Complex Integration (Line Integral)

Exercise:6 (part-2)

Objective:

Finding the path of integration in different form and evaluating some line integrals for a simple path

Methodologies:

By using the path and reducing the integrands into a single valued function, line integrals will be evaluated

Problem: (i) Sketch and represent the Line segment parametrically from $-1+2i$ to $4-2i$.

Solution: Equation of the line from $-1+2i$ or $(-1,2)$ to $4-2i$ or $(4,-2)$ is:

$$\frac{y-y_1}{y_1-y_2}=\frac{x-x_1}{x_1-x_2} \Rightarrow \frac{y-2}{2+2}=\frac{x+1}{-1-4}$$

$$\Rightarrow \frac{y-2}{4}=\frac{x+1}{-5}$$

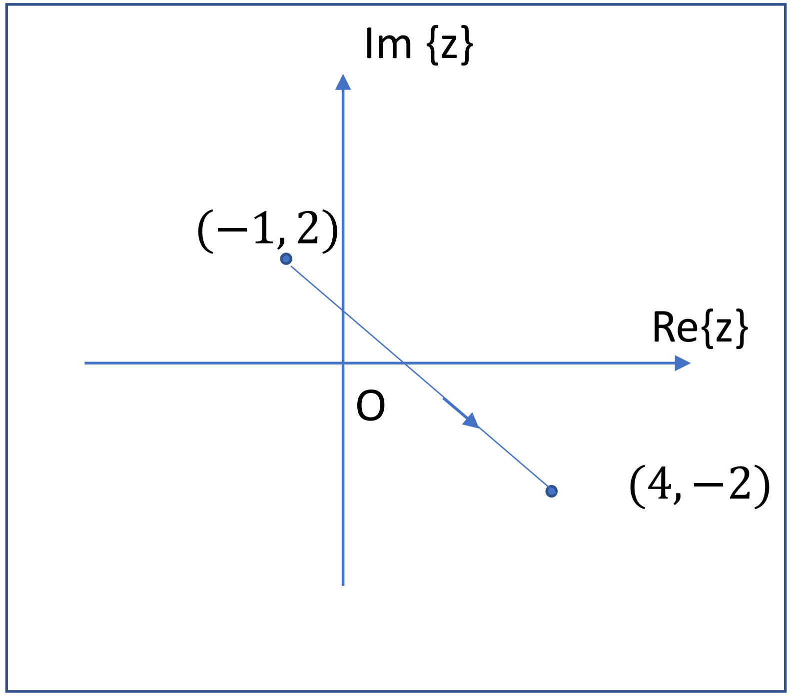
$$\Rightarrow y-2=-\frac{4}{5}(x+1)$$

$$\Rightarrow y=-\frac{4}{5}x-\frac{4}{5}+2$$

$$\Rightarrow y=-\frac{4}{5}x+\frac{6}{5}$$

Let $x=t$

$$\therefore y=-\frac{4}{5}t+\frac{6}{5}$$



So, $z=x+iy=t+i(-\frac{4}{5}t+\frac{6}{5})$ where $t=-1$ to $t=4$.

Which is a parametric representation of the given line segment.

Problem: (ii) Sketch and represent unit circle (counter-clockwise) parametrically.

Solution: Equation of unit circle (counterclockwise) is,

$|z| = 1$ (counterclockwise)

