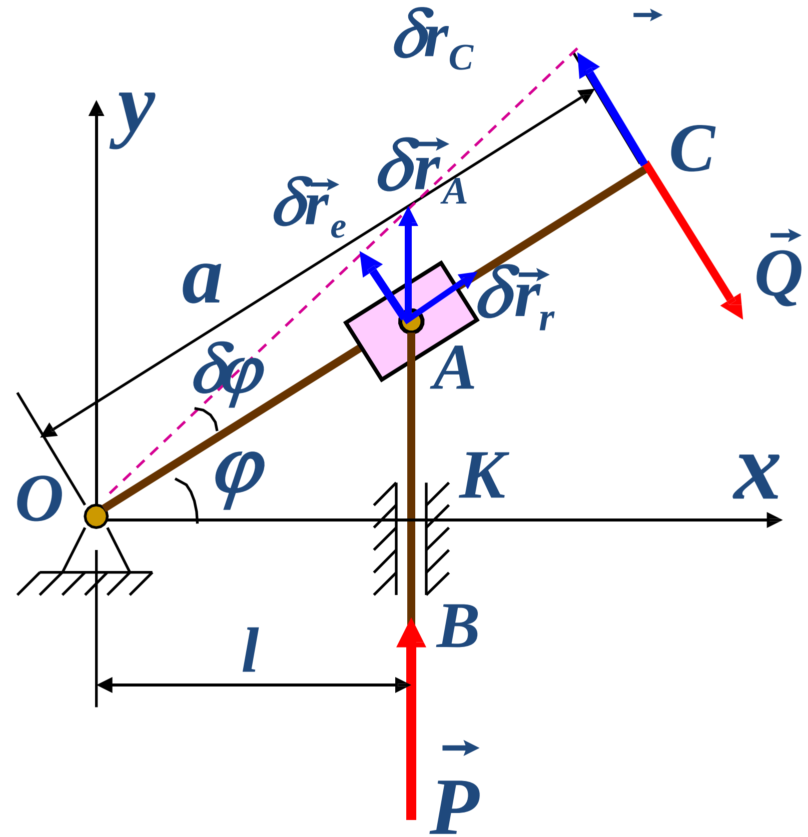
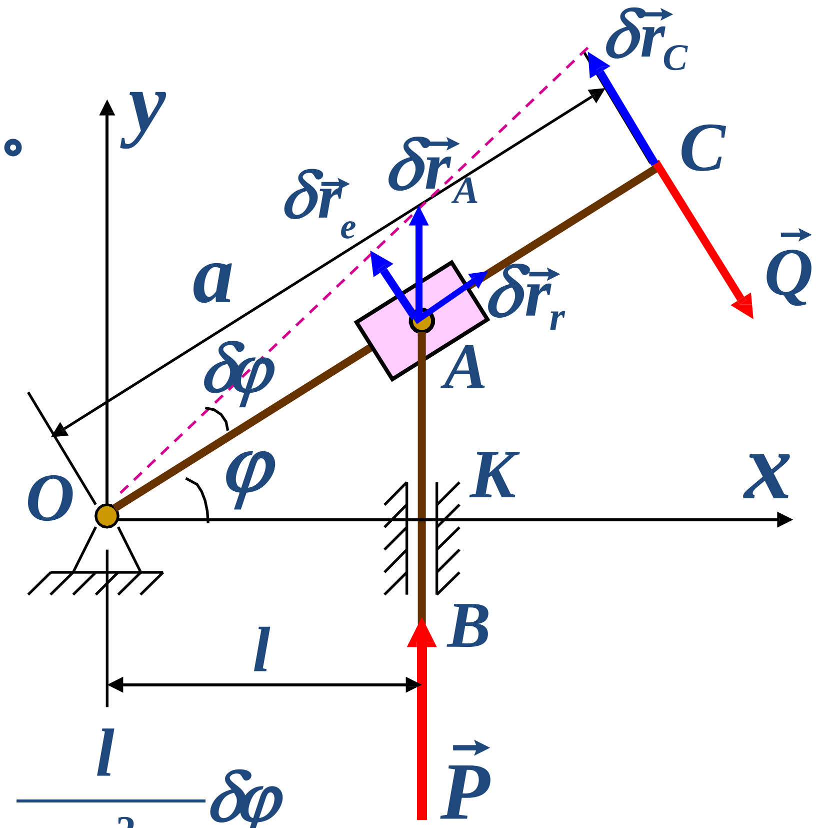
思考：图示机构中，当曲柄*OC*绕轴摆动时，滑块*A*沿曲柄自由滑动，从而带动杆*AB*在铅垂导槽*K*内移动。已知*OC=a*，*OK=l*，在C点垂直于曲柄作用一力*Q*，而在*B*点沿*BA*作用一力*P*。求机构平衡时，力*P*与*Q*的关系。



