# Weak solutions and invariant measures of stochastic Oldroyd-B type model driven by jump noise

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## Abstract

In this work, we consider sub-critical and critical models for viscoelastic flows driven by pure jump Lévy noise. Due to the elastic property, the noise in the equation for the stress tensor is considered in the Marcus canonical form. We investigate existence of a weak martingale solution for stochastic Oldroyd-B models, with full dissipation in whole of $\mathbb{R}^d$, $d = 2, 3$. The key ingredients of the proof are classical Faedo-Galerkin approximation, the compactness method and the Jakubowski version of the Skorokhod Theorem for nonmetric spaces. Pathwise uniqueness, existence of a strong solution and uniqueness in law for the two-dimensional model are also shown. We also prove, in a Poincaré domain in two-dimensions, existence of an invariant measure using *bw*-Feller property of the associated Markov semigroup.
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## 1. Introduction

Over the past few years, there have been many works devoted to viscoelastic fluids in dimensions two and three. Most of these works are concerned about local existence of strong solutions, global existence of weak solutions, necessary condition for blow-up (in the spirit of well-known Beale-Kato-Majda criterion [4]) and global well-posedness for smooth solutions with small initial data.

### 1.1. Description of the deterministic model

Let us focus upon the classical Oldroyd type models for viscoelastic fluids (see Oldroyd [61]) in $\mathbb{R}^d, d = 2, 3$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \nu \Delta \mathbf{v} + \nabla p = \nu_1 \nabla \cdot \tau \quad \text{in} \quad \mathbb{R}^d \times (0, T), \qquad (1.1)$$

$$\frac{\partial \tau}{\partial t} + (\mathbf{v} \cdot \nabla)\tau - \kappa \Delta \tau(t) + a\tau + \mathbf{Q}(\tau, \nabla \mathbf{v}) = \nu_2 \mathcal{D}(\mathbf{v}) \quad \text{in} \quad \mathbb{R}^d \times (0, T), \qquad (1.2)$$

$$\nabla \cdot \mathbf{v} = 0 \quad \text{in} \quad \mathbb{R}^d \times (0, T), \qquad (1.3)$$

$$\mathbf{v}(\cdot, 0) = \mathbf{v}_0, \ \tau(\cdot, 0) = \tau_0 \quad \text{in} \quad \mathbb{R}^d. \qquad (1.4)$$

Here $\mathbf{v}$ is the velocity vector field which is assumed to be divergence free, $\tau$ is the non-Newtonian part of the stress tensor (i.e., $\tau(x, t)$ is a $(d, d)$ symmetric matrix), $p$ is the pressure of the fluid, which is a scalar. The parameters $\nu$ (the viscosity of the fluid), $a$ (the reciprocal of the relaxation time), $\nu_1$ and $\nu_2$ (determined by the dynamical viscosity of the fluid, the retardation time and $a$) are assumed to be non-negative. $\mathcal{D}(\mathbf{v})$ is called the deformation tensor and is the symmetric part of the velocity gradient

$$\mathcal{D}(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} + \nabla^t \mathbf{v}).$$

$\mathbf{Q}$ is a quadratic form in $(\tau, \nabla \mathbf{v})$. As remarked in Chemin and Masmoudi [19], since the equation for the stress tensor should be invariant under coordinate transformation, $\mathbf{Q}$ cannot be most general quadratic form, and for Oldroyd fluids one usually chooses

$$\mathbf{Q}(\tau, \nabla \mathbf{v}) = \tau \mathcal{W}(\mathbf{v}) - \mathcal{W}(\mathbf{v})\tau - b\left(\mathcal{D}(\mathbf{v})\tau + \tau \mathcal{D}(\mathbf{v})\right),$$

where $b \in [-1, 1]$ is a constant and $\mathcal{W}(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} - \nabla^t \mathbf{v})$ is the vorticity tensor, and is the skew-symmetric part of velocity gradient. We impose an idealized condition (or far-field condition) on the fluid, i.e.,

$$\mathbf{v}(x, t) \to 0 \quad \text{and} \quad \tau(x, t) \to 0 \quad \text{as} \quad |x| \to \infty.$$

### 1.2. Deterministic Oldroyd-B system

There is growing literature devoted to these systems and it is almost impossible to provide a complete review on the topic. We shall restrict ourselves to a few significant works which are relevant to our paper.

### 1.2.1. The case $\nu > 0$ and $\kappa = 0$

Lack of diffusion in $\tau$ equation (1.2) is a classical issue and analysis of the above model under this situation is quite challenging. To be a little more precise, in this case, one of the key difficulties in proving local existence with diffusion only in the **v** equation, stems from the nonlinear terms. Even the semigroup method to mild solutions may fail and also the local $m$-accretivity property is unavailable due to the absence of a diffusive term in $\tau$ equation. However, similar complexities can be seen in Euler equation or in semi-dissipative/ideal magnetohydrodynamic (MHD) systems. In [28] and [19], authors have suggested some ideas to tackle this issue.

Such parabolic-hyperbolic coupling of the system and its corresponding stationary problem along with the special structure of **Q** is also interesting and has been studied by Renardy [67]. Authors have established existence and uniqueness of local strong solutions in $H^m$ ($m \in \mathbb{N}$) in [32] and existence of global solutions (provided the coupling between the two equations is weak as well as the initial data are small) in [33]. Furthermore, in [29], authors have obtained existence of $L^s - L^r$ solutions. Global existence of weak solutions for the corotational case (i.e., when $b = 0$) and the classical case (i.e., when $\nu > 0$ and $\kappa = 0$) has been proved in [49], [50]. Under similar classical case, for a simple survey regarding the proof of global wellposedness when the data are small, we refer to Masmoudi et al. [19,47], Lin et al. [48], Lei et al. [45,46].

### 1.2.2. The case $\nu \geq 0$ and $\kappa > 0$

When both $\nu$ and $\kappa$ are positive, the problem is subcritical in dimension two, and global wellposedness of strong solutions can be obtained in [20] due to the smoothing effect provided by the fully parabolic system. In three dimensions only partial results are available due to lack of global solvability result of the Navier-Stokes equations.

Authors in [25] have considered the case when $\nu = 0$ and $\kappa > 0$ in two dimension, when **Q** has a special structure (that is, in the corotational case) and have proved global existence of weak solutions (without uniqueness) and have obtained global existence of strong solutions with large data when **Q** is zero. As an improvisation of [25], the authors in [26] have exploited the method of transferring dissipation and have obtained the existence of a unique global solution in whole three dimensional space provided the initial data is small in the $H^s$ Sobolev-norm with $s > \frac{5}{2}$. It is worth noting that even when **Q** $= 0$, the coupling is critical with respect to the smoothing effect provided by the partial parabolic regularisation (see, for instance, [25] for details). Summarising, in dimension two, for **Q** $= 0$, the case

(1) $\nu > 0$, $\kappa > 0$ is sub-critical,
(2) $\nu = 0$, $\kappa > 0$, is critical,
(3) $\nu > 0$, $\kappa = 0$, is critical.

In dimension two, for **Q** $\neq 0$, all possible cases (1),(2),(3) are critical. In dimension three, irrespective of **Q** being zero or non-zero, all possible cases are critical. We refer the readers to [25,26] for further details.

### 1.3. Related deterministic problems

Let us now briefly describe the connection between some deterministic system with critical coupling which are close to our model. In particular, a critical coupling for the Boussinesq system has been studied in Hmidi et al. [35]-[36], Manna and Panda [51] (see also [52]), for the MHD system in Caflisch et al. [17], for the liquid crystal model in Lin and Liu [48], to name a few. The

coupling in the Boussinesq system is simpler than the one in the MHD system (or in the liquid crystal model or viscoelastic fluid of Oldroyd type considered in this work) in the sense that the vorticity equation is forced by the gradient of the temperature but then the temperature solves an unforced convection-diffusion equation. Consequently, in the case of critical coupling, if one can find a combination of the vorticity and the temperature that has better regularity properties, it is rather easy to deduce an estimate on each individual quantity (see [25]). This is not the case for the MHD system, the liquid crystal model or for the Oldroyd model (even if $\mathbf{Q} = 0$), since they are strongly coupled. Many other interesting results are devoted to the Oldroyd-B model and related models and are available in [27,37,43,44,71] and the references therein.

### 1.4. Related stochastic problems

Stochastic analysis of the above critical coupled systems influenced by random forcing is very limited and quite recent (e.g. see Yamazaki [70] for Boussinesq system with zero dissipation, Manna et al. [53] for non-resistive MHD system, Brzeźniak et al. [7] for liquid crystal model). Existence and asymptotic behaviour of a linear viscoelastic fluid equation perturbed by additive or multiplicative Wiener processes has been studied, in Barbu et al. [3] and Razafimandimby [66]. Due to absence of any critical nonlinear coupling, linear viscoelastic fluid flow takes the form of an integro-differential equation (of Volterra type) consisting of the Navier-Stokes equation and a hereditary (or memory) term as the integral of a linear kernel. Therefore, the standard techniques of the stochastic Navier-Stokes equation can be borrowed to establish the well-posedness and regularity of solutions.

In this context, it is worthy to mention some recent works have been devoted to understand fluid behaviours in the micro-macro regimes. In order to build a micro-macro model, one needs to go down to the microscopic scale and make use of kinetic theory to obtain a mathematical model for the evolution of the microstructures of the fluid (e.g. configurations of the polymer chains in the case of polymeric fluid). In other words, this micro-macro approach translates into a coupled multiscale system (simplest example of such a model is the dumbbell model) in which the polymers are modelled as dumbbells each of which consists of two beads connected by a spring (see [40] and [63] for detailed introduction on the subject). For a quick survey and more details, we refer to our earlier paper [54] and references therein.

### 1.5. Description of the problem considered in this work

The viscoelastic property demands that the material must return to its original shape after any deforming external force has been removed (i.e., it will show an elastic response) even though it may take time to do so. Noting the fact that the equation modelling stress tensor (i.e. the equation for $\tau$) is invariant under coordinate transformation, a critically coupled Oldroyd-B system (when $\nu > 0$, $\kappa = 0$, $\mathbf{Q} \neq 0$) under the influence of a Wiener process in Stratonovich form has been studied [54] by the authors of this paper and existence and uniqueness of the local maximal strong solution of have been proved in $\mathbb{R}^d$, $d = 2, 3$.

The natural but intriguing property of invariance under diffeomorphism of Stratonovich differential equations makes such noise a obvious choice for the study of stochastic differential equations (SDEs) on manifolds or constrained SDEs. However, this property does not automatically translate to SDEs with Lévy noise or in particular to SDEs with jump noise. Work of Marcus [55] (see also Chechkin and Pavyukevich [18], Applebaum [2] and Kunita [41]) provides an answer in this direction (see also Appendix A of this paper; in particular Theorem A.1).

However, there are only a very few recent work available on constrained stochastic partial differential equations (SPDEs) driven by Lévy noise in the "Marcus" canonical form, and that too was initiated by the first author of this paper and his collaborators. In particular, in [9,10,12], the authors have studied Landau-Lifshitz-Gilbert equations to understand phase transitions between different equilibrium states of ferromagnetic material under the effect of random discontinuous fluctuations (i.e. pure jump noise in the Marcus canonical form) of the effective field. In another recent paper [11], the first author of this paper and his collaborators have studied weak solution for a system governing dynamics of nematic liquid crystals driven by a pure jump noise in the Marcus canonical form in both two and three dimensions.

The current work is motivated by this question and we believe similar questions are yet unanswered for many other constrained partial differential equations (e.g. harmonic map flow, nonlinear Schrödinger equation on a compact Riemannian manifold etc.) driven by jump noise or Lévy noise. We hope this work will contribute to the understanding of these questions, and open up directions for theoretical and (possibly) numerical study of various constrained SPDEs perturbed by jump or Lévy noise.

In this work we consider the following stochastic viscoelastic equations (1.1)-(1.4) with pure jump noise (in Itô-Lévy sense for the velocity equation and Marcus canonical sense for the stress tensor equation)

$$d\mathbf{v}(t) + [(\mathbf{v}(t) \cdot \nabla)\mathbf{v}(t) - \nu\Delta\mathbf{v}(t) + \nabla p]dt = \nu_1\nabla \cdot \tau(t)dt + \int_Z F(\mathbf{v}(t-), z)\tilde{N}_1(dt, dz), \quad (1.5)$$

$$d\tau(t) + [(\mathbf{v}(t) \cdot \nabla)\tau(t) + a\tau(t) - \kappa\Delta\tau(t)]dt = \nu_2\mathcal{D}(\mathbf{v}(t))dt + \sum_{i=1}^{k}(h_i \otimes \tau(t)) \diamond dL_i(t), \quad (1.6)$$

$$\nabla \cdot \mathbf{v} = 0, \ \mathbf{v}(\cdot, 0) = \mathbf{v}_0, \ \tau(\cdot, 0) = \tau_0, \quad (1.7)$$

where each $h_i; i = 1, \ldots, k$ is bounded function in $\mathbb{R}^{d \times d}$, $L(t) := (L_1(t), \cdots, L_k(t))$ is a $\mathbb{R}^k$-valued pure jump Lévy process with the intensity measure $\lambda_2$ such that $supp\lambda_2 \subset B$, where $B := \mathbb{B}(0, 1) \subset \mathbb{R}^k$; $l \in \mathbb{R}^k$ i.e.,

$$L(t) = \int_0^t \int_B l \, \tilde{N}_2(ds, dl)$$

where $\tilde{N}_2$ represents time homogeneous compensated Poisson random measure. Here $\tilde{N}_1$ represents time homogeneous compensated Poisson random measure on $Z$, a measurable subspace of $L^2(\mathbb{R}^d; \mathbb{R}^d)$, where the solutions of the above system have its paths. Precise definition of $\diamond$ will be stated in the next section. The tensor product $\otimes$ denotes usual matrix multiplication.

The aims of this paper is manifold and the results new in the context of Oldroyd-B fluids. Here we investigate existence of weak martingale solution for the system (1.5)-(1.7) when $\nu > 0$ and $\kappa > 0$ in whole of $\mathbb{R}^d$, $d = 2, 3$. Existence and uniqueness of a strong solution and uniqueness in law for the two-dimensional model are also shown. We go one step further and prove that, in a Poincaré domain in $\mathbb{R}^2$, the corresponding Markov semigroup is sequentially weakly Feller and has an invariant probability measure. We propose to study weak solution for the critical coupled system (1.5)-(1.7) in the absence of dissipation in the velocity equation, i.e., when $\nu = 0$ and $\kappa > 0$, $\mathbf{Q} \neq 0$ in whole of $\mathbb{R}^2$. In a future work as it has independent interest and difficulty.

Let us briefly describe the content of the paper. In Section 2, we have listed detailed description of the functional spaces, operators involved, the Marcus map and assumptions of this paper. In Section 3, statement of the main result and the key ideas have been discussed. In Section 4, we introduce the finite dimensional approximation and prove the existence of the global solutions $(\mathbf{v}_n, \tau_n)$ of the approximate system of equations of (1.5)-(1.7). Uniform bounds on the approximated solution in some suitable spaces have been also obtained. Section 5 is one of the main technical sections, where tightness of the laws of the solutions have been obtained on a suitable path space, which has been followed by application of Skhorokhod's Theorem to construct a new probability space and some processes $(\bar{\mathbf{v}}_n, \bar{\tau}_n)$ with the same laws as $(\mathbf{v}_n, \tau_n)$ such that $(\bar{\mathbf{v}}_n, \bar{\tau}_n)$ converges a.s. to a limit process $(\mathbf{v}, \tau)$ which finally is the weak martingale solution of the stochastic system. Existence and uniqueness of strong solution and uniqueness in law have been shown to the problem (3.1)-(3.3) in two-dimensions in Section 6 whereas existence of an invariant measure has been studied for the same problem in Section 7. Section 8 comprises of the proof of the Main Result 1. Finally, certain useful compactness and tightness criterion for cádlag functions and predictable processes have been mentioned in the Appendix.

## 2. Functional settings, assumptions and the Marcus map

Let $C_c^\infty(\mathbb{R}^d; \mathbb{R}^d)$ be the space of all $\mathbb{R}^d$-valued compactly supported $C^\infty$ functions in $\mathbb{R}^d$. Let us denote

$$\mathcal{V} := \{\mathbf{v} \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^d) : Div\, \mathbf{v} = 0\}, \quad \tilde{\mathcal{V}} := \{\tau \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^{d \times d}) : \tau^T = \tau\}.$$

Let $H, V, \tilde{H}$ and $\tilde{V}$ be the closure of $\mathcal{V}$ in $L^2(\mathbb{R}^d; \mathbb{R}^d), H^1(\mathbb{R}^d; \mathbb{R}^d), L^2(\mathbb{R}^d; \mathbb{R}^{d \times d})$ and $H^1(\mathbb{R}^d; \mathbb{R}^{d \times d})$ respectively. For $s > 0$, let $V_s$ and $\tilde{V}_s$ be the closure of $\mathcal{V}$ and $\tilde{\mathcal{V}}$ in $H^s(\mathbb{R}^d; \mathbb{R}^d)$ and $H^s(\mathbb{R}^d; \mathbb{R}^{d \times d})$ respectively. It is clear from the definitions that $V_1 = V$ and $\tilde{V}_1 = \tilde{V}$.

**Notation 2.1.** Throughout this paper, we denote $X'$ as the dual of $X$ where $X$ is a Hilbert space.

We note the following:

(i) $H$ is a Hilbert space with scalar product $(\mathbf{u}, \mathbf{v})_H := (\mathbf{u}, \mathbf{v})_{L^2}, \ \forall \mathbf{u}, \mathbf{v} \in H$ and $V$ is a Hilbert space with scalar product $(\mathbf{u}, \mathbf{v})_V := (\mathbf{u}, \mathbf{v})_{L^2} + (\nabla \mathbf{u}, \nabla \mathbf{v})_{L^2}, \ \forall \mathbf{u}, \mathbf{v} \in V$.

(ii) $\tilde{H}$ is a Hilbert space with scalar product $(\mathbf{u}, \mathbf{v})_{\tilde{H}} := (\mathbf{u}, \mathbf{v})_{L^2} \ \forall \mathbf{u}, \mathbf{v} \in \tilde{H}$ and $\tilde{V}$ is a Hilbert space with scalar product $(\mathbf{u}, \mathbf{v})_{\tilde{V}} := (\mathbf{u}, \mathbf{v})_{L^2} + (\nabla \mathbf{u}, \nabla \mathbf{v})_{L^2}, \quad \forall \mathbf{u}, \mathbf{v} \in \tilde{V}$.

### 2.1. Bilinear operators

#### 2.1.1. Trilinear operator b

Let us define the trilinear operator $b : L^p(\mathbb{R}^d; \mathbb{R}^d) \times W^{1,q}(\mathbb{R}^d; \mathbb{R}^d) \times L^r(\mathbb{R}^d; \mathbb{R}^d) \to \mathbb{R}$ defined by

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \sum_{i,j=1}^d \int_{\mathbb{R}^d} \mathbf{u}^{(i)} \partial_{x_i} \mathbf{v}^{(j)} \mathbf{w}^{(j)} dx, \ \mathbf{u} \in L^p(\mathbb{R}^d; \mathbb{R}^d), \ \mathbf{v} \in W^{1,q}(\mathbb{R}^d; \mathbb{R}^d), \ \mathbf{w} \in L^r(\mathbb{R}^d; \mathbb{R}^d)$$

with $\dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r} = 1$ and $p, q, r \in [1, \infty]$.

Let us recall some fundamental properties of the form $b$ which will be useful in the later sections and are valid for both bounded and unbounded domains (see Temam [69]).

(i) $b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}), \quad \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$. In particular $b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0, \quad \mathbf{u}, \mathbf{v} \in V$.

(ii) Using Sobolev embedding Theorem and Hölder's inequality, we obtain

$$|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq c|\mathbf{u}|_{L^4}|\nabla\mathbf{v}|_{L^2}|\mathbf{w}|_{L^4} \leq c|\mathbf{u}|_V|\mathbf{w}|_V|\mathbf{v}|_V, \qquad \mathbf{u}, \mathbf{w}, \mathbf{v} \in V \qquad (2.1)$$

for some positive constant $c$. Hence the form $b$ is continuous on $V$.

(iii) Let us define a bilinear map $B : V \times V \to V'$ by $B(\mathbf{u}, \mathbf{v}) = b(\mathbf{u}, \mathbf{v}, \cdot)$. Then by inequality (2.1), we have $B(\mathbf{u}, \mathbf{v}) \in V'$ and $|B(\mathbf{u}, \mathbf{v})|_{V'} \leq c|\mathbf{u}|_V|\mathbf{v}|_V$ for $\mathbf{u}, \mathbf{v} \in V$. Hence $B : V \times V \to V'$ is continuous.

Moreover, for all $\mathbf{u}, \mathbf{v} \in V$, we have (Section 1.2 of Temam [69])

$$|B(\mathbf{u}, \mathbf{v})|_{V'} \leq c \, |\mathbf{u}|_H^{1-\frac{d}{4}} |\nabla\mathbf{u}|_{L^2}^{\frac{d}{4}} |\mathbf{v}|_H^{1-\frac{d}{4}} |\nabla\mathbf{v}|_{L^2}^{\frac{d}{4}}, \quad d \in \{2, 3\} \qquad (2.2)$$

(iv) Let $\mathbf{u}, \mathbf{v} \in V$ and $\mathbf{w} \in V_s$ for $s > \dfrac{d}{2} + 1$. Then

$$|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| = |b(\mathbf{u}, \mathbf{w}, \mathbf{v})| \leq |\mathbf{u}|_{L^2}|\mathbf{v}|_{L^2}|\nabla\mathbf{w}|_{L^\infty} \leq c|\mathbf{u}|_{L^2}|\mathbf{v}|_{L^2}|\mathbf{w}|_{V_s}$$

for some positive constant $c$. Hence $b$ can be uniquely extended to a trilinear form (denoted by the same) $b : H \times H \times V_s \to \mathbb{R}$ and $|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq c|\mathbf{u}|_H|\mathbf{v}|_H|\mathbf{w}|_{V_s}$, for $\mathbf{u}, \mathbf{v} \in H$ and $\mathbf{w} \in V_s$.

(v) The operator $B$ can also be extended uniquely to a bounded operator $B : H \times H \to V_s'$ and $|B(\mathbf{u}, \mathbf{v})|_{V_s'} \leq c|\mathbf{u}|_H|\mathbf{v}|_H$ for $\mathbf{u}, \mathbf{v} \in H$.

(vi) Let us define $B : V \to V'$ by $B(\mathbf{u}) := B(\mathbf{u}, \mathbf{u}), \mathbf{u} \in V$. Then $B$ is locally Lipschitz, i.e., for every $r > 0$ there exists $L_r > 0$ such that

$$|B(\mathbf{u}_1) - B(\mathbf{u}_2)|_{V'} \leq L_r|\mathbf{u}_1 - \mathbf{u}_2|_V, \ \mathbf{u}_1, \mathbf{u}_2 \in V \quad \text{and} \quad |\mathbf{u}_1|_V, |\mathbf{u}_2|_V \leq r.$$

### 2.1.2. Trilinear operator $\tilde{b}$

Let us consider another trilinear map $\tilde{b} : L^p(\mathbb{R}^d; \mathbb{R}^d) \times W^{1,q}(\mathbb{R}^d; \mathbb{R}^{d \times d}) \times L^r(\mathbb{R}^d; \mathbb{R}^{d \times d}) \to \mathbb{R}$ defined by

$$\tilde{b}(\mathbf{v}, \tau, \theta) := \sum_{i,j,k=1}^{d} \int_{\mathbb{R}^d} \mathbf{v}^{(i)} \partial_{x_i} \tau^{(jk)} \theta^{(jk)} dx,$$

$$\mathbf{v} \in L^p(\mathbb{R}^d; \mathbb{R}^d), \ \tau \in W^{1,q}(\mathbb{R}^d; \mathbb{R}^{d \times d}), \ \theta \in L^r(\mathbb{R}^d; \mathbb{R}^{d \times d})$$

with $\dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r} = 1$ and $p, q, r \in [1, \infty]$.

Likewise for $b$, let us state some basic properties of the form $\tilde{b}$ which will be useful in the later sections and are valid for both bounded and unbounded domains.

(i) $\tilde{b}(\mathbf{v}, \tau, \theta) = -\tilde{b}(\mathbf{v}, \theta, \tau)$, $\quad \mathbf{v} \in V$, $\tau, \theta \in \tilde{V}$.
   In particular $\tilde{b}(\mathbf{v}, \tau, \tau) = 0$, $\quad \mathbf{v} \in V$, $\tau \in \tilde{V}$.
(ii) Using Sobolev embedding Theorem and Hölder's inequality, we obtain

$$|\tilde{b}(\mathbf{v}, \tau, \theta)| \leq c|\mathbf{v}|_{\mathrm{L}^4}|\tau|_{\tilde{V}}|\theta|_{\mathrm{L}^4} \leq c|\mathbf{v}|_V|\tau|_{\tilde{V}}|\theta|_{\tilde{V}},$$

for some positive constant c. Hence the form $\tilde{b}$ is continuous on $V \times \tilde{V} \times \tilde{V}$.

(iii) Let us define a bilinear map $\tilde{B} : V \times \tilde{V} \to \tilde{V}'$ by $\tilde{B}(\mathbf{v}, \tau) = \tilde{b}(\mathbf{v}, \tau, \cdot)$. Then by inequality (2.1) we have $\tilde{B}(\mathbf{v}, \tau) \in \tilde{V}'$ and $|\tilde{B}(\mathbf{v}, \tau)|_{\tilde{V}'} \leq c|\mathbf{v}|_V|\tau|_{\tilde{V}}$ for $\mathbf{v} \in V$, $\tau \in \tilde{V}$. Hence $\tilde{B} : V \times \tilde{V} \to \tilde{V}'$ is linear and continuous.

   Moreover using the Gagliardo-Nirenberg inequalities, one can show there exists a positive constant $C$ such that

$$\left| \tilde{B}(\mathbf{v}, \tau) \right| \leq C \, |\mathbf{v}|_H^{1-\frac{d}{4}} |\nabla \mathbf{v}|_{\mathrm{L}^2}^{\frac{d}{4}} |\nabla \tau|_{\mathrm{L}^2}^{1-\frac{d}{4}} |\Delta \tau|_{\tilde{H}}^{\frac{d}{4}}, \qquad \mathbf{u} \in V, \tau \in H^2, \; d \in \{2, 3\}. \quad (2.3)$$

(iv) Let $\mathbf{v} \in V$, $\tau \in \tilde{V}$ and $w \in \tilde{V}_s$ for $s > \dfrac{d}{2} + 1$. Then

$$|\tilde{b}(\mathbf{v}, \tau, \theta)| = |\tilde{b}(\mathbf{v}, \theta, \tau)| \leq |\mathbf{v}|_{\mathrm{L}^2}|\tau|_{\mathrm{L}^2}|\nabla \theta|_{\mathrm{L}^\infty} \leq c|\mathbf{v}|_{\mathrm{L}^2}|\tau|_{\mathrm{L}^2}|\theta|_{\tilde{V}_s}$$

for some positive constant $c$. Hence $\tilde{b}$ can be uniquely extended to a trilinear form (denoted by the same) $\tilde{b} : H \times \tilde{H} \times \tilde{V}_s \to \mathbb{R}$ and $|\tilde{b}(\mathbf{v}, \tau, \theta)| \leq c|\mathbf{v}|_H|\tau|_{\tilde{H}}|\theta|_{\tilde{V}_s}$, for $\mathbf{v} \in V$, $\tau \in \tilde{H}$, $\theta \in \tilde{V}_s$.

(v) The operator $\tilde{B}$ can also be extended uniquely to a bounded linear operator $\tilde{B} : H \times \tilde{H} \to \tilde{V}_s'$ and $|B(\mathbf{v}, \tau)|_{\tilde{V}_s'} \leq c|\mathbf{v}|_H|\tau|_{\tilde{H}}$.

