COMPLEX VARIABLE

LECTURE 9

OBJECTIVE:

Surveying the algebraic and geometric structure of the complex number system through

- Complex number
- Graphical representation
- Fundamental operations
- Conjugates
- Absolute value/modulus
- Power of imaginary unit
- Polar form and argument

Complex Numbers:

$z = a + ib$; where a and b are any real number and i is the imaginary unit.

- $i = \sqrt{-1}$ and $i^2 = -1$.
- If $a = 0$, the number $z = ib$ is called purely imaginary,
- If $b = 0$, the number $z = a$ is called real.
- Real part of z is: Re{z}$= a$
- Imaginary pat of z is: Im{z}$= b$.

Example: For $z = 2 - 4i$, Re{z}$= 2$ and Im{z}$= -4$.

Example: For $z = \frac{-1+2i}{3}$, Re{z}$= -\frac{1}{3}$ and Im{z}$= \frac{2}{3}$.

Conjugate:

- Conjugate of a complex number $z = a + ib$ is $\bar{z} = a - ib$.
- The geometric interpretation of a complex conjugate is the reflection along the real axis.

Example: If $z = 2 + 3i$ then conjugate of z will be $\bar{z} = 2 - 3i$.

Example: If $z = -2 - i$ then conjugate of z will be $\bar{z} = -2 + i$.

Example: If $z = -\frac{i}{3}$ then conjugate of z will be $\bar{z} = \frac{i}{3}$.

Example: If $z = 5$ then conjugate of z will be $\bar{z} = 5$.

Absolute value/Modulus:

- The distance from the origin to any complex number is the **absolute value** or **modulus.**
- Absolute value of a complex number $z = a + ib$ denoted by mod z or $|z|$

$$\text{mod } z = |z| = \sqrt{a^2 + b^2}$$

Example: If $z = 4 - 3i$ then

$$\text{mod } z = |z| = \sqrt{(4)^2 + (-3)^2} = 5.$$

Some properties of conjugate:

1. $\bar{\bar{z}} = z$

2. $\overline{z + w} = \bar{z} + \bar{w}$

3. $\overline{zw} = \bar{z}\bar{w}$

4. $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}, w \neq 0.$

Some properties of modulus:

1. $|z_1 . z_2| = |z_1||z_2|$

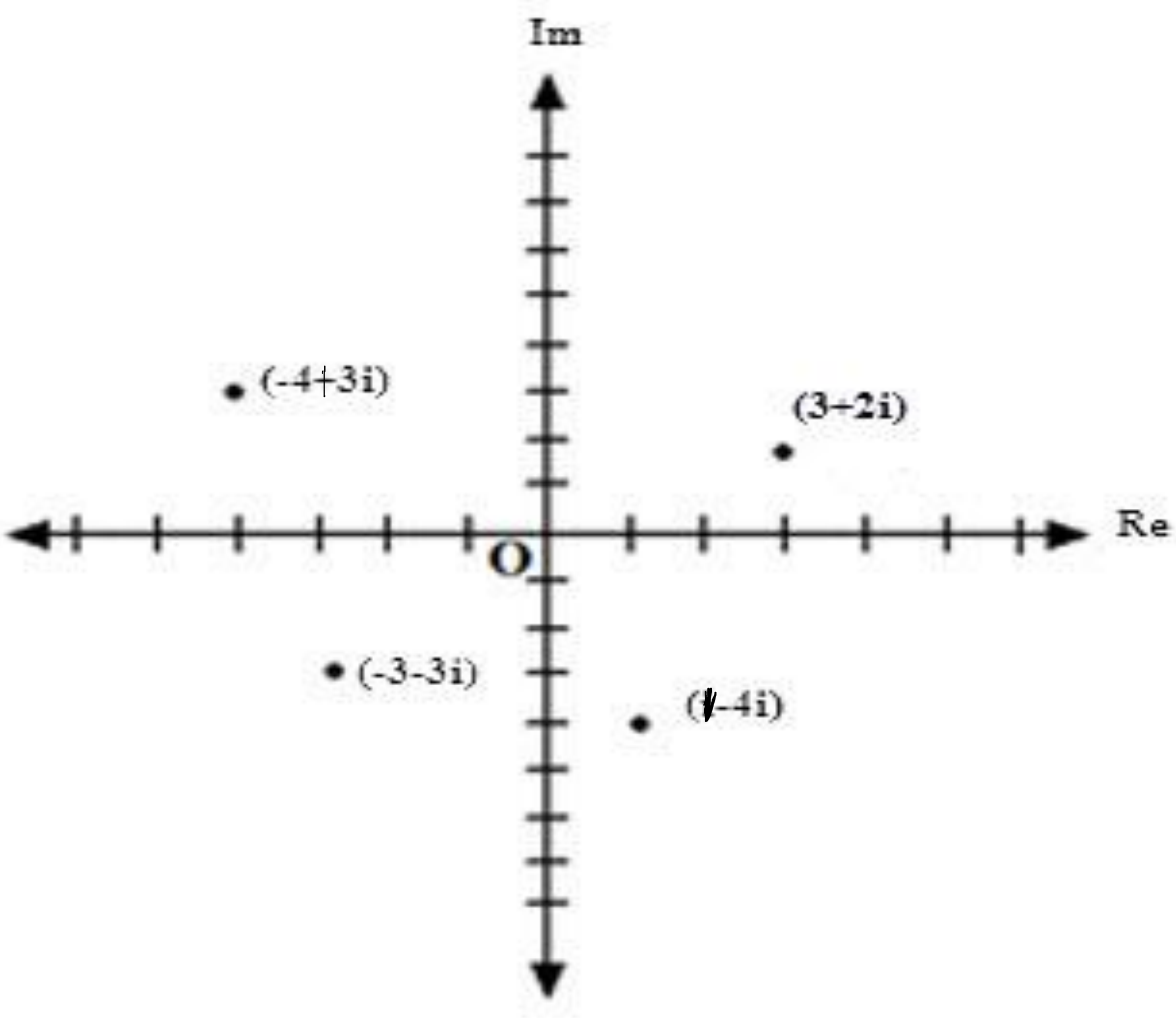
2. $|z|^2 = z.\bar{z}$

3. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$

Graphical Representation of Complex Number/ Argand Diagram:

- Mathematician Argand represented a complex number in a diagram known as **Argand diagram**.
- A complex number $z = a + ib$ can be represented as an ordered pair of real number (a, b).
- A complex number can be represented by points in a xy plane which is called **complex plane/ Argand diagram**.
- The horizontal axis is called the **real axis** and the vertical axis is called the **imaginary** axis.

Complex Numbers in complex plane:



Fundamental operations with complex number:

➢ Addition and Subtraction:

The **sum** and **difference** of complex numbers is defined by adding or subtracting their real components where $a, b \in \mathbb{R}$ i.e.:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a - bi) + (c - di) = (a - c) + (b - d)i$$

Example: Let , $z_1 = (3 + i)$ and $z_2 = (1 - 7i)$

$$\therefore z_1 + z_2 = (3 + 1) + (1 - 7)\, i = 4 - 6i$$

And, $z_1 - z_2 = (3 - 1) + (1 + 7)\, i = 2 - 8i.$

➢ Product: The commutative and distributive properties hold for the **product** of complex numbers:

$$(a + bi)(c + di) = a(c + di) + bi(c + di)$$
$$= ac + adi + bci + bdi^2 = (ac - bd) + i\,(ad + bc).$$

Example: Let , $z_1 = (3 + i)$ and $z_2 = (1 - 7i)$.

$$\therefore z_1 * z_2 = (3 + i)\,(1 - 7i) = 3 - 21i + i - 7i^2 = 3 - 20i + 7 = 10 - 20i.$$

Division:

For the division of two complex numbers to rationalize the denominator, multiply both the numerator and denominator by the conjugate of the denominator.

$$\frac{(a+bi)}{(c+di)} = \frac{(a+bi)}{(c+di)} \cdot \frac{(c-di)}{(c-di)} = \frac{(ac+bd)}{(c^2+d^2)} + i\,\frac{(bc-ad)}{(c^2+d^2)}.$$

Problem: Express $\frac{-3+i}{7-3i}$ in terms of $a+ib$.

Solution: $\frac{-3+i}{7-3i} = \frac{-3+i}{7-3i} * \frac{7+3i}{7+3i} = \frac{-21+7i-9i+3i^2}{7^2+(3)^2} = \frac{-21-2i-3}{49+9}$

$$= \frac{-24-2i}{58} = -\frac{12}{29} - i\,\frac{1}{29}.$$

Problem: Find Re{z} and Im{z} where $z = \frac{3-2i}{1-2i}$.

Solution: Here, $z = \frac{3-2i}{1-2i} = \frac{3-2i}{1-2i} * \frac{1+2i}{1+2i} = \frac{3+6i-2i+4}{1+4} = \frac{7+4i}{5} = \frac{7}{5} + \frac{4}{5}i.$

$\therefore \mathrm{Re}\{z\} = \frac{7}{5}$ and $\mathrm{Im}\{z\} = \frac{4}{5}$.

Powers of imaginary unit i:

Power of imaginary unit i are:

$i^0 = 1, i^1 = i, i^2 = -1, i^3 = i^2.i = (-1).i,\ i^4 = i^3.i = (-i).i = 1$

$i^5 = i^4.i = i;\quad i^6 = i^5.i = -1;\quad i^7 = i^6.i = -i.$

∴By induction, for any positive integer n:

$i^{4n} = 1;\ i^{4n+1} = i;\ i^{4n+2} = -1;\ i^{4n+3} = -i.$

If n is a negative integer, then

$$i^{-n} = (i^{-1})^n = \left(\frac{1}{i}\right)^n = \left(\frac{i}{i.i}\right)^n = (-i)^n.$$

Problem: Evaluate $i^{105} + i^{23} + i^{20} - i^{34}$.

Solution: $i^{105} + i^{23} + i^{20} - i^{34}$

$= i^{4\cdot 26+1} + i^{4\cdot 5+3} + i^{4\cdot 5} - i^{4\cdot 8+2}$

$= i - i + 1 + 1 = 2.$

Problem: Evaluate $i^{-27} + i^{-8} + i^{17}$.

Solution: $i^{-27} + i^{-8} + i^{17}$

$= (-i)^{27} + (-i)^8 + (i)^{17}$

$= -i^{4.6+3} + i^{4.2} + i^{4.4+1}$

$= -i^3 + 1 + i$

$= i + 1 + i = 1 + 2i$