Continuous Random Variables

02 October 2025 06:22

- Continuous r.v. take values in intervals.
- Example : ① time taken to complete assignment.
  ② Weight of a randomly chosen individual.
- Counting no. of students in a class is discrete, but counting time/ weight/ height is cont.
- Every cont. r.v. has a function f associated with it, called probability density function.
- Let X is cont. r.v. & f is its density then :

$$P[X \in (a,b)] = \int_a^b f(x)\,dx$$

Normal Random Variables

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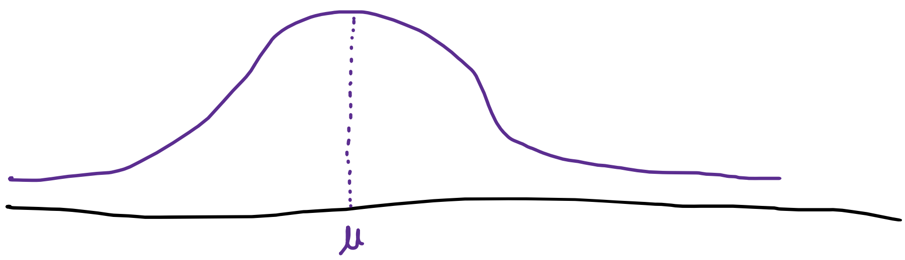
- A cont. r.v. X with density f s.t. f is determined by parameters μ & σ^2 s.t.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) \quad \forall\, x \in \mathbb{R}$$

then X is called Normal r.v. with parameters μ & σ^2.

$\underline{X \sim N}(\mu, \sigma^2)$

- A plot of normal density look like this :



Curve is symmetric along $x = \mu$.

Larger the value of σ, more the spread of f.

- $E(X) = \mu$ & $Var(X) = \sigma^2$

Pf : $E(X) = \sum_x x P(X = x)$

$$= \int_{-\infty}^{\infty} x f(x)\, dx \qquad \because X \text{ is continuous.}$$

$$= \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

$\frac{x-\mu}{\sigma} = t \Rightarrow dx = \sigma\, dt$

$$\frac{1}{} \int^{\infty} (\sigma t + \mu) \exp\left(-\frac{t^2}{}\right) \cancel{\sigma}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{(\sigma t + \mu)}{\cancel{\sigma}} \exp\left(-\frac{t^2}{2}\right) \cancel{\sigma}\, dt$$

$$\sqrt{2\pi}\, E(x) = \underbrace{\sigma \int_{-\infty}^{\infty} \cancel{t \exp\left(\frac{t^2}{2}\right)}}_{\text{odd function}} \xrightarrow{} 0 + \mu \int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{2}\right) dt$$

$$\frac{\sqrt{2\pi}}{\mu} E(x) = \int_{-\infty}^{\infty} e^{-t^2/2}\, dt = 2\int_{0}^{\infty} e^{-t^2/2}\, dt$$

$$\frac{t^2}{2} = x \Rightarrow dt = \frac{dx}{\sqrt{2x}}$$

$$\frac{\cancel{\sqrt{2}}\sqrt{\pi}}{\mu} E(x) = \frac{\sqrt{2}}{\cancel{2}} \int_0^{\infty} \frac{e^{-x}\, dx}{\sqrt{2}\, x} = \int_0^{\infty} x^{-1/2} e^{-x}\, dx$$

$$\frac{\sqrt{\pi}\, E(x)}{\mu} = \int_0^{\infty} x^{1/2 - 1} e^{-x}\, dx = \Gamma(1/2) = \sqrt{\pi}$$

$$\Rightarrow E(x) = \mu$$

$\blacksquare$

$Var(X) = \sigma^2.$

Pf: $Var(X) = E(X^2) - E^2(X)$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

$$\frac{x-\mu}{\sigma} = t \Rightarrow dx = \sigma\, dt$$

$$E(X^2) = \frac{1}{\sqrt{2\pi}} \int^{\infty} \frac{(\sigma t + \mu)^2}{} \exp\left(-\frac{t^2}{2}\right) \cdot \cancel{\sigma}\, dt$$