(vi) $\tilde{B}$ is locally Lipschitz on $V \times \tilde{V}$, i.e., for every $r > 0$ there exists $L_r > 0$ such that

$$|\tilde{B}(\mathbf{v}_1, \tau_1) - \tilde{B}(\mathbf{v}_2, \tau_2)|_{\tilde{V}'} \leq L_r \Big( |\mathbf{v}_1 - \mathbf{v}_2|_V + |\tau_1 - \tau_2|_{\tilde{V}} \Big), \; \mathbf{v}_1, \mathbf{v}_2 \in V, \; \tau_1, \tau_2 \in \tilde{V},$$

$$\text{and} \quad |\mathbf{v}_1|_V, |\mathbf{v}_2|_V, |\tau_1|_{\tilde{V}}, |\tau_2|_{\tilde{V}} \leq r.$$

## 2.2. Some additional operators

The content of this subsection is motivated from [13,59]. We borrow certain notations and ideas from Section 2.3 of [13]. If $s > \dfrac{d}{2} + 1$, then by Sobolev embedding theorem

$$H^{s-1}(\mathbb{R}^d; \mathbb{R}^d) \subset C_b(\mathbb{R}^d; \mathbb{R}^d) \subset \mathrm{L}^\infty(\mathbb{R}^d; \mathbb{R}^d)$$

where $C_b(\mathbb{R}^d; \mathbb{R}^d)$ denotes the space of continuous and bounded $\mathbb{R}^d$ valued functions defined on $\mathbb{R}^d$. We have the following conclusions:

$$V_s \hookrightarrow_{j_s} V \hookrightarrow_{j} H \cong H' \hookrightarrow_{j'} V' \hookrightarrow_{j'_s} V'_s, \quad \text{and} \quad \tilde{V}_s \hookrightarrow_{\tilde{j}_s} \tilde{V} \hookrightarrow_{\tilde{j}} \tilde{H} \cong \tilde{H}' \hookrightarrow_{\tilde{j}'} \tilde{V}' \hookrightarrow_{\tilde{j}'_s} \tilde{V}_s{}'. \quad (2.4)$$

The following embeddings $j, j_s, \tilde{j}, \tilde{j}_s$ are continuous and dense. By Lemma C.1 in Appendix C of [13], there exist Hilbert spaces $\mathcal{U}, \tilde{\mathcal{U}}$ such that $\mathcal{U} \hookrightarrow V_s$ and $\tilde{\mathcal{U}} \hookrightarrow \tilde{V}_s$ and

the natural embeddings $l_s : \mathcal{U} \hookrightarrow V_s, \quad \text{and} \quad \tilde{l}_s : \tilde{\mathcal{U}} \hookrightarrow \tilde{V}_s$ are dense and compact. (2.5)

Combining (2.4) and (2.5) we have

$$\mathcal{U} \hookrightarrow_{l_s} V_s \hookrightarrow_{j_s} V \hookrightarrow_{j} H \cong H' \hookrightarrow_{j'} V' \hookrightarrow_{j'_s} V'_s \hookrightarrow_{l'_s} \mathcal{U}', \quad \text{and}$$

$$\tilde{\mathcal{U}} \hookrightarrow_{\tilde{l}_s} \tilde{V}_s \hookrightarrow_{\tilde{j}_s} \tilde{V} \hookrightarrow_{\tilde{j}} \tilde{H} \cong \tilde{H}' \hookrightarrow_{\tilde{j}'} \tilde{V}' \hookrightarrow_{\tilde{j}'_s} \tilde{V}_s{}' \hookrightarrow_{\tilde{l}'_s} \tilde{\mathcal{U}}'.$$

Let us consider the adjoint of the following maps $j, j_s, l_s, \tilde{j}, \tilde{j}_s, \tilde{l}_s$ denoted by $j^*, j_s^*, l_s^*, (\tilde{j})^*$, $(\tilde{j}_s)^*, (\tilde{l}_s)^*$ respectively and are defined as:

$$j^* : H \to V, \ j_s^* : V \to V_s, \ l_s^* : V_s \to \mathcal{U}, \quad \text{and} \quad \tilde{j}^* : \tilde{H} \to \tilde{V}, \ \tilde{j}_s^* : \tilde{V} \to \tilde{V}_s, \ \tilde{l}_s^* : \tilde{V}_s \to \tilde{\mathcal{U}}.$$

Since all the embeddings $j, j_s, l_s, \tilde{j}, \tilde{j}_s, \tilde{l}_s$ are dense so $j^*, j_s^*, l_s^*, (\tilde{j})^*, (\tilde{j}_s)^*, (\tilde{l}_s)^*$ are one-one. Let us define the map $i : \mathcal{U} \hookrightarrow H$ by $i := j \circ j_s \circ l_s$ and $\tilde{i} : \tilde{\mathcal{U}} \hookrightarrow \tilde{H}$ by $\tilde{i} := \tilde{j} \circ \tilde{j}_s \circ \tilde{l}_s$. Note that $i, \tilde{i}$ are compact and range of $i$ and $\tilde{i}$ are dense in $H$ and $\tilde{H}$. Let us consider the adjoint of $i, \tilde{i}$ defined as $i^* : H \to \mathcal{U}, \ (\tilde{i})^* : \tilde{H} \to \tilde{\mathcal{U}}$ are one-one.

Let us define the following operators

$$(A, D(A)), (A_s, D(A_s)), (\mathfrak{A}_s, D(\mathfrak{A}_s)), (\mathfrak{A}, D(\mathfrak{A})), (\tilde{A}, D(\tilde{A})), (\tilde{A}_s, D(\tilde{A}_s)),$$

$$(\tilde{\mathfrak{A}}_s, D(\tilde{\mathfrak{A}}_s)), (\tilde{\mathfrak{A}}, D(\tilde{\mathfrak{A}}))$$

by

$$D(A) := j^*(H) \subset V, \ A\mathbf{v} := (j^*)^{-1}\mathbf{v}, \quad \forall \, \mathbf{v} \in D(A),$$

$$D(A_s) := j_s^*(V) \subset V_s, \ A_s\mathbf{v} := (j_s^*)^{-1}\mathbf{v}, \quad \forall \, \mathbf{v} \in D(A_s),$$

$$D(\mathfrak{A}) := i^*(H) \subset \mathcal{U}, \ \mathfrak{A}\mathbf{v} := (i^*)^{-1}\mathbf{v}, \quad \forall \, \mathbf{v} \in D(\mathfrak{A}),$$

$$D(\mathfrak{A}_s) := i_s^*(V_s) \subset \mathcal{U}, \ \mathfrak{A}_s\mathbf{v} := (i_s^*)^{-1}\mathbf{v}, \quad \forall \, \mathbf{v} \in D(\mathfrak{A}_s),$$

$$D(\tilde{A}) := (\tilde{j})^*(\tilde{H}) \subset \tilde{V}, \ \tilde{A}\tau := \left((\tilde{j})^*\right)^{-1}\tau, \quad \forall \, \tau \in D(\tilde{A}),$$

$$D(\tilde{A}_s) := (\tilde{j}_s)^*(\tilde{V}) \subset \tilde{V}_s, \ \tilde{A}_s\tau := \left((\tilde{j}_s)^*\right)^{-1}\tau, \quad \forall \, \tau \in D(\tilde{A}_s),$$

$$D(\tilde{\mathfrak{A}}) := (\tilde{i})^*(H) \subset \tilde{\mathcal{U}}, \ \tilde{\mathfrak{A}}\tau := \left((\tilde{i})^*\right)^{-1}\tau, \quad \forall \, \tau \in D(\tilde{\mathfrak{A}}),$$

$$D(\tilde{\mathfrak{A}}_s) := (\tilde{i}_s)^*(\tilde{V}_s) \subset \tilde{\mathcal{U}}, \ \tilde{\mathfrak{A}}_s\tau := \left((\tilde{i}_s)^*\right)^{-1}\tau, \quad \forall \, \tau \in D(\tilde{\mathfrak{A}}_s).$$

We note that $\mathfrak{A} = A \circ A_s \circ \mathfrak{A}_s$ and $\tilde{\mathfrak{A}} = \tilde{A} \circ \tilde{A}_s \circ \tilde{\mathfrak{A}}_s$ and

$$(\mathfrak{A}\mathbf{v}, \mathbf{u})_H = (\mathbf{v}, \mathbf{u})_{\mathcal{U}}, \quad \forall\, \mathbf{v} \in D(\mathfrak{A}), \mathbf{u} \in \mathcal{U},$$

$$(\tilde{\mathfrak{A}}\tau, \mathbf{w})_{\tilde{H}} = (\tau, \mathbf{w})_{\tilde{\mathcal{U}}}, \quad \forall\, \tau \in D(\tilde{\mathfrak{A}}), \mathbf{w} \in \tilde{\mathcal{U}},$$

$$(A\mathbf{v}, \mathbf{u})_H = (\mathbf{u}, \mathbf{v})_V, \quad \forall\, \mathbf{v} \in D(A), \mathbf{u} \in V$$

$$(\tilde{A}\tau, \mathbf{w})_{\tilde{H}} = (\tau, \mathbf{w})_{\tilde{V}}, \quad \forall\, \tau \in D(\tilde{A}), \mathbf{w} \in \tilde{V}.$$

Note that for each $\mathbf{v} \in V$ the function $\mathbf{u} \mapsto (\nabla\mathbf{v}, \nabla\mathbf{u})_H$ is linear and continuous and

$$|(\nabla\mathbf{v}, \nabla\mathbf{u})_H| \leq |\nabla\mathbf{v}|_H |\nabla\mathbf{u}|_H \leq |\nabla\mathbf{v}|_H (|\mathbf{u}|_H^2 + |\nabla\mathbf{u}|_H^2)^{\frac{1}{2}} = |\nabla\mathbf{v}|_H |\mathbf{u}|_V, \quad \forall\, \mathbf{u} \in V.$$

Therefore, there exist an element $\mathcal{A}\mathbf{v} \in V'$ such that

$$_{V'}\langle \mathcal{A}\mathbf{v}, \mathbf{u}\rangle_V = (\nabla\mathbf{v}, \nabla\mathbf{u})_H, \quad \forall\, \mathbf{v}, \mathbf{u} \in V.$$

Hence, $|\mathcal{A}\mathbf{v}|_{V'} \leq |\nabla\mathbf{v}|_H, \ \forall\, \mathbf{v} \in V$. Similarly we can define $\tilde{\mathcal{A}} : \tilde{V} \to \tilde{V}'$ by

$$_{\tilde{V}'}\langle \tilde{\mathcal{A}}\tau, \mathbf{w}\rangle_{\tilde{V}} = (\nabla\tau, \nabla\mathbf{w})_H, \quad \forall\, \tau, \mathbf{w} \in \tilde{V}$$

$$\text{and} \quad |\tilde{\mathcal{A}}\tau|_{\tilde{V}'} \leq |\nabla\tau|_{\tilde{H}}, \ \forall\, \tau \in \tilde{V}.$$

**Lemma 2.2.**

(i). *Let* $\mathbf{v} \in D(A), \mathbf{u} \in V$. *Then we have*

$$((A - I)\mathbf{v}, \mathbf{u})_H = (\nabla\mathbf{v}, \nabla\mathbf{u})_H =_{V'} \langle \mathcal{A}\mathbf{v}, \mathbf{u}\rangle_V, \ \text{and} \quad |\mathcal{A}\mathbf{v}|_{V'} \leq |A\mathbf{v} - \mathbf{v}|_H. \quad (2.6)$$

(ii). *Let* $\tau \in D(\tilde{A}), \mathbf{w} \in \tilde{V}$. *Then we have*

$$((\tilde{A} - I)\tau, \mathbf{w})_{\tilde{H}} = (\nabla\tau, \nabla\mathbf{w})_{\tilde{H}} =_{\tilde{V}'} \langle \tilde{\mathcal{A}}\tau, \mathbf{w}\rangle_{\tilde{V}}, \ \text{and} \quad |\tilde{\mathcal{A}}\tau|_{\tilde{V}'} \leq |\tilde{A}\tau - \tau|_{\tilde{H}}. \quad (2.7)$$

For proof see [13].

**Lemma 2.3.** *Let* $\mathfrak{A}, \tilde{\mathfrak{A}}$ *be defined above. Then both* $\mathfrak{A}, \tilde{\mathfrak{A}}$ *are self adjoint and* $\mathfrak{A}^{-1}, (\tilde{\mathfrak{A}})^{-1}$ *are compact.*

For proof see [13].

*2.3. Assumptions*

Below we list the assumptions made throughout this paper.

**Assumption 2.4.**

(A.1) Assume that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space, where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, is the filtration, satisfying the usual conditions, i.e.,

(a) $\mathbb{P}$ is complete on $(\Omega, \mathcal{F})$,
(b) for each $t \geq 0$, $\mathcal{F}_t$ contains all $(\mathbb{F}, \mathbb{P})$-null sets,
(c) the filtration $\mathcal{F}_t$ is right-continuous.

(A.2) Assume that $N_1$ is a $L^2$-valued, $(\mathcal{F}_t)$-adapted time homogeneous Poisson random measure defined on the above probability space with intensity measure $\lambda_1$ (and compensator $Leb \otimes \lambda_1$).

(A.3) Assume that $(L(t))_{t \geq 0}$ is a $\mathbb{R}^k$-valued, $(\mathcal{F}_t)$-adapted Lévy process of pure jump type defined on the above probability space with drift 0 and the corresponding time homogeneous Poisson random measure $N_2$.

(A.4) Assume that the intensity measure $\lambda_2$ of $N_2$ (with compensator $Leb \otimes \lambda_2$) is such that $supp\lambda_2 \subset B$, where $B$ is the closed unit ball in $\mathbb{R}^k$.

(A.5) Assume $N_1$ and $N_2$ are independent.

(A.6) Assume that $h_i \in W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^{d \times d})$, for each $i = 1, 2, \ldots k$.

(A.7) Assume $F : [0, T] \times H \times Z \to H$ is a measurable function such that there exists a positive constant $L$ such that

$$\int_Z \big|F(\mathbf{v}_1(t), z) - F(\mathbf{v}_2(t), z)\big|_H^2 \lambda_1(dz) \leq L\big|\mathbf{v}_1(t) - \mathbf{v}_2(t)\big|_H^2, \quad \mathbf{v}_1, \mathbf{v}_2 \in H, \ t \in [0, T] \tag{2.8}$$

and for each $p \geq 1$ there exists a positive constant $C_p$ such that

$$\int_Z \big|F(\mathbf{v}(t), z)\big|_H^p \lambda_1(dz) \leq C_p\big(1 + |\mathbf{v}(t)|_H^p\big), \quad \mathbf{v} \in H, \quad t \in [0, T], \tag{2.9}$$

## 2.4. The Marcus map

We provide a general discussion about this topic in the Appendix A.

Define a map $g_i : \tilde{H} \to \tilde{H}$ by $g_i(\tau) = h_i \otimes \tau$. Since $h_i \in W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^{d \times d})$, the above map is bounded linear. Note, $g_i : \tilde{V} \to \tilde{V}$ is also a bounded linear map.

Let us define a generalized Marcus mapping $\Phi : \mathbb{R}_+ \times \mathbb{R}^k \times \tilde{H} \to \tilde{H}$ such that for each fixed $l \in \mathbb{R}^k$, $\tau_0 \in \tilde{H}$, the function $t \mapsto \Phi(t, l, \tau_0)$ is the continuously differentiable solution of the ordinary differential equation

$$\frac{dy(t)}{dt} = \sum_{i=1}^k l_i g_i(y(t)) \quad t \geq 0, \quad y(0) = \tau_0.$$

Therefore, we can write

$$\Phi(t, l, \tau_0) = \Phi(0, l, \tau_0) + \sum_{i=1}^k \int_0^t l_i g_i(\Phi(s, l, \tau_0)) ds, \quad t \geq 0.$$

**Notation:** Let us fix $t = 1$ and denote $\Phi(l, \cdot) = \Phi(1, l, \cdot)$.

Equation (1.6) with notation $\diamond$ is defined in the integral form as follows:

$$\tau(t) = \tau_0 - \int_0^t \left[ (\mathbf{v}(s) \cdot \nabla)\tau(s) + a\tau(s) + \kappa \Delta \tau(s) - \nu_2 \mathcal{D}(\mathbf{v}(s)) \right] ds$$

$$+ \int_0^t \int_B \left[ \Phi(l, \tau(s-)) - \tau(s-) \right] \tilde{N}_2(ds, dl)$$

$$+ \int_0^t \int_B \left\{ \Phi(l, \tau(s)) - \tau(s) - \sum_{i=1}^k l_i g_i(\tau(s)) \right\} \lambda_2(dl) ds, \qquad (2.10)$$

where $\tau_0$ is a $\mathcal{F}_0$-measurable random variable.

For $z \in \tilde{H}$, we denote $G(l, z) := \Phi(l, z) - z$, $K(l, z) := \Phi(l, z) - z - \sum_{i=1}^k l_i g_i(z)$, and $b(z) := \int_B K(l, z) \lambda_2(dl)$. With these above notations, (2.10) can be written as:

$$\tau(t) = \tau_0 - \int_0^t \left[ (\mathbf{v}(s) \cdot \nabla)\tau(s) + a\tau(s) + \kappa \Delta \tau(s) - \nu_2 \mathcal{D}(\mathbf{v}(s)) \right] ds$$

$$+ \int_0^t \int_B G(l, \tau(s-)) \tilde{N}_2(ds, dl) + \int_0^t b(\tau(s)) ds.$$

Now, define a linear operator

$$\mathcal{R} : \tilde{H} \ni \tau \mapsto \sum_{i=1}^k l_i g_i(\tau) \in \tilde{H}.$$

Then $\|\mathcal{R}\|_{\mathcal{L}(\tilde{H})} \leq |l|_{\mathbb{R}^k} \|g\|_{\mathcal{L}(\tilde{H})}$.

Thus, if we denote $y(t) := \Phi(t, l, x)$, then $y$ satisfies $\dfrac{dy}{dt} = \mathcal{R}y$, $y(0) = x$. Hence $y(t) = e^{t\mathcal{R}} x = \sum_{j=0}^{\infty} \dfrac{t^j}{j!} \mathcal{R}^j x$.

We then have the following useful result. We postpone the proof to Appendix A.

**Lemma 2.5.** *Let* $\psi : \tilde{H} \to \mathbb{R}$ *defined by* $\psi(\tau) = |\tau|_{\tilde{H}}^p$, $p \geq 1$. *If* $\mathfrak{N} := \int_0^1 \|e^{s\mathcal{R}}\|^p ds$, *then*

(1) $|\psi(\Phi(l, \tau)) - \psi(\tau)| \leq \mathfrak{N} \, p \, |l|_{\mathbb{R}^k} \|g\|_{\mathcal{L}(\tilde{H})} |\tau|_{\tilde{H}}^p$.
(2) $|\psi(\Phi(l, \tau)) - \psi(\tau) - \psi'(\tau) \mathcal{R}\tau| \leq \mathfrak{N} \, p^2 \, |l|_{\mathbb{R}^k}^2 \|g\|_{\mathcal{L}(\tilde{H})}^2 |\tau|_{\tilde{H}}^p$.

## 3. Main result and key ideas

Two notions of solutions (i.e. martingale and strong) and two notions of uniqueness (pathwise and in law) have been brought in this section to state the main result. We introduce the definitions

of such notions in relevant sections (see e.g. Definition 5.1 for martingale solution, Definition 6.1 for strong solution, Definition 6.2 for pathwise uniqueness and Definition 6.3 for uniqueness in law).

### 3.1. Main result

Under the above abstract setting as explained in the Section 2, system of equations (1.5)-(1.7) defined on the space $H \times \tilde{H}$ becomes:

$$d\mathbf{v}(t) + \left[\nu \mathcal{A}\mathbf{v}(t) + B(\mathbf{v}(t)) - \nu_1 Div\,\tau(t)\right]dt = \int_Z F(\mathbf{v}(t-), z)\tilde{N}_1(dt, dz), \quad t \geq 0 \quad (3.1)$$

$$d\tau(t) + \left[\kappa \tilde{\mathcal{A}}\tau(t) + \tilde{B}(\mathbf{v}(t), \tau(t)) + a\tau(t) - \nu_2\mathcal{D}(\mathbf{v}(t))\right]dt$$

$$= \int_B G(l, \tau(t-))\tilde{N}_2(dt, dl) + b(\tau(t))dt, \quad t \geq 0 \tag{3.2}$$

$$\mathbf{v}(0) = \mathbf{v}_0; \ \tau(0) = \tau_0. \tag{3.3}$$

**Main Result 1.**

(i) Let $d = 2, 3$, and the Assumption 2.4 hold. Let $(\mathbf{v}_0, \tau_0) \in H \times \tilde{H}$. Then there exists a weak martingale solution $\left(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}, \bar{\mathbf{v}}, \bar{\tau}, \bar{N}_1, \bar{N}_2\right)$ to the system (3.1)-(3.3).

(ii) Let $d = 2$. If $\left(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}, \bar{\mathbf{v}}, \bar{\tau}, \bar{N}_1, \bar{N}_2\right)$ is a weak martingale solution of (3.1)-(3.3), then for $\bar{\mathbb{P}}-$ almost all $\omega \in \bar{\Omega}$ the trajectories $\bar{\mathbf{v}}(\cdot, \omega)$ and $\bar{\tau}(\cdot, \omega)$ are equal almost everywhere to càdlàg $H$-valued and $\tilde{H}$-valued functions defined on $[0, T]$. Moreover, for every $t \in [0, T]$, $\bar{\mathbb{P}}$-a.s.

$$\bar{\mathbf{v}}(t) = \mathbf{v}_0 - \int_0^t \left[\nu \mathcal{A}\bar{\mathbf{v}}(s) + B(\bar{\mathbf{v}}(s)) + \nu_1 Div\,\bar{\tau}(s)\right]ds + \int_0^t \int_Z F(\bar{\mathbf{v}}(s-), z)\tilde{N}_1(ds, dz),$$

$$\tag{3.4}$$

$$\bar{\tau}(t) = \tau_0 - \int_0^t \left[\kappa \tilde{\mathcal{A}}\bar{\tau}(s) + \tilde{B}(\bar{\mathbf{v}}(s), \bar{\tau}(s)) + a\bar{\tau}(s) + \nu_2\mathcal{D}(\bar{\mathbf{v}}(s))\right]ds$$

$$+ \int_0^t \int_B G(l, \bar{\tau}(s-))\tilde{N}_2(ds, dl) + \int_0^t b(\bar{\tau}(s))ds. \tag{3.5}$$

(iii) Let $d = 2$, and let Assumption 2.4 and Assumption 6.1 be satisfied. Let $(\mathbf{v}_0, \tau_0) \in H \times \tilde{H}$. Then there exists a pathwise unique strong solution of (3.1)-(3.3).

(iv) Let $d = 2$. Then the martingale solution of (3.1)-(3.3) is unique in law.

(v) Let Assumption 2.4 be satisfied. Let $(\mathbf{v}_0, \tau_0) \in H \times \tilde{H}$. Then, in a Poincaré domain in $\mathbb{R}^2$, the semigroup $\{\mathscr{P}_t\}_{t \geq 0}$, defined in (3.6), is sequentially weakly Feller. Moreover, there exists an invariant measure of this semigroup.

### 3.1.1. Main ideas of the proof

We first consider the classical Faedo-Galerkin solution $(\mathbf{v}_n, \tau_n)$ of the corresponding approximated system and prove suitable uniform bounds for $\mathbf{v}_n$ and $\tau_n$ in Section 4.4, namely,

$$\sup_n \mathbb{E} \left( \sup_{0 \leq t \leq T} |\mathbf{v}_n(t)|^2_H \right) < \infty, \quad \sup_n \mathbb{E} \left( \int_0^T |\mathbf{v}_n(t)|^2_V \, dt \right) < \infty,$$

$$\sup_n \mathbb{E} \left( \sup_{0 \leq t \leq T} |\tau_n(t)|^2_{\tilde{H}} \right) < \infty, \quad \sup_n \mathbb{E} \left( \int_0^T |\tau_n(t)|^2_{\tilde{V}} \, dt \right) < \infty.$$

It is worth to mention here that in order to prove the tightness of family of the law of the solutions of stochastic evolution equation, one usually adopts the classical powerful result of Flandoli and Gatarek [31]. However, this method is not applicable when the noise is of jump type. On the other hand, the approach due to Métivier [57], Brzeźniak *et al.* [13], Motyl [59,60] in proving tightness relies on a deterministic compactness result, which depends on certain energy bounds and a stochastic version of an equicontinuity result of Arzelà-Ascoli type, called the Aldous condition. We adopt similar methods in this work, and with the help of certain deterministic compactness results (see Theorem B.1), we are able to establish, in Lemma 5.1, the Aldous condition (see Definition B.2) to obtain tightness criterion for the laws $\{\mathcal{L}(\mathbf{v}_n, \tau_n), n \in \mathbb{N}\}$ on an appropriate functional (product) space denoted as $\mathcal{Z}$ (see (5.1)), which is not metrizable (hence not a Polish space). We further employ Jakubowski's version of the Skorokhod representation theorem for nonmetric spaces, which allows us to construct a stochastic processes $\bar{\mathbf{v}}, \bar{\tau}$ with trajectories in the space $\mathcal{Z}$, time homogeneous Poisson random measures $\bar{N}_1, \bar{N}_2$ defined on some filtered probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$, such that the system $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}, \bar{\mathbf{v}}, \bar{\tau}, \bar{N}_1, \bar{N}_2)$ is a weak martingale solution of the problem (3.1)-(3.3) satisfying (5.4) and (5.7). One of the essential ingredients in this approach is the partial generalisation from the case of weakly-continuous (see Lemma 4.2 in [14]) to weakly càdlàg functions with respect to application of Kuratowski's theorem for non-Polish spaces. The complete proof of part (i) is given in Theorem 5.8.

To prove part (ii), we first observe that, according to the definition of martingale solution, the equations are understood in the weak sense. We first show that each term in the representation of the solution in the integral (strong) form i.e., the right hand side of (3.4) and (3.5) are well-defined in the spaces $V'$ and $\tilde{V}'$ respectively. Then with the help of some additional regularity of the stochastic integrals, and the Gyöngy-Krylov lemma (see Theorem 2 of [34]), we conclude, in Lemma 6.2, that the trajectories $\bar{\mathbf{v}}(\cdot, \omega)$ and $\bar{\tau}(\cdot, \omega)$ are almost everywhere equal to càdlàg $H$-valued and càdlàg $\tilde{H}$-valued functions, respectively.

For (iii), to prove pathwise uniqueness, we follow Schmalfuss [68] idea of choosing suitable exponential weighted function in order to apply Gyöngy-Krylov version [34] (see also Theorem A.1 from [8]) of infinite-dimensional Itô's formula. The remaining part of (iii) and (iv) are the by product of the infinite dimensional generalisation of the classical Yamada-Watanabe theory (see e.g. Theorem 2 and Theorem 11 of Ondreját [62]. We prove these results in Lemma 6.3 and Theorem 6.4.

It is well-known (see e.g. [23,24]) that Krylov-Bogoliubov method is a classical and the most standard method for proving existence of an invariant measure for a Markov process on an infinite dimensional Hilbert or Banach spaces, which was successfully implemented in the celebrated paper of Flandoli [30] for the two-dimensional Navier-Stokes equation with additive Gaussian

noise and also by Brzeźniak and Gatarek [6] for stochastic reaction-diffusion equations in Banach spaces. In the works of Da Prato and Debussche [21,22], somewhat a reverse problem of the above approach is investigated, where given an *a-priori* invariant measure whether one can construct a Markov process with certain properties.

However, our approach is different from all the above works, and is built upon the theory developed by Maslowski and Seidler [56]. In order to explain this theory briefly, let us assume that $K$ be a separable Hilbert space and $\mathscr{P}$ be a transition probability of a homogeneous Markov process in $K$. Let $\{\mathscr{P}_t\}_{t \geq 0}$ be the corresponding transition semigroup on the space of bounded Borel functions on $K$. The Maslowski and Seidler method of establishing existence of an invariant measure for $\mathscr{P}_t$ requires $\mathscr{P}_t$ to be $bw$-Feller and to find a Borel probability measure $\rho$ on $K$ such that the set of measures

$$\left\{ \frac{1}{T_n} \int_0^{T_n} \mathscr{P}_t^\star \rho(\cdot) \, dt; \; n \geq 1 \right\}$$

is tight on $K_w$, where $K_w$ is a locally convex space endowed with the weak topology, and $\{\mathscr{P}_t^\star\}_{t \geq 0}$ is the dual semigroup acting on a finite Borel measure on $K$. Note, when we say the semigroup $\mathscr{P}_t$ is $bw$-Feller, we mean that if $\varphi : K \to \mathbb{R}$ is a bounded sequentially weakly continuous function and $t > 0$, then $\mathscr{P}_t \varphi : K \to \mathbb{R}$ is also a bounded sequentially weakly continuous function. However, as remarked in [14], it is usually difficult to apply this approach as identification of stochastic partial differential equations, for which the associated transition semigroups are $bw$-Feller, is not at all straightforward. Inspired by the Maslowski and Seidler approach and to investigate its possible scope for application, recently this method has been applied to hyperbolic type SPDEs, such as beam and nonlinear wave equations, in [16] and to stochastic Navier-Stokes equations in two-dimensional unbounded domains driven by cylindrical Wiener process in [14]. Thus, our approach takes obvious inspiration from [56] and follows methods developed in [14,16]. However, it is noteworthy that, our result is new to Oldroyd-B fluids and also extends to the jump noise case.

By part (iii), for any fixed initial data $(\mathbf{v}_0, \tau_0) \in H \times \tilde{H}$, there exists a unique global solution $((\mathbf{v}, \tau)(t, \mathbf{v}_0, \tau_0))$, $t \geq 0$ to the above problem (3.1) - (3.3), defined on a given stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$. We define the corresponding semigroup $\{\mathscr{P}_t\}_{t \geq 0}$ in (7.8) as,

$$(\mathscr{P}_t \varphi)(\mathbf{v}_0, \tau_0) := \mathbb{E}\left[ \varphi((\mathbf{v}, \tau)(t, \mathbf{v}_0, \tau_0)) \right], \quad t \geq 0 \, (\mathbf{v}_0, \tau_0) \in H \times \tilde{H}, \tag{3.6}$$

for any bounded Borel functions $\varphi \in \mathscr{B}_b(H \times \tilde{H})$. It is straightforward to prove that this semigroup is Markov (see Proposition 7.5). Next, making use of continuous dependence result from Theorem 7.4, we prove, in Proposition 7.6, that this semigroup is $bw$-Feller. Further, we show certain uniform boundedness in probability in Lemma 7.7. Both of these results together finally give, in Theorem 7.8, the existence of an invariant measure.

