and the choice $C = 0$ gives

$$f(y) = 4y(1-y),$$

which is the function in Example 3.30 with $a = 4$. Other values of D lead to more complicated expressions.

Let's solve

$$y(t+1) = 4y(t)\,(1 - y(t)).$$

From Eq. (3.23), $z = 2\sin^{-1}\sqrt{y}$, so $y = \sin^2\frac{z}{2}$. This change of variable in the difference equation results in

$$\begin{aligned}\sin^2\frac{z(t+1)}{2} &= 4\sin^2\frac{z(t)}{2}\cos^2\frac{z(t)}{2},\\ &= \sin^2 z(t)\end{aligned}$$

or

$$z(t+1) = 2z(t).$$

Then $z(t) = A \cdot 2^t$, so

$$y(t) = \sin^2(B \cdot 2^t),$$

where B is an arbitrary constant. Of course, this solution is valid only for $0 \leq y \leq 1$.

Additional examples using Eq. (3.22), as well as a generalization, are contained in the exercises.

3.7 The z-Transform

The z-transform is a mathematical device similar to a generating function which provides an alternate method for solving linear difference equations as well as certain summation equations. In this section we will define the z-transform, derive several of its properties, and consider an application. The z-transform is important in the analysis and design of digital control systems. Jury [146] is a good source of information on this topic.

Definition 3.4. The "z-transform" of a sequence $\{y_k\}$ is a function $Y(z)$ of a complex variable defined by

$$Y(z) = Z(y_k) = \sum_{k=0}^{\infty}\frac{y_k}{z^k},$$

and we say that the z-transform "exists" provided there is a number $R > 0$ such that $\sum_{k=0}^{\infty}\frac{y_k}{z^k}$ converges for $|z| > R$. The sequence $\{y_k\}$ is said to be "exponentially bounded" if there is an $M > 0$ and a $c > 1$ such that

$$|y_k| \leq Mc^k$$

for $k \geq 0$.

Theorem 3.10. If the sequence $\{y_k\}$ is exponentially bounded, then the z-transform of $\{y_k\}$ exists.

Proof. Assume that the sequence $\{y_k\}$ is exponentially bounded. Then there is an $M > 0$ and a $c > 1$ such that

$$|y_k| \leq Mc^k$$

for $k \geq 0$. We have

$$\sum_{k=0}^{\infty} \left| \frac{y_k}{z^k} \right| \leq \sum_{k=0}^{\infty} \frac{|y_k|}{|z|^k} \leq M \sum_{k=0}^{\infty} \left| \frac{c}{z} \right|^k,$$

and the last sum converges for $|z| > c$. It follows that the z-transform of the sequence $\{y_k\}$ exists. ■

In this section we will frequently use, without reference, the following theorem.

Theorem 3.11. If the sequence $\{f_k\}$ is exponentially bounded, each solution of the n^{th} order difference equation

$$y_{k+n} + p_1 y_{k+n-1} + p_2 y_{k+n-2} + \cdots + p_n y_k = f_k$$

is exponentially bounded and hence its z-transform exists.

Proof. We will give the proof of this theorem just for the case $n = 2$. Assume y_k is a solution of the second order equation

$$y_{k+2} + p_1 y_{k+1} + p_2 y_k = f_k$$

and $\{f_k\}$ is exponentially bounded. Since $\{f_k\}$ is exponentially bounded, there is an $M > 0$ and a $c > 1$ such that

$$|f_k| \leq Mc^k$$

for $k \geq 0$. Since y_k is a solution of the above second order difference equation, we have that

$$|y_{k+2}| \leq |p_1||y_{k+1}| + |p_2||y_k| + Mc^k. \tag{3.24}$$

Let

$$B = \max\{|p_1|, |p_2|, |y_0|, |y_1|, M, c\}.$$

We now prove by induction that

$$|y_k| \leq 3^{k-1} B^k \tag{3.25}$$

for $k = 1, 2, 3 \cdots$. It is easy to see that the inequality (3.25) is true for $k = 1$. Now assume that $k_0 \geq 1$ and that the inequality (3.25) is true for $1 \leq k \leq k_0$. Letting $k = k_0 - 1$ in (3.24), we have that

$$|y_{k_0+1}| \leq |p_1||y_{k_0}| + |p_2||y_{k_0-1}| + Mc^{k_0-1}.$$

Using the induction hypothesis and the definition of B we get that

$$|y_{k_0+1}| \leq B3^{k_0-1}B^{k_0} + B3^{k_0-2}B^{k_0-1} + BB^{k_0-1}.$$

It follows that

$$|y_{k_0+1}| \leq 3^{k_0-1}B^{k_0+1} + 3^{k_0-1}B^{k_0+1} + 3^{k_0-1}B^{k_0+1} = 3^{k_0}B^{k_0+1},$$

which completes the induction. From the inequality (3.25),

$$|y_k| \leq (3B)^k$$

for $k = 1, 2, 3 \cdots$, so y_k is exponentially bounded. By Theorem 3.10, the z-transform of y_k exists. ■

Example 3.32. Find the z-transform of the sequence $\{y_k = 1\}$.

$$\begin{aligned} Y(z) = Z(1) &= \sum_{k=0}^{\infty} \frac{1}{z^k} \\ &= \frac{1}{1 - z^{-1}} \\ &= \frac{z}{z - 1}, \qquad |z| > 1. \end{aligned}$$

Example 3.33. Find the z-transform of the sequence $\{u_k = a^k\}$.

$$\begin{aligned} U(z) = Z(a^k) &= \sum_{k=0}^{\infty} \frac{a^k}{z^k} \\ &= \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k = \frac{1}{1 - \frac{a}{z}} \\ &= \frac{z}{z - a}, \qquad |z| > |a|. \end{aligned}$$

Example 3.34. Find the z-transform of $\{v_k = k\}_{k=0}^{\infty}$.

$$
\begin{aligned}
V(z) = Z(k) &= \sum_{k=0}^{\infty} \frac{k}{z^k} \\
&= \sum_{k=0}^{\infty} \frac{k+1}{z^{k+1}} \\
&= \frac{1}{z} \sum_{k=0}^{\infty} \frac{k+1}{z^k} \\
&= \frac{1}{z} V(z) + \frac{1}{z} Z(1), \qquad |z| > 1,
\end{aligned}
$$

so, by rearrangement,

$$
\begin{aligned}
\frac{z-1}{z} V(z) &= \frac{1}{z} \frac{z}{z-1}, \qquad |z| > 1, \\
V(z) &= \frac{z}{(z-1)^2}, \qquad |z| > 1.
\end{aligned}
$$

These formulas for z-transforms along with some others are collected in Table 3.1 at the end of this section. Of course, this table is easily converted into a table of generating functions by the substitution $z = \frac{1}{x}$.

Theorem 3.12. (Linearity Theorem) If a and b are constants, then

$$Z(au_k + bv_k) = aZ(u_k) + bZ(v_k)$$

for those z in the common domain of $U(z)$ and $V(z)$.

Proof. Simply compute

$$
\begin{aligned}
Z(au_k + bv_k) &= \sum_{k=0}^{\infty} \frac{au_k + bv_k}{z^k} \\
&= a \sum_{k=0}^{\infty} \frac{u_k}{z^k} + b \sum_{k=0}^{\infty} \frac{v_k}{z^k} \\
&= aZ(u_k) + bZ(v_k).
\end{aligned}
$$

■

Example 3.35. Find the z-transform of $\{\sin ak\}_{k=0}^{\infty}$.

The following calculation makes use of the Linearity Theorem:

$$Z(\sin ak) = Z\left(\frac{1}{2i} e^{iak} - \frac{1}{2i} e^{-iak}\right)$$

$$
\begin{aligned}
&= \frac{1}{2i} Z(e^{iak}) - \frac{1}{2i} Z(e^{-iak}) \\
&= \frac{1}{2i} \frac{z}{z - e^{ia}} - \frac{1}{2i} \frac{z}{z - e^{-ia}} \\
&= \frac{z^2 - ze^{-ia} - z^2 + ze^{ia}}{2i[z^2 - (e^{ia} + e^{-ia})z + 1]} \\
&= \frac{z \sin a}{z^2 - 2(\cos a)z + 1}.
\end{aligned}
$$

Similarly, one can show that

$$
Z(\cos ak) = \frac{z^2 - z\cos a}{z^2 - 2z\cos a + 1}.
$$

Theorem 3.13. If $Y(z) = Z(y_k)$ for $|z| > r$, then

$$
Z\left((k+n-1)^{\underline{n}} y_k\right) = (-1)^n z^n \frac{d^n Y}{dz^n}(z)
$$

for $|z| > r$.

