Lecture Notes on Group Theory

History

- The term group was coined by Galois around 1830 to described sets functions on finite sets that could be grouped together to form a closed set. The modern definition of the group given by both Heinrich Weber and Walter Von Dyck in 1882, it did not gain universal acceptance until the twentieth century.

Binary Operation :-

- Let G be a set. A binary operation on G is a function that assigns each order pair of elements of G an element of G.

$$f : G \times G \rightarrow G$$

- Remark :

  o is a binary operation on G iff $a o b \in G$.

Algebraic Structure :-

- A non empty set together with one or more than one binary operation is called algebraic structure.

Examples :-

1. $(R, +, \cdot)$ is an algebraic structure.
2. $(N, +)$, $(Z, +)$, $(Q, +)$ are algebraic structures.

Group :-

A non empty set G together with an operation o is called a group if the following conditions are satisfied :

- Closure axiom,

  $\forall\, a, b \in G \Rightarrow aob \in G.$

- Associative axiom,

  $(aob)oc = ao(boc) \quad \forall\, a, b, c \in G$

- Existence of identity,

  $\exists$ an element $e \in G$, called identity

  $aoe = eoa = a \quad \forall\, a \in G.$

- Existence of inverse,

  $a \in G\,,\ \exists\ a^{-1} \in G\ \ s.t$

  $a^{-1}oa = aoa^{-1} = e$

  This a^{-1} is called inverse of a.

Abelian Group :-

A group (G, o) is called abelian group or commutative group if $aob = boa \;\; \forall\, a, b \in G$.

Examples :-

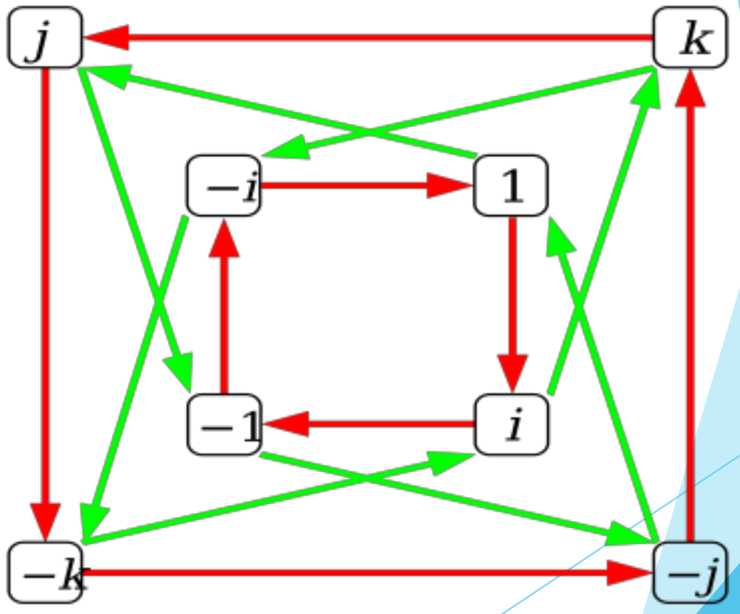
1. $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$ all are commutative group.
2. $(\mathbb{Q}_0, \cdot)$, $(\mathbb{R}_0, \cdot)$ are commutative group. 1 is an identity ,$\frac{1}{a}$ is the inverse of a in each case.
3. The set of all $m \times n$ matrics (real and complex) with matrix addition as a binary operation is commutative group. The zero matric is the identity element and the inverse of matric of A is –A.

Quaternion Group :-

$G = \{\pm 1, \pm i, \pm j, \pm k\}$ define a binary operation of multiplication as

$$i^2 = j^2 = k^2 = -1,\ ij = -jk = k, ki = -ik = j,\ jk = -kj = i.$$

The red arrows represent multiplication on the right by i, and the green arrows represent multiplication on the right by j.



This is non abelian group for this operation.

This is called Quaternion group.

Klein's four group

- Let $G = (e, a, b, c)$ with operation o defined by the following table :

o	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Theorem :-

Uniqueness of identity

- The identity e in a group always unique.

