Complex Variable, Laplace & Z- transformation

Lecture 01

Applications of Laplace Transformation

1. Laplace Transformation Is widely used by electronic engineers to solve quickly differential equations occurring in the analysis of electronic circuits.

2. We employ Laplace Transform to solve digital signal processing problem.

3. In order to get true form of radioactive decay Laplace Transform is used. It makes easy to study analytic part of nuclear physics.

4. Laplace Transform is used to simplify calculations in system modeling, where large number of differential equations are used.

5. Ordinary Differential equations can be easily solved by the Laplace Transform method without finding the general solution and the arbitrary constants.

Definition of Laplace Transformation

Let the function $f(t)$ be defined for all positive values of t, then multiply $f(t)$bye^{-st} and integrate it with respect to t from zero to infinity. If the resulting integral exists (i.e., has some finite value), it is a function of s,s may be real or complex, say $F(s)$.

$$F(s) = \int_0^\infty e^{-st} f(t) dt .$$

This function $F(s)$ of variable s is called Laplace Transformation of the original function $f(t)$ and will be denoted by $\mathcal{L}\{f(t)\}$, where $\mathcal{L}$ denotes the Laplace transform operator.

Proof of some selective formulae:

- 1. $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, (s > a)$

Proof: From the definition of Laplace transformation we know that,

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

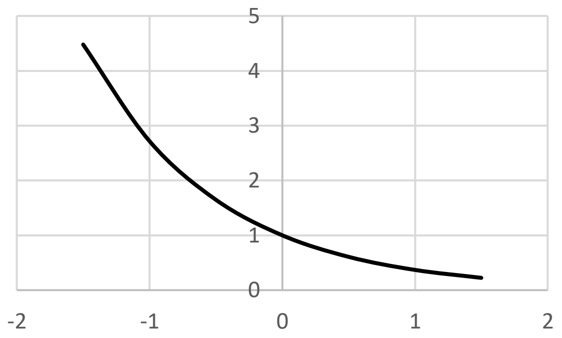
$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt$$

$$= \lim_{p\to\infty} \int_0^p e^{-(s-a)t} dt$$

$$= \lim_{p\to\infty} \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]\Bigg|_0^p; \qquad \left[\int e^{ax} dx = \frac{1}{a} e^{ax} + C\right]$$

$$= \lim_{p\to\infty} \frac{1 - e^{-(s-a)p}}{(s-a)}; \qquad \left[\lim_{x\to\infty} e^{-x} = 0\right]$$

$$= \frac{1}{s-a} \text{ if } s > a.$$



Proof of some selective formulae:

- 1. $\mathcal{L}\{t\} = \frac{1}{s^2}, (s > 0)$

Proof: From the definition of Laplace transformation we know that,

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st} t\, dt$$

$$= \lim_{p\to\infty} \int_0^p te^{-st} dt$$

$$= \lim_{p\to\infty} \left[\frac{-t}{s} e^{-st} - \frac{1}{s^2} e^{-st}\right]\Bigg|_0^p;$$

$$= \lim_{p\to\infty} \left[\frac{-p}{s} e^{-sp} - \frac{1}{s^2} e^{-sp} + \frac{1}{s^2}\right]$$

$= \frac{1}{s^2}$ if $s > 0$.

sign	u	v
+	t	e^{-st}
-	1	$\frac{-1}{s} e^{-st}$
+	0	$\frac{1}{s^2} e^{-st}$

Important Notation of Laplace Transformation

$$\mathcal{L}\{f(t)\} = F(s)$$

The original function $f(t)$ is called the inverse transform or inverse of $F(s)$ and will be denoted by

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

Important Formulae of Laplace Transformation

1. $\mathcal{L}\{c\} = \frac{c}{s}$, c is any constant, $(s > 0)$
2. $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, when $n = 0,1,2,3,..$
3. $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, $(s > a)$
4. $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$
5. $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$
6. $\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$
7. $\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$

Some workout Examples of Laplace Transformation using direct formula

$$2\cos^2 t = 1 + \cos 2t$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Example: 1

$$\mathcal{L}\{(t^2+1)^2\}$$
$$= \mathcal{L}\{t^4 + 2t^2 + 1\}$$
$$= \mathcal{L}\{t^4\} + 2\mathcal{L}\{t^2\} + \mathcal{L}\{1\}$$
$$= \frac{4!}{s^{4+1}} + 2\frac{2!}{s^{2+1}} + \frac{1}{s}$$
$$= \frac{24}{s^5} + \frac{4}{s^3} + \frac{1}{s}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\{\cosh at\} = \frac{s}{s^2 - a^2}$$

Example: 2

$$\mathcal{L}\{\mathrm{e}^{-3t} + 5\cosh t\}$$
$$= \mathcal{L}\{\mathrm{e}^{-3t}\} + 5\,\mathcal{L}\{\cosh\ t\}$$
$$= \frac{1}{\mathrm{s}-(-3)} + 5\frac{\mathrm{s}}{\mathrm{s}^2 - 1^2}$$
$$= \frac{1}{\mathrm{s}+3} + \frac{5\mathrm{s}}{\mathrm{s}^2 - 1}$$

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}$$

Example: 3

$$\mathcal{L}\{\cos^2 3t\}$$
$$= \mathcal{L}\left\{\frac{1}{2}(1+\cos 6t)\right\}$$
$$= \mathcal{L}\left\{\frac{1}{2}\right\} + \frac{1}{2}\,\mathcal{L}\{\cos 6t\}$$
$$= \frac{1}{2s} + \frac{1}{2}\left(\frac{s}{s^2+36}\right)$$

Sample Exercise Set on Laplace Transformation using direct formula

Find the Laplace Transforms and also sketch (if free hand sketching is getting complex then use MATLAB) the following functions (1-8):

1. $f(t) = 3t + 12$,
2. $f(t) = e^{5t}$,
3. $f(t) = e^{-2t}$,
4. $f(t) = (a - bt)^2$,
5. $f(t) = \cos \pi t$,
6. $f(t) = \cos^2 \omega t$,
7. $f(t) = \sin(\omega t + \theta)$,
8. $f(t) = 1.5 \sin\left(3t - \frac{\pi}{2}\right)$,

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

Learning Outcomes of Laplace Transformation

The original main use of Laplace transform is to solve initial value problems for linear ordinary and partial differential equations. They can reduce ordinary differential equations to algebraic equations, and partial differential equations to ordinary differential equations. The transformed equations are easier to solve, and then the solution in the Laplace domain is transformed back to the time domain

Sample MCQ

1. $\mathcal{L}\{\cos at\}=$

(a) $\frac{a}{s^2+a^2}$ (b) $\frac{a}{s^2-a^2}$ (c) $\frac{s}{s^2+a^2}$ (d) $\frac{s}{s^2-a^2}$

2. $\mathcal{L}\{t\} =$

(a) $\frac{1}{s^2}$ (b) $\frac{1}{s}$ (c) $\frac{1}{s+1}$ (d) $\frac{1}{s-1}$

3. $\mathcal{L}\{\sin(\omega t + \theta)\} =$

(a) $\frac{\omega \cos\theta + s\ \sin\theta}{\omega^2 + s^2}$ (b) $\frac{\omega \cos\theta + s\ \sin\theta}{\omega^2 + s^2}$ (c) $\frac{\omega \cos\theta + s\ \sin\theta}{\omega^2 + s^2}$ (d) $\frac{\omega \cos\theta + s\ \sin\theta}{\omega^2 + s^2}$