Proof. By definition,

$$
Y(z) = \sum_{k=0}^{\infty} \frac{y_k}{z^k} = \sum_{k=0}^{\infty} y_k z^{-k}
$$

for $|z| > r$. The n^{th} derivative is

$$
\begin{aligned}
\frac{d^n Y}{dz^n}(z) &= (-1)^n \sum_{k=0}^{\infty} k(k+1)\cdots(k+n-1) y_k z^{-k-n} \\
&= \frac{(-1)^n}{z^n} \sum_{k=0}^{\infty} \frac{(k+n-1)^{\underline{n}} y_k}{z^k}.
\end{aligned}
$$

Hence

$$
Z\left((k+n-1)^{\underline{n}} y_k\right) = (-1)^n z^n \frac{d^n Y}{dz^n}(z). \qquad \blacksquare
$$

For $n = 1$ in Theorem 3.13 we get the special case

$$
Z(ky_k) = -zY'(z).
$$

Example 3.36. Find $Z(ka^k)$.

$$\begin{aligned} Z(ka^k) &= -z\frac{d}{dz}Z(a^k) \\ &= -z\frac{d}{dz}\left(\frac{z}{z-a}\right) \\ &= \frac{az}{(z-a)^2}. \end{aligned}$$

Example 3.37. Find $Z(k^2)$.

$$\begin{aligned} Z(k^2) &= Z(k \cdot k) \\ &= -z\frac{d}{dz}Z(k) \\ &= -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right] \\ &= \frac{z(z+1)}{(z-1)^3}. \end{aligned}$$

Define the unit step sequence $u(n)$ by

$$u_k(n) = \begin{cases} 0, & 0 \le k \le n-1 \\ 1, & n \le k. \end{cases}$$

Note that the unit step sequence has a single "step" of unit height located at $k = n$. The following result is known as a "shifting theorem."

Theorem 3.14. For n a positive integer

$$\begin{aligned} Z(y_{k+n}) &= z^n Z(y_k) - \sum_{m=0}^{n-1} y_m z^{n-m}, \\ Z(y_{k-n}u_k(n)) &= z^{-n}Z(y_k). \end{aligned}$$

In Fig. 3.6 the various sequences used in this theorem are illustrated.

Proof. First observe that

$$\begin{aligned} Z(y_{k+n}) &= \sum_{k=0}^{\infty} y_{k+n}z^{-k} \\ &= \sum_{k=n}^{\infty} y_k z^{-k+n} \end{aligned}$$

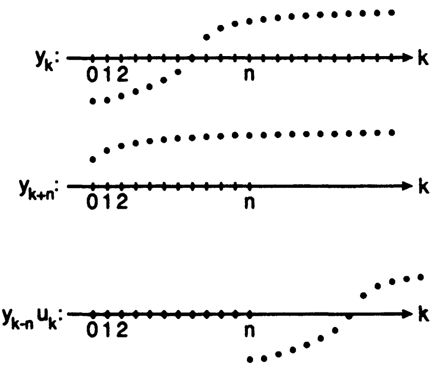

Fig. 3.6 Sequences used in Theorem 3.14

$$= z^n \left[\sum_{k=0}^{\infty} y_k z^{-k} - \sum_{m=0}^{n-1} y_m z^{-m} \right]$$

$$= z^n Z(y_k) - \sum_{m=0}^{n-1} y_m z^{n-m}.$$

For the second part, we have

$$\begin{aligned} Z\left(y_{n-k}u_k(n)\right) &= \sum_{k=0}^{\infty} y_{k-n} u_k(n) z^{-k} \\ &= \sum_{k=n}^{\infty} y_{k-n} z^{-k} \\ &= \sum_{k=0}^{\infty} y_k z^{-k-n} \\ &= z^{-n} Z(y_k). \end{aligned}$$

■

Example 3.38. Find $Z\left(u_k(n)\right)$.

$$\begin{aligned} Z\left(u_k(n)\right) &= z^{-n} Z(1) \\ &= \frac{z^{1-n}}{z-1}. \end{aligned}$$

Example 3.39. Find $Z(y_k)$ if $y_k = 2$, $0 \le k \le 99$, $y_k = 5$, $100 \le k$.

We write the given sequence in terms of u_k and apply Theorem 3.14.

$$Z(y_k) = Z\left(2 + 3u_k(100)\right)$$

$$= \frac{2z}{z-1} + \frac{3z^{-99}}{z-1}$$
$$= \frac{2z^{100}+3}{z^{99}(z-1)}.$$

Theorem 3.15. For any integer $n \geq 0$,

$$Z\left((k+n-1)^{\underline{n}}\right) = \frac{n!z^n}{(z-1)^{n+1}}$$
$$Z\left(k^{\underline{n}}\right) = \frac{n!z}{(z-1)^{n+1}}$$

for $|z| > 1$.

Proof. Letting $y_k = 1$ in Theorem 3.13, we have

$$Z\left((k+n-1)^{\underline{n}}\right) = (-1)^n z^n \frac{d^n}{dz^n}\frac{z}{(z-1)}$$
$$= (-1)^n z^n \frac{(-1)^n n!}{(z-1)^{n+1}}$$
$$= \frac{n!z^n}{(z-1)^{n+1}},$$

which is the first formula. Now using Theorem 3.14, we get

$$Z\left((k+n-1)^{\underline{n}}\right) = z^{n-1}Z(k^{\underline{n}}) - \sum_{m=0}^{n-2} m^{\underline{n}} z^{n-1-m}.$$

Hence

$$Z\left(k^{\underline{n}}\right) = \frac{1}{z^{n-1}}\left(\frac{n!z^n}{(z-1)^{n+1}}\right)$$
$$= \frac{n!z}{(z-1)^{n+1}}.$$

■

Theorem 3.16. (initial value and final value theorem)

(a) If $Y(z)$ exists for $|z| > r$, then

$$y_0 = \lim_{z\to\infty} Y(z).$$

(b) If $Y(z)$ exists for $|z| > 1$ and $(z-1)Y(z)$ is analytic at $z = 1$, then

$$\lim_{k\to\infty} y_k = \lim_{z\to 1}(z-1)Y(z).$$

Proof. Part (a) follows immediately from the definition of the z-transform. To prove part (b), consider

$$
\begin{aligned}
Z(y_{k+1} - y_k) &= \sum_{k=0}^{\infty} y_{k+1} z^{-k} - \sum_{k=0}^{\infty} y_k z^{-k} \\
&= \lim_{n\to\infty} \left[\sum_{k=0}^{n} y_{k+1} z^{-k} - \sum_{k=0}^{n} y_k z^{-k} \right] \\
&= \lim_{n\to\infty} \Big[-y_0 + y_1(1 - z^{-1}) + y_2(z^{-1} - z^{-2}) \\
&\qquad + \cdots + y_n \left(z^{-n+1} - z^{-n} \right) + y_{n+1} z^{-n} \Big].
\end{aligned}
$$

Thus

$$\lim_{z\to 1} (Z(y_{k+1}) - Z(y_k)) = \lim_{n\to\infty} (y_{n+1} - y_0).$$

From the shifting theorem,

$$\lim_{z\to 1} [zY(z) - zy_0 - Y(z)] = \lim_{k\to\infty} y_k - y_0.$$

Hence

$$\lim_{k\to\infty} y_k = \lim_{z\to 1} (z-1) Y(z).$$

■

Example 3.40. Verify directly the last theorem for the sequence $y_k = 1$.

$$
\begin{aligned}
1 = y_0 = \lim_{z\to\infty} Z(1) &= \lim_{z\to\infty} \frac{z}{z-1} = 1, \\
1 = \lim y_k &= \lim_{z\to 1} (z-1) Z(1) \\
&= \lim_{z\to 1} \frac{(z-1)z}{z-1} = 1.
\end{aligned}
$$

Theorem 3.17. If $Z(y_k) = Y(z)$ for $|z| > r$, then for constants $a \neq 0$,

$$Z\left(a^k y_k\right) = Y\left(\frac{z}{a}\right)$$

for $|z| > r|a|$.

Proof. Observe that

$$
\begin{aligned}
Z(a^k y_k) &= \sum_{k=1}^{\infty} \frac{a^k y_k}{z^k} = \sum_{k=1}^{\infty} y_k \left(\frac{z}{a}\right)^{-k} \\
&= Y\left(\frac{z}{a}\right).
\end{aligned}
$$

■

Example 3.41. Find $Z(3^k \sin 4k)$.

$$\begin{aligned} Z(3^k \sin 4k) &= Z(\sin 4k)]_{\frac{z}{3}} \\ &= \frac{\frac{z}{3}\sin 4}{\frac{z^2}{9} - 2(\cos 4)\frac{z}{3} + 1} \\ &= \frac{3z\sin 4}{z^2 - 6z\cos 4 + 9}. \end{aligned}$$

Example 3.42. Solve the following initial value problem using z-transforms:

$$\begin{aligned} y_{k+1} - 3y_k &= 4, \\ y_0 &= 1. \end{aligned}$$

Taking the z-transform of both sides of the difference equation, we have

$$\begin{aligned} zY(z) - zy_0 - 3Y(z) &= 4\frac{z}{z-1} \\ (z-3)Y(z) &= z + \frac{4z}{z-1} = \frac{z^2+3z}{z-1} \\ Y(z) &= z \cdot \frac{z+3}{(z-1)(z-3)} \\ &= z\left[\frac{-2}{z-1} + \frac{3}{z-3}\right] \\ &= -2\frac{z}{z-1} + 3\frac{z}{z-3}. \end{aligned}$$

From Table 3.1 we find the solution

$$y_k = -2 + 3^{k+1}.$$

Example 3.43. Solve the initial value problem

$$\begin{aligned} y_{k+1} - 3y_k &= 3^k, \\ y_0 &= 2. \end{aligned}$$

Since 3^k is a solution of the homogeneous equation, we expect the solution of this problem to involve the function $k3^k$.

$$\begin{aligned} zY(z) - 2z - 3Y(z) &= \frac{z}{z-3} \\ (z-3)Y(z) &= 2z + \frac{z}{z-3} = \frac{2z^2-5z}{z-3} \end{aligned}$$

$$\begin{aligned} Y(z) &= z\frac{2z-5}{(z-3)^2} \\ &= z\left[\frac{2}{z-3}+\frac{1}{(z-3)^2}\right] \\ &= 2\frac{z}{z-3}+\frac{1}{3}\frac{3z}{(z-3)^2}. \end{aligned}$$

Then

$$y_k = 2\cdot 3^k + \frac{1}{3}k3^k.$$

We can use the z-transform to solve some difference equations with variable coefficients.

