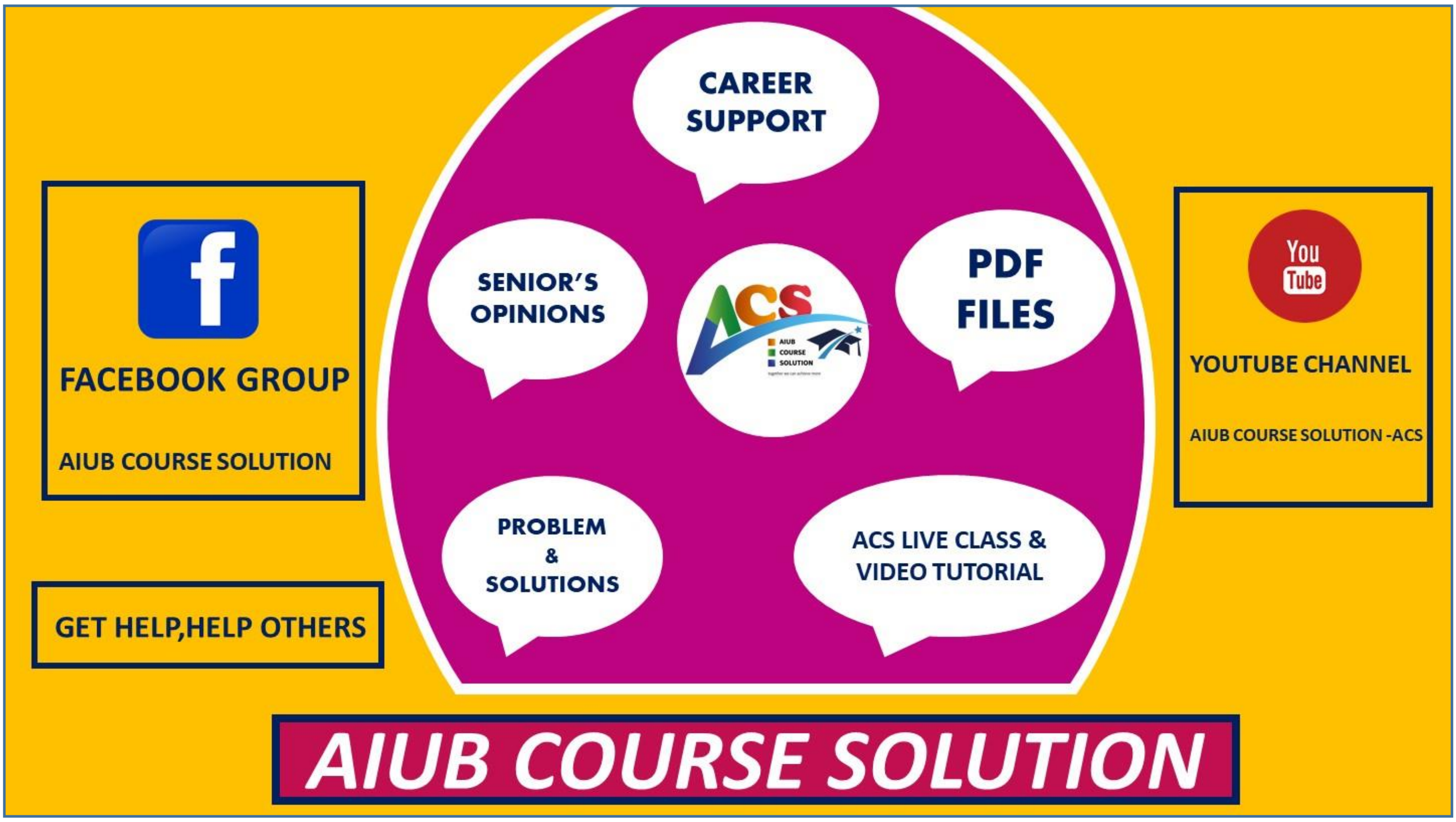

TOGETHER WE CAN ACHIEVE MORE

COURSE NAME: Complex variables, Laplace and Z-transformation.

LECTURE: 1-5

SOLVED BY

NAME: NAZMUS SHAKIB

Lecture -1

Course title : Complex Variables, Laplace and Z- transformation.

Lecture: 1

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

If $f(t)$ is a function defined for all $t \geq 0$ (all positive values of t), its Laplace transform is the integral of $f(t)$ times e^{-st} from $t=0$ to ∞. It is a function of s, say $F(s)$.

This function $F(s)$ of variable s is called Laplace transformation of the original function $f(t)$ and denoted by $\mathcal{L}\{f(t)\}$, where $\mathcal{L}$ is denoted by Laplace transform operator.

Important formulas of Laplace Trans-
-formation:

1. $\mathcal{L}\{constant\ (c)\} = \frac{c}{s}$

$\mathcal{L}\{7\} = \frac{7}{s}$

2. $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

$\mathcal{L}\{t^5\} = \frac{5!}{s^{5+1}} = \frac{5!}{s^6}$

3. $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

$\mathcal{L}\{e^{5t}\} = \frac{1}{s-5}$

4. $\mathcal{L}\{e^{-at}\} = \frac{1}{s-(-a)} = \frac{1}{s+a}$

$\mathcal{L}\{e^{-5t}\} = \frac{1}{s+5}$

5. $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$

$\mathcal{L}\{\cos 5t\} = \frac{s}{s^2+5^2} = \frac{s}{s^2+25}$

6. $L\{\cosh at\} = \dfrac{s}{s^2 - a^2}$

(−) for hyperbolic

$L\{\cosh 2t\} = \dfrac{s}{s^2 - 2^2} = \dfrac{s}{s^2 - 4}$

7. $L\{\sin at\} = \dfrac{a}{s^2 + a^2}$

$L\{\sin 6t\} = \dfrac{6}{s^2 + \cancel{3}6^2} = \dfrac{6}{s^2 + 36}$

8. $L\{\sinh at\} = \dfrac{a}{s^2 - a^2}$

$L\{\sinh t\} = \dfrac{1}{s^2 - 1^2} = \dfrac{1}{s^2 - 1}$

* $L\{t^0\} = \dfrac{0!}{s^{0+1}} = \dfrac{1}{s}$

or

$L\{t^0\} = L\{1\} = \dfrac{1}{s}$

Important Notation of Laplace - - Transformation:

$$\mathcal{L}\{f(t)\} = F(s)$$

The original function $f(t)$ is called the inverse transform or inverse of $F(s)$ and will be denoted by,

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

---

1. $\mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} \text{ or } \mathcal{L}\left\{\frac{1}{s^{2+1}}\right\} = \frac{t^2}{2!} = \frac{t^2}{2}$$

2. $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}$$

3. $\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$

$\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$.