## 4. Faedo-Galerkin approximation and energy estimate

### 4.1. Orthogonal projection

Since $\mathfrak{A}, \tilde{\mathfrak{A}}$ are self adjoint and $\mathfrak{A}^{-1}, (\tilde{\mathfrak{A}})^{-1}$ are compact, there exist an orthonormal basis $\{e_i\}, \{\tilde{e}_i\}$ of $H, \tilde{H}$ respectively consist of eigenvectors of operators $\mathfrak{A}, \tilde{\mathfrak{A}}$. Let $\{\lambda_i\}, \{\tilde{\lambda}_i\}$ be the

corresponding eigenvalues of $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$ respectively. Therefore, we have $\mathfrak{A}e_i = \lambda_i e_i$ and $\tilde{\mathfrak{A}}\tilde{e}_i = \tilde{\lambda}_i \tilde{e}_i$, $\forall i \in \mathbb{N}$. Let us denote $H_n = span\{e_1, \ldots, e_n\}$, $\tilde{H}_n = span\{\tilde{e}_1, \ldots, \tilde{e}_n\}$. Let $P_n$, $\tilde{P}_n$ be the orthogonal projection from $\mathcal{U}'$ to $H_n$ and $\tilde{\mathcal{U}}'$ to $\tilde{H}_n$ respectively. Let us denote $f_i := \dfrac{e_i}{|e_i|_{\mathcal{U}}}$, $\tilde{f}_i := \dfrac{\tilde{e}_i}{|\tilde{e}_i|_{\tilde{\mathcal{U}}}}$, $\forall i \in \mathbb{N}$.

**Lemma 4.1.**

(i) *The system $\{f_i\}_{i \in \mathbb{N}}$, $\{\tilde{f}_i\}_{i \in \mathbb{N}}$ forms orthonormal basis of $(\mathcal{U}, (\cdot, \cdot)_{\mathcal{U}})$ and $(\tilde{\mathcal{U}}, (\cdot, \cdot)_{\tilde{\mathcal{U}}})$ respectively.*

(ii) *For every $\mathbf{v} \in \mathcal{U}$, $\tau \in \tilde{\mathcal{U}}$,*

$$(a) \quad \lim_{n \to \infty} |P_n \mathbf{v} - \mathbf{v}|_{\mathcal{U}} = 0, \quad \lim_{n \to \infty} |\tilde{P}_n \tau - \tau|_{\tilde{\mathcal{U}}} = 0,$$

$$(b) \quad \lim_{n \to \infty} |P_n \mathbf{v} - \mathbf{v}|_{V_s} = 0, \quad \lim_{n \to \infty} |\tilde{P}_n \tau - \tau|_{\tilde{V}_s} = 0,$$

$$(c) \quad \lim_{n \to \infty} |P_n \mathbf{v} - \mathbf{v}|_V = 0, \quad \lim_{n \to \infty} |\tilde{P}_n \tau - \tau|_{\tilde{V}} = 0.$$

*4.2. Approximate system*

Let $T > 0$ and $n \in \mathbb{N}$ be fixed. For $\mathbf{v} \in H_n$, $\tau \in \tilde{H}_n$ and $z \in Z$, let us denote $B_n(\mathbf{v}) := P_n B(\mathbf{v})$, $\tilde{B}_n(\mathbf{v}, \tau) := \tilde{P}_n \tilde{B}(\mathbf{v}, \tau)$, $F_n(\mathbf{v}, z) := P_n F(\mathbf{v}, z)$, $g_{in}(\tau) := \tilde{P}_n(h_i \otimes \tau)$, $\mathbf{v}_n(0) = P_n \mathbf{v}_0$ and $\tau_n(0) = \tilde{P}_n \tau_0$. Let $\Phi_n(t, l, \mathbf{v}_n(0))$ be a flow on $\tilde{H}_n$ associated to the vector field $\sum_{i=1}^{k} l_i g_{in}$ For $\tau_n \in \tilde{H}_n$, let us define $G_n(l, \tau_n) := \Phi_n(l, \tau_n) - \tau_n$, $K_n(l, \tau_n) := \Phi_n(l, \tau_n) - \tau_n - \sum_{i=1}^{k} l_i g_{in}(\tau_n)$, and $b_n(\tau_n) := \int_B K_n(l, \tau_n) \lambda_2(dl)$. Keeping the above notations in mind, we now consider the following $H_n \times \tilde{H}_n$-valued approximated system:

$$d\mathbf{v}_n(t) + \left[\nu P_n \mathcal{A} \mathbf{v}_n(t) + B_n(\mathbf{v}_n(t)) - \nu_1 Div \, \tau_n(t)\right] dt = \int_Z F_n(\mathbf{v}_n(t-), z) \tilde{N}_1(dt, dz), \quad t \geq 0 \tag{4.1}$$

$$d\tau_n(t) + \left[\kappa \tilde{P}_n \tilde{\mathcal{A}} \tau_n(t) + \tilde{B}_n(\mathbf{v}_n(t), \tau_n(t)) + a\tau_n(t) - \nu_2 \mathcal{D}(\mathbf{v}_n(t))\right] dt$$
$$= \int_B G_n(l, \tau_n(t-)) \tilde{N}_2(dt, dl) + b_n(\tau_n(t)) dt, \quad t \geq 0 \tag{4.2}$$

$$\mathbf{v}_n(0) = \mathbf{v}_{0n}; \; \tau_n(0) = \tau_{0n}.$$

We now state some useful results (see Lemma 3.3–3.5 in [9]) obtained by the first author and his collaborator.

**Lemma 4.2.**

- *There exists $M_1 > 0$ such that for any $l \in B$ and $\tau \in \tilde{H}_n$,*

$$|G_n(l, \tau)|_{\tilde{H}_n} \leq M_1 |l|_{\mathbb{R}^k} (1 + |\tau|_{\tilde{H}_n}).$$

- *There exists $M_2 > 0$ such that for any $l \in B$ and $\tau_1, \tau_2 \in \tilde{H}_n$,*

$$|G_n(l, \tau_2) - G_n(l, \tau_1)|_{\tilde{H}_n} \leq M_2 |l|_{\mathbb{R}^k} |\tau_2 - \tau_1|_{\tilde{H}_n}.$$

- *There exists $M_3 > 0$ such that for any $l \in B$ and $\tau \in \tilde{H}_n$,*

$$|K_n(l, \tau)|_{\tilde{H}_n} \leq M_3 |l|_{\mathbb{R}^k}^2 (1 + |\tau|_{\tilde{H}_n}).$$

- *There exists $M_4 > 0$ such that for any $l \in B$ and $\tau_1, \tau_2 \in \tilde{H}_n$,*

$$|K_n(l, \tau_2) - K_n(l, \tau_1)|_{\tilde{H}_n} \leq M_4 |l|_{\mathbb{R}^k}^2 |\tau_2 - \tau_1|_{\tilde{H}_n}.$$

**Lemma 4.3.** *There exists a constant $C_1 > 0$ such that for any $\tau_1, \tau_2 \in \tilde{H}_n$,*

$$\left| b_n(\tau_2) - b_n(\tau_1) \right|_{\tilde{H}_n}^2 + \int_B \left| G_n(l, \tau_2) - G_n(l, \tau_2) \right|_{\tilde{H}_n}^2 \lambda_2(dl) \leq C_1 \left| \tau_2 - \tau_1 \right|_{\tilde{H}_n}^2.$$

**Lemma 4.4.** *There exists a constant $C_2 > 0$ such that for any $\tau \in \tilde{H}_n$,*

$$\left| b_n(\tau) \right|_{\tilde{H}_n}^2 + \int_B \left| G_n(l, \tau) \right|_{\tilde{H}_n}^2 \lambda_2(dl) \leq C_2 |\tau|_{\tilde{H}_n}^2. \tag{4.3}$$

## 4.3. Existence of solutions for the approximate system

The result in this section is true for both dimensions two and three. Consider the following mappings $B'_n : H_n \to H_n$ and $\tilde{B}'_n : H_n \times \tilde{H}_n \to \tilde{H}_n$ by

$$B'_n(\mathbf{v}) := P_n B(\chi_n(\mathbf{v}), \mathbf{v}), \ \ \tilde{B}'_n(\mathbf{v}, \tau) := P_n \tilde{B}(\chi_n(\mathbf{v}), \tau), \quad \mathbf{v} \in H_n, \tau \in \tilde{H}_n,$$

where $\chi_n : H \to H$ is defined by $\chi_n(\mathbf{v}) = \theta_n(|\mathbf{v}|_{V'})\mathbf{v}$, where $\theta_n : \mathbb{R} \to [0, 1]$ of class $\mathcal{C}^\infty$ such that

$$\theta_n(r) = 1 \quad if \quad r \leq n \quad and \quad \theta_n(r) = 0 \quad if \quad r \geq n + 1.$$

Moreover, $B'_n$ and $\tilde{B}'_n$ are globally Lipschitz continuous.

We now consider the following $H_n \times \tilde{H}_n$-valued *modified* approximated system:

$$d\mathbf{v}_n(t) + \left[ \nu P_n \mathcal{A} \mathbf{v}_n(t) + B'_n(\mathbf{v}_n(t)) - \nu_1 Div \, \tau_n(t) \right] dt = \int_Z F_n(\mathbf{v}_n(t-), z) \tilde{N}_1(dt, dz), \quad t \geq 0$$

$$\tag{4.4}$$

$$d\tau_n(t) + \left[ \kappa \tilde{P}_n \tilde{\mathcal{A}} \tau_n(t) + \tilde{B}'_n(\mathbf{v}_n(t), \tau_n(t)) + a\tau_n(t) - \nu_2 \mathcal{D}(\mathbf{v}_n(t)) \right] dt$$

$$= \int_B G_n(l, \tau_n(t-)) \tilde{N}_2(dt, dl) + b_n(\tau_n(t)) dt. \tag{4.5}$$

Since all relevant maps are globally Lipschitz, we have the following standard result, see e.g. [1] for a reference.

**Lemma 4.5.** *For each $n \in \mathbb{N}$, there exists a unique global, $\mathbb{F}$-progressively measurable, $H_n \times \tilde{H}_n$-valued càdlàg process $(\mathbf{v}_n, \tau_n)$ satisfying the modified Galerkin approximated system* (4.4)-(4.5).

It follows easily, see for instance [1], that for each $n \in \mathbb{N}$, the equations (4.1)-(4.2) have a unique local maximal solution. By a combination of the proof of [1, Theorem 3.1] with the proofs of Theorem 4.7 below (in the case $p = 2$), we infer that (4.1)-(4.2) has a unique global solution. In other words, we have the following result.

**Theorem 4.6.** *For each $n \in \mathbb{N}$, the system* (4.1)-(4.2) *has a unique global solution.*

### 4.4. A priori energy estimate

In this subsection, we will obtain uniform bounds for $\mathbf{v}_n$ and $\tau_n$ in some suitable spaces.

**Theorem 4.7.** *Let $p \geq 1$. Let $\mathbf{v}_0 \in \mathrm{L}^{2p}(\Omega; H)$ and $\tau_0 \in \mathrm{L}^{2p}(\Omega; \tilde{H})$. Let $(\mathbf{v}_n, \tau_n)$ be the solution of* (4.1)-(4.2). *Then for every $T > 0$ there exists $C > 0$, depending on $T$, $p$ and on the parameters of the problem (which is independent of n) such that*

$$\sup_{n \in \mathbb{N}} \mathbb{E}\left[ \sup_{0 \leq s \leq T} \left( \nu_2 |\mathbf{v}_n(s)|_H^2 + \nu_1 |\tau_n(s)|_{\tilde{H}}^2 \right)^p \right]$$

$$+ \sup_{n \in \mathbb{N}} \mathbb{E}\left[ \left( 2\nu_2 \nu \int\limits_0^T |\nabla \mathbf{v}_n(s)|_H^2 ds + 2\nu_1 \kappa \int\limits_0^T |\nabla \tau_n(s)|_{\tilde{H}}^2 ds + 2a\nu_1 \int\limits_0^T |\tau_n(s)|_{\tilde{H}}^2 ds \right)^p \right] \leq C(p). \tag{4.6}$$

**Proof.** Let us define stopping time by

$$\rho_N^n = \inf_{t \geq 0} \left\{ t : \nu_2 |\mathbf{v}_n(t)|_H^2 + \nu_1 |\tau_n(t)|_{\tilde{H}}^2 + 2\nu \int\limits_0^t |\nabla \mathbf{v}_n(r)|_H^2 dr + 2\kappa \int\limits_0^t |\nabla \tau_n(r)|_{\tilde{H}}^2 dr > N \right\}.$$

We apply finite-dimensional Itô's formula (see Brzeźniak et al. [8], Ikeda and Watanabe [38]) to the function $x \mapsto \frac{1}{2}|x|_H^2$ (for the process $\mathbf{v}_n(t)$) and $\frac{1}{2}|x|_{\tilde{H}}^2$ (for the process $\tau_n(t)$) to the first equation and the second equation respectively. Hence using $\langle A\mathbf{v}_n(t), \mathbf{v}_n(t) \rangle = |\nabla \mathbf{v}_n(t)|_H^2$, $\langle B_n(\mathbf{v}_n(t)), \mathbf{v}_n(t) \rangle = 0$, $\langle \tilde{A}\tau_n(t), \tau_n(t) \rangle = |\nabla \tau_n(t)|_{\tilde{H}}^2$ and $\langle \tilde{B}_n(\mathbf{v}_n(t), \tau_n(t)), \tau_n(t) \rangle_{\tilde{H}} = 0$, and then multiplying the first equation by $\nu_2$ and second equation by $\nu_1$ and then on adding and integrating on $[0, t \wedge \rho_N^n]$ we obtain,

$$\left[ \nu_2 |\mathbf{v}_n(t \wedge \rho_N^n)|_H^2 + \nu_1 |\tau_n(t \wedge \rho_N^n)|_{\tilde{H}}^2 \right] + 2\nu_2 \nu \int\limits_0^{t \wedge \rho_N^n} |\nabla \mathbf{v}_n(s)|_H^2 ds + 2\nu_1 \kappa \int\limits_0^{t \wedge \rho_N^n} |\nabla \tau_n(s)|_{\tilde{H}}^2 ds$$

$$+ 2a\nu_1 \int_0^{t \wedge \rho_N^n} |\tau_n(s)|_{\tilde{H}}^2 \, ds$$

$$= \underbrace{\nu_2 |\mathbf{v}_{0n}|_H^2 + \nu_1 |\tau_{0n}|_{\tilde{H}}^2}_{I_0} + \underbrace{\nu_2 \int_0^{t \wedge \rho_N^n} \int_Z \left\{ |\mathbf{v}_n(s-) + F_n(\mathbf{v}_n(s-), z)|_H^2 - |\mathbf{v}_n(s-)|_H^2 \right\} \tilde{N}_1(ds, dz)}_{I_1(t)}$$

$$+ \underbrace{\nu_2 \int_0^{t \wedge \rho_N^n} \int_Z \left\{ |\mathbf{v}_n(s) + F_n(\mathbf{v}_n(s), z)|_H^2 - |\mathbf{v}_n(s)|_H^2 - 2\langle F_n(\mathbf{v}_n(s), z), \mathbf{v}_n(s) \rangle_H \right\} \lambda_1(dz) ds}_{I_2(t)}$$

$$+ \underbrace{\nu_1 \int_0^{t \wedge \rho_N^n} \int_B \left\{ |\Phi_n(l, \tau_n(s-))|_{\tilde{H}}^2 - |\tau_n(s-)|_{\tilde{H}}^2 \right\} \tilde{N}_2(ds, dl)}_{I_3(t)}$$

$$+ \underbrace{\nu_1 \int_0^{t \wedge \rho_N^n} \int_B \left\{ |\Phi_n(l, \tau_n(s))|_{\tilde{H}}^2 - |\tau_n(s)|_{\tilde{H}}^2 - 2\sum_{i=1}^k l_i \langle g_{in}(\tau_n(s)), \tau_n(s) \rangle_{\tilde{H}} \right\} \lambda_2(dl) ds}_{I_4(t)} . \qquad (4.7)$$

We note that $I_1$ and $I_3$ are local martingales with zero averages, i.e., $\mathbb{E}[I_1] = \mathbb{E}[I_3] = 0$. Now using Assumption (2.9) and Lemma 2.5 (ii), we have

$$I_2(t) \leq C\nu_2 T + C\nu_2 \int_0^{t \wedge \rho_N^n} |\mathbf{v}_n(s)|_H^2 \, ds, \qquad (4.8)$$

$$\text{and} \quad I_4(t) \leq 4\mathfrak{N} \|g\|_{\mathcal{L}(\tilde{H})}^2 \left( \int_B |l|_{\mathbb{R}^k}^2 \lambda_2(dl) \right) \left( \nu_1 \int_0^{t \wedge \rho_N^n} |\tau_n(s)|_{\tilde{H}}^2 \, ds \right). \qquad (4.9)$$

Using (4.8), (4.9) and taking expectation on both sides of (4.7) and using $\mathbb{E}[I_1] = \mathbb{E}[I_3] = 0$, we get

$$\mathbb{E}\left[ \nu_2 |\mathbf{v}_n(t \wedge \rho_N^n)|_H^2 + \nu_1 |\tau_n(t \wedge \rho_N^n)|_{\tilde{H}}^2 + 2\nu_2\nu \int_0^{t \wedge \rho_N^n} |\nabla \mathbf{v}_n(s)|_H^2 \, ds + 2\nu_1 \kappa \int_0^{t \wedge \rho_N^n} |\nabla \tau_n(s)|_{\tilde{H}}^2 \, ds \right.$$

$$\left. + 2a\nu_1 \int_0^{t \wedge \rho_N^n} |\tau_n(s)|_{\tilde{H}}^2 \, ds \right]$$

$$\leq \mathbb{E}\Big[\nu_2|\mathbf{v}_{0n}|_H^2 + \nu_1|\tau_{0n}|_{\tilde{H}}^2 + C\nu_2 T\Big] + C\mathbb{E}\Big[\int_0^{t \wedge \rho_N^n}\Big(\nu_2|\mathbf{v}_n(s)|_H^2 + \nu_1|\tau_n(s)|_{\tilde{H}}^2\Big)ds\Big].$$

Using Gronwall inequality and as letting $N \to \infty$ we have $t \wedge \rho_N^n \to t$, hence

$$\mathbb{E}\Big[\nu_2|\mathbf{v}_n(t)|_H^2 + \nu_1|\tau_n(t)|_{\tilde{H}}^2 + 2\nu_2\nu\int_0^t |\nabla\mathbf{v}_n(s)|_H^2 ds + 2\nu_1\kappa\int_0^t |\nabla\tau_n(s)|_{\tilde{H}}^2 ds$$

$$+2a\nu_1\int_0^t |\tau_n(s)|_{\tilde{H}}^2 ds\Big] \leq \mathbb{E}\Big[\nu_2|\mathbf{v}_0|_H^2 + \nu_1|\tau_0|_{\tilde{H}}^2 + C\nu_2 T\Big]e^{CT}. \quad (4.10)$$

Now to get the required $p$th estimate, we first raise the power $p \geq 1$. Taking supremum over $[0, T \wedge \rho_N^n]$ and then expectation we obtain,

$$\mathbb{E}\Big[\sup_{0 \leq s \leq T \wedge \rho_N^n}\Big(\nu_2|\mathbf{v}_n(s)|_H^2 + \nu_1|\tau_n(s)|_{\tilde{H}}^2\Big)^p\Big] + \mathbb{E}\Big[\Big(2\nu_2\nu\int_0^{T \wedge \rho_N^n}|\nabla\mathbf{v}_n(s)|_H^2 ds$$

$$+2\nu_1\kappa\int_0^{T \wedge \rho_N^n}|\nabla\tau_n(s)|_{\tilde{H}}^2 ds + 2a\nu_1\int_0^{T \wedge \rho_N^n}|\tau_n(s)|_{\tilde{H}}^2 ds\Big)^p\Big]$$

$$\leq 4^{p-1}\Big(\mathbb{E}\Big[\nu_2|\mathbf{v}_0|_H^2 + \nu_1|\tau_0|_{\tilde{H}}^2 + C\nu_2 T\Big]^p + \mathbb{E}\Big[C\int_0^{t \wedge \rho_N^n}\Big(\nu_2|\mathbf{v}_n(s)|_H^2 + \nu_1|\tau_n(s)|_{\tilde{H}}^2\Big)ds\Big]^p\Big)$$

$$+\mathbb{E}\Big[\sup_{0 \leq s \leq T \wedge \rho_N^n}|I_1(s)|^p\Big] + \mathbb{E}\Big[\sup_{0 \leq s \leq T \wedge \rho_N^n}|I_3(s)|^p\Big]. \quad (4.11)$$

Using Burkholder-Davis-Gundy inequality, Assumption 2.4, Young's inequality, we have

$$\mathbb{E}\Big[\sup_{0 \leq s \leq T \wedge \rho_N^n}|I_1(s)|^p\Big]$$

$$\leq C_{p,\nu_2}\mathbb{E}\Big[\Big(\int_0^{T \wedge \rho_N^n}\int_Z\Big\{|\mathbf{v}_n(s) + F_n(\mathbf{v}_n(s), z)|_H^2 - |\mathbf{v}_n(s)|_H^2\Big\}^2 \lambda_1(dz)ds\Big)^{p/2}\Big]$$

$$\leq C_p\nu_2^p T^{p/2} + \frac{1}{2}\mathbb{E}\Big[\sup_{0 \leq s \leq T \wedge \rho_N^n}\nu_2|\mathbf{v}_n(s)|_H^2\Big]^p + C_p T^{p-1}\mathbb{E}\Big[\int_0^{T \wedge \rho_N^n}\Big(\nu_2|\mathbf{v}_n(s)|_H^2\Big)^p ds\Big]. \quad (4.12)$$

Again using Burkholder-Davis-Gundy inequality, Lemma 2.5 and Young's inequality, we have

$$\mathbb{E}\left[\sup_{0\le s\le T\wedge\rho_N^n}|I_3(s)|^p\right]\le\frac{1}{2}\mathbb{E}\left[\left(\sup_{0\le s\le T\wedge\rho_N^n}|\tau_n(s)|_{\tilde{H}}^2\right)^p\right]+C_pT^{p-1}\mathbb{E}\left[\int_0^{T\wedge\rho_N^n}\left(\nu_1|\tau_n(s)|_{\tilde{H}}^2\right)^pds\right].$$

$$(4.13)$$

Substituting (4.12) and (4.13) in (4.11) we have

$$\frac{1}{2}\mathbb{E}\left[\sup_{0\le s\le T\wedge\rho_N^n}\left(\nu_2|\mathbf{v}_n(s)|_H^2+\nu_1|\tau_n(s)|_{\tilde{H}}^2\right)^p\right]+\mathbb{E}\left[\left(2\nu_2\nu\int_0^{T\wedge\rho_N^n}|\nabla\mathbf{v}_n(s)|_H^2ds\right.\right.$$

$$+2\nu_1\kappa\int_0^{T\wedge\rho_N^n}|\nabla\tau_n(s)|_{\tilde{H}}^2ds+2a\nu_1\int_0^{T\wedge\rho_N^n}|\tau_n(s)|_{\tilde{H}}^2ds\Bigg)^p\Bigg]$$

$$\le C_p\mathbb{E}\left[\nu_2|\mathbf{v}_0|_H^2+\nu_1|\tau_0|_{\tilde{H}}^2+C\nu_2T\right]^p+C_pT^{p-1}\mathbb{E}\left[\int_0^{T\wedge\rho_N^n}\left(\nu_2|\mathbf{v}_n(s)|_H^2+\nu_1|\tau_n(s)|_{\tilde{H}}^2\right)^pds\right].$$

Using Gronwall inequality and letting $N\to\infty$, $T\wedge\rho_N^n\to T$, so we have

$$\mathbb{E}\left[\sup_{0\le s\le T}\left(\nu_2|\mathbf{v}_n(s)|_H^2+\nu_1|\tau_n(s)|_{\tilde{H}}^2\right)^p\right]+\mathbb{E}\left[\left(2\nu_2\nu\int_0^T|\nabla\mathbf{v}_n(s)|_H^2ds+2\nu_1\kappa\int_0^T|\nabla\tau_n(s)|_{\tilde{H}}^2ds\right.\right.$$

$$+2a\nu_1\int_0^T|\tau_n(s)|_{\tilde{H}}^2ds\Bigg)^p\Bigg]\le C_p\mathbb{E}\left[\left(\nu_2|\mathbf{v}_0|_H^2+\nu_1|\tau_0|_{\tilde{H}}^2+C\nu_2T\right)^p\right]e^{C_pT^p}.$$

This completes the proof.   □

**Lemma 4.8.** *Let* $p\ge 2$. *Let* $\mathbf{v}_0\in L^p(\Omega;H)$ *and* $\tau_0\in L^p(\Omega;\tilde{H})$. *Let* $(\mathbf{v}_n,\tau_n)$ *be the solution of* (4.1)-(4.2). *Then for every* $T>0$ *there exists* $C>0$, *depending on* $T$, $p$ *and on the parameters of the problem (but is independent of n) such that*

$$\sup_{n\in\mathbb{N}}\mathbb{E}\left[\sup_{0\le s\le T}\left(\nu_2|\mathbf{v}_n(s)|_H^p+\nu_1|\tau_n(s)|_{\tilde{H}}^p\right)\right]+\sup_{n\in\mathbb{N}}\mathbb{E}\left[p\nu_2\nu\int_0^T|\nabla\mathbf{v}_n(s)|_H^2|\mathbf{v}_n(s)|_H^{p-2}ds\right.$$

$$+p\nu_1\kappa\int_0^T|\tau_n(s)|_{\tilde{H}}^{p-2}|\nabla\tau_n(s)|_{\tilde{H}}^2ds\Bigg]\le C(p).$$

$$(4.14)$$

**Proof.** Borrowing some ideas from Theorem 4.7, (4.14) can be proved by applying Itô's Lemma to the function $x\mapsto\dfrac{1}{p}|x|_H^p$ (for the process $\mathbf{v}_n(t)$) and $\dfrac{1}{p}|x|_{\tilde{H}}^p$ (for the process $\tau_n(t)$) to the system of equation (4.4)-(4.5).   □

**Remark 1.** A direct consequence of Theorem 4.7 (or Lemma 4.8) gives

$$\sup_{n \in \mathbb{N}} \mathbb{E}\Big[\int_0^T |\mathbf{v}_n(s)|_V^2\, ds\Big] \leq C, \quad \sup_{n \in \mathbb{N}} \mathbb{E}\Big[\int_0^T |\tau_n(s)|_{\tilde{V}}^2\, ds\Big] \leq C.$$

## 5. Existence of a weak martingale solution to (3.1)-(3.3)

We first introduce the following functional spaces endowed with the respective topologies:

$$D([0, T]; \mathcal{U}') := \text{ the space of càdlàg functions } \mathbf{v} : [0, T] \to \mathcal{U}'$$

with the topology $\mathcal{T}_1$ induced by the Skorokhod metric $\delta_{T, \mathcal{U}'}$ (see (B.1) for definition);

$$\mathrm{L}_w^2(0, T; V) := \text{ the space } \mathrm{L}^2(0, T; V) \text{ with the weak topology } \mathcal{T}_2;$$

$$\mathrm{L}^2(0, T; H_{loc}) := \text{ the space of measurable functions } \mathbf{v} : [0, T] \to H \text{ such that for all } R > 0,$$

$$p_{T,R}(\mathbf{v}) := ||\mathbf{v}||_{\mathrm{L}^2(0,T;H_{B_R})} := \left(\int_0^T \int_{B_R} |\mathbf{v}(x, t)|_H^2\, dx\, dt\right)^{1/2} < \infty,$$

with the topology $\mathcal{T}_3$ generated by the seminorms $(p_{T,R})_{R>0}$, where $B_R$ is a closed ball of radius $R$;

$$H_w := \text{ the Hilbert space } H \text{ endowed with the weak topology;}$$

$$D([0, T]; H_w) := \text{ the space of all weakly càdlàg functions } \mathbf{v} : [0, T] \to H$$

with the weakest topology $\mathcal{T}_4$ such that for all $h \in H$ the mappings

$$D([0, T]; H_w) \ni \mathbf{v} \mapsto (\mathbf{v}(\cdot), h)_H \in D([0, T]; \mathbb{R})$$

are continuous. In particular, $\mathbf{v}_n \to \mathbf{v}$ in $D([0, T]; H_w)$ iff for all $h \in H$

$$(\mathbf{v}_n(\cdot), h)_H \to (\mathbf{v}(\cdot), h)_H \text{ in } D([0, T]; \mathbb{R}).$$

Similarly, one can define the spaces $D([0, T]; \tilde{\mathcal{U}}')$, $\mathrm{L}_w^2(0, T; \tilde{V})$, $\mathrm{L}^2(0, T; \tilde{H}_{loc})$ and $D([0, T]; \tilde{H}_w)$.

Now let us consider the spaces $\mathcal{Z}_i; \; i = 1, 2$ defined by

$$\begin{cases} \mathcal{Z}_1 := D([0, T]; \mathcal{U}') \cap D([0, T]; H_w) \cap \mathrm{L}_w^2(0, T; V) \cap \mathrm{L}^2(0, T; H_{loc}), \\ \mathcal{Z}_2 := D([0, T]; \tilde{\mathcal{U}}') \cap D([0, T]; \tilde{H}_w) \cap \mathrm{L}_w^2(0, T; \tilde{V}) \cap \mathrm{L}^2(0, T; \tilde{H}_{loc}), \end{cases} \tag{5.1}$$

and let $\mathcal{T}_i$ be the supremum of the corresponding topologies in the spaces $\mathcal{Z}_i, i = 1, 2$. Let $\mathcal{Z} := \mathcal{Z}_1 \times \mathcal{Z}_2$ and $\mathcal{T}$ be the supremum of $\mathcal{T}_1$ and $\mathcal{T}_2$.

Let us now recall the definition of a weak martingale solution.

