FFE : Problem Set-2

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Problem-1 : Intertemporal optimality

2 period : t and $t+1$

endowments : e_t and e_{t+1}

Consumption : c_t and c_{t+1}.

risk free asset : ~~ ~~ $1 \longrightarrow R_f$
at t at $t+1$

investment in risk free asset : θ.

Utility f^n : $u(c) = \dfrac{c^{1-\gamma}}{1-\gamma}$, with $\gamma > 0$ & $\neq 1$

disc rate : β

① Budget constraint at t:

$$e_t = c_t + \theta. \Rightarrow c_t = e_t - \theta \longrightarrow (1)$$

Budget constraint at $t+1$

$$e_{t+1} = c_{t+1} - \theta \times R_f. \Rightarrow c_{t+1} = e_{t+1} + \theta R_f \longrightarrow (2)$$

②. Optimal condition

Agent is utility maximiser.

So agent problem is, $\longrightarrow (3)$

$$\boxed{\arg\max_{\theta} \; u(c_t) + \beta u(c_{t+1}) \quad \text{s.t.} \quad c_t = e_t - \theta \quad \text{and } c_{t+1} = e_{t+1} + \theta R_f.}$$

$$L = u(C_t) + \beta\, u(C_{t+1})$$

$$= \frac{C_t^{1-\gamma}}{1-\gamma} + \beta \times \frac{C_{t+1}^{1-\gamma}}{1-\gamma}$$

$$= \frac{(e_t - \theta)^{1-\gamma} + \beta \times \left[e_{t+1} + \theta R_t \right]^{1-\gamma}}{1-\gamma}$$

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$$\frac{dL}{d\theta} = 0$$

$$\Rightarrow \frac{(1-\gamma)\cdot(e_t-\theta)^{-\gamma} \times (-1) + \beta(1-\gamma)\,(e_{t+1} + \theta R_f)^{-\gamma} \times R_t}{(1-\gamma)} = 0$$

$$\Rightarrow (e_t-\theta)^{-\gamma} = \beta R_t (e_{t+1} + \theta R_t)^{-\gamma}$$

$$\Rightarrow \frac{1}{(e_t-\theta)^{\gamma}} = \frac{\beta R_t}{(e_{t+1} + \theta R_t)^{\gamma}}$$

$$\Rightarrow \cancel{(\beta R_t)^{(1/\gamma)} \times (e_{t+1} + \theta R_t) = e_t - \theta}$$

$$\Rightarrow \cancel{\theta\left[1 + (\beta R_t)^{(1/\gamma)}\right.}.$$

$$\Rightarrow (\beta R_t)^{(1/\gamma)} (e_t - \theta) = (e_{t+1} + \theta R_t).$$

$$\Rightarrow \theta\left(R_t + (\beta R_t)^{(1/\gamma)}\right) = e_t \times (\beta R_t)^{(1/\gamma)} - e_{t+1}$$

$$\Rightarrow \boxed{\theta^* = \frac{\left(e_t \times (\beta R_f)^{(1/r)} - e_{t+1}\right)}{\left(R_f + (\beta R_f)^{(1/r)}\right)}} \longrightarrow (4)$$

Using θ^* in budget condition.

$$c_t^* = e_t - \theta^*$$

$$= e_t - \frac{\left[e_t \times (\beta R_f)^{(1/r)} - e_{t+1}\right]}{\left[R_f + (\beta R_f)^{(1/r)}\right]}.$$

$$=$$

and $\cancel{c_{t+1}^* = e_{t+1} + \theta^* \times R_f}$

$$=$$

$$\Rightarrow \boxed{c_t = \frac{\left(e_t R_f + e_{t+1}\right)}{\left(R_f + (\beta R_f)^{(1/r)}\right)}} \longrightarrow (5)$$

Similarly $c_{t+1}^* = e_{t+1} + \theta^* \times R_f$

$$= e_{t+1} + \frac{e_t \times (\beta R_f)^{(1/r)} \times R_f - e_{t+1} \times R_f}{\left(R_f + (\beta R_f)^{(1/r)}\right)}$$

$$\Rightarrow \boxed{c_{t+1}^* = \frac{(\beta R_f)^{(1/r)} \times \left[e_{t+1} + e_t \times R_f\right]}{\left[R_f + (\beta R_f)^{(1/r)}\right]}} \longrightarrow (6)$$

For log utility f^n:

$$\gamma = 1$$

So $$c_t^* = \frac{e_t R_f + e_{t+1}}{R_f}.$$

i.e. $$c_t^* = e_t + \frac{e_{t+1}}{R_f}$$

and $$c_{t+1}^* = \left(\frac{\beta}{1+\beta}\right) \times \left(e_{t+1} + e_t \times R_f\right)$$

$\longrightarrow$ (7)

(3) Consumption growth rate.

$$\frac{c_{t+1}^*}{c_t^*} = \frac{\dfrac{(\beta R_f)^{(1/\gamma)} \times (e_{t+1} + e_t R_f)}{(R_f + (\beta R_f)^{(1/\gamma)})}}{\dfrac{(e_{t+1} + e_t R_f)}{(R_f + (\beta R_f)^{(1/\gamma)})}}$$

$$\Rightarrow \quad \frac{c_{t+1}^*}{c_t^*} = (\beta R_f)^{1/\gamma}. \longrightarrow (8)$$

So if $\beta R_f > 1 \Rightarrow$ consumption decreases as γ increases.

if $\beta R_f < 1 \Rightarrow$ consumption ~~decreases~~ increases as γ increases.

Problem - 2. Refer excel sheet "Consumption data_Srinivas"

(4)

CRRA preferences $u(C) = \frac{C^{1-\gamma}}{1-\gamma}$

$\beta = 0.99$.

For representative agent SDF is given by.

$$M_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)}$$

$$= \beta \times \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} \longrightarrow (1).$$

~~C_{t+1} & C_t are available in the columns Total_growth & Total_lag_growth.~~

~~Time series of M_{t+1} $\rightarrow$ please refer spreadsheet~~

Here $\frac{C_{t+1}}{C_t}$ i.e. the growth rate is available (calculated) ~~in the~~ column Total growth rat

Price of risk (Normality assumption of g_c).

(6) assume $g_c = \log\left(\frac{C_{t+1}}{C_t}\right) \sim N(\mu, \sigma_c^2)$.

$\because$ g_c is normally distributed.

So $\frac{C_{t+1}}{C_t}$ is log-normally distributed.

Thus $E\left(\frac{C_{t+1}}{C_t}\right) = \exp\{\mu + \frac{1}{2}\sigma_c^2\}$.

and $Var\left(\frac{C_{t+1}}{C_t}\right) = (e^{\sigma_c^2} - 1) \times \exp\{2\mu + \sigma_c^2\}$.

~~Thus~~ $\cancel{E(m_t) = \beta \times E\left(\frac{C_{t+1}}{C_t}\right)}$.

Thus $E(m) = \beta \times \exp\{\gamma(-\mu + \gamma\frac{\sigma_c^2}{2})]$.

$\therefore$ Price of risk $\boxed{\varphi = \frac{\sigma(m)}{E(m)} = [\exp(\sigma_c^2\gamma^2) - 1]^{1/2}}$