Math Sample Question

1. (a) Sketch the path and its orientation given by $z(t) = 2 - 4e^{it}$ $(0 \leq t \leq 2\pi)$

= We know,

$$z = x + iy$$

$$\Rightarrow x + iy = 2 - 4e^{it}$$

$$\Rightarrow x + iy = 2 - 4 \cdot (\cos t + i \sin t)$$

$$\Rightarrow x + iy = 2 - 4\cos t - 4i\sin t$$

$$x = 2 - 4\cos t \qquad y = -4\sin t$$

$$\Rightarrow \frac{x-2}{-4} = \cos t \qquad \Rightarrow \frac{y}{-4} = \sin t$$

We know,

$$\cos^2 t + \sin^2 t = 1$$

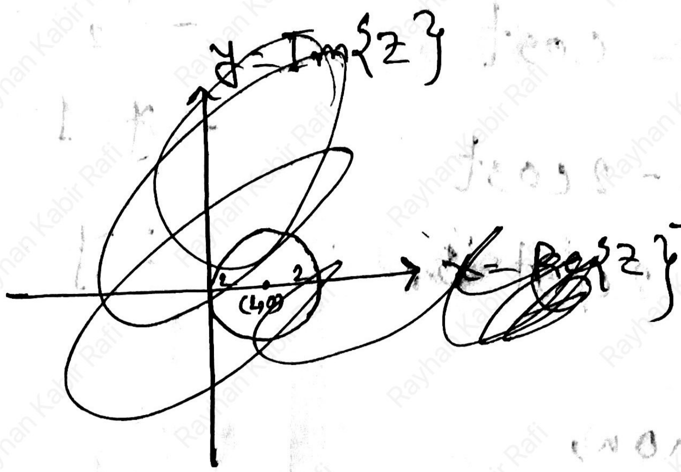
$$\Rightarrow \left(\frac{x-2}{-4}\right)^2 + \left(\frac{y}{-4}\right)^2 = 1$$

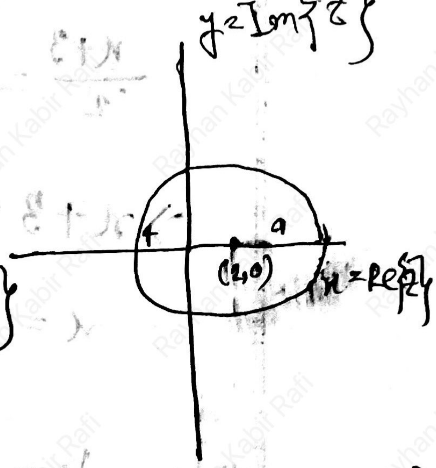
$$\Rightarrow \frac{(x-2)^2}{(-4)^2} + \frac{y^2}{(-4)^2} = 1$$

$$\Rightarrow (x-2)^2 + y^2 = (-4)^2$$

$$\therefore (x-2)^2 + y^2 = 4^2$$

centre $(2,0)$ and radius 2. $0 \le t \le 2\pi$





$z(t)$ represents a circle with centre $(2,0)$ and radius 2.

(b) Sketch and represents $|z+3-i| = 2$ Parametrically, counter clockwise.