$$E(X^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{(\sigma t + \mu)^2}{\cancel{\sigma}} \exp\left(-\frac{t^2}{2}\right) \cdot \cancel{\sigma}\, dt$$

$$\sqrt{2\pi}\, E(X^2) = \int_{-\infty}^{\infty} (\sigma^2 t^2 + 2\mu\sigma t + \mu^2) \exp\left(-\frac{t^2}{2}\right) dt$$

$$= 2\sigma^2 \int_0^{\infty} t^2 e^{-t^2/2}\, dt + \cancel{2\mu\sigma \int_{-\infty}^{\infty} t e^{-t^2/2}\, dt} \to 0 + 2\mu^2 \int_0^{\infty} e^{-\frac{t^2}{2}} dt$$

$$\frac{t^2}{2} = x \Rightarrow dt = \frac{dx}{\sqrt{2x}}$$

$$= 2\sigma^2 \int_0^{\infty} \frac{2x e^{-x}}{\sqrt{2x}} dx + 2\mu^2 \int_0^{\infty} \frac{e^{-x}}{\sqrt{2x}} dx$$

$$= 2\sqrt{2}\sigma^2 \int_0^{\infty} x^{1/2} e^{-x} dx + \sqrt{2}\mu^2 \int_0^{\infty} x^{-1/2} e^{-x} dx$$

$$= 2\sqrt{2}\sigma^2\, \Gamma(3/2) + \sqrt{2}\mu^2\, \Gamma(1/2)$$

$$\Gamma(n+1) = n\Gamma(n) \Rightarrow \Gamma(3/2) = \frac{1}{2}\Gamma(1/2)$$

$$= \frac{\cancel{2}\sqrt{2}\sigma^2}{\cancel{2}} \Gamma(1/2) + \sqrt{2}\mu^2\, \Gamma(1/2)$$

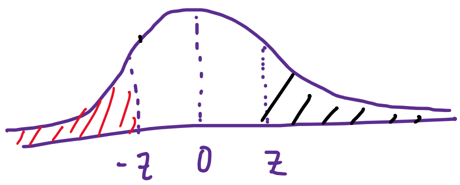
$$\cancel{\sqrt{2\pi}}\, E(X^2) = \cancel{\sqrt{2\pi}}(\sigma^2 + \mu^2)$$

$$Var(X) = E(X^2) - E^2(X) \Rightarrow (\sigma^2 + \mu^2) - \mu^2 = \underline{\underline{\sigma^2}} \quad \blacksquare$$

- A normal random variable with $\mu = 0$ & $\sigma = 1$ is called a standard normal r. variable.
- Z is a standard Normal or $Z \sim N(0,1)$, then

$$\Phi(x) := P(Z \le x).$$

- Area(red shaded) = Area(black shaded) =

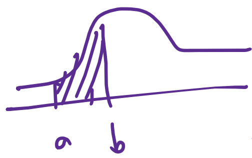


$$P(Z \le -z) = P(Z > z)$$

$$\Phi(-z) = 1 - \Phi(z)$$

Example 2.2a Let Z be a standard normal random variable. For $a < b$, express $P\{a < Z \le b\}$ in terms of Φ.

$P(a < Z \le b) =$ Shaded area (



)

$$= P(Z \le b) - P(Z \le a)$$

$$= \underline{\Phi(b) - \Phi(a)}$$

OR

$$P(a < Z \le b) = P(Z > a \text{ and } Z \le b)$$

$$= P(Z > a) + P(Z \le b) - P(Z \le a \text{ or } Z > b)$$

$$= \cancel{1} - \Phi(a) + \Phi(b) - \cancel{1}$$

$$= \underline{\Phi(b) - \Phi(a)}$$

Table 2.1: $\Phi(x) = P\{Z \leq x\}$

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

- For higher Accuracy :

