MM 225: AI and Data Science

Conditional probabilities

M P Gururajan and Hina A Gokhale

August 24, 2023

Lecture 6: Conditional probabilities

Outline

Discrete conditional probability

Question

Let X be the outcome of an experiment that consists of rolling a die. Let F be the event $X = 6$. What is P(F)?

Answer

What is P(F)?

- **Link:** https://www.menti.com/al5bh8pjsno3
- **Code:** Go to menti.com and use code 8623 7695

Results

Results of the poll

Question

Let X be the outcome of an experiment that consists of rolling a die. Let F be the event $X = 6$. Let E be the event $X > 4$. Suppose if you know the event E occured, what is P(F)?

Answer

What is P(F)?

- **Link:** https://www.menti.com/al2z9np5yhj4
- **Code:** Go to menti.com and use code 2800 8237

Results

▸ Results of the poll

Conditional probability

- We assign a distribution function to a sample space
- We learn that an event (E) has occurred
- How do we change the probabilities of the remaining events?
- The new probability for the event F given E: conditional probability
- Conditional probability: $P(F|E)$

Question

Suppose in a population of 100000 females, 89.835% are expected to live to age 60 and 57.062% are expected to live to age 80. Given that a woman is 60, what is the probability that she lives to age 80?

Answer

The probability that a woman is 60, the probability that she lives to 80 is

- **Link:** https://www.menti.com/aljfyf45g66t
- **Code:** Go to menti.com and use code 4763 5780

Results

▸ Results of the poll

Questions

Suppose three candidates G, H and S are running for office.

- Suppose H and S have an equal chance of winning while G only half as likely to win as H. What are the probabilities of win?
- Suppose H drops out of race; what are the probabilities of win?
- Suppose H drops out of race, and, in the absence of H in the race, most voters will favour G. What are the probabilities of win?

Conditional probability

$$P(F|E) = \frac{P(F \cap E)}{P(E)} \tag{1}$$

Bayes probability

A problem

- We have two urns, say Urn I and Urn II.
- Urn I contains 1 black ball and 1 white ball. Urn II contains 2 black balls and 3 white balls.
- If a black ball is drawn, what is the probability that it came from Urn I?

Solution using tree diagram

Bayes probability

- $P(I|B) = \frac{P(I \cap B)}{P(B)}$
- $P(I \cap B) = \frac{1}{4}$
- $P(B) = P(B \cap I) + P(B \cap II)$
- $P(B) = \frac{1}{5} + \frac{1}{4}$
- $P(I|B) = \frac{5}{9}$
- Original tree: probability of black given urn chosen
- Bayes: reverse; given colour, probability for the urn from which it came

Monty Hall

There are three doors. Behind one is a car and behind the others are goats. You pick a door. The game show host opens another and shows you that there is a goat. Should you switch or stick to your original door? Can you see that this is a conditional probability problem? Can you write a python script to do a simulation to check your intuition?

Independent events

Independent events

Theorem

If $P(E) > 0$ and $P(F) > 0$, then E and F are independent events if and only if $P(E \cap F) = P(E)P(F)$.

A question

Suppose a coin, when tossed, comes up heads with proabbility p and tails with probability q. Let this coin be tossed twice. Let E be the event that the heads turns up in the first toss and F be the event that tails turns up in the second toss. Are these two events independent?

Another question

Consider a fair coin; that is, when tossed, it comes up heads with proabbility 0.5 and tails with probability 0.5. Let this coin be tossed twice. Let A be the event that the heads turns up in the first toss and B be the event that the two outcomes are the same. Are these two events independent?

One more question

Consider a fair coin; that is, when tossed, it comes up heads with proabbility 0.5 and tails with probability 0.5. Let this coin be tossed twice. Let A be the event that the heads turns up in the first toss and B be the event that two heads turn up. Are these two events independent?

Remarks

- First problem: our probability assignment makes sure that independent events give the right probability
- Second and third problems: the formula helps determine whether two events are independent or not

Independent events: generalisation

Theorem

A set of events $\{A_1, A_2, A_3, \cdots A_n\}$ *is said to be mutually independent if for any subset* $\{A_1, A_2, A_3, \cdots A_m\}$ *of these events we have* $P(A_1 \cap A_2 \cap \cdots \cap A_m) = P(A_1)P(A_2)\cdots P(A_m)$.