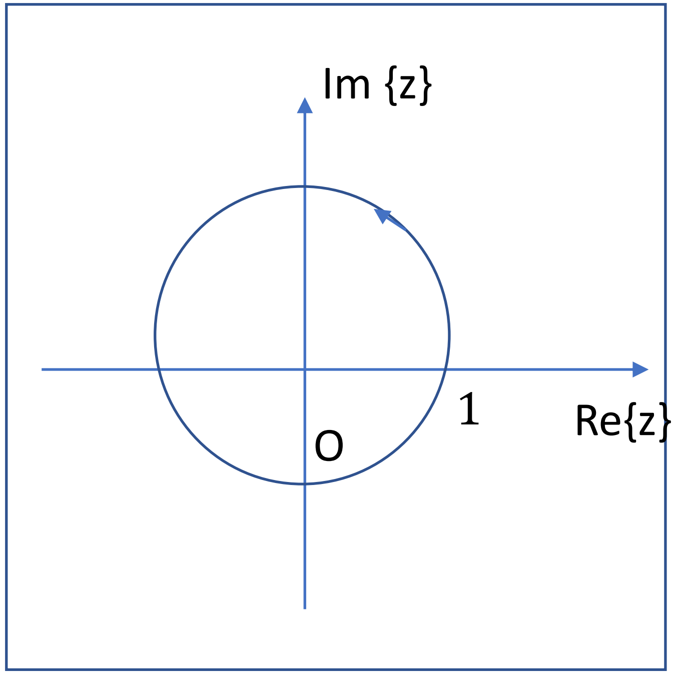
Or, $|x + i\,y| = 1 \Rightarrow \sqrt{x^2 + y^2} = 1 \Rightarrow x^2 + y^2 = 1$.

Let, $x = \cos\ t$ and $y = \sin\ t$,

So, $z = x + iy$

$\Rightarrow z = \cos t + i \sin t = e^{it}$, where $t = 0$ to $t = 2\pi$.

Which is a parametric representation of the unit circle.



Problem: (iii) Sketch and represent the curve $|z-5+i|=4$ (clock-wise) parametrically. Also identify where the point $(1,2)$ is interior, exterior or on the boundary of the curve.

Solution: Given equation of the curve is,

$|z-5+i|=4$ (clock-wise)

$\Rightarrow |x+i\,y-5+i|=4$

$\Rightarrow |(x-5)+i\,(y+1)|=4$

$\Rightarrow \sqrt{(x-5)^2+(y+1)^2}=4$

$\Rightarrow (x-5)^2+(y+1)^2=16$.

Let, $x-5=4\cos t \Rightarrow x=5+4\cos t$

and $y+1=4\sin t \Rightarrow y=-1+4\sin t$.

So, $z=x+iy$

$\Rightarrow z=5+4\cos t+i\,(-1+\sin t)$

$\Rightarrow z=5-i+4e^{it}$, where $t=2\pi$ to $t=0$.

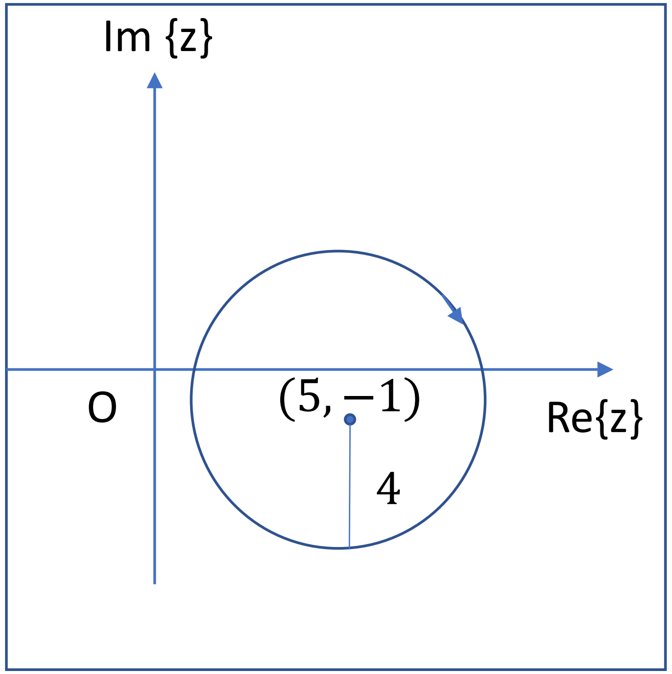
Which is a parametric representation of the given curve which is a circle of radius 4 and center at $(5,-1)$.

Or, $|z-5+i|=4$

$\Rightarrow z-5+i=4e^{it}$

$\Rightarrow z=5-i+4e^{it}$

where $t=2\pi$ to $t=0$.