$$\Phi(x) \approx 1 - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \left(a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4 + a_5 y^5\right)$$

where $y = \dfrac{1}{1 + 0.2316419\,x}$

where $y = \dfrac{1}{1 + 0.2316419\,x}$

$a_1 = 0.319381530$

$a_2 = -0.356563782$

$a_3 = 1.781477937$

$a_4 = -1.821255978$

$a_5 = 1.330274429$

Properties of Normal Random Variables

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- If X is normal then $aX+b$ is also normal $\forall a,b \in \mathbb{R}$ $a \neq 0$.
  & $X \sim N(\mu, \sigma^2)$ then $aX+b \sim N(a\mu+b, a^2\sigma^2)$

Pf: $E(aX+b) = a\mu + b$
$Var(aX+b) = a^2\sigma^2$ $\boxtimes$

- For any Normal r.v. X with μ & σ^2,
  $Z = \dfrac{X-\mu}{\sigma}$, Z is standard normal r.v.

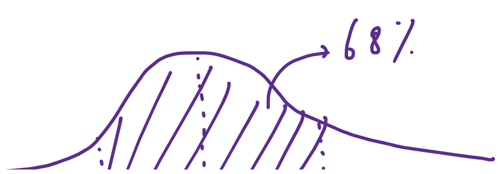
Example 2.3a IQ examination scores for sixth-graders are normally distributed with mean value 100 and standard deviation 14.2. What is the probability that a randomly chosen sixth-grader has an IQ score greater than 130?

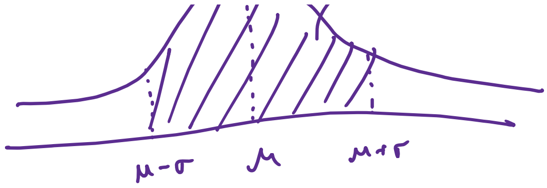
$\mu = 100$, $\sigma = 14.2$, X is r.v. of marks s.t. $X \sim N(\mu, \sigma^2)$

$P(X > 130)$ $\quad Z = \dfrac{X-100}{14.2}$

$$P\left(\frac{X-100}{14.2} > \frac{130-100}{14.2}\right) = P(Z > 2.11)$$
$$= 1 - \Phi(2.11)$$
$$= 1 - 0.9821$$
$$= \underline{\underline{0.018}}$$

- Around 68% of the time Normal will be between 1 s.d.
  95% between 2 s.d. 99.74% between 3 s.d.





- Sum of independent normal r.v.'s is also a normal r.v.

$X_i \sim N(\mu_i, \sigma_i^2)$

$X = \sum_{i=1}^{n} X_i$, $X \sim N\left(\sum_{i=1}^{n} \mu_i , \sum_{i=1}^{n} \sigma_i^2\right)$

Example 2.3c The annual rainfall in Cleveland, Ohio, is normally distributed with mean 40.14 inches and standard deviation 8.7 inches. Find the probabiity that the sum of the next two years' rainfall exceeds 84 inches.

X_1 is rainfall for this year

X_2 is —— " —— next year

$P(X_1 + X_2 > 84) = ?$

$\overset{iid}{\sim} N(40.14, 8.7)$

$X = X_1 X_2 \sim N(80.28, 23.19)$

$P\left(\frac{X - 80.28}{23.19} > \frac{84 - 80.28}{23.19}\right) = 1 - \Phi(0.1604)$

≈ 0.3812

- A r.v. Y is said to be lognormal (μ, σ^2) if

$\log(Y) \sim N(\mu, \sigma^2)$

Let $X \sim N(\mu, \sigma^2)$ & $Y = e^X$

$E(Y) = E(e^X) = \int_{-\infty}^{\infty} e^x \cdot \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma}\right) dx$