Equivalently, if for any sequence $\bar{A}_1,\ \bar{A}_2\ \cdots\ \bar{A}_n$ with $\bar{A}_i = A_i$ or $\bar{A}_i$ $P(\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_m) = P(\bar{A}_1)P(\bar{A}_2)\cdots P(\bar{A}_m)$ then, the events are mutually independent.

Remark

Any sequence of possible outcomes of a Bernoulli trials process forms a sequence of mutually independent events.

Remark

Independent, Identical Distribution (i.i.d): usual assumption in data (except in time series)

Example 1

Remark

First example: $P(A_1 \cap A_2 \cap \cdots \cap A_m) = P(A_1)P(A_2)\cdots P(A_m)$ does not imply $A_1,\ A_2,\ \cdots\ A_n$ are mutually independent.

Let $\Omega = \{a, b, c, d, e, f\}$. Assume that $m(a) = m(b) = \frac{1}{8}$ and $m(c) = m(d) = m(e) = m(f) = \frac{3}{16}$. Let A, B and C be the events $A = \{d, e, a\}$, $B = \{c, e, a\}$ and $C = \{c, d, a\}$. Show that $P(A \cap B \cap C) = P(A)P(B)P(C)$ but no two of these events are independent.

Example 2

Remark

Second example: If all pairs of a set are independent does not necessarily imply that the whole set is mutually independent.

A coin is tossed twice. Consider the events: A – heads on the first toss; B – Heads on the second toss; C – The two tosses come out the same. Show that A, B and C are pairwise independent, but A, B, C are not independent.

Joint and marginal distributions

Coin toss

- Let us toss a coin thrice
- The random variable X_i: outcome of the i-th toss ($i = 1, 2, 3$)
- Joint random variable $\bar{X}$: (X_1, X_2, X_3)
- Y_i: number of heads in the first i tosses
- Joint random variable $\bar{Y}$ =$(0 \leq a_1 < 1, 0 \leq a_2 \leq 2, 0 \leq a_3 \leq 3)$ with $a_1 < a_2 < a_3$
- Probability of outcomes for $\bar{X}$: $\frac{1}{8}$
- Probability of outcomes for $\bar{Y}$: $\frac{1}{8}$ (for allowed combinations) and zero othwewise
- Joint distribution function: probability of function all outcomes of $\bar{X}$

Definitions

Definition

Let X_1, X_2, $\cdots$, X_n be the random variables associated with an experiment. Suppose that the sample space (i.e., the set of possible outcomes) of X_i is the set R. Then, the joint random variable $\bar{X} = (X_1, X_2, \cdots, X_n)$ is defined to be the random variable whose outcomes consist of ordered n-tuples of outcomes, wth the i-th coordinate lying in the set R_i. The sample space Ω of $\bar{X}$ is the Cartesian product of the R_i's: $\Omega = R_1 \times R_2 \times \cdots \times R_n$. The joint distribution function of $\bar{X}$ is the function which gives the probability of each of the outcomes of $\bar{X}$.

Definition

The random variables X_1, X_2, $\cdots$, X_n are mutually independent (or, independent) if $P(X_1 = r_1, X_2 = r_2, \cdots, X_n = r_n) = P(X_1 = x_1)P(X_2 = r_2) \cdots P(X_n = r_n)$ for any choice of $r_1, r_2, \cdots r_n$.

Joint and Marginal distributions

S (Smoking?)

C (Cancer?)	0 (No)	1 (Yes)	Marginal
0 (No)	$\frac{40}{60}$	$\frac{10}{60}$	$\frac{50}{60}$
1 (Yes)	$\frac{7}{60}$	$\frac{3}{60}$	$\frac{10}{60}$
Marginal	$\frac{47}{60}$	$\frac{13}{60}$	

- Joint distribution (C,S)
- Marginial: distribution of individual random variables
- Note: S and C are not independent; $P(C = 1|S = 1) > P(C = 1)$

THANK YOU!!!