4. $\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}$ or $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} = \cos 2t$

5. $\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

$\mathcal{L}^{-1}\left\{\frac{s}{s^2-4^2}\right\} = \cosh 4t$.

6. $\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$

$\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\} = \sin 3t$

7. $\mathcal{L}^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at$

$\mathcal{L}^{-1}\left\{\frac{5}{s^2-5^2}\right\} = \sinh 5t$

*. Linearity of Laplace Transform:

$$\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

proof:

$$\mathcal{L}\{af(t)+bg(t)\} = \int_0^\infty e^{-st}\left[af(t)+bg(t)\right]dt.$$

$$;\left[\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt\right]$$

$$= a\int_0^\infty e^{-st} f(t)\,dt + b\int_0^\infty e^{-st} g(t)\,dt$$

$$= \boxed{a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}}$$

$$= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

1

Exercise:

1. $\mathcal{L}\{3t+12\}$

$= \mathcal{L}(3t) + \mathcal{L}(12)$

$= 3\mathcal{L}(t) + \mathcal{L}(12)$

$= 3 \times \frac{1}{s^2} + \frac{12}{s}$

Ans:

2. $\mathcal{L}\{e^{5t}\} = \frac{1}{s-5}$

Ans:

3. $\mathcal{L}\{e^{-3t}\} = \frac{1}{s-(-3)} = \frac{1}{s+3}$

Ans!

4. $\mathcal{L}\{(b-ct)^2\}$

$= \mathcal{L}\{b^2 - 2 \cdot b \cdot c \cdot t + c^2 t^2\}$

$= \mathcal{L}\{b^2\} - 2bc\,\mathcal{L}\{t\} + c^2\,\mathcal{L}\{t^2\}$

$= \frac{b^2}{s} - 2bc\frac{1}{s^2} + c^2\,\frac{2!}{s^3}$ Ans

5. $\mathcal{L}\{\cos \pi t\}$

$$= \frac{s}{s^2+\pi^2}$$

Ans:

6. $\mathcal{L}\{\cos^2 \omega t\}$

$$= \mathcal{L}\{\tfrac{1}{2}\, 2\cos^2 \omega t\}$$

$$= \frac{1}{2}\mathcal{L}\{1+\cos 2\omega t\} \quad ; \left[2\cos^2\theta = (1+\cos 2\theta)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4\omega^2}\right]$$

Ans:

7. $\cancel{f(t)}$ $\mathcal{L}\{\sin(\underset{A}{\omega t} + \underset{B}{\theta})\}$

$$= \mathcal{L}\{(\sin \omega t \cdot \cos\theta + \cos \omega t \sin\theta)\}$$

$$; \left[\sin(A+B) = \sin A \cos B + \cos A \sin B\right]$$

$$= \cos\theta\, \mathcal{L}\{\sin \omega t\} + \sin\theta\, \mathcal{L}\{\cos \omega t\}$$

$$= \cos\theta\left(\frac{\omega}{s^2+\omega^2}\right) + \sin\theta\left(\frac{s}{s^2+\omega^2}\right)$$

Ans:

8.

$$\mathcal{L}\left\{1.5\sin\left(3t-\frac{\pi}{2}\right)\right\}$$

$$= 1.5\,\mathcal{L}\left\{\sin\left(3t-\pi/2\right)\right\} \;;\; \left[\sin(A-B)=\sin A\cos B - \cos A\sin B\right]$$

$$= 1.5\,\mathcal{L}\left[\sin 3t\cdot\cos\frac{\pi}{2} - \cos 3t\cdot\sin\frac{\pi}{2}\right]$$

$$= 1.5\left[\cos\frac{\pi}{2}\,\mathcal{L}\{\sin 3t\} - \sin\frac{\pi}{2}\,\mathcal{L}\{\cos 3t\}\right]$$

$$= 1.5\left[\cos\frac{\pi}{2}\left(\frac{3}{s^2+3^2}\right) - \sin\frac{\pi}{2}\left(\frac{s}{s^2+3^2}\right)\right]$$

$$= 1.5\left[0\cdot\left(\frac{3}{s^2+3^2}\right) - 1\left(\frac{s}{s^2+3^2}\right)\right]$$

$$= 1.5\left(-\frac{s}{s^2+3^2}\right)$$

$$= -\frac{1.5s}{s^2+3^2}$$

Ans:

Lecture-2

Course Title: Complex Variabe, Laplace and Z-Transformation

Lecture 2:

S-shifting: Replacing s by (s-a) in the transform.

⇒ we know,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

Similarly,

$$\int_0^\infty e^{-pt} f(t) dt = F(p)$$

$$\int_0^\infty e^{-(s+a)t} f(t) dt = \cancel{F(s)}\ F(s+a)$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} \cdot e^{at} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt.$$

$$= F(s-a)$$

- $\mathcal{L}\{f(t)\} = F(s)$
- $\mathcal{L}\{f(t)e^{at}\} = F(s-a)$
- $\mathcal{L}\{e^{-at} f(t)\} = F(s+a)$

Example:

1. $\mathcal{L}\{e^{2t}\, t^n\}$.

$= F(s-2)$

Now,

$F(s) = \mathcal{L}\{t^n\}$

$= \frac{n!}{s^{n+1}}$

$F(s-2) = \frac{n!}{(s-2)^{n+1}}$

So, $\mathcal{L}\{e^{2t} t^n\} = \frac{n!}{(s-2)^{n+1}}$

Ans.

2. $\mathcal{L}\{e^{-2t}\, t^2\}$

$= F[s-(-2)] = F(s+2)$

Now,

$F(s) = \mathcal{L}\{t^2\}$

$= \frac{2!}{s^{2+1}} = \frac{2!}{s^3}$

(PTO)

$$F(s+2) = \frac{2}{(s+2)^3}$$

So, $\mathcal{L}\{e^{-2t} \cdot t^2\} = \frac{2}{(s+2)^3}$

Ans.

3. $\mathcal{L}\{t^3 e^{t}\}$

$= F(s-1)$

Now,

$F(s) = \mathcal{L}\{t^3\}$

$$= \frac{3!}{s^{3+1}} = \frac{3!}{s^4}$$

$$F(s-1) = \frac{3!}{(s-1)^4}$$

So, $\mathcal{L}\{t^3 e^t\} = \frac{3!}{(s-1)^4}$

Ans.

