Complex variable, Laplace & Z- transformation

Lecture 06

This Lecture Covers-

1. Process of Solving Differential Equations using Laplace transformation.

2. Some Important formulae.

3. Example and exercises of solving differential equations using Laplace Transformation.

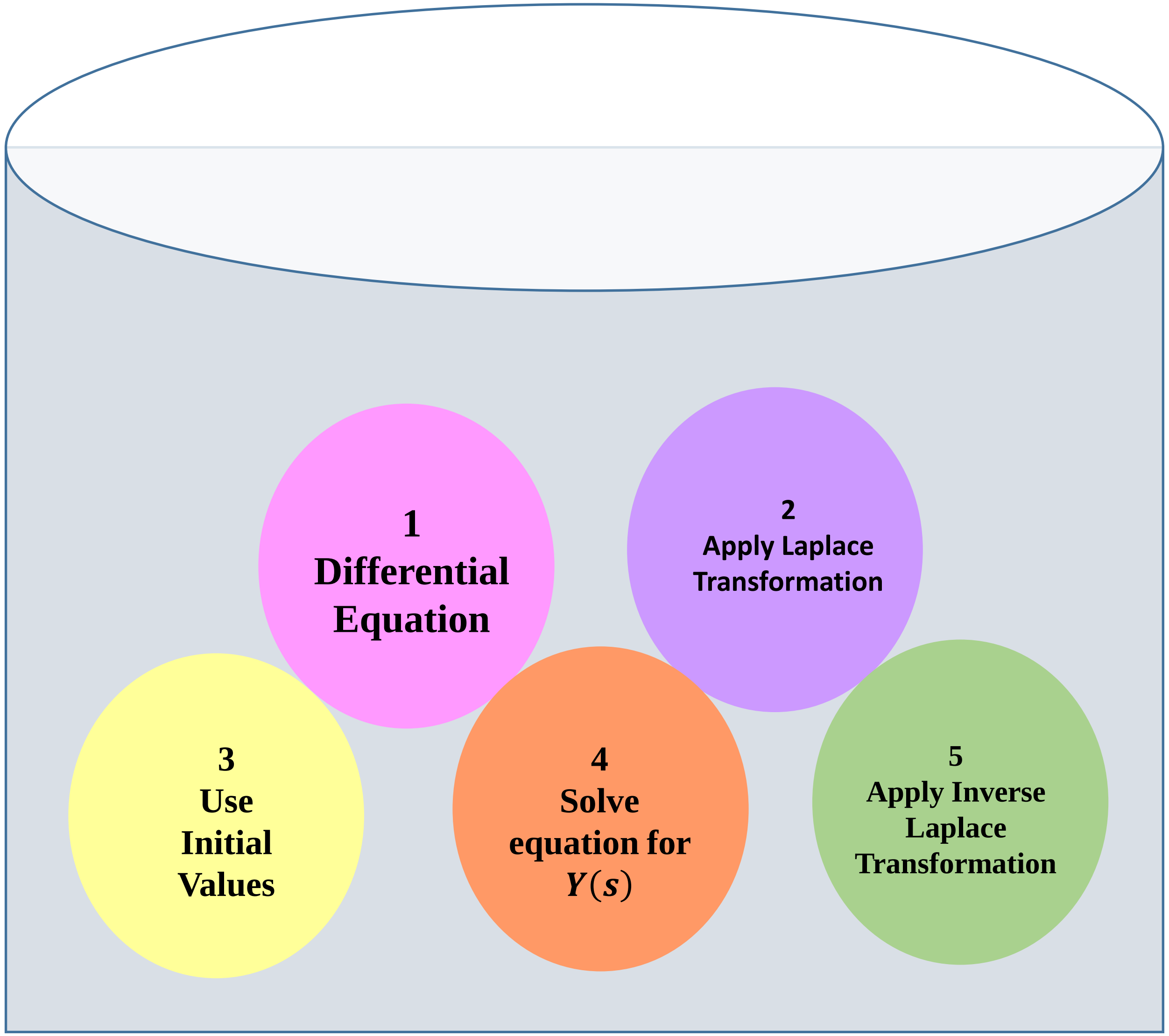
$$\mathcal{L}\{\delta(t)\} = 1 \qquad \mathcal{L}^{-1}\{1\} = \delta(t)$$

$$\mathcal{L}\{\delta(t-a)\} = e^{-as} \qquad \mathcal{L}^{-1}\{e^{-as}\} = \delta(t-a)$$

$$\delta(t) = \begin{cases} 1; & t = 0 \\ 0; & otherwise \end{cases}$$

$$\delta(t-a) = \begin{cases} 1; & t = a \\ 0; & otherwise \end{cases}$$

Process of Solving Differential Equations using Laplace Transformation



Important Formulae

1. $\mathcal{L}\{\dot{f}(t)\} = \mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0).$

2. $\mathcal{L}\{\ddot{f}(t)\} = \mathcal{L}\left\{\frac{d^2 f(t)}{dt^2}\right\} = s^2F(s) - sf(0) - \dot{f}(0)$ where $f(0)$, and $\dot{f}(0)$are the initial values of f and$\dot{f}$.