解1：（几何法）以系统为研究对象，受的主动力有P、Q。系统给组虚位移如图。



其中 $\delta\vec{r}_A = \delta\vec{r}_e + \delta\vec{r}_r$

由虚位移原理 $\sum \vec{F}_i \cdot \delta\vec{r}_i = 0$，得

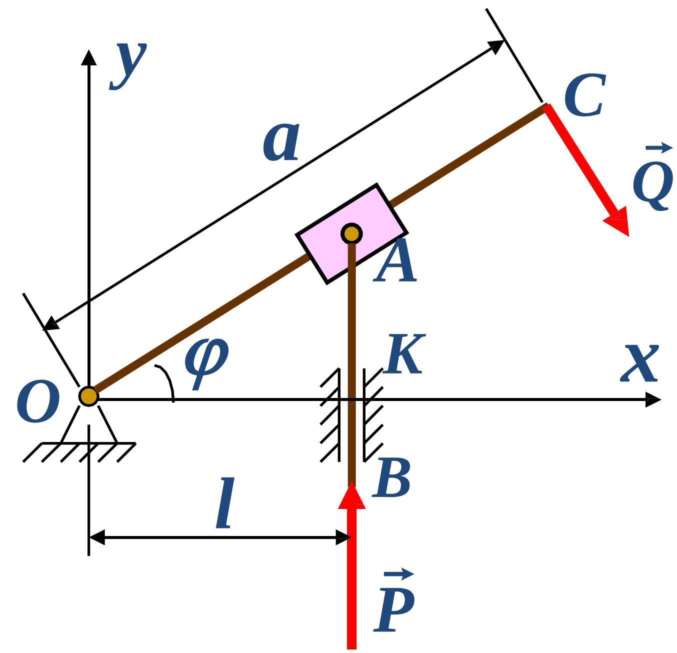
$$P\delta r_A - Q\delta r_C = 0$$

式中 $\delta r_C = a\delta\varphi$ $\delta r_A = \dfrac{\delta r_e}{\cos\varphi} = \dfrac{l}{\cos^2\varphi}\delta\varphi$

故有 $P\dfrac{l}{\cos^2\varphi}\delta\varphi - Qa\delta\varphi = 0$ 由于$\delta\varphi \neq 0$，于是得

$$Q = \frac{l}{a\cos^2\varphi}P$$

解2　解析法:建立如图坐标。　主



动力作用点的坐标及其变分为

$$y_A = l\operatorname{tg}\varphi \Rightarrow \delta y_A = \frac{l}{\cos^2\varphi}\delta\varphi$$

$$x_C = a\cos\varphi \Rightarrow \delta x_C = -a\sin\varphi\delta\varphi$$

$$y_C = a\sin\varphi \Rightarrow \delta y_C = a\cos\varphi\delta\varphi$$

主动力在坐标方向上的投影为

$$F_{Ay} = P \qquad F_C = Q\sin\varphi \qquad F_C = -Q\sin\varphi$$

由　$$\sum(F_{xi}\overset{x}{\delta} x_i + F_{yi}\delta y_i + F_{zi}\overset{y}{\delta} z_i) = 0$$

