Introduction to game theory by example

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Introduction

"L'art de découvrir en mathématique est plus précieux que la plupart des choses qu'on découvre"

FONTENELLE

Game theory is a discipline that studies interactive decision-making, where multiple individuals, or players, must make decisions and choose actions that will result in an outcome. The interests of the players are often divergent, which can lead to conflicts or competition.

While mathematicians have always shown a strong interest in games of chance and strategy, the first theoretical works on strategic games appeared in the early 20th century with Zermelo [Zer12], Borel [Bor21], and von Neumann [Neu28]. Game theory truly emerged as a discipline thanks to the work of mathematician John von Neumann and economist Oskar Morgenstern, who wrote the foundational book *Games and Economic Behavior* in 1944 [NM44]. The contributions of John Nash [Nas50] in providing a solution concept for non-zero-sum games further solidified the foundations of game theory. Since then, the field has seen significant mathematical development and has found numerous applications in various disciplines, including biology, computer science, and economics. Its success in economics is particularly remarkable, with several game theorists being awarded the Nobel Prize : John C. Harsanyi, John Nash, and Reinhardt Selten in 1994 [HNS94], and Robert J. Aumann and Thomas C. Schelling in 2005 [AS05].

Today, game theory includes several branches, such as cooperative games, strategic games, games with incomplete information, dynamic games, and differential games.

Classification by Number of Moves
Classification by the Information Available to Players
Classification by Relations Between Players
Other Classes of Games

1. Classification of Games

1.1 Classification by Number of Moves

Games in Normal Form (Strategic Form)

These are games that occur in a single move. The normal form of a game can be used when players act simultaneously. In the case of finite games (where players' strategy sets are finite), the representation is done via a payoff matrix, where the rows and columns correspond to different strategies, and the table shows the payoffs for each possible outcome.

Games in Extensive Form

These are games that involve multiple moves. The extensive form is used in games where the rules specify that certain players act multiple times. The representation of such a game is done via a Kuhn tree, which clarifies the sequence of actions taken by the players and the information available to them at each node.

Remarque 1.

In sequential games, the player is required to predict future moves of the opponent(s). The choice of strategy is determined by calculating the future consequences, a concept known as **anticipation**.

1.2 Classification by the Information Available to Players

Games with Complete Information

A game is said to have complete information if each participant knows :

1. Their set of strategies,
2. The set of strategies available to other players,
3. All possible outcomes and their associated payoffs,
4. The motives of other players (as well as their own).

This class includes games where all relevant information for the game is observable by all. Example : Chess.

Games with Incomplete Information

A game is said to have incomplete information if players lack information about :

1. The available strategies,
2. The payoff functions (the outcomes resulting from choosing various strategies).

1.2.1 Classification by Relations Between Players

Non-Cooperative Games

A non-cooperative game is one where players cannot form coalitions, though they may agree on certain outcomes provided there is no enforceable binding agreement. No player will seek to manipulate the others ; they aim only to maximize their own payoff.

Cooperative Games

A game is cooperative when players communicate with each other and can form coalitions through binding agreements. In other words, players can hand over their decision-making power to a collective entity they create together.

Remarque 2.