$$|z+3-i| = 2$$

$$\Rightarrow |x+iy+3-i| = 2$$

$$\Rightarrow |(x+3)+i(y-1)| = 2$$

$$\Rightarrow \sqrt{(x+3)^2+(y-1)^2} = 2$$

$$\Rightarrow (x+3)^2+(y-1)^2 = 2^2$$

$$\therefore \frac{(x+3)^2}{2^2} + \frac{(y-1)^2}{2^2} = 1$$

Let,

$$\frac{x+3}{2} = \cos t \qquad \frac{y-1}{2} = \sin t$$

$$\Rightarrow x+3 = 2\cos t \qquad \Rightarrow y-1 = 2\sin t$$

$$\therefore x = 2\cos t - 3 \qquad \therefore y = 2\sin t + 1$$

We know,

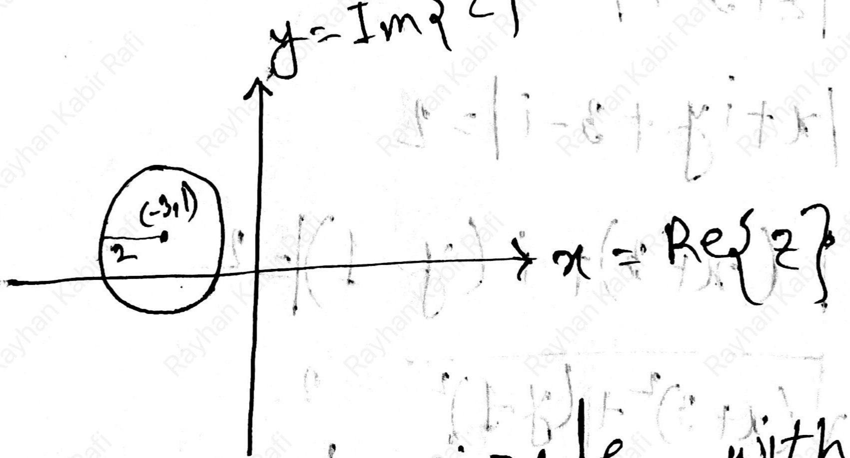
$$z = x+iy$$

$$= 2\cos t - 3 + i\,(2\sin t + 1)$$

$$= 2\cos t - 3 + 2i\sin t + i$$

$$= (i-3) + 2(\cos t + i\sin t)$$

$$= i - 3 + 2e^{it}$$



This represents a circle with centre (-3,1) and radius 2. $0 \leq t \leq 2\pi$

c) Test whether the point $(1,-1)$ is interior or exterior or boundary of $|z-2+3i|=1$

$$= \quad |z-2+3i|=1$$

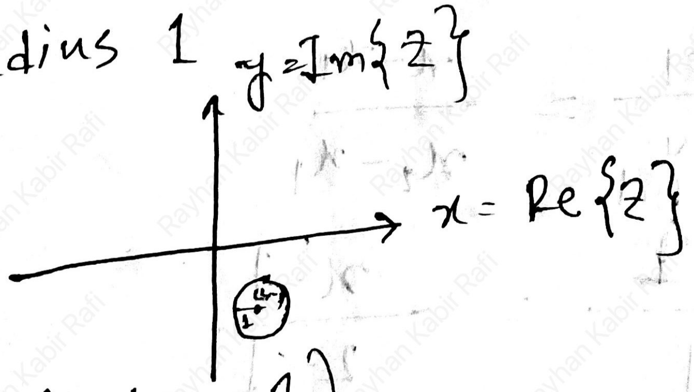
$$\Rightarrow |x+iy-2+3i|=1$$

$$\Rightarrow |(x-2)+i(y+3)|=1$$

$$\Rightarrow \sqrt{(x-2)^2+(y+3)^2}=1$$

$$\Rightarrow (x-2)^2+(y+3)^2=1^2$$

This represents a circle with centre $(2,-3)$ and radius 1



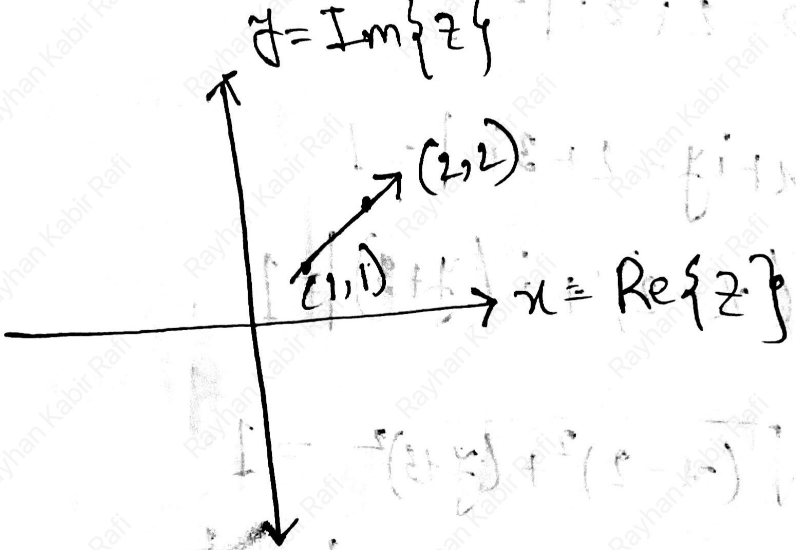
Now, For Point $(1,-1)$,

$$(1-2)^2+(-1+3)^2=1$$

$$\Rightarrow 5>1$$

$\therefore$ The point will be exterior.

d) Find the equation of path C which is the line segment from $z = 1+i$ to $z = 2+2i$.



