TOGETHER WE CAN ACHIEVE MORE

COURSE NAME: MATH – 3 LECTURE -7(7.1 & 7.3)

SEMESTER: FALL 2021-2022

SOLVED BY

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7.1

Singular point,

$$f(z)=\frac{1}{(z+1)(z-3)}$$

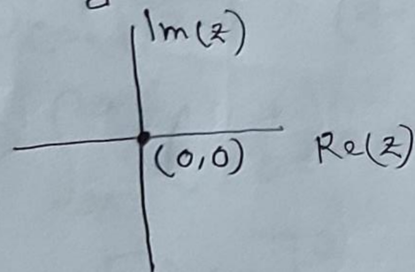
For singular point, $(z+1)(z-3)=0$

$\Rightarrow z+1=0 \quad | \quad z-3=0$

$\Rightarrow z=-1 \quad | \quad \Rightarrow z=3$

1. a) i) $f(z)=\frac{1}{2z}$.

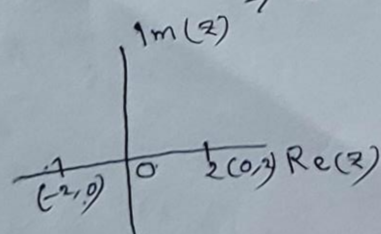
Singular point, $2z=0 \Rightarrow z=0$.



2 ii) $\frac{1}{z^2-4}$.

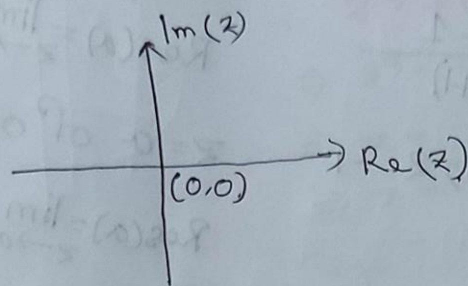
Singular point, $z^2-4=0$

$\Rightarrow z^2=4 \Rightarrow z=\pm 2$ (2,0) (−2,0)



iii) $\frac{\sin z}{z}$

Singular point, $z = 0$



iv) $f(z) = \cot(z)$.

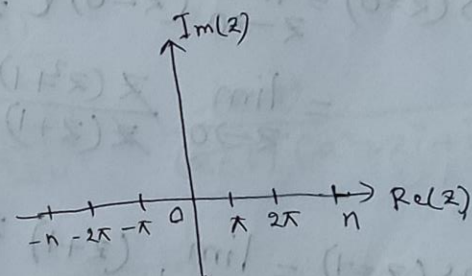
$= \frac{\cos(z)}{\sin(z)}$.

Singular point;

$\sin z = 0$

$\Rightarrow \sin z = \sin n\pi$

$\therefore z = n\pi$



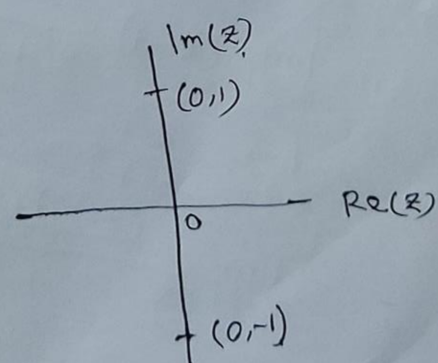
v) $f(z) = \frac{1}{z^6 + 1}$.

Singular point,

$z^6 + 1 = 0$

$\Rightarrow z^6 = -1$.

$\Rightarrow z^3 = \pm i$



b)

i) $f(z) = \dfrac{z^2+1}{z^2+z}$

$= \dfrac{z^2+1}{z(z+1)}$

$z=a$ of order 1 then,

$\text{Res}(a) = \lim_{z \to a} (z-a) f(z)$

Singular point; $z=0$, $z=-1$

$z=a$ of order m

$\text{Res}(a) = \lim_{z \to a} \dfrac{1}{(m-1)!} \dfrac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m f(z) \right\}$

Residue at $z=0$:

$\text{Res}(z=0) = \lim_{z \to 0} (z-0) \left(\dfrac{z^2+1}{z^2+z} \right)$

$= \lim_{z \to 0} \dfrac{\cancel{z}(z^2+1)}{\cancel{z}(z+1)} = 1.$

$\text{Res}(z=-1) = \lim_{z \to -1} (z+1) \dfrac{z^2+1}{z^2+z}$

$= \lim_{z \to -1} \dfrac{\cancel{(z+1)}(z^2+1)}{\cancel{(z+1)}\, z} = \dfrac{z^2+1}{z} = \dfrac{2}{-1} = -2.$

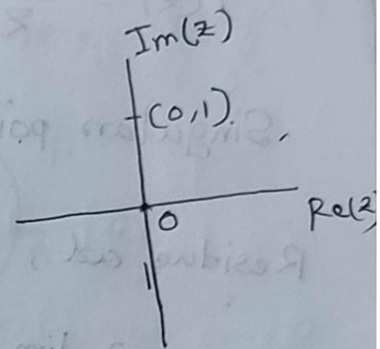
ii) $f(z) = \frac{1}{z^3+i}$

Singular point, $z^3 + i = 0$

$\Rightarrow z^3 = -i$

$\Rightarrow z = -i^3$

$\Rightarrow z = i$.



Residue at $(z=i)$.

$$\text{Res}(z=i) = \lim_{z \to i} (z-i)\frac{1}{z^3+i}$$

$$\Rightarrow \lim_{z \to i} \frac{(z-i)}{z^3+i}$$

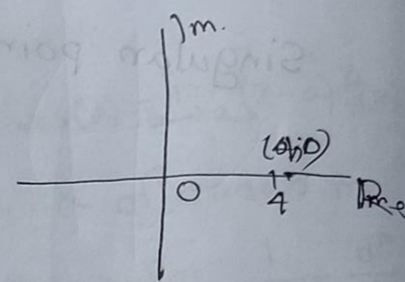
$$\Rightarrow \lim_{z \to i} \frac{(z-i)}{z^3-(i)^3} = \frac{\cancel{(z-i)}}{\cancel{(z-i)}(z^2+zi+i^2)}$$

$$\Rightarrow \lim_{z \to i} \frac{1}{z^2+iz+i^2}$$

$$= \frac{1}{i^2+i^2+i^2} = \frac{1}{-1-1-1} = -\frac{1}{3}$$

iii) $f(z) = \frac{z^2+2}{z-4}$

Singular point, $z = 4$.



Residue at, $(z=4)$

$$\text{Res}(z=4) = \lim_{z \to 4} \cancel{(z-4)} \frac{z^2+2}{\cancel{z-4}}.$$

$$\Rightarrow \lim_{z \to 4} z^2+2$$

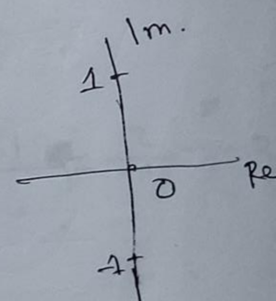
$$\Rightarrow 16+2 = 18$$

iv) $f(z) = \frac{1}{z^6+1}$

Singular point, $z^6 = -1$

$$\Rightarrow z^3 = \pm i^3$$

$$\Rightarrow z = i \;,\; z = -i$$



Residue at $(z=i)$

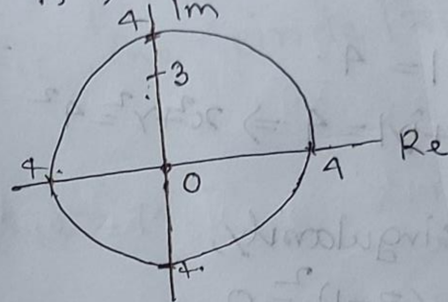
$$\text{Res}(z=i) = \lim_{z \to i} (z-i) \frac{1}{z^6+1}.$$

$$= \lim_{z \to i} \frac{z-i}{z^6+1}.$$

$$= \lim_{z \to i} \frac{\frac{(z-i)(z+i)}{(z+i)}}{z^6+1} = \frac{\frac{z^2-i^2}{z+i}}{z^6+1}.$$

$$= \lim_{z \to i} \frac{z^2+1}{(z+i)(z^6+1)} \Rightarrow \lim_{z \to i} \frac{z^2+1}{(z+i)\left((z^2)^3+1^3\right)}$$

