Set-12: Modeling Economic Growth Using the Solow Model and Nonlinear Dynamics

Natansh Shah (202201445)* and Tirth Modi (202201513)†
Dhirubhai Ambani Institute of Information & Communication Technology,
Gandhinagar, Gujarat 382007, India
SC401, Introduction to Nonlinear Dynamics

This assignment explores the mathematical principles underlying Robert Solow's theory of economic growth, emphasizing the application of nonlinear dynamics to economic modeling. The study focuses on analyzing the growth dynamics of the capital-to-labor ratio $r(t)$ under the Solow growth model, incorporating Harrod's natural growth model for labor and a Cobb-Douglas production function. Using analytical and numerical techniques, key growth trends and their transitions are examined.

I. THEORETICAL MODEL

The Solow growth model describes how economic output evolves over time through the accumulation of capital and the growth of labor. It assumes a neoclassical production function where output $Y(t)$ depends on capital $C(t)$ and labor $L(t)$. A commonly used functional form is the Cobb-Douglas production function:

$$Y(t) = C(t)^{\alpha} L(t)^{1-\alpha}, \quad 0 < \alpha < 1 \tag{1}$$

In this setup, labor grows exponentially over time according to Harrod's natural growth model:

$$\frac{dL}{dt} = nL \Rightarrow L(t) = L_0 e^{nt} \tag{2}$$

Here, n is the constant rate of labor growth. The model assumes a constant fraction s of output is reinvested into capital:

$$\frac{dC}{dt} = sY \tag{3}$$

To analyze long-term behavior, we define the capital-to-labor ratio $r(t) = \frac{C(t)}{L(t)}$, which simplifies the model. Using the quotient rule and substituting the above expressions, the evolution of $r(t)$ becomes:

$$\frac{dr}{dt} = sr^{\alpha} - nr \tag{4}$$

This is a first-order nonlinear differential equation. The fixed point, or steady-state, is found by setting $\frac{dr}{dt} = 0$:

$$sr^{\alpha} = nr \Rightarrow r^{\alpha-1} = \frac{n}{s} \Rightarrow r^* = \left(\frac{s}{n}\right)^{\frac{1}{1-\alpha}} \tag{5}$$

We solve this equation analytically to find:

$$r(t) = \left(\frac{s}{n}\right)^{\frac{1}{1-\alpha}} \left[1 + Ae^{-n(1-\alpha)t}\right]^{\frac{1}{1-\alpha}} \tag{6}$$

where,

$$A = \left(\frac{n}{s}\right)^{\frac{1}{1-\alpha}} r_0^{1-\alpha} - 1 \tag{7}$$

This shows that $r(t)$ grows initially and asymptotically approaches the fixed point r^*. The rate of convergence depends on n, α, and the initial condition r_0. The transition time to reach saturation is defined by:

$$t_{\text{trans}} = \frac{1}{n(1-\alpha)} \ln\left(\frac{-A}{1-\alpha}\right) \tag{8}$$

This theoretical framework illustrates how nonlinear dynamics govern long-term economic behavior. Depending on the values of s, n, and α, economies with low initial capital per worker may catch up to the steady state at different rates.

II. RESULTS

A. Phase Diagram: $\dot{r}$ vs r

We analyze the dynamical behavior by plotting $\dot{r} = sr^{\alpha} - nr$ along with the individual curves $y_1 = sr^{\alpha}$ and $y_2 = nr$.

* 202201445@daiict.ac.in
† 202201513@daiict.ac.in

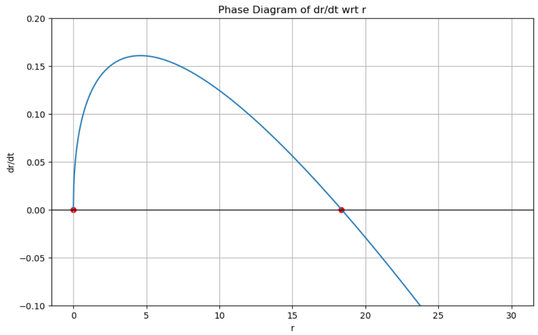

FIG. 1. Phase diagram of $\dot{r}$ vs r showing nonlinear growth and decay. Fixed points occur at intersections of y_1 and y_2.