4. $\mathcal{L}\{e^{-2t}\sin 3t\}$

$= F(s-(-2))$

$= F(s+2)$

Now, $F(s) = \mathcal{L}\{\sin 3t\}$

$= \dfrac{3}{s^2+3^2}$

$= \dfrac{3}{s^2+9}$

$F(s+2) = \dfrac{3}{(s+2)^2+9}$

So, $\mathcal{L}\{e^{-2t}\sin 3t\} = \dfrac{3}{(s+2)^2+9}$

Ans:

5. $\mathcal{L}\{e^{5t}.\cos 2t\} = F(s-5)$, where ~~$F(s) =$~~

$F(s) = \mathcal{L}\{\cos 2t\} = \dfrac{s}{s^2+2^2} = \dfrac{s}{s^2+4}$

$F(s-5) = \dfrac{(s-5)}{(s-5)^2+4}$ Ans:

Exercise:

1. $\mathcal{L}\{e^{2t}\sinh 3t\}$

$= F(s-2)$

Now,

$F(s) = \mathcal{L}\{\sinh 3t\}$

$= \frac{3^2}{s^2-3^2}$

$= \frac{3}{s^2-9}$

$F(s-2) = \frac{3}{(s-2)^2-9}$

$\mathcal{L}\{e^{2t}\sinh 3t\} = \frac{3}{(s-2)^2-9}$

Ans:

2. $\mathcal{L}\{e^{-t}\sinh 4t\}$

$= F(s+1)$; $[e^{-t} = e^{-1t}]$

Now,

$F(s) = \mathcal{L}\{\sinh 4t\}$

$= \frac{4}{s^2-4^2} = \frac{4}{s^2-16}$ (PTO)

$$F(s+1) = \frac{4}{(s+1)^2-16}$$

Ans:

3. $\mathcal{L}\{e^{2t}\cos 3t\}$

$= F(s-2)$

Now,

$F(s) = \mathcal{L}\{\cos 3t\}$

$$= \frac{s}{s^2+3^2} = \frac{s}{s^2+9}$$

$$F(s-2) = \frac{(s-2)}{(s-2)^2+9}$$

So, $\mathcal{L}\{e^{2t}.\cos 3t\} = \dfrac{(s-2)}{(s-2)^2+9}$

Ans:

4. $\mathcal{L}\{t^{10}\, e^{-7t}\}$

$= F\{s-(-7)\} = F(s+7)$

Now,

$F(s) = \mathcal{L}\{t^{10}\}$

$$= \frac{10!}{s^{10+1}} = \frac{10!}{s^{11}}$$

$$f(s+7) = \frac{10!}{(s+7)^{11}}$$

Now,

$$\mathcal{L}\{t^{10} e^{-7t}\} = \frac{10!}{(s+7)^{11}}$$

Ans.

5. $\mathcal{L}\{e^{5t} \cosh 6t\}$

$= F(s-5)$

Now,

$$F(s) = \mathcal{L}\{\cosh 6t\}$$

$$= \frac{s}{s^2 - 36^2}$$

$$= \frac{s}{s^2 - 36}$$

$$f(s-5) = \frac{(s-5)}{(s-5)^2 - 36}$$

So, $\mathcal{L}\{e^{5t} \cosh 6t\} = \dfrac{(s-5)}{(s-5)^2 - 36}$

Ans.

Property of multiplication by t^n:

if $\mathcal{L}\{f(t)\} = F(s)$ then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} \mathcal{L}\{f(t)\}$$

Example:

$$\mathcal{L}\{t^2 e^{3t}\} = (-1)^2 \frac{d^2}{ds^2} \mathcal{L}\{e^{3t}\}$$

$$= 1 \frac{d^2}{ds^2}\left(\frac{1}{s-3}\right)$$

$$= \frac{d^2}{ds^2}(s-3)^{-1}$$

$$= \frac{d}{ds}\left[(-1)(s-3)^{-2}\right] ; \left[\frac{d}{dx}x^n = nx^{n-1}\right]$$

$$= (-1)(-2)(s-3)^{-3}$$

$$= 2(s-3)^{-3}$$

$$= \frac{2}{(s-3)^3}$$

Ans:

(a). $\mathcal{L}\{(t+e^{2t})^2\}$

$= \mathcal{L}\{t^2 + 2 \cdot t \cdot e^{2t} + (e^{2t})^2\}$

$= \mathcal{L}\{t^2 + 2 \cdot t \cdot e^{2t} + e^{4t}\}$

$= \mathcal{L}\{t^2\} + 2\mathcal{L}\{t \cdot e^{2t}\} + \mathcal{L}\{e^{4t}\}$

$= \frac{2!}{s^{2+1}} + 2(-1)\frac{d}{ds}\mathcal{L}\{e^{2t}\} + \frac{1}{s-4}$

$= \frac{2!}{s^3} - 2\frac{d}{ds}\left(\frac{1}{s-2}\right) + \frac{1}{s-4}$

$= \frac{2}{s^3} - 2\frac{d}{ds}(s-2)^{-1} + \frac{1}{s-4}$

$= \frac{2}{s^3} - 2(-1)(s-2)^{-2} + \frac{1}{s-4}$

$= \frac{2}{s^3} + 2(s-2)^{-2} + \frac{1}{s-4}$

$= \frac{2}{s^3} + \frac{2}{(s-2)^2} + \frac{1}{s-4}$

Ans.

$\mathcal{L}\{(t+\cos t)^2\}$

$= \mathcal{L}\{t^2 + 2t\cdot\cos t + \cos^2 t\}$

$= \mathcal{L}\{t^2\} + 2\cdot\mathcal{L}\{t\cdot\cos t\} + \mathcal{L}\{\cos^2 t\}$

$= \frac{2!}{s^{2+1}} + 2(-1)^1 \frac{d}{ds}[\mathcal{L}(\cos t)] + \mathcal{L}\{\frac{1}{2}\cdot 2\cos^2 t\}$

$= \frac{2}{s^3} - 2\frac{d}{ds}\left(\frac{s}{s^2-1}\right) + \frac{1}{2}\mathcal{L}\{1+\cos 2t\}$

$\therefore [1+\cos 2t = 2\cos^2 t]$

$= \frac{2}{s^3} - 2\left[\frac{(s^2-1)\frac{d}{ds}s - s\frac{d}{ds}(s^2-1)}{(s^2-1)^2}\right] + \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+2^2}\right]$

