Complex Integration (Line Integral)

Exercise:6 (Part-1)

Objective:

Finding the path of integration in different form

Methodologies:

By separating real and imaginary part equation of the path will be evaluated.

Line Integral:

- Complex definite integrals are called (complex) **line integrals**.
- Line integral is written as $\int_C f(z)\,dz$. The integrand f(z)is integrated over the curve C. This curve C in the complex plane is called the **path of integration**.
- If C is a **closed path** (one whose terminal point coincides with its initial point), then it is denoted by $\oint_C f(z)dz$.
- C can be represented parametrically as $z(t) = x(t) + i\,y(t) \quad a \le t \le b$.
- If C is a combination of $C_1$and C_2then $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$ and the line integral becomes $\int_C f(z)\,dz = \int_C f(z(t))\,z'(t)dt$.

Problem:(i) Find and sketch the path $z(t) = (1+2i)t$ and its orientation is $(1 \leq t \leq 4)$

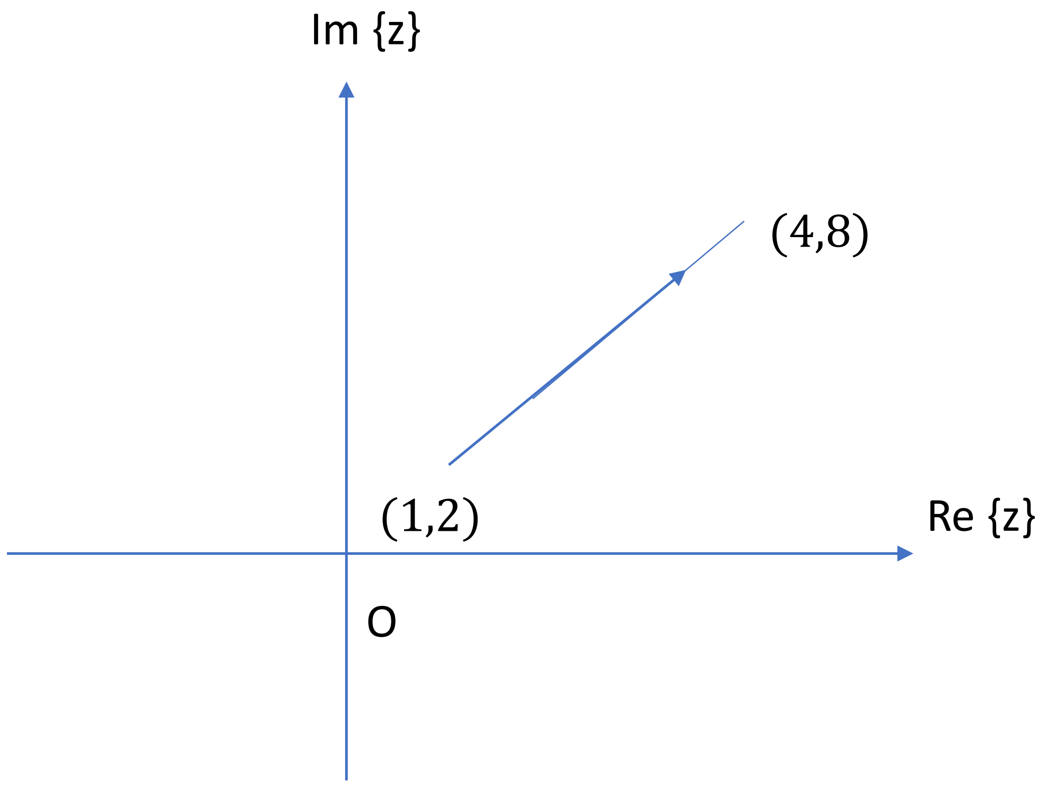
Solution: $z(t) = (1+2i)t \ (1 \leq t \leq 4)$

$$\Rightarrow x(t) + i\,y(t) = t + i\,2t$$

Comparing real and imaginary part, we get

$x(t) = t\,,\, y(t) = 2t \ \ (1 \leq t \leq 4).$

t	x	y	(x, y)
1	1	2	(1,2)
4	4	8	(4,8)



Problem: (ii) Find and sketch the path $z(t) = 4 + i + 2e^{it}$ and its orientation $(0 \leq t \leq 2\pi)$.