**Definition 5.1.** A **weak martingale solution** of the problem (3.1)-(3.3) is a system $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}, \bar{\mathbf{v}}, \bar{\tau}, \bar{N}_1, \bar{N}_2)$, where

(1) $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$ is a filtered probability space with a filtration $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_t)_{t\geq 0}$,

(2) $\bar{N}_1$ is a time homogeneous Poisson random measure on $(Z, \mathcal{B}(Z))$ over $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$ with the intensity measure $\lambda_1$ and $\bar{N}_2$ is a time homogeneous Poisson random measure on $(B, \mathcal{B}(B))$ with the intensity measure $\lambda_2$.

(3) $\bar{\mathbf{v}} : [0, T] \times \bar{\Omega} \to H$ is an $\bar{\mathbb{F}}$-progressively measurable process with $\bar{\mathbb{P}}$-a.e. paths

$$\bar{\mathbf{v}}(\cdot, \omega) \in D([0, T]; H_w) \cap L^2(0, T; V) \tag{5.2}$$

such that for all $t \in [0, T]$ and all $\phi \in V$

$$(\bar{\mathbf{v}}(t), \phi)_H + \nu \int_0^t \langle \mathcal{A}\bar{\mathbf{v}}(s), \phi \rangle ds + \int_0^t \langle B(\bar{\mathbf{v}}(s)), \phi \rangle ds + \nu_1 \int_0^t \langle Div\, \bar{\tau}(s), \phi \rangle ds$$

$$= (\mathbf{v}_0, \phi)_H + \int_0^t \int_Z \left( F(\bar{\mathbf{v}}(s), z), \phi \right)_H \tilde{\bar{N}}_1(ds, dz), \qquad \bar{\mathbb{P}} - \text{a.s.} \tag{5.3}$$

$$\text{and} \quad \bar{\mathbb{E}} \left( \sup_{0 \leq t \leq T} |\bar{\mathbf{v}}(t)|_H^2 + 2\nu \bar{\mathbb{E}} \int_0^T |\bar{\mathbf{v}}(t)|_V^2 \, dt \right) < \infty, \tag{5.4}$$

where $\bar{\mathbb{E}}$ denotes the expectation with respect to $\bar{\mathbb{P}}$.

(4) $\bar{\tau} : [0, T] \times \bar{\Omega} \to \tilde{H}$ is an $\bar{\mathbb{F}}$-progressively measurable process with $\bar{\mathbb{P}}$-a.e. paths

$$\bar{\tau}(\cdot, \omega) \in D([0, T]; \tilde{H}_w) \cap L^2(0, T; \tilde{V}) \tag{5.5}$$

such that for all $t \in [0, T]$ and all $\psi \in \tilde{V}$,

$$(\bar{\tau}(t), \psi)_{\tilde{H}} + \kappa \int_0^t \langle \tilde{\mathcal{A}}\bar{\tau}(s), \psi \rangle ds + \int_0^t \langle \tilde{B}(\bar{\mathbf{v}}(s), \bar{\tau}(s)), \psi \rangle ds + a \int_0^t \langle \bar{\tau}(s), \psi \rangle ds$$

$$= \left( \bar{\tau}(0), \psi \right)_{\tilde{H}} + \nu_2 \int_0^t \langle \mathcal{D}(\bar{\mathbf{v}}(s)), \psi \rangle ds + \int_0^t \langle b(\bar{\tau}(s)), \psi \rangle ds$$

$$+ \int_0^t \int_B \left( G(l, \bar{\tau}(s)), \psi \right)_{\tilde{H}} \tilde{\bar{N}}_2(ds, dl), \qquad \bar{\mathbb{P}} - \text{a.s.} \tag{5.6}$$

$$\text{and} \quad \bar{\mathbb{E}} \left( \sup_{0 \leq t \leq T} |\bar{\tau}(t)|_{\tilde{H}}^2 + 2\kappa \bar{\mathbb{E}} \int_0^T |\bar{\tau}(t)|_{\tilde{V}}^2 \, dt \right) < \infty. \tag{5.7}$$

## 5.1. Tightness criteria

We note that the solutions $(\mathbf{v}_n, \tau_n)$ obtained from Galerkin approximation induces sequence of measures $\mathcal{L}(\mathbf{v}_n, \tau_n)$ on the space $(\mathcal{Z}, \mathcal{T})$. Then we have the following result:

**Lemma 5.1.** *(Tightness of law) The sequence $\{\mathcal{L}(\mathbf{v}_n, \tau_n)\}_{n \in \mathbb{N}}$ of measures is tight on the space* $(\mathcal{Z}, \mathcal{T})$.

**Proof.** We observe from Theorem 4.7 and Remark 1 that there exists constant $C_T = C_T(\nu_1, \nu_2, \nu, \kappa, a) > 0$ such that for all $t \in [0, T]$, $\forall\ n \in \mathbb{N}$

$$\sup_{n \in \mathbb{N}} \mathbb{E}\left[\sup_{t \in [0,T]} |\mathbf{v}_n(t)|_H^2\right] \leq C_T, \quad \sup_{n \in \mathbb{N}} \mathbb{E}\left[\int_0^T |\mathbf{v}_n(t)|_V^2 dt\right] \leq C_T, \tag{5.8}$$

$$\sup_{n \in \mathbb{N}} \mathbb{E}\left[\sup_{t \in [0,T]} |\tau_n(t)|_{\tilde{H}}^2\right] \leq C_T, \quad \sup_{n \in \mathbb{N}} \mathbb{E}\left[\int_0^T |\tau_n(t)|_{\tilde{V}}^2 dt\right] \leq C_T. \tag{5.9}$$

Therefore from (5.8)-(5.9) we conclude that first two conditions of Theorem B.3 for $(\mathbf{v}_n, \tau_n)$ are satisfied. Now we are left to prove that $(\mathbf{v}_n)_{n \in \mathbb{N}}$ and $(\tau_n)_{n \in \mathbb{N}}$ satisfy Aldous condition [A] (see Definition B.2) in the space $\mathcal{U}'$ and $\tilde{\mathcal{U}}'$ respectively.

Let $(\rho_n)_{n \in \mathbb{N}}$ be a sequence of stopping times such that $0 \leq \rho_n \leq T$. Then for $t \in (0, \rho_n \wedge T)$ we get

$$\mathbf{v}_n(t) = \mathbf{v}_{0n} - \nu \int_0^t \mathcal{A}\mathbf{v}_n(s)ds - \int_0^t B_n(\mathbf{v}_n(s))ds + \nu_1 \int_0^t Div\, \tau_n(s)ds$$

$$+ \int_0^t \int_Z F_n(\mathbf{v}_n(s), z)\tilde{N}_1(ds, dz) := \sum_{i=1}^5 J_i^n(t).$$

We will show that each of $\{J_i^n(t)\}_{i=1}^5$ will satisfy condition (B.2) in Lemma B.2 for suitable choices of $\alpha$ and $\zeta$. Let us fix $\theta > 0$. $J_n^1$ being independent of time clearly satisfies (B.2). Now exploiting the fact $\mathcal{A} : V \to V'$ given by $|\mathcal{A}\mathbf{v}|_{V'} \leq |\mathbf{v}|_V$, the embedding $V' \hookrightarrow \mathcal{U}'$ is continuous, then by Hölder's inequality and (5.8), $J_2^n$ can be estimated as

$$\mathbb{E}\left[|J_2^n(\rho_n + \theta) - J_2^n(\rho_n)|_{\mathcal{U}'}\right] \leq C_1 \mathbb{E}\left[\theta^{\frac{1}{2}}\left(\int_0^T |\mathbf{v}_n(s)|_V^2 ds\right)^{\frac{1}{2}}\right] \leq c_2 \theta^{\frac{1}{2}}.$$

Thus $J_2^n$ satisfies (B.2) with $\alpha = 1$ and $\zeta = \frac{1}{2}$.

Using $B : V \times V \to V'$ is continuous and the continuous embedding $V' \hookrightarrow \mathcal{U}'$, and (5.8), we estimate $J_3^n$ as

$$\mathbb{E}\Big[|J_3^n(\rho_n+\theta)-J_3^n(\rho_n)|_{\mathcal{U}'}\Big] \leq C\mathbb{E}\Big[\int_0^T |\mathbf{v}_n(s)|_V^2 ds\Big]\cdot\theta =: c_3\theta.$$

Thus $J_3^n$ satisfies (B.2) with $\alpha=1$ and $\zeta=1$.

Using (5.9) and the embedding $H \hookrightarrow V' \hookrightarrow \mathcal{U}'$ we have

$$\mathbb{E}\Big[|J_4^n(\rho_n+\theta)-J_4^n(\rho_n)|_{\mathcal{U}'}\Big] \leq \nu_1 \mathbb{E}\Big[\Big(\int_{\rho_n}^{\rho_n+\theta} |\nabla\tau_n(s)|_{\tilde{H}}^2 ds\Big)^{\frac{1}{2}} \theta^{\frac{1}{2}}\Big]$$

$$\leq \nu_1\, C\Big(\mathbb{E}\Big[\sup_{0\leq s\leq T} |\nabla\tau_n(s)|_{\tilde{H}}^2\Big]\Big)^{\frac{1}{2}} \theta^{\frac{1}{2}} := c_4\theta^{\frac{1}{2}}.$$

Therefore, $J_4^n$ satisfies (B.2) with $\alpha=1$ and $\zeta=\frac{1}{2}$.

Using (5.8), the continuous embedding $H \hookrightarrow \tilde{\mathcal{U}}'$, Itô-Lévy isometry and Assumption 2.4, we have

$$\mathbb{E}\Big[|J_5^n(\rho_n+\theta)-J_5^n(\rho_n)|_{\mathcal{U}'}^2\Big] \leq C\mathbb{E}\Big[\int_{\rho_n}^{\rho_n+\theta}\int_Z \Big|F_n(\mathbf{v}_n(s),z)\Big|_H^2 \lambda_1(dz)ds\Big]$$

$$\leq C\theta\Big(1+\mathbb{E}\Big[\sup_{0\leq s\leq T} |\mathbf{v}_n(s)|_H^2\Big]\Big) := c_5\theta.$$

Thus $J_5^n$ satisfies (B.2) with $\alpha=2$ and $\zeta=1$.

Let us consider the equation for $\tau_n$ for $t \in (0, \rho_n\wedge T]$,

$$\tau_n(t) = \tau_{0n} - \kappa\int_0^t \tilde{A}\tau_n(s)ds - \int_0^t \tilde{B}_n(\mathbf{v}_n(s),\tau_n(s))ds - a\int_0^t \tau_n(s)ds + \nu_2\int_0^t \mathcal{D}(\mathbf{v}_n(s))ds$$

$$+ \int_0^t\int_B G_n(l,\tau_n(s))\tilde{N}_2(dl,ds) + \int_0^t b_n(\tau_n(s))ds := \sum_{i=1}^7 K_i^n(t).$$

$K_n^1$ being independent of time, (B.2) is automatically satisfied for any $\alpha,\beta>0$. Now exploiting the fact $\tilde{A}:\tilde{V}\to\tilde{V}'$ given by $|\tilde{A}\tau|_{\tilde{V}'}\leq |\tau|_{\tilde{V}}$, the continuous embedding $\tilde{V}'\hookrightarrow\tilde{\mathcal{U}}'$, Hölder's inequality and (5.8), $K_2^n$ can be estimated as

$$\mathbb{E}\Big[|K_2^n(\rho_n+\theta)-K_2^n(\rho_n)|_{\tilde{\mathcal{U}}'}\Big] \leq C_1\mathbb{E}\Big[\theta^{\frac{1}{2}}\Big(\int_0^T |\tau_n(s)|_{\tilde{V}}^2 ds\Big)^{\frac{1}{2}}\Big] \leq \tilde{c}_2\theta^{\frac{1}{2}}.$$

Thus $K_2^n$ satisfies (B.2) with $\alpha=1$ and $\zeta=\frac{1}{2}$.

Using $|\tilde{B}(\mathbf{v}, \tau)|_{\tilde{V}_s{}'} \le |\mathbf{v}|_H |\tau|_{\tilde{H}}$ and the embedding $\tilde{V}_s{}' \hookrightarrow \tilde{\mathcal{U}}'$ we have

$$\mathbb{E}\Big[|K_3^n(\rho_n + \theta) - K_3^n(\rho_n)|_{\tilde{\mathcal{U}}'}\Big] \le C \mathbb{E}\Big[\Big(\int\limits_{\rho_n}^{\rho_n+\theta} |\mathbf{v}_n(s)|_H^2 ds\Big)^{\frac{1}{2}} \Big(\int\limits_{\rho_n}^{\rho_n+\theta} |\tau_n(s)|_{\tilde{H}}^2 ds\Big)^{\frac{1}{2}}\Big] \le \tilde{c}_3 \theta.$$

Thus $K_3^n$ satisfies (B.2) with $\alpha = 1$ and $\zeta = 1$.

Using the embedding $\tilde{H} \hookrightarrow \tilde{\mathcal{U}}'$ and (5.9) we have

$$\mathbb{E}\Big[|K_4^n(\rho_n + \theta) - K_4^n(\rho_n)|_{\tilde{\mathcal{U}}'}\Big] \le C\, a\, T^{\frac{1}{2}} \Big(\mathbb{E}[\sup_{0 \le s \le T} |\tau_n(s)|_{\tilde{H}}^2]\Big)^{\frac{1}{2}} \theta^{\frac{1}{2}} := \tilde{c}_4 \theta^{\frac{1}{2}}.$$

Thus $K_4^n$ satisfies (B.2) with $\alpha = 1$ and $\zeta = \frac{1}{2}$.

Using the embedding $\tilde{H} \hookrightarrow \tilde{\mathcal{U}}'$ and (5.9) we have

$$\mathbb{E}\Big[|K_5^n(\rho_n + \theta) - K_5^n(\rho_n)|_{\tilde{\mathcal{U}}'}\Big] \le C\, \nu_2 \mathbb{E}\Big[\int\limits_{\rho_n}^{\rho_n+\theta} |\mathcal{D}(\mathbf{v}_n(s))|_{\tilde{H}} ds\Big] \le \tilde{c}_5 \theta^{\frac{1}{2}}.$$

Thus $K_5^n$ satisfies (B.2) with $\alpha = 1$ and $\zeta = \frac{1}{2}$.

Using Itô-Lévy isometry, embedding $\tilde{H} \hookrightarrow \tilde{\mathcal{U}}'$, Lemma 4.4 and (5.9) we have

$$\mathbb{E}\Big[|K_6^n(\rho_n + \theta) - K_6^n(\rho_n)|_{\tilde{\mathcal{U}}'}^2\Big] \le C \mathbb{E}\Big[\int\limits_{\rho_n}^{\rho_n+\theta} \int\limits_B |G(l, \tau_n(s))|_{\tilde{H}}^2 \lambda_2(dl) ds\Big]$$

$$\le C \mathbb{E}\Big[\int\limits_{\rho_n}^{\rho_n+\theta} |\tau_n(s)|_{\tilde{H}}^2 ds\Big] \le \tilde{c}_6 \theta.$$

Thus $K_6^n$ satisfies (B.2) with $\alpha = 2$ and $\zeta = 1$.

Since $\tilde{H} \hookrightarrow \tilde{\mathcal{U}}'$ is continuous, using Schwarz's inequality, from Lemma 4.4 and (5.9) we get,

$$\mathbb{E}\big[|K_7^n(\rho_n + \theta) - K_7^n(\rho_n)|_{\tilde{\mathcal{U}}'}\big] \le C\, \theta^{\frac{1}{2}}\, \mathbb{E}\Big[\Big(\int\limits_{\rho_n}^{\rho_n+\theta} |b_n(\tau_n(s))|_{\tilde{H}}^2 ds\Big)^{\frac{1}{2}}\Big] \le \tilde{c}_7 \theta.$$

Thus $K_7^n$ satisfies the condition (B.2) with $\alpha = 1$ and $\zeta = 1$. Thus Lemma B.2 ensures that the Aldous condition [A] is satisfied by the sequences $(\mathbf{v}_n)_{n \in \mathbb{N}}$ and $(\tau_n)_{n \in \mathbb{N}}$ in the space $\mathcal{U}'$ and $\tilde{\mathcal{U}}'$ respectively. Thus the proof is complete. $\quad\square$

### 5.2. Construction of new probability space and processes

Lemma 5.1 ensures the tightness of $\{\mathcal{L}(\mathbf{v}_n, \tau_n),\, n \in \mathbb{N}\}$ on the space $\mathcal{Z}$, which is a non-metric locally convex space. The set of measures $\{\mathcal{L}(N_1^n, N_2^n)\}$ is tight on $M_{\bar{\mathbb{N}}}([0, T] \times Z) \times M_{\bar{\mathbb{N}}}([0, T] \times B)$ where $N_i^n := N_i$ for $i = 1, 2$. Thus the set $\{\mathcal{L}(\mathbf{v}_n, \tau_n, N_1^n, N_2^n),\, n \in \mathbb{N}\}$ is tight on $\mathcal{Z} \times M_{\bar{\mathbb{N}}}([0, T] \times Z) \times M_{\bar{\mathbb{N}}}([0, T] \times B)$. By the Skorokhod-Jakubowski theorem (see Proposition B.4 and Corollary B.5, there exists a subsequence $(n_k)_{k \in \mathbb{N}}$, a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ and on this space, $\mathcal{Z} \times M_{\bar{\mathbb{N}}}([0, T] \times Z) \times M_{\bar{\mathbb{N}}}([0, T] \times B)$-valued random variables $(\mathbf{v}_*, \tau_*, N_1^*, N_2^*), (\bar{\mathbf{v}}_k, \bar{\tau}_k, \bar{N}_1^k, \bar{N}_2^k), k \in \mathbb{N}$ such that

- $\mathcal{L}\big((\bar{\mathbf{v}}_k, \bar{\tau}_k, \bar{N}_1^k, \bar{N}_2^k)\big) = \mathcal{L}\big((\mathbf{v}_{n_k}, \tau_{n_k}, N_1^{n_k}, N_2^{n_k})\big)$ for all $k \in \mathbb{N}$;
- $(\bar{\mathbf{v}}_k, \bar{\tau}_k, \bar{N}_1^k, \bar{N}_2^k) \to (\mathbf{v}_*, \tau_*, N_1^*, N_2^*)$ in $\mathcal{Z} \times M_{\bar{\mathbb{N}}}([0, T] \times Z) \times M_{\bar{\mathbb{N}}}([0, T] \times B)$ with probability 1 on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ as $k \to \infty$;
- $(\bar{N}_1^k(\bar{\omega}), \bar{N}_2^k(\bar{\omega})) = (N_1^*(\bar{\omega}), N_2^*(\bar{\omega}))$ for all $\bar{\omega} \in \bar{\Omega}$.

We will denote these sequences again by $\big((\mathbf{v}_n, \tau_n, N_1^n, N_2^n)\big)_{n \in \mathbb{N}}$ and $\big((\bar{\mathbf{v}}_n, \bar{\tau}_n, \bar{N}_1^n, \bar{N}_2^n)\big)_{n \in \mathbb{N}}$.

Using the definition of $\mathcal{Z}$, we have $\bar{\mathbb{P}}$-a.s.

$$\bar{\mathbf{v}}_n \to \mathbf{v}_* \text{ in } \mathrm{L}_w^2(0, T; V) \cap \mathrm{L}^2(0, T; H_{loc}) \cap D([0, T]; \mathcal{U}') \cap D([0, T]; H_w) \tag{5.10}$$

$$\text{and} \quad \bar{\tau}_n \to \tau_* \text{ in } \mathrm{L}_w^2(0, T; \tilde{V}) \cap \mathrm{L}^2(0, T; \tilde{H}_{loc}) \cap D([0, T]; \tilde{\mathcal{U}}') \cap D([0, T]; \tilde{H}_w). \tag{5.11}$$

### 5.3. Properties of the new processes and the limiting processes

It is easy to verify that the spaces $\mathcal{Z}_i, i = 1, 2$ are not a Polish spaces. So the following result cannot be deduced directly from the Kuratowski theorem [42]. We adopt here a method from Lemma 4.2 in [14], and generalise for Skorokhod spaces.

**Proposition 5.2.** *The set $D([0, T]; H_n) \cap \mathcal{Z}_1$ is a Borel subset of $\mathcal{Z}_1$ and the corresponding embedding transforms Borel sets into Borel subsets. A similar result is true for $\mathcal{Z}_2$.*

**Proof.** The space $D([0, T]; \mathcal{U}') \cap \mathrm{L}^2(0, T; H)$ is a Polish space. Then by Kuratowski theorem, $D([0, T]; H_n)$ is a Borel subset of $D([0, T]; \mathcal{U}') \cap \mathrm{L}^2(0, T; H)$. Hence $D([0, T]; H_n) \cap \mathcal{Z}_1$ is a Borel subset of $D([0, T]; \mathcal{U}') \cap \mathrm{L}^2(0, T; H) \cap \mathcal{Z}_1$, which happens to be equal to $\mathcal{Z}_1$. $\quad\square$

The above result leads us to the following conclusion:

**Corollary 5.3.** *$\bar{\mathbf{v}}_n$ and $\bar{\tau}_n$ take values in $H_n$ and $\tilde{H}_n$ respectively. The laws of $\mathbf{v}_n$ and $\bar{\mathbf{v}}_n$ are equal on $D([0, T]; H_n)$ and the laws of $\tau_n$ and $\bar{\tau}_n$ are equal on $D([0, T]; \tilde{H}_n)$.*

In view of the above, it is straightforward to show that the sequence $(\bar{\mathbf{v}}_n)_{n \in \mathbb{N}}$ satisfies the same estimates as the original sequence $(\mathbf{v}_n)_{n \in \mathbb{N}}$. In particular, for any $p \geq 2$, we have

$$\sup_{n \in \mathbb{N}} \bar{\mathbb{E}}\Big[\sup_{s \in [0,T]} |\bar{\mathbf{v}}_n(s)|_H^p\Big] \leq C_1(p), \quad \sup_{n \in \mathbb{N}} \bar{\mathbb{E}}\Big[\int_0^T |\bar{\mathbf{v}}_n(s)|_V^2 ds\Big] \leq C_2. \tag{5.12}$$

By (5.12), $\bar{\mathbf{v}}_n$ is uniformly bounded in the space $L^2(\bar{\Omega}; L^2(0, T; V))$, hence there exists a weak convergent subsequence, still denoted by $\bar{\mathbf{v}}_n$ such that $\bar{\mathbf{v}}_n \xrightarrow{w} \mathbf{v}_*$ in $L^2([0, T] \times \bar{\Omega}; V)$ (using (5.10)) and $\mathbf{v}_* \in L^2([0, T] \times \bar{\Omega}; V)$. Arguing as above, inequality (5.12) with $p := 2$ we can further extract a subsequence of $\bar{\mathbf{v}}_n$ such that $\bar{\mathbf{v}}_n \xrightarrow{w^*} \mathbf{v}_*$ in $L^2(\bar{\Omega}; L^\infty(0, T; H))$ (using (5.10)). Similarly, using (5.11) and arguing in similar fashion for $\mathbf{v}_n$, for any $p \geq 2$, we have

$$\sup_{n\in\mathbb{N}} \bar{\mathbb{E}}\left[ \sup_{s\in[0,T]} |\bar{\tau}_n(s)|^p_{\tilde{H}} \right] \leq C_1(p), \quad \sup_{n\in\mathbb{N}} \bar{\mathbb{E}}\left[ \int_0^T |\bar{\tau}_n(s)|^2_{\tilde{V}} ds \right] \leq C_2.$$

**Proposition 5.4.** *Let $\mathbf{v}_*$ and $\tau_*$ be the limiting processes defined above. Then for every $r \geq 2$, $T > 0$ there exists $C > 0$, depending on $T$, $r$ and on the parameters of the problem (which is independent of n) such that*

$$\mathbb{E}\left[ \sup_{s\in[0,T]} |\mathbf{v}_*(s)|^r_H \right] \leq C_r, \qquad \mathbb{E}\left[ \sup_{s\in[0,T]} |\tau_*(s)|^r_{\tilde{H}} \right] \leq C_r. \tag{5.13}$$

**Proof.** We now prove estimate of $\mathbf{v}_*$. Estimate of $\tau_*$ can be followed in similar lines.

By (5.12), we have that $\{\bar{\mathbf{v}}_n\}_{n\geq 1}$ is uniformly bounded in $L^r(\bar{\Omega}; L^\infty(0, T; H))$ for $r \geq 2$. Since $L^r(\bar{\Omega}; L^\infty(0, T; H))$ is isomorphic to the space $(L^{\frac{r}{r-1}}(\bar{\Omega}; L^1(0, T; H)))^*$, by Banach Alaoglu Theorem, there exists a subsequence, still denoted by $(\bar{\mathbf{v}}_n)_{n\in\mathbb{N}}$, and $\mathbf{v} \in L^r(\bar{\Omega}; L^\infty(0, T; H))$ such that $\bar{\mathbf{v}}_n \xrightarrow{w^*} \mathbf{u}$ in $L^r(\bar{\Omega}; L^\infty(0, T; H))$, i.e.,

$$\bar{\mathbb{E}}\left[ \int_0^T (\bar{\mathbf{v}}_n(t, \omega), \varphi(t, \omega))_H dt \right] \rightarrow \bar{\mathbb{E}}\left[ \int_0^T (\mathbf{u}(t, \omega), \varphi(t, \omega))_H dt \right] \quad \forall \varphi \in L^{\frac{r}{r-1}}(\bar{\Omega}; L^1(0, T; H)).$$

Also as $\bar{\mathbf{v}}_n \xrightarrow{w} \mathbf{v}_*$ in $L^2(\bar{\Omega}; L^2(0, T; V))$, we have

$$\bar{\mathbb{E}}\left[ \int_0^T \langle \bar{\mathbf{v}}_n(t, \omega), \varphi(t, \omega) \rangle dt \right] \rightarrow \bar{\mathbb{E}}\left[ \int_0^T \langle \mathbf{v}_*(t, \omega), \varphi(t, \omega) \rangle dt \right] \quad \forall \varphi \in L^2(\bar{\Omega}; L^2(0, T; V')).$$

Exploiting the Gelfand triple $V \subset H \subset V'$ we therefore have,

$$\bar{\mathbb{E}}\left[ \int_0^T (\bar{\mathbf{v}}_n(t, \omega), \varphi(t, \omega))_H dt \right] \rightarrow \bar{\mathbb{E}}\left[ \int_0^T (\mathbf{v}_*(t, \omega), \varphi(t, \omega))_H dt \right] \quad \forall \varphi \in L^2(\bar{\Omega}; L^2(0, T; H)).$$

For $r \geq 2$, $L^2(\bar{\Omega}; L^2(0, T; H))$ is a dense subspace of $L^{\frac{r}{r-1}}(\bar{\Omega}; L^1(0, T; H))$. Hence we have

$$\bar{\mathbb{E}}\left[ \int_0^T (\mathbf{u}(t, \omega), \varphi(t, \omega))_H dt \right] = \bar{\mathbb{E}}\left[ \int_0^T (\mathbf{v}_*(t, \omega), \varphi(t, \omega))_H dt \right] \quad \forall \varphi \in L^2(\bar{\Omega}; L^2(0, T; H)).$$

Thus we have, $\mathbf{u}(t, \omega) = \mathbf{v}_*(t, \omega)$ for *Leb*-almost all $t \in [0, T]$ and $\bar{\mathbb{P}}$-almost all $\omega \in \bar{\Omega}$. Since $\mathbf{u} \in$ $\mathrm{L}^r(\bar{\Omega}; \mathrm{L}^\infty(0, T; H))$, we infer $\mathbf{v}_* \in \mathrm{L}^r(\bar{\Omega}; \mathrm{L}^\infty(0, T; H))$, i.e., $\bar{\mathbb{E}}\left[\sup_{s \in [0,T]} |\mathbf{v}_*(s)|^r\right] \leq C_r$, for some constant $C_r$ depending on $T$, $r$ and on the parameters of the problem (which is independent of $n$). $\quad\square$

### 5.4. *Convergence of the new processes to the corresponding limiting processes*

Let us define the following functionals for $t \in [0, T]$, and for all $\phi \in \mathcal{U}$, $\psi \in \tilde{\mathcal{U}}$,

$$\mathfrak{S}_n(\bar{\mathbf{v}}_n, \bar{\tau}_n, \bar{N}_1^n, \phi)(t) := \left(\bar{\mathbf{v}}_n(0), \phi\right)_H - \nu \int_0^t \left\langle \mathcal{A}\bar{\mathbf{v}}_n(s), \phi\right\rangle ds - \int_0^t \left\langle B_n(\bar{\mathbf{v}}_n(s)), \phi\right\rangle ds$$

$$+ \nu_1 \int_0^t \left\langle Div\, \bar{\tau}_n(s), \phi\right\rangle ds + \int_0^t \int_Z \left(F_n(\bar{\mathbf{v}}_n(s), z), \phi\right)_H \tilde{\bar{N}}_1(ds, dz),$$

$$(5.14)$$

$$\Lambda_n(\bar{\mathbf{v}}_n, \bar{\tau}_n, \bar{N}_2^n, \psi)(t) := \left(\bar{\tau}_n(0), \psi\right)_{\tilde{H}} - \kappa \int_0^t \left\langle \tilde{\mathcal{A}}\bar{\tau}_n(s), \psi\right\rangle ds - \int_0^t \left\langle \tilde{B}_n(\bar{\mathbf{v}}_n(s), \bar{\tau}_n(s)), \psi\right\rangle ds$$

$$- a \int_0^t \left\langle \bar{\tau}_n(s)), \psi\right\rangle ds + \nu_2 \int_0^t \left\langle \mathcal{D}(\bar{\mathbf{v}}_n(s)), \psi\right\rangle ds$$

$$+ \int_0^t \int_B \left(G_n(l, \bar{\tau}_n(s)), \psi\right)_{\tilde{H}} \tilde{\bar{N}}_2^n(ds, dl)$$

$$+ \int_0^t \left\langle b_n(\bar{\tau}_n(s)), \psi\right\rangle ds,$$

$$(5.15)$$

$$\mathfrak{S}(\mathbf{v}_*, \tau_*, N_1^*, \phi)(t) := \left(\mathbf{v}_*(0), \phi\right)_H - \nu \int_0^t \left\langle \mathcal{A}\mathbf{v}_*(s), \phi\right\rangle ds - \int_0^t \left\langle B(\mathbf{v}_*(s)), \phi\right\rangle ds$$

$$+ \nu_1 \int_0^t \left\langle Div\, \tau_*(s), \phi\right\rangle ds + \int_0^t \int_Z \left(F(\mathbf{v}_*(s), z), \phi\right)_H \tilde{N}_1^*(ds, dz),$$

$$(5.16)$$

$$\Lambda(\mathbf{v}_*, \tau_*, N_2^*, \psi)(t) := \left(\tau_*(0), \psi\right)_{\tilde{H}} - \kappa \int_0^t \left\langle \tilde{\mathcal{A}}\tau_*(s), \psi\right\rangle ds - \int_0^t \left\langle \tilde{B}(\mathbf{v}_*(s), \tau_*(s)), \psi\right\rangle ds$$

$$- a \int_0^t \left\langle \tau_*(s)), \psi\right\rangle ds + \nu_2 \int_0^t \left\langle \mathcal{D}(\mathbf{v}_*(s)), \psi\right\rangle ds$$

$$+ \int\limits_0^t \int\limits_B \big( G_n(l, \tau_*(s)), \psi \big)_{\tilde{H}} \, \tilde{N}_2^*(ds, dl) + \int\limits_0^t \big\langle b(\tau_*(s)), \psi \big\rangle ds, \quad (5.17)$$