$= \frac{2}{s^3} - 2\left[\frac{(s^2-1) - s\cdot 2s}{(s^2-1)^2}\right] + \frac{1}{2s} + \frac{s}{2(s^2+2^2)}$

$= \frac{2}{s^3} - 2\left[\frac{s^2-1-2s^2}{(s^2-1)^2}\right] + \frac{1}{2s} + \frac{s}{2(s^2+2^2)}$

$= \frac{2}{s^3} + \frac{2s^2+2}{(s^2-1)^2} + \frac{1}{2s} + \frac{s}{2(s^2+2^2)}$

Ans:

Exercise:

1. $\mathcal{L}\{t\sin 2t\} = (-1)^1 \frac{d}{ds}\left[\mathcal{L}\{\sin 2t\}\right]$

$= -1 \frac{d}{ds}\left(\frac{2}{s^2+4}\right)$

$= -1 \frac{d}{ds}\left[(s^2+4)^{-2}\right]$

$= (-1)(-2)(s^2+4)^{-3} \frac{d}{ds}(s^2+4)$

$= 2(s^2+4)^{-3} \cdot 2s$

$= 4s(s^2+4)^{-3}$

Ans:

2. $\mathcal{L}\{t\sinh 3t\} = (-1)\frac{d}{ds}\left[\mathcal{L}\{\sinh 3t\}\right]$

$= -1 \frac{d}{ds} \frac{3}{s^2-3^2}$

$= (-1) \frac{d}{ds}(s^2-9)^{-3}$

$= (-1)(-3)(s^2-9)^{-4}\frac{d}{ds}(s^2-9)$

$= 3(s^2-9)^{-4} \cdot 2s$

$= 6s(s^2-9)^{-4}$ Ans:

$$3.\ \mathcal{L}\{t\cos bt\} = (-1)\frac{d}{ds}\left[\mathcal{L}\{\cos bt\}\right]$$

$$= (-1)\frac{d}{ds}\left(\frac{s}{s^2+b^2}\right)$$

$$\cancel{= (-1)\frac{d}{ds}(s^2+b^2)}$$

$$= (-1)\left[\frac{(s^2+b^2)\frac{d}{ds}s - s\frac{d}{ds}(s^2+b^2)}{(s^2+b^2)^2}\right]$$

$$;\left[\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}u - u\frac{d}{dx}v}{v^2}\right]$$

$$= (-1)\left[\frac{(s^2+b^2) - s\cdot 2s}{(s^2+b^2)^2}\right]$$

$$= -1\left[\frac{s^2+b^2-2s^2}{(s^2+b^2)^2}\right]$$

$$= \frac{s^2-b^2}{(s^2+b^2)^2}$$

Ans:

9. $\mathcal{L}\{t^2 e^{-4t}\} = (-1)^2 \frac{d^2}{ds^2}\left[\mathcal{L}\{e^{-4t}\}\right]$

$= 1 \frac{d^2}{ds^2}\left(\frac{1}{s+4}\right)$

$= \frac{d^2}{ds^2}(s+4)^{-1}.$

$= \frac{d}{ds}\left[(-1)(s+4)^{-2}\right]$

$= (-1)(-2)(s+4)^{-3}$

$= 2(s+4)^{-3}$

Ans:

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5. $\mathcal{L}\{t\cosh 2t\} = (-1)\frac{d}{ds}\left[\mathcal{L}\{\cosh 2t\}\right]$

$$= (-1)\frac{d}{ds}\left(\frac{s}{s^2-2^2}\right)$$

$$= (-1)\frac{d}{ds}\left(\frac{s}{s^2-4}\right)$$

$$= (-1)\left[\frac{(s^2-4)\frac{d}{ds}s - s\frac{d}{ds}(s^2-4)}{(s^2-4)^2}\right]$$

$$= (-1)\left[\frac{(s^2-4) - s\cdot 2s}{(s^2-4)^2}\right]$$

$$= (-1)\left[\frac{-s^2-4}{(s^2-4)^2}\right]$$

$$= \frac{s^2+4}{(s^2-4)^2}$$

Ans:

Lecture -3

Course Title: Complex Variable, Laplace ~~and~~ and Z-Transformation

Lecture 3: unit step function: $u(t-a)$

$$u(t-a) = \begin{cases} 0 & \text{if } t<a \\ 1 & \text{if } t>a \end{cases}$$

The unit step function $u(t-a)$ is 0 for $t<a$ and 1 for $t>a$

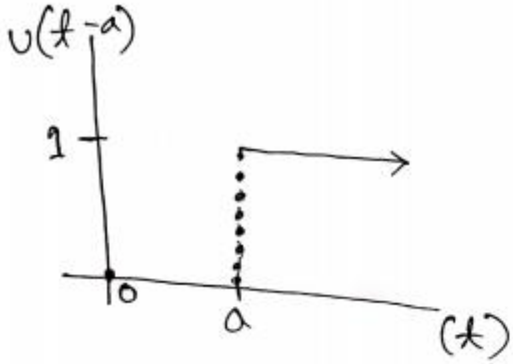
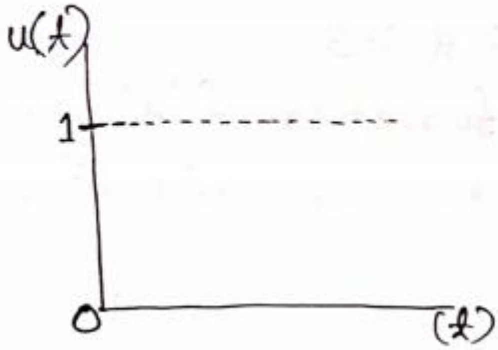


figure: unit step function $u(t-a)$.

$$u(t) = u(t-0) = \begin{cases} 0; & t<0 \\ 1; & t>0 \end{cases}$$



* write this $f(t)=t^2$; $1<t<3$, function into unit step function.

$$\boxed{t^2[u(t-1)-u(t-3)]}$$

unit step function.

$$f(t)=\begin{cases} t; & (0<t<3) \\ t^2; & (t>3)\end{cases}$$

, express $f(t)$ in terms of unit step function.