The equation of straight line passing through (1,1) and (2,2) is,

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\Rightarrow \frac{y-1}{2-1} = \frac{x-1}{2-1}$$

$$\Rightarrow y-1 = x-1$$

$$\Rightarrow y = x-1+1$$

$$\therefore y = x$$

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Let $x = t$ $\therefore y = t$

t varies from 1 to 2

$\therefore z(t) = t + it$

$= t(1+i)$ $1 \le t \le 2$

e) Find the singular points and corresponding order of $f(z) = \dfrac{1-z}{(z-2)^2+(z+3)^3}$

$\Rightarrow$ For singular points,

$(z-2)^2 = 0$ $(z+3)^2 = 0$

$\Rightarrow z = 2$ $\Rightarrow z = -3$

$z = 2$ is a singular points of order 2

$z = -3$ is a singular points of order 3

$$\text{Res}(z=a) = \lim_{z\to a} (z-a) f(z)$$

f) Find the residue at the singular point of $f(z) = \frac{z}{(z+2)}$ for $|z| = 3$.

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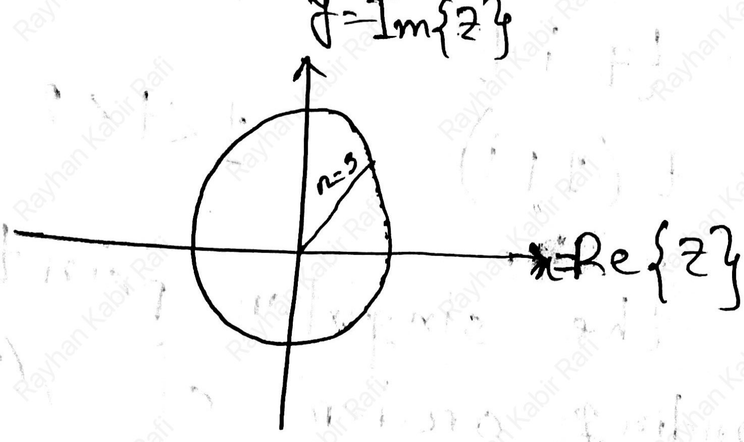

For singular points,

$z+2=0$

$\Rightarrow z=-2$ is the singular point of order 1 which is inside the circle $|z|=3$

$$\text{Res}(z=-2) = \lim_{z\to -2} (z+2)\frac{z}{(z+2)}$$

$$= \lim_{z\to -2} z$$

$$= -2 \text{ (Ans.)}$$

g) Evaluate $Z\{(4)^n u[n]\}$, $u[n]$ is the discrete time unit step function.

$= Z\{4^n u[n]\}$

$= \dfrac{1}{1-4z^{-1}}$ $\quad \left[Z\{a^n u[n]\} = \dfrac{1}{1-az^{-1}}\right]$

h) Evaluate $Z\{3\delta[n-2]\}$, $\delta[n]$ is the Kronecker delta function.

$= Z\{3\delta[n-2]\}$

$= 3z^{-2}$

i) Evaluate $Z\{((-2)^n+4)u[n]\}$. Also find ROC (Region of convergence).