**Proposition 5.5.** *If $\mathfrak{S}_n$, $\Lambda_n$, $\mathfrak{S}$, $\Lambda$ are defined by* (5.14), (5.15), (5.16) *and* (5.17) *respectively, then we have the following convergences:*

1. $\displaystyle \lim_{n\to\infty} \| \mathfrak{S}_n(\bar{\mathbf{v}}_n, \bar{\tau}_n, \bar{N}_1^n, \phi) - \mathfrak{S}(\mathbf{v}_*, \tau_*, N_1^*, \phi) \|_{\mathrm{L}^2([0,T]\times\bar{\Omega})} = 0,$        (5.18)

2. $\displaystyle \lim_{n\to\infty} \| \Lambda_n(\bar{\mathbf{v}}_n, \bar{\tau}_n, \bar{N}_2^n, \phi) - \Lambda(\mathbf{v}_*, \tau_*, N_2^*, \phi) \|_{\mathrm{L}^2([0,T]\times\bar{\Omega})} = 0.$        (5.19)

**Proof.** We now see using Fubini's Theorem that

$$\| \mathfrak{S}_n(\bar{\mathbf{v}}_n, \bar{\tau}_n, \bar{N}_1^n, \phi) - \mathfrak{S}(\mathbf{v}_*, \tau_*, N_1^*, \phi) \|_{\mathrm{L}^2([0,T]\times\bar{\Omega})}^2$$

$$= \int\limits_0^T \int\limits_{\bar{\Omega}} | \mathfrak{S}_n(\bar{\mathbf{v}}_n, \bar{\tau}_n, \bar{N}_1^n, \phi)(t) - \mathfrak{S}(\mathbf{v}_*, \tau_*, N_1^*, \phi)(t) |^2 \, d\bar{\mathbb{P}}(\omega) \, dt$$

$$= \int\limits_0^T \bar{\mathbb{E}} \big[ | \mathfrak{S}_n(\bar{\mathbf{v}}_n, \bar{\tau}_n, \bar{N}_1^n, \phi)(t) - \mathfrak{S}(\mathbf{v}_*, \tau_*, N_1^*, \phi)(t) |^2 \big] \, dt. \quad (5.20)$$

Lemma 5.6 (the following Lemma) ensures that each term on the right hand side of (5.14) converges to the right hand side of corresponding term in (5.16) in $L^2([0,T]\times\bar{\Omega})$ which further assures that right hand side of (5.20) goes to zero as $n\to\infty$. This verifies (5.18). Lemma 5.7 (which is proven below) ensures that each term on the right hand side of (5.15) converges to the right hand side of corresponding term in (5.17) in $\mathrm{L}^2([0,T]\times\bar{\Omega})$ which further verifies (5.19). This completes the proof. □

**Lemma 5.6.** *For all $\phi \in \mathcal{U}$,*

(i) $\lim_{n\to\infty} \bar{\mathbb{E}} \big[ \int_0^T \big| (\bar{\mathbf{v}}_n(t) - \mathbf{v}_*(t), \phi)_H \big|^2 \, dt \big] = 0,$

(ii) $\lim_{n\to\infty} \int_0^T \bar{\mathbb{E}} \big[ | (\bar{\mathbf{v}}_n(0) - \mathbf{v}_*(0), \phi)_H |^2 \big] dt = 0,$

(iii) $\lim_{n\to\infty} \int_0^T \bar{\mathbb{E}} \big[ | \int_0^t \langle \mathcal{A}\bar{\mathbf{v}}_n(s) - \mathcal{A}\mathbf{v}_*(s), \phi \rangle \, ds |^2 \big] dt = 0,$

(iv) $\lim_{n\to\infty} \int_0^T \bar{\mathbb{E}} \big[ | \int_0^t \big\langle B_n\big(\bar{\mathbf{v}}_n(s)\big) - B\big(\mathbf{v}_*(s)\big), \phi \big\rangle ds |^2 \big] dt = 0,$

(v) $\lim_{n\to\infty} \int_0^T \bar{\mathbb{E}} \big[ | \int_0^t \big\langle Div\, \bar{\tau}_n(s) - Div\, \tau_*(s), \phi \big\rangle ds |^2 \big] dt = 0,$

(vi) $\lim_{n\to\infty} \int_0^T \bar{\mathbb{E}} \big[ | \int_0^t \int_z \big\langle F_n\big(\bar{\mathbf{v}}_n(s), z\big) - F\big(\mathbf{v}_*(s), z\big), \phi \big\rangle \lambda_1(dz) ds |^2 \big] dt = 0,$

(vii) $\lim_{n\to\infty} \int_0^T \bar{\mathbb{E}} \big[ | \int_0^t \int_z \big\langle F_n\big(\bar{\mathbf{v}}_n(s), z\big) - F\big(\mathbf{v}_*(s), z\big), \phi \big\rangle \tilde{N}_1^*(ds, dz) |^2 \big] dt = 0.$

**Proof.** Let $\phi \in \mathcal{U}$ be fixed.

(i) Owing to (5.10) we have $\bar{\mathbf{v}}_n \to \mathbf{v}_*$ in $D([0,T]; H_w)$, $\bar{\mathbb{P}}$ − a.s. Hence, in particular for almost all $t \in [0,T]$, $\lim_{n\to\infty} (\bar{\mathbf{v}}_n(t), \phi)_H = (\mathbf{v}_*(t), \phi)_H$ $\bar{\mathbb{P}}$ − a.s. Also by (5.12), $\sup_{t\in[0,T]} |\bar{\mathbf{v}}_n(t)|^2 < \infty$, and so by employing Vitali Theorem we have

$$\lim_{n\to\infty} \|(\bar{\mathbf{v}}_n, \phi)_H - (\mathbf{v}_*, \phi)_H\|^2_{\mathrm{L}^2([0,T]\times\bar{\Omega})} = \lim_{n\to\infty} \bar{\mathbb{E}}\Big[\int_0^T |(\bar{\mathbf{v}}_n(t) - \mathbf{v}_*(t), \phi)_H|^2 dt\Big] = 0.$$

(ii) By (5.10) we have $\bar{\mathbf{v}}_n \to \mathbf{v}_*$ in $D([0,T]; H_w)$, $\bar{\mathbb{P}}$-a.s. and $\mathbf{v}_*$ is right continuous at $t = 0$, we infer that

$$(\bar{\mathbf{v}}_n(0), \phi)_H \to (\mathbf{v}_*(0), \phi)_H \quad \bar{\mathbb{P}} - \text{-a.s.}$$

(5.12) and Vitali Theorem gives $\lim_{n\to\infty} \bar{\mathbb{E}}[|(\bar{\mathbf{v}}_n(0) - \mathbf{v}_*(0), \phi)_H|^2] = 0$. Hence

$$\lim_{n\to\infty} \|(\bar{\mathbf{v}}_n(0) - \mathbf{v}_*(0), \phi)_H\|^2_{\mathrm{L}^2([0,T]\times\bar{\Omega})} = 0.$$

(iii) Since $\bar{\mathbf{v}}_n \to \mathbf{v}_*$ in $\mathrm{L}^2_w(0,T;V)$, $\bar{\mathbb{P}}$-a.s. (by (5.10)) so for any $\tilde{\phi} \in \mathrm{L}^2(0,T;V)$ we have

$$\lim_{n\to\infty} \int_0^T \langle \mathcal{A}\bar{\mathbf{v}}_n(s) - \mathcal{A}\mathbf{v}_*(s), \tilde{\phi}(s)\rangle ds = 0. \tag{5.21}$$

Let $\phi \in \mathcal{U}$. Let $t \in [0,T]$ be fixed. We choose $\tilde{\phi}(s) = \chi_{(0,t)}(s)\phi$ and note that $\tilde{\phi} \in \mathrm{L}^2(0,T;V)$, indeed,

$$\int_0^T |\tilde{\phi}(s)|^2_V ds \leq C \int_0^T \chi_{(0,t)}(s)|\phi|^2_{\mathcal{U}} ds \leq C|\phi|^2_{\mathcal{U}} T < \infty, \quad \bar{\mathbb{P}} - \text{a.s.}$$

Hence, using (5.21) we have

$$0 = \lim_{n\to\infty} \int_0^T \langle \mathcal{A}\bar{\mathbf{v}}_n(s) - \mathcal{A}\mathbf{v}_*(s), \tilde{\phi}(s)\rangle ds = \lim_{n\to\infty} \int_0^t \langle \mathcal{A}\bar{\mathbf{v}}_n(s) - A\mathbf{v}_*(s), \phi\rangle ds. \tag{5.22}$$

Hence by Hölder's inequality and (5.12) we achieve for all $t \in [0,T]$, and $n \in \mathbb{N}$. Let $r > 2$.

$$\bar{\mathbb{E}}\Big[|\int_0^t \langle \mathcal{A}\bar{\mathbf{v}}_n(s), \phi\rangle ds|^r\Big] \leq \bar{\mathbb{E}}\Big[\int_0^t |(\bar{\mathbf{v}}_n(s), \phi)_V|^r ds\Big] \leq 2^r T^r \|\phi\|^r_{\mathcal{U}} \bar{\mathbb{E}}\Big[\sup_{0\leq s\leq t} |\bar{\mathbf{v}}_n(s)|^r_V\Big] \leq c_r,$$

$$\tag{5.23}$$

for some positive constant $c_r$ (depending on $r$). Therefore using (5.22) and (5.23) and by Vitali Theorem, we conclude for all $t \in [0,T]$

$$\lim_{n\to\infty} \bar{\mathbb{E}}\Big[|\int_0^t \langle \mathcal{A}\bar{\mathbf{v}}_n(s) - \mathcal{A}\mathbf{v}_*(s), \phi\rangle ds|^2\Big] = 0. \tag{5.24}$$

Hence (5.23), (5.24) and the Dominated Convergence Theorem gives (iii).

(iv) Let $\phi \in \mathcal{U}$. We have, $B(\bar{\mathbf{v}}_n) - B(\mathbf{v}_*) := B(\bar{\mathbf{v}}_n, \bar{\mathbf{v}}_n) - B(\mathbf{v}_*, \mathbf{v}_*) = B(\bar{\mathbf{v}}_n - \mathbf{v}_*, \bar{\mathbf{v}}_n) + B(\mathbf{v}_*, \bar{\mathbf{v}}_n - \mathbf{v}_*)$. Now, using Hölder's inequality, we obtain

$$
\left| \int_0^t \langle B(\bar{\mathbf{v}}_n(s), \bar{\mathbf{v}}_n(s)), \phi \rangle ds - \int_0^t \langle B(\mathbf{v}_*(s), \mathbf{v}_*(s)), \phi \rangle ds \right|
$$

$$
\leq C \|\bar{\mathbf{v}}_n - \mathbf{v}_*\|_{L^2(0,T;H)} \left( \|\bar{\mathbf{v}}_n\|_{L^2(0,T;H)} + \|\mathbf{v}_*\|_{L^2(0,T;H)} \right) |\phi|_{\mathcal{U}},
$$

where $C > 0$ is a constant. Since $\bar{\mathbf{v}}_n \to \mathbf{v}_*$ in $L^2(0, T; H)$, we have

$$
\lim_{n \to \infty} \int_0^t \langle B(\bar{\mathbf{v}}_n(s)) - B(\mathbf{v}_*(s)), \phi \rangle ds = 0 \quad \bar{\mathbb{P}} - \text{a.s.}
$$

For every $\phi \in \mathcal{U}$, by (c) of second part of Lemma 4.1, we have $P_n \phi \to \phi$ in $V_s$. Since $\mathcal{U} \subset V_1$, we conclude that for all $\phi \in \mathcal{U}$ and all $t \in [0, T]$,

$$
\lim_{n \to \infty} \int_0^t \langle B_n(\bar{\mathbf{v}}_n(s)) - B(\mathbf{v}_*(s)), \phi \rangle ds = 0 \quad \bar{\mathbb{P}} - \text{a.s.} \tag{5.25}
$$

Using Hölder's inequality and (5.12) we obtain for all $t \in [0, T]$, $r > 2$ and $n \in \mathbb{N}$,

$$
\bar{\mathbb{E}} \left[ \left| \int_0^t \langle B_n(\bar{\mathbf{v}}_n(s)), \phi \rangle ds \right|^r \right] \leq C |\phi|_{\mathcal{U}}^r t^{r-1} \bar{\mathbb{E}} \left[ \int_0^t |\bar{\mathbf{v}}_n(s)|_H^{2r} ds \right] \leq C C_1(r), \tag{5.26}
$$

for some constant $C > 0$. Considering (5.25) and (5.26) and by the Vitali Theorem we achieve for all $t \in [0, T]$

$$
\lim_{n \to \infty} \bar{\mathbb{E}} \left[ \left| \int_0^t \langle B_n(\bar{\mathbf{v}}_n(s)) - B(\mathbf{v}_*(s)), \phi \rangle ds \right|^2 \right] = 0. \tag{5.27}
$$

Hence, in view of (5.26), (5.27) and the Dominated Convergence Theorem, we infer that

$$
\lim_{n \to \infty} \int_0^T \bar{\mathbb{E}} \left[ \left| \int_0^t \langle B_n(\bar{\mathbf{v}}_n(s)) - B(\mathbf{v}_*(s)), \phi \rangle ds \right|^2 \right] dt = 0. \tag{5.28}
$$

Hence we have (iv).

(v) As $\phi \in \tilde{\mathcal{U}}$, so $\nabla \phi \in \tilde{H}$. Now using (5.11) we have

$$\left| \int_0^t \langle Div \, \bar{\tau}_n(s) - Div \, \tau_*(s), \phi \rangle ds \right|^2 \leq |\nabla \phi|_{\tilde{H}}^2 \, T \left( \int_0^T |\bar{\tau}_n(s) - \tau_*(s)|_{\tilde{H}}^2 ds \right) \to 0 \quad \text{as } n \to \infty.$$

(5.29)

Let $r \geq 2$. Now using (5.11) and Proposition 5.4 we have

$$\int_0^T \bar{\mathbb{E}} \left[ \left| \int_0^t \langle Div \, \bar{\tau}_n(s) - Div \, \tau_*(s), \phi \rangle ds \right|^r \right]$$

$$\leq \int_0^T \bar{\mathbb{E}} \left[ \left( \int_0^t |\bar{\tau}_n(s) - \tau_*(s)|_{\tilde{H}} |\nabla \phi|_{\tilde{H}} ds \right)^r \right] dt \leq C_r,$$

(5.30)

for some positive constant $C_r$. Hence by Vitali Theorem, using (5.29) and (5.30) we have (v).

(vi) Let us consider $\phi \in \mathcal{U}$. Using Assumption 2.4 and $\bar{\mathbf{v}}_n \to \mathbf{v}_*$ in $\mathrm{L}^2(0, T; H)$, $\bar{\mathbb{P}} - $ a.s. we have

$$\int_0^t \int_Z \left| (F(\bar{\mathbf{v}}_n(s), z) - F(\mathbf{v}_*(s), z), \phi)_H \right|^2 \lambda_1(dz) ds$$

$$\leq CL \, |\phi|_{\mathcal{U}}^2 \int_0^t |\bar{\mathbf{v}}_n(s) - \mathbf{v}_*(s)|_H^2 ds \to 0 \quad \text{as} \quad n \to \infty.$$

(5.31)

Furthermore, using Assumption 2.4 and (5.12) for every $t \in [0, T]$, $r > 2$ and $n \in \mathbb{N}$, we have the following inequality

$$\bar{\mathbb{E}} \left[ \int_0^t \int_Z \left| (F(\bar{\mathbf{v}}_n(s), z) - F(\mathbf{v}_*(s), z), \phi)_H \right|^2 \lambda_1(dz) ds |^r \right] \leq \tilde{C}_r,$$

(5.32)

for some constant $\tilde{C}_r > 0$. Hence by (5.31), (5.32) and by the Vitali Theorem we infer that for all $t \in [0, T]$, $\forall \, \phi \in \mathcal{U}$.

$$\lim_{n \to \infty} \bar{\mathbb{E}} \left[ \int_0^t \int_Z \left| (F(\bar{\mathbf{v}}_n(s), z) - F(\mathbf{v}_*(s), z), \phi)_H \right|^2 \lambda_1(dz) ds \right] = 0.$$

Since the restriction of $P_n$ to the space $H$ is the $(\cdot, \cdot)_H$-projection onto $H_n$, we conclude that

$$\lim_{n\to\infty} \bar{\mathbb{E}}\left[\int_0^t \int_Z \left|(F_n(\bar{\mathbf{v}}_n(s), z) - F(\mathbf{v}_*(s), z), \phi)_H\right|^2 \lambda_1(dz)ds\right] = 0, \quad \forall\, \phi \in H.$$

(5.33)

Since $\mathcal{U} \subset H$, (5.33) holds for all $\phi \in \mathcal{U}$.

Moreover, Assumption 2.4 and (5.12) yield the following inequality

$$\bar{\mathbb{E}}\left[\int_0^t \int_Z \left|(F_n(\bar{\mathbf{v}}_n(s), z) - F(\mathbf{v}_*(s), z), \phi)_H\right|^2 \lambda_1(dz)ds\right] \leq \tilde{C}_2.$$

(5.34)

Now (5.33), (5.34) and the Dominated Convergence Theorem assures assertion (vi).

(vii) Using Itô-Lévy isometry, and the fact that $\bar{N}_i^n = N_i^*$ for $i = 1, 2$, we have

$$\bar{\mathbb{E}}\left[\left|\int_0^t \int_Z (F_n(\bar{\mathbf{v}}_n(s), z) - F(\mathbf{v}_*(s), z), \phi)_H \tilde{N}_1^*(ds, dz)\right|^2\right]$$

$$= \bar{\mathbb{E}}\left[\int_0^t \int_Z \left|(F_n(\bar{\mathbf{v}}_n(s), z) - F(\mathbf{v}_*(s), z), \phi)_H\right|^2 \lambda_1(dz)ds\right].$$

(5.35)

Thus $\forall\, \phi \in \mathcal{U}$, combining (5.33), (5.34), (5.35) and then exploiting Dominated Convergence Theorem we ensure assertion (vii). $\quad\square$

**Lemma 5.7.** *For all $\psi \in \tilde{\mathcal{U}}$,*

(i) $\lim_{n\to\infty} \bar{\mathbb{E}}\left[\int_0^T \left|(\bar{\tau}_n(t) - \tau_*(t), \psi)_{\tilde{H}}\right|^2 dt\right] = 0$,

(ii) $\lim_{n\to\infty} \int_0^T \bar{\mathbb{E}}\left[\left|(\bar{\tau}_n(0) - \tau_*(0), \psi)_{\tilde{H}}\right|^2\right] dt = 0$,

(iii) $\lim_{n\to\infty} \int_0^T \bar{\mathbb{E}}\left[\left|\int_0^t \langle \tilde{A}\bar{\tau}_n(s) - \tilde{A}\tau_*(s), \psi\rangle\, ds\right|^2\right] dt = 0$,

(iv) $\lim_{n\to\infty} \int_0^T \bar{\mathbb{E}}\left[\left|\int_0^t \langle \tilde{B}_n(\bar{\mathbf{v}}_n(s), \bar{\tau}_n(s)) - \tilde{B}(\mathbf{v}_*(s), \tau_*(s)), \psi\rangle ds\right|^2\right] dt = 0$,

(v) $\lim_{n\to\infty} \int_0^T \bar{\mathbb{E}}\left[\left|\int_0^t \langle \mathcal{D}(\bar{\mathbf{v}}_n(s)) - \mathcal{D}(\mathbf{v}_*(s)), \psi\rangle ds\right|^2\right] dt = 0$,

(vi) $\lim_{n\to\infty} \int_0^T \bar{\mathbb{E}}\left[\left|\int_0^t \int_B \langle G_n(l, \bar{\tau}_n(s)) - G(l, \tau_*(s)), \psi\rangle \tilde{N}_2^*(dl, ds)\right|^2 dt\right] = 0$,

(vii) $\lim_{n\to\infty} \int_0^T \bar{\mathbb{E}}\left[\left|\int_0^t \langle b_n(\bar{\tau}_n(s)) - b(\tau_*(s)), \psi\rangle ds\right|^2\right] dt = 0$.

**Proof.** Proofs of (i)-(iv) are similar to the proofs of Lemma 5.6 (i)-(iv), and hence omitted.

(v) Let $\psi \in \tilde{\mathcal{U}}$. Now using integration by parts we note that

$$\langle \mathcal{D}\bar{\mathbf{v}}_n(s) - \mathcal{D}\mathbf{v}_*(s), \psi\rangle = -\frac{1}{2}\langle \bar{\mathbf{v}}_n(s) - \mathbf{v}_*(s), Div\,(\psi + \psi^T)\rangle.$$

(5.36)

So using (5.36) and (5.10) we have

$$\left| \int_0^t \langle \mathcal{D}(\bar{\mathbf{v}}_n(s)) - \mathcal{D}(\mathbf{v}_*(s)), \psi \rangle ds \right|^2 \le \frac{T}{2} |\psi|_{\tilde{V}}^2 \left( \int_0^T |\bar{\mathbf{v}}_n(s) - \mathbf{v}_*(s)|_H^2 ds \right) \to 0 \quad \text{as} \quad n \to \infty.$$

(5.37)

Let $r \ge 2$. Now using (5.10) and Proposition 5.4 we have

$$\int_0^T \bar{\mathbb{E}}\left[ \left| \int_0^t \langle \mathcal{D}(\bar{\mathbf{v}}_n(s)) - \mathcal{D}(\mathbf{v}_*(s)), \psi \rangle ds \right|^r \right] dt$$

$$\le |\psi|_{\tilde{V}}^r 2^r T^{r+1} \bar{\mathbb{E}}\left[ \sup_{0 \le s \le T} |\bar{\mathbf{v}}_n(s) - \mathbf{v}_*(s)|_H^r \right] \le C_r,$$

(5.38)

for some positive constant $C_r$ depending on $r$. Hence by Vitali Theorem, using (5.37) and (5.38) we ensure assertion (v).

(vi) Using Lipschitz property of $G$ and (5.11), we obtain for all $\psi \in \tilde{H}$, for all $t \in [0, T]$, $\bar{\mathbb{P}}$-a.s.

$$\int_0^t \int_B \left| \left( G(l, \bar{\tau}_n(s)) - G(l, \tau_*(s)), \psi \right)_{\tilde{H}} \right|^2 \lambda_2(dl)ds$$

$$\le C |\psi|_{\tilde{H}}^2 \int_0^t \left| \bar{\tau}_n(s) - \tau_*(s) \right|_{\tilde{H}}^2 ds \to 0 \quad \text{as} \quad n \to \infty.$$

(5.39)

Moreover, from (5.13), for every $t \in [0, T]$, every $r \ge 1$ and every $n \in \mathbb{N}$,

$$\bar{\mathbb{E}}\left[ \left| \int_0^t \int_B \left| \left( G(l, \bar{\tau}_n(s)) - G(l, \tau_*(s)), \psi \right)_{\tilde{H}} \right|^2 \lambda_2(dl)ds \right|^r \right]$$

$$\le C \left( \bar{\mathbb{E}}\left[ \int_0^t |\bar{\tau}_n(s)|_{\tilde{H}}^{2r} \right] + \bar{\mathbb{E}}\left[ \int_0^t |\tau_*(s)|_{\tilde{H}}^{2r} \right] \right) \le C.$$

(5.40)

Then by (5.39), (5.40) and by Vitali's Theorem, for all $t \in [0, T]$, $\forall \, \psi \in \tilde{H}$

$$\lim_{n \to \infty} \bar{\mathbb{E}}\left[ \int_0^t \int_B \left| \left( G(l, \bar{\tau}_n(s)) - G(l, \tau_*(s)), \psi \right)_{\tilde{H}} \right|^2 \lambda_2(dl)ds \right] = 0.$$

Since the restriction of $\tilde{P}_n$ to the space $\tilde{H}$ is the $(\cdot, \cdot)_{\tilde{H}}$-projection onto $\tilde{H}_n$, we obtain

$$\lim_{n \to \infty} \bar{\mathbb{E}}\left[ \int_0^t \int_B \left| \left( G_n(l, \bar{\tau}_n(s)) - G(l, \tau_*(s)), \psi \right)_{\tilde{H}} \right|^2 \lambda_2(dl)ds \right] = 0, \quad \psi \in \tilde{H}. \text{ (5.41)}$$

Since $\tilde{\mathcal{U}} \hookrightarrow \tilde{H}$, (5.41) holds for all $\psi \in \tilde{\mathcal{U}}$.

