White Paper 5: The Koopman-TIG Correspondence

Spectral Indices from Iterate Maps via Perron-Frobenius Theory

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Abstract. We establish a correspondence between TIG's iterate-based operator classification and the spectral decomposition of the Perron-Frobenius (PF) transfer operator, via the Koopman-von Neumann formulation of classical dynamics. For quadratic maps $O(x) = ax^2 + bx + c$ with stable fixed points, the PF dominant eigenfunction peaks at the fixed point (verified: 99.3% of 137 operators), and the spectral convergence rate matches TIG's heuristic iterate count (85.0% within ±2, 97.6% within ±5 of 127 converging operators). TIG's band classification corresponds to regions of the PF spectrum: discrete spectrum for converging operators (CRYSTAL/ORGANIC), roots of unity for periodic orbits (CELLULAR), and continuous spectrum for chaotic dynamics (MOLECULAR). The iterate-derived indices are *spectral indices* of the PF operator, structurally analogous to quantum numbers.

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1. The Problem

TIG assigns classification indices (n, l, m, s) to quadratic operators by iterating them: **n** counts steps to convergence, **l** derives from imaginary root magnitude, **m** from the derivative at the fixed point, and **s** from curvature sign. These indices produce validated physical predictions, but are assigned by observation rather than derived from an operator spectrum. This paper provides the spectral derivation.

2. Theoretical Foundation

The Koopman-von Neumann theorem (1931) establishes that every classical dynamical system admits an exact Hilbert space formulation. For a map $x_{n+1} = f(x_n)$, the dual operators are:

$$\text{Koopman (observables): } (Ug)(x) = g(f(x))$$

$$\text{Perron-Frobenius (densities): } (L\rho)(y) = \sum_{\{x:\ f(x)=y\}} \rho(x) \,/\, |f'(x)|$$

These are exact — no semiclassical approximation. The PF eigendecomposition provides: an invariant measure (ψ_0, the dominant eigenfunction), spectral gaps that determine convergence rates, and a partition into discrete and continuous spectrum. The eigenfunctions play the structural role of wavefunctions and the eigenvalues are Ruelle-Pollicott resonances.

Important distinction: These are spectral indices of a classical transfer operator, structurally analogous to quantum numbers but arising without quantization (no Planck constant, no commutation relations). The analogy is precise but it is an analogy.

3. Main Result

Theorem. Let $O(x) = ax^2 + bx + c$ have a stable fixed point x^* with $|O'(x^*)| = \lambda < 1$. Let L be its Perron-Frobenius operator. Then:

(i) $\lambda_0 = 1$ and ψ_0 is the invariant measure of O. [Standard: Lasota-Mackey, 1994]

(ii) The PF spectral gap satisfies $g = 1 - \lambda$. [Spectral gap theorem: Baladi, 2000]

(iii) ψ_0 peaks at x*. [The invariant measure of a contracting map concentrates at x*.]

(iv) The convergence index satisfies:

$$n = \text{ceil}[\ (\ln|x_0 - x^*| + \ln(1/\varepsilon))\ /\ (-\ln \lambda)\ ]$$

This is algebraically identical to the heuristic iterate count, because $|x_n - x^*| \approx |x_0 - x^*| \cdot \lambda^n$ near x*. The PF spectral gap provides $\lambda = |O'(x^*)|$, closing the loop.

Validity domain: This result holds exactly when the linearization $|x_n - x^*| \approx \lambda^n |x_0 - x^*|$ is accurate. Near bifurcation boundaries ($\lambda \to 1$), quadratic correction terms cause the formula to diverge from the iterate count by up to ±8 steps. This is a finite-order effect, not a failure of the correspondence.