$= Z\{((-2)^n+4)u[n]\}$ $\quad Z\{a^n u[n]\}$

$= Z\{(-2)^n u[n]\} + Z\{4u[n]\}$ $\quad = \dfrac{1}{1-az^{-1}}$

$= \dfrac{1}{1+2}$

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$$= \frac{1}{1+2z^{-1}} + \frac{4}{1-z^{-1}}$$

$$= \cancel{(1+2z^{-1})^{-1}}$$

$$= \frac{1}{1+\frac{2}{z}} + \frac{4}{1-\frac{1}{z}}$$

(j) Evaluate $Z^{-1}\left\{2z^{-3}\right\}$

$= Z^{-1}\left\{2z^{-3}\right\}$

$= 2\,\delta[n-3]$

k) Evaluate $Z^{-1}\left\{\frac{3}{1+2z^{-1}}\right\}, |z|>2$

$= Z^{-1}\left\{\frac{3}{1+2z^{-1}}\right\}$

$= 3\,(-2)^n u[n]$

l) Evaluate $Z^{-1}\left\{\frac{2z^{-1}}{1+z^{-1}}\right\} \quad |z|>1$

$\frac{2}{1} =$

$\frac{2(1+z^{-1})-2}{1+z^{-1}}$

$$\begin{array}{r|l} & 2 \\ 1+z^{-1} & 2z^{-1} \\ & 2+2z^{-1} \\ & (-)\ (-) \\ \hline & -2 \end{array}$$

$\frac{3z^{-1}}{1+2z^{-1}}$

$\frac{\frac{3}{2}(1+2z^{-1})-\frac{3}{2}}{1+2z^{-1}}$

$Z^{-1}\left\{2-\frac{2}{1+z^{-1}}\right\}$

$= 2\,\delta[n] - 2\,(-1)^n u[n]$

m) Evaluate $Z^{-1}\left\{\frac{z^{-1}}{(1-3z^{-1})^2}\right\}$ $|z|>3$

$$= Z^{-1}\left\{\frac{z^{-1}}{(1-3z^{-1})^2}\right\}$$

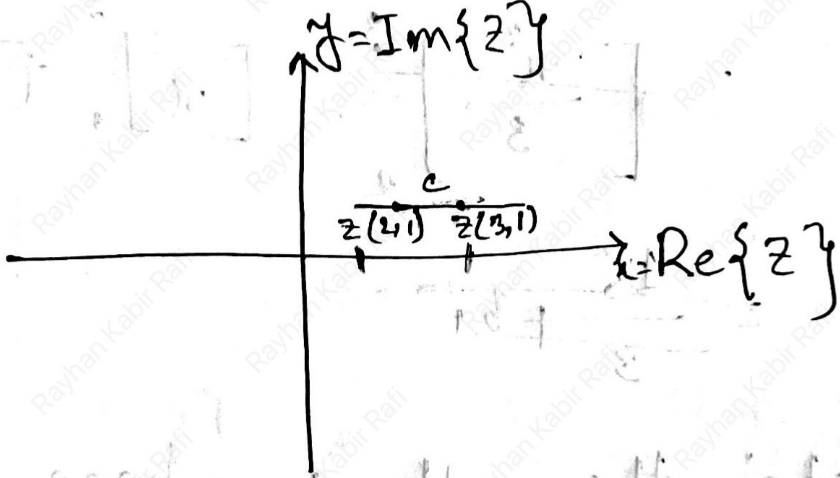
$$= Z^{-1}\left\{\frac{\frac{1}{3}\times 3z^{-1}}{(1-3z^{-1})^2}\right\}$$

$$= \frac{1}{3}n(3)^n u[n]$$

$$\frac{az^{-1}}{(1-az^{-1})^2} = na^n u[n]$$

Ans. to the ques. no. 2

a) Sketch the path C, where C consists of line segment from $z = 2+i$ to $z = i+3$ and hence evaluate $\int_C \mathrm{Re}\{z^2\}\, dz$



Along C, $y = 1$, x varies from 2 to 3

$$z = x + iy$$
$$= x + i$$
$$dz = dx$$

$$f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$$
$$= (x^2 - 1) + 2ix$$

$$\int_C \mathrm{Re}\{z^2\}\, dz = \int_2^3 (x^2 - 1)\, dx$$
$$= \left[\frac{x^3}{3}\right]_2^3 - \left[x\right]_2^3$$
$$= \frac{16}{3} \text{ (Ans.)}$$

b) Sketch the path C, where C is the circle $|z-2i|=3$ and evaluate $\int_C \left[\frac{1}{z-2i} - \frac{2}{(z-2i)^2}\right] dz$