2. a) $\oint_C \frac{dz}{z-3i}$, C is the circle $|z|=4$



$|z| = 4$

$\Rightarrow |x+yi| = 4$

$\Rightarrow x^2+y^2=(4)^2$

For singularity,

$$z-3i=0$$

$$\Rightarrow z=3i$$

(0,3)

Residue at $(z=3i)$.

$$Re(z=3i) = \lim_{z\to 3i} \cancel{(z-3i)} \frac{1}{\cancel{(z-3i)}}$$

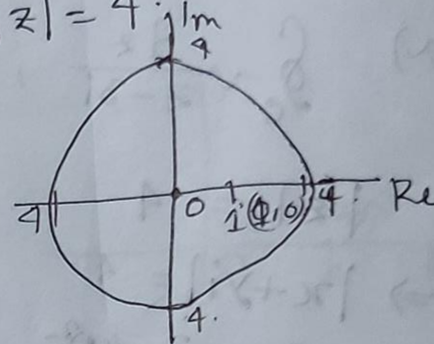
$$=1.$$

Now, Applying CRT, we have,

$$\oint \frac{dz}{z-3i} = 2\pi i \times 1 \quad \left[\oint_C \frac{dz}{z-3i} = 2\pi i\,[\text{Res}(z=3i)]\right]$$

$$=2\pi i$$

b) $\oint_C \frac{e^{-z}}{(z-1)^2}$, C consists of $|z| = 4$.



$|z| = 4$.

$\Rightarrow |x+iy| = 4 \Rightarrow x^2+y^2 = 4^2$

For singularity,

$(z-1)^2 = 0$

$\therefore z = 1$.

Order $= 2$,

As $(z=1)$ lie inside C, so CRT possible.

Residue $(z=1)$

$$Res(z=1) = \lim_{z\to 1} \frac{1}{(2-1)!} \frac{d}{dz}\left\{\cancel{(z-1)^2} \cdot \frac{e^{-z}}{\cancel{(z-1)^2}}\right\}$$

$$= \lim_{z\to 1} \frac{d}{dz}(e^{-z})$$

$$= \lim_{z\to 1}(-e^{-z}) \Rightarrow -e^{-1} = -\frac{1}{e}$$

we know that CRT is.

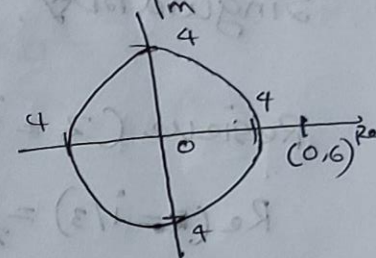
$$\oint \frac{e^{-z}}{(z-1)^2} dz = 2\pi i\left[-\frac{1}{e}\right] = -\frac{2\pi i}{e} \text{ Ans.}$$

~~c.) $\oint_C \frac{e^{-z}}{(z-6)^{10}}$, C is~~

c) $\oint_C \frac{dz}{(z-6)^{10}}$, C is the circle $|z|=4$

$|z| = 4$

$|x+iy| = 4 \Rightarrow x^2+y^2 = (4)^2$

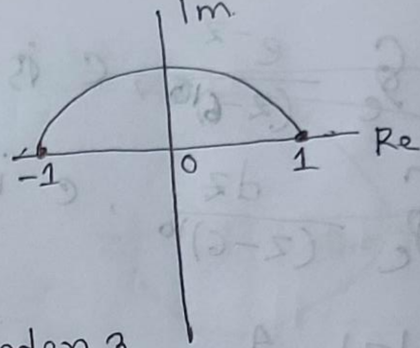


For singularity, $(z-6)^{10} = 0$

$\Rightarrow z = 6$

As $z=6$ lie outside C, so, CRT is not possible.