Example 3.44. Solve the initial value problem

$$(k+1)y_{k+1}-(50-k)y_k = 0, \quad y_0 = 1.$$

Taking the z-transform of both sides,

$$\begin{aligned} zZ(ky_k) - 50Y(z) - zY'(z) &= 0 \\ -z^2Y'(z) - zY'(z) &= 50Y(z) \\ \frac{Y'(z)}{Y(z)} &= \frac{-50}{z(z+1)} \\ &= -\frac{50}{z}+\frac{50}{z+1} \\ \log Y(z) &= -50\log z + 50\log(z+1) + C \\ Y(z) &= \left(\frac{z+1}{z}\right)^{50}. \end{aligned}$$

By Exercise 3.118(b),

$$y_k = \binom{50}{k}.$$

Example 3.45. Solve the second order initial value problem

$$\begin{aligned} y_{k+2}+y_k &= 10\cdot 3^k \\ y_0 &= 0, \quad y_1 = 0. \end{aligned}$$

By the shifting theorem,

$$z^2Z(y_k) - y_0z^2 - y_1z + Z(y_k) = \frac{10z}{z-3}$$

$$(z^2+1)Z(y_k) = \frac{10z}{z-3},$$

so we have

$$\begin{aligned} Z(y_k) &= \frac{10z}{(z-3)(z^2+1)} \\ &= z\left[\frac{A}{z-3} + \frac{Bz+C}{z^2+1}\right] \\ &= z\left[\frac{1}{z-3} - \frac{z+3}{z^2+1}\right] \\ &= \frac{z}{z-3} - \frac{z^2}{z^2+1} - 3\frac{z}{z^2+1} \\ &= \frac{z}{z-3} - \frac{z^2 - z\cos\frac{\pi}{2}}{z^2 - 2z\cos\frac{\pi}{2} + 1} - 3\frac{z\sin\frac{\pi}{2}}{z^2 - 2z\cos\frac{\pi}{2}+1}. \end{aligned}$$

Hence

$$y_k = 3^k - \cos(\frac{\pi}{2}k) - 3\sin(\frac{\pi}{2}k).$$

Example 3.46. Solve the system

$$\begin{aligned} u_{k+1} - v_k &= 3k3^k, \\ u_k + v_{k+1} - 3v_k &= k3^k, \\ u_0 = 0, \qquad v_0 &= 3. \end{aligned}$$

By Theorem 3.12

$$\begin{aligned} zU(z) - zu_0 - V(z) &= \frac{9z}{(z-3)^2}, \\ U(z) + zV(z) - zv_0 - 3V(z) &= \frac{3z}{(z-3)^2}, \end{aligned}$$

or

$$\begin{aligned} zU(z) - V(z) &= \frac{9z}{(z-3)^2}, \\ U(z) + (z-3)V(z) &= 3z + \frac{3z}{(z-3)^2}. \end{aligned}$$

Multiplying both sides of the first equation by $z-3$ and adding, we get

$$(z^2 - 3z + 1)U(z) = 3z + \frac{3z}{(z-3)^2} + \frac{9z^2 - 27z}{(z-3)^2}$$

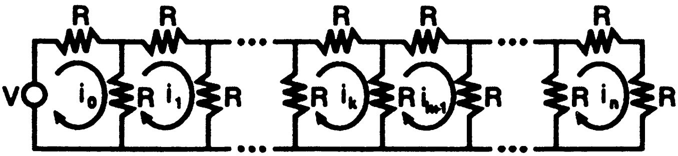

Fig. 3.7 A ladder network

$$= \frac{3z(z^2 - 3z + 1)}{(z-3)^2}$$

$$U(z) = \frac{3z}{(z-3)^2},$$

and one of the unknowns is given by

$$u_k = k3^k.$$

Furthermore, the other unknown is

$$\begin{aligned} v_k &= u_{k+1} - 3k3^k \\ &= (k+1)3^{k+1} - 3k3^k \\ &= 3^{k+1}. \end{aligned}$$

Example 3.47. Find the currents i_k, $0 \le k \le n$, in the ladder network shown in Fig. 3.7.

We begin by applying Kirchhoff's Law to the initial loop in Fig. 3.7:

$$V = Ri_0 + R(i_0 - i_1).$$

Solving for i_1, we obtain

$$i_1 = 2i_0 - \frac{V}{R}.$$

Now we apply Kirchhoff's Law to the loop corresponding to i_{k+1} and obtain

$$R(i_{k+1} - i_{k+2}) + R(i_{k+1} - i_k) + Ri_{k+1} = 0.$$

Simplifying, we have

$$i_{k+2} - 3i_{k+1} + i_k = 0$$

for $0 \le k \le n-2$. If we apply the z-transform to both sides of the preceding equation, we get

$$(z^2 I(z) - z^2 i_0 - zi_1) - 3(zI(z) - zi_0) + I(z) = 0$$

or

$$(z^2 - 3z + 1)I(z) = i_0 z^2 + (i_1 - 3i_0)z.$$

Using the equation for i_1, we have

$$I(z) = i_0 \frac{z^2 - \left(1 + \frac{V}{i_0 R}\right) z}{z^2 - 3z + 1}.$$

Let a be the positive solution of $\cosh a = \frac{3}{2}$; then $\sinh a = \frac{\sqrt{5}}{2}$. Note that

$$I(z) = i_0 \frac{z^2 - z\cosh a}{z^2 - 2z\cosh a + 1} + \left(\frac{i_0}{2} - \frac{V}{R}\right) \frac{2}{\sqrt{5}} \frac{z \sinh a}{z^2 - 2z\cosh a + 1}.$$

It follows that

$$i_k = i_0 \cosh(ak) + \left(\frac{i_0}{2} - \frac{V}{R}\right) \frac{2}{\sqrt{5}} \sinh(ak),$$

for $0 \leq k \leq n$. Using Kirchhoff's Law for the last loop in Fig. 3.6, we get that $i_{n-1} = 3i_n$. This additional equation uniquely determines i_0 and hence all the i_k's for $0 \leq k \leq n$.

We now define the unit impulse sequence $\delta(n)$, $n \geq 1$, by

$$\delta_k(n) = \begin{cases} 1, & k = n \\ 0, & k \neq n. \end{cases}$$

It follows immediately from the definition of the z-transform that

$$Z\left(\delta_k(n)\right) = \frac{1}{z^n}.$$

Example 3.48. Solve the initial value problem

$$y_{k+1} - 2y_k = 3\delta_k(4), \qquad y_0 = 1.$$

Taking the z-transform of both sides, we have

$$\begin{aligned} zY(z) - z - 2Y(z) &= \frac{3}{z^4} \\ (z-2)Y(z) &= z + \frac{3}{z^4} \\ Y(z) &= \frac{z}{z-2} + \frac{3}{z^4(z-2)} \\ &= \frac{z}{z-2} + 3z^{-5}\frac{z}{z-2}. \end{aligned}$$

An application of the inverse z-transform results in

$$y_k = 2^k + 3 \cdot 2^{k-5} u_k(5).$$

We can also write this in the form

$$y_k = \begin{cases} 2^k, & 0 \le k \le 4 \\ 2^k + 3 \cdot 2^{k-5}, & k \ge 5. \end{cases}$$

We define the convolution of two sequences, $\{u_k\}$ and $\{v_k\}$, by

$$\{u_k\} * \{v_k\} = \left\{ \sum_{m=0}^{k} u_{k-m} v_m \right\}.$$

Briefly we write

$$u_k * v_k = \sum_{m=0}^{k} u_{k-m} v_m.$$

Theorem 3.18. (Convolution Theorem) If $U(z)$ exits for $|z| > a$ and $V(z)$ exists for $|z| > b$, then

$$Z(u_k * v_k) = U(z)V(z)$$

for $|z| > \max\{a, b\}$.

Proof. For $|z| > \max\{a, b\}$,

$$\begin{aligned} U(z)V(z) &= \sum_{k=0}^{\infty} \frac{u_k}{z^k} \sum_{k=0}^{\infty} \frac{v_k}{z^k} \\ &= \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{u_{k-m} v_m}{z^k} \\ &= Z(u_k * v_k). \end{aligned}$$

■

Since $\sum_{m=0}^{k} y_m = 1 * y_k$, Theorem 3.18 gives us

$$\begin{aligned} Z\left(\sum_{m=0}^{k} y_m \right) &= Z(1)Z(y_k) \\ &= \frac{z}{z-1} Z(y_k). \end{aligned}$$

Corollary 3.3. If $Z(y_k)$ exists for $|z| > r$, then

$$Z\left(\sum_{m=0}^{k} y_m\right) = \frac{z}{z-1} Z(y_k)$$

for $|z| > \max\{1, r\}$.

