

Motivating Example (1)

- Consider the renovation of the building of a firm.
In this process several tasks are undertaken
 - Remove asbestos
 - Replace windows
 - Paint walls
 - Refinish floors
 - Assign offices
 - Move in office furniture
 - ...

Motivating Example (2)

- Clearly, some things had to be done before others could begin
 - Asbestos had to be removed before anything (except assigning offices)
 - Painting walls had to be done before refinishing floors to avoid ruining them, etc.
- On the other hand, several things could be done concurrently:
 - Painting could be done while replacing the windows
 - Assigning offices could be done at anytime before moving in office furniture
- This scenario can be nicely modeled using partial orderings

Partial Orderings: Definitions

- **Definitions:**
 - A relation R on a set S is called a partial order if it is
 - Reflexive
 - Antisymmetric
 - Transitive
 - A set S together with a partial ordering R is called a partially ordered set (poset, for short) and is denoted (S, R)
- Partial orderings are used to give an order to sets that may not have a natural one
- In our renovation example, we could define an ordering such that $(a, b) \in R$ if 'a must be done before b can be done'

Partial Orderings: Notation

- We use the notation:
 - $a \prec b$, when $(a,b) \in R$
 - $a \not\prec b$, when $(a,b) \in R$ and $a \neq b$
- The notation \prec is not to be mistaken for “less than” (\prec versus \leq)
- The notation \prec is used to denote any partial ordering

Comparability: Definition

- **Definition:**
 - The elements a and b of a poset (S, \prec) are called comparable if either $a \prec b$ or $b \prec a$.
 - When for $a, b \in S$, we have neither $a \prec b$ nor $b \prec a$, we say that a, b are incomparable
- Consider again our renovation example
 - Remove Asbestos $\prec a_i$ for all activities a_i except assign offices
 - Paint walls \prec Refinish floors
 - Some tasks are incomparable: Replacing windows can be done before, after, or during the assignment of offices

Total orders: Definition

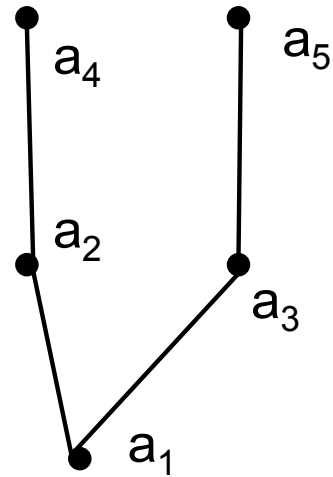
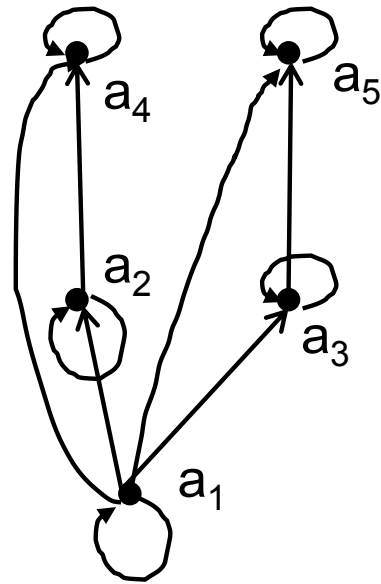
- **Definition:**
 - If (S, \prec) is a poset and every two elements of S are comparable, S is called a totally ordered set.
 - The relation \prec is said to be a total order
- Example
 - The relation “less than or equal to” over the set of integers (\mathbb{Z}, \leq) since for every $a, b \in \mathbb{Z}$, it must be the case that $a \leq b$ or $b \leq a$
 - What happens if we replace \leq with $<$?