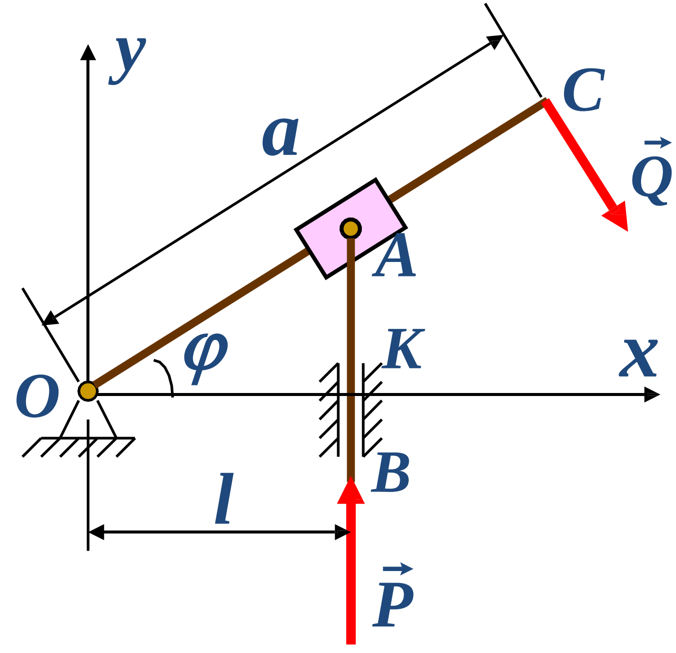
即 $$F_{Ay}\delta y_A + F_{Cx}\delta x_C + F_{Cy}\delta y_C = 0$$

得 $$P\frac{l}{\cos^2\varphi}\delta\varphi + Q\sin\varphi(-a\sin\varphi\delta\varphi) + (-Q\cos\varphi)a\cos\varphi\delta\varphi = 0$$