Solution: $z(t) = 4 + i + 2e^{it}\ (0 \leq t < 2\pi)$

$$\Rightarrow x(t) + i\,y(t) = 4 + i + 2\cos(\mathrm{t}) + i\,2\sin(t)$$

$$\Rightarrow x(t) + i\,y(t) = (4 + 2\cos(\mathrm{t})) + i\,(1 + 2\sin(t))$$

Comparing real and imaginary part, we get

$$x(t) = 4 + 2\cos(t)$$

$$\Rightarrow x - 4 = 2\cos t \Rightarrow \frac{x-4}{2} = \cos t$$

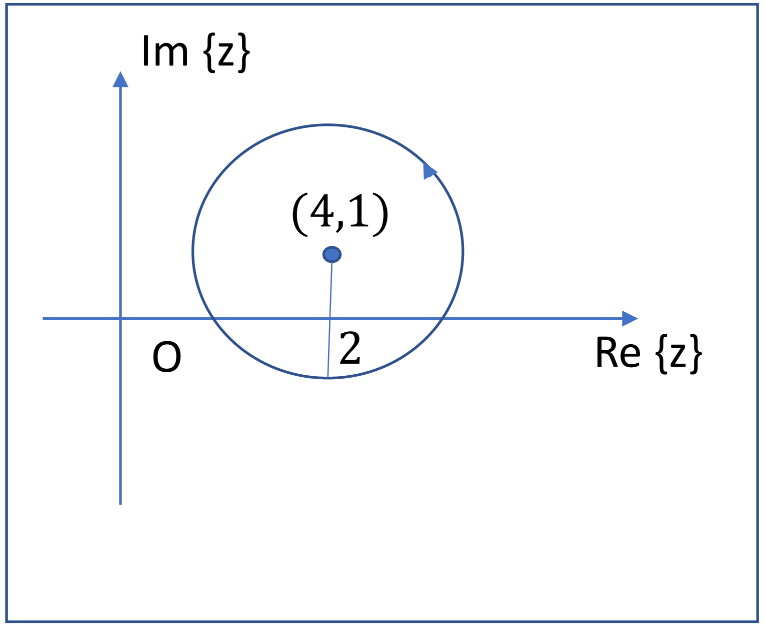
$$y(t) = 1 + 2\sin(t)$$

$$\Rightarrow y - 1 = 2\sin t \Rightarrow \frac{y-1}{2} = \sin t$$

$$\Rightarrow \left(\frac{x-4}{2}\right)^2 + \left(\frac{y-1}{2}\right)^2 = 1$$

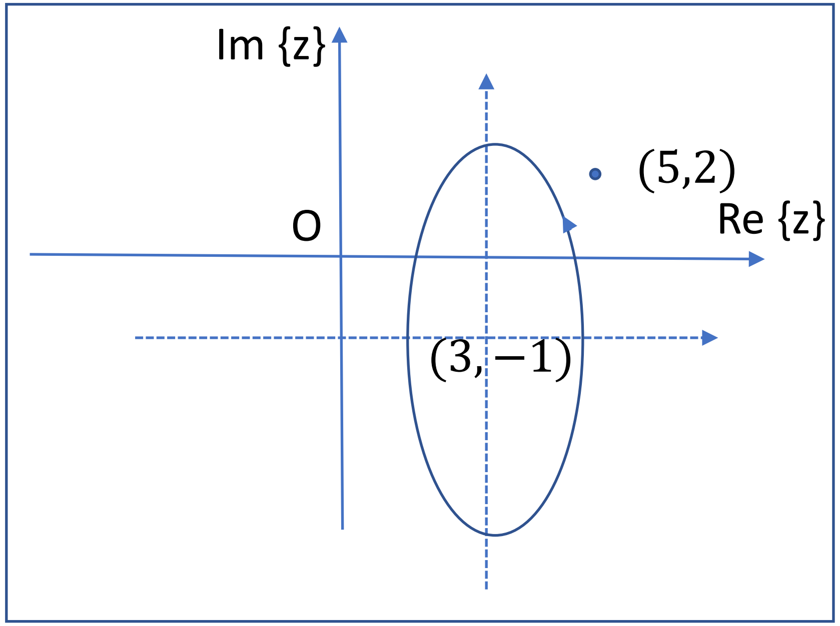
$$\Rightarrow (x-4)^2 + (y-1)^2 = 2^2$$

So, $z(t) = 4 + i + 2e^{it}\ (0 \leq t < 2\pi)$ represents a circle of radius 2 with center $(4, 1)$.



Problem: (iii) Find and sketch the path $z(t) = 3 - i + 2\sin(t) + i\,3\cos(t), (0 \leq t \leq 2\pi)$. Also test whether the point $(5,2)$ is interior, exterior or boundary of this curve.