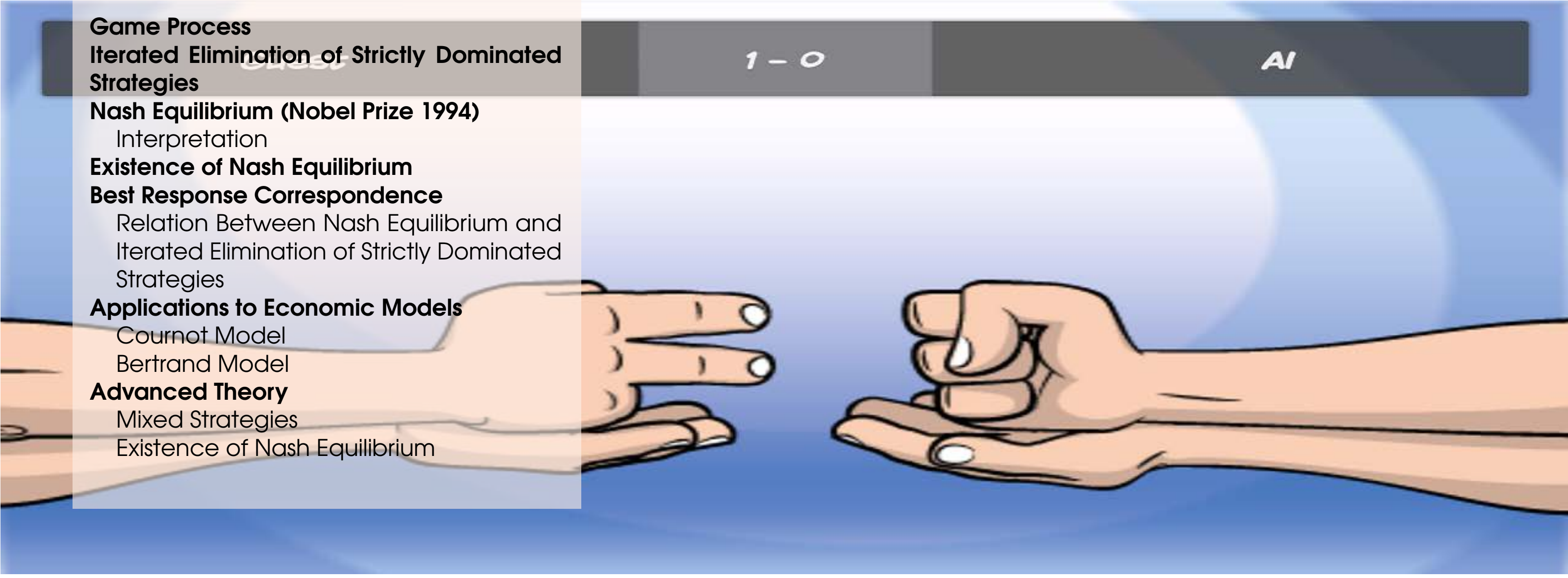
Cooperative and non-cooperative games can further be divided into two subcategories :

1. Games with side payments : these are games where utility transfers between players are possible, allowing players to transfer part of their gain to others to form coalitions and increase their total payoffs.
2. Games without side payments : players receive only the payoffs determined by the game's rules and are prohibited from transferring any portion of their payoffs to others.

1.2.2 Other Classes of Games

There are other game classifications that we can mention in this course :

- Two-person games (duels) : these constitute the majority of common games.
- N-person games : the number of participants is greater than two.
- Zero-sum games : the total payoff at the end of the game is zero, meaning the amount won by one player equals the amount lost by others. Example : Business situations, political life, or the Prisoner's Dilemma.
- Single-criterion and multi-criteria games : depending on the number of criteria or attributes considered by each player.



Game Process
Iterated Elimination of Strictly Dominated Strategies
Nash Equilibrium (Nobel Prize 1994)
Interpretation
Existence of Nash Equilibrium
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Cournot Model
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Existence of Nash Equilibrium

2. Non-Cooperative Strategic Games

We define a strategic game (where players act simultaneously) as :

$$\langle I, X, f(x) \rangle$$

where :
$I = \{1, .., n\}$ is the set of players,
$X = \prod_{i=1}^{n} X_i$ is the set of action profiles, $X \subset \mathbb{R}^s$,
X_i is the set of strategies for player i, $X_i \subset \mathbb{R}^{n_i}, \sum_{i=1}^{n} n_i = s$,
$x_i \in X_i$ is a strategy for player i,
$x = (x_1, ..., x_n) \in X$ is a profile of actions for the game,
$f = (f_1, ..., f_n), f_i : X \to \mathbb{R}$ is a payoff function for player i.

Notation : Let i be any player. The rest of the players $I \setminus \{i\}$ will be denoted by $I - i$, and X_{I-i} denotes $\prod_{j \in I-i} X_j$. Let x and x^0 be two action profiles of the game, and x_i be player i's strategy in outcome x. The profile of actions obtained by replacing player i's strategy x_i^0 in profile x^0 with strategy x_i in profile x is denoted as (x_i, x_{I-i}^0). That is :

$$(x_i, x_{I-i}^0) = (x_1^0, ..., x_{i-1}^0, x_i, x_{i+1}^0, ..., x_n^0).$$

In particular, $(x^0, x_{I-i}^0) = x^0$.