The relation $<$ is not reflexive, and $(\mathbb{Z}, <)$ is not a poset

Hasse Diagrams

- Like relations and functions, partial orders have a convenient graphical representation: Hasse Diagrams
 - Consider the digraph representation of a partial order
 - Because we are dealing with a partial order, we know that the relation must be reflexive and transitive
 - Thus, we can simplify the graph as follows
 - Remove all self loops
 - Remove all transitive edges
 - Remove directions on edges assuming that they are oriented upwards
 - The resulting diagram is far simpler

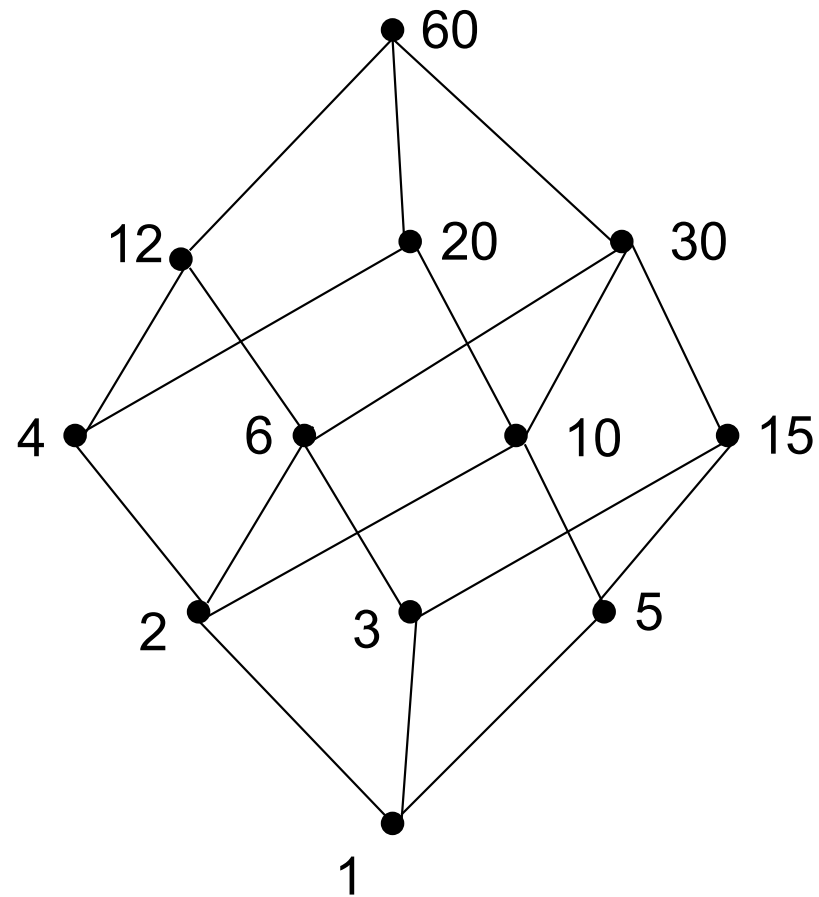
Hasse Diagram: Example



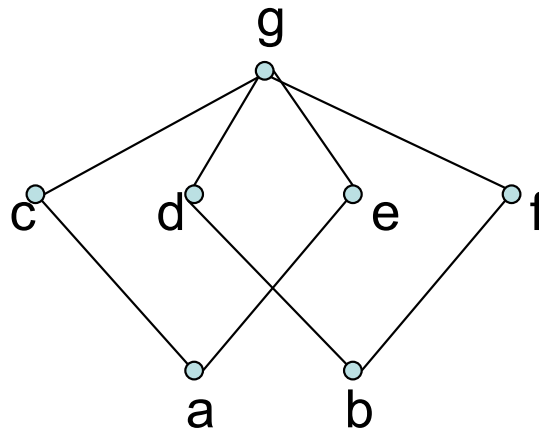
Hasse Diagrams: Example (1)

- Of course, you need not always start with the complete relation in the partial order and then trim everything.
- Rather, you can build a Hasse Diagram directly from the partial order
- Example: Draw the Hasse Diagram
 - for the following partial ordering: $\{(a,b) \mid a|b\}$
 - on the set $\{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$
 - (these are the divisors of 60 which form the basis of the ancient Babylonian base-60 numeral system)

Hasse Diagram: Example (2)