3. i) $\oint \frac{2z}{(3z-i)^3}\, dz$ (Fig.1)



$$\Rightarrow \oint \frac{2z}{6(z-i/3)^3}\, dz$$

Singularity, $z = i/3$ of order 3

Residue $(z = i/3)$

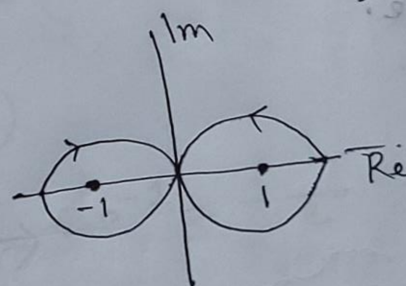
$$Re(z=i/3) = \lim_{z\to i/3} \frac{1}{(3-1)!} \frac{d^{3-1}}{dz^{3-1}}\left[(z-i/3)^3 \frac{2z}{6(z-i/3)^3}\right]$$

$$= \lim_{z\to i/3} \left(\frac{1}{2!}\right) \frac{d^2}{dz^2}\left[\frac{2z}{6}\right]$$

$$= \lim_{z\to i/3} \frac{1}{2}\cdot 0 = 0$$

Ans.

ii) $\oint_c \frac{dz}{z^2-1}$ (Fig.2)



For singularity,

$$z^2-1=0$$
$$\Rightarrow z^2=1$$
$$\Rightarrow z=\pm 1.$$

$$Re[z=1] = \lim_{z\to 1} (z-1)\frac{1}{z^2-1^2}$$

$$= \lim_{z\to 4} \cancel{(z-1)} \frac{1}{\cancel{(z-1)}(z+1)}$$

$$= \lim_{z\to 4} \frac{1}{z+1} = \frac{1}{1+1} = \frac{1}{2}$$

$$Re\,(z=1) = \lim_{z \to -1} \cancel{(z+1)} \frac{1}{\cancel{(z+1)}(z-1)}$$

$$= \frac{1}{-1-1} = \frac{1}{-2}$$

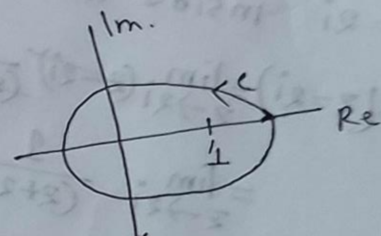
Applying CRT,

$$\oint \frac{dz}{z^2-1}\,dz = 2\pi i\left[Res(z=1) + Res(z=-1)\right]$$

$$= 2\pi i\left[\frac{1}{2} - \frac{1}{2}\right] = 2\pi i \times 0 = 0.$$

Ans.

iii) $\oint_C \frac{2z-1}{z^2-z}\,dz$ (Fig. 3)



$$\Rightarrow \oint_C \frac{2z-1}{z(z-1)}\,dz$$

For singularity,

$$z=0,\ z=1.$$

$$Res[z=0] = \lim_{z \to 0} (z-0)\frac{2z-1}{z(z-1)}$$

$$= \lim_{z \to 0} \cancel{z}\frac{2z-1}{\cancel{z}(z-1)}$$

$$= \frac{-1}{-1} = 1$$

$$Res[z=1] = \lim_{z \to 1} \cancel{(z-1)}\frac{2z-1}{\cancel{(z-1)}z}$$

$$= \lim_{z \to 1} \frac{2z-1}{z} \Rightarrow \frac{2-1}{1} = 1$$

Applying CRT,

$$\oint \frac{2z-1}{z^2-z} dz = 2\pi i \, (1+1) = 4\pi i \quad \text{Ans.}$$

4. a) $\oint_C \frac{dz}{z^2+4}$, C is the contour.

i) $|z+2i| = 1$

For singularity,

$z^2+4=0$

$\Rightarrow z^2 = -4$

$\Rightarrow z^2 = 4i^2$

$\Rightarrow z = \pm 2i$

$\Rightarrow z = 2i, -2i$

$z = 2i$ inside the circle,

$$\text{Res}(z=2i) = \lim_{z \to 2i} \cancel{(z-2i)} \frac{1}{\cancel{(z-2i)}(z+2i)}$$

$$= \lim_{z \to 2i} \frac{1}{(z+2i)}$$

$$= \frac{1}{2i+2i} = \frac{1}{4i}$$

ii) $|z-2i| = 1$

$z = -2i$ inside the circle,

$$\text{Res}(z=-2i) = \lim_{z \to -2i} \cancel{(z+2i)} \frac{1}{\cancel{(z+2i)}(z-2i)}$$

$$= \lim_{z \to 2i} \frac{1}{(z-2i)} \Rightarrow \frac{1}{-2i-2i} = \frac{1}{-4i}$$