2.1 Game Process

Consider the game (J) and suppose that :

1. The players are rational, i.e., they choose their strategies to maximize their payoffs,
2. Each player knows the set of strategies and the payoff functions of all other players,
3. No binding agreements exist between players, i.e., the game is non-cooperative,
4. Side payments are prohibited.

The lack of cooperation between players may be due to one of the following reasons :

1. The players' interests are contradictory,
2. Lack of communication,
3. Legal prohibition.

Définition 1.

$\alpha_i = \sup_{x_i \in X_i} \inf_{x_{I-i} \in X_{I-i}} f_i(x_i, x_{I-i}), \forall i \in I$ is called the security level of player i, or the minimum guaranteed payoff for player i.

Définition 2.

A profile of actions x^0 for game (J) satisfies the principle of individual rationality if it provides each player i with a payoff greater than or equal to α_i. In other words :

$$\forall i \in I, \alpha_i \leq f_i(x^0).$$

Examples of Mathematical Game Models

Exemple 1 (Prisoner's Dilemma)**.**

"The police have arrested two suspects who committed a crime together and are interrogating them separately. Each is given the following deal : if your accomplice confesses and you stay silent, you will get ten years in prison, and your accomplice will go free with a suspended sentence. If the reverse happens, you will go free while your accomplice serves ten years. If both confess, the sentence will be split (five years each)." If both remain silent, each will serve three years.

The possible choices for the prisoners (denoted P_1 and P_2) can be represented by the following table :

		Prisoner 1	
		Remain Silent	Confess
Prisoner 2	Remain Silent	(-3,-3)	(-10,0)
	Confess	(0,-10)	(-5,-5)

In this situation, both prisoners (players) face two options : confess or remain silent. Each prisoner will choose the strategy that best suits them, considering the other's potential action. This situation can be modeled as a two-person, non-cooperative (since they are separated), strategic-form game (since it occurs in a single round).

1. The strategy sets for both players are $X_i = \{\text{confess, remain silent}\}$ for $i = 1, 2$.
2. The set of action profiles for the game is $X = X_1 \times X_2$

$$X = \{(\text{confess, confess}), (\text{remain silent, confess}), (\text{confess, remain silent}), (\text{remain silent, remain silent})\}$$

3. The payoffs for the players represent the number of years in prison based on their choices.
4. Each player's goal is to maximize their gain (minimize the number of years in prison).
5. The payoff function for player i, $f_i : X_i \to \mathbb{R}$, is defined as :

$$f_1(\text{remain silent, remain silent}) = f_2(\text{remain silent, remain silent}) = -3$$

$$f_1(\text{remain silent, confess}) = f_2(\text{confess, remain silent}) = -10$$
$$f_1(\text{confess, remain silent}) = f_2(\text{remain silent, confess}) = 0$$
$$f_1(\text{confess, confess}) = f_2(\text{confess, confess}) = -5$$

The most suitable situation for both prisoners is one in which neither has an incentive to deviate. According to the principle of individual rationality, the best strategy for each player is to confess. The outcome (confess, confess) does not represent the best result for both players. The optimal outcome is (remain silent, remain silent), where both players are freed, but this can only be achieved if they cooperate.

Remarque 3.

The Prisoner's Dilemma shows that there are games where :

1. The optimal situation according to the principle of individual rationality does not guarantee the best result.
2. The minimal payoff can only be achieved through cooperation between the players.

Exemple 2 (Cournot Duopoly 1938)**.**

Consider two firms, A and B, producing the same product without cost (or with negligible cost), and their production capacities are infinite. The profit of these two firms results from the total quantity presented to the market.
This example can be modeled as a two-person, non-cooperative, strategic-form game (A and B produce simultaneously) as follows :

1. $I = \{A, B\} = \{1, 2\}$ is the set of players.
2. $X_i = [0, +\infty], i = 1, 2$ is the strategy set for player i. A player's strategy (or a firm's action) represents the quantity of the product they produce.
3. The payoffs for both players A and B, resulting from strategies (actions) $x_i \in X_i, i = 1, 2$, are represented by $f_i(x_1, x_2), i = 1, 2$, defined as follows :

$$f_1(x_1, x_2) = [p - (x_1 + x_2)]x_1$$

$$f_2(x_1, x_2) = [p - (x_1 + x_2)]x_2$$

According to Cournot, the solution to this game is a pair of outputs (x_1, x_2), where x_1 is for firm A, and x_2 is for firm B, such that each firm maximizes its profit, considering the output of the other.