$E(Y) = E(e^{X}) = \int_{-\infty} \ldots \frac{1}{\sqrt{2\pi\sigma^2}}$

$$\boxed{\begin{aligned} E(Y) &= e^{\mu + \frac{\sigma^2}{2}} \\ Var(Y) &= e^{2\mu + \sigma^2}\left(e^{\sigma^2} - 1\right) \end{aligned}}$$

Pf: WLOG assume $N(0,1)$

$$E(Y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{x} \cdot e^{-\frac{x^2}{2}}\, dx \qquad \text{Claim: } E(Y) = e^{1/2}.$$

$$\sqrt{2\pi}\, E(Y) = \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2} + x\right) dx$$

$$= \int_{-\infty}^{\infty} \exp\left(-\left(\frac{x^2}{2} - x + \frac{1}{2}\right) + \frac{1}{2}\right)$$

$$= e^{1/2} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-1)^2}{2}\right)$$

$$= \quad x - 1 \equiv t$$

$$= e^{+1/2} \int_{-\infty}^{\infty} e^{-t^2/2}\, dt = e^{1/2} \cdot \sqrt{2\pi}$$

$$E(Y) = e^{1/2} \qquad \blacksquare$$

$$E(Y^2) = \int_{-\infty}^{\infty} e^{2x} \cdot \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx \qquad \text{Claim } V(X) = e(e-1)$$

$$\sqrt{2\pi}\, E(Y^2) = \int_{-\infty}^{\infty} \exp\left(-\left(\frac{x^2}{2} - 2x\right)\right) dx$$

$$\int^{\infty} \exp\left(-\frac{1}{2}\left(x^2 - 4x + 4 - 4\right)\right) dx$$

$$= \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left(x^2 - 4x + 4 - 4\right)\right) dx$$

$$= e^2 \int_{-\infty}^{\infty} \exp\left(\frac{(x-2)^2}{2}\right) dx$$

$$= e^2 \sqrt{2\pi}$$

$$E(Y^2) = e^2$$

$$Var(Y^2) = e^2 - (e^{1/2})^2 = e^2 - e = e(e-1) \quad \boxtimes$$

Example 2.3d Starting at some fixed time, let $S(n)$ denote the price of a certain security at the end of n additional weeks, $n \geq 1$. A popular model for the evolution of these prices assumes that the price ratios $S(n)/S(n-1)$ for $n \geq 1$ are independent and identically distributed (i.i.d.) lognormal random variables. Assuming this model, with lognormal parameters $\mu = .0165$ and $\sigma = .0730$, what is the probability that

(a) the price of the security increases over each of the next two weeks;
(b) the price at the end of two weeks is higher than it is today?

$$X_n = \frac{S_n}{S_{n-1}} \overset{iid}{\sim} \log Norm(\mu, \sigma^2)$$

$t = 0$ week. (now)

(a). $X_1 = \dfrac{S_1}{S_0}$, $X_2 = \dfrac{S_2}{S_1}$

$$P(X_1 > 1 \ \& \ X_2 > 1) = P(X_1 > 1) . P(X_2 > 1)$$
$$= P^2(X_1 > 1) \quad (iid)$$

$X_1 \sim \log Nor(\mu, \sigma^2)$

$X_1 = e^Y$ where $Y \sim N(\mu, \sigma^2)$

$$= P^2(e^Y > 1) = P^2(Y > \ln(1))$$

$X_1 = e^Y$ where $Y \sim N(\mu, \sigma^2)$

$$P^2(X_1 > 1) = P^2(e^Y > 1) = P^2(Y > \ln(1))$$

$$= P^2(Y > 0)$$

$$= P^2\left(\frac{Y-\mu}{\sigma} > \frac{-0.0165}{0.730}\right)$$

$$= P^2(Z > -0.226)$$

$$= P^2(Z < 0.226)$$

$$= \Phi^2(0.226)$$

$$= (0.589)^2$$

$$\approx 0.35$$

(b) $P(S_2 > S_0) \Rightarrow P\left(\frac{S_2}{S_0} > 1\right) \Rightarrow P\left(\frac{S_2}{S_1} \cdot \frac{S_1}{S_0} > 1\right)$