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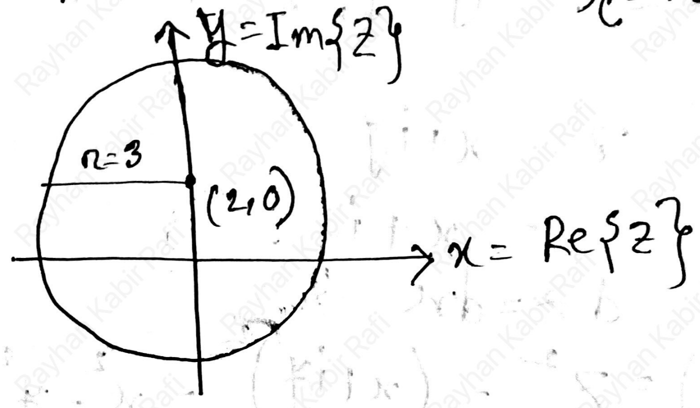


$|z-2i| = 3$

$\Rightarrow z - 2i = 3e^{it}$ $t \to 0$ to 2π (If clockwise)

$\Rightarrow \frac{d}{dt}(z-2i) = \frac{d}{dt}(3e^{it})$ $t \to 2\pi$ to 0 (If anti-clockwise

$$\Rightarrow \frac{dz}{dt} = 3ie^{it}$$

$$\therefore dz = 3ie^{it}\,dt$$

Now, $I = \int_c \left[\frac{1}{z-2i} - \frac{2}{(z-2i)^2}\right] dz$

$$= \int_0^{2\pi} \left[\frac{1}{3e^{it}} - \frac{2}{(3e^{it})^2}\right] 3ie^{it}\,dt$$

$$= i\int_0^{2\pi}\left(1 - \frac{2}{3}e^{-it}\right)dt$$

$$= i\left[t - \frac{2}{3}\frac{e^{-it}}{-i}\right]_0^{2\pi}$$

$$= i\left[(2\pi - 0) + \frac{2}{3i}\left(e^{-i2\pi} - 1\right)\right]$$

$$= i\left[2\pi + \frac{2}{3i}(1-1)\right]$$

$$= 2\pi i \text{ (Ans.)}$$

Optional

If anti clockwise $= i\left[t - \frac{2}{3}\frac{e^{-it}}{-i}\right]_{2\pi}^{0}$

$$= i\left[(0-2\pi) + \frac{2}{3i}\left(1 - e^{-i2\pi}\right)\right]$$

$$= i\left[-2\pi + \frac{2}{3i}(1-1)\right]$$

$$= -2\pi i \text{ (Ans.)}$$

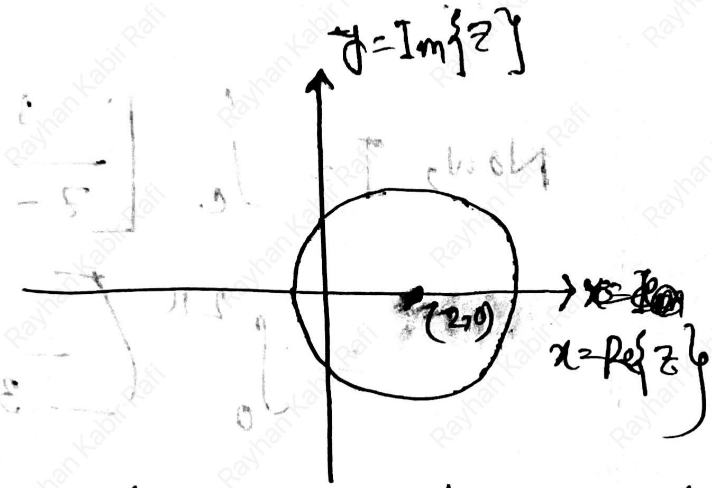
c) Evaluate $\oint \frac{z-1}{z^2-4} dz$, $c: |z-2|=3$, using the Cauchy residue theorem.