2.2 Iterated Elimination of Strictly Dominated Strategies

Définition 3.

In a normal-form game, consider two possible actions for player i, denoted x_i' and x_i'' ($\in X_i$). The strategy x_i' is strictly dominated by strategy x_i'' if, for every possible combination of strategies chosen by the other players, the utility of playing x_i' is strictly less than that of playing x_i'' :

$$u_i(x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_n) < u_i(x_1, \dots, x_{i-1}, x_i'', x_{i+1}, \dots, x_n)$$

$$\forall (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \{X_1 \times \cdots \times X_n\}$$

Remarque 4.

A rational player will never play a strictly dominated strategy.

Exemple 3.

Consider the following example We can iterate the elimination process :

⁕ ⁕ ⁕	Player 1			
	⁕ ⁕ ⁕	Left	Middle	Right
Player 2	Top	(1,0)	(1,2)	(0,1)
	Bottom	(0,3)	(0,1)	(2,0)

We eliminate Right, then Bottom, then Left.
This process is called : ***Iterated elimination of strictly dominated strategies.***

Disadvantages

This method has a few disadvantages that need to be considered :

✈ **Limitation 1 (Player Rationality)** : At each step, an additional assumption is needed about what players know regarding each other's rationality. For an arbitrary number of steps, we need to assume common knowledge of rationality (i.e., "I know that you know that I know that you know...").

✈ **Limitation 2 : Sometimes imprecise prediction of the game's solution.**

⊠ ⊠ ⊠	L	C	R
T	0,4	4,0	5,3
M	4,0	0,4	5,3
B	3,5	3,5	6,6

There are no strictly dominated strategies to eliminate !

2.3 Nash Equilibrium (Nobel Prize 1994)

Définition 4.

A profile of actions $x^0 = (x_1^0, x_2^0, \ldots, x_n^0) \in X$ in (J) is a Nash equilibrium if it satisfies :

$$\forall i \in I, \forall x_i \in X, f_i(x_i, x_{I-i}^0) \leq f_i(x^0)$$

2.3.1 Interpretation

1. The concept of Nash equilibrium describes the stability of the solution x^0, in the sense that each player i, $i \in I$, has no incentive to unilaterally change their choice x_i^0 if the other players choose their strategies from x^0.
2. Nash equilibrium is a no-regret situation, as in this equilibrium, each player does not regret the choice they made after observing the choices of others.
3. Nash equilibrium is an individually rational situation. In other words, if x^0 is a Nash equilibrium, then $f_i(x^0) \geq \alpha_i, \forall i \in I$.

Remarque 5.

1. In the Prisoner's Dilemma, the situation (confess, confess) is a Nash equilibrium.
2. Every dominant strategy equilibrium is a Nash equilibrium, but the converse is not always true.
3. Nash equilibrium is not always unique (Example : Battle of the Sexes).
4. Nash equilibrium generalizes Cournot's equilibrium (corresponding to the intersection of reaction curves where each firm produces to maximize its profit given the output of its competitor).
5. The existence problem of this equilibrium has been deeply studied using powerful tools from mathematical analysis such as Ky Fan's inequality, Brouwer's fixed-point theorem, and Kakutani's fixed-point theorem.

2.4 Existence of Nash Equilibrium

Définition 5.

For each player i, we say that x_i^0 is a best response for player i against a profile of actions of their opponents x_{I-i} if :

$$\forall x_i \in X_i, f_i(x_i, x_{I-i}) \leq f_i(x_i^0, x_{I-i})$$

2.5 Best Response Correspondence

Définition 6.