As $\bar{N}_2^n = N_2^*$, for all $n \in \mathbb{N}$. From (5.41) and the Itô isometry we have,

$$\lim_{n \to \infty} \bar{\mathbb{E}} \left[ \left| \int_0^t \int_B \left\langle G_n(l, \bar{\tau}_n(s)) - G(l, \tau_*(s)), \psi \right\rangle \tilde{N}_2^*(ds, dl) \right|^2 \right] = 0. \qquad (5.42)$$

Moreover, from (5.40) and the Itô isometry, with $r = 1$, we obtain,

$$\bar{\mathbb{E}} \left[ \left| \int_0^t \int_B \left\langle G_n(l, \bar{\tau}_n(s)) - G(l, \tau_*(s)), \psi \right\rangle \tilde{N}_2^*(ds, dl) \right|^2 \right]$$

$$= \bar{\mathbb{E}} \left[ \int_0^t \int_B \left| \left( G_n(l, \bar{\tau}_n(s)) - G(l, \tau_*(s)), \psi \right)_{\tilde{H}} \right|^2 \lambda_2(dl) ds \right] \leq C. \qquad (5.43)$$

Finally, from (5.42), (5.43) and using Dominated Convergence Theorem we obtain,

$$\lim_{n \to \infty} \int_0^T \bar{\mathbb{E}} \left[ \left| \int_0^t \int_B \left\langle G_n(l, \bar{\tau}_n(s)) - G(l, \tau_*(s)), \psi \right\rangle \tilde{N}_2^*(ds, dl) \right|^2 \right] dt = 0.$$

(vii) Since, $\bar{\tau}_n \to \tau_*$ in $\mathrm{L}^2(0, T; \tilde{V})$, exploiting the Lipschitz property of $b$, we obtain,

$$\lim_{n \to \infty} \int_0^t \left\langle b_n(\bar{\tau}_n(s)) - b(\tau_*(s)), \psi \right\rangle ds = \lim_{n \to \infty} \int_0^t \left\langle b(\bar{\tau}_n(s)) - b(\tau_*(s)), \tilde{P}_n \psi \right\rangle ds = 0.$$

$$(5.44)$$

Now using Lemma 4.3 and (5.13), for $r \geq 1$ we get,

$$\bar{\mathbb{E}} \left[ \left| \int_0^t \left\langle b_n(\bar{\tau}_n(s)), \psi \right\rangle ds \right|^r \right] \leq C \, T^{r-1} |\psi|_{\tilde{H}}^r \, \bar{\mathbb{E}} \left[ \int_0^t \left( |b_n(\bar{\tau}_n(s))|_{\tilde{H}}^2 \right)^{\frac{r}{2}} ds \right] \leq C_r. \quad (5.45)$$

So from (5.44), (5.45) and using Vitali's theorem we get,

$$\lim_{n \to \infty} \bar{\mathbb{E}} \left[ \left| \int_0^t \left\langle b_n(\bar{\tau}_n(s)) - b(\tau_*(s)), \psi \right\rangle ds \right|^2 \right] = 0. \qquad (5.46)$$

Finally from (5.46) and Dominated convergence theorem we get

$$\lim_{n \to \infty} \int_0^T \bar{\mathbb{E}} \left[ \left| \int_0^t \left\langle b_n(\bar{\tau}_n(s)) - b(\tau_*(s)), \psi \right\rangle ds \right|^2 \right] dt = 0. \quad \square$$

## 5.5. Existence of weak martingale solution

**Theorem 5.8.** *Let (A.1)-(A.7) of Assumption 2.4 are satisfied. Then there exists a weak martingale solution $\left(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}, \bar{\mathbf{v}}, \bar{\tau}, \bar{N}_1, \bar{N}_2\right)$ to the system (3.1)-(3.3).*

**Proof.** Lemma 5.6 and Proposition 5.5 ensure that

$$\lim_{n \to \infty} \left\| (\bar{\mathbf{v}}_n(\cdot), \phi)_H - (\mathbf{v}_*(\cdot), \phi)_H \right\|_{L^2([0,T] \times \bar{\Omega})} = 0, \tag{5.47}$$

and $\quad \lim_{n \to \infty} \left\| \mathfrak{S}_n(\bar{\mathbf{v}}_n, \bar{\tau}_n, \bar{N}_1^n, \phi) - \mathfrak{S}(\mathbf{v}_*, \tau_*, N_1^*, \phi) \right\|_{L^2([0,T] \times \bar{\Omega})} = 0. \tag{5.48}$

Lemma 5.7 and Proposition 5.5 ensure that

$$\lim_{n \to \infty} \left\| (\bar{\tau}_n(\cdot), \psi)_{\tilde{H}} - (\tau_*(\cdot), \psi)_{\tilde{H}} \right\|_{L^2([0,T] \times \bar{\Omega})} = 0, \tag{5.49}$$

and $\quad \lim_{n \to \infty} \left\| \Lambda_n(\bar{\mathbf{v}}_n, \bar{\tau}_n, \bar{N}_2^n, \psi) - \Lambda(\mathbf{v}_*, \tau_*, N_2^*, \psi) \right\|_{L^2([0,T] \times \bar{\Omega})} = 0. \tag{5.50}$

Since $(\mathbf{v}_n, \tau_n)$ is a solution of the Galerkin approximation equations (4.1)-(4.2) for all $t \in [0, T]$, we have for $\mathbb{P}$-a.s.

$$(\mathbf{v}_n(t), \phi)_H = \mathfrak{S}_n(\mathbf{v}_n, \tau_n, N_1^n, \phi)(t) \quad \text{and} \quad (\tau_n(t), \psi)_{\tilde{H}} = \Lambda_n(\mathbf{v}_n, \tau_n, N_2^n, \psi)(t).$$

In particular,

$$\int_0^T \mathbb{E}\left[ \left| (\mathbf{v}_n(t), \phi)_H - \mathfrak{S}_n(\mathbf{v}_n, \tau_n, N_1^n, \phi)(t) \right|^2 \right] dt = 0$$

$$= \int_0^T \mathbb{E}\left[ \left| (\tau_n(t), \psi)_{\tilde{H}} - \Lambda_n(\mathbf{v}_n, \tau_n, N_2^n, \psi)(t) \right|^2 \right] dt.$$

Since $\mathcal{L}(\mathbf{v}_n, \tau_n, N_1^n, N_2^n) = \mathcal{L}(\bar{\mathbf{v}}_n, \bar{\tau}_n, \bar{N}_1^n, \bar{N}_2^n)$, we conclude

$$\int_0^T \bar{\mathbb{E}}\left[ \left| (\bar{\mathbf{v}}_n(t), \phi)_H - \mathfrak{S}_n(\bar{\mathbf{v}}_n, \bar{\tau}_n, \bar{N}_1^n, \bar{N}_2^n, \phi)(t) \right|^2 \right] dt = 0$$

$$= \int_0^T \bar{\mathbb{E}}\left[ \left| (\bar{\tau}_n(t), \psi)_{\tilde{H}} - \Lambda_n(\bar{\mathbf{v}}_n, \bar{\tau}_n, \bar{N}_2^n, \psi)(t) \right|^2 \right] dt.$$

From (5.47), (5.48), (5.49) and (5.50), we have

$$\int_0^T \bar{\mathbb{E}}\left[ \left| (\mathbf{v}_*(t), \phi)_H - \mathfrak{S}(\mathbf{v}_*, \tau_*, N_1^*, \phi)(t) \right|^2 \right] dt = 0$$

$$= \int_0^T \bar{\mathbb{E}}\left[\left|(\tau_*(t), \psi)_{\tilde{H}} - \Lambda(\mathbf{v}_*, \tau_*, N_2^*, \psi)(t)\right|^2\right] dt.$$

Hence for *Leb*-almost all $t \in [0, T]$ and $\bar{\mathbb{P}}$-almost all $\omega \in \bar{\Omega}$, we obtain

$$(\mathbf{v}_*(t), \phi)_H - \mathfrak{S}(\mathbf{v}_*, \tau_*, N_1^*, \psi)(t) = 0 = (\tau_*(t), \psi)_{\tilde{H}} - \Lambda(\mathbf{v}_*, \tau_*, N_2^*, \psi)(t).$$

In particular,

$$(\mathbf{v}_*(t), \phi)_H + \nu \int_0^t \langle \mathcal{A}\mathbf{v}_*(s), \phi \rangle ds + \int_0^t \langle B(\mathbf{v}_*(s)), \phi \rangle ds + \nu_1 \int_0^t \langle Div\, \tau_*(s), \phi \rangle ds$$

$$= (\mathbf{v}_*(0), \phi)_H + \int_0^t \int_Z (F(\mathbf{v}_*(s), z), \phi)_H \tilde{N}_1^*(ds, dz), \qquad (5.51)$$

and $\quad (\tau_*(t), \psi)_{\tilde{H}} + \kappa \int_0^t \langle \tilde{\mathcal{A}}\tau_*(s), \psi \rangle ds + \int_0^t \langle \tilde{B}(\mathbf{v}_*(s), \tau_*(s)), \psi \rangle ds + a \int_0^t \langle \tau_*(s), \psi \rangle ds$

$$= (\tau_*(0), \psi)_{\tilde{H}} + \nu_2 \int_0^t \langle \mathcal{D}(\mathbf{v}_*(s)), \psi \rangle ds + \int_0^t \langle b(\tau_*(s)), \psi \rangle ds$$

$$+ \int_0^t \int_B (G(l, \tau_*(s)), \psi)_{\tilde{H}} \tilde{N}_2^*(ds, dl). \qquad (5.52)$$

Since $(\mathbf{v}_*, \tau_*)$ is $\mathcal{Z}_1 \times \mathcal{Z}_2$-valued random variable, and $\mathbf{v}_*, \tau_*$ are weakly càdlàg, we obtain that the equalities (5.51) and (5.52) hold for all $t \in [0, T]$ and all $\phi \in \mathcal{U}$ and $\psi \in \tilde{\mathcal{U}}$ respectively. Since $\mathcal{U}$ is dense in $V$ and $\tilde{\mathcal{U}}$ is dense in $\tilde{V}$, we infer that equalities (5.51) and (5.52) hold for all $t \in [0, T]$ and all $\phi \in V$ and $\psi \in \tilde{V}$ respectively. Finally, putting $\bar{\mathbf{v}} := \mathbf{v}_*$, $\bar{\tau} := \tau_*$ and $\bar{N}_1 := N_1^*$ and $\bar{N}_2 := N_2^*$, we infer that the system $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}, \bar{\mathbf{v}}, \bar{\tau}, \bar{N}_1, \bar{N}_2)$ is a weak martingale solution of (3.1)-(3.3). $\quad \square$

## 6. Pathwise uniqueness and existence of strong solution to (3.1)-(3.3) in two-dimensions

### 6.1. Some definitions

For the convenience of the readers let us first recall the notion of strong solution and two basic concepts of uniqueness, i.e., pathwise uniqueness and uniqueness in law. For this let us formulate the following assumptions on the stochastic basis.

**Assumption 6.1.** Assume that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a stochastic basis with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, $N_1 = (N_1(t))_{t \geq 0}$ is an $L^2$-valued Poisson random measure and $L = (L(t))_{t \geq 0}$ is an $\mathbb{R}^k$-valued pure jump Lévy process, both defined on the stochastic basis.

**Definition 6.1.** Assume that $(\mathbf{v}_0, \tau_0) \in H \times \tilde{H}$. Assume Assumption 6.1. We say the problem (3.1)-(3.3) has a **strong solution** if and only if there exist $\mathbb{F}$-progressively measurable processes $\mathbf{v} : [0, T] \times \Omega \to H$ and $\tau : [0, T] \times \Omega \to \tilde{H}$ with $\mathbb{P}$-a.e. paths

$$\mathbf{v}(\cdot, \omega) \in D([0, T]; H_w) \cap L^2(0, T; V), \quad \tau(\cdot, \omega) \in D([0, T]; \tilde{H}_w) \cap L^2(0, T; \tilde{V}),$$

such that for all $t \in [0, T]$ and $\phi \in V$, $\psi \in \tilde{V}$ the following identities hold $\mathbb{P}$-a.s.

$$(\mathbf{v}(t), \phi)_H + \nu \int_0^t \langle \mathcal{A}\mathbf{v}(s), \phi \rangle ds + \int_0^t \langle B(\mathbf{v}(s)), \phi \rangle ds + \nu_1 \int_0^t \langle Div\, \tau(s), \phi \rangle ds$$

$$= (\mathbf{v}_0, \phi)_H + \int_0^t \int_Z \big( F(\mathbf{v}(s), z), \phi \big)_H \tilde{N}_1(ds, dz),$$

$$(\tau(t), \psi)_{\tilde{H}} + \kappa \int_0^t \langle \tilde{\mathcal{A}}\tau(s), \psi \rangle ds + \int_0^t \langle \tilde{B}(\mathbf{v}(s), \tau(s)), \psi \rangle ds + a \int_0^t \langle \tau(s), \psi \rangle ds$$

$$= (\tau_0, \psi)_{\tilde{H}} + \nu_2 \int_0^t \langle \mathcal{D}(\mathbf{v}(s)), \psi \rangle ds + \int_0^t \langle b(\tau(s)), \psi \rangle ds + \int_0^t \int_B \big( G(l, \tau(s)), \psi \big)_{\tilde{H}} \tilde{N}_2(ds, dl),$$

and for all $T > 0$

$$\mathbb{E} \left( \sup_{0 \le t \le T} |\mathbf{v}(t)|_H^2 + 2\nu \mathbb{E} \int_0^T |\mathbf{v}(t)|_V^2 \, dt \right) < \infty, \quad \text{and}$$

$$\mathbb{E} \left( \sup_{0 \le t \le T} |\tau(t)|_{\tilde{H}}^2 + 2\nu \mathbb{E} \int_0^T |\tau(t)|_{\tilde{V}}^2 \, dt \right) < \infty.$$

**Definition 6.2.** It is said that the solutions to problem (3.1)-(3.3) are **pathwise unique** iff for any two strong solutions $\mathbf{v}_i : [0, T] \times \Omega \to H$ and $\tau_i : [0, T] \times \Omega \to \tilde{H}$, $i = 1, 2$, to problem (3.1)-(3.3) defined on the same stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying Assumption 6.1, and

$$\big( \mathbf{v}_1(0), \tau_1(0) \big) = \big( \mathbf{v}_2(0), \tau_2(0) \big), \quad \mathbb{P} - \text{a.s.},$$

$$\text{then} \quad (\mathbf{v}_1(t), \tau_1(t)) = (\mathbf{v}_2(t), \tau_2(t)) \quad \mathbb{P} - \text{a.s.} \quad \text{for all } t \in (0, T].$$

**Definition 6.3.** We say that the problem (3.1)-(3.3) has uniqueness in law property if and only if for any two weak martingale solutions $\big( \Omega^i, \mathcal{F}^i, \mathbb{F}^i, \mathbb{P}^i, \{\mathbf{v}_i(t)\}_{t \ge 0}, \{\tau_i(t)\}_{t \ge 0}, \{\tilde{N}_1^i(t, \cdot), \}_{t \ge 0}, \{\tilde{N}_2^i(t, \cdot), \}_{t \ge 0} \big)$, $i = 1, 2$, of problem (3.1)-(3.3) with

$$\mathcal{L}_{\mathbb{P}^1}(\mathbf{v}_1(0)) = \mathcal{L}_{\mathbb{P}^2}(\mathbf{v}_2(0)) \quad \text{on} \quad H \quad \text{and} \quad \mathcal{L}_{\mathbb{P}^1}(\tau_1(0)) = \mathcal{L}_{\mathbb{P}^2}(\tau_2(0)) \quad \text{on} \quad \tilde{H},$$

then the laws of the solutions are also equal, i.e.

$$\mathcal{L}_{\mathbb{P}^1}(\mathbf{v}_1) = \mathcal{L}_{\mathbb{P}^2}(\mathbf{v}_2) \quad \text{on} \quad L^2(0, T; V) \cap D([0, T]; H_w),$$

$$\text{and} \quad \mathcal{L}_{\mathbb{P}^1}(\tau_1) = \mathcal{L}_{\mathbb{P}^2}(\tau_2) \quad \text{on} \quad L^2(0, T; \tilde{V}) \cap D([0, T]; \tilde{H}_w),$$

where $\mathcal{L}_{\mathbb{P}^i}(\mathbf{v}_i)$ and $\mathcal{L}_{\mathbb{P}^i}(\tau_i)$ for $i = 1, 2$ are probability measures on $L^2(0, T; V) \cap D([0, T]; H_w)$ and $L^2(0, T; \tilde{V}) \cap D([0, T]; \tilde{H}_w)$ respectively.

**Lemma 6.2.** *Let $d = 2$. Let $(\mathbf{v}_0, \tau_0) \in L^2(\bar{\Omega}; H) \times L^2(\bar{\Omega}; \tilde{H})$ and (A.1)-(A.7) of Assumption 2.4 are satisfied. Let $\left(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}, \bar{\mathbf{v}}, \bar{\tau}, \bar{N}_1, \bar{N}_2\right)$ be a weak martingale solution of (3.1)-(3.3). In particular,*

$$\bar{\mathbb{E}} \left( \sup_{0 \leq t \leq T} |\bar{\mathbf{v}}(t)|_H^2 \right) + 2\nu \bar{\mathbb{E}} \left( \int_0^T |\bar{\mathbf{v}}(t)|_V^2 \, dt \right) < \infty,$$

$$\bar{\mathbb{E}} \left( \sup_{0 \leq t \leq T} |\bar{\tau}(t)|_{\tilde{H}}^2 \right) + 2\kappa \bar{\mathbb{E}} \left( \int_0^T |\bar{\tau}(t)|_{\tilde{V}}^2 \, dt \right) < \infty. \tag{6.1}$$

*Then for $\bar{\mathbb{P}}$-almost all $\omega \in \bar{\Omega}$, the trajectories $\bar{\mathbf{v}}(\cdot, \omega)$ and $\bar{\tau}(\cdot, \omega)$ are almost everywhere equal to càdlàg $H$-valued and $\tilde{H}$-valued functions respectively. Moreover, for every $t \in [0, T]$, $\bar{\mathbb{P}}$-a.s.*

$$\bar{\mathbf{v}}(t) = \mathbf{v}_0 - \int_0^t \left[ \nu \mathcal{A} \bar{\mathbf{v}}(s) + B(\bar{\mathbf{v}}(s)) + \nu_1 \, Div \, \bar{\tau}(s) \right] ds + \int_0^t \int_Z F(\bar{\mathbf{v}}(s-), z) \tilde{N}_1(ds, dz), \tag{6.2}$$

$$\bar{\tau}(t) = \tau_0 - \int_0^t \left[ \kappa \tilde{\mathcal{A}} \bar{\tau}(s) + \tilde{B}(\bar{\mathbf{v}}(s), \bar{\tau}(s)) + a \bar{\tau}(s) + \nu_2 \mathcal{D}(\bar{\mathbf{v}}(s)) \right] ds$$

$$+ \int_0^t \int_B G(l, \bar{\tau}(s-)) \tilde{N}_2(ds, dl) + \int_0^t b(\bar{\tau}(s)) ds. \tag{6.3}$$

**Proof.** We first claim that if $\left(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}, \bar{\mathbf{v}}, \bar{\tau}, \bar{N}_1, \bar{N}_2\right)$ is a weak martingale solution to the problem (3.1)-(3.3), then $\bar{\mathbb{P}}$-a.e. paths of the processes $\bar{\mathbf{v}}(t)$ and $\bar{\tau}(t)$, $t \in [0, T]$, are respectively $V'$-valued and $\tilde{V}'$-valued càdlàg functions, i.e., for $\bar{\mathbb{P}}$-a.e., $\omega \in \bar{\Omega}$

$$\bar{\mathbf{v}}(\cdot, \omega) \in D([0, T]; V'), \quad \bar{\tau}(\cdot, \omega) \in D([0, T]; \tilde{V}'),$$

and the equalities (6.2) and (6.3) are to be understood in the spaces $V'$ and $\tilde{V}'$ respectively.

To prove the claim, let us fix any $T > 0$.

Exploiting the fact $\mathcal{A} : V \to V'$ given by $|\mathcal{A}\bar{\mathbf{v}}|_{V'} \leq |\bar{\mathbf{v}}|_V$ and $|B(\bar{\mathbf{v}})|_{V'} \leq c|\bar{\mathbf{v}}|_V^2$ for $\bar{\mathbf{v}} \in V$, and then by Hölder's inequality and (6.1) we have the following bound:

$$\mathbb{E}\Big(\Big|\int_0^t \mathcal{A}\bar{\mathbf{v}}(s)ds\Big|_{V'}^2\Big) < \infty, \qquad \text{and} \qquad \mathbb{E}\Big(\Big|\int_0^t B(\bar{\mathbf{v}}(s))ds\Big|_{V'}^2\Big) < \infty.$$

Using Hölder's inequality and (6.1), we have

$$\mathbb{E}\Big[|\nu_1 \int_0^T |Div\,\bar{\tau}(s)|_{V'}ds\Big] \le \nu_1 \, C\Big(\mathbb{E}\Big[\sup_{0\le s\le T}|\nabla\bar{\tau}(s)|_{\tilde{H}}^2\Big]\Big)^{\frac{1}{2}} < \infty.$$

Using (6.1), the embedding $H \hookrightarrow V'$ is continuous, Itô-Lévy isometry and Assumption 2.4, we have

$$\mathbb{E}\Big[\Big|\int_0^T \int_Z F(\bar{\mathbf{v}}(s),z)\tilde{N}_1(ds,dz)\Big|_{V'}^2\Big] \le C\Big(1 + \mathbb{E}\Big[\sup_{0\le s\le T}|\bar{\mathbf{v}}(s)|_H^2\Big]\Big) < \infty.$$

Now exploiting the fact $\tilde{\mathcal{A}} : \tilde{V} \to \tilde{V}'$ given by $|\tilde{\mathcal{A}}\bar{\tau}|_{\tilde{V}'} \le |\bar{\tau}|_{\tilde{V}}$, and $|\tilde{B}(\bar{\mathbf{v}},\bar{\tau})|_{\tilde{V}'} \le |\bar{\mathbf{v}}|_H|\bar{\tau}|_{\tilde{H}}$, then by Hölder's inequality, (6.1) and (6.1), we have

$$\mathbb{E}\Big(\Big|\int_0^t \tilde{\mathcal{A}}\bar{\mathbf{v}}(s)ds\Big|_{\tilde{V}'}^2\Big) < \infty, \qquad \text{and} \qquad \mathbb{E}\Big(\Big|\int_0^t \tilde{B}(\bar{\mathbf{v}}(s),\bar{\tau}(s))ds\Big|_{\tilde{V}'}^2\Big) < \infty.$$

Using (6.1) and Hölder's inequality, we directly have

$$\mathbb{E}\Big[|\int_0^t |\bar{\tau}(s)|_{\tilde{H}}^2\Big] \le C\Big(\mathbb{E}[\sup_{0\le s\le T}|\bar{\tau}(s)|_{\tilde{H}}^2]\Big)^{\frac{1}{2}} < \infty.$$

Using (6.1), we have

$$\mathbb{E}\Big[\int_0^T |\mathcal{D}(\bar{\mathbf{v}}(s))|_{\tilde{V}'}ds\Big] \le C\,\nu_2\Big(\mathbb{E}[\int_0^T |\bar{\mathbf{v}}(s)|_V^2 ds]\Big)^{\frac{1}{2}} < \infty.$$

Using Itô-Lévy isometry, embedding $\tilde{H} \hookrightarrow \tilde{V}'$, Lemma 4.4 and (6.1) we have

$$\mathbb{E}\Big[|\int_0^T \int_B G(l,\bar{\tau}(s))\tilde{N}_2(dl,ds)|_{\tilde{V}'}^2\Big] = C\mathbb{E}\Big[\int_0^T \int_B |G(l,\bar{\tau}(s))|_{\tilde{H}}^2 \lambda_2(dl)ds\Big] < \infty.$$

Since $\tilde{H} \hookrightarrow \tilde{V}'$ is continuous, using Schwarz's inequality, from Lemma 4.4 and (6.1) we get,

$$\mathbb{E}\Big[\Big|\int_0^T b(\bar{\tau}(s))\,ds\Big|_{\tilde{V}'}\Big] \le C\mathbb{E}\Big[\Big(\int_0^T |b(\bar{\tau}(s))|_{\tilde{H}}^2\,ds\Big)^{\frac{1}{2}}\Big] \le C\,\theta\Big\{\mathbb{E}\Big[\sup_{s\in[0,T]}|\bar{\tau}(s)|_{\tilde{H}}^2\,ds\Big]\Big\}^{\frac{1}{2}} < \infty.$$

Thus the right hand sides of the equalities (6.2) and (6.3) are well-defined in the spaces $V'$ and $\tilde{V}'$ respectively. Now using (5.3) and (5.6), we infer that for $\bar{\mathbb{P}}$-a.e., $\omega \in \bar{\Omega}$

$$\bar{\mathbf{v}}(\cdot, \omega) \in D([0, T]; V'), \quad \bar{\tau}(\cdot, \omega) \in D([0, T]; \tilde{V}'),$$

and for every $t \in [0, T]$ equalities (6.2) and (6.3) hold. This proves the claim.

Next note that using (6.1), Itô-Lévy isometry and Assumption 2.4, we can deduce

$$\mathbb{E}\left[\left| \int\limits_0^T \int\limits_Z F(\bar{\mathbf{v}}(s), z)\tilde{N}_1(ds, dz)\right|_H^2\right] \le C\left(1 + \mathbb{E}\left[\sup_{0 \le s \le T} |\bar{\mathbf{v}}(s)|_H^2\right]\right) < \infty,$$

and likewise exploiting Lemma 4.4 and (6.1) we deduce

$$\mathbb{E}\left[|\int\limits_0^T \int\limits_B G(l, \bar{\tau}(s))\tilde{N}_2(dl, ds)|_{\tilde{H}}^2\right] = C\mathbb{E}\left[\int\limits_0^T \int\limits_B |G(l, \bar{\tau}(s))|_{\tilde{H}}^2 \lambda_2(dl)ds\right] < \infty.$$

Thus applying the Gyöngy-Krylov result (see Theorem 2 in [34]), we conclude that the trajectories $\bar{\mathbf{v}}(\cdot, \omega)$ and $\bar{\tau}(\cdot, \omega)$ are almost everywhere equal to càdlàg $H$-valued and $\tilde{H}$-valued functions respectively. $\quad \square$

**Lemma 6.3.** *Let $d = 2$. Then solutions to problem* (3.1)-(3.3) *are pathwise unique.*

**Proof.** Let $(\mathbf{v}_1, \tau_1)$ and $(\mathbf{v}_2, \tau_2)$ be two solutions defined on the same stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying (3.1)-(3.3) with the same initial conditions.

Let us denote $\tilde{\mathbf{v}}(t) = \mathbf{v}_1(t) - \mathbf{v}_2(t)$, $\tilde{\tau}(t) = \tau_1(t) - \tau_2(t)$. Then $(\tilde{\mathbf{v}}(0), \tilde{\tau}(0)) = (\tilde{\mathbf{v}}_0, \tilde{\tau}_0) = (0, 0)$. Therefore, we note that $(\tilde{\mathbf{v}}, \tilde{\tau})$ satisfies

$$d\tilde{\mathbf{v}}(t) + [(\tilde{\mathbf{v}}(t) \cdot \nabla)\mathbf{v}_1(t) + (\mathbf{v}_2(t) \cdot \nabla)\tilde{\mathbf{v}}(t) - \nu\Delta\tilde{\mathbf{v}}(t)]dt$$

$$= \nu_1 \nabla \cdot \tilde{\tau}(t)dt + \int\limits_Z \tilde{F}(t-, z)\tilde{N}_1(dt, dz), \tag{6.4}$$

$$d\tilde{\tau}(t) + [(\tilde{\mathbf{v}}(t) \cdot \nabla)\tau_1(t) + (\mathbf{v}_2(t) \cdot \nabla)\tilde{\tau}(t) + a\tilde{\tau}(t) - \kappa\Delta\tilde{\tau}(t)]dt$$

$$= \nu_2 \mathcal{D}(\tilde{\mathbf{v}}(t))dt + \int\limits_B \tilde{G}(l, t-)\tilde{N}_2(dt, dl) + \tilde{b}(t)dt, \tag{6.5}$$

where $\tilde{F}(t, z) := F(\mathbf{v}_1(t), z) - F(\mathbf{v}_2(t), z)$, $\tilde{G}(l, t) := G(l, \tau_1(t)) - G(l, \tau_2(t))$, $\tilde{b}(t) := b(\tau_1(t)) - b(\tau_2(t))$. Now let us define $\Theta(t) := \exp\{-2\int_0^t \vartheta(s)ds\}$ where $\vartheta$ will be determined later. We now apply infinite dimensional version of Itô's formula from Gyöngy and Krylov (see Theorem 2 in [34]) to $\frac{1}{2}\Theta(t)|\tilde{\mathbf{v}}(t)|_H^2$ and to $\frac{1}{2}\Theta(t)|\tilde{\tau}(t)|_{\tilde{H}}^2$ to equations (6.4) and (6.5) respectively and then multiplying first equation by $\nu_2$ and the second equation by $\nu_1$ and then on adding we obtain

$$\frac{1}{2}d\Big[\Theta(t)\Big(\nu_2|\tilde{\mathbf{v}}(t)|_H^2 + \nu_1|\tilde{\tau}(t)|_{\tilde{H}}^2\Big)\Big] + \Theta(t)\Big[a\nu_1|\tau(t)|_{\tilde{H}}^2 + \nu_1\kappa|\nabla\tilde{\tau}(t)|_{\tilde{H}}^2 + \nu\nu_2|\nabla\tilde{\mathbf{v}}(t)|_H^2\Big]dt$$

$$= \Theta(t)\Big[\underbrace{-\nu_2\langle(\tilde{\mathbf{v}}(t)\cdot\nabla)\mathbf{v}_1(t),\tilde{\mathbf{v}}(t)\rangle}_{J_1(t)} \underbrace{-\nu_1\langle(\tilde{\mathbf{v}}(t)\cdot\nabla)\tau_1(t),\tilde{\tau}(t)\rangle}_{J_2(t)} + \underbrace{\nu_1\langle\tilde{b}(t),\tilde{\tau}(t)\rangle}_{J_3(t)}\Big]dt$$

$$+\Theta(t)\Big[\underbrace{\nu_2\int_Z\langle\tilde{F}(t-,z),\tilde{v}(t)\rangle_H\tilde{N}_1(dt,dz) + \nu_2\int_Z|\tilde{F}(t,z)|_H^2\lambda_1(dz)dt}_{J_4(t)\qquad\qquad\qquad\qquad J_5(t)}$$

$$+\underbrace{\nu_1\int_B\langle\tilde{G}(l,t),\tilde{\tau}(t)\rangle_{\tilde{H}}\tilde{N}_2(dt,dl)}_{J_6(t)} + \underbrace{\nu_1\int_B|\tilde{G}(l,t)|_{\tilde{H}}^2\lambda_2(dl)dt}_{J_7(t)}\Big]$$

$$+\frac{\Theta'(t)}{2}\Big(\nu_2|\tilde{\mathbf{v}}(t)|_H^2 + \nu_1|\tilde{\tau}(t)|_{\tilde{H}}^2\Big). \tag{6.6}$$

Now using (2.2) and (2.3) and Young's inequality we obtain

$$|J_1(t)| \le \nu_2|\langle(\tilde{\mathbf{v}}(t)\cdot\nabla)\mathbf{v}_1(t),\tilde{\mathbf{v}}(t)\rangle| \le \bar{\kappa}_1|\nabla\tilde{\mathbf{v}}(t)|_H^2 + C(\bar{\kappa}_1)|\mathbf{v}_1(t)|_H^2|\nabla\mathbf{v}_1(t)|_H^2|\tilde{\mathbf{v}}(t)|_H^2 \tag{6.7}$$

$$\text{and}\quad |J_2(t)| \le \bar{\kappa}_2|\nabla\tilde{\tau}(t)|_{\tilde{H}}^2 + \bar{\kappa}_3|\nabla\tilde{\mathbf{v}}(t)|_H^2 + C(\bar{\kappa}_2,\bar{\kappa}_3)|\tilde{\mathbf{v}}(t)|_H^2|\nabla\tau_1(t)|_{\tilde{H}}^2|\tau_1(t)|_{\tilde{H}}^2. \tag{6.8}$$

Using Cauchy-Schwartz, Young's inequalities and Lipschitz property of $b$ (i.e., Lemma 4.3), we have

$$|J_3(t)| \le \nu_1|\tilde{\tau}(t)|_{\tilde{H}}^2. \tag{6.9}$$

Similarly using Lipschitz property of $G$, (i.e., Lemma 4.3) we have

$$|J_7(t)| \le C_1|\tilde{\tau}(t)|_{\tilde{H}}^2, \tag{6.10}$$

and using Assumption 2.4 we estimate $J_5$ as

$$J_5(t) \le L|\tilde{v}(t)|_H^2. \tag{6.11}$$

Substituting (6.7)-(6.11) in (6.6) and choosing $\bar{\kappa}_1 = \frac{\nu\nu_1}{4}, \bar{\kappa}_2 = \frac{\kappa}{2}, \bar{\kappa}_3 = \frac{\nu\nu_1}{4}$, we have

$$\frac{1}{2}d\Big[\Theta(t)\Big(\nu_2|\tilde{\mathbf{v}}(t)|_H^2 + \nu_1|\tilde{\tau}(t)|_{\tilde{H}}^2\Big)\Big] + \Theta(t)\Big[a\nu_1|\tau(t)|_{\tilde{H}}^2 + \frac{\nu_1\kappa}{2}|\nabla\tilde{\tau}(t)|_{\tilde{H}}^2 + \frac{\nu\nu_2}{2}|\nabla\tilde{\mathbf{v}}(t)|_H^2\Big]dt$$

$$\le \Theta(t)\Big(C(\bar{\kappa}_1)|\mathbf{v}_1(t)|_H^2|\nabla\mathbf{v}_1(t)|_H^2|\tilde{\mathbf{v}}(t)|_H^2 + C(\bar{\kappa}_2,\bar{\kappa}_3)|\tilde{\mathbf{v}}(t)|_H^2|\nabla\tau_1(t)|_{\tilde{H}}^2|\tau_1(t)|_{\tilde{H}}^2\Big)$$

$$+ \Theta(t)\Big(C\nu_1 + C_1\Big)|\tilde{\tau}(t)|_{\tilde{H}}^2 + L\Theta(t)|\tilde{\mathbf{v}}(t)|_H^2 + \Theta(t)\Big(J_4(t) + J_6(t)\Big)$$

$$+ \frac{\Theta'(t)}{2}\Big(\nu_2|\tilde{\mathbf{v}}(t)|_H^2 + \nu_1|\tilde{\tau}(t)|_{\tilde{H}}^2\Big).$$

We now choose $\vartheta$ as:

$$\vartheta(t) = \frac{C(\bar\kappa_1)}{v_2}|\mathbf{v}_1(t)|_H^2|\nabla\mathbf{v}_1(t)|_H^2 + \frac{C(\bar\kappa_2,\bar\kappa_3)}{v_1}|\nabla\tau_1(t)|_{\tilde{H}}^2|\tau_1(t)|_{\tilde{H}}^2,$$

so that $\mathbb{E}\int_0^t \vartheta(s)\,ds < \infty$ for every $t \in [0,T]$.