$$P\left(\log\left(\frac{S_2}{S_1}\right) + \log\left(\frac{S_1}{S_0}\right) > 0\right)$$

$$P(Y_1 + Y_2 > 0) \qquad Y_1 + Y_2 \sim N(2\mu, 2\sigma^2)$$

$$P\left(Z > \frac{-2\mu}{\sqrt{2}\,\sigma}\right) = \Phi\left(\frac{\sqrt{2}\,\mu}{\sigma}\right)$$

$$= \Phi(0.258) = \underline{\underline{0.603}}$$

The Central Limit Theorem

03 October 2025 03:17

- This Thm states that the sum of a large number of indp random variables, all having same distribution, will itself be approximately a normal random variable.

Theorem: Let $X_1, X_2, \ldots$ be iid r.v. & $S_n = \sum_{i=1}^{n} X_i$. For large n, S_n will approximately be a normal r.v. with expected value $n\mu$ & variance $n\sigma^2$.

$$\lim_{n \to \infty} \left(P\left(\frac{S_n - n\mu}{\sqrt{n}\,\sigma} \le x \right) \right) = \Phi(x)$$

Example 2.4a A fair coin is tossed 100 times. What is the probability that heads appears fewer than 40 times?

$$X_i = \begin{cases} 1 & \text{if Head is outcome of } i^{th} \text{ toss} \\ 0 & \text{else} \end{cases} \quad X_i \overset{iid}{\sim} \text{Bern}(0.5)$$

$$X = \sum_{i=1}^{100} X_i .$$

$$P(X < 40) = ?$$

By CLT, $\frac{X - \mu n}{\sqrt{n}\sigma} \sim N(0,1)$ $\mu = 0.5$, $n = 100$, $\sqrt{n} = 10$, $\sigma = \sqrt{0.25} = 0.5$

$$P\left(\frac{X - 50}{5} < \frac{-10}{5} \right) = 1 - \Phi(+2) = 1 - 0.977$$
$$= 0.023$$

[Exact Solution By Computer Program is $\underline{\overline{0.176}}$]

Exercises

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Exercise 2.1 For a standard normal random variable Z, find:

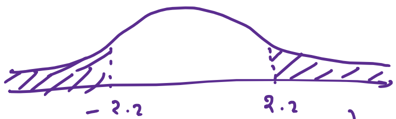
(a) $P\{Z < -.66\}$;
(b) $P\{|Z| < 1.64\}$;
(c) $P\{|Z| > 2.20\}$.

(a) $P(Z < -0.66) = \Phi(-0.66) = 1 - \Phi(0.66) = 1 - 0.7454$
$= \underline{0.2546}$

(b) $P\{|Z| < 1.64\} =$

$\Phi(1.64) - \Phi(-1.64)$
$2\Phi(1.64) - 1 = 2(0.9484) - 1 = 1.8968 - 1 = \underline{0.8968}$