= For singular point,

$z^2-4=0$

$\Rightarrow z^2=4$

$\therefore z=\pm 2$



$z=2$ ~~and~~ is the singular points of order 1 which is inside the circle $c: |z-2|=3$

$z=-2$ is the singular points of order 1 which is outside the circle $c: |z-2|=3$

$$\operatorname{Res}(z=2) = \lim_{z \to 2} (z-2) \frac{z-1}{(z+2)(z-2)}$$

$$= \lim_{z \to 2} \frac{z-1}{z+2}$$

$$= \frac{2-1}{2+2}$$

$$= \frac{1}{4}$$

Using Cauchy residue theorem,

$$I = \oint \frac{z-1}{z^2-4}\,dz$$

$$= 2\pi i \times \frac{1}{4}$$

$$= \frac{\pi i}{2} \text{ (Ans.)}$$

d) Expand $f(z) = \dfrac{z+5}{(z-3)(2+z)}$ in a Laurent series valid for $2 < |z| < 3$.

⇒ We have,

$$\frac{z+5}{(z-3)(2+z)} = \frac{A}{z-3} + \frac{B}{2+z}$$

$$\Rightarrow z+5 = A(2+z) + B(z-3)$$

If $z=3$,

$8 = A(2+3)$

$\Rightarrow 5A = 8$

$\therefore A = \frac{8}{5}$

If $z=-2$,

$3 = B(-2-3)$

$\Rightarrow -5B = 3$

$\therefore B = -\frac{3}{5}$

$$\therefore \frac{z+5}{(z-3)(2+z)} = \frac{\frac{8}{5}}{z-3} + \frac{-\frac{3}{5}}{2+z}$$

$2 < |z| < 3$

$2 < |z|$

$\Rightarrow |z| > 2$

$\Rightarrow \frac{|z|}{2} > 1$

$\Rightarrow \frac{2}{|z|} < 1$

and, $|z| < 3$

$\Rightarrow \frac{|z|}{3} < 1$

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$$f(z) = \frac{\frac{8}{5}}{z-3} + \frac{-\frac{3}{5}}{2+z}$$

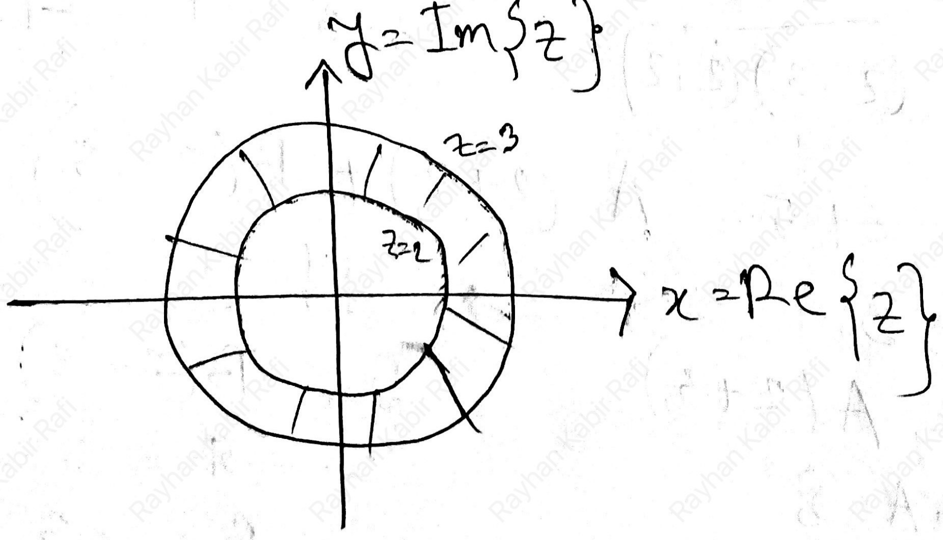
$$= \frac{8}{5} \times \frac{1}{z-3} - \frac{3}{5} \times \frac{1}{2+z}$$

$$= \frac{8}{5} \times \frac{1}{-3\left(1-\frac{z}{3}\right)} - \frac{3}{5} \times \frac{1}{z\left(1+\frac{2}{z}\right)}$$

$$= \frac{8}{5} \times -\frac{1}{3} \times \frac{1}{\left(1-\frac{z}{3}\right)} - \frac{3}{5z} \times \frac{1}{1+\frac{2}{z}}$$

$$= -\frac{8}{15}\left(1-\frac{z}{3}\right)^{-1} - \frac{3}{5z}\left(1+\frac{2}{z}\right)^{-1}$$

$$= -\frac{8}{15}\left(1+\frac{z}{3}+\frac{z^2}{9}+\cdots\right) - \frac{3}{5z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\cdots\right)$$