3. $\mathcal{L}\{\dddot{f}(t)\} = \mathcal{L}\left\{\frac{d^3 f(t)}{dt^3}\right\} = s^3F(s) - s^2f(0) - s\dot{f}(0) - \ddot{f}(0).$

The general case for the Laplace transform of an n^{th} derivative is

$$\mathcal{L}\{f^n(t)\} = \mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

Examples

1. Solve the differential equation $\dot{y}(t) = 3t + 2;\ \ y(0) = -1;$

Solution: Given,

$$\dot{y}(t) = 3t + 2$$
$$\Rightarrow \mathcal{L}\{\dot{y}(t)\} = \mathcal{L}\{3t + 2\}$$
$$\Rightarrow s\,Y(s) - y(0) = \frac{3}{s^2} + \frac{2}{s}$$
$$\Rightarrow s\,Y(s) + 1 = \frac{3}{s^2} + \frac{2}{s}$$
$$\Rightarrow s\,Y(s) = \frac{3}{s^2} + \frac{2}{s} - 1$$
$$\Rightarrow\ Y(s) = \frac{3}{s^3} + \frac{2}{s^2} - \frac{1}{s}$$
$$\Rightarrow y(t) = \frac{3}{2}t^2 + 2t - 1.$$

2. Solve the differential equation

$$\ddot{y}(t) - \dot{y}(t) - 2y(t) = 0\ ; \quad y(0) = 1, \dot{y}(0) = -2;$$

Solution:

Given,

$$\ddot{y}(t) - \dot{y}(t) - 2y(t) = 0$$
$$\Rightarrow \mathcal{L}\{\ddot{y}(t) - \dot{y}(t) - 2y(t) = 0\} = \mathcal{L}\{0\}$$
$$\Rightarrow s^2Y(s) - sy(0) - \dot{y}(0) - \{s\,Y(s) - y(0)\} - 2Y(s) = 0$$
$$\Rightarrow s^2Y(s) - s + 2 - s\,Y(s) + 1 - 2Y(s) = 0$$
$$\Rightarrow (s^2 - s - 2)Y(s) = s - 3$$
$$\Rightarrow\ Y(s) = \frac{s-3}{(s-2)(s+1)}$$
$$\Rightarrow\ Y(s) = \frac{-\frac{1}{3}}{(s-2)} + \frac{\frac{4}{3}}{(s+1)}$$
$$\Rightarrow y(t) = -\frac{1}{3}e^{2t} + \frac{4}{3}e^{-t}.$$

$$A = \frac{(2-3)}{(2+1)} = -\frac{1}{3}$$
$$B = \frac{(-1-3)}{-1-2} = \frac{4}{3}$$

3. Solve the differential equation $\dot{y}(t) - y(t) = 3$ for $y(0) = 1$;

Solution: Given,

$$\dot{y}(t) - y(t) = 3$$

$$\Longrightarrow \mathcal{L}\{\dot{y}(t)\} - \mathcal{L}\{y(t)\} = \mathcal{L}\{3\}$$

$$\Longrightarrow s\,Y(s) - y(0) - Y(s) = \frac{3}{s}$$

$$\Longrightarrow s\,Y(s) - 1 - Y(s) = \frac{3}{s}$$

$$\Longrightarrow (s-1)\,Y(s) = \frac{3}{s} + 1$$

$$\Longrightarrow (s-1)\,Y(s) = \frac{3+s}{s}$$

$$\Longrightarrow Y(s) = \frac{(s+3)}{s\,(s-1)}$$

$$\Longrightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{-3}{s} + \frac{4}{s-1}\right\}$$

$$\Longrightarrow y(t) = -3 + 4\,e^{t}$$

Ans.

$$A = \frac{(0+3)}{0-1} = -3$$

$$B = \frac{(1+3)}{1} = 4$$

Exercise

Apply Laplace transform to solve the following ordinary differential equations and hence justify your answer, where $\dot{y} \equiv \frac{dy(t)}{dt}$ and $\ddot{y} \equiv \frac{d^2y(t)}{dt^2}$:

1. $\dot{y}(t) = 3\delta(t)$; $y(0) = 2$.
2. $\dot{y}(t) = 4t$; $y(0) = 1$.
3. $\dot{y}(t) = 2t - 1$; $y(0) = 3$.
4. $\dot{y}(t) = t^2$; $y(0) = 4$.
5. $\dot{y}(t) = e^{2t}$; $y(0) = 2$.
6. $\dot{y}(t) + y(t) = 2$; $y(0) = 0$.
7. $\ddot{y}(t) = 5$; $y(0) = 1, \dot{y}(0) = 2$.
8. $\ddot{y}(t) - 4\,\dot{y}(t) = \cosh t$; $y(0) = 0, \dot{y}(0) = 1$.
9. $\ddot{y}(t) + 3\,\dot{y}(t) - 4y(t) = e^{-t}$; $y(0) = \dot{y}(0) = 0$.
10. $\ddot{y}(t) - 7\dot{y}(t) + 12y(t) = 0, y(0) = 2, \dot{y}(0) = 1$.
11. $\ddot{y}(t) + y(t) = \begin{cases} t, & 0 < t < 1 \\ 0, & t > 1 \end{cases}, y(0) = 0, \dot{y}(0) = 0$.

Learning Outcomes

After completing this lecture student will learn solving differential equation using Laplace transformation.

Sample MCQ

For $\dot{y}(t) = 3t$; $y(0) = 2$ answer the following questions:

1. What is the Laplace transformation of given differential equation?

(a) $sY(s) - y(0)$ (b) $sY(s) - 2$ (c) Only a   (d) Both a and b

2. Which one of the following is the term of $Y(s)$ for given differential equation?

(a) $\frac{1}{s^3} + \frac{2}{s}$ (a) $\frac{3}{s^3} + \frac{1}{s}$ (a) $\frac{3}{s^3} + \frac{2}{s}$ (a) $\frac{3}{s^3} - \frac{2}{s}$

3. What is the Inverse Laplace transformation of $Y(s)$ for given differential equation?

(a) $t^2 + 2$ (b) $\frac{3}{2}\,t^2 - 2$ (c) $\frac{3}{2}\,t^2 + 2$ (d) $-\frac{3}{2}\,t^2 + 2$