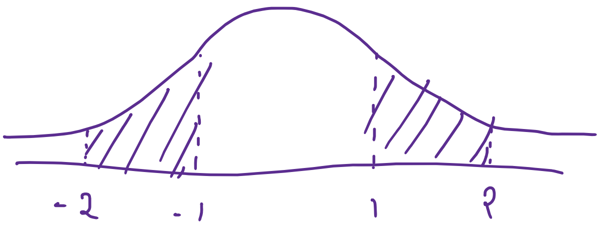
(c) $P\{|Z| > 2.20\} =$



$2(1 - \Phi(2.2)) = 2(1 - 0.9861) = 2 \times 0.0139$
$= \underline{0.0278}$

Exercise 2.2 Find the value of x when Z is a standard normal random variable and

$$P\{-2 < Z < -1\} = P\{1 < Z < x\}.$$

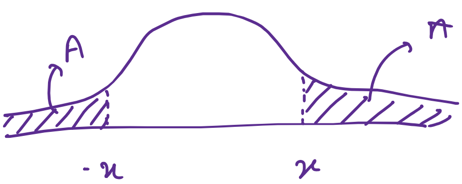


$x = 2$

Exercise 2.3 Argue (a picture is acceptable) that

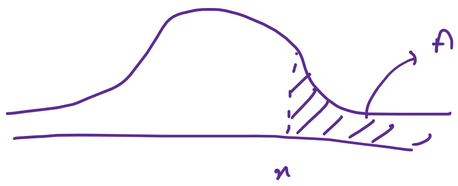
$$P\{|Z| > x\} = 2P\{Z > x\},$$

where $x > 0$ and Z is a standard normal random variable.



$= \underline{2A}$

$2x$



$= 2A$

Exercise 2.4 Let X be a normal random variable having expected value μ and variance σ^2, and let $Y = a+bX$. Find values a, b ($a \neq 0$) that give Y the same distribution as X. Then, using these values, find $\mathrm{Cov}(X, Y)$.

$X \sim N(\mu, \sigma^2)$

$Y = a + bX$.

If $X \overset{d}{=} Y$, then $E(X) = E(Y)$ & $Var(X) = Var(Y)$

$a + b\mu = \mu$ & $b^2 \cancel{\mu} = \cancel{\mu}\ 1$

Case 1 : $\mu = 0$

$b \in \mathbb{R}$, $a = \mu(1-b)$

Case 2 : $\mu \neq 0$

$b = \pm 1$, $a = \cancel{0}, \mu$ $\because a \neq 0$

$\therefore \underline{\underline{Y = \mu - X}}$

$Cov(X, Y) = Cov(X, \mu - X) = Cov(X, \mu) - Cov(X, X)$

$= 0 - Var(X)$

$= -\sigma^2$ //

Exercise 2.5 The systolic blood pressure of male adults is normally distributed with a mean of 127.7 and a standard deviation of 19.2.

(a) Specify an interval in which the blood pressures of approximately 68% of the adult male population fall.

(b) Specify an interval in which the blood pressures of approximately 95% of the adult male population fall.

(c) Specify an interval in which the blood pressures of approximately 99.7% of the adult male population fall.

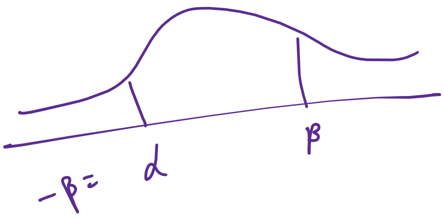
$P(a < X < b) = 68\%$

$P(a < X < b) = 68\%$

$P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right) = 0.68$

$P(\alpha < Z < \beta) = 0.68$

Assume $\alpha = -\beta$



$P(-\beta < Z < \beta) = \Phi(\beta) - \Phi(-\beta)$

$= 2\Phi(\beta) - 1 = 0.68$

$\Phi(\beta) = \frac{1.68}{2} = 0.84$

$\underline{\beta = 1}$

$\therefore \frac{a - 127.7}{19.2} = -1$

$\Rightarrow a = 108.5$

$\frac{b - 127.7}{19.2} = 1$

$\Rightarrow b = 146.9$

$\therefore X \in \underline{[108.5, 146.9]}$

(b) Similarly

$\beta = \Phi^{-1}\left(\frac{.95 + 1}{2}\right) = \Phi^{-1}(0.98) = 2$

$X \in [87.3, 166.1]$

(c) $\beta = \Phi^{-1}\left(\frac{.997 + 1}{2}\right) = \Phi^{-1}(.998) = 3$

$X \in [68.1, 185.3]$

Exercise 2.6 Suppose that the amount of time that a certain battery functions is a normal random variable with mean 400 hours and stan-

Exercise 2.6 Suppose that the amount of time that a certain battery functions is a normal random variable with mean 400 hours and standard deviation 50 hours. Suppose that an individual owns two such batteries, one of which is to be used as a spare to replace the other when it fails.