Steps to evaluate line integral $\int_C f(z)dz$:

Step 1: Find the equation of the path C in cartesian or polar (for circular path) form

Step 2: Find the integrand $f(z)$ using equation of C

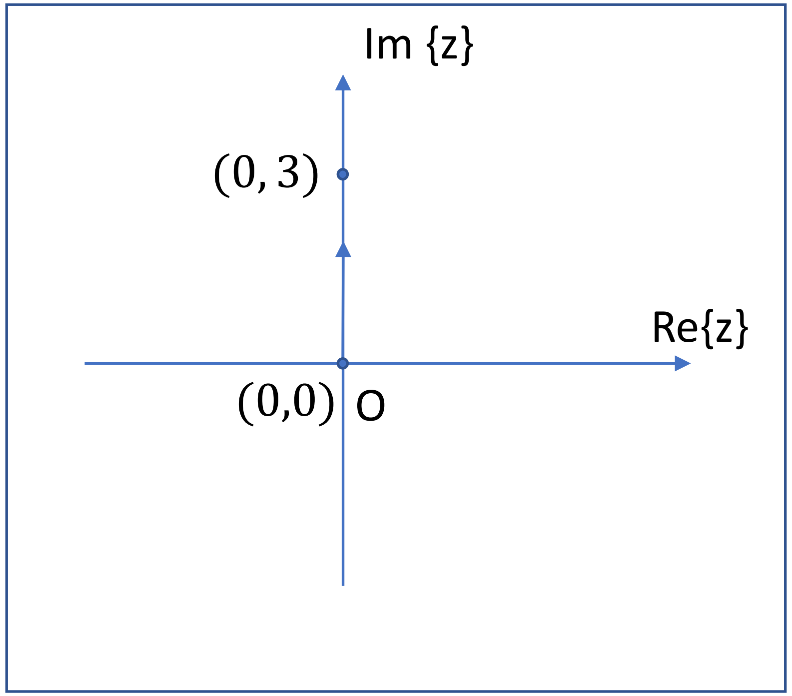
Step 3: Find dz from $z = x + iy$ or $z = re^{i\theta}$ (for circular path)

Step 4: Evaluate the line integral $\int_C f(z)dz$ using the information from step 1, step 2 and step 3.

Problem: (i) Sketch the path C, which is a line segment from $z = 0$ to $z = 3i$ and hence evaluate $\int_C \text{Im}\,\{z\}\, dz$.

Solution: Here C is the line segment from $z = 0$ or (0,0) to $z = 3i$ or (0,3).

$\therefore$Equation of the path C is: $x = 0$



Now $f(z) = \text{Im}\,\{z\} = \text{Im}\,\{x + iy\} = y$

And $z = x + iy = iy \; (\because x = 0)$

$$\therefore dz = i dy$$

So, $\int_C \text{Im}\,\{z\}\, dz = \int_0^3 y\, i dy = i \left[\frac{y^2}{2}\right]_0^3 = \frac{9}{2} i.$

Problem: (ii) Sketch the path C, which is a line segment from $z = 0$ to $z = 3$ and hence evaluate $\int_C \text{Re } z^2 \, dz$.

Solution: Here C is the line segment from $z = 0$ or (0,0) to $z = 3$ or (3,0).