Proof-

If possible, suppose that e and e' are two identity elements in a group G.

e is an identity element

$\Rightarrow ee' = e'e = e'$ $\quad [ae = ea = a]$

e' is an identity element

$\Rightarrow ee' = e'e = e$ $\quad [ae' = e'a = a]$

these statements prove that $e = ee' = e'e = e'$

from which, we get $e = e'$.

Theorem :-

The cancellation laws

- Suppose, a, b, c are arbitrary elements of a group G. Then

  1. $ab = ac \Rightarrow b = c$ (left cancellation)
  2. $ba = ca \Rightarrow b = c$ (right cancellation)

  Proof :-

  Let e be the identity element in a group G. Let $a, b, c \in G$ be arbitrary

  $ab = ac$

  $\Rightarrow a^{-1}(ab) = a^{-1}(ac)$

  $\Rightarrow (a^{-1}a)b = (a^{-1}a)c$ [by associative law]

  $\Rightarrow eb = ec$

  $\Rightarrow b = c$

Again

$$ba = ca$$
$$\Rightarrow \quad (ba)a^{-1} = (ca)a^{-1}$$
$$\Rightarrow \quad b(aa^{-1}) = c(aa^{-1})$$
$$\Rightarrow \quad be = ce$$
$$\Rightarrow \quad b = c$$

Example :-

1. The positive integer form a cancellative semigroup under addition.

2. The non-negative integers form a cancellative monoid under addition.

3. The cross product of two vectors does not obey the cancellation law.

   if $a \times b = a \times c$, then it does not follow that $b = c$ even if $a \neq 0$.

4. Matrix multiplication also does not necessary obey the cancellation law.

$AB = AC$ and $A \neq 0$

Consider the set of all 2×2 matrices with integer coefficients. The matrix multiplication is defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix}$$

It is associative, and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is identity but the cancellation law does not follow

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This implies

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \text{ but} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$$

Theorem :-

Uniqueness of inverse

- The inverse of each element of a group is unique.

Proof :-

If possible, let a and b be two elements of a group $G,$ so that

$ba = ab = e$ ...(1)

$ca = ac = e$ ...(2)

e be an identity in G.

$ba = e = ca$

or $ba = ca$

$b = c$ [by right cancellation law.]

Theorem :-

- The left identity is also the right identity.

Proof:-

Let e be the left identity of a group G and let $a \in G$ be arbitrary. Then

$ea = a$ .... (1)

To prove that e is also that right identity. It suffices to show that

$ae = a$

suppose a^{-1} is the left inverse of a, then

a^{-1} is the left inverse of a, then

$a^{-1}a = e$ .... (2)

by associative law in G.

$a^{-1}(ae) = (a^{-1}a)e = e$ [by (2)]

$= e = a^{-1}a$ [again by (2)]

$a^{-1}(ae) = a^{-1}a \Rightarrow ae = a$ [by left cancellation law]

Theorem :-

Reverse rule

- Let a and b be the elements of a group G. Then

  then $(ab)^{-1} = b^{-1}a^{-1}$.

Proof:-

consider arbitrary elements a and b of a group G.

Since b^{-1} and a^{-1} are inverse of a and b respectively.

$b^{-1}b = bb^{-1} = e,$ and $a^{-1}a = aa^{-1} = e$ ....(1)

Hence, by associativity law,

$(ab)(b^{-1}a^{-1}) = a[(bb^{-1})a^{-1}] = a[ea^{-1}] = aa^{-1} = e$ ....(2)

This $\Rightarrow$ $(ab)^{-1} = b^{-1}a^{-1}$.

Generalizing this result, we obtain

$(abc \dots)^{-1} = \dots c^{-1}b^{-1}a^{-1}$

Theorem:-

- If let G be a group and $a \in G$ then $(a^{-1})^{-1} = a$.

Proof:-

let a^{-1} be the inverse of an element a of a group G, then

$a^{-1}a = e$ ....(1)

to prove that the inverse of a^{-1} is a,

premultiplying (1) by $(a^{-1})^{-1}$,

$[(a^{-1})^{-1}\, a^{-1}]\, a = (a^{-1})^{-1}e$, by associative law

$ea = (a^{-1})^{-1}$

$a = (a^{-1})^{-1}$

Remark - $e^{-1} = e$.

THANK YOU

?