Lecture 03

Complex Variable, Laplace & Z-transformation

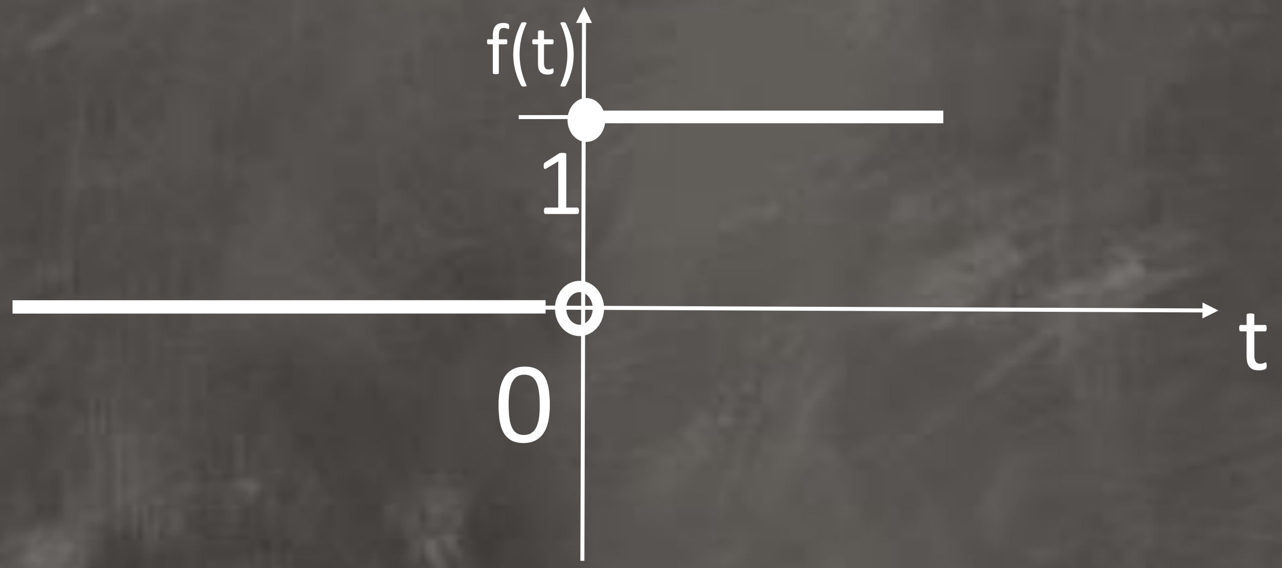
This Lecture Covers -

1. Definition of Unit Step Function.
2. Rectangular Pulse.
4. Laplace Transformation of Unit Step Function.
5. Examples & Exercises on Laplace Transformation of Unit Step Function.

Definition of Unit Step Function

The Unit Step or Heaviside's function is defined as follows:

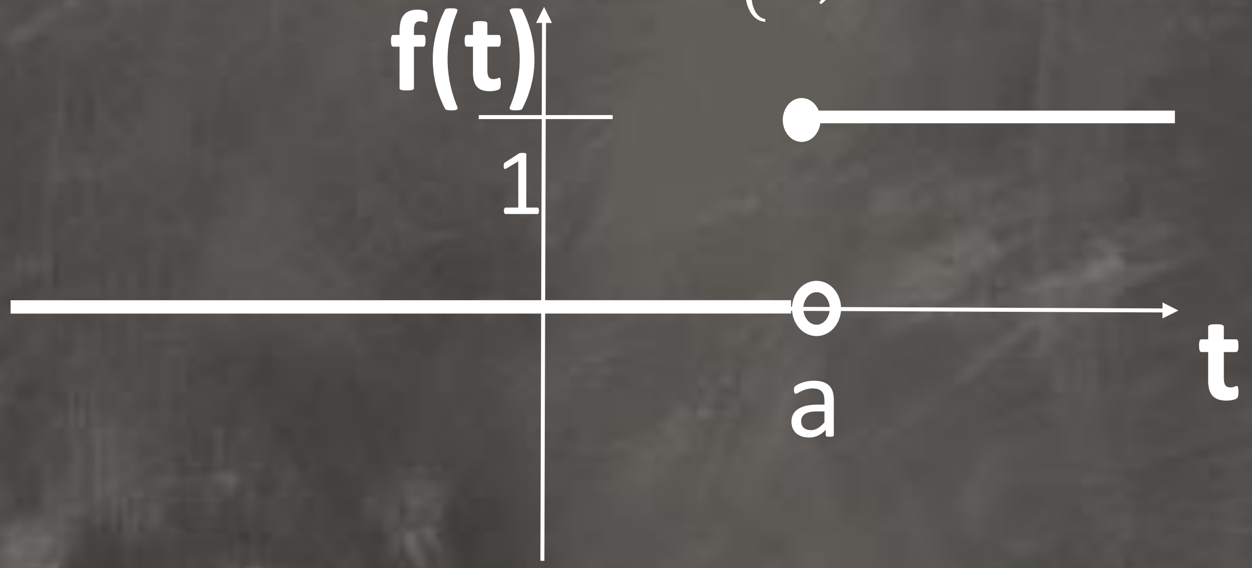
$$f(t) = u(t) = \begin{cases} 0\,; & t < 0 \\ 1\,; & t \geq 0 \end{cases}$$



Shifted Unit Step Function

The Unit Step or Heaviside's function is defined as follows:

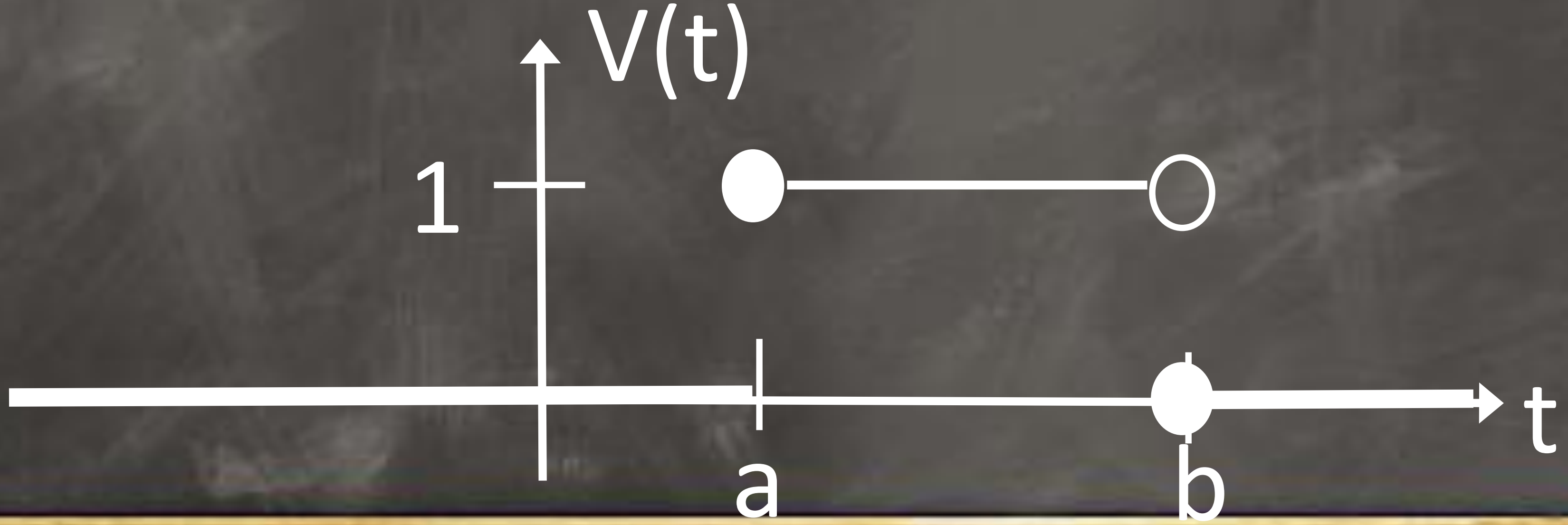
$$u_a(t) = u(t-a) = \begin{cases} 0\,; & t < a \\ 1\,; & t \geq a \end{cases}$$