$\therefore$Equation of the path C is: $y = 0$

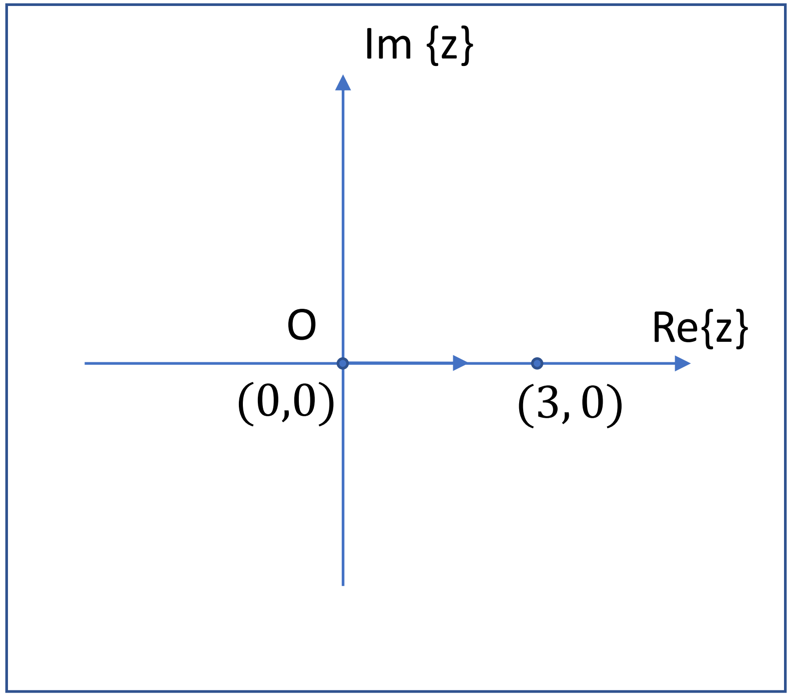
Now $f(z) = \text{Re}\,\{z^2\} = \text{Re}\,\{(x+iy)^2\}$

$$= \text{Re}\{x^2 + i2xy - y^2\} = x^2 - y^2 = x^2$$

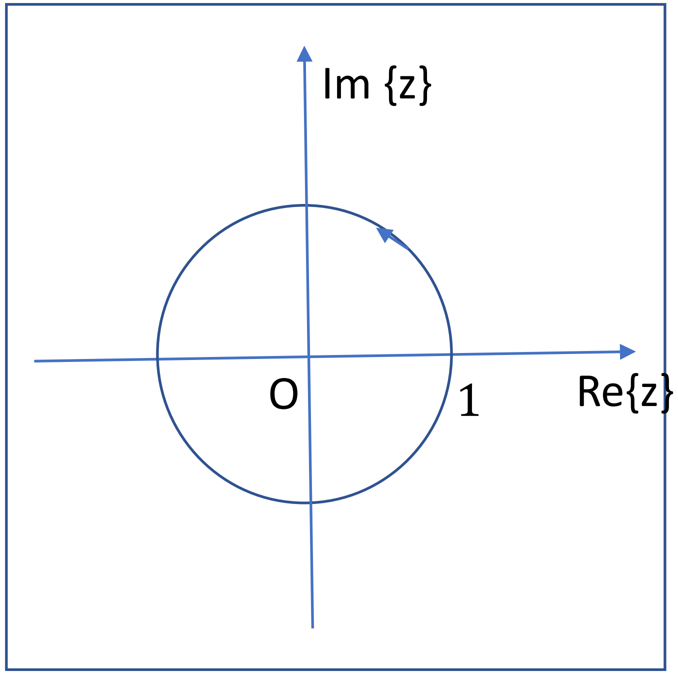
And $z = x + iy = x \;(\because y = 0)$

$\therefore dz = dx$

So, $\int_C \text{Re}\,\{z^2\} dz = \int_0^3 x^2 \, dx = \left[\frac{x^3}{3}\right]_0^3 = 9.$



Problem: (iii) Sketch the path C, which is the unit circle $|z| = 1$(counter clock-wise) and hence evaluate $\int_C (z + \bar{z})dz$



Solution: Equation of the path C is:$|z| = 1$ or, $z = e^{i\theta}$

Now $f(z) = (z + \bar{z}) = e^{i\theta} + e^{-i\theta}$

And $z = re^{i\theta} = e^{i\theta} (\because r = 1)$

$$\therefore dz = ie^{i\theta} d\theta$$

So,$\int_C (z + \bar{z})dz = \int_0^{2\pi} (e^{i\theta} + e^{-i\theta})\, ie^{i\theta} d\theta$

$$= i\int_0^{2\pi} (e^{2i\theta} + 1)\, d\theta = i\left[\frac{e^{2i\theta}}{2i} + \theta\right]_0^{2\pi} = i\left[\frac{e^{i4\pi}}{2i} + 2\pi - \frac{1}{2i}\right] = 2\pi i \;[\because e^{i4\pi} = 1]$$

Problem: (iv) Sketch the path C, which is shortest path from $z = 2$ to $z = 2 + i$ and hence evaluate $\int_C e^z dz$.