(a) What is the probability that the total life of the batteries will exceed 760 hours?
(b) What is the probability that the second battery will outlive the first by at least 25 hours?
(c) What is the probability that the longer-lasting battery will outlive the other by at least 25 hours?

X_1 = Life of first battery
X_2 = Life of second Battery

sd = 50
$\sigma^2 = 2500$

100

$X = X_1 + X_2 \Rightarrow X \sim N(800, 5000)$

$P(X > 760) = 1 - P(X \le 760) = 1 - P\left(\frac{X-800}{100\sqrt{2}} \le \frac{-\cancel{40}\,2\sqrt{2}}{\cancel{100}\,5}\right)$

$= 1 - \Phi\left(-\frac{2\sqrt{2}}{5}\right) = \Phi(0.4 \times 1.414) \Rightarrow \Phi(0.5657)$

$= 0.7142$

(b) $P(X_2 - X_1 > 25)$ $\quad X_2 - X_1 \sim N(0, 5000)$

$1 - \Phi\left(\frac{\cancel{25}\sqrt{2}}{\cancel{100}\,4}\right) = 1 - \Phi\left(\frac{1.414}{4}\right) = 1 - \Phi(0.353)$

$= 1 - 0.637 = \underline{0.363}$

(c) $P(|X_2 - X_1| > 25) = 2\big(P(|X_2 - X_1| > 25)\big) = \underline{0.726}$

X **Exercise 2.7** The time it takes to develop a photographic print is a random variable with mean 18 seconds and standard deviation 1 second. Approximate the probability that the total amount of time that it takes to process 100 prints is

(a) more than 1,710 seconds;
(b) between 1,690 and 1,710 seconds.

$T_i \overset{iid}{\sim} \text{Dist}(\mu = 18, \sigma^2 = 1)$

$T = \sum_{i=1}^{100} T_i$ $\quad T \sim \text{Dist}(\mu = 1800, \sigma^2 = 100)$

$$\frac{T - 100 \times 18}{10 \times 10} \sim N(0,1)$$

$$\frac{T - 1800}{100} \sim N(0,1)$$

(a) $P(T > 1710) = P\left(\frac{T-1800}{10} > -.9\right)$

$= 1 - \Phi(-.9) = \Phi(.9) = \underline{\underline{.8159}}$

(b) $P(1690 < T < 1710)$

$P(-1.10 < Z < -.9) = P(.9 < Z < 1.1)$

$= \Phi(1.1) - \Phi(.9)$

$= .8643 - 0.8159 = \underline{\underline{0.0484}}$

Exercise 2.8 Frequent fliers of a certain airline fly a random number of miles each year, having mean and standard deviation of 25,000 and 12,000 miles, respectively. If 30 such people are randomly chosen, approximate the probability that the average of their mileages for this year will

(a) exceed 25,000;
(b) be between 23,000 and 27,000.

$A = \frac{1}{30} \sum_{i=1}^{30} F_i \quad , \quad \sum_{1}^{30} F_i = F$

$F \sim N(30 \times 25000, \; 30 \times (12000)^2)$

$A \sim N\left(25000, \frac{(12000)^2}{30}\right)$

(a) $P(A > 25000) = \underline{\underline{1/2}}$

(b) $P(23000 < A < 27000) = P\left(\frac{-2000\sqrt{30}}{12000} < Z < \frac{2000\sqrt{30}}{12000}\right)$

$= \frac{\sqrt{30}}{6} \approx \frac{5 + \frac{5}{10}}{6} = 5.5 \times \frac{1}{6} = 0.913$

$2\Phi(0.913) - 1 = \underline{\underline{0.6388}}$

Exercise 2.9 A model for the movement of a stock supposes that, if the present price of the stock is s, then – after one time period – it will either be us with probability p or ds with probability $1 - p$. Assuming that successive movements are independent, approximate the probability that the stock's price will be up at least 30% after the next 1,000 time periods if $u = 1.012$, $d = .990$, and $p = .52$.