Solution:



$$z(t) = 3 - i + 2\sin(t) + i\,3\cos(t)$$

$$\Rightarrow x(t) + i\,y(t) = (3 + 2\sin(t)) + i\,(-1 + 3\cos(t))$$

Comparing real and imaginary part, we get

$x(t) = 3 + 2\sin(t)$

$\Rightarrow x - 3 = 2\sin t \Rightarrow \frac{x-3}{2} = \sin t$

$y(t) = -1 + 3\cos(t)$

$\Rightarrow y + 1 = 3\cos(t) \Rightarrow \frac{y+1}{3} = \cos t$

$\Rightarrow \left(\frac{x-3}{2}\right)^2 + \left(\frac{y+1}{3}\right)^2 = 1$

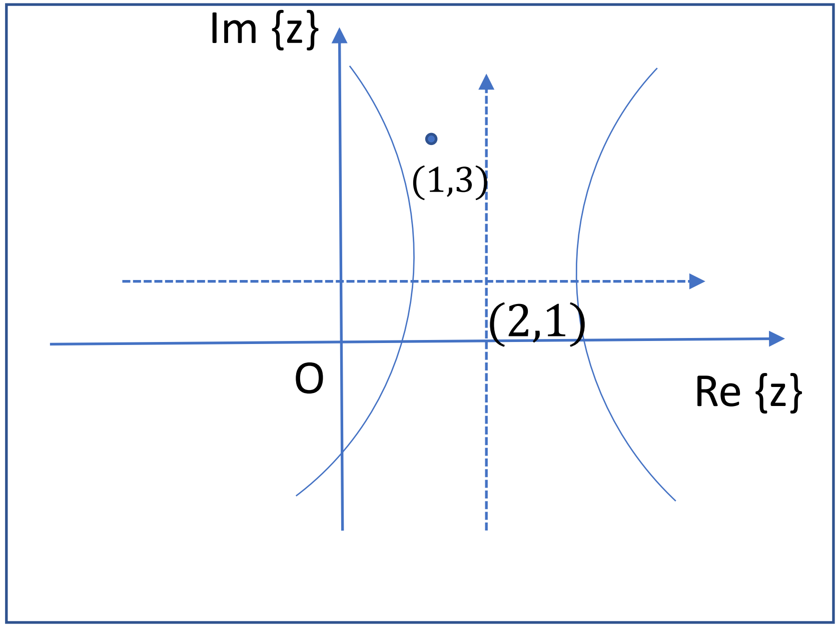
So the given path represents an ellipse with center at $(3,-1)$ and vertices $(\pm 3 - 1, 3) = (2,3), (-4,3)$.

Now at $(5,2)$, $\left(\frac{x-3}{2}\right)^2 + \left(\frac{y+1}{3}\right)^2 = 2 > 1$

So the point $(5,2)$ will be exterior to the ellipse.

Problem: (iv) Find and sketch the path $z(t) = 2 + i + (\cosh t + i \sinh t)$. Also test whether the point $(1, 3)$ is interior, exterior or boundary of this curve.