The best response correspondence of player i is the subset of strategies $C(i) \subset X_i$ that maximizes their utility given the strategies of the other players x_{I-i} :

$$C_i : X_{I-i} \longrightarrow X_i; \quad x_{I-i} \longrightarrow C_i(x_{I-i}) \tag{2.1}$$

with

$$C_i = \{x_i^0 \in X_i \mid \sup_{x_i \in X_i} f_i(x_i, x_{I-i}) = f_i(x_i^0, x_{I-i})\}$$

In other words, $C_i(x_{I-i}) = \{x_i^0 \in X_i \mid f_i(x_i, x_{I-i}) \leq f_i(x_i^0, x_{I-i}), \forall x_i \in X_i\}$.

This means that the best response correspondence for a player gives their optimal choices in response to all possible strategies. We can combine the best responses of all players by defining the correspondence :

$$C : X \longrightarrow X; \quad x \longrightarrow C(x) = \prod_{i \in I} C_i(x_{I-i}) \tag{2.2}$$

The relationship between the Nash equilibrium and the best response correspondence of game (J) is presented in the following proposition :

Définition 7.

We say that $x \in X$ is a fixed point of the correspondence $C(.)$ if $x \in C(x)$.

Proposition 1.

A profile of actions x^0 in (J) is a Nash equilibrium if and only if x^0 is a fixed point of $C(.)$.

Preuve 1. *x^0 is a Nash equilibrium :*

$$\begin{aligned}
&\Leftrightarrow \quad \forall i \in I, \forall x_i \in X_i, f_i(x_i, x_{I-i}^0) \leq f_i(x_i^0, x_{I-i}^0) = f_i(x^0) \\
&\Leftrightarrow \quad \forall i \in I, x_i^0 \in C_i(x^0) \\
&\Leftrightarrow \quad x^0 \in C(x^0).
\end{aligned}$$

Remarque 6.

1. The above proposition gives us a characterization of Nash equilibria using the best response correspondence $C(.)$. This characterization allows us to present sufficient conditions for the existence of a Nash equilibrium. To do this, it suffices to gather the conditions for the existence of a fixed point of the correspondence $C(.)$ (Kakutani's fixed-point theorem).
2. Sometimes, we will denote the best response correspondence of a player $i \in I$ by $MR_i(x_{I-i})$.

In this section, we will limit ourselves to stating a single existence theorem, whose proof is based on the concept of a fixed point of a correspondence.

Théorème 8 (Nash's Theorem)**.**

Suppose that in (J), the following conditions are satisfied :

1. $X_i, i = 1, \ldots, n$ are non-empty, convex, and compact ;
2. The functions $x_i \longrightarrow f_i(x_i, x_{I-i})$ are quasi-concave for all $x_{I-i} \in X_{I-i}, i = 1, \ldots, n$; the functions $x \longrightarrow f_i(x)$ are continuous for all $i = 1, \ldots, n$.

Then (J) has at least one Nash equilibrium.

How to find the Nash equilibrium ? : A brute-force application

We underline the best responses of each player given what the other player does.

Let's revisit the example of the Prisoner's Dilemma :

		Prisoner 1	
		Silent	Confess
Prisoner 2	Silent	(-1,-1)	(-9,**0**)
	Confess	(**0**,-9)	**(-6,-6)**

(Confess, Confess) is the unique Nash equilibrium, meaning both prisoners have an incentive to confess !

Exemple 4.

⁂	D	C
D	**1,1**	**5**,0
C	0,**5**	4,4

(D, D) is the unique Nash equilibrium.

2.5.1 Relation Between Nash Equilibrium and Iterated Elimination of Strictly Dominated Strategies

Proposition 2.

In an n-player game, if IESDS (Iterated Elimination of Strictly Dominated Strategies) eliminates all strictly dominated strategies except $(s_1^*, \ldots, s_n^*)$, then $(s_1^*, \ldots, s_n^*)$ is the unique Nash equilibrium.

Proposition 3.

In an n-player game, if the strategies $(s_1^*, \ldots, s_n^*)$ form a Nash equilibrium, then they survive IESDS.

The Nash equilibrium is a stronger solution concept than iterated elimination of strictly dominated strategies, in the sense that strategies in a Nash equilibrium survive the elimination process, while the converse is not always true.