Solution: Here C is the line segment from $z = 2$ or (2,0) to $z = 2 + i$ or (2,1).

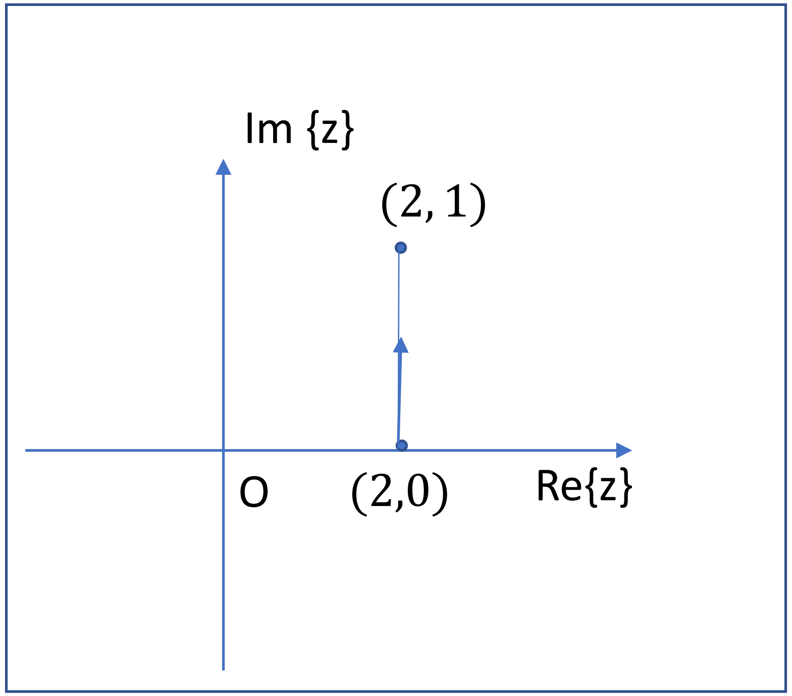
$\therefore$ Equation of the path C is: $x = 2$

Now $f(z) = e^z = e^{(x+iy)} = e^{(2+iy)}$

And $z = x + iy = 2 + iy \; (\because x = 2)$

$$\therefore dz = idy$$

So, $\int_C e^z dz = \int_0^1 e^{(2+iy)}\, idy = i\left[\frac{e^{(2+iy)}}{i}\right]_0^1 = e^{2+i} - e^2.$



Exercises:

- Sketch the path C from $z = i$ to $z = i + 2$ and hence evaluate $\int_C \text{Im}\,\{z^2\}\, dz$.
- Sketch the path C, which is shortest path from $z = i$ to $z = 1 + i$ and hence evaluate $\int_C \sin z\, dz$.
- Sketch the path C, which is shortest path from $z = 0$ to $z = 3i$ and hence evaluate $\int_C e^{2z} dz$.
- Sketch the path C from $z = 0$ to $z = 2i$ and hence evaluate $\int_C z^2\, dz$.
- Sketch the path C, which is the circle $|z| = 2$ and hence evaluate $\int_C 2\bar{z}\, dz$.

Sample MCQ:

❖Parametric representation of line segment from $z = 1 + i$ to $z = 4 - 2i$ is:

(a) $x = t, y = t + 2$ (b) $x = -t, y = -t + 2$ (c) $x = t, y = -t + 2$.

❖If $|z| = 2$, then which of the following is true?

(a) radius of circle 2^2 (b) $z = 2e^{i\theta}$ (c) $z^{-1} = 2\,e^{-i\theta}$.

❖Parametric representation of $|z| = 3$ is:

(a) $z = 3e^{it}$

(b) $z = e^{it}$

(c) $z = 3$.

❖Evaluate $\int_c \bar{z}\,dz$; where C is the line segment from $z = 0$ to $z = 3$.

(a) $\frac{9}{2}$ (b) $\frac{5}{2}$ (c) $\frac{7}{2}$.

❖Evaluate $\int_c z^2 dz$; where C is the line segment from $z = 0$ to $z = 1$.

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) 1 .