Applying CRT,

$$\oint_C \frac{dz}{z^2+4} = 2\pi i \left[\frac{1}{4i} - \frac{1}{4i} \right] = 0 \quad \text{Ans}$$

Sample Exercise set: 7.3

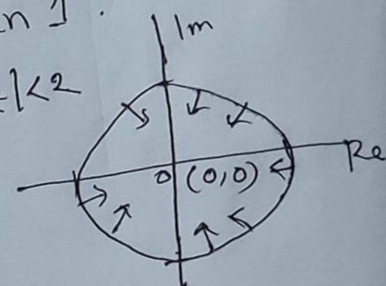
Laurents Theorem:

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \frac{b_3}{(z-z_0)^3} + \dots$$

1. $f(z) = \frac{3z}{(z-1)(2-z)}$

a) $|z| < 1$.

That means a circle center $(0,0)$ which radius is less than 1.



if $|z|<1$, then it must be $|z|<2$

$|z|<1$

so, $\frac{|z|}{1} < 1$

$|z| < 2$

so, $\frac{|z|}{2} < 1$.

$$f(z) = \frac{3z}{(z-1)(2-z)}$$

so, $\frac{3z}{(z-1)(2-z)} = \frac{A}{z-1} + \frac{B}{2-z}$

$$= \frac{3}{z-1} + \frac{6}{2-z}.$$

$$\cancel{= \frac{3}{z(1-\frac{1}{z})} + \frac{6}{2(1-z/2)}}$$

$$\cancel{= \frac{3}{z}\left(1-\frac{1}{z}\right)^{-1} + 3\left(1-z/2\right)^{-1}}$$

For, $z=1$,
$A = \frac{3 \cdot 1}{2-1} = 3$

For, $z=2$,
$B = \frac{3 \cdot 2}{2-1} = 6$

$$\Rightarrow \cancel{\frac{3}{z}\left[1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\cdots\right]} + 3\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right]$$

~~b) $1<|z|<2$.~~

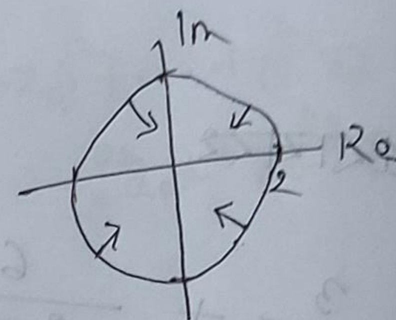
$$\Rightarrow \frac{3}{z-1} + \frac{6}{2-z}$$

$$\Rightarrow \frac{3}{-1(1-z)} + \frac{6}{2(1-z/2)}$$

$$\Rightarrow -3(1-z)^{-1} + 3(1-z/2)^{-1}$$

$$\Rightarrow -3\left[1+z+z^2+z^3+\cdots\right] + 3\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right]$$

2. $f(z) = \frac{1}{z(z-2)}$ in a Laurent Series valid for.



a) $0 < |z| < 2.$

$|z| < 2$

$\Rightarrow \frac{|z|}{2} < 1.$

$$f(z) = \frac{A}{z} + \frac{B}{(z-2)}.$$

$$= \frac{-\frac{1}{2}}{z} + \frac{\frac{1}{2}}{(z-2)}.$$

$$= -\frac{1}{2z} + \frac{1}{2(z-2)}$$

$$= -\frac{1}{2z} + \frac{1}{2}\left(\frac{1}{-2\left(1-\frac{z}{2}\right)}\right)$$

$$\Rightarrow -\frac{1}{2z} - \frac{1}{4}\left(1-\frac{z}{2}\right)^{-1}.$$

$$\Rightarrow -\frac{1}{2z} - \frac{1}{4}\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right]$$

Ans.

For $z = 0$

$A = \frac{1}{0-2} = \frac{1}{-2}$

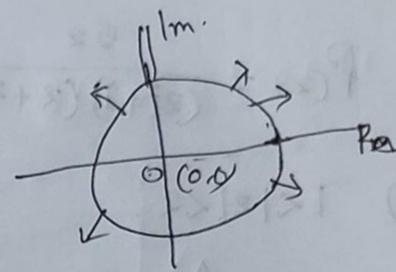
For, $z = 2$

$B = \frac{1}{2}.$

b) $|z|>2$.