Multiplicity of Equilibria

Exemple 5 (The Battle of the Sexes or Coordination Game)**.**

	Man		
		Opera	Boxing
Woman	Opera	(2,1)	(0,0)
	Boxing	(0,0)	(1,2)

This table illustrates the "Battle of the Sexes," a classic example of a coordination game. The scores indicate the respective preferences of Woman and Man for the activities of Opera and Boxing.
(Opera, Opera) and **(Boxing, Boxing)** are Nash equilibria, where neither party has an incentive to change their strategy unilaterally.

2.6 Applications to Economic Models

2.6.1 Cournot Model

The Cournot duopoly (Antoine Augustin Cournot, 1838) corresponds to a situation where each firm independently chooses the quantities to produce for the market. These quantities are decided with knowledge of the market structure (number of competitors = 1) and the demand function. Neither firm can learn in advance the production of its competitor.

In this case, firm 1 must calculate the quantities that maximize its profit for every possible level of production by its competitor (q_2), so as to determine in advance the best response it can give to each of its competitor's strategies.

Firm 1 must also neglect the repercussions of its own production on these quantities, as they will not be observed in advance by its competitor.

In summary :

- Cournot 1838 (Spring Water)
- Two producers, 1 and 2, produce respective quantities q_1 and q_2.
- They make their choices simultaneously.
- Total quantity on the market : $Q = q_1 + q_2$.
- Inverse demand : $P(Q) = a - Q$ for $Q < a$ and $P(Q) = 0$ otherwise.
- No fixed costs and constant marginal cost : $C_i(q_i) = cq_i$.
- Each producer must maximize : $\max_{q_i} q_i P(q_i + q_j) - cq_i$.
- The Nash equilibrium is found by solving a system of two equations with two unknowns :
  - $q_1^* = \frac{(a-q_2^*-c)}{2}$
  - $q_2^* = \frac{(a-q_1^*-c)}{2}$

  This results in : $q_1^* = q_2^* = \frac{a-c}{3}$, which is the ***Cournot-Nash equilibrium***.