Ans. to the ques. no. 3

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3. a) Determine the sequence $x[n]$ by using inverse Z-Transform of $X(z) = (1+4z)(1-4z^{-1})(2+3z^{-2})$. Also sketch the sequence $x[n]$

$$\Rightarrow \quad X(z) = (1+4z)(1-4z^{-1})(2+3z^{-2})$$

$$= (1+4z)(2+3z^{-2} - 8z^{-1} - 12z^{-3})$$

$$= 2 + 3z^{-2} - 8z^{-1} - 12z^{-3} + 8z + 12z^{-1} - \cancel{32} - 48z^{-2}$$

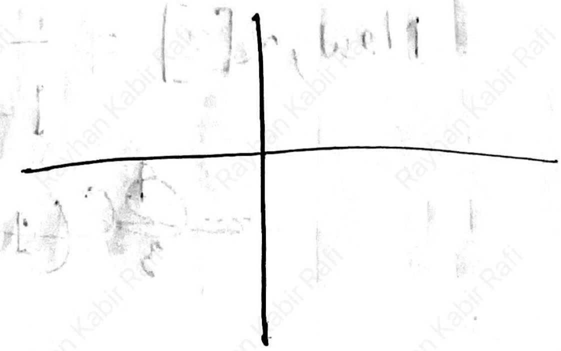
$$\cancel{-30 + 4z^{-1}}$$

$$= -30 + 8z + 4z^{-1} - 45z^{-2} - 12z^{-3}$$

Taking Inverse Z-Transformation,

$$x(n) = -30\,u[n] + 8u[n+1] + 4u[n-1] - 45u[n-2] - 12u[n-3]$$

$$x(n) = \begin{cases} -30 & n=0 \\ 8 & n=-1 \\ 4 & n=1 \\ -45 & n=2 \\ -12 & n=3 \end{cases}$$



Sketch

(b) Determine inverse Z-Transform of

$$x[z] = \frac{z^{-1}}{(1-0.25z^{-1})(1+0.5z^{-1})} \; ; \; |z|>0.5$$

$$= x[z] = \frac{z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)}$$

$$= \frac{A}{1-\frac{1}{4}z^{-1}} + \frac{B}{1+\frac{1}{2}z^{-1}}$$

$$A = \left[\frac{z^{-1}}{1+\frac{1}{2}z^{-1}}\right]_{z^{-1}=4} = \frac{4}{3}$$

$$B = \left[\frac{z^{-1}}{1-\frac{1}{4}z^{-1}}\right]_{z^{-1}=-2} = -\frac{4}{3}$$

$$\text{Now, } x[z] = \frac{\frac{4}{3}}{1-\frac{1}{4}z^{-1}} \cancel{B} - \frac{\frac{4}{3}}{1+\frac{1}{2}z^{-1}}$$

Taking inverse Z- Transformation,

$$\frac{4}{3}\left(\frac{1}{4}\right)^n u[n] - \frac{4}{3}\left(-\frac{1}{2}\right)^n u[n]$$

ROC:

$R_1 : |z| > \frac{1}{4}$

$R_2 : |z| > \frac{1}{2}$

$\therefore R : |z| > \frac{1}{2}$

(c) Use the Z-Transform to determine $y[n]$; $n \geq 0$ for

$$y[n] + 2y[n-1] - 15y[n-2] = 0 \; ; \; y[-1] = 1, \; y[-2] = 1$$

We have,

$$y[n] + 2y[n-1] - 15y[n-2] = 0$$

Taking Z-Transformation,

$$Y(z) + 2\left[z^{-1}\{Y(z) + y[-1]z\}\right] - 15z^{-2}\left[Y(z) + y[-1]z + y[-2]z^{2}\right] = 0$$

$$\Rightarrow Y(z) + 2z^{-1}\left[Y(z) + z\right] - 15z^{-2}\left[Y(z) + z + z^{2}\right] = 0$$

$$\Rightarrow Y(z) + 2z^{-1}Y(z) + 2 - 15z^{-2}Y(z) - 15z^{-1} - 15 = 0$$

$$\Rightarrow Y(z)\left(1 + 2z^{-1} - 15z^{-2}\right) - 15z^{-1} - 13 = 0$$

$$\therefore Y(z) = \frac{15z^{-1}+13}{1+2z^{-1}-15z^{-2}}$$

$$= \frac{15z^{-1}+13}{1+5z^{-1}-3z^{-1}-15z^{-2}}$$

$$= \frac{15z^{-1}+13}{1(1+5z^{-1})-3z^{-1}(1+5z^{-1})}$$

$$= \frac{15z^{-1}+13}{(1+5z^{-1})(1-3z^{-1})}$$

$$= \frac{A}{1+5z^{-1}} + \frac{B}{1-3z^{-1}}$$

$$A = \left[\frac{15z^{-1}+13}{1-3z^{-1}}\right]_{z^{-1}=-\frac{1}{5}} = \frac{25}{4}$$

$$B = \left[\frac{15z^{-1}+13}{1+5z^{-1}}\right]_{z^{-1}=\frac{1}{3}} = \frac{27}{4}$$

Now taking inverse Z-transformation.

$$y(n) = \frac{25}{4}(-5)^n u[n] + \frac{27}{4}(3)^n u[n]$$

ROC: $R_1 : |z| > 5$

$R_2 : |z| > 3$ $\quad \therefore R : |z| > 5$

(d) Consider an LTI system with input $x[n]$ and output $y[n]$ that satisfies the difference equation with zero initial conditions

$$y[n] - 2y[n-1] + y[n-2] = x[n] - x[n-1]$$

compute

(a) The transfer function $H(z)$.

(b) The discrete-time impulse response $h(n)$.

(c) The response when the input is the discrete unit step function $u[n]$

$x(n) \longleftrightarrow X(z)$

$x[n-1] \longleftrightarrow z^{-1} X(z)$

$y[n-2] \longleftrightarrow z^{-2} Y(z)$

(a)

$$y[n] - 2y[n-1] + y[n-2] = x[n] - x[n-1]$$

$$\Rightarrow Y(z) - 2z^{-1} Y(z) + z^{-2} Y(z) = X(z) - z^{-1} X(z)$$

$$\Rightarrow Y(z) \left[1 - 2z^{-1} + z^{-2}\right] = X(z) \left(1 - z^{-1}\right)$$

$\therefore$ Transfer function, $H(z) = \dfrac{Y(z)}{X(z)}$

$$= \frac{1 - z^{-1}}{1 - 2z^{-1} + z^{-2}}$$

$$= \frac{1 - z^{-1}}{(1 - z^{-1})^2}$$

$$= \frac{1}{1 - z^{-1}}$$ (Ans)

(b)

We have,

$$H(z) = \frac{1}{1-z^{-1}} \; ; [\text{From (a)}]$$

Taking inverse Z-Transformation,

$$h(n) = (1)^n u[n]$$

$$\Rightarrow h(n) = u[n]$$ (Ans.)

(c)

$$x[n] = u[n]$$

Taking Z-Transformation,

$$X(z) = \frac{1}{1-z^{-1}}$$

We have,

$$Y(z)\left[1-2z^{-1}+z^{-2}\right] = X(z)\left(1-z^{-1}\right)$$

$$\Rightarrow Y(z)\left[1-2z^{-1}+z^{-2}\right] = \frac{1}{(1-z^{-1})}\left(1-z^{-1}\right)$$

$$\Rightarrow Y(z) = \frac{1}{1-2z^{-1}+z^{-2}}$$

$$\therefore Y(z) = \frac{1}{(1-z^{-1})^2}$$

Taking inverse z-transformation,

$$y[n] = (n+1)\,u[n]$$