Example 3.49. Find

$$Z\left(\sum_{m=0}^{k} 3^m\right).$$

By Corollary 3.3,

$$\begin{aligned} Z\left(\sum_{m=0}^{k} 3^m\right) &= \frac{z}{z-1} Z(y_k), \\ &= \frac{z^2}{(z-1)(z-3)}, \qquad (|z| > 3). \end{aligned}$$

Now consider the Volterra summation equation of convolution type

$$y_k = f_k + \sum_{m=0}^{k-1} u_{k-m-1} y_m \qquad (k \geq 0), \tag{3.26}$$

where f_k and u_{k-m-1} are given. The term u_{k-m-1} is called the kernel of the summation equation. The equation is said to be homogeneous if $f_k \equiv 0$ and nonhomogeneous otherwise. Such an equation can often be solved by use of the z-transform.

To see this, replace k by $k+1$ in Eq. (3.26) to get

$$y_{k+1} = f_{k+1} + \sum_{m=0}^{k} u_{k-m} y_m$$

or

$$y_{k+1} = f_{k+1} + u_k * y_k.$$

Taking the z-transform of both sides and using the fact that $y_0 = f_0$, we have

$$zY(z) = zF(z) + U(z)Y(z).$$

Hence

$$Y(z) = \frac{zF(z)}{z - U(z)}.$$

The desired solution y_k is then obtained if we can compute the inverse transform. The next example is of this type.

Example 3.50. Solve the Volterra summation equation

$$y_k = 1 + 16\sum_{m=0}^{k-1}(k-m-1)y_m, \qquad k \geq 0.$$

Replacing k by $k+1$, we have

$$\begin{aligned} y_{k+1} &= 1 + 16\sum_{m=0}^{k}(k-m)y_m \\ &= 1 + 16k * y_k. \end{aligned}$$

Taking the z-transform of both sides, we obtain $Y(z)$ as follows:

$$\begin{aligned} zY(z) - z &= \frac{z}{z-1} + 16\frac{z}{(z-1)^2}Y(z) \\ \left[1 - \frac{16}{(z-1)^2}\right]Y(z) &= 1 + \frac{1}{z-1} \\ \frac{z^2-2z-15}{(z-1)^2}Y(z) &= \frac{z}{z-1} \\ Y(z) &= \frac{z(z-1)}{(z-5)(z+3)} \\ &= z\left[\frac{\frac{1}{2}}{z-5} + \frac{\frac{1}{2}}{z+3}\right] \\ &= \frac{1}{2}\frac{z}{z-5} + \frac{1}{2}\frac{z}{z+3}. \end{aligned}$$

Then

$$y_k = \frac{1}{2}5^k + \frac{1}{2}(-3)^k.$$

A related equation is the Fredholm summation equation

$$y_k = f_k + \sum_{m=a}^{b} K_{k,m}y_m \qquad (a \leq k \leq b). \tag{3.27}$$

Here a and b are integers, and the kernel $K_{k,m}$ and the sequence f_k are given. Since this equation is actually a linear system of $b-a+1$ equations in $b-a+1$ unknowns $y_a, \ldots, y_b$, it can be solved by matrix methods. If $b-a$ is large, this might not be the best way to solve this equation. If $K_{k,m}$ is separable, the following procedure may yield a more efficient method of solution.

We say $K_{k,m}$ is separable provided that

$$K_{k,m} = \sum_{i=1}^{p}\alpha_i(k)\beta_i(m), \qquad (a \leq k, m \leq b).$$

Substituting this expression into Eq. (3.27) we obtain

$$y_k = f_k + \sum_{i=1}^{p} \alpha_i(k) \left(\sum_{m=a}^{b} \beta_i(m) y_m \right).$$

Hence

$$y_k = f_k + \sum_{i=1}^{p} c_i \alpha_i(k), \qquad a \le k \le b, \tag{3.28}$$

where

$$c_i = \sum_{m=a}^{b} \beta_i(m) y_m.$$

By multiplying both sides of Eq. (3.28) by $\beta_j(k)$ and summing from a to b, we obtain

$$\sum_{k=a}^{b} \beta_j(k) y_k = \sum_{k=a}^{b} \beta_j(k) f_k + \sum_{i=1}^{p} c_i \left(\sum_{k=a}^{b} \alpha_i(k) \beta_j(k) \right).$$

Hence

$$c_j = u_j + \sum_{i=1}^{p} a_{ji} c_i, \qquad 1 \le j \le p, \tag{3.29}$$

where

$$u_j = \sum_{k=a}^{b} f_k \beta_j(k)$$

and

$$a_{ij} = \sum_{k=a}^{b} \alpha_j(k) \beta_i(k).$$

Let A be the p by p matrix $A = (a_{ij})$, let $\vec{c} = [c_1, \cdots, c_p]^T$, and let $\vec{u} = [u_1, \cdots, u_p]^T$. Then Eq. (3.29) becomes

$$\vec{c} = \vec{u} + A\vec{c}.$$

But this equation is equivalent to

$$(I - A)\vec{c} = \vec{u},, \tag{3.30}$$

where I is the p by p identity matrix. We have essentially proved the following theorem.

Theorem 3.19. The Fredholm equation (3.27) with a separable kernel has a solution y_k if and only if Eq. (3.30) has a solution $\vec{c}$. If $\vec{c} = (c_1, \cdots, c_p)^T$ is a solution of Eq. (3.30), then a corresponding solution y_k of Eq. (3.27) is given by Eq. (3.28).

Example 3.51. Solve the Fredholm summation equation

$$y_k = 1 + \sum_{m=0}^{19} (1 + km) y_m, \qquad 0 \le k \le 19.$$

Here we have the separable kernel

$$K_{k,m} = 1 + km.$$

Take

$$\begin{aligned} \alpha_1(k) &= 1, & \beta_1(m) &= 1, \\ \alpha_2(k) &= k, & \beta_2(m) &= m. \end{aligned}$$

Then

$$\begin{aligned} a_{11} &= \sum_{k=0}^{19} 1 = 20 \\ a_{12} = a_{21} &= \sum_{k=0}^{19} k = 190 \\ a_{22} &= \sum_{k=0}^{19} k^2 = 2470. \end{aligned}$$

Furthermore,

$$\begin{aligned} u_1 &= \sum_{k=0}^{19} 1 = 20 \\ u_2 &= \sum_{k=0}^{19} k = 190. \end{aligned}$$

Equation (3.30) in this case is

$$\begin{bmatrix} -19 & -190 \\ -190 & -2469 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 190 \end{bmatrix}.$$

Solving for c_1 and c_2, we obtain

$$c_1 = \frac{-13,280}{10,811}, \qquad c_2 = \frac{10}{569}.$$

From Theorem 3.19 we obtain the unique solution

$$\begin{aligned} y_k &= 1 + \frac{-13,280}{10,811} \cdot 1 + \frac{10}{569} \cdot k \\ &= \frac{-2469}{10,811} + \frac{10}{569}k, \qquad 0 \le k \le 19. \end{aligned}$$

Example 3.52. Solve the following Fredholm summation equation for all values of λ:

$$y_k = 2 + \lambda \sum_{m=0}^{29} \frac{m}{29} y_m, \qquad 0 \le k \le 29.$$

Take

$$\alpha_1(k) = \lambda, \qquad \beta_1(m) = \frac{m}{29};$$

then

$$\begin{aligned} a_{11} &= \sum_{m=0}^{29} \lambda \frac{m}{29} = 15\lambda, \\ u_1 &= \sum_{m=0}^{29} \frac{2m}{29} = 30. \end{aligned}$$

Hence Eq. (3.30) is

$$(1 - 15\lambda)c = 30.$$

For $\lambda = \frac{1}{15}$ there is no solution of this summation equation. For $\lambda \neq \frac{1}{15}$ $c = \frac{30}{1-15\lambda}$. The corresponding solution is

$$y_k = \frac{2}{1 - 15\lambda}, \qquad 0 \le k \le 29.$$

Now consider the homogeneous Fredholm equation

$$y_k = \lambda \sum_{m=a}^{b} K_{k,m} y_m, \qquad a \le k \le b, \tag{3.31}$$

where λ is a parameter. We say that λ_0 is an eigenvalue of this equation, provided that for this value of λ, there is a nontrivial solution y_k, called an eigensequence. We say that (λ_0, y_k) is an eigenpair for Eq. (3.31). Note that $\lambda = 0$ is not an eigenvalue. We say that $K_{k,m}$ is symmetric provided that

$$K_{k,m} = K_{m,k}$$

for $a \leq k, m \leq b$. Several properties of eigenpairs for Eq. (3.31) with a symmetric kernel are given in the following theorem.