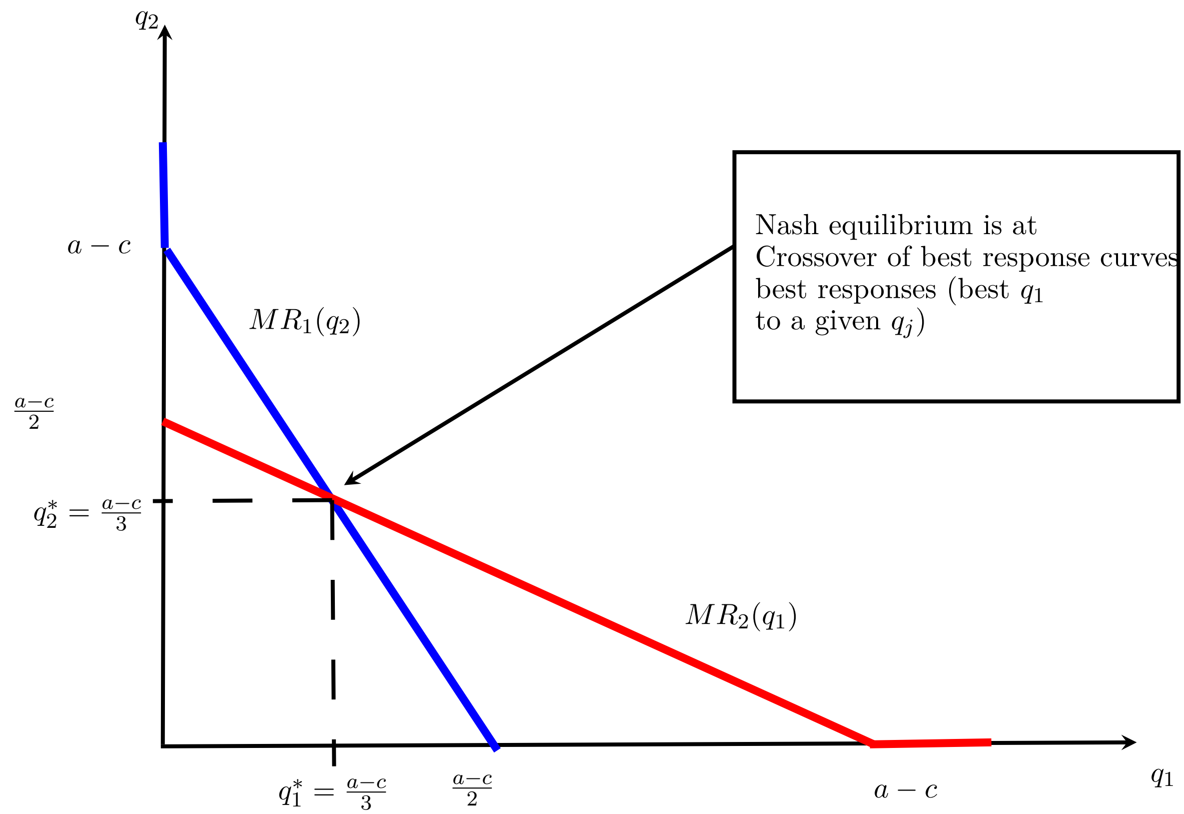

FIGURE 2.1 – Graphical representation of the Cournot-Nash equilibrium

2.6.2 Bertrand Model

Also known as price competition : the Bertrand duopoly (1883) is a critique of Cournot's work. It corresponds to a situation where firms compete through prices. Each firm seeks to maximize its profit by adjusting its price.

The characteristics of this model are :

- ☞ The two players choose prices p_1 and p_2, not quantities.
- ☞ Differentiated products.
- ☞ They make their choices simultaneously.
- ☞ The quantity demanded by consumers from producer i :

$$q_i(p_i, p_j) = a - p_i + bp_j$$

  where b measures the substitutability between the two products.
- ☞ No fixed costs and constant marginal cost : $C_i(q_i) = cq_i$.
- ☞ Each producer must maximize : $\max_{p_i}(p_i - c)q_i(p_i, p_j)$.
- ☞ Bertrand-Nash equilibrium (complementary strategies) :

$$p_1^* = p_2^* = \frac{a+c}{2-b}$$

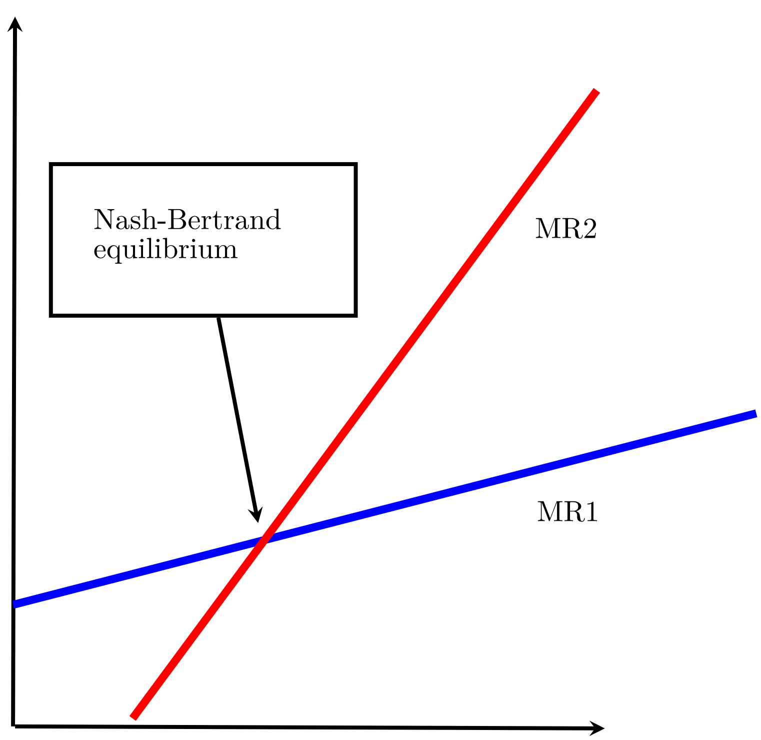

FIGURE 2.2 – Graphical representation of the Bertrand-Nash equilibrium

2.7 Advanced Theory

2.7.1 Mixed Strategies

Some games do not have a Nash equilibrium.

Exemple 6.

Heads or Tails Game

	Player 1		
		Heads	Tails
Player 2	Heads	(-1,1)	(1,-1)
	Tails	(1,-1)	(-1,1)

Due to this type of situation, the notion of mixed strategies is introduced, which can be interpreted in terms of uncertainty about what the other player will do (Harsanyi, 1973).