亦即 $$P\frac{l}{\cos^2\varphi}\delta\varphi - Qa\delta\varphi = 0$$

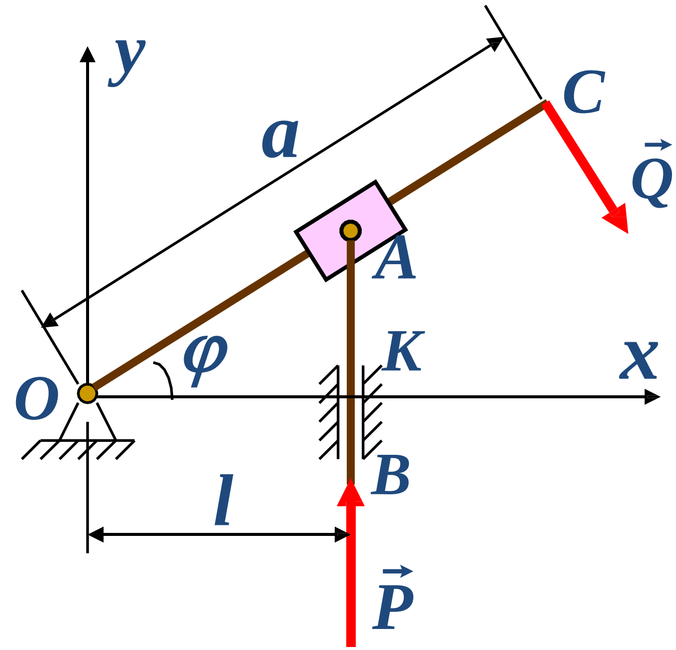
由于$\delta\varphi \neq 0$ 于是得

$$Q = \frac{l}{a\cos^2\varphi}P$$



解3：综合法。

本题用解析法计算$\vec{P}$力的虚功，用几何法计算$\vec{Q}$力的虚功，此时虚功方程可以写为



$$F_{Ay}\delta y_A+\vec{Q}\cdot\delta\vec{r}_C=0$$

将 $F_{Ay}=P$ ，$y_A=l\mathrm{tg}\varphi$ ，$\delta r_C=a\delta\varphi$

代入上式，得 $P\delta(l\mathrm{tg}\varphi)-Q\delta r_C=0$

解析法中，广义坐标的增量总是取增大的方向。本例中$\delta\varphi$取为增大的方向，即为逆时针转向。

即 $P\dfrac{l}{\cos^2\varphi}\delta\varphi-Qa\delta\varphi=0$ 可得同样的结果。

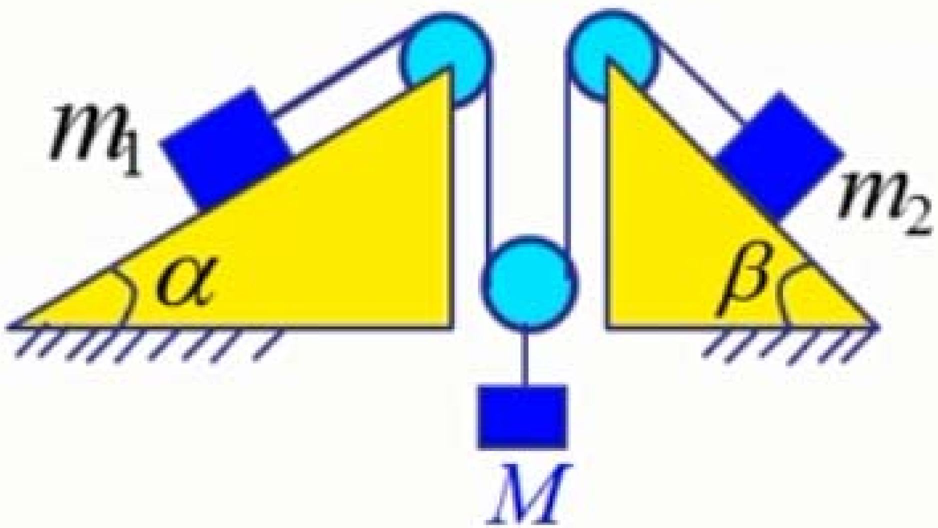
总结广义力的求法：

①直接用广义力的定义式：

$$Q_j = \sum_{i=1}^{3n} F_i \frac{\partial x_i}{\partial q_j} \quad or \quad Q_j = \sum_{i=1}^{n} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \quad (j = 1, 2, \cdots, s)$$

②遵照导出式的步骤，先将 δx_i表示成$\delta q_1, \delta q_2, .., \delta q_s$的函数，代入$\sum_{i=1}^{3n} F_i \delta x_i$中，再与 $\sum_{j=1}^{s} Q_j \delta q_j$ 比较。

已知： $m_1, m_2, M, \alpha, \beta$, 且接触面光滑。
求：平衡时， m_1, m_2, M 的关系。



§6.3 拉格朗日方程

- （一）达朗贝尔原理

研究**n**个质点组成的体系，每个质点的运动都服从牛顿定律：

$$m_i \ddot{\vec{r}}_i = \vec{F}_i + \vec{F}_i' \qquad (1,2,\cdots,n)$$

$$\Downarrow$$

$$\vec{F}_i + \vec{F}_i' - m_i \ddot{\vec{r}}_i = 0 \qquad (1,2,\cdots,n)$$

意义：如果把$-m_i\ddot{\vec{r}}_i$当作作用在质点上的力看待，那么任何瞬时作用在体系中任意质点***i***上的主动力 $\vec{F}_i$，约束力$\vec{F}_i'$，和力$-m_i\ddot{\vec{r}}_i$总是平衡的，质点的动力学方程转化为静力学方程，此平衡原则称为**达朗贝尔原理**

$-m_i\ddot{\vec{r}}_i$ 称为逆效力或达朗贝尔惯性力

以静制动！

- 达朗贝尔-拉格朗日方程

根据虚功原理，体系的静平衡条件为：

$$\delta W=\sum_{i=1}^{n}\left(\vec{F}_i+\vec{F}_i{}'-m_i\ddot{\vec{r}}_i\right)\cdot\delta\vec{r}_i=0$$