After 1000 time periods, final price $= s\,u^{t} d^{1000-t}$.

where t = no. of times stock increased.

$T_i = \begin{cases} 1 & \text{stock increased} \\ 0 & \text{else} \end{cases} \qquad T_i \sim \text{Bern}(p)$

$t = \sum_{i=1}^{1000} T_i \Rightarrow t \sim N(1000p,\ 1000p(1-p))$

$t \sim N(520, (15.799)^2)$

$$P\left(\frac{s u^{t} d^{1000-t} - s}{s} > 0.3\right) = ?$$

$$\Rightarrow P\left(u^{t} d^{1000-t} > 1.3\right)$$

$$= P\left(t(\ln u - \ln d) + 1000 \ln d > \ln(1.3)\right)$$

$$= P\left(t > \frac{\ln(1.3) - 1000 \ln d}{\ln u - \ln d}\right)$$

$$= P\left(t > \frac{0.2624 + 10.0503}{0.0220}\right)$$

$$= P(t > 468.7) = P(t > 468)$$

$$= 1 - P(t \le 468)$$

$$= 1 - P\left(\frac{t - 520}{15.8} \le \frac{468 - 520}{15.8}\right)$$

$$= \Phi(3.29) \approx 0.9995$$

Exercise 2.10 In each time period, a certain stock either goes down 1 with probability .39, remains the same with probability .20, or goes up 1 with probability .41. Assuming that the changes in successive time pe-

Exercise 2.10 In each time period, a certain stock either goes down 1 with probability .39, remains the same with probability .20, or goes up 1 with probability .41. Asuming that the changes in successive time periods are independent, approximate the probability that, after 700 time periods, the stock will be up more than 10 from where it started.

find price $= s + a - b$

$$\text{Change} = \begin{cases} +1 & 0.41 \\ 0 & 0.20 \\ -1 & 0.39 \end{cases}$$

$$A_i = \begin{cases} 1 & \text{stock increased} \\ 0 & \text{else} \end{cases} \qquad A_i \overset{iid}{\sim} \text{Ber}(0.41)$$

$$B_i = \begin{cases} 1 & \text{Stock decreased} \\ 0 & \text{else} \end{cases} \qquad B_i \overset{iid}{\sim} \text{Ber}(0.39)$$

$$a = \sum_{i=1}^{700} A_i, \quad b = \sum_{i=1}^{700} B_i$$

$$a \sim N(700(0.41),\ 700(0.41)(0.59))$$
$$b \sim N(700(0.39),\ 700(0.39)(0.61))$$

$$a \sim N(287,\ (13)^2)$$
$$b \sim N(273,\ (12.9)^2)$$

$$P(\text{Increase} > 10) = P(a - b > 10)$$

$$a - b \sim N(14,\ (18.33)^2) \qquad \{a \text{ \& } b \text{ are not indp}\}$$

$$P(a-b > 10) = 1 - P(a-b \le 10)$$
$$= 1 - P\left(\frac{(a-b)-6}{18.33} \le \frac{-4}{18.33}\right)$$
$$= 1 - \Phi(-.21) = \Phi(0.21)$$
$$= \underline{0.5832}$$

OR

$$C_i = \begin{matrix}\text{Change} \\ \text{for } i\end{matrix} = \begin{cases} -1 & 0.39 \\ 0 & 0.2 \\ +1 & 0.41 \end{cases}$$

$$E(C_i) = 0.02$$

$$Var(C_i) = 0.7996$$

$$C = \text{Total Change} = \sum_{i=1}^{700} C_i$$

$$C \sim N(14, (23.66)^2)$$

$$P(C > 10) = 1 - P(C \leq 10)$$

$$= 1 - P\left(\frac{C-14}{23.66} \leq \frac{-4}{23.66}\right)$$

$$= \Phi(0.17) = \underline{\underline{0.5675}} \text{ Ans}$$