$1<|z|$ and $2<|z|$.

$\Rightarrow \frac{1}{z}<1$ and $\Rightarrow \frac{2}{z}<1$.



$$f(z) = -\frac{1}{2z} + \frac{1}{2(z-2)}$$

$$= -\frac{1}{2z} + \frac{1}{2}\cdot\frac{1}{z\left(1-\frac{2}{z}\right)}$$

$$= -\frac{1}{2z} + \frac{1}{2z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= -\frac{1}{2z} + \frac{1}{2z}\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\cdots\right]$$

Ans.

4.

a) $1+z+z^2+z^3+\cdots$

$\boxed{(1-x)^{-1} = 1+x+x^2\cdots}$

$\Rightarrow (1-z)^{-1}$

$= \dfrac{1}{1-z}$

$\therefore |z|<1$

b) $1-z+z^2-z^3+\cdots$

$\Rightarrow (1+z)^{-1}$

$= \dfrac{1}{1+z}$

$\therefore |z|<1$

c) $1+2z+3z^2+4z^3+\cdots$

$= (1-z)^{-2}$

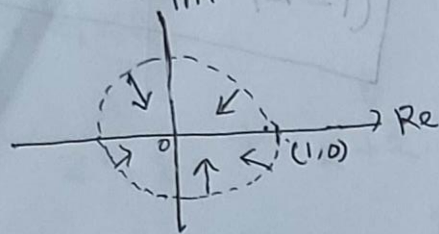
$= \dfrac{1}{(1-z)^2}$

$\therefore |z|<1$

5. i) $f(z) = \dfrac{z}{(z-1)(3-z)} = \dfrac{A}{z-1} + \dfrac{B}{3-z}$

$= \dfrac{1/2}{z-1} + \dfrac{3/2}{3-z}$

a)



$|z| < 1$ $\qquad$ $|z| < 3$

$= \dfrac{|z|}{1} < 1$ $\qquad$ $\Rightarrow \dfrac{|z|}{3} < 1$

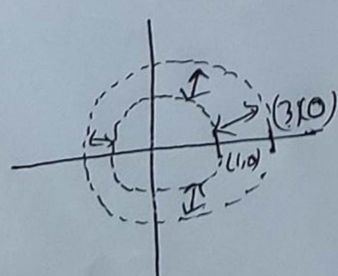
$f(z) = \dfrac{1/2}{z-1} + \dfrac{3/2}{3-z}$

$= \dfrac{1/2}{-1(1-z)} + \dfrac{3/2}{3\left(1-\frac{z}{3}\right)}$

$\Rightarrow -\dfrac{1}{2}(1-z)^{-1} + \dfrac{1}{2}\left(1-\frac{z}{3}\right)^{-1}$

$\Rightarrow -\dfrac{1}{2}\left[1+z+z^2+z^3+\cdots\right] + \dfrac{1}{2}\left[1+\left(\frac{z}{3}\right)+\left(\frac{z}{3}\right)^2+\left(\frac{z}{3}\right)^3 + \cdots\right]$

b)



$|z| > 1$ $\qquad$ $|z| < 3$

$\Rightarrow \dfrac{1}{|z|} < 1$ $\qquad$ $\dfrac{|z|}{3} > 3$

$\therefore f(z) = \dfrac{1/2}{z-1} + \dfrac{3/2}{3-z}$

$= \dfrac{1/2}{z\left(1-\frac{1}{z}\right)} + \dfrac{3/2}{3\left(1-\frac{z}{3}\right)}$

$$\Rightarrow \frac{1}{2z}\left(1-\frac{1}{z}\right)^{-1}+\frac{1}{2}\left(1-\frac{z}{3}\right)^{-1}$$

$$\Rightarrow \frac{1}{2z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^{2}+\left(\frac{1}{z}\right)^{3}+\cdots\right]+\frac{1}{2}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^{2}+\left(\frac{z}{3}\right)^{3}+\cdots\right]$$

Ans:

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