Theorem 3.20. If $K_{k,m}$ is real and symmetric, then all the eigenvalues of Eq. (3.31) are real. If (λ_i, u_k) (λ_j, v_k) are eigenpairs with $\lambda_i \neq \lambda_j$, then u_k and v_k are orthogonal; that is,

$$\sum_{k=a}^{b} u_k v_k = 0.$$

We can always pick a real eigensequence that corresponds to each eigenvalue.

Proof. Let (μ, u_k), (ν, v_k) be eigenpairs of Eq. (3.31). Then $\mu, \nu \neq 0$. Since (μ, u_k) is an eigenpair for Eq. (3.31),

$$u_k = \mu \sum_{m=a}^{b} K_{k,m} u_m.$$

Multiplying by v_k and summing from a to b, we obtain

$$\begin{aligned} \sum_{k=a}^{b} u_k v_k &= \mu \sum_{k=a}^{b} \sum_{m=a}^{b} K_{k,m} u_m v_k \\ &= \mu \sum_{m=a}^{b} \left(\sum_{k=a}^{b} K_{m,k} v_k \right) u_m \\ &= \frac{\mu}{\nu} \sum_{m=a}^{b} v_m u_m, \end{aligned}$$

since (ν, v_k) is an eigenpair for Eq. (3.31). It follows that

$$(\nu - \mu) \sum_{k=a}^{b} u_k v_k = 0. \tag{3.32}$$

If $\mu \neq \nu$, we get the orthogonality result

$$\sum_{k=a}^{b} u_k v_k = 0.$$

If (λ_i, y_k) is an eigenpair of Eq. (3.31), then $(\overline{\lambda}_i, \overline{y}_k)$ is an eigenpair of Eq. (3.31). With $(\mu, u_k) = (\lambda_i, y_k)$ and $(\nu, v_k) = (\overline{\lambda}_i, \overline{y}_k)$, Eq. (3.32) becomes

$$(\overline{\lambda} - \lambda) \sum_{k=a}^{b} y_k \overline{y}_k = 0.$$

It follows that $\lambda = \overline{\lambda}$, and hence every eigenvalue of Eq. (3.31) is real. The last statement of the theorem is left as an exercise. ■

Table 3.1. z-Transforms

Sequence	z-transform
1	$\frac{z}{z-1}$
a^k	$\frac{z}{z-a}$
k	$\frac{z}{(z-1)^2}$
k^2	$\frac{z(z+1)}{(z-1)^3}$
$k^{\underline{n}}$	$\frac{n!z}{(z-1)^{n+1}}$
$\sin(ak)$	$\frac{z \sin a}{z^2-2z\cos a+1}$
$\cos(ak)$	$\frac{z^2-z\cos a}{z^2-2z\cos a+1}$
$\sinh(ak)$	$\frac{z \sinh a}{z^2-2z\cosh a+1}$
$\cosh(ak)$	$\frac{z^2-z\cosh a}{z^2-2z\cosh a+1}$
$\delta_k(n)$	$\frac{1}{z^n}$
$u_k(n)$	$\frac{z^{1-n}}{z-1}$
ky_k	$-zY'(z)$
$u_k * v_k$	$U(z)V(z)$
$\sum_{m=0}^{k} y_m$	$\frac{z}{z-1}Y(z)$
$a^k y_k$	$Y\left(\frac{z}{a}\right)$
y_{k+n}	$z^n Y(z) - \sum_{m=0}^{n-1} y_m z^{n-m}$
$y_{k-n}u_k(n)$	$z^{-n}Y(z)$

Exercises

Section 3.1

3.1 Show that the equation $\Delta y(t) + y(t) = e^t$ cannot be put in the form of Eq. (3.1) and so is not a first order linear difference equation.

3.2 Solve by iteration for $t = 1, 2, 3, \cdots$:
(a) $u(t+1) = \frac{t}{t+1} u(t)$.
(b) $u(t+1) = \frac{3t+1}{3t+7} u(t)$.

3.3 Find all solutions:
(a) $u(t+1) - e^{3t} u(t) = 0$.
(b) $u(t+1) - e^{\cos 2t} u(t) = 0$.

3.4 Show that a general solution of the constant coefficient first order difference equation $u(t+1) - cu(t) = 0$ is $u(t) = Ac^t$. Use this result to solve these nonhomogeneous equations:
(a) $y(t+1) - 2y(t) = 5$.
(b) $y(t+1) - 4y(t) = 3 \cdot 2^t$.
(c) $y(t+1) - 5y(t) = 5^t$.

3.5 Let $y(t)$ represent the total number of squares of all dimensions on a t by t checkerboard.
(a) Show that $y(t)$ satisfies

$$y(t+1) = y(t) + t^2 + 2t + 1.$$

(b) Solve for $y(t)$.

3.6 Suppose $y(1) = 2$ and find the solution of

$$y(t+1) - 3y(t) = e^t \qquad (t = 1, 2, 3, \cdots).$$

3.7 Solve for $t = 1, 2, \cdots$:

$$y(t+1) - \frac{3t+1}{3t+7} y(t) = \frac{t}{(3t+4)(3t+7)}.$$

3.8 Consider for $t = 1, 2, \cdots$ the equation $y(t+1) - ty(t) = 1$.
(a) Show that the solution is

$$y(t) = (t-1)! \left[\sum_{k=1}^{t-1} \frac{1}{k!} + y(1) \right].$$

(b) Given that $\sum_{k=1}^{\infty} \frac{1}{k!} = e - 1$, derive another expression for $y(t)$.

3.9

(a) Show that the solutions of the equation $y_{n+1}(x) + \frac{x}{n} y_n(x) = \frac{e^{-x}}{n}$ $(n = 1, 2, \cdots)$ are

$$y_n(x) = \frac{(-x)^{n-1}}{(n-1)!}\left[C + e^{-x}\sum_{k=1}^{n-1}(-1)^k(k-1)!\left(\frac{1}{x}\right)^k\right].$$

(b) For what value of C is $y_n(x) = E_n(x)$ the exponential integral? (See Exercise 1.15.)

3.10 Solve the following equations:

(a) $y(t+1) - 3y(t) = t6^t$.

(b) $y(n+1) - \frac{n+2}{n+1}y(n) = (n+2)^2$.

(c) $y(n+1) - \frac{n}{n+1}y(n) = \frac{n}{n+1}$.

(d) $y(n+1) - (n+2)y(n) = (n+3)!$.

3.11 If we invest \$1000 at an annual interest rate of 10% for 10 years, how much money will we have if the interest is compounded at each of the following intervals?

(a) Annually.

(b) Semiannually.

(c) Quarterly.

(d) Monthly.

(e) Daily.

3.12 Assume we invest a certain amount of money at 8% a year compounded annually. How long does it take for our money to double? triple?

3.13 What is the present value of an annuity in which we deposited \$900 at the beginning of each year for nine years at the annual interest rate of 9%?

3.14 A man aged 40 wishes to accumulate a fund for retirement by depositing \$1200 at the beginning of each year for 25 years until he retires at age 65. If the annual interest rate is 7%, how much will he accumulate for his retirement?

3.15 In Example 3.2, suppose we are allowed to increase our deposit by 5% each year. How much will we have in the IRA at the end of the t^{th} year? How much will we have after 20 years?

3.16 Let y_n denote the number of multiplications needed to compute the determinant of an n by n matrix by cofactor expansion.

(a) Show that $y_{n+1} = (n+1)(y_n + 1)$.

(b) Compute y_n.

3.17 In an elementary economics model of the marketplace, the price p_n of a product after n years is related to the supply s_n after n years by $p_n = a - bs_n$, where a and b are positive constants, since a large supply causes the price to be low in a given year. Assume that price and supply in alternate years are proportional: $kp_n = s_{n+1}$ $(k > 0)$.

(a) Show that p_n satisfies $p_{n+1} + bkp_n = a$.

(b) Solve for p_n.

(c) If $bk < 1$, show that the price stabilizes. In other words, show that p_n converges to a limit as $n \to \infty$. What happens if $bk > 1$?

3.18 Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$ in the differential equation $y'(x) = y(x) + e^x$.

(a) Show that $\{a_n\}$ satisfies the difference equation

$$a_{n+1} = \frac{a_n}{n+1} + \frac{1}{(n+1)!}.$$

(b) Use the solution of the equation in (a) to compute $y(x)$.

3.19 Show by substitution that the function

$$u(t) = C(t)a^t \frac{\Gamma(t-r_1)\cdots\Gamma(t-r_n)}{\Gamma(t-s_1)\cdots\Gamma(t-s_m)},$$

with $\Delta C(t) = 0$, satisfies the equation

$$u(t+1) = a\frac{(t-r_1)\cdots(t-r_n)}{(t-s_1)\cdots(t-s_m)}u(t).$$

3.20 Solve $u(t+1) = \frac{2t^3}{3(t+1)^2}u(t)$ in terms of the gamma function. Simplify your answer.

3.21 Here is an example of a "full history" difference equation:

$$y_n = n + \sum_{k=1}^{n-1} y_k \qquad (n = 2, 3, \cdots).$$

Solve for y_n, assuming $y_1 = 1$. (Hint: compute $y_{n+1} - y_n$.)

3.22 Find a solution of

$$y(t+1) - ty(t) = -t$$

that has the form of a factorial series. Show that the series converges for all $t \neq 0, -1, -2, \cdots$.

