Complex Integration (Line Integral)

Exercise:6 (part-3)

Objective:

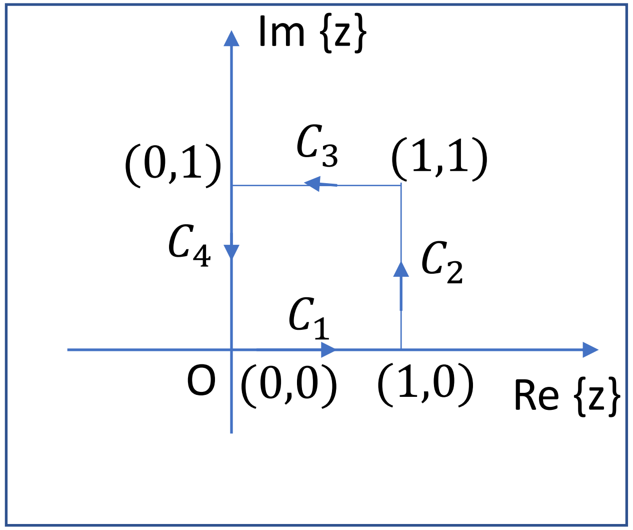
Evaluating line integrals for the paths which consists of multiple paths (line segments).

Methodologies:

Find the line integrals for each of the path separately and then take their summation for the total line integral consisting of multiple paths.

Problem: (i) Sketch the path C which is around the square with vertices $0, \ 1, \ 1+i, i$ and hence evaluate $\int_C |z|^2 \, dz$.

Solution: Here $f(z) = |z|^2 = |x+iy|^2 = \left(\sqrt{x^2+y^2}\right)^2 = x^2+y^2$



For C_1: Equation of the path is $y = 0$

$f(z) = x^2$ and $z = x + iy = x \Longrightarrow dz = dx \ [\because y = 0]$

$\int_{C_1} |z|^2 \, dz = \int_0^1 x^2 \, dx = \frac{1}{3}$.

For C_2: Equation of the path is $x = 1$

$f(z) = 1 + y^2$ and $z = x + iy = 1 + iy \Longrightarrow dz = idy \ [\because x = 1]$

$\int_{C_2} |z|^2 \, dz = \int_0^1 (1+y^2) \, idy = i\frac{4}{3}$.

For C_3: Equation of the path is $y = 1$

$f(z) = x^2 + 1$ and $z = x + iy = x + i \Longrightarrow dz = dx \ [\because y = 1]$

$\int_{C_3} |z|^2 \, dz = \int_1^0 (x^2+1) \, dx = -\frac{4}{3}$.

For C_4: Equation of the path is $x = 0$

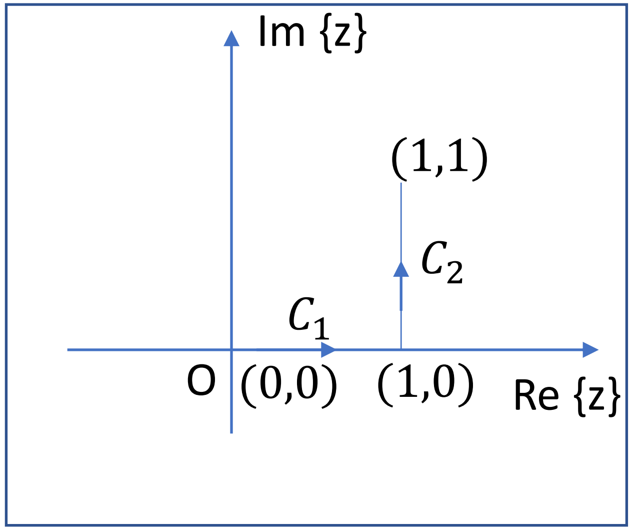
$f(z) = y^2$ and $z = x + iy = iy \Longrightarrow dz = idy \ [\because x = 0]$

$\int_{C_4} |z|^2 \, dz = \int_1^0 y^2 \, idy = -i\frac{1}{3}$.

So, $\int_C |z|^2 \, dz = \frac{1}{3} + i\frac{4}{3} - \frac{4}{3} - i\frac{1}{3} = -1 + i$.

Problem: (ii) Evaluate $\int_C f(z)dz$ where $f(z) = x - y + i\,x^2$ along C which consists two line segments one from $z = 0$ to $z = 1$ and another one from $z = 1$ to $z = 1 + i$.

Solution: Here $f(z) = x - y + i\,x^2$



For C_1: Equation of the path is $y = 0$

$f(z) = x + i\,x^2$ and $z = x + iy = x \Longrightarrow dz = dx$ $[\because y = 0]$

$\int_{C_1} f(z)\,dz = \int_0^1 (x + ix^2)\,dx = \frac{1}{2} + i\frac{1}{3}$.