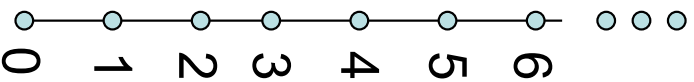


Example



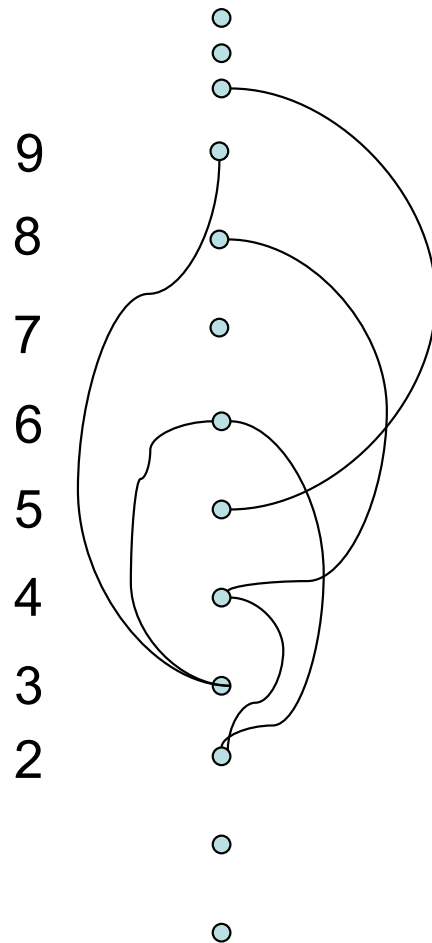
- $L = \{a, b, c, d, e, f, g\}$
- $\prec = \{(a, c), (a, d), (b, d), (b, e), (c, g), (d, g), (e, g), (f, g)\}^{\text{RT}}$
- (L, \prec) is a partial order

Example



- ⌊ $L = \mathbb{N}$ (natural numbers)
- ⌋ $\prec = \{(0,1), (1,2), (2,3), (3,4), (4,5), \dots\}_{\text{RT}}$
- ⌋ (L, \prec) is a totally ordered set (infinite)

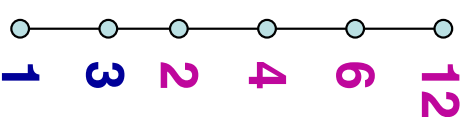
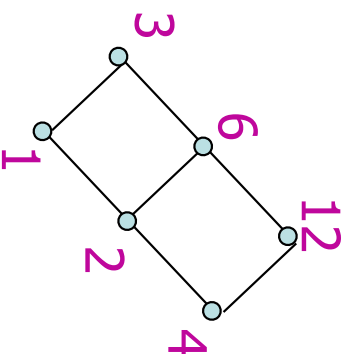
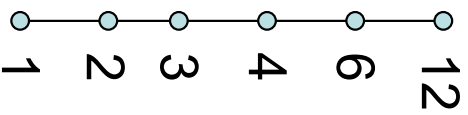
Example



- ☛ $L = \mathbb{N}$ (natural numbers)
- ☛ $\prec = \{(n, m) : \exists k \text{ such that } m = n * k\}$
- ☛ (L, \prec) is a partially ordered set (infinite)

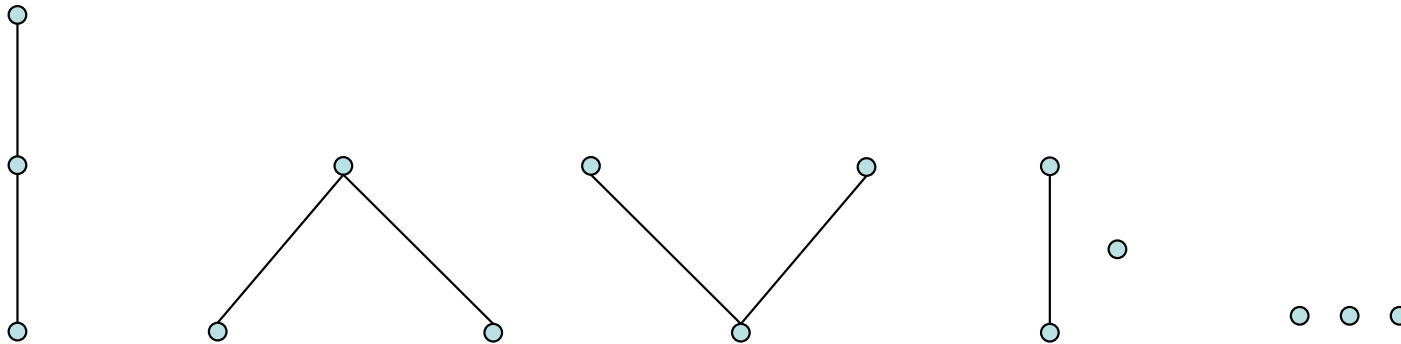
Example

- On the same set $E = \{1, 2, 3, 4, 6, 12\}$ we can define different partial orders:



Example

- All possible partial orders on a set of three elements (modulo renaming)



Extremal Elements: Summary

We will define the following terms:

- A maximal/minimal element in a poset (S, \prec)
- The maximum (greatest)/minimum (least) element of a poset (S, \prec)
- An upper/lower bound element of a subset A of a poset (S, \prec)
- The greatest lower/least upper bound element of a subset A of a poset (S, \prec)

Extremal Elements: Maximal

- **Definition:** An element a in a poset (S, \prec) is called maximal if it is not less than any other element in S . That is: $\neg(\exists b \in S (a \prec b))$
- If there is one unique maximal element a , we call it the maximum element (or the greatest element)

Extremal Elements: Minimal

- **Definition:** An element a in a poset (S, \prec) is called minimal if it is not greater than any other element in S . That is: $\neg(\exists b \in S (b \prec a))$
- If there is one unique minimal element a , we call it the minimum element (or the least element)

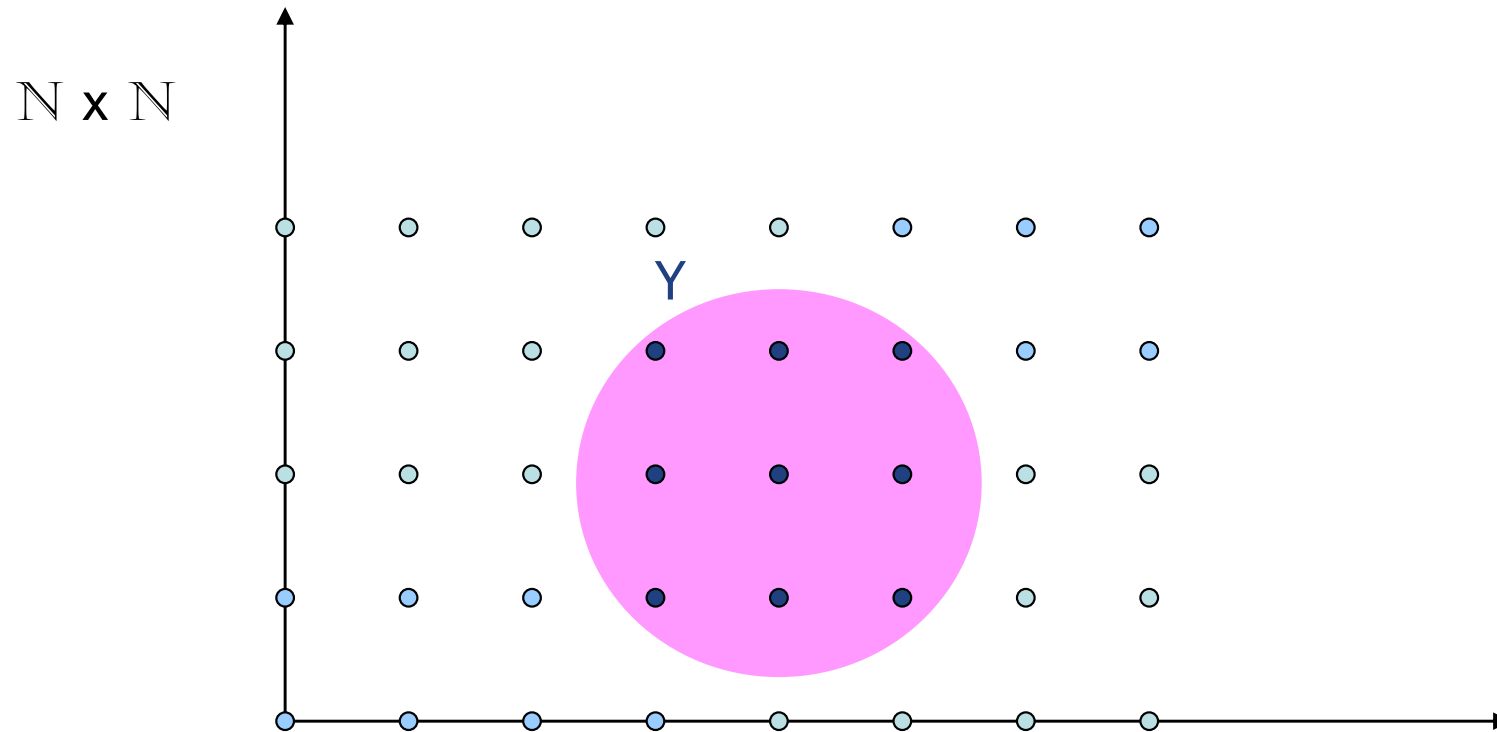
Extremal Elements: Upper Bound

- **Definition:** Let (S, \prec) be a poset and let $A \subseteq S$. If u is an element of S such that $a \prec u$ for all $a \in A$ then u is an upper bound of A
- An element x that is an upper bound on a subset A and is less than all other upper bounds on A is called the least upper bound on A . We abbreviate it as lub.

Extremal Elements: Lower Bound

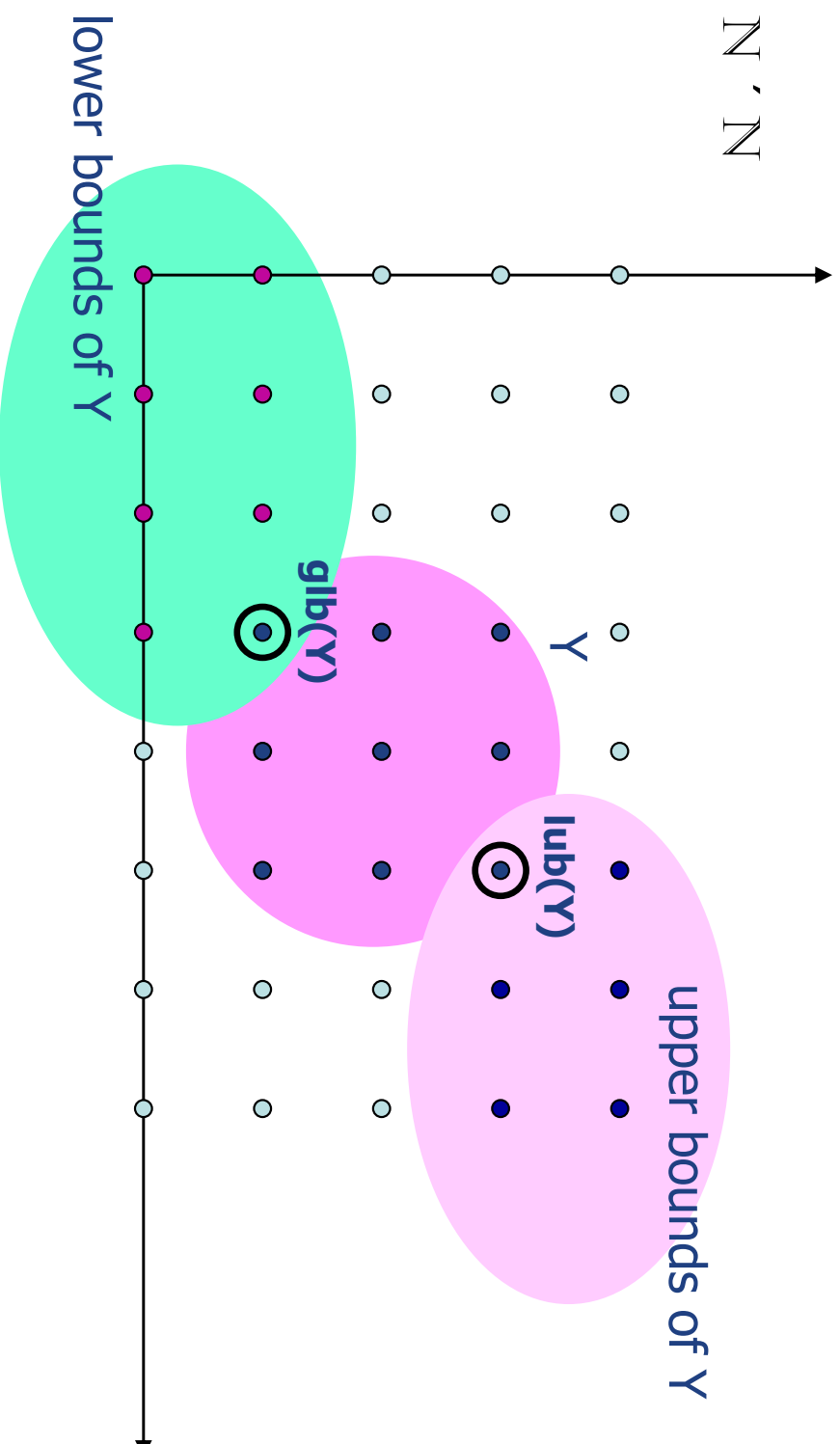
- **Definition:** Let (S, \prec) be a poset and let $A \subseteq S$. If l is an element of S such that $l \prec a$ for all $a \in A$ then l is an lower bound of A
- An element x that is a lower bound on a subset A and is greater than all other lower bounds on A is called the greatest lower bound on A . We abbreviate it glb.