Solution:



$$z(t) = 2 + i + (\cosh t + i \sinh t)$$

$$\Rightarrow x(t) + i\, y(t) = 2 + i + \cosh t + i \sinh t$$

$$\Rightarrow x(t) + i\, y(t) = (2 + \cosh(\mathrm{t})) + i\,(1 + \sinh(t))$$

Comparing real and imaginary part, we get

$x(t) = 2 + \cosh(\mathrm{t}) \Rightarrow x - 2 = \cosh t$

$y(t) = 1 + \sinh(t) \Rightarrow y - 1 = \sinh t$

$\Rightarrow (x-2)^2 - (y-1)^2 = 1$

So the given path represents a hyperbola with center at $(2,1)$ and vertices $(\pm 1 + 2, 1) = (3,1), (1,1)$.

Now at $(1,3)$, $(x-2)^2 - (y-1)^2 = -3 < 1$

So the point $(1,3)$ will be interior to the hyperbola.

<table>
<tr><th>Horizontal Ellipse</th><th>Vertical Ellipse</th></tr>
<tr><td>

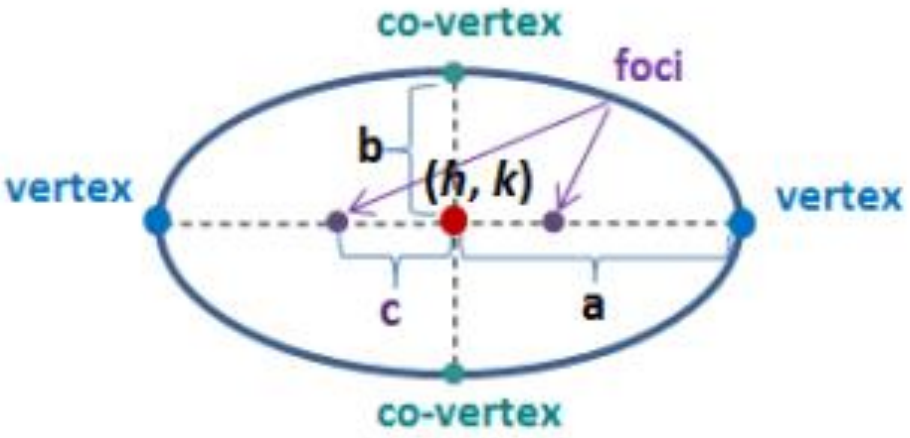
At (0, 0): $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

General: $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

$a^2-b^2=c^2$

Center: (h,k) **Foci:** $(h\pm c,k)$

Vertices: $(h\pm a,k)$ **Co-Vertices:** $(h,k\pm b)$



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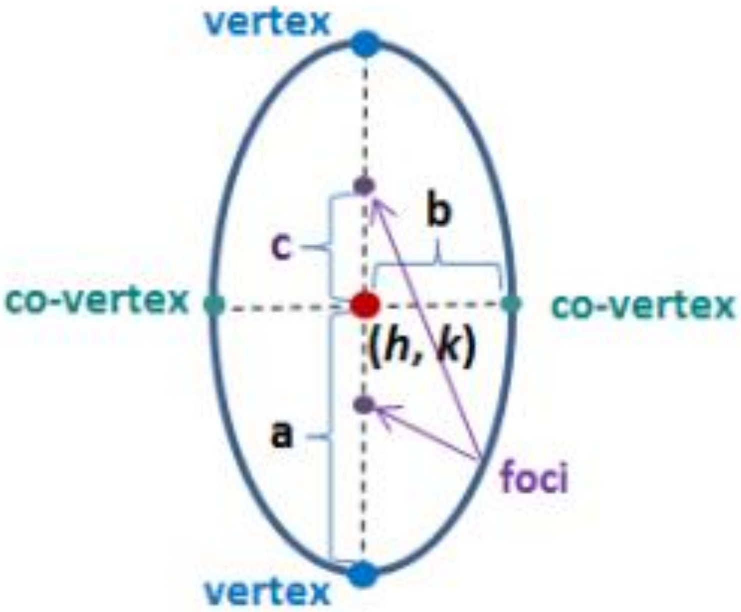