Section 3.2

3.23 What is the order of this equation

$$\Delta^3 y(t) + \Delta^2 y(t) - \Delta y(t) - y(t) = 0$$

3.24 Give proofs of Theorem 3.3 and Corollary 3.1.

3.25 Show that $u_1(t) = 2^t$ and $u_2(t) = 3^t$ are linearly independent solutions of

$$u(t+2) - 5u(t+1) + 6u(t) = 0.$$

3.26 Use the result of Exercise 3.25 to find the unique solution of the following initial value problem,

$$u(t+2) - 5u(t+1) + 6u(t) = 0,$$

$$u(3) = 0, \qquad u(4) = 12,$$

where $t = 3, 4, 5, \cdots$.

3.27 Verify that the Casoratian satisfies Eq. (3.5).

3.28

(a) Show that $u_1(t) = t^2 + 2$, $u_2(t) = t^2 - 3t$ and $u_3(t) = 2t - 1$ are solutions of $\Delta^3 u(t) = 0$.

(b) Compute the Casoratian of the functions in (a) and determine whether they are linearly independent.

3.29 Are $u_1(t) = 2^t \cos \frac{2\pi t}{3}$ and $u_2(t) = 2^t \sin \frac{2\pi t}{3}$ linearly independent solutions of $u(t+2) + 2u(t+1) + 4u(t) = 0$?

Section 3.3

3.30 In the case that the characteristic roots $\lambda_1, \cdots, \lambda_n$ are distinct, show that the solutions $\lambda_1^t, \cdots, \lambda_n^t$ of Eq. (3.6) are linearly independent. (Hint: use the value of the Vandermonde determinant:

$$\det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ c_1 & c_2 & \cdots & c_n \\ c_1^2 & c_2^2 & \cdots & c_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ c_1^{n-1} & c_2^{n-1} & \cdots & c_n^{n-1} \end{bmatrix} = \prod_{j>i}(c_j - c_i).)$$

3.31 Solve the following equations:

(a) $(E-6)^5 u(t) = 0$.

(b) $u(t+2) + 6u(t+1) + 3u(t) = 0$.

(c) $u(t+3) - 4u(t+2) + 5u(t+1) - 2u(t) = 0$.

(d) $u(t+4) - 8u(t+2) + 16u(t) = 0$.

3.32 Find all real solutions:

(a) $u(t+2) + u(t) = 0$.

(b) $u(t+2) - 8u(t+1) + 32u(t) = 0$.

(c) $u(t+4) + 2u(t+2) + u(t) = 0$.

(d) $u(t+6) + 2u(t+3) + u(t) = 0$.

3.33 Compute the sequence of coefficients $\{a_n\}_{n=0}^{\infty}$ so that

$$\frac{2-3t}{1-3t+2t^2} = \sum_{n=0}^{\infty} a_n t^n$$

on some open interval about $t = 0$. Find the radius of convergence of the infinite series.

3.34 Find a homogeneous equation with constant coefficients for which one solution is
(a) $(t+\sqrt{2})^2$.
(b) t^5.
(c) $t(-3)^t$.
(d) $\frac{\sin\frac{2\pi}{3}t}{2^t}$.

3.35 Find the total number of downward pointing triangles of all sizes in Example 2.4.

3.36 Solve by the annihilator method
(a) $8y(t+2) - 6y(t+1) + y(t) = 2^t$.
(b) $y(t+2) - 2y(t+1) + y(t) = 3t + 5$.
(c) $y(t+2) + y(t+1) - 12y(t) = t3^t$.

3.37 Solve by the annihilator method

$$y(t+2) + 4y(t) = \cos t.$$

3.38 Solve the initial value problem

$$y_{n+2} - 4y_{n+1} + 3y_n = n4^n, \qquad y_1 = \frac{2}{9}, \qquad y_2 = \frac{1}{9}.$$

3.39 Use the annihilator method to solve

$$(E^2 - E + 2)y(t) = 3^t + t3^t.$$

3.40 Use the annihilator method to solve:
(a) $y(t+2) - 7y(t+1) + 10y(t) = 4^t$.
(b) $y(t+2) - 6y(t+1) + 8y(t) = 4^t$.
(c) $y(t+2) - y(t+1) - 2y(t) = 4 + 2^t$.
(d) $y(t+2) - 4y(t+1) + 4y(t) = 2^t$.

3.41 Solve the homogeneous system

$$u(t+1) - 3u(t) + v(t) = 0,$$

$$-u(t) + v(t+1) - v(t) = 0.$$

3.42 Find all $u(t)$ and $v(t)$ that satisfy

$$u(t+2) - 3u(t) + 2v(t) = 0,$$
$$u(t) + v(t+2) - 2v(t) = 0.$$

3.43 Use the annihilator method to solve

$$u(t+1) - 4u(t) - v(t) = 3^t,$$
$$u(t+1) - 2u(t) + v(t+1) - 2v(t) = 2.$$

3.44 Use the method of variation of parameters to solve the following equations:

(a) $y(t+2) - 7y(t+1) + 10y(t) = 4^t$.

(b) $y(t+2) - 5y(t+1) + 6y(t) = 3$.

(c) $y(n+2) - y(n+1) - 2y(n) = n2^n$.

(d) $y(n+2) - 7y(n+1) + 12y(n) = 5^n$.

3.45 Use Theorem 3.8 to solve the first order equation

$$y(t+1) - 2y(t) \;=\; 2^t \binom{t}{5}.$$

3.46 Find all solutions of

$$y(t+2) - 7y(t+1) + 6y(t) = 2t - 1.$$

3.47 Use variation of parameters to solve

$$y(t+3) - 2y(t+2) - y(t+1) + 2y(t) = 8 \cdot 3^t.$$

3.48

(a) Show that

$$a_1(t) = -\sum_{k=a}^{t-1} \frac{r(k)}{p_2(k)} \frac{u_2(k+1)}{w(k+1)}$$

and

$$a_2(t) = \sum_{k=a}^{t-1} \frac{r(k)}{p_2(k)} \frac{u_1(k+1)}{w(k+1)}$$

are solutions of Eqs. (3.10) and (3.11).

(b) Use part (a) to prove Corollary 3.2.

3.49 For $n = 2$, define the Cauchy function $K(t, k)$ for Eq. (3.4) to be the function defined for $t, k = a, a+1, \cdots$, such that for each fixed k, $K(t, k)$ is the solution of Eq. (3.4′), satisfying $K(k+1, k) = 0$, $K(k+2, k) = (p_2(k))^{-1}$.

(a) Show that

$$K(t, k) = \frac{-1}{p_2(k)w(k+1)} \det \begin{bmatrix} u_1(t) & u_2(t) \\ u_1(k+1) & u_2(k+1) \end{bmatrix},$$

where $u_1(t), u_2(t)$ are linearly independent solutions of Eq. (3.4′) with Casoratian $w(t)$.

(b) Use Corollary 3.2 to show that the solution of the initial value problem in Eq. (3.4), $y(a) = y(a+1) = 0$, is given by

$$y(t) = \sum_{k=a}^{t-1} K(t, k)r(k).$$

(A related formulation is given in Chapter 6.)

3.50 Use Corollary 3.2 to solve

$$y(t+2) - 5y(t+1) + 6y(t) = 2^t, \qquad y(1) = y(2) = 0.$$

Section 3.4

3.51 Find a formula for the sum of the first n Fibonacci numbers.

3.52 Show that the generating function for the Fibonacci sequence is $\frac{x}{1-x-x^2}$.

3.53 If F_n is the n^{th} Fibonacci number, show that

$$F_{n+1}F_{n-1} - F_n{}^2 = (-1)^n \qquad (n \geq 1).$$

3.54 Show that

(a) $F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$.

(b) F_{mn} is an integral multiple of F_n.

3.55 A strip is one unit wide by n units long. We want to paint this strip with one by one squares that are red or blue. In how many ways can we paint the strip if we do not allow consecutive red squares?

3.56 In how many ways can a 1 by n hallway be tiled if we use one by one blue tiles and one by two red tiles?

3.57

(a) Solve the difference equation in Example 3.15 for the case $\frac{w^2 m}{4k} > 1$.

(b) Show that most of the solutions in part (a) are unbounded as $n \to \infty$.

3.58 Solve the problem

$$\Delta x(t) = -.5x(t) - .3y(t)$$
$$\Delta y(t) = -.2x(t) - .6y(t)$$

if $x(0) = 2$, $y(0) = 5$.

3.59

(a) Use the method of Section 3.3 to solve

$$u_{n+2} - 2xu_{n+1} + u_n = 0, \qquad u_0 = 1, \qquad u_1 = x.$$

(b) Show that the u_n obtained in part (a) is the same as $T_n(x)$.

3.60 Show that $T_n(x)$ is a polynomial of degree n.

3.61 Show that the generating function for $\{T_n(x)\}$ is $\frac{1-xt}{1-2xt+t^2}$.

3.62 The Chebyshev polynomials of the second kind are defined by

$$U_n(x) = \frac{\sin\left((n+1)\cos^{-1} x\right)}{\sqrt{1-x^2}} \qquad (n \geq 0).$$

Show that $U_n(x)$ satisfies the same difference equation as $T_n(x)$.