只考虑理想约束体系： $\sum_{i=1}^{n}\vec{F}_i{}'\cdot\delta\vec{r}_i=0$

得到 $\delta W=\sum_{i=1}^{n}\left(\vec{F}_i-m_i\ddot{\vec{r}}_i\right)\cdot\delta\vec{r}_i=0$

在理想约束下，运动的每一瞬间系统所受主动力和逆效力的虚功之和为零

- 基本形式的拉格朗日方程

考虑n个质点组成的自由度为s的体系：

$$\vec{r}_i = \vec{r}_i(q_\alpha, t) \qquad (\alpha = 1,2,\cdots,s)$$

先证明下述两个恒等式

$$\boxed{\frac{\partial \vec{r}_i}{\partial q_\alpha} = \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_\alpha}} \qquad \boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \vec{r}_i}{\partial q_\alpha}\right) = \frac{\partial}{\partial q_\alpha}\left(\frac{\mathrm{d}\vec{r}_i}{\mathrm{d}t}\right) = \frac{\partial \dot{\vec{r}}_i}{\partial q_\alpha}}$$

$$\boxed{\delta\vec{r}_i = \sum_{\alpha=1}^{s}\frac{\partial\vec{r}_i}{\partial q_\alpha}\delta q_\alpha}$$

$$\sum_{i=1}^{n}\left(\vec{F}_i - m_i\ddot{\vec{r}}_i\right)\cdot\delta\vec{r}_i = 0$$

$$\Downarrow$$

$$\sum_{i=1}^{n}\left(\vec{F}_i - m_i\ddot{\vec{r}}_i\right)\cdot\sum_{\alpha=1}^{s}\frac{\partial\vec{r}_i}{\partial q_\alpha}\delta q_\alpha = 0$$

$$\Downarrow$$

$$\sum_{i=1}^{n}\sum_{\alpha=1}^{s}\vec{F}_i\cdot\frac{\partial\vec{r}_i}{\partial q_\alpha}\delta q_\alpha - \sum_{i=1}^{n}\sum_{\alpha=1}^{s}m_i\ddot{\vec{r}}_i\cdot\frac{\partial\vec{r}_i}{\partial q_\alpha}\delta q_\alpha = 0$$

$$\sum_{\alpha=1}^{s}\sum_{i=1}^{n}\left(\vec{F}_i\cdot\frac{\partial\vec{r}_i}{\partial q_\alpha} - m_i\ddot{\vec{r}}_i\cdot\frac{\partial\vec{r}_i}{\partial q_\alpha}\right)\delta q_\alpha = 0$$

$$\sum_{\alpha=1}^{s}\sum_{i=1}^{n}\left(\vec{F}_i\cdot\frac{\partial\vec{r}_i}{\partial q_\alpha}-m_i\ddot{\vec{r}}_i\cdot\frac{\partial\vec{r}_i}{\partial q_\alpha}\right)\delta q_\alpha=0$$

可写为 $$\sum_{\alpha=1}^{s}(Q_\alpha-P_\alpha)\delta q_\alpha=0$$

其中 $$Q_\alpha=\sum_{i=1}^{n}\vec{F}_i\cdot\frac{\partial\vec{r}_i}{\partial q_\alpha}$$

$$P_\alpha=\sum_{i=1}^{n}m_i\ddot{\vec{r}}_i\cdot\frac{\partial\vec{r}_i}{\partial q_\alpha}$$

$$P_\alpha = \sum_{i=1}^{n} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} = \frac{d}{dt}\sum_{i=1}^{n} m_i \left(\dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} \right) - \sum_{i=1}^{n} m_i \left(\dot{\vec{r}}_i \cdot \frac{d}{dt}\frac{\partial \vec{r}_i}{\partial q_\alpha} \right)$$

$$\boxed{\dot{\vec{r}}_i = \frac{d\vec{r}_i}{dt} = \sum_{\alpha=1}^{s} \frac{\partial \vec{r}_i}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial \vec{r}_i}{\partial t}} \Rightarrow \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_\alpha} = \frac{\partial \vec{r}_i}{\partial q_\alpha}$$