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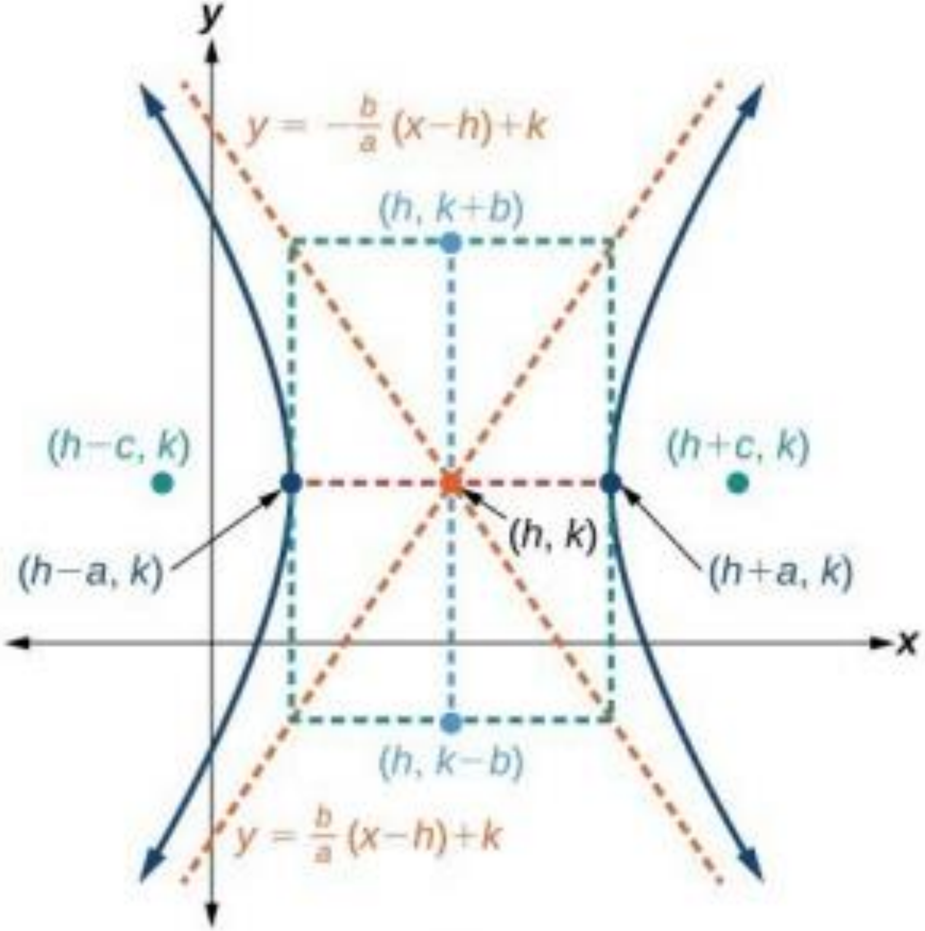
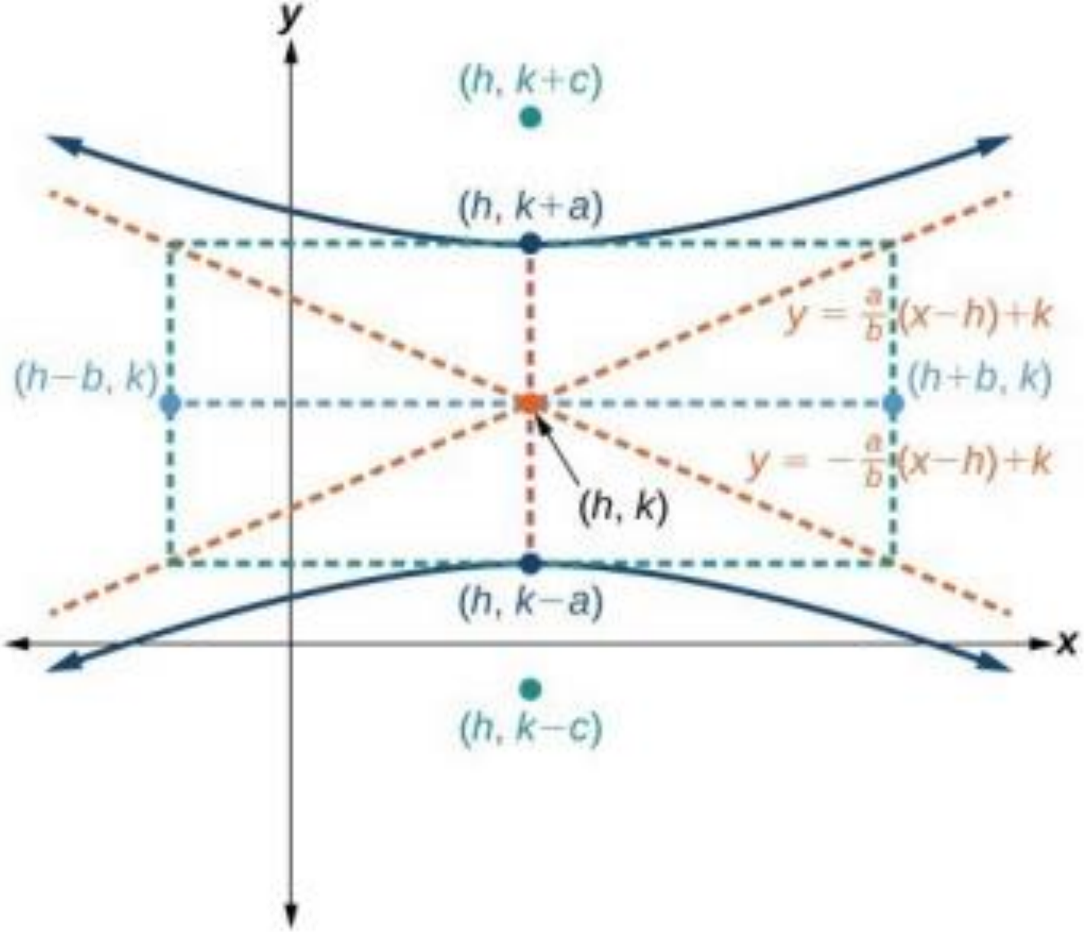
$$\text{(a)}\ \frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \qquad \text{(b)}\ \frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$