For C_2: Equation of the path is $x = 1$

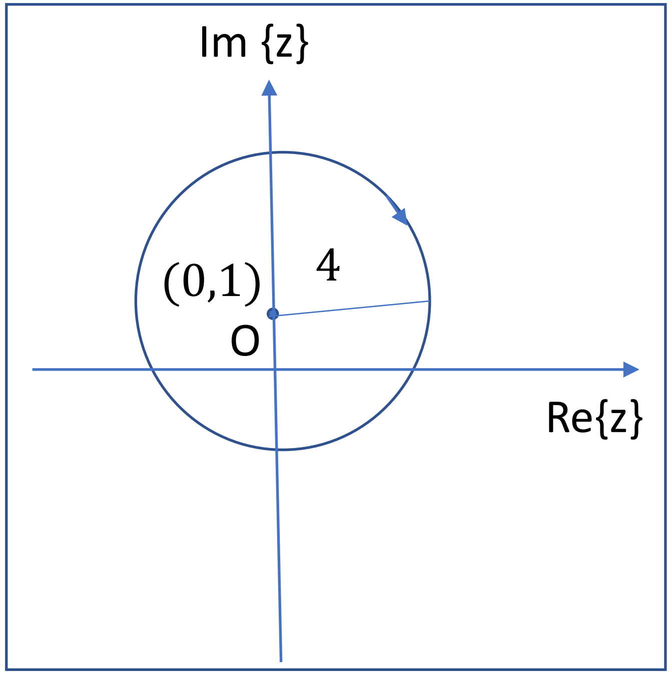
$f(z) = (1 - y + i)$ and $z = x + iy = 1 + iy \Longrightarrow dz = idy$ $[\because x = 1]$

$\int_{C_2} f(z)\,dz = \int_0^1 1 - y + i)\,idy = \frac{1}{2}i - 1.$

So, $\int_C f(z)\,dz = \frac{1}{2} + i\frac{1}{3} + \frac{1}{2}i - 1 = -\frac{1}{2} + i\frac{5}{6}.$

Problem: (iii) Evaluate $\int_C \left(\frac{1}{z-i} - \frac{2}{(z-i)^2}\right) dz$ along C which is the circle $|z-i| = 4$, clockwise.

Solution: Here equation of the path is $|z-i| = 4 \Rightarrow z - i = 4e^{i\theta}$



$$\therefore f(z) = \left(\frac{1}{z-i} - \frac{2}{(z-i)^2}\right) = \left(\frac{1}{4e^{i\theta}} - \frac{2}{16e^{i2\theta}}\right)$$

and $z = re^{i\theta} = 4e^{i\theta} \Rightarrow dz = 4ie^{i\theta} d\theta$ $[\because r = 4]$

$$\int_C f(z)\, dz = \int_{2\pi}^{0} \left(\frac{1}{4e^{i\theta}} - \frac{2}{16e^{i2\theta}}\right) 4ie^{i\theta} d\theta$$

$$= i \int_{2\pi}^{0} \left(1 - \frac{1}{2e^{i\theta}}\right) d\theta = i \int_{2\pi}^{0} \left(1 - \frac{1}{2} e^{-i\theta}\right) d\theta$$

$$= i \left[\theta - \frac{1}{2} \frac{e^{-i\theta}}{(-i)}\right]_{2\pi}^{0} = i \left[\frac{1}{2i} - 2\pi - \frac{e^{-i2\pi}}{2i}\right] = -2\pi i \ \left[\because e^{-i2\pi} = 1\right].$$

Exercises:

- Sketch the path C which is around the rectangle with vertices $0,\ 2,\ 2+i, i$ and hence evaluate $\int_C (z.\bar{z})dz$.
- Evaluate $\int_C f(z)dz$ where $f(z) = 3x + 2xy - i\,y^2$ along C which consists two line segments one from $z = 0$ to $z = 2$ and another one from $z = 2$ to $z = 2 + 2i$.
- Evaluate $\int_C \left(\frac{1}{z+2i} + \frac{1}{(z+2i)^2}\right) dz$ along C which is the circle $|z + 2i| = 4$, counter-clockwise.

Sample MCQ

❖Evaluate $\oint_C \; z^2 \, dz; C: |z| = 1.$

(a) 0 (b) $2\pi i$ (c) not determined .

❖Evaluate $\int_C \; \frac{1}{z-i} \, dz; \; C: |z - i| = 3,$ clockwise.

(a) $2\pi i$ (b) $-2\pi i$ (c) 0 .

❖Evaluate $\int_C f(\,z\,)dz$ where $f(z) = x - iy$ along C which consists two line segments one from $z = 0$ to $z = 1$ and another one from $z = 1$ to $z = 1 + i$.

(a) $1 + i$ (b) $1 - i$ (c) None.

❖ Evaluate $\int_C f(\,z\,)dz$ where $f(z) = x^2 y - i$ along C which consists two line segments one from $z = 0$ to $z = i$ and another one from $z = i$ to $z = 1 + i$.

(a) $\frac{4}{3} - i$ (b) $\frac{4}{3} + i$ (c) None.