Example



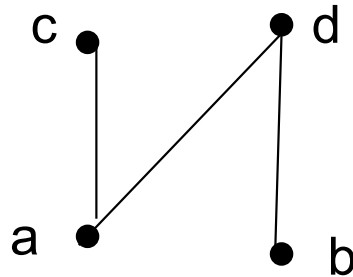
$$(x_1, y_1) \leq_{\mathbb{N} \times \mathbb{N}} (x_2, y_2) \Leftrightarrow x_1 \leq_{\mathbb{N}} x_2 \wedge y_1 \leq_{\mathbb{N}} y_2$$

Example



$$(x_1, y_1) \leq_N (x_2, y_2) \Leftrightarrow x_1 \leq_N x_2 \wedge y_1 \leq_N y_2$$

Extremal Elements: Example 1



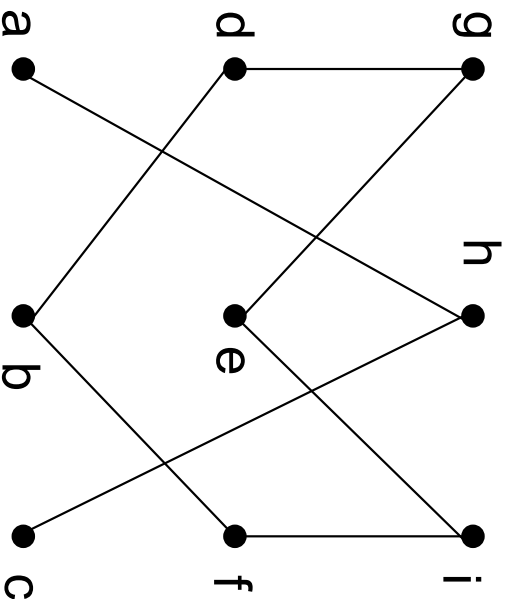
What are the minimal, maximal, minimum, maximum elements?

- Minimal: $\{a, b\}$
- Maximal: $\{c, d\}$
- There are no unique minimal or maximal elements, thus no minimum or maximum

Extremal Elements: Example 2

Give lower/upper bounds & glb/lub of the sets:

$\{d, e, f\}$, $\{a, c\}$ and $\{b, d\}$



$\{d, e, f\}$

- Lower bounds: \emptyset , thus no glb
- Upper bounds: \emptyset , thus no lub

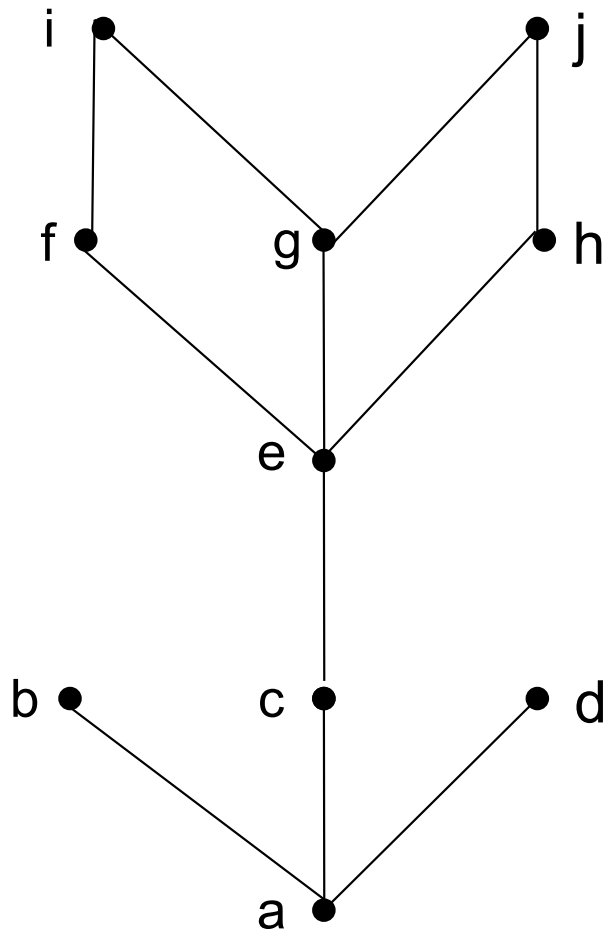
$\{a, c\}$

- Lower bounds: \emptyset , thus no glb
- Upper bounds: $\{h\}$, lub: h

$\{b, d\}$

- Lower bounds: $\{b\}$, glb: b
- Upper bounds: $\{d, g\}$, lub: d because $d < g$

Extremal Elements: Example 3



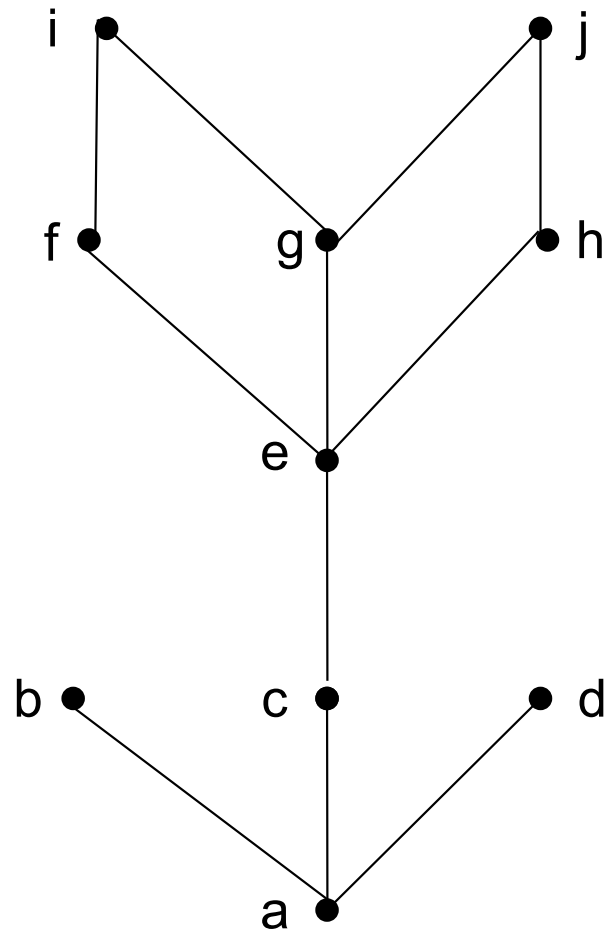
- Minimal/Maximal elements?
 - Minimal & Minimum element: a
 - Maximal elements: b,d,i,j
- Bounds, glb, lub of {c,e}?
 - Lower bounds: {a,c}, thus glb is c
 - Upper bounds: {e,f,g,h,i,j}, thus lub is e
- Bounds, glb, lub of {b,i}?
 - Lower bounds: {a}, thus glb is c
 - Upper bounds: \emptyset , thus lub DNE

Lattices

- A special structure arises when every pair of elements in a poset has an lub and a glb
- **Definition:** A lattice is a partially ordered set in which every pair of elements has both
 - a least upper bound and
 - a greatest lower bound