(a)



(b)

Exercises:

Find and sketch the path and its orientation. Also classify whether the indicated points are interior, exterior or boundary of the following curves.

- $z(t) = (1 - 6i)t \ (0 \le t \le 3)$
- $z(t) = (5 - i)t \ (-2 \le t \le 2)$
- $z(t) = 4e^{it} \ (0 \le t \le \pi)$
- $z(t) = 2\cos t + i\, 2\sin t \quad (0 \le t \le \pi)$
- $z(t) = -4 - i + e^{it} \ (0 \le t \le 2\pi)$
- $z(t) = 6\sin(t) + i\, 4\cos(t) \ (0 \le t \le 2\pi)$; (5,1)
- $z(t) = 2\cosh(t) + i\, 3\sinh(t)$.
- $z(t) = 2 + i + (\cosh t + i\sinh t)$ (2,3).

Sample MCQ:

❖Equation of unit circle is :

(a) $|z| = r$ (b) $|z| = 2$ (c) $|z| = 1$.

❖ What is the equation of the path C, passing through the points $z = 0$ to $z = 2$.

(a) $y = 0$ (b) $x = 0$ (c) $y = 2$.

❖ Mention whether the point $(1,2)$ is interior, exterior or boundary of $|z - 5 + i| = 4$.

(a) interior (b) exterior (c) on boundary .

❖ $|z - 2| = 4$; center of the circle is

(a) $(-2,0)$ (b) $(2,0)$ (c) $(0,2)$.

❖The vertices $0, 2, 2 + i, i$ represents which of the following shape?

(a) Rectangle (b) Triangle (c) Square .

❖ Which of the following line is parallel to the imaginary axis?

(a) $z = 1$ to $z = 1 + i$

(b) $z = 1$ to $z = 1 + 2i$

(c) $z = 1 + i$ to $z = i$.