$$\Rightarrow f(t) = t[\cancel{t}\; u(t-0) - u(t-3)] + t^2[u(t-3)]$$

$$= t\,u(t-0) - t\,u(t-3) + t^2 u(t-3)$$

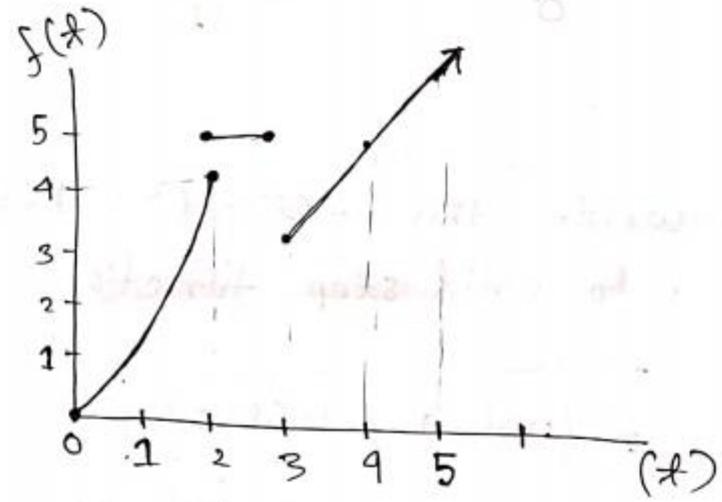
$$= t\,u(t-0) + (t-3)u(t^2-t)$$

↳ unit step function.

$$f(t) = \begin{cases} t^2 \,; & 0 < t < 2 \\ 5 \,; & 2 < t < 3 \\ t \,; & t > 3 \end{cases}$$

Sketch the function $f(t)$ and express $f(t)$ in terms of unit step function.

$\Rightarrow$



Unit step function:

$$f(t) = t^2[u(t-0) - u(t-2)] + 5[u(t-2) - u(t-3)] + t[u(t-3)]$$

$$= t^2 u(t-0) - t^2 u(t-2) + 5u(t-2) - 5u(t-3) + t\,u(t-3)$$

$$= t^2 u(t+0) + u(t-2)(5-t^2) + u(t-3)(t-5)$$

Ans.

formula : $\mathcal{L}\{f(t)\,u(t-a)\} = e^{-as}\,\mathcal{L}\{f(t+a)\}$

Question:

$$\mathcal{L}\{t\,u(t-2)\} = e^{-2s}\,\mathcal{L}\{t+2\}$$

$$= e^{-2s}\,\mathcal{L}\{t\} + \mathcal{L}\{2\}$$

$$= e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right)$$

Ans:

$$\cancel{\circledast\ \mathcal{L}\{t^2\,u(t-1)\}}$$

$$\cancel{= e^{-s}\,\mathcal{L}\{(t+a)^2\}}$$

$$\cancel{= e^{-s}\,\mathcal{L}\{t^2 + 2at + 1}$$

$$\circledast\ \mathcal{L}\{t^2\,u(t-1)\}$$

$$= e^{-s}\,\mathcal{L}\{(t+1)^2\}$$

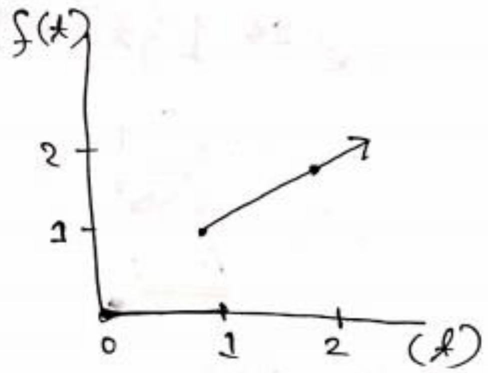
$$= e^{-s}\,\mathcal{L}\{t^2 + 2t + 1\}$$

$$= e^{-s}\left[\frac{2!}{s^3} + 2\cdot\frac{1}{s^2} + \frac{1}{s}\right]$$ Ans.

Exercise

Sketch the following function & find their Laplace transformation:

1. $f(t) = t\,u(t-1) = \begin{cases} 0 & ; t<1 \\ t & ; t>1 \end{cases}$



$\mathcal{L}\{f(t)\} = \mathcal{L}\{t\,u(t-1)\}$

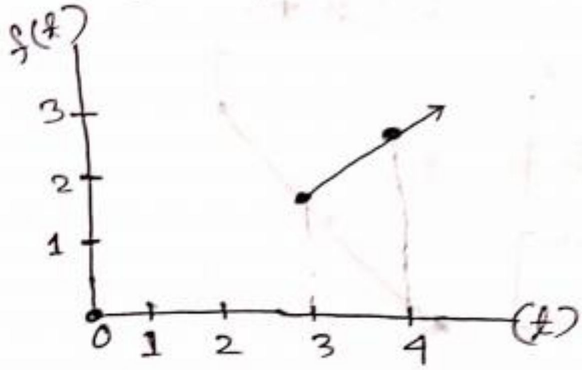
$= e^{-s}\,\mathcal{L}\{t+1\}$

$= e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$

Ans:

2. $f(t) = (t-1)\,u(t-3)$

$$= \begin{cases} 0 ; & t < 3 \\ (t-1) ; & t > 3 \end{cases}$$



$$\mathcal{L}\{f(t)\} = \mathcal{L}\{(t-1)\,u(t-3)\}$$

$$= e^{-3s}\,\mathcal{L}\{(t+3-1)\}$$

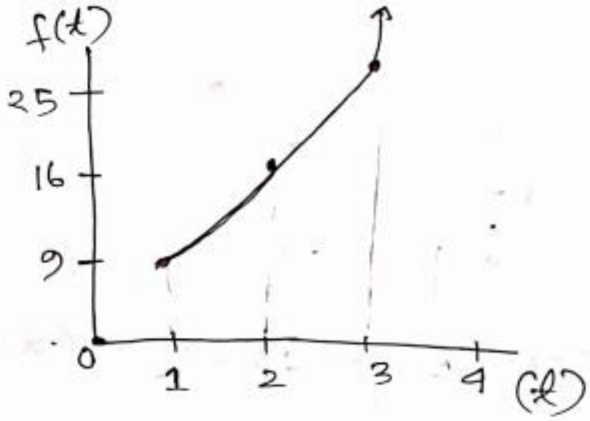
$$= e^{-3s}\,\mathcal{L}\{(t+2)\}$$

$$= e^{-3s}\left[\frac{1}{s^2} + \frac{2}{s}\right]$$

Ans

3. $f(t) = (t+2)^2 u(t-1)$

$$= \begin{cases} 0 \;; & t < 1 \\ (t+2)^2 \;; & t > 1 \end{cases}$$



$$\mathcal{L}\{(t+2)^2 u(t-1)\} = e^{-s} \mathcal{L}\{(t+1+2)^2\}$$

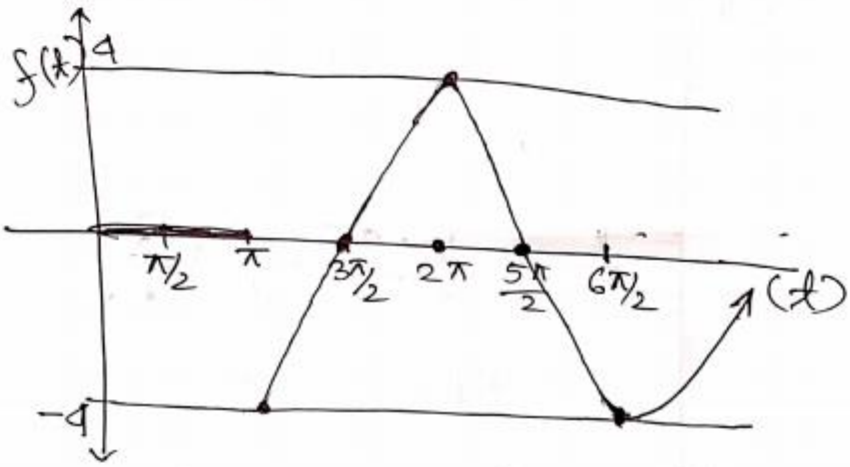
$$= e^{-s} \mathcal{L}\{(t+3)^2\}$$

$$= e^{-s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]$$

Ans.

4. $f(t) = 4\cos t \; u(t-\pi)$

$$= \begin{cases} 0 ; & t<\pi \\ 4\cos t ; & t>\pi \end{cases}$$



$\mathcal{L}\{4\cos t \; u(t-\pi)\}$

$= 4e^{-\pi s}\,\mathcal{L}\{\cos(t+\pi)\}$

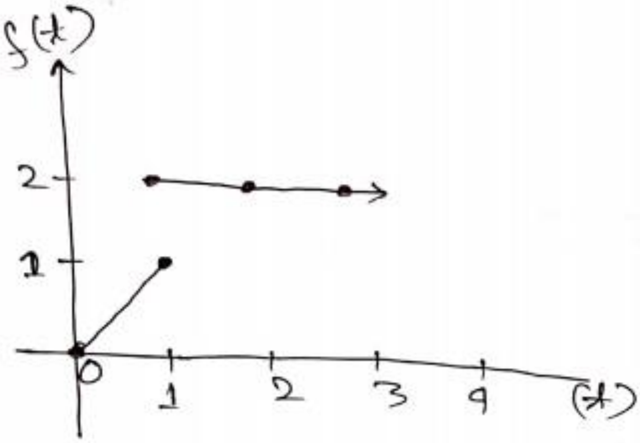
$= 4e^{-\pi s}\,\mathcal{L}\{-\cos t\}$

$= -4e^{-\pi s}\left(\dfrac{s}{s^2+1}\right)$

Ans:

Sketch the following question and also express f(t) in unit step function and find its laplace transformation.

$$f(t) = \begin{cases} t & ; 0 < t < 1 \\ 2 & ; t > 1 \end{cases}$$



$$f(t) = t\left[u(t-0) - u(t-1)\right] + 2\left[u(t-1)\right]$$

$$= t\,u(t-0) + u(t-1)(2-t)$$

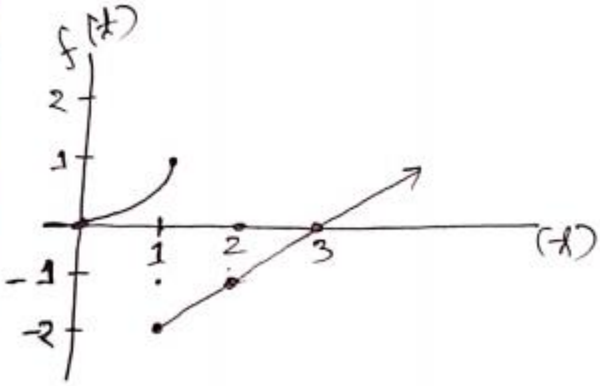
$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t\,u(t-0)\} + \mathcal{L}\{(2-t)\,u(t-1)\}$$

$$= e^{-0.s}\,\mathcal{L}\{t\} + e^{-s}\,\mathcal{L}\{(3-t)\}$$

$$= 1\cdot\frac{1}{s^2} + e^{-s}\left[\frac{3}{s} - \frac{1}{s^2}\right]$$

Ans:

$$* \; f(t) = \begin{cases} t^2 ; & 0<t<1 \\ t-3 ; & t>1 \end{cases}$$



$$f(t) = t^2\left[u(t-0) - u(t-1)\right] + (t-3)\left[u(t-1)\right]$$

$$= t^2 u(t-0) - t^2 u(t-1) + (t-3)\,u(t-1)$$

$$= t^2 u(t-0) + u(t-1)\,(t-3-t^2)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2 u(t-0)\} + \mathcal{L}\{(t-3-t^2)\,u(t-1)\}$$

$$= e^{-0}\mathcal{L}\{t^2\} + e^{-s}\mathcal{L}\{t+1-3-(t+1)^2\}$$

$$= \frac{2!}{s^3} + e^{-s}\mathcal{L}\{t-2-t^2-2t-1\}$$

$$= \frac{2}{s^3} + e^{-s}\mathcal{L}\{-t-t^2-3\}$$

$$= \frac{2}{s^3} - e^{-s}\left[\frac{1}{s^2} + \frac{2}{s^3} + \frac{3}{s}\right]$$

Ans.

Lecture-4

Course : Complex Variable, Laplace and Z Tramformation.

Lecture 4 : Inverse Laplace Transformation.

Formula of Inverse Laplace Tramformation

$\Rightarrow$:

1. $\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$

2. $\mathcal{L}^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}$

3. $\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

4. $\mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

5. $\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$

6. $\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

7. $\mathcal{L}^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at$

Exercise: find the inverse Laplace Transformation of the following functions:

1. $\mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\}$

$= e^{5t}$.