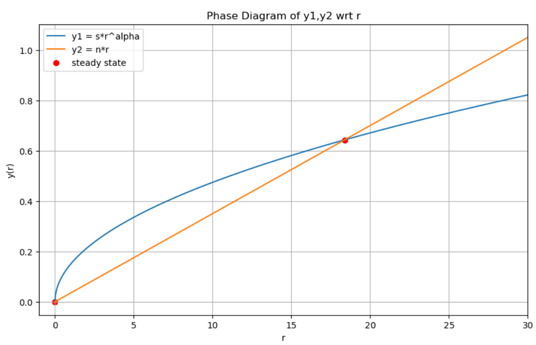

FIG. 2. Curves $y_1 = sr^{\alpha}$ and $y_2 = nr$ plotted against r. Their intersection determines the steady-state value of r.

B. Numerical vs Analytical (Initial: $r_0 = 0$)

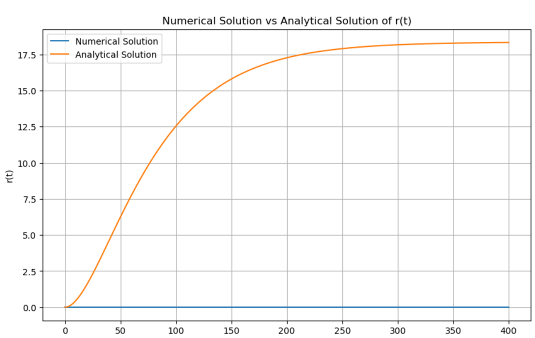

FIG. 3. Numerical and analytical solutions for $r(t)$ with $r_0 = 0$. Only the analytical solution shows growth. $T_{\text{trans}} = 46.2098$ s.

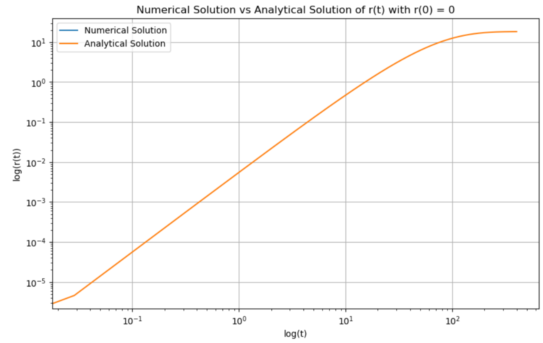

FIG. 4. Log-log plot of analytical solution for $r_0 = 0$. Early-time power-law growth is evident before saturation. $T_{\text{trans}} = 46.2098$ s.

C. Numerical vs Analytical (Initial: $r_0 = 0.001$)

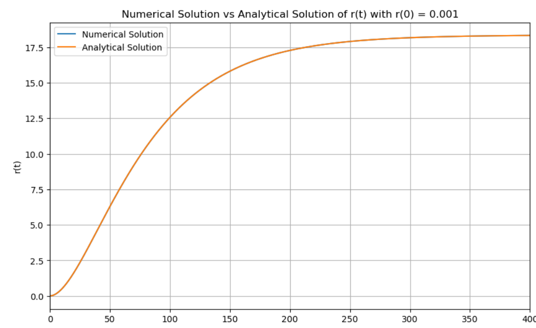

FIG. 5. With $r_0 = 0.001$, both methods agree well. $T_{\text{trans}} = 45.7868$ s.

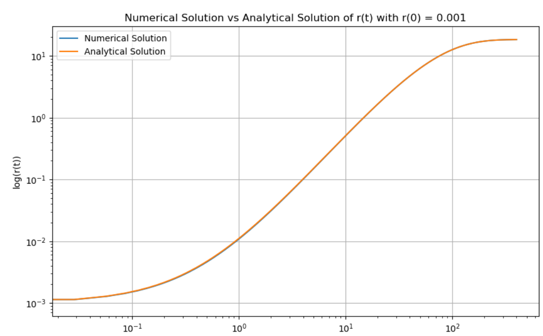

FIG. 6. Log-log plot with $r_0 = 0.001$ showing transition from early growth to stabilization. $T_{\text{trans}} = 45.7868$ s.

III. CONCLUSIONS

1. The Solow model describes both the increase and eventual stabilization of the capital-to-labor ratio over time.

2. The exact analytical solution clearly shows how the

system evolves, whereas the numerical approach requires a positive starting value for capital.

3. The idea of transition time effectively marks the phase shift from rapid growth to a steady economic state.

4. The fixed point r^* is identified at the intersection of $y_1 = sr^\alpha$ and $y_2 = nr$, aligning with the theoretical expectations.

[1] Arnab K. Ray, *Lecture Notes.*