Définition 9.
A mixed strategy for a player is a probability distribution over their set of (pure) strategies.

Mixed or random strategies allow for the representation of possibilities such as bluffing or playing randomly.

In a mixed strategy equilibrium, each player is indifferent between playing their mixed strategy or one of their pure strategies that supports their mixed strategy. The expected payoffs are the same.

※※※	Player 2			
Player 1	※※※	Left	Middle	Right
	Top	(1,0)	(1,2)	(0,1)
	Bottom	(0,3)	(0,1)	(2,0)

For Player 2, $(q, r, 1-q-r)$ is a mixed strategy. The pure strategies are Left, Middle, and Right, which coincide with the probability vectors $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, respectively.

In mixed strategies :

$$U_{J_1}(\text{Top}/q,r) = U_{J_1}(\text{Bottom}/q,r) \Leftrightarrow 1.q + 1.r + 0.(1-q-r) = 2.(1-q-r)$$

$$\Leftrightarrow 1.q + 1.r + 0.(1-q-r) = 2.(1-q-r)$$

$$U_{J_2}(\text{Left}/p) = U_{J_2}(\text{Middle}/p) = U_{J_2}(\text{Right}/p) \Leftrightarrow p = p = 2(1-p) \Rightarrow p = \frac{1}{2}.$$

An Interest in Mixed Strategies

If a strategy x_i is strictly dominated, then there is no belief that player j could have about the strategies of the other players that would make choosing x_i optimal.

The inverse is true if mixed strategies are used (while it is false if we restrict ourselves to pure strategies).

A pure strategy can be the best response to a mixed strategy, even if it is not the best response to any pure strategy of the opponent.

⁂	Player 1		
Player 1		L	R
	T	3,-	0,-
	M	0,-	3,-
	B	2,-	2,-

B is not dominated by a pure strategy, but it is dominated by a mixed strategy.

B is not an optimal response to L or R ; however, it is optimal for Player 1 for Player 2's mixed strategy $(q, 1-q)$ with $\frac{1}{3} < q < \frac{2}{3}$.

2.7.2 Existence of Nash Equilibrium

Définition 10.

In the n-player game $G = \{S_1, ..., S_n; u_1, ..., u_n\}$, the mixed strategies (p_1^*, p_2^*) are a Nash equilibrium if each player's strategy is an optimal response to the mixed strategies of the others.

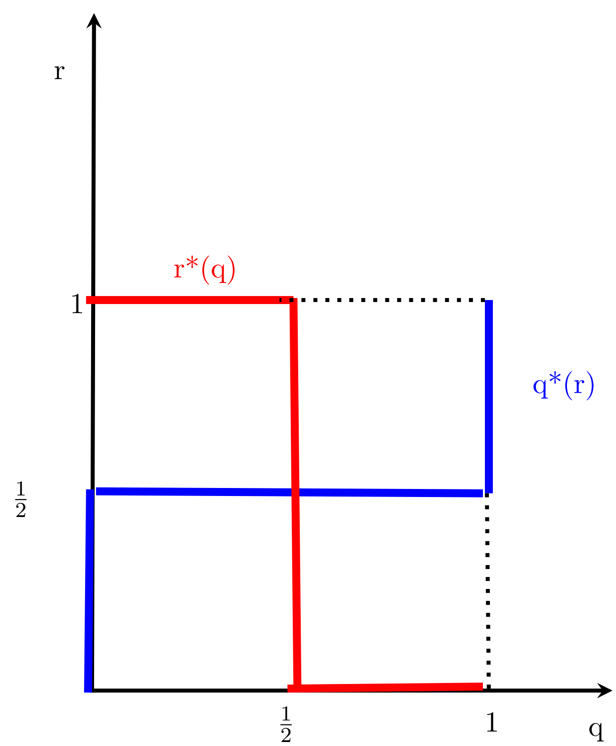

FIGURE 2.3 – Graphical representation of the Heads or Tails game (existence of Nash equilibrium)

Théorème 11 (Nash, 1950).

Any finite game $G = \{S_1, ..., S_n; u_1, ..., u_n\}$ (with n finite and the S_i finite) admits at least one Nash equilibrium, possibly in mixed strategies.