Ans:

2. $\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s^{4+1}}\right\}$

$= \frac{t^4}{4!}$

Ans:

3. $\mathcal{L}^{-1}\left\{\frac{s^3-5s^2+6}{s^4}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 5\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 6\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}$

$= 1 - 5t + 6\frac{t^3}{3!}$

Ans:

4. $L^{-1}\left\{\frac{2+4s}{s^2+25}\right\}$

$= L^{-1}\left\{\frac{2}{s^2+5^2}\right\} + L^{-1}\left\{\frac{4s}{s^2+25}\right\}$

$= 2\, L^{-1}\left\{\frac{5}{5(s^2+5^2)}\right\} + 4\, L^{-1}\left\{\frac{s}{s^2+5^2}\right\}$

$= 2\times\frac{1}{5}\, L^{-1}\left\{\frac{5}{s^2+5^2}\right\} + 4\cos 5t$

$= \frac{2}{5}\sin 5t + 4\cos 5t.$

Ans:

5. $L^{-1}\left\{\frac{3}{s^2-4}\right\}$

$= L^{-1}\left\{\frac{3\times 2}{2(s^2-2^2)}\right\}$

$= \frac{3}{2} L^{-1}\left\{\frac{2}{s^2-2^2}\right\}$

$= \frac{3}{2}\sinh 2t$

Ans:

* find Inverse Laplace of the following function:

1. $F(s) = \frac{1}{(s-3)^4}$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-3)^4}\right\}$$

$$= \frac{t^3}{3!} \cdot e^{3t}.$$

Ans.

2. $F(s) = \frac{3}{(s+2)^2+9}$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{(s+2)^2+3^2}\right\}$$

$$= e^{-2t}\sin 3t.$$

Ans.

3. $F(s) = \frac{(s-2)}{(s-2)^2-16}$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2-16}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{(s-2)}{(s-2)^2-4^2}\right\}$$

$$= e^{2t}\cosh 4t$$

Ans:

4. $\mathcal{L}^{-1}\left\{\frac{s}{s^2+4s-9}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{s+2-2}{s^2+2.s.2+2^2-13}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+2-2}{(s+2)^2-13}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2-(\sqrt{13})^2}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2-(\sqrt{13})^2}\right\}$$

$$= e^{-2t}\cosh\sqrt{13} - 2\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2-(\sqrt{13})^2}\right\}$$

$$= e^{-2t}\cosh\sqrt{13} - \frac{2e^{-2t}}{\sqrt{13}}\sinh\sqrt{13}$$

Ans:

5. $f(s) = \left\{ \frac{s}{s^2+2s+10} \right\}$

$$\mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{ \frac{s}{s^2+2s+10} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{s}{s^2+2\cdot s\cdot 1+1^2+9} \right\}$$

5.

$$= \mathcal{L}^{-1}\left\{ \frac{s}{(s+1)^2+3^2} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{s+1-1}{(s+1)^2+3^2} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{(s+1)}{(s+1)^2+3^2} \right\} - \mathcal{L}^{-1}\left\{ \frac{1}{(s+1)^2+3^2} \right\}$$

$$= e^{-t}\cos 3t - \frac{e^{-t}\sin 3t}{3}$$

Ans:

Lecture-5

2

Lecture 5 : Inverse Laplace Transformation by partial fraction.

1. $\mathcal{L}^{-1}\left\{\frac{s+1}{s(s-2)(s+3)}\right\}$

Let,

$$\frac{s+1}{s(s-2)(s+3)} = \frac{A}{s} + \frac{B}{(s-2)} + \frac{C}{(s+3)} \quad \text{(1)}$$

$\Rightarrow s+1 = A(s-2)(s+3) + B(s+3)\cdot s + C\cdot s(s-2)$

$s=2,$

$2+1 = A\cdot 0 + B\cdot 5\cdot 2 + C\cdot 0$

$\Rightarrow 3 = 10B.$

$\Rightarrow B = \frac{3}{10}$

$s=-3,$

$(-3+1) = A\cdot 0 + B\cdot 0 + C(-3)(-5)$

$\Rightarrow -2 = 15C$

$\Rightarrow C = \frac{-2}{15}$

(PTO)

$s=0,$

$(0+1)=A(0-2)(0+3)+B\cdot 0+C\cdot 0$

$\Rightarrow 1=-6A.$

$\Rightarrow A=\frac{-1}{6}$

put the value of A, B & C in equation (1)

$$\frac{(s+1)}{s(s-2)(s+3)}=\frac{A}{s}+\frac{B}{s-2}+\frac{C}{s+3}$$

$$=\frac{-1}{6s}+\frac{3}{10(s-2)}-\frac{2}{15(s+3)}$$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s(s-2)(s+3)}\right\}=\mathcal{L}^{-1}\left\{-\frac{1}{6s}+\frac{3}{10(s-2)}-\frac{2}{15(s+3)}\right\}$$

$$\cancel{=-\frac{1}{6}+\frac{10}{3}e^{2t}}$$

$$=-\frac{1}{6}+\frac{3}{10}e^{2t}-\frac{2}{15}e^{-3t}$$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s(s-2)(s+3)}\right\}=-\frac{1}{6}+\frac{3}{10}e^{2t}-\frac{2}{15}e^{-3t}$$

Ans.

a3

2. $\mathcal{L}^{-1}\left\{\frac{6s-17}{s^2-5s+6}\right\}$

~~Let,~~

$\frac{6s-17}{}$

$\mathcal{L}^{-1}\left\{\frac{6s-17}{s^2-3s-2s+6}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{6s-17}{(s-3)(s-2)}\right\}$

Let,

$\frac{6s-17}{(s-3)(s-2)} = \frac{A}{(s-3)} + \frac{B}{(s-2)}$ —— (i)

Now,

$(6s-17) = A(s-2) + B(s-3)$ —— (ii)

$s = 2, \Rightarrow$ (ii)

$(6\times 2-17) = A\cdot 0 + B(-1)$

$\Rightarrow B = 5$

$s = 3, \Rightarrow$ (ii)

$(6\times 3-17) = A(3-2) + B\cdot 0$

$\Rightarrow A = 1$

(PTO)

~~putting the value of A, B in~~

put the value of A & B in Equation (1),

$$\frac{6s-17}{(s-3)(s-2)} = \frac{1}{s-3} + \frac{5}{s-2}$$

so,

$$\mathcal{L}^{-1}\left\{\frac{6s-17}{(s-3)(s-2)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s-3} + \frac{5}{s-2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + 5\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= e^{3t} + 5\,e^{2t}$$

$$\mathcal{L}^{-1}\left\{\frac{6s-17}{(s-3)(s-2)}\right\} = e^{3t} + 5\cdot e^{2t}$$

Ans:

1. $f(s) = \dfrac{20}{(s^2+4s+1)(s+1)}$

Let,

$$\frac{20}{(s^2+4s+1)(s+1)} = \frac{A}{(s+1)} + \frac{(Bs+C)}{s^2+4s+1} \quad \text{--- (i)}$$

Now,

$$20 = A(s^2+4s+1) + (Bs+C)(s+1) \quad \text{--- (ii)}$$

$s = -1$,

$$20 = A[(-1)^2 + 4(-1)+1] + (Bs+C)\cdot 0$$

$$\Rightarrow 20 = A(1-4+1) \text{ or } -2A$$

$$\Rightarrow A = -10$$

$s = 0$,

$$20 = A(0+0+1) + (B.0+C)(0+1)$$

$$\Rightarrow 20 = A + C$$

$\Rightarrow C = 20 - A$.