By choice of $\Theta$, we have

$$\Theta(t)\Big(C(\bar\kappa_1)|\mathbf{v}_1(t)|_H^2|\nabla\mathbf{v}_1(t)|_H^2|\tilde{\mathbf{v}}(t)|_H^2 + C(\bar\kappa_2,\bar\kappa_3)|\tilde{\mathbf{v}}(t)|_H^2|\nabla\tau_1(t)|_{\tilde{H}}^2|\tau_1(t)|_{\tilde{H}}^2\Big)$$

$$+ \frac{\Theta'(t)}{2}\Big(v_2|\tilde{\mathbf{v}}(t)|_H^2 + v_1|\tilde\tau(t)|_{\tilde{H}}^2\Big) \leq 0.$$

Hence, we have

$$\frac{1}{2}d\Big[\Theta(t)\Big(v_2|\tilde{\mathbf{v}}(t)|_H^2 + v_1|\tilde\tau(t)|_{\tilde{H}}^2\Big)\Big] + \Theta(t)\Big[av_1|\tau(t)|_{\tilde{H}}^2 + \frac{v_1\kappa}{2}|\nabla\tilde\tau(t)|_{\tilde{H}}^2 + \frac{vv_2}{2}|\nabla\tilde{\mathbf{v}}(t)|_H^2\Big]dt$$

$$\leq C\Theta(t)\Big(v_2|\tilde{\mathbf{v}}(t)|_H^2 + v_1|\tilde\tau(t)|_{\tilde{H}}^2\Big) + \Theta(t)\Big(J_4(t) + J_6(t)\Big),$$

for some large constant $C$.

Integrating in $[0,t]$ and then taking expectation and further using that $\int_0^t \Theta(s)J_4(s)ds$ and $\int_0^t \Theta(s)J_6(s)ds$ are local martingales with zero averages, we have

$$\frac{1}{2}\mathbb{E}\Big[\Theta(t)\Big(v_2|\tilde{\mathbf{v}}(t)|_H^2 + v_1|\tilde\tau(t)|_{\tilde{H}}^2\Big)\Big]$$

$$+ \mathbb{E}\Big[\int_0^t \Theta(s)\Big(av_1|\tau(s)|_{\tilde{H}}^2 + \frac{v_1\kappa}{2}|\nabla\tilde\tau(t)|_{\tilde{H}}^2 + \frac{vv_2}{2}|\nabla\tilde{\mathbf{v}}(t)|_H^2\Big)ds\Big]$$

$$\leq C\,\mathbb{E}\Big[\int_0^t \Theta(s)\Big(v_2|\tilde{\mathbf{v}}(s)|_H^2 + v_1|\tilde\tau(s)|_{\tilde{H}}^2\Big)ds\Big].$$

Using Gronwall's inequality and $\tilde{\mathbf{v}}_0 = \tilde\tau_0 = 0$, we have

$$(\mathbf{v}_1(t), \tau_1(t)) = (\mathbf{v}_2(t), \tau_2(t)), \quad \mathbb{P} - \text{a.s.} \quad t \in [0,T]. \quad \square$$

Finally, we have the following results.

**Theorem 6.4.** *Let $d = 2$ and $(\mathbf{v}_0, \tau_0) \in L^2(\bar\Omega; H) \times L^2(\bar\Omega; \tilde{H})$. Let Assumption 2.4 and Assumption 6.1 be satisfied. Then*

(1) *There exists a pathwise unique strong solution of (3.1)-(3.3).*
(2) *Moreover, if $(\bar\Omega, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}, \bar{\mathbf{v}}, \bar\tau, \bar{N}_1, \bar{N}_2)$ is a weak martingale solution of (3.1)-(3.3), then for $\bar{\mathbb{P}} -$ almost all $\omega \in \bar\Omega$ the trajectories $\bar{\mathbf{v}}(\cdot, \omega)$ and $\bar\tau(\cdot, \omega)$ are equal almost everywhere to càdlàg $H$-valued and $\tilde{H}$-valued functions defined on $[0,T]$.*
(3) *The weak martingale solution of (3.1)-(3.3) is unique in law.*

**Proof.** By Theorem 5.8, there exists a weak martingale solution to the system (3.1)-(3.3) and by Lemma 6.3, the solution is pathwise unique. Hence assertion (1) follows from infinite dimensional generalisation of the classical Yamada-Watanabe result (see e.g. Theorems 2 of Ondreját [62]). Assertion (2) is a direct consequence of Lemma 6.2. Assertion (3) follows from Theorem 11 of Ondreját [62]. □

## 7. Existence of an invariant measure to the problem (3.1)-(3.3) in two-dimensions

This section is built upon the recent results by [14] and [16]. One can look into the introduction section in [14] for discussion on other approaches. We begin the section with an observation that instead of the system (3.1) - (3.3), we could have alternatively considered the same system of equations (3.1) - (3.2) with initial conditions (3.3) replaced by the initial distributions, i.e.

$$\mathcal{L}(\mathbf{v}(0), \tau(0)) = \rho_0. \tag{7.1}$$

We assume the following about the laws of the initial conditions:

**Assumption 7.1.** The Borel probability measure $\rho_0$ defined on $H \times \tilde{H}$ is such that for every $p \geq 2$, $\int_{H \times \tilde{H}} |x|^p \rho_0(dx) < \infty$.

Then under Assumption 2.4 and Assumption 7.1, there exists a weak martingale solution $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}, \bar{\mathbf{v}}, \bar{\tau}, \bar{N}_1, \bar{N}_2)$ to the system (3.1)-(3.2) with (7.1). In fact, we can establish the following result.

**Lemma 7.2.** *Let $d = 2$ and Assumption 2.4 and Assumption 7.1 hold. We assume $\nu = \kappa = \nu_1 = \nu_2 = 1$. Then for every $T > 0$ and $R > 0$, there exist constants $C_1$ and $C_2$ depending on $T$, $R$ and $p$ such that if $\rho_0$ is a Borel probability measure on $H \times \tilde{H}$ satisfying $\int_{H \times \tilde{H}} |x|^p \rho_0(dx) < R$, then every martingale solution $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}, \bar{\mathbf{v}}, \bar{\tau}, \bar{N}_1, \bar{N}_2)$ of the system (3.1)-(3.2) with (7.1) satisfies the following estimates:*

$$\bar{\mathbb{E}}\left[ \sup_{s \in [0,T]} |\bar{\mathbf{v}}(s)|_H^p + \sup_{s \in [0,T]} |\bar{\tau}(s)|_{\tilde{H}}^p \right] \leq C_1(p), \tag{7.2}$$

$$\bar{\mathbb{E}}\left[ \int_0^T |\nabla \bar{\mathbf{v}}(s)|_H^2 ds + \int_0^T |\nabla \bar{\tau}(s)|_{\tilde{H}}^2 ds \right] \leq C_2. \tag{7.3}$$

The proof of the above Lemma is similar to the proof of Theorem 4.7, and hence is omitted. However it is worthy to note that in Theorem 4.7, Itô's formula was applied to the processes $\mathbf{v}_n$ and $\tau_n$ taking values in finite-dimensional Hilbert spaces $H_n$ and $\tilde{H}_n$, respectively, where these processes were the Galerkin (in strong or classical sense) solutions of the equations (4.1)-(4.2). On the other hand, in the above Lemma 7.2, $\bar{\mathbf{v}}$ and $\bar{\tau}$ are the martingale solutions of the equations (3.1)-(3.2), and these are processes taking values in $H$ and $\tilde{H}$ respectively (see Lemma 6.2). Hence to prove the above Lemma we need to apply infinite dimensional version of Itô's formula due to Gyöngy and Krylov (see Theorem 2 in [34]). Thus as a first step, one needs to verify whether all the conditions of the Gyöngy and Krylov Theorem are satisfied, and this has already been done in the proof of Lemma 6.2.

Owing to the fact that existence of an invariant measure is still an open problem in the whole of $\mathbb{R}^2$, we restrict ourselves to a Poincaré domain $\mathscr{O}$ in $\mathbb{R}^2$, where positive results are available for stochastic Navier-Stokes equations, see [16] and the references therein. For the remaining part of the discussions in this section, instead of the whole space $\mathbb{R}^2$, we work on $\mathscr{O}$.

### 7.0.1. Function spaces

Now we introduce the following notations to avoid confusion with the previous part of the paper. Let $C_c^\infty(\mathscr{O}; \mathbb{R}^d)$ be the space of all $\mathbb{R}^d$-valued compactly supported $C^\infty$ functions in $\mathscr{O}$. Let us denote

$$\mathcal{V}_{\mathscr{O}} := \{\mathbf{v} \in C_c^\infty(\mathscr{O}; \mathbb{R}^d) : Div\, \mathbf{v} = 0\}, \quad \tilde{\mathcal{V}}_{\mathscr{O}} := \{\tau \in C_c^\infty(\mathscr{O}; \mathbb{R}^{d \times d}) : \tau^T = \tau\}.$$

Let $H_{\mathscr{O}}, V_{\mathscr{O}}, \tilde{H}_{\mathscr{O}}$ and $\tilde{V}_{\mathscr{O}}$ be the closure of $\mathcal{V}_{\mathscr{O}}$ in $L^2(\mathscr{O}; \mathbb{R}^d), H^1(\mathscr{O}; \mathbb{R}^d), L^2(\mathscr{O}; \mathbb{R}^{d \times d})$ and $H^1(\mathscr{O}; \mathbb{R}^{d \times d})$ respectively. In the space $\mathcal{V}_{\mathscr{O}}$, we consider the inner product inherited from $H^1(\mathscr{O}, \mathbb{R}^d)$

$$(\mathbf{u}, \mathbf{v})_{\mathcal{V}_{\mathscr{O}}} := (\mathbf{u}, \mathbf{v})_{L^2} + ((\mathbf{u}, \mathbf{v}))_{\mathscr{O}},$$

where $((\mathbf{u}, \mathbf{v}))_{\mathscr{O}} = (\nabla \mathbf{u}, \nabla \mathbf{v})_{L^2}, \quad \mathbf{u}, \mathbf{v} \in \mathcal{V}_{\mathscr{O}}$. For $s > 0$, let $V_s(\mathscr{O})$ and $\tilde{V}_s(\mathscr{O})$ be the closure of $\mathcal{V}_{\mathscr{O}}$ and $\tilde{\mathcal{V}}_{\mathscr{O}}$ in $H^s(\mathscr{O}; \mathbb{R}^d)$ and $H^s(\mathscr{O}; \mathbb{R}^{d \times d})$ respectively. As the domain $\mathscr{O}$ is Poincaré, there exists $C > 0$ such that

$$C \int_{\mathscr{O}} |\varphi|^2 dx \leq \int_{\mathscr{O}} |\nabla \varphi|^2 dx, \quad \forall \varphi \in H^1(\mathscr{O}, \mathbb{R}^d). \tag{7.4}$$

The bilinear operators $b_{\mathscr{O}}$ and $\tilde{b}_{\mathscr{O}}$ can be defined as in Subsection 2.1 where the domain $\mathbb{R}^d$ is replaced by $\mathscr{O}$. The construction of the embeddings in Subsection 2.2 holds true in $\mathscr{O}$. By Lax-Milgram Theorem, there exists unique bounded linear operator

$$\mathcal{A}_{\mathscr{O}} : V_{\mathscr{O}} \to V'_{\mathscr{O}}$$

such that

$$_{V'_{\mathscr{O}}}\langle \mathcal{A}_{\mathscr{O}} \mathbf{u}, \mathbf{v} \rangle_{V_{\mathscr{O}}} = ((\mathbf{u}, \mathbf{v}))_{\mathscr{O}}, \quad \forall\, \mathbf{v}, \mathbf{u} \in V_{\mathscr{O}}. \tag{7.5}$$

Similarly, there exists bounded linear operator $\tilde{\mathcal{A}}_{\mathscr{O}} : \tilde{V}_{\mathscr{O}} \to \tilde{V}'_{\mathscr{O}}$ such that

$$_{\tilde{V}'_{\mathscr{O}}}\langle \tilde{\mathcal{A}}_{\mathscr{O}} \tau, \mathbf{w} \rangle_{\tilde{V}_{\mathscr{O}}} = ((\tau, \mathbf{w}))_{\mathscr{O}}, \quad \forall\, \tau, \mathbf{w} \in \tilde{V}_{\mathscr{O}}.$$

For the functions in the space $L^p(\mathscr{O}, \mathbb{R}^d), H^1(\mathscr{O}; \mathbb{R}^d), \mathcal{V}_{\mathscr{O}}, \tilde{\mathcal{V}}_{\mathscr{O}}, V_{\mathscr{O}}, \tilde{V}_{\mathscr{O}}, H_{\mathscr{O}}, \tilde{H}_{\mathscr{O}}, L^2(\mathscr{O}; \mathbb{R}^{d \times d})$ and $H^1(\mathscr{O}; \mathbb{R}^{d \times d})$, we can extend the functions by defining zero outside $\mathscr{O}$. As a result, the functions lie in the space $L^p(\mathbb{R}^d, \mathbb{R}^d), H^1(\mathbb{R}^d; \mathbb{R}^d), \mathcal{V}, \tilde{\mathcal{V}}, V, \tilde{V}, H, \tilde{H}, L^2(\mathbb{R}^d; \mathbb{R}^{d \times d})$ and $H^1(\mathbb{R}^d; \mathbb{R}^{d \times d})$. Therefore we can borrow results of the previous sections for the functions defined on $\mathscr{O}$. Additionally, the Poincaré inequality (7.4) holds for the elements in $V_{\mathscr{O}}, \tilde{V}_{\mathscr{O}}$.

**Remark 2.** Theorem 5.8, Theorem 6.4 and Lemma 7.2 are true in any Poincaré domain.

As a consequence of the Remark 2, we have the following result in the Poincaré domain $\mathscr{O}$.

**Corollary 7.3.** *Let $\mathscr{O}$ be a Poincaré domain in $\mathbb{R}^2$. The under the same assumptions as in Lemma 7.2, we have the following additional estimate:*

$$\int_0^t \bar{\mathbb{E}}\Big[|\bar{\mathbf{v}}(s)|^2_{H_{\mathscr{O}}} + |\bar{\tau}(s)|^2_{\tilde{H}_{\mathscr{O}}}\Big]ds \leq \frac{1}{2C}\bar{\mathbb{E}}\Big[|\mathbf{v}_0|^2_{H_{\mathscr{O}}} + |\tau_0|^2_{\tilde{H}_{\mathscr{O}}} + Ct\Big]e^{Ct}, \qquad t \geq 0. \quad (7.6)$$

**Proof.** First note that, if we define a stopping time

$$\mathfrak{T}_N = \inf_{t\in[0,\infty)}\Big\{t : |\bar{\mathbf{v}}(t)|^2_{H_{\mathscr{O}}} + |\bar{\tau}(t)|^2_{\tilde{H}_{\mathscr{O}}} > N\Big\},$$

then apply Itô's formula due to Gyöngy and Krylov (see Theorem 2 in [34]), and proceed as in Theorem 4.7, we obtain the following estimate similar to (4.10) $\forall\, t \geq 0$,

$$\bar{\mathbb{E}}\Big[|\bar{\mathbf{v}}(t)|^2_{H_{\mathscr{O}}} + |\bar{\tau}(t)|^2_{\tilde{H}_{\mathscr{O}}} + 2\int_0^t |\nabla\bar{\mathbf{v}}(s)|^2_{H_{\mathscr{O}}}ds + 2\int_0^t |\nabla\bar{\tau}(s)|^2_{\tilde{H}_{\mathscr{O}}}ds + 2a\int_0^t |\bar{\tau}(s)|^2_{\tilde{H}_{\mathscr{O}}}ds\Big]$$

$$\leq \bar{\mathbb{E}}\Big[|\mathbf{v}_0|^2_{H_{\mathscr{O}}} + |\tau_0|^2_{\tilde{H}_{\mathscr{O}}} + Ct\Big]e^{Ct}. \qquad (7.7)$$

Now let us fix $t \geq 0$. By the use of Poincaré inequality, for almost all $s \in [0, t]$, we have

$$|\bar{\mathbf{v}}(s)|^2_{H_{\mathscr{O}}} + |\bar{\tau}(s)|^2_{\tilde{H}_{\mathscr{O}}} \leq \frac{1}{C}\Big(|\nabla\bar{\mathbf{v}}(s)|^2_{H_{\mathscr{O}}} + |\nabla\bar{\tau}(s)|^2_{\tilde{H}_{\mathscr{O}}}\Big).$$

Hence, using (7.7), we infer

$$\int_0^t \bar{\mathbb{E}}\Big[|\bar{\mathbf{v}}(s)|^2_{H_{\mathscr{O}}} + |\bar{\tau}(s)|^2_{\tilde{H}_{\mathscr{O}}}\Big]ds \leq \frac{1}{C}\int_0^t \bar{\mathbb{E}}\Big(|\nabla\bar{\mathbf{v}}(s)|^2_{H_{\mathscr{O}}} + |\nabla\bar{\tau}(s)|^2_{\tilde{H}_{\mathscr{O}}}\Big)$$

$$\leq \frac{1}{2C}\bar{\mathbb{E}}\Big[|\mathbf{v}_0|^2_{H_{\mathscr{O}}} + |\tau_0|^2_{\tilde{H}_{\mathscr{O}}} + Ct\Big]e^{Ct}, \quad t \geq 0. \quad \square$$

### 7.1. Continuous dependence with respect to the initial data

Owing to Theorem 5.8, Theorem 6.4 and Lemma 7.2, we can now obtain continuous dependence of the solution to the system (3.1) - (3.3) with respect to the initial data.

**Theorem 7.4.** *Let $d = 2$ and Assumption 2.4 be satisfied. Let $(\mathbf{v}_0, \tau_0) \in H \times \tilde{H}$. Suppose that there is an $H \times \tilde{H}$-valued sequence $(\mathbf{v}_{0,n}, \tau_{0,n})_{n\in\mathbb{N}}$ weakly converging to $(\mathbf{v}_0, \tau_0)$ in $H \times \tilde{H}$. Let $\big(\Omega_n, \mathcal{F}_n, \mathbb{F}_n, \mathbb{P}_n, \mathbf{v}_n, \tau_n, N_1^n, N_2^n\big)$ be a weak martingale solution of the system (3.1) - (3.3) on $[0, \infty)$ with the initial data $\big(\mathbf{v}_{0,n}, \tau_{0,n}\big)$. Then for every $T > 0$ there exist*

(i) *a subsequence $(n_k)_k$,*

(ii) *a stochastic basis $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}})$, with a filtration $\hat{\mathbb{F}} = (\hat{\mathcal{F}}_t)_{t \geq 0}$,*

(iii) *two independent time homogeneous Poisson random measures $\hat{N}_i(t, \cdot), i = 1, 2, \, t \in [0, \infty)$ defined on this stochastic basis*

(iv) *$\hat{\mathbb{F}}$-progressively measurable processes $\hat{\mathbf{v}}(t), \hat{\tau}(t), (\hat{\mathbf{v}}_{n_k}(t))_{k \in \mathbb{N}}, (\hat{\tau}_{n_k}(t))_{k \in \mathbb{N}}, \, t \in [0, T]$, defined on this basis with laws supported in $\mathcal{Z}$ such that*

$$\mathcal{L}\big((\hat{\mathbf{v}}_{n_k}, \hat{\tau}_{n_k})\big) = \mathcal{L}\big((\mathbf{v}_{n_k}, \tau_{n_k})\big) \quad and \quad (\hat{\mathbf{v}}_{n_k}, \hat{\tau}_{n_k}) \to (\hat{\mathbf{v}}, \hat{\tau}) \text{ in } \mathcal{Z}, \quad \hat{\mathbb{P}} - a.s.$$

*Moreover, the system $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}}, \hat{\mathbf{v}}, \hat{\tau}, \hat{N}_1, \hat{N}_2)$ is a weak martingale solution (according to Definition 5.1) to the problem (3.1) - (3.3) on $[0, T]$ with the initial law $\delta_0 := (\delta_{\mathbf{v}_0}, \delta_{\tau_0})$.*

**Proof.** We first observe that, a weak martingale solution to the problem (3.1) - (3.2) with deterministic initial data $(\mathbf{v}_0, \tau_0) \in H \times \tilde{H}$, is also a weak martingale solution to the problem (3.1) - (3.2) with initial law $\rho_0 = \delta_0$.

Since the sequences $(\mathbf{v}_{0,n})_{n \in \mathbb{N}} \subset H$ and $(\tau_{0,n})_{n \in \mathbb{N}} \subset \tilde{H}$ converge weakly in $H$ and $\tilde{H}$, respectively, we infer that there exist $R_i > 0, i = 1, 2$ such that

$$\sup_{n \in \mathbb{N}} |\mathbf{v}_{0,n}|_H \leq R_1, \quad and \quad \sup_{n \in \mathbb{N}} |\tau_{0,n}|_{\tilde{H}} \leq R_2.$$

Therefore by Lemma 7.2, we infer that the processes $(\mathbf{v}_n, \tau_n), n \in \mathbb{N}$, satisfy inequalities (7.2) and (7.3). Therefore all the assertions follow similar to the proof of Theorem 5.8. □

**Remark 3.** Theorem 7.4 is true in any Poincaré domain.

In the next section, we will prove existence of an invariant measure in the Poincaré domain $\mathscr{O}$.

## 7.2. Existence of an invariant measure

Let us denote, for any fixed initial data $(\mathbf{v}_0, \tau_0) \in H_{\mathscr{O}} \times \tilde{H}_{\mathscr{O}}$, by $((\mathbf{v}, \tau)(t, \mathbf{v}_0, \tau_0)), \, t \geq 0$, the unique solution to the problem (3.1) - (3.3), defined on a given stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying Assumption 6.1. Note that, the question of uniqueness in two-dimensions has already been resolved in Lemma 6.3.

For any bounded Borel functions $\varphi \in \mathscr{B}_b(H_{\mathscr{O}} \times \tilde{H}_{\mathscr{O}})$ and $t \geq 0$, we define

$$(\mathscr{P}_t \varphi)(\mathbf{v}_0, \tau_0) := \mathbb{E}\left[\varphi((\mathbf{v}, \tau)(t, \mathbf{v}_0, \tau_0))\right], \quad (\mathbf{v}_0, \tau_0) \in H_{\mathscr{O}} \times \tilde{H}_{\mathscr{O}}. \tag{7.8}$$

We aim to prove, exploiting the weak convergence of the solutions $(\mathbf{v}, \tau)$ of the problem (3.1) - (3.3), that the family $\{\mathscr{P}_t\}_{t \geq 0}, i = 1, 2$ is sequentially weak Feller. Finally using this property, along with certain a-priori estimates, we prove existence of an invariant measure by means of Maslowski-Seidler Theorem [56]. For a similar approach, see [14], where existence of an invariant measure for two-dimensional stochastic Navier-Stokes equations driven by a cylindrical Wiener process has been established. However our work extends to jump noise.

It follows from [1] that $\mathscr{P}_t \varphi \in \mathscr{B}_b(H_{\mathscr{O}} \times \tilde{H}_{\mathscr{O}})$, and $\{\mathscr{P}_t\}_{t \geq 0}$ is a semigroup on $\mathscr{B}_b(H_{\mathscr{O}} \times \tilde{H}_{\mathscr{O}})$. Thus, as a consequence of Theorem 6.4, we have the following standard result (see Proposition 4.2 of [1], Section 9.6 of [65]).

**Proposition 7.5.** *The family* $((\mathbf{v}, \tau)(t, \mathbf{v}_0, \tau_0))$, $t \geq 0$, $(\mathbf{v}_0, \tau_0) \in H_{\mathscr{O}} \times \tilde{H}_{\mathscr{O}}$ *is Markov. In particular,* $\mathscr{P}_{t+s} = \mathscr{P}_t \mathscr{P}_s$ *for* $t, s \geq 0$.

Moreover, by Proposition 4.3 of [1], $\{\mathscr{P}_t\}_{t \geq 0}$ is a Feller semigroup, i.e. if $\varphi \in C_b(H_{\mathscr{O}} \times \tilde{H}_{\mathscr{O}})$ and $t \geq 0$, then $\mathscr{P}_t \varphi \in C_b(H \times \tilde{H})$.

Now we will prove one of the main results of this section.

**Proposition 7.6.** *The semigroup* $\{\mathscr{P}_t\}_{t \geq 0}$ *is sequentially weakly Feller, i.e. if* $\varphi : H_{\mathscr{O}} \times \tilde{H}_{\mathscr{O}} \to \mathbb{R}$ *is a bounded sequentially weakly continuous function, then for* $t > 0$, $\mathscr{P}_t \varphi : H_{\mathscr{O}} \times \tilde{H}_{\mathscr{O}} \to \mathbb{R}$ *is also a bounded sequentially weakly continuous function. In particular, if* $(\mathbf{v}_{0n}, \tau_{0n}) \to (\mathbf{v}_0, \tau_0)$ *weakly in* $H_{\mathscr{O}} \times \tilde{H}_{\mathscr{O}}$, *then*

$$\left(\mathscr{P}_t \varphi\right)(\mathbf{v}_{0n}, \tau_{0n}) \to \left(\mathscr{P}_t \varphi\right)(\mathbf{v}_0, \tau_0).$$

**Proof.** Let us choose and fix $0 < t \leq T$, $(\mathbf{v}_0, \tau_0) \in H_{\mathscr{O}} \times \tilde{H}_{\mathscr{O}}$. Let $\varphi : H_{\mathscr{O}} \times \tilde{H}_{\mathscr{O}} \to \mathbb{R}$ be bounded sequentially weakly continuous functions. Let us also choose an $H_{\mathscr{O}} \times \tilde{H}_{\mathscr{O}}$-valued sequence $(\mathbf{v}_{0n}, \tau_{0n})$ which is weakly convergent to $(\mathbf{v}_0, \tau_0)$ in $H_{\mathscr{O}} \times \tilde{H}_{\mathscr{O}}$.

Since the function $\mathscr{P}_t \varphi : H_{\mathscr{O}} \times \tilde{H}_{\mathscr{O}} \to \mathbb{R}$ is already bounded, we are left to prove that it is sequentially weakly continuous. Let $\left((\mathbf{v}_n, \tau_n)(\cdot)\right) = \left((\mathbf{v}, \tau)(\cdot, \mathbf{v}_{0,n}, \tau_{0,n})\right)$ (resp. $\left((\mathbf{v}, \tau)(\cdot)\right) = \left((\mathbf{v}, \tau)(\cdot, \mathbf{v}_0, \tau_0)\right)$) be a strong solution of the system (3.1) - (3.3) on $[0, T]$ with the initial data $\left(\mathbf{v}_{0,n}, \tau_{0,n}\right)$ (resp. $\left(\mathbf{v}_0, \tau_0\right)$), and these processes are defined on the same stochastic basis $\left(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, N_1, N_2\right)$, which exist due to Theorem 6.4. Now, by the continuous dependence on the initial data (see Theorem 7.4), for every $T > 0$, there exist

   (i) a subsequence $(n_k)_k$,
  (ii) a stochastic basis $\left(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}}\right)$, with a filtration $\hat{\mathbb{F}} = \left(\hat{\mathcal{F}}_s\right)_{s \in [0,T]}$,
 (iii) two independent time homogeneous Poisson random measures $\hat{N}_i(s, \cdot)$, $i = 1, 2$, $s \in [0, T]$ defined on this stochastic basis
 (iv) $\hat{\mathbb{F}}$-progressively measurable processes $\hat{\mathbf{v}}(s)$, $\hat{\tau}(s)$, $\left(\hat{\mathbf{v}}_{n_k}(s)\right)_{k \in \mathbb{N}}$, $\left(\hat{\tau}_{n_k}(s)\right)_{k \in \mathbb{N}}$, $s \in [0, T]$, defined on this basis with laws supported in $\mathcal{Z}$ such that

$$\mathcal{L}\left((\hat{\mathbf{v}}_{n_k}, \hat{\tau}_{n_k})\right) = \mathcal{L}\left((\mathbf{v}_{n_k}, \tau_{n_k})\right) \quad \text{and} \quad \left(\hat{\mathbf{v}}_{n_k}, \hat{\tau}_{n_k}\right) \to \left(\hat{\mathbf{v}}, \hat{\tau}\right) \text{ in } \mathcal{Z}, \quad \hat{\mathbb{P}} - a.s. \quad (7.9)$$

Moreover, the system $\left(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}}, \hat{\mathbf{v}}, \hat{\tau}, \hat{N}_1, \hat{N}_2\right)$ is a weak martingale solution to the problem (3.1) - (3.3) on $[0, T]$ with the initial data $(\mathbf{v}_0, \tau_0)$.