3.63 Show that the Chebyshev polynomials of the second kind are orthogonal on $[-1, 1]$ with respect to the weight function $\sqrt{1-x^2}$.

3.64 Suppose that in Example 3.18 we want to compute only the quantity of water in the topsoil at 9 P.M. each day. Find the solution by solving a first order equation.

3.65 Solve Example 3.18 with the assumption that only a quarter of the total amount of water in the topsoil is lost between 9 A.M. and 9 P.M.

3.66 Let D_n be the value of an n by n tridiagonal determinant with 4's down the diagonal, 3's down the superdiagonal, and 1's down the subdiagonal. By solving an appropriate initial value problem, find a formula for D_n.

3.67 Compute the determinant in Example 3.19 for the case $a^2 - 4bc > 0$.

3.68 Compute the determinant in Example 3.19 for the case $a^2 = 4bc$.

3.69 Solve the equation $y_{n+1} = \sum_{k=0}^{n}(\varepsilon + y_{n-k})A_k$ in Example 3.20 if $A_0 = A_1 = c > 0$ and $A_k = 0$ for $k \geq 2$.

3.70 Use the method of generating functions to solve this equation

$$u_{n+1} = \sum_{k=0}^{n} \frac{u_{n-k}}{2^k}$$

if $u_0 = 1$.

3.71 A binary tree is a tree with a node at the top, such that each node is attached by line segments to at most two nodes below it.

(a) Show that the number T_n of binary trees having n nodes satisfies

$$T_n = \sum_{i=0}^{n-1} T_i T_{n-i-1} \qquad (n \geq 1),$$

where we use the convention $T_0 = 1$.

(b) Show that the generating function for T_n is

$$G(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

(c) Use the binomial series to show:

$$G(x) = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n+1} (-1)^n 2^{2n+1} x^n,$$

and obtain a formula for T_n.

(d) Show that the formula for T_n obtained in part (c) can be simplified to $T_n = \frac{1}{n+1}\binom{2n}{n}$.

3.72 Let $f(x)$ be the exponential generating function for a_n and let $g(x)$ be the exponential generating function for b_n. Show that $f(x)g(x)$ is the exponential generating function for

$$\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.$$

3.73 Use an exponential generating function to solve this equation

$$\sum_{k=0}^{n} \binom{n}{k} a_k a_{n-k} = (-1)^n, \qquad (n = 1, 2, \cdots)$$

if $a_0 = 1$.

3.74 Let $x(n)$ be the number of ways a 4 by n hallway can be tiled using 2 by 1 tiles.

(a) Find a system of three equations in three unknowns (one of which is $x(n)$) that models the problem.

(b) Use your equations iteratively to find the number of ways to tile a four by ten hallway.

3.75 Three products A, B, and C compete for the same (fixed) market. Let $x(t)$, $y(t)$, and $z(t)$ be the respective percentages of the market for these products after t months. If the changes in the percentages are given by

$$\begin{aligned}\Delta x(t) &= -x(t) + \frac{1}{3}y(t) + \frac{2}{3}z(t),\\ \Delta y(t) &= \frac{1}{3}x(t) - \frac{1}{3}y(t) + \frac{1}{3}z(t),\\ \Delta z(t) &= \frac{2}{3}x(t) - z(t),\end{aligned}$$

and if initially product A has 50% of the market, product B has 30% of the market, and product C has 20% of the market, find the percentages for each product after t months.

3.76 Consider a game with two players A and B, where player A has probability p of winning a chip from B and player B has probability $1-p$ of winning a chip from A on each turn. The game ends when one player has all the chips.

(a) Let $u(t)$ be the probability that A will win the game given that A has t chips. Show that $u(t)$ satisfies

$$u(t) = pu(t+1) + (1-p)u(t-1).$$

(b) Suppose that at the beginning of the game A has a chips and B has b chips. Find the probability that A will win the game.

3.77 Let

$$I_n = \int_0^{\pi} \frac{\cos n\theta - \cos n\phi}{\cos\theta - \cos\phi}\,d\theta, \qquad (n = 0, 1, \cdots).$$

(a) Show that I_n satisfies the equation

$$I_{n+2} - 2(\cos\phi)I_{n+1} + I_n = 0, \qquad (n = 0, 1, \cdots).$$

(b) Compute I_n for $n = 0, 1, 2, \cdots$.

Section 3.5

3.78 Find the general solution of

$$(E - (t+1))\,(E+1)u(t) = 0.$$

3.79 Solve by the method of factoring

(a) $u_{n+2} - (2n+1)u_{n+1} + n^2u_n = 0.$

(b) $u_{n+2} - (e^n+1)u_{n+1} + e^n u_n = 0.$

3.80 Solve each equation by factoring
(a) $y(t+2)+(2t-1)y(t+1)-6ty(t)=0$.
(b) $y(t+2)-(2t+4)y(t+1)+4ty(t)=0$.
(c) $y(n+2)+y(n+1)-n^2y(n)=0$.
(d) $y(n+2)-(n+4)y(n+1)+(2n+2)y(n)=0$.

3.81 Factor and solve

$$u_{n+2}-\frac{3n-2}{n-1}u_{n+1}+\frac{2n}{n-1}u_n=n2^n.$$

3.82 Use the method of reduction of order to solve the difference equation $u_{n+2}-5u_{n+1}+6u_n=0$, given that $u_n=3^n$ is a solution.

3.83 In the n^{th} order equation $\sum_{k=0}^{n}p_k(t)u(t+k)=0$, suppose a solution $u_1(t)$ is known. Make the substitution $u=u_1v$ and use Theorem 2.8 with $a_k=p_k(t)u_1(t+k)$, $b_k=v(t+k)$ to obtain an $(n-1)^{\text{st}}$ order equation with unknown Δv.

3.84 Find general solutions of
(a) $2t(t+1)\Delta^2u(t)+8t\Delta u(t)+4u(t)=0$.
(b) $t(t+1)\Delta^2u(t)-3t\Delta u(t)+4u(t)=0$.

3.85 Solve the following Euler-Cauchy difference equations:
(a) $t(t+1)\Delta^2y(t)-7t\Delta y(t)+16y(t)=0$.
(b) $t(t+1)\Delta^2y(t)-3t\Delta y(t)+4y(t)=0$.
(c) $t(t+1)\Delta^2y(t)+3ty\Delta y(t)+y(t)=0$.

3.86 Solve the equation

$$t(t+1)\Delta^2u-2t\Delta u+2u=t.$$

3.87 Verify Eqs. (3.15) and (3.16).

3.88 Use the method of generating functions to solve

$$3(n+2)u_{n+2}-(3n+4)u_{n+1}+u_n=0$$

if $u_0=3u_1$.

3.89 One solution of $(n+1)u_{n+2}+(2n-1)u_{n+1}-3nu_n=0$ is easy to find. What is the general solution?

3.90 Check that $u_n=2^n$ solves

$$nu_{n+2}-(1+2n)u_{n+1}+2u_n=0,$$

and find a second independent solution.

3.91 Use generating functions to solve $(n+2)(n+1)u_{n+2}-3(n+1)u_{n+1}+2u_n=0$.

3.92 Given the initial value $u_{-1} = 0$, find solutions of the equation

$$2(k+1)u_{k+1} - (1+2k)u_k + u_{k-1} = 0 \qquad (k \geq 0).$$

3.93 In Example 3.14 we introduced the Fibonacci numbers F_n.

(a) Compute the exponential generating function for F_n. (Note: it satisfies a second order differential equation.)

(b) Use your answer in part (a) to rederive the formula for F_n obtained in Example 3.14.

3.94 Find a factorial series solution of the form $\sum_{k=0}^{\infty} a_k t^{\underline{-k}}$ for

$$u(t+2) - 3(t+2)(t+1)u(t+1) + 3(t+2)(t+1)u(t) \;=\; 0.$$

3.95

(a) Compute a formal series solution $u(t) \;=\; \sum_{k=0}^{\infty} a_k t^{\underline{-k+\frac{1}{2}}}$ for $t\Delta u(t) - \frac{1}{2}u(t) = 0$. (Hint: Use the identity $tt^{\underline{r}} = t^{\underline{r+1}} + rt^{\underline{r}}$.)

(b) Show that the trial solution $u(t) = \sum_{k=0}^{\infty} a_k t^{\underline{-k}}$ leads to the zero solution.

Section 3.6

3.96 Solve the Riccati equations

(a) $y(t+1)y(t) + 2y(t+1) + 7y(t) + 20 = 0.$

(b) $y(t+1)y(t) - 2y(t) + 2 = 0.$

3.97 For the following Riccati equations, write your answers in terms of only one arbitrary constant:

(a) $y(t+1)y(t) + 7y(t+1) + y(t) + 15 = 0.$

(b) $y(t+1)y(t) + 3y(t+1) - 3y(t) = 0.$

(c) $y(t+1)y(t) + y(t+1) - 3y(t) + 1 = 0.$

3.98 Use the change of variable $v(t) = \frac{1}{y(t)}$ to solve the Riccati equation

$$ty(t+1)y(t) + y(t+1) - y(t) = 0.$$

3.99 Use a logarithm to solve

(a) $\frac{y_{n+1}}{y_n} = 2y_n^{\frac{1}{n}}.$

(b) $y_{n+2} = y_{n+1}y_n^2.$

3.100 Solve $(t+1)y^2(t+1) - ty^2(t) = 1.$

3.101 Use the change of variable $y_n = \sin z_n$ to solve $y_{n+1} = 2y_n\sqrt{1-y_n^2}.$

3.102 Solve the equation

$$y(t+1) \;=\; y(t)\left(y^2(t) + 3y(t) + 3\right)$$

by trying $\xi(y) = cy + d$ in Eq. (3.22).

