MM 225: AI and Data Science

Discrete probability distributions

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Lecture 2: Introduction to probability

Outline

Terminology

Recall

- Experiment: Toss a coin
- Outcome: H or T
- Outcome: decided by chance
- Random variable (X): outcome of an experiment

Sample space

- Sample space: set of all possible outcomes of the experiment
- Sample space for a single coin toss: $\{H,T\}$
- X: random variable for toss of a single coin
- Sample space (Ω) of X: $\{H,T\}$
- Event: subset of a sample space
- X: discrete if sample space is finite or countably infinite

Distribution function and probability

- A real valued function (m) with domain Ω: distribution function for X such that
  1. $m(\omega) \geq 0$ for all $\omega \in \Omega$;
  2. $\sum_{\omega \in \Omega} m(\omega) = 1$
- For any subset E of Ω, we define, probability of E, $P(E) = \sum_{\omega \in E} m(\omega)$

Toss of two coins

Event space for the toss of two coins

- X: random variable for the toss of two (fair) coins
- What is the event space Ω of X?
- Assume we record the order:
  $\Omega(X) = \{HH, HT, TH, TT\}$
- Assume we do not record the order:
  $\Omega(X) = \{HH, HT/TH, TT\}$
- Assume we record the number of heads:
  $\Omega(X) = \{0, 1, 2\}$

Distribution function and probability

- What is the distribution function?
  Recording order: $m(HH) = m(HT) = m(TH) = m(TT) = 0.25$
  Without recording order: $m(HH) = m(TT) = 0.25$; $m(HT/TH) = 0.5$
  For number of heads: $m(0) = m(2) = 0.25$, $m(1) = 0.5$
- Probability: For every $\omega \in \Omega$, probability of the elementary event $\{\omega\}$ is the distribution function $m(\omega)$:

$$P\{\omega\} = m(\omega) \tag{1}$$

for all $\omega \in \Omega$

Recall: set theoretic notation

Need

- Set theory: easier to describe events
- Venn diagrams: useful in proving some of the results
- Recall the set notation
- Venn diagrams: look up and you should be able to prove theorems and solve problems

Set notations

- A, B: two sets
- Union: $A \cup B = \{x | x \in A \; or \; x \in B\}$
- Intersection: $A \cap B = \{x | x \in A \; and \; x \in B\}$
- Difference: $A - B = \{x | x \in A \; and \; x \notin B$
- Complement: $\bar{A} = \Omega - A$
- $A \subset B$: A is a subset of B

Probability: properties and theorems

Postulates and theorems of probability P

- $P(E) \geq 0$ for every $E \subset \Omega$
- $P(\Omega) = 1$
- If A and B are disjoint subsets of Ω then $P(A \cup B) = P(A) + P(B)$
- If $E \subset F \subset \Omega$ then, $P(E) \leq P(F)$
- Complement: $\bar{A} = \Omega - A$
- $P(\bar{A}) = 1 - P(A)$ for every $A \subset \Omega$

Theorems

Theorem

If $A_1, A_2, \ldots A_n$ *are pairwise disjoint subsets of* Ω, $P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} P(A_i)$

Theorem

If $A_1, A_2, \ldots A_n$ *are pairwise disjoint subsets of* Ω *such that* $A_1 \cup A_2 \ldots \cup A_n = \Omega$, *and, if* E *be any event,* $P(E) = \sum_{i=1}^{n} P(A_i \cap E)$

Corollary

For any two events A *and* B, $P(A) = P(A \cap B) + P(A \cap \bar{B})$

Theorems

Theorem

For any two $A, B \in \Omega$, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Uniform distribution

Uniform distribution

Definition

On a sample space Ω containing n elements, $m(\omega) = \frac{1}{n}$ for every $\omega \in \Omega$

Remark

Note that for the two coin toss problem, this works for the sample space where we keep track of the order but does not work when we keep track of the number of heads.

Odds

Odds

Definition

If $P(E) = p$ the odds in favour of te event E occuring are r:s where $\frac{r}{s} = \frac{p}{1-p}$

Remark

Note that this implies $p = \frac{r}{r+s}$

Countably infinite sample spaces

First H

Question

Let a fair coin be tossed till the first time a head turns up. What is the sample space?

$\Omega = \{1, 2, 3, \ldots\}$

Question

Why 0 is not part of Ω?

Head will turn up eventually and the probability of all T in an infinite tosses is zero.

Distribution function

- Head in first toss: $m(1) = \frac{1}{2}$
- Head after a tail: $m(2) = \frac{1}{4}$
- Head after two tails: $m(3) = \frac{1}{8}$
- This m can be used if we can show $\sum_{\omega} m(\omega) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 1$
- Home work: show!!

A question

Question

What are the probabilities that an H turns up after (i) an even number, and (ii) an odd number, of tosses?

Calculation

What is the probability that an H turns up after an even number of tosses?

- **Link:** https://www.menti.com/alrgdgpq84jq
- **Code:** Go to menti.com and use code 1687 5803

Results

Results of calculation

Answer

The probability that an H turns up after an even number of tosses

$$\sum_{\omega} m(\omega) = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3}$$

This implies the probability that an H turns up after an odd number of tosses is $\frac{2}{3}$!

Homework Simulate!

Problem of points

Fermat and Pascal

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By unknown; a copy of the painting of François II Quesnel, which was made for Gérard Edelinck en 1691[réf. nécessaire].

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The problem

Suppose G and H are playing several games. G needs three more wins and H needs two more wins to get the stakes. If the games are stopped at this point, what is the fair division of stakes? Assume that the stakes are Rs. 64.

Answers

Pascal: If G wins one more game, they would split the stakes in half. So, at this point, H can ask for Rs. 32 + Rs 16 (since there is equal chance of winning or losing the next game). So, the fair division is 48 for H and 16 for G.
Fermat: If G and H can play another 4 games, the decision will be absolute. Listing all possible game outcomes in 4 games, we see that H wins in 11 cases and G wins in 5 cases. Hence, the fair division is (11/16) of 64 for H and the rest for G.

Remark

Answer is the same. Pascal's solution is algorithmic; Fermat's is dependent on combinatorics. We will use more such combinatoric problems to decide on probabilities in the sessions to come.

THANK YOU!!!