$\Rightarrow C = 20 - (-10)$

$= 20 + 10$

$= 30$

Equating the cooefficient of s^2 from equation (II)

$0 = A + B$.

$\Rightarrow B = 10$

putting these value in equation (I)

$$\frac{20}{(s^2+4s+1)(s+1)} = \frac{-10}{(s+1)} + \frac{10s+30}{s^2+4s+1}$$

Now,

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{20}{(s^2+4s+1)(s+1)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{-10}{(s+1)} + \frac{10s+30}{s^2+4s+1}\right\}$$

$$= -10\,\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + \mathcal{L}^{-1}\left\{\frac{10(s+3)}{s^2+2\cdot s\cdot 2+2^2-2^2+1}\right\}$$

9L

$$= -10e^{-t} + 10\mathcal{L}^{-1}\left\{\frac{s+2+1}{(s+2)^2-3}\right\}$$

$$= -10e^{-t} + \left[10\mathcal{L}^{-1}\left\{\frac{(s+2)}{(s+2)^2-(\sqrt{3})^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2-(\sqrt{3})^2}\right\}\right.$$

$$= -10e^{-t} + 10e^{-2t}\cosh(\sqrt{3})t + \frac{e^{-2t}}{\sqrt{3}} \times \sinh\sqrt{3}t.$$

$$= -10e^{-t} + 10e^{-2t}\cosh(\sqrt{3})t + \frac{e^{-2t}}{\sqrt{3}} \times \sinh 3t$$

Ans:

2. $F(s) = \dfrac{s}{(s^2+4)(s-1)}$

Let,

$$\frac{s}{(s^2+4)(s-1)} = \frac{A}{(s-1)} + \frac{Bs+c}{s^2+4} \quad \text{--- (i)}$$

$$\Rightarrow s = A(s^2+4) + (Bs+c)(s-1) \quad \text{--- (ii)}.$$

$s = 1$, $\Rightarrow$ (ii)

$1 = A(1+4) + (B.1+c).0$

$\Rightarrow 1 = 5A$, $\Rightarrow A = 1/5$

$s = 0$, $\Rightarrow$ (ii)

$0 = A(0+4) + (B.0+c)(0-1)$

$\Rightarrow 0 = 4A - c.$

$\Rightarrow c = 4A$

$\Rightarrow c = \dfrac{4}{5}$

Equatting the coeefficient of s^2 from equation $\Rightarrow$ (ii)

$0 = A + B$

$\Rightarrow B = -A \Rightarrow B = -\dfrac{1}{5}$

Putting the value of A, B, C in equation → (I)

$$\frac{s}{(s^2+4)(s-1)} = \frac{1}{5(s-1)} - \frac{\frac{1}{5}s+\frac{1}{5}\cdot 4}{s^2+4}$$

Now,

$$\mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)(s-1)}\right\}$$

$$= \mathcal{L}^{-1}\left[\frac{1}{5(s-1)} - \frac{\frac{1}{5}(s+4)}{s^2+4}\right]$$

$$= \frac{1}{5}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) - \left[\frac{1}{5}\mathcal{L}^{-1}\left(\frac{s+4}{s^2+4}\right)\right]$$

$$= \frac{1}{5}e^t - \left[\frac{1}{5}\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) + \frac{1}{5}\mathcal{L}^{-1}\left(\frac{4}{s^2+4}\right)\right]$$

$$= \frac{1}{5}e^t - \frac{1}{5}\cos 2t - \frac{1}{5}\sin 2t$$

So, $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)(s-1)}\right\} = \frac{1}{5}e^t - \frac{1}{5}\cos 2t - \frac{1}{5}\sin 2t$

Ans.

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3. $f(s) = \dfrac{3s-4}{(s-2)(s+1)^2}$

Let,

$$\frac{3s-4}{(s-2)(s+1)^2} = \frac{A}{s-2} + \frac{B}{s+1} + \frac{C}{(s+1)^2} \quad \text{——(i)}$$

$$(3s-4) = A(s+1)^2 + B(s-2)(s+1) + C(s-2) \quad \text{——(ii)}$$

putting, $s = -1$,

$\Rightarrow -7 = A \cdot 0 + B \cdot 0 + C(-3)$

$\Rightarrow C = \frac{7}{3}$

putting,
$s = 2$,

$2 = 9A + B \cdot 0 + C \cdot 0$

$\Rightarrow A = \cancel{\frac{9}{2}}\ 2/9$

putting $s = 0$,

$-4 = A - 2B - 2C$

$\Rightarrow -4 = \frac{9}{2} - 2B + 2 \times \cancel{7}(-7/3)$

$$\Rightarrow -4 = \frac{9}{2} - 2B - \frac{14}{3}$$

$$\Rightarrow -4 = \frac{7}{3} - 2B.$$

$$\Rightarrow \frac{19}{3} = 2B.$$

$$\Rightarrow 19/6 = B.$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{3s-4}{(s-2)(s+1)^2}\right\}$$

$$= L^{-1}\left\{\frac{A}{s-2} + \frac{B}{s+1} + \frac{C}{(s+1)^2}\right\}$$

$$= L^{-1}\left\{\frac{2}{9(s-2)} + \frac{19}{6(s+1)} + \frac{7}{3(s+1)^2}\right\}$$

$$\cancel{= \frac{9}{2} L^{-1}\left(\frac{1}{s-2}\right) + \frac{19}{} L^{-1}}$$

$$= \frac{2}{9} L^{-1}\left(\frac{1}{s-2}\right) + \frac{19}{6} L^{-1}\left(\frac{1}{s+1}\right) + \frac{7}{3} L^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

$$= \frac{2}{9} e^{2t} + \frac{19}{6} e^{-t} + \frac{7}{3} e^{-t} \cdot t$$

Ans

The End