but through the immanent development of internal tensions. The system's own resources provide the ladder out of its constraints.

Third, this process isn't merely negative or destructive—it's generative. The sublation of limitations produces richer, more adequate concepts that preserve the truth of earlier understandings while transcending their partiality.

Finally, recognition plays a crucial role in this dialectic. The Sneetches recognize each other beyond their stars; consciousness recognizes universality within particularity; mathematicians recognize that certain infinities exceed others. In each case, recognition enables transformation.

Beyond Formalism to Living Truth

Throughout this chapter, I have employed some formal notation to clarify the structural similarities between mathematical proofs and philosophical insights. My hope is that these formal elements have illuminated rather than obscured the deep connections we've explored.

Recall the title of this chapter: "A Foundational Star." I chose this title partly to keep readers oriented toward the story of the Sneetches, but also as an expression of the inherent paradoxes of the understanding. Stars are 'up there'—hardly 'foundations' in common language. Yet this apparent contradiction points to something profound: what grounds our understanding often transcends our immediate context, just as the stars have guided travelers since ancient times.

As we conclude this exploration, I invite you to consider how these patterns of incompleteness, self-reference, and transcendence might illuminate your own intellectual journey. The dialectical movement we've traced isn't merely an abstract structure—it's a living process that unfolds whenever we engage seriously with the limits of our understanding.

And perhaps most importantly, as readers emerge from this exploration of how mathematical, philosophical, and narrative incompatibilities accrue into transformative insights, I invite you to forget ”about stars; And whether they had one, or not, upon thars.“ For in the end, the most profound insight may be that our certainties, like the Sneetches’ stars, are not what define us—it’s our capacity to move beyond them while carrying forward what was true in what we leave behind.

References

Brandom, R. B. (2008). *Between saying and doing: Towards an analytic pragmatism*. Oxford University Press.

Brandom, R. B. (2019). *A spirit of trust: A reading of Hegel's phenomenology*. The Belknap Press of Harvard University Press.

Cantor, G. (1891). Ueber eine elementare frage der mannigfaltigkeitslehre. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, *1*, 75–78.

Euclid. (2007, December). *Euclid's Elements of Geometry* (J. L. Heiberg, Ed.; R. Fitzpatrick, Trans.; Revised and corrected edition). Richard Fitzpatrick.

Franzén, T. (2004, September). *Inexhaustibility: A Non-Exhaustive Treatment: Lecture Notes in Logic 16* (1st edition). A K Peters/CRC Press.

Frege, G. (1948). Sense and reference. *The Philosophical Review*, *57*(3), 209–230. http://www.jstor.org/stable/2181485

Gaifman, H. (2005). *The easy way to Gödel's proof and related matters* [Unpublished manuscript. Available online].

Gaifman, H. (2006). Naming and diagonalization, from cantor to gödel to kleene. *Logic Journal of the IGPL*, *14*(5), 709–728. https://doi.org/10.1093/jigpal/jzl006

Hardy, M., & Woodgold, C. (2009). Prime simplicity. *The Mathematical Intelligencer*, *31*, 44–52.

Hegel, G. W. F. (1977). *Phenomenology of spirit* (A. V. Miller, Trans.; Vol. 632). Oxford University Press Oxford.

Inwood, M. J. (1992). *A hegel dictionary*. Blackwell Oxford.

Kanamori, A. (1996). The mathematical development of set theory from cantor to cohen. *Bulletin of Symbolic Logic*, *2*(1), 1–71.

Maybee, J. E. (2020). Hegel's Dialectics. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Winter 2020). Metaphysics Research Lab, Stanford University.

Savich, T. M. (2022). *Towards a critical mathematics*. Indiana University.

Seuss, D. (1961, August). *The Sneetches and Other Stories* (First edition.). Random House Books for Young Readers.