$\because \dot{\vec{r}}_i = \dot{\vec{r}}_i(q_\alpha, \dot{q}_\alpha, t)$, $\dfrac{\partial \vec{r}_i}{\partial q_\alpha}, \dfrac{\partial \vec{r}_i}{\partial t}$ 不是 $\dot{q}_\alpha$ 的函数

而 $$\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_\alpha} \right) = \frac{\partial}{\partial q_\alpha}\left(\frac{d\vec{r}_i}{dt} \right) = \frac{\partial \dot{\vec{r}}_i}{\partial q_\alpha}$$

得 $$P_\alpha = \frac{d}{dt}\sum_{i=1}^{n} m_i \left(\dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_\alpha} \right) - \sum_{i=1}^{n} m_i \left(\dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_\alpha} \right)$$

注意 $T = \dfrac{1}{2}\sum_{i=1}^{n} m_i \dot{\vec{r}}_i^{\,2}$ 得到 $P_\alpha = \dfrac{d}{dt}\dfrac{\partial T}{\partial \dot{q}_\alpha} - \dfrac{\partial T}{\partial q_\alpha}$

$$\sum_{\alpha=1}^{s}(Q_\alpha - P_\alpha)\delta q_\alpha = 0$$

$$\sum_{\alpha=1}^{s}\left(Q_\alpha - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\alpha} + \frac{\partial T}{\partial q_\alpha}\right)\delta q_\alpha = 0$$

基本形式的拉格朗日方程

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = Q_\alpha \qquad (\alpha = 1, 2, \cdots s)$$

广义速度

广义动量

拉格朗日力

广义力

$$\sum_{\alpha=1}^{s}\left(Q_\alpha-\sum_{i=1}^{n}m_i\ddot{\vec{r}}_i\cdot\frac{\partial\vec{r}_i}{\partial q_\alpha}\right)\delta q_\alpha=0$$

由于s 个广义坐标的变分各自独立，得到

$$\boxed{\sum_{i=1}^{n}m_i\ddot{\vec{r}}_i\cdot\frac{\partial\vec{r}_i}{\partial q_\alpha}=Q_\alpha}$$

$$\sum_{i=1}^{n}m_i\ddot{\vec{r}}_i\cdot\frac{\partial\vec{r}_i}{\partial q_\alpha}=\sum_{i=1}^{n}m_i\frac{\mathrm{d}\dot{\vec{r}}_i}{\mathrm{d}t}\cdot\frac{\partial\vec{r}_i}{\partial q_\alpha}$$

$$=\sum_{i=1}^{n}m_i\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\vec{r}}_i\cdot\frac{\partial\vec{r}_i}{\partial q_\alpha}\right)-\sum_{i=1}^{n}m_i\dot{\vec{r}}_i\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\vec{r}_i}{\partial q_\alpha}\right)$$

- 有势系的拉格朗日方程

对于有势体系，广义力为

$$Q_\alpha = \sum_{i=1}^{n} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} = -\sum_{i=1}^{n} \frac{\partial V}{\partial \vec{r}_i} \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} = -\frac{\partial V}{\partial q_\alpha}$$

则拉格朗日方程变为

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = Q_\alpha = -\frac{\partial V}{\partial q_\alpha}$$

$$\frac{\partial V}{\partial \dot{q}_\alpha} = 0$$

移项整理得

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial(T-V)}{\partial\dot{q}_\alpha}-\frac{\partial(T-V)}{\partial q_\alpha}=0$$