Lattices: Example 1

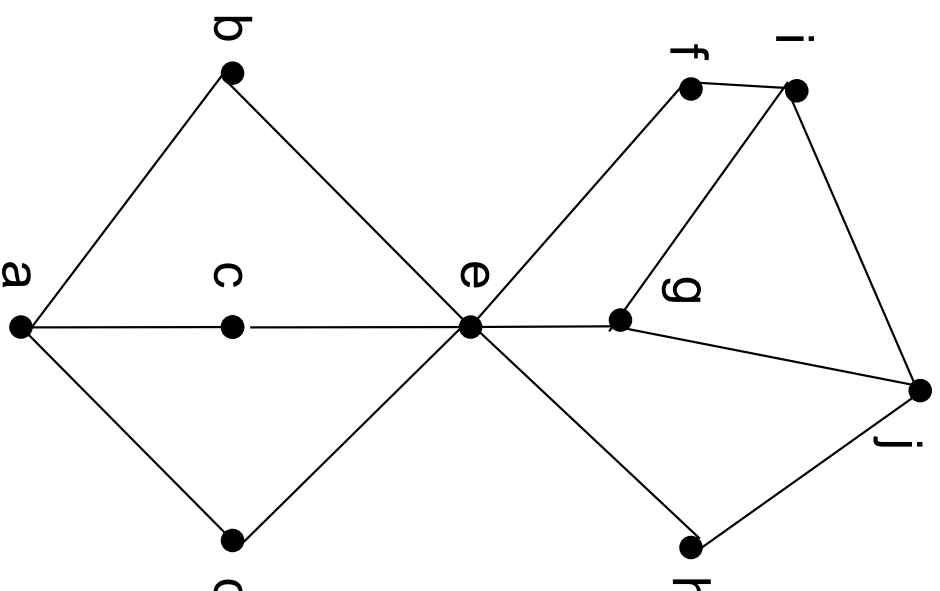
- Is the example from before a lattice?
- **No, because the pair $\{b,c\}$ does not have a least upper bound**



Lattices: Example 2

- What if we modified it as shown here?

- **Yes, because for any pair, there is an lub & a glb**



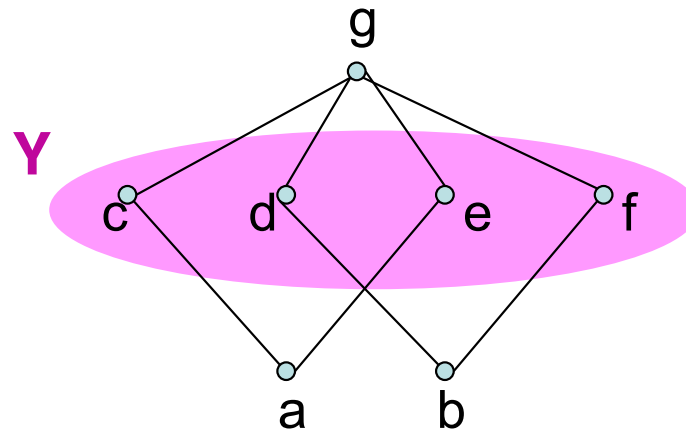
A Lattice Or Not a Lattice?

- To show that a partial order is not a lattice, it suffices to find a pair that does not have an lub or a glb (i.e., a counter-example)
- For a pair not to have an lub/glb, the elements of the pair must first be incomparable (Why?)
- You can then view the upper/lower bounds on a pair as a sub-Hasse diagram: If there is no maximum/minimum element in this sub-diagram, then it is not a lattice

Complete lattices

- Definition:
A lattice A is called a complete lattice if every subset S of A admits a glb and a lub in A .
- Exercise:
Show that for any (possibly infinite) set E , $(P(E), \subseteq)$ is a complete lattice
($P(E)$ denotes the powerset of E , i.e. the set of all subsets of E).

Example



- $L = \{a, b, c, d, e, f, g\}$
- $\leq = \{(a, c), (a, d), (b, d), (b, e), (c, g), (d, g), (e, g), (f, g)\}^T$
- (L, \leq) is not a lattice:
a and b are lower bounds of Y, but a and b are not comparable

Exercise

- Prove that “Every finite lattice is a complete lattice”.

Example