In particular, by (7.9), $\left(\hat{\mathbf{v}}_{n_k}(t), \hat{\tau}_{n_k}(t)\right) \to \left(\hat{\mathbf{v}}(t), \hat{\tau}(t)\right)$ weakly in $H_{\mathscr{O}} \times \tilde{H}_{\mathscr{O}}$ $\hat{\mathbb{P}}$-a.s. Since $\varphi : H_{\mathscr{O}} \times \tilde{H}_{\mathscr{O}} \to \mathbb{R}$ is sequentially weakly continuous functions, we conclude that $\hat{\mathbb{P}}$-a.s.,

$$\left(\varphi\left(\hat{\mathbf{v}}_{n_k}(t), \hat{\tau}_{n_k}(t)\right)\right) \to \left(\varphi\left(\hat{\mathbf{v}}(t), \hat{\tau}(t)\right)\right) \text{ in } \mathbb{R} \times \mathbb{R}.$$

Since the function $\varphi$ is bounded, by the Lebesgue dominated convergence theorem, we infer that

$$\lim_{k \to \infty} \hat{\mathbb{E}}\left[\varphi\left(\hat{\mathbf{v}}_{n_k}(t), \hat{\tau}_{n_k}(t)\right)\right] = \hat{\mathbb{E}}\left[\varphi\left(\hat{\mathbf{v}}(t), \hat{\tau}(t)\right)\right]. \quad (7.10)$$

Since by (7.9), $\mathcal{L}\big((\hat{\mathbf{v}}_{n_k}, \hat{\tau}_{n_k})\big) = \mathcal{L}\big((\mathbf{v}_{n_k}, \tau_{n_k})\big)$, $k \in \mathbb{N}$, on the space $\mathcal{Z}$, we infer that the laws are also same on $H_w \times \tilde{H}_w$. Hence

$$\hat{\mathbb{E}}\Big[\varphi\big(\hat{\mathbf{v}}_{n_k}(t), \hat{\tau}_{n_k}(t)\big)\Big] = \mathbb{E}\Big[\varphi\big(\mathbf{v}_{n_k}(t), \tau_{n_k}(t)\big)\Big] =: (\mathscr{P}_t\varphi)\big(\mathbf{v}_{0n_k}, \tau_{0n_k}\big). \tag{7.11}$$

By assumption, the system $\big(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \mathbf{v}, \tau, N_1, N_2\big)$ is a weak martingale solution to the problem (3.1) - (3.3) on $[0, T]$ with the initial data $(\mathbf{v}_0, \tau_0)$. Also by above (assertion (iv)), the system $\big(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}}, \hat{\mathbf{v}}, \hat{\tau}, \hat{N}_1, \hat{N}_2\big)$ is a weak martingale solution to the problem (3.1) - (3.3) on $[0, T]$ with the initial data $(\mathbf{v}_0, \tau_0)$. Hence by uniqueness in law (see Theorem 6.4), we assert that the processes $(\mathbf{v}, \tau)$ and $(\hat{\mathbf{v}}, \hat{\tau})$ have the same laws on $\mathcal{Z}$. Thus

$$\hat{\mathbb{E}}\Big[\varphi\big(\hat{\mathbf{v}}(t), \hat{\tau}(t)\big)\Big] = \mathbb{E}\Big[\varphi\big(\mathbf{v}(t), \tau(t)\big)\Big] =: (\mathscr{P}_t\varphi)(\mathbf{v}_0, \tau_0). \tag{7.12}$$

Hence by (7.10), (7.11), and (7.12), we obtain as $k \to \infty$

$$(\mathscr{P}_t\varphi)\big(\mathbf{v}_{0n_k}, \tau_{0n_k}\big) \to (\mathscr{P}_t\varphi)(\mathbf{v}_0, \tau_0).$$

Finally, using the standard subsequence argument, we conclude that the whole sequence $\big((\mathscr{P}_t\varphi)(\mathbf{v}_{0n}, \tau_{0n})\big)_{n \in \mathbb{N}}$ is also convergent, and as $n \to \infty$

$$(\mathscr{P}_t\varphi)(\mathbf{v}_{0n}, \tau_{0n})) \to (\mathscr{P}_t\varphi)(\mathbf{v}_0, \tau_0)).$$

Thus the proof is now complete.  $\square$

Let us denote by $\big\{\mathscr{P}_t^\star\big\}_{t \geq 0}$ the dual semigroup acting on finite Borel measures on $H_\mathscr{O} \times \tilde{H}_\mathscr{O}$.

**Lemma 7.7.** *Let* $(\mathbf{v}_0, \tau_0) \in H_\mathscr{O} \times \tilde{H}_\mathscr{O}$, *and let* $\big(\mathbf{v}(t), \tau(t)\big)$, $t \geq 0$, *be the unique solution to the system* (3.1) - (3.3). *Then there exists* $T_0 \geq 0$ *such that for every* $\varepsilon > 0$ *there exists* $R > 0$ *such that*

$$\sup_{T \geq T_0} \frac{1}{T} \int_0^T \big(\mathscr{P}_s^\star\delta_0\big) \Big((H_\mathscr{O} \times \tilde{H}_\mathscr{O}) \backslash \bar{\mathbb{B}}_{H_\mathscr{O} \times \tilde{H}_\mathscr{O}, R}\Big) \, ds \leq \varepsilon,$$

*where* $\bar{\mathbb{B}}_{H_\mathscr{O} \times \tilde{H}_\mathscr{O}, R} := \Big\{(\mathbf{v}, \tau) \in H_\mathscr{O} \times \tilde{H}_\mathscr{O} : |\mathbf{v}|_{H_\mathscr{O}} + |\tau|_{\tilde{H}_\mathscr{O}} \leq R\Big\}$.

**Proof.** By direct use of Chebyshev inequality and inequality (7.6) from Corollary 7.3, we obtain for every $T \geq 0$ and $R > 0$,

$$\frac{1}{T} \int_0^T \big(\mathscr{P}_s^\star\delta_{\mathbf{v}_0}\big) \Big((H_\mathscr{O} \times \tilde{H}_\mathscr{O}) \backslash \bar{\mathbb{B}}_{H_\mathscr{O} \times \tilde{H}_\mathscr{O}, R}\Big) \, ds = \frac{1}{T} \int_0^T \mathbb{P}\big(|\mathbf{v}(s)|_{H_\mathscr{O}} + |\tau(s)|_{\tilde{H}_\mathscr{O}} > R\big) \, ds$$

$$\leq \frac{1}{TR^2} \frac{1}{2C}\big[|\mathbf{v}_0|^2_{H_\mathscr{O}} + |\tau_0|^2_{\tilde{H}_\mathscr{O}} + CT\big]e^{CT}.$$

Therefore for fixed $T > 0$, the above expression can be made sufficiently small by suitable choice of $R$. Hence the assertion follows. □

We now state the main result of this section.

**Theorem 7.8.** *Let $\mathscr{O}$ be a Poincaré domain in $\mathbb{R}^2$ and Assumption 2.4 be satisfied. Then there exists an invariant measure of the semigroup $\left(\mathscr{P}_t\right)_{t \geq 0}$ defined by (7.8), i.e., a probability measure $\rho$ on $H_{\mathscr{O}} \times \tilde{H}_{\mathscr{O}}$ such that*

$$\mathscr{P}_t^\star \rho = \rho.$$

The proof follows from Proposition 7.6, Lemma 7.7, and Maslowski-Seidler Theorem [56].

## 8. Proof of the Main Result 1

**Proof.** From Remark 2, Proposition 7.6 and Theorem 7.8, we conclude the proof of Main Result 1. □

## Acknowledgments & disclosures

## Appendix A. Marcus canonical SDEs

In this section, we provide some details outlining an analogue of Stratonovich integral in the case of stochastic integral with respect to compensated Poisson Random Measure. We borrow certain details about Marcus Canonical SDEs from the joint work of the first named author of this paper and his collaborator [9]. For more details about this topic, interested readers may look into the works of Marcus [55], Sections 4.4.5 and 6.10 of Applebaum [2], Chechkin and Pavlyukevich [18] and Kunita [41].

Let $\left(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}\right)$ be a filtered probability space, where $\mathbb{F} = \left(\mathcal{F}_t\right)_{t \geq 0}$ is the filtration, satisfying the usual hypotheses. Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k : \mathbb{R}^d \to \mathbb{R}^d$ be complete $C^1$-vector fields. Define $\mathbf{v} : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^k, \mathbb{R}^d)$ such that $\mathbf{v}(y)(h) := \sum_{j=1}^k \mathbf{v}_j(y)h_j$, $h \in \mathbb{R}^k$, $y \in \mathbb{R}^d$.

Let $L(t) := (L_1(t), \cdots, L_k(t))$ be a $\mathbb{R}^k-$ valued Lévy process with pure jump,

$$L(t) = \int_0^t \int_B l \, \tilde{N}(ds, dl) + \int_0^t \int_{B^c} l \, N(ds, dl),$$

where $B := \mathbb{B}(0, 1) \subset \mathbb{R}^k$, $l = (l_1, \ldots, l_k) \in \mathbb{R}^k$; $N$, $\tilde{N}$ represent time homogeneous Poisson random measure with $\lambda$ being the corresponding intensity measure of $N$ and the compensated one with the compensator $Leb \otimes \lambda$ respectively. We always assume that $N$ is independent of $\mathcal{F}_0$.

Let us consider the following 'Marcus' stochastic differential equation:

$$dY(t) = \mathbf{v}_0(Y(t))\,dt + \mathbf{v}(Y(t-)) \diamond dL(t) = \mathbf{v}_0(Y(t))\,dt + \sum_{j=1}^{k} \mathbf{v}_j(Y(t-)) \diamond dL_j(t), \quad \text{(A.1)}$$

which is defined in the integral form as follows

$$Y(t) = Y_0 + \int_0^t \mathbf{v}_0(Y(s))\,ds + \int_0^t \int_B \left[\Phi(1,l,Y(s-)) - Y(s-)\right]\tilde{N}(ds,dl)$$

$$+ \int_0^t \int_{B^c} \left[\Phi(1,l,Y(s-)) - Y(s-)\right]N(ds,dl)$$

$$+ \int_0^t \int_B \left[\Phi(1,l,Y(s)) - Y(s) - \sum_{j=1}^{k} l_j \mathbf{v}_j(Y(s))\right]\lambda(dl)ds,$$

where $y(t) := \Phi(t,l,y_0)$ solves $\frac{dy}{dt} = \sum_{j=1}^{k} l_j \mathbf{v}_j(y)$, with initial condition $y(0) = y_0$. Then we have the following interesting result from Appendix B of [9].

**Theorem A.1.** *Let $\varphi : \mathbb{R}^d \to \mathbb{R}^k$ be a $C^1$-class function. If $Y$ is an $\mathbb{R}^d$-valued process a solution to* (A.1), *then*

$$\varphi(Y(t)) - \varphi(Y_0) = \int_0^t \varphi'(Y(s))(\mathbf{v}_0(Y(s)))\,ds$$

$$+ \int_0^t \int_{B^c} \left[\varphi(\Phi(1,l,Y(s-))) - \varphi(Y(s-))\right]N(ds,dl)$$

$$+ \int_0^t \int_B \left[\varphi(\Phi(1,l,Y(s-))) - \varphi(Y(s-))\right]\tilde{N}(ds,dl)$$

$$+ \int_0^t \int_B \left[\varphi(\Phi(1,l,Y(s))) - \varphi(Y(s)) - \sum_{j=1}^{k} l_j \varphi'(Y(s))(\mathbf{v}_j(Y(s)))\right]\lambda(dl)ds.$$
$$\text{(A.2)}$$

*Moreover, when $k = d$ and $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ is a $C^1$-diffeomorphism, we define for each $j = 0, 1, \ldots, k$, the "Push-forward" of the vector fields $\mathbf{v}_j$ by $\varphi'$ as $\hat{\mathbf{v}}_j : \mathbb{R}^d \to \mathbb{R}^d$ such that*

$$z \mapsto (d_{\varphi^{-1}(z)}\varphi)(\mathbf{v}_j(\varphi^{-1}(z))) := \varphi'(\varphi^{-1}(z))(\mathbf{v}_j(\varphi^{-1}(z))).$$

*Let $\hat{\mathbf{v}} : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^k, \mathbb{R}^d)$ be as before.*

*Then $Y$ is a solution to* (A.1) *iff $Z(t) := \varphi(Y(t))$ is a solution to*

$$dZ = \hat{\mathbf{v}}_0(Z(t))\, dt + \hat{\mathbf{v}}(Z(t)) \diamond dL(t), \quad Z_0 = \varphi(Y_0). \tag{A.3}$$

One can write down a similar result when the vector fields and the process $Y$ are Hilbert space-valued, see e.g. Appendix B of [9].

We now provide the proof of Lemma 2.5, which can be deduced from Lemma 2.2 of [9]. However, we sketch the proof below.

**Proof of Lemma 2.5.** Since $\psi(z) = |z|_{\tilde{H}}^p$, for all $g, k \in \tilde{H}$,

$$\psi'(z)g = p|z|_{\tilde{H}}^{p-2} \langle z, g \rangle_{\tilde{H}}, \text{ and } \psi''(z)(g, k) = \left[ p(p-2)|z|_{\tilde{H}}^{p-4} (z \otimes z) + p|z|_{\tilde{H}}^{p-2} \right]\langle g, k \rangle_{\tilde{H}}.$$

(1) Note that

$$\psi(\Phi(l, \tau)) - \psi(\tau) = \psi(y(1)) - \psi(y(0)) = \int_0^1 \frac{d}{ds}[\psi \circ y](s)ds = \int_0^1 \psi'(y(s))(\mathcal{R}y(s))ds.$$

Hence

$$|\psi(\Phi(l, \tau)) - \psi(\tau)| \le p \int_0^1 |y(s)|_{\tilde{H}}^{p-1}|\mathcal{R}y(s)|_{\tilde{H}}ds$$

$$\le p\|\mathcal{R}\|_{\mathcal{L}(\tilde{H})} \int_0^1 |y(s)|_{\tilde{H}}^p ds \le \mathfrak{N}p|l|_{\mathbb{R}^k}\|g\|_{\mathcal{L}(\tilde{H})}|\tau|_{\tilde{H}}^p.$$

This proves first part of the Lemma.

(2) We begin with the observation that

$$\psi(\Phi(l, \tau)) - \psi(\tau) - \psi'(\tau)\mathcal{R}\tau = \psi(y(1)) - \psi(y(0)) - \psi'(y(0))\mathcal{R}(y(0))$$

$$:= \int_0^1 (\psi \circ y)'(s)ds - (\psi \circ y)'(0) = \int_0^1 \int_0^s \psi''(y(r))(\mathcal{R}y(r), \mathcal{R}y(r))dr ds$$

$$+ \int_0^1 \int_0^s \psi'(y(r))(\mathcal{R}^2 y(r))dr ds := I_1 + I_2. \tag{A.4}$$

Hence as $0 \le s \le 1$,

$$|I_1| \le p(p-1) \int_0^1 |y(s)|_{\tilde{H}}^{p-2} |\mathcal{R}y(s)|_{\tilde{H}}^2 ds \le p(p-1)\|\mathcal{R}\|_{\mathcal{L}(\tilde{H})}^2 \int_0^1 |y(s)|_{\tilde{H}}^p ds$$

$$\leq p(p-1)\|\mathcal{R}\|^2_{\mathcal{L}(\tilde{H})}|\tau|^p_{\tilde{H}} \int_0^1 \|e^{s\mathcal{R}}\|^p ds \leq \mathfrak{N} p(p-1)\|\mathcal{R}\|^2_{\mathcal{L}(\tilde{H})}|\tau|^p_{\tilde{H}}. \tag{A.5}$$

Similarly, one can obtain

$$|I_2| \leq p\|\mathcal{R}\|^2_{\mathcal{L}(\tilde{H})} \int_0^1 |y(s)|^p_{\tilde{H}} ds \leq \mathfrak{N} p\|\mathcal{R}\|^2_{\mathcal{L}(\tilde{H})}|\tau|^p_{\tilde{H}}. \tag{A.6}$$

Therefore using the estimates (A.5) and (A.6) in (A.4) we have

$$|\psi(\Phi(l,\tau)) - \psi(\tau) - \psi'(\tau)\mathcal{R}\tau| \leq \mathfrak{N} p^2 \|\mathcal{R}\|^2_{\mathcal{L}(\tilde{H})}|\tau|^p_{\tilde{H}} \leq \mathfrak{N} p^2 |l|^2_{\mathbb{R}^k}\|g\|^2_{\mathcal{L}(\tilde{H})}|\tau|^p_{\tilde{H}}. \quad \Box$$

## Appendix B. Compactness and tightness criteria

Let $\mathbb{S}$ be a complete separable metric space with a metric $\rho$. Let us denote by $D([0,T];\mathbb{S})$, the set of all $\mathbb{S}$-valued functions defined on $[0,T]$, which are right continuous and have left limits (càdlàg functions) for every $t \in [0,T]$. The space $D([0,T];\mathbb{S})$ is endowed with the Skorokhod $J$-topology, which is defined below.

**Definition B.1.** The $J$-topology on $D([0,T];\mathbb{S})$ is generated by the following metric $\delta_{T,\mathbb{S}}$, see [5, formulae (12.13) and (12.16)]

$$\delta_{T,\mathbb{S}}(x,y) := \inf_{\lambda \in \Lambda_T} \left[ \sup_{t \in [0,T]} \rho(x(t), y(\lambda(t))) + \sup_{t \in [0,T]} |t - \lambda(t)| \right.$$
$$\left. + \sup_{s<t} \left| \log\left( \frac{\lambda(t) - \lambda(s)}{t-s} \right) \right| \right], \tag{B.1}$$

where $\Lambda_T$ is the set of all increasing homeomorphisms of $[0,T]$.

It follows from Definition B.1, see also top of page 112 in [5], that a sequence $(x_n)_{n\in\mathbb{N}}$ converges in $D([0,T];\mathbb{S})$ to $x$ if and only if there exists a sequence $(\lambda_n)$ in $\Lambda_T$ such that with id being the identity map of $[0,T]$,

$$x_n \circ \lambda_n \to x, \quad \text{uniformly on } [0,T].$$

In particular, if an $D([0,T];\mathbb{S})$-valued sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x$ uniformly, then it converges to $x$ in $D([0,T];\mathbb{S})$ as well.

For more details see Métivier's [57, Chapter II] and Billingsley's [5, Chapter 3].

Now we state the compactness criteria for $\mathbf{v}$ and $\tau$ in the space of cádlag functions.

**Theorem B.1.** *(Compactness criteria for* $(\mathbf{v},\tau)$*) Let* $\mathcal{Z}_1 := D([0,T];\mathcal{U}') \cap D([0,T];H_w) \cap \mathrm{L}^2_w(0,T;V) \cap \mathrm{L}^2(0,T;H_{loc})$ *and* $\mathcal{T}_1$ *be the supremum of the corresponding topologies. Let* $\mathcal{Z}_2 := D([0,T];\tilde{\mathcal{U}}') \cap D([0,T];\tilde{H}_w) \cap \mathrm{L}^2_w(0,T;\tilde{V}) \cap \mathrm{L}^2(0,T;\tilde{H}_{loc})$ *and* $\mathcal{T}_2$ *be the supremum of the corresponding topologies. Define* $\mathcal{Z} := \mathcal{Z}_1 \times \mathcal{Z}_2$ *and* $\mathcal{T}$ *as the supremum of* $\mathcal{T}_1$ *and* $\mathcal{T}_2$*. Then* $(\mathcal{K}_1, \mathcal{K}_2) \subset \mathcal{Z}$ *is* $\mathcal{T}$*-relatively compact if the following three conditions are satisfied:*

(a) $\forall\,(\mathbf{v},\tau)\in\mathcal{K}_1\times\mathcal{K}_2$ and all $t\in[0,T]$, $(\mathbf{v}(t),\tau(t))\in H\times\tilde{H}$, and

$$\sup_{(\mathbf{v},\tau)\in\mathcal{K}_1\times\mathcal{K}_2}\left(\sup_{s\in[0,T]}|\mathbf{v}(s)|_H+\sup_{s\in[0,T]}|\tau(s)|_{\tilde{H}}\right)<\infty,$$

(b) $\sup_{(\mathbf{v},\tau)\in\mathcal{K}_1\times\mathcal{K}_2}\int_0^T\left(|\mathbf{v}(s)|_V^2+|\tau(s)|_{\tilde{V}}^2\right)ds<\infty$, i.e. $(\mathcal{K}_1,\mathcal{K}_2)$ is bounded in $\mathrm{L}^2(0,T;V)\times$
$\mathrm{L}^2(0,T;\tilde{V})$,

(c) $\lim_{\delta\to 0}\sup_{(\mathbf{v},\tau)\in\mathcal{K}_1\times\mathcal{K}_2}\left(w_{[0,T],V'}(\mathbf{v},\delta)+w_{[0,T],\tilde{V}'}(\tau,\delta)\right)=0.$

Proof of Theorem B.1 can be found in Lemma 3.3 in Brzeźniak and Motyl [13], Lemma 4.1 in Motyl [60], Theorem 2 of Motyl [59], Lemma 2.7 in Mikulevicius and Rozovskii [58].

### B.1. Tightness criterion for predictable process

Assume that $\left(\Omega,\mathcal{F},\mathbb{F},\mathbb{P}\right)$ is a filtered probability space, where $\mathbb{F}=\left(\mathcal{F}_t\right)_{t\geq 0}$ is the filtration satisfies the usual conditions.

**Definition B.2.** Let $(\mathbf{u}_n)_{n\in\mathbb{N}}$ be a sequence of càdlàg, $\mathbb{F}$-adapted stochastic processes in a Banach space $E$. Assume that for every $\varepsilon>0$ and $\eta>0$ there is $\delta>0$ such that for every sequence $(\rho_n)_{n\in\mathbb{N}}$ of $\mathbb{F}$-stopping times with $\rho_n\leq T$ one has

$$\sup_{n\in\mathbb{N}}\sup_{0<\theta\leq\delta}\mathbb{P}\left\{\|\mathbf{u}_n(\rho_n+\theta)-\mathbf{u}_n(\rho_n)\|_E\geq\eta\right\}\leq\varepsilon.$$

In this case, we say that $(\mathbf{u}_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition $[A]$.

We now quote the following lemma given in Métivier [57] and Motyl [59], which ensures the Aldous condition $[A]$ in a separable Banach space for the sequence $(\mathbf{u}_n)_{n\in\mathbb{N}}$.

**Lemma B.2.** *Let $(\mathbf{u}_n)_{n\in\mathbb{N}}$ be a sequence of càdlàg, $\mathbb{F}$-adapted stochastic processes in a separable Banach space $E$. Assume that for every sequence $(\rho_n)_{n\in\mathbb{N}}$ of $\mathbb{F}$-stopping times with $\rho_n\leq T$ and for every $n\in\mathbb{N}$ and $\theta\geq 0$ the following condition holds*

$$\mathbb{E}\left[\|\mathbf{u}_n(\rho_n+\theta)-\mathbf{u}_n(\rho_n)\|_E^{\alpha}\right]\leq C\theta^{\zeta},\tag{B.2}$$

*for some $\alpha,\zeta>0$ and some constant $C>0$. Then the sequence $(\mathbf{u}_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition $[A]$ in $E$.*

The following results are useful consequences of the Aldous condition $[A]$.

**Theorem B.3.** *Let $(\mathbf{v}_n,\tau_n)_{n\in\mathbb{N}}$ be a sequence of càdlàg $\mathbb{F}$-adapted $V'\times\tilde{V}'$-valued processes such that*

(a) *there exists a positive constant $C_1$ such that*

$$\sup_{n\in\mathbb{N}}\mathbb{E}\left[\sup_{s\in[0,T]}\left(|\mathbf{v}_n(s)|_H+|\tau_n(s)|_{\tilde{H}}\right)\right]\leq C_1,$$

(b) *there exists a positive constant $C_2$ such that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\int_0^T \left(|\mathbf{v}_n(s)|_V^2 + |\tau_n(s)|_{\tilde{V}}^2\right)ds\right] \leq C_2,$$

(c) $(\mathbf{v}_n)_{n\in\mathbb{N}}$ *(respectively $(\tau_n)_{n\in\mathbb{N}}$) satisfies the Aldous condition $[A]$ in $V'$ (respectively $\tilde{V}'$).*

*Let $(\mathbb{P}_n^1, \mathbb{P}_n^2)$ be the law of $(\mathbf{v}_n, \tau_n)_{n\in\mathbb{N}}$ on $\mathcal{Z} := \mathcal{Z}_1 \times \mathcal{Z}_2$. Then for every $\epsilon > 0$ there exist compact subset $(K_\epsilon^1, K_\epsilon^2)$ of $\mathcal{Z}$ such that $\mathbb{P}_n^i(K_\epsilon^1) \geq 1 - \epsilon$, for $i = 1, 2$ and the sequence of measures $\{(\mathbb{P}_n^1, \mathbb{P}_n^2), n \in \mathbb{N}\}$ is said to be tight on $(\mathcal{Z}, \mathcal{T})$.*

For a proof see Corollary 1, Motyl [59]. In metric spaces, one can apply Prokhorov Theorem (see [64], Theorem II.6.7) and Skorokhod Theorem (see [5], Theorem 6.7) to obtain convergence from tightness. Since the space $\mathcal{Z}$ is a non-metrizable locally convex space, we use the following generalization of Skorokhod's Theorem to nonmetric spaces.

**Proposition B.4** *(Skorokhod-Jakubowski). Let $\mathcal{X}$ be a topological space such that there is a sequence of continuous functions $f_m : \mathcal{X} \to \mathbb{C}$ that separates points of $\mathcal{X}$. Let $\mathcal{A}$ be the $\sigma$-algebra generated by $(f_m)_m$. Then, we have the following assertions:*

a) *Every compact set $K \subset \mathcal{X}$ is metrizable.*
b) *Let $(\chi_n)_{n\in\mathbb{N}}$ be a tight sequence of probability measures on $(\mathcal{X}, \mathcal{A})$. Then, there are a subsequence $(\chi_{n_k})_{k\in\mathbb{N}}$, random variables $\mathbf{v}_k$, $\mathbf{v}$ for $k \in \mathbb{N}$ on a common probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$, where $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_t)_{t\geq 0}$, with $\bar{\mathbb{P}}^{\mathbf{v}_k} = \chi_{n_k}$ for $k \in \mathbb{N}$, and $\mathbf{v}_k \to \mathbf{v}$ $\bar{\mathbb{P}}$-almost surely for $k \to \infty$.*

We stated Proposition B.4 in the form of [15] (see also [39]) where it was first used to construct martingale solutions for stochastic evolution equations. We apply the above result to a concrete situation in this paper and is stated below.

**Corollary B.5.** *Let $(\mathbf{v}_n, \tau_n)_{n\in\mathbb{N}}$ be a sequence of adapted $\mathcal{U}' \times \tilde{\mathcal{U}}'$-valued processes satisfying the Aldous condition $[A]$ in $\mathcal{U}' \times \tilde{\mathcal{U}}'$ and*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\|\mathbf{v}_n\|_{L^\infty(0,T;H)}^2 + \|\tau_n\|_{L^\infty(0,T;\tilde{H})}^2\right] < \infty, \quad \sup_{n\in\mathbb{N}} \mathbb{E}\left[\|\mathbf{v}_n\|_{L^2(0,T;V)}^2 + \|\tau_n\|_{L^2(0,T;\tilde{V})}^2\right] < \infty.$$

*Then, there are a subsequence $(\mathbf{v}_{n_k}, \tau_{n_k})_{k\in\mathbb{N}}$ and random variables $(\bar{\mathbf{v}}_k, \bar{\tau}_k)$ and $(\bar{\mathbf{v}}, \bar{\tau})$ for $k \in \mathbb{N}$ on a common probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$ with $\bar{\mathbb{P}}^{\bar{\mathbf{v}}_k} = \mathbb{P}^{\mathbf{v}_{n_k}}$ and $\bar{\mathbb{P}}^{\bar{\tau}_k} = \mathbb{P}^{\tau_{n_k}}$ for $k \in \mathbb{N}$, and $(\bar{\mathbf{v}}_k, \bar{\tau}_k) \to (\bar{\mathbf{v}}, \bar{\tau})$ $\bar{\mathbb{P}}$-almost surely in $\mathcal{Z}$ for $k \to \infty$.*

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