Rectangular Pulse

A common situation in a circuit is for a voltage v(t), to be applied at a particular time (say $t = a$) and removed later at $t = b$ (say). We write such a situation using unit step function as:

$$v(t) = u(t-a) - u(t-b) = \begin{cases} 1\,; a \le t < b \\ 0\,; \text{otherwise} \end{cases}$$



Laplace Transformation of Unit Step Function and Examples

Formulae:

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

$$\mathcal{L}\{f(t)\, u(t-a)\} = e^{-as}\, \mathcal{L}\{f(t+a)\}$$

Example 1:

$$\mathcal{L}\{t^2\, u(t-3)\}$$
$$= e^{-3s}\, \mathcal{L}\{f(t+3)\}$$
$$= e^{-3s}\, \mathcal{L}\{(t+3)^2\}$$
$$= e^{-3s}\, \mathcal{L}\{t^2+6t+9\}$$
$$= e^{-3s}\, [\mathcal{L}\{t^2\} + 6\,\mathcal{L}\{t\} + \mathcal{L}\{9\}]$$
$$= e^{-3s}\, \left[\frac{2!}{s^3} + 6\frac{1}{s^2} + 9\frac{1}{s}\right].$$

Ans.

Laplace Transformation of Unit Step Function and Examples

Example 1:

$$\mathcal{L}\{\sin t \ u(t)\}$$
$$= e^{0\times s}\,\mathcal{L}\{f(t)\}$$
$$= \ \mathcal{L}\{\sin t\}$$
$$= \ \frac{1}{s^2+1}$$

Ans.

Example 2:

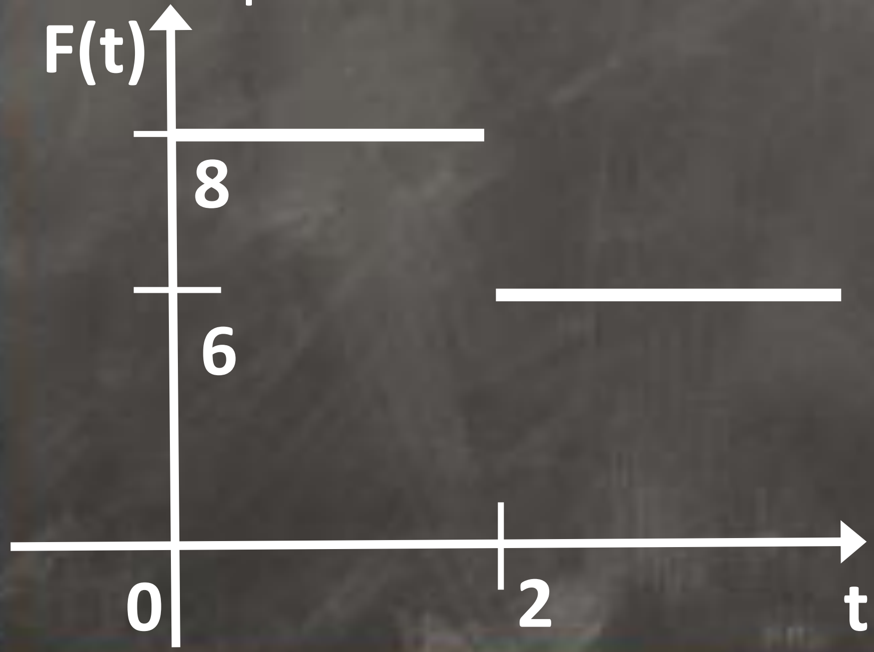
$$\mathcal{L}\{e^{-2t}\,u_\pi(t)\}$$
$$= \mathcal{L}\{e^{-2t}\,u(t-\pi)\}$$
$$= e^{-\pi s}\,\mathcal{L}\{f(t+\pi)\}$$
$$= e^{-\pi s}\,\mathcal{L}\left\{e^{-2(t+\pi)}\right\}$$
$$= e^{-\pi s}\,[\,\mathcal{L}\{e^{-2t}\,e^{-2\pi}\}]$$
$$= e^{-\pi s}\ \ e^{-2\pi}\,\mathcal{L}\{e^{-2t}\}$$
$$= e^{-\pi(s+2)}\,\frac{1}{s+2}.$$

Ans.