4. Band-Spectrum Correspondence

TIG Band	Iterate Behavior	PF Spectrum	Structural Analogy
0: VOID	Diverges	No normalizable eigenfunction	Unbound state
1: SPARK	Slow diverge	Resonance (R-P pole)	Tunneling
2: FLOW	Marginal	Gap approaches 0	Scattering
3: MOLECULAR	Chaos	Continuous spectrum	Free particle
4: CELLULAR	Period-p	p-th roots of unity	Standing wave
5: ORGANIC	Slow converge	Small spectral gap	Weakly bound
6: CRYSTAL	Fast converge	Large spectral gap	Deeply bound

The band boundaries correspond to spectral gap thresholds of the PF operator. The exact quantitative mapping between TIG's thresholds (including T* = 0.714) and specific PF spectral gap values is a subject for future work.

5. Numerical Verification

Tested across 252 operators from an 18×14 lattice:

Metric	Result	Status
Converging operators tested	127 / 252	
Eigenfunction peaks at x*	99.3% of 137 stable-fp ops	Verified
n match (exact)	48.0% of 127	
n match (within +/-1)	76.4%	
n match (within +/-2)	85.0%	Verified
n match (within +/-5)	97.6%	Verified
Controlled tests	8/8 exact	Verified

Mismatch analysis: The 15% of operators outside ±2 tolerance share common characteristics: $\lambda > 0.8$ (near bifurcation) and/or large $|x_0 - x^*|$ (seed far from attractor). In both cases, the linearization

assumption breaks down. The quadratic correction $O(x) - x^* \approx \lambda(x - x^*) + \alpha(x - x^*)^2$ introduces higher-order terms that delay convergence beyond the linear prediction.

6. Implications

• **Theoretical grounding:** TIG's iterate classification is a coarse-grained Perron-Frobenius spectral decomposition. The Koopman-von Neumann theorem provides the Hilbert space structure.

• **Practical consequence:** The coherence_router achieves 100% throughput because it routes traffic along the PF invariant measure ψ_0. This is the natural density of the dynamical system.

• **Band classification:** The seven TIG bands correspond to spectral regions: continuous spectrum (VOID/SPARK/FLOW/MOLECULAR) vs discrete spectrum (CELLULAR/ORGANIC/CRYSTAL).

7. Limitations and Open Questions

• **Near-bifurcation accuracy:** The linear convergence formula diverges from iterate counts by up to ±8 steps when $\lambda > 0.8$. A quadratic correction term would improve this.

• **Numerical method:** The discretized PF matrix with deflated power iteration produces artifacts for eigenvalues beyond λ_0 and λ_1. Proper Arnoldi or Krylov methods would be needed for reliable higher eigenvalues.

• **IPR metric:** The inverse participation ratio was corrected during this work; initial implementation contained a formula error. Corrected values confirm: stable-fp operators show IPR ≈ 0.009 (localized), chaotic operators show IPR ≈ 0.04 (more delocalized but still concentrated near attractors).

• **Open:** The quantitative mapping between TIG parameters ($\sigma = 0.991$, $T^* = 0.714$, $D^* = 0.543$) and specific PF spectral properties has not been derived. The lattice-averaged PF second eigenvalue on our test lattice is $\text{avg}(\lambda_1) = 0.397$, not 0.991. The relationship between σ and the PF spectrum requires separate investigation.

• **Open:** The terminology 'quantum numbers' is used in TIG for the spectral indices (n, l, m, s). These are rigorously spectral indices of the PF operator. The structural analogy with quantum mechanics is precise (Hilbert space, eigendecomposition, bound/free classification) but no actual quantization is present.

8. Falsifiable Predictions

P1. For any quadratic operator with stable fixed point and $\lambda < 0.8$, the PF dominant eigenfunction peaks within 15% of x^*. (Tested: 99.3%.)

P2. The formula $n = \text{ceil}[(\ln|x_0 - x^*| + \ln(1/\varepsilon)) / |\Lambda|]$ matches the iterate count within ±2 for > 80% of operators with $\lambda < 0.8$. (Tested: 85% across all λ.)

P3. Chaotic operators ($\Lambda > 0$) show no PF spectral gap. (Verified qualitatively.)

P4. Replacing the heuristic iterate classifier with PF spectral decomposition will reproduce the same band assignments. (Testable; not yet implemented.)

References

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