3.103 Find the most general equation $y(t+1) = f(y(t))$ that can be solved using $\xi(y) = cy + d$ in Eq. (3.22).

3.104 Let a be a positive constant. Then Newton's Method for computing $\sqrt{a^2} = a$ is

$$y_{n+1} = \frac{1}{2}\left(y_n + \frac{a^2}{y_n}\right).$$

(a) Find D so that $\xi(y) = a - \frac{y^2}{a}$ solves Eq. (3.22) for this difference equation.
(b) Use the change of variable Eq. (3.23) to solve the difference equation.
(c) Show the solution $y_n \to a$ as $n \to \infty$.

3.105 Solve the difference equation

$$y_{n+1} = \frac{1}{2}\left(y_n - \frac{a^2}{y_n}\right).$$

(Hint: try $\xi(y) = -a - \frac{y^2}{a}$.)

3.106 Solve $y(t+1) = (1 - 2y(t))^2$. (Hint: see Example 3.31.)

3.107 Consider the equation $y(t+1) = f(t, y(t))$. Suppose that $\xi(t, y)$ and $D(t)$ satisfy

$$D(t)\xi(t+1, f(t, y)) = \xi(t, y)\frac{tialf}{tialy}(t, y).$$

Show that a change of variable transforms the difference equation into a first order linear equation.

3.108 Solve the equation $y(t+1) = (y(t) + t - 1)^t - t$ by choosing $\xi = y + t - 1$ in the last exercise.

3.109 Use $\xi = \sqrt{y(t-y)}$ to solve $y(t) = \frac{4(1+t)}{t^2} y(t)(t - y(t))$.

Section 3.7

3.110 Find the z-transform of each of the following:
(a) $y_k = 2 + 3k$.
(b) $u_k = 3^k \cos 2k$.
(c) $v_k = \sin(2k - 3)$.
(d) $y_k = k^3$.
(e) $u_k = 3y_{k+3}$.
(f) $v_k = k \cos \frac{k\pi}{2}$.
(g) $y_k = \frac{1}{k!}$.

(h) $u_k = \begin{cases} \frac{(-1)^{\frac{k}{2}}}{(k+1)!}, & k \text{ even} \\ 0, & k \text{ odd.} \end{cases}$

3.111 Find $Z(\cosh at)$ using Theorem 3.12.

3.112 Find $Z(\cos at)$ using Theorem 3.12.

3.113 Find the sequences whose z-transforms are

(a) $Y(z) = \frac{2z^2-3z}{z^2-3z-4}$.

(b) $U(z) = \frac{3z^2-4z}{z^2-3z+2}$.

(c) $V(z) = \frac{2z^2+z}{(z-1)^2}$.

(d) $Y(z) = \frac{z}{2z^2-2\sqrt{2}z+2}$.

(e) $U(z) = \frac{2z^2-z}{2z^2-2z+2}$.

(f) $V(z) = \frac{z^2+3z}{(z-3)^2}$.

(g) $W(z) = \frac{3z^2+5}{z^4}$.

(h) $Y(z) = e^{\frac{1}{z^2}}$.

3.114 Use Theorem 3.13 to show that

$$Z(k^n) = (-1)^n \left(z\frac{d}{dz}\right)^n \frac{z}{z-1}.$$

Use this formula to find $Z(k^3)$.

3.115 Use Theorem 3.15 to find $Z(k^2)$ and $Z(k^3)$.

3.116 Derive the formula for $Z(\delta_k(n))$ by expressing $\delta_k(n)$ in terms of step functions.

3.117 Find the z-transform of each of the following sequences:

(a) $y_1 = 1, \quad y_3 = 4, \quad y_5 = 2, \quad y_k = 0$ otherwise.

(b) $y_{2k+1} = 0, \quad y_{2k} = 1, \quad k = 0, 1, 2, \cdots$.

(c) $y_{2k} = 0, \quad y_{2k+1} = 1, \quad k = 0, 1, 2, \cdots$.

3.118

(a) Use Theorem 3.15 to show that for n a positive integer

$$Z\left(\binom{k}{n}\right) = \frac{z}{(z-1)^{n+1}}, \qquad |z| > 1.$$

(b) Use the Binomial Theorem to show that

$$Z\left(\binom{r}{k}\right) = \frac{(z+1)^r}{z^r}, \qquad |z| > 1.$$

3.119 Solve the following first order initial value problems using z-transforms.

(a) $y_{k+1} - 3y_k = 4^k, \qquad y_0 = 0.$

(b) $y_{k+1} + 4y_k = 10, \qquad y_0 = 3.$

(c) $y_{k+1} - 5y_k = 5^{k+1}, \qquad y_0 = 0.$

(d) $y_{k+1} - 2y_k = 3 \cdot 2^k, \qquad y_0 = 3.$

(e) $y_{k+1} + 3y_k = 4\delta_k(2), \qquad y_0 = 2.$

3.120 Solve the following second order initial value problems using z-transforms:

(a) $y_{k+2} - 5y_{k+1} + 6y_k = 0, \qquad y_0 = 1, \qquad y_1 = 0.$

(b) $y_{k+2} - y_{k+1} - 6y_k = 0, \qquad y_0 = 5, \qquad y_1 = -5.$

(c) $y_{k+2} - 8y_{k+1} + 16y_k = 0, \qquad y_0 = 0, \qquad y_1 = 4.$

(d) $y_{k+2} - y_k = 16 \cdot 3^k, \qquad y_0 = 2, \qquad y_1 = 6.$

(e) $y_{k+2} - 3y_{k+1} + 2y_k = u_k(4), \qquad y_0 = 0, \qquad y_1 = 0.$

3.121 Solve the following systems using z-transforms:

(a)
$$\begin{aligned} u_{k+1} - 2v_k &= 2 \cdot 4^k \\ -4u_k + v_{k+1} &= 4^{k+1} \\ u_0 = 2, \quad v_0 &= 3. \end{aligned}$$

(b)
$$\begin{aligned} u_{k+1} - v_k &= 0 \\ u_k + v_{k+1} &= 0 \\ u_0 = 0, \quad v_0 &= 1. \end{aligned}$$

(c)
$$\begin{aligned} u_{k+1} - v_k &= 2k \\ -u_k + v_{k+1} &= 2k + 2 \\ u_0 = 0, \quad v_0 &= 1. \end{aligned}$$

(d)
$$\begin{aligned} u_{k+1} - v_k &= -1 \\ -u_k + v_{k+1} &= 3 \\ u_0 = 0, \quad v_0 &= 2. \end{aligned}$$

3.122 Use Theorem 3.14 to prove Corollary 3.3.

3.123 Prove that the convolution product is commutative ($u_k * v_k = v_k * u_k$) and associative ($(u_k * v_k) * w_k = u_k * (v_k * w_k)$).

3.124 Calculate the following convolutions:

(a) $1 * 1$.

(b) $1 * k$.

(c) $k * k$.

3.125 Solve the following summation equations for $k \geq 0$:

(a) $y_k = 3 \cdot 5^k - 4 \sum_{m=0}^{k-1} 5^{k-m-1} y_m$.

(b) $y_k = k + 4 \sum_{m=0}^{k-1} (k - m - 1) y_m$.

(c) $y_k = 3 + 12\sum_{m=0}^{k-1}\left(2^{k-m-1} - 1\right)y_m$.

3.126 Solve the following equations for $k \geq 0$:

(a) $y_k = 2^k + \sum_{m=0}^{k-1} 2^{k-m-1} y_m$.

(b) $y_k = 3 + 9\sum_{m=0}^{k-1}(k - m - 1)y_m$.

(c) $y_k = 2^k + 12\sum_{m=0}^{k-1}\left(3^{k-m-1} - 2^{k-m-1}\right)y_m$.

3.127 Solve

$$y_k = 2 + \lambda \sum_{m=0}^{24} \frac{k}{50} y_m$$

for all values of λ for which the equation has a solution.

3.128 Solve the following Fredholm summation equations:

(a) $y_k = 10 + \sum_{m=0}^{20} kmy_m$.

(b) $y_k = k + \sum_{m=1}^{15} my_m$.

(c) $y_k = k + \sum_{m=1}^{15} ky_m$.

(d) $y_k = \lambda \sum_{m=1}^{19} kmy_m$.

3.129 Solve the following Fredholm summation equations:

(a) $y_k = 1 + \sum_{m=1}^{15}(1 - km)y_m$.

(b) $y_k = k + \sum_{m=1}^{10}(m^2 + km)y_m$.

3.130 Prove the last statement in Theorem 3.20.

3.131 Find the currents in the ladder network obtained from Fig. 3.7 by replacing the resistor at the top of each loop with a resistor having resistance $R_0 \neq R$.