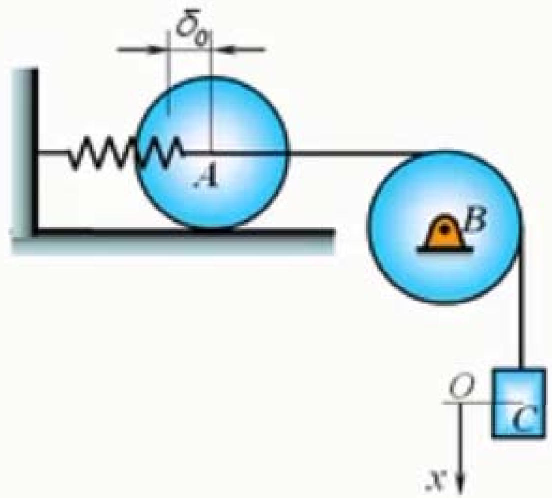
把 $L(q,\dot{q};t)=T(q,\dot{q};t)-V(q;t)$ 定义为拉格朗日函数，则拉格朗日方程变为

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_\alpha}-\frac{\partial L}{\partial q_\alpha}=0 \quad (\alpha=1,2,\cdots,s)$$

$\boxed{\text{受理想约束的完整有势系的拉格朗日方程}}$

例 1 质量为m_1的物块C以细绳跨过定滑轮B联于点A，A，B两轮皆为均质圆盘，半径为R，质量为m_2，弹簧刚度为k，质量不计。

求：当弹簧较软，在细绳能始终保持张紧的条件此系统的运动微分方程。



解：此系统具有一个自由度，以物块平衡位置为原点，取x为广义坐标。以平衡位置为重力零势能点。

取弹簧原长处为弹性力零势能点。

系统在任意位置x处的势能为：

$$V=\frac{1}{2}k(\delta_0+x)^2-m_1gx$$

其中δ_0为平衡位置处弹簧的伸长

系统的动能为：

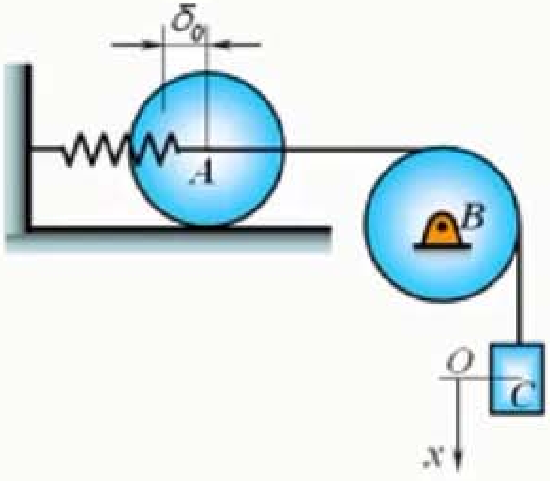
$$T=\frac{1}{2}m_1\dot{x}^2+\frac{1}{2}\cdot\frac{1}{2}m_2R^2(\frac{\dot{x}}{R})^2+\frac{1}{2}m_2\dot{x}^2+\frac{1}{2}\cdot\frac{1}{2}m_2R^2(\frac{\dot{x}}{R})^2=(m_2+\frac{1}{2}m_1)\dot{x}^2$$

系统的主动力为有势力，为保守系统，系统的拉格朗日函数为：

$$L=T-V=(m_2+\frac{1}{2}m_1)\dot{x}^2-\frac{1}{2}k(\delta_0+x)^2+m_1gx$$

代入保守系统的拉格朗日方程:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0$$



得 $(2m_2+m_1)\ddot{x}+k\delta_0+kx-m_1g=0$

注意 $k\delta_0=m_1g$

则系统的运动微分方程为:

$$(2m_2+m_1)\ddot{x}+kx=0$$

这是自由振动的微分方程，其振动周期为:

$$T=2\pi\sqrt{\frac{2m_2+m_1}{k}}$$

运用拉格朗日方程求解完整约束系统的动力学问题，特别是保守系统的动力学问题，不比分析受力，也不用分析运动，只需用广义坐标表示出系统的动能和势能，形式简洁，便于计算，应用非常简单方便