Laplace Transformation of Unit Step Function and Examples

Example 3. Given $f(t) = \begin{cases} 8\,; 0 < t < 2 \\ 6\,; \quad\quad t > 2 \end{cases}$, Sketch the function $f(t)$, also express $f(t)$ in terms of

Unit step function and hence find it's Laplace transformation.



$$\begin{aligned} f(t) &= 8[u(t) - u(t-2)] + 6\,u(t-2) \\ &= 8\,u(t) - 8\,u(t-2) + 6\,u(t-2) \\ &= 8\,u(t) \; - 2\,u(t-2) \end{aligned}$$

$$\begin{aligned} F(s) &= 8\,\mathcal{L}\{u(t)\} - 2\,\mathcal{L}\{u(t-2)\} \\ &= 8\frac{e^{0\times s}}{s} - 2\frac{e^{-2s}}{s} \\ &= 8\frac{1}{s} - 2\frac{e^{-2s}}{s}. \end{aligned}$$

Exercise Set on Laplace Transformation of Unit Step function

Sketch the following function and find their Laplace Transformations:

1. $f(t) = t\,u(t-1)$,

2. $f(t) = (t-1)\,u(t-3)$,

3. $f(t) = (t+2)^2\,u(t-1)$,

4. $f(t) = e^{-2t}\,u(t-3)$,

5. $f(t) = 4\cos t\;u(t-\pi)$.

Sketch the following function, also express $f(t)$ in terms of unit step function and find it's Laplace Transformation:

6. $f(t) = \begin{cases} t\,; 0 < t < 1 \\ 2\,; \quad t > 1 \end{cases}$

7. $f(t) = \begin{cases} t^2\,;\; 0 \le t < 1 \\ t-3\,; t \ge 1 \end{cases}$

Learning Outcomes:

In engineering applications, we frequently encounter functions whose values change abruptly at specified values of time *t*. One common example is when a voltage is switched on or off in an electrical circuit at a specified value of time *t.* The switching process can be described mathematically by the function called the **Unit Step Function.**

In this lecture we overviewed the general concept of unit step function and also discussed the process of Laplace Transformation of unit step function.

Sample MCQ

1\. If $f(t) = \begin{cases} 1-t & ;0<t<1 \\ 0\ ; & t>1 \end{cases}$ then what is $F(s)$?

(a) $\frac{2s+e^{-s}-1}{s^2}$ (b) $\frac{s+e^{-s}-1}{s^2}$ (c) $\frac{s-e^{-s}-1}{s^2}$ (d) $\frac{s^2+e^{-s}-1}{s^2}$

2\. If $V(t) = \begin{cases} 0, & t<3 \\ 2t+8, & 3<t<5 \\ 0, & t>5 \end{cases}$. then which of the following is corresponding unit step function?

(a) $(2t+8)\cdot[u(t-3)-u(t-5)]$ (b) $[u(t-3)-u(t-5)]$

(c) $[u(t+3)-u(t-5)]$ (d) $[u(t+3)-u(t+5)]$

3 . $\mathcal{L}\{t^2\, u(t-3)\} =?$

(a) $e^{-3s}\left[\frac{2}{s^3}+\frac{1}{s^2}+9\frac{1}{s}\right]$ (b) $e^{-3s}\left[\frac{2}{s^3}+6\frac{1}{s^2}+9\frac{1}{s}\right]$ (c) $e^{-3s}\left[\frac{2}{s^3}+6\frac{1}{s^2}+\frac{1}{s}\right]$ (d) $e^{-3s}\left[\frac{1}{s^3}+6\frac{1}{s^2}+9\frac{1}{s}\right]$