# Complex Variable, Laplace & Z- transformation

## Lecture 05

# This Lecture Covers-

1. Inverse Laplace transformation using partial fraction with type unrepeated, repeated and complex or irrational factors.

2. Inverse Laplace transformation associated with unit step function.

# Inverse Laplace Transformation Using Partial Fraction

## Example on type Unrepeated Factors (Linear Factors)

**Example: 01**

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-5s+6}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{1}{(s-3)(s-2)}\right\}$$

Let, $\frac{1}{(s-3)(s-2)} \equiv \frac{A}{s-3}+\frac{B}{s-2}$

$$\Rightarrow 1 = A(s-2)+B(s-3)$$

$If\ s=2, B=-1$ and $if\ s=3, A=1$

Now,

$$=\mathcal{L}^{-1}\left\{\frac{1}{(s-3)(s-2)}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{1}{(s-3)}-\frac{1}{(s-2)}\right\}$$

$$=e^{3t}-e^{2t}.$$

***<u>Short Technique for Linear factors</u>***

$$A=\frac{1}{s-2}\,at\ s=3; A=1$$

$$B=\frac{1}{s-3}\,at\ s=2; B=-1$$

To Practice-

1. $F(s)=\frac{s+1}{s\,(s-2)\,(s+3)}$ ,

2. $F(s)=\frac{6}{(s+2)\,(s-4)}$ ,

3.. $F(s)=\frac{6\,s-17}{s^2-5s+6}$.

# Inverse Laplace Transformation Using Partial Fraction

## Example on type Repeated Factors

**Example: 01**

$$\mathcal{L}^{-1}\left\{\frac{4s+5}{(s-1)^2(s+2)}\right\}$$

Let, $\frac{4s+5}{(s-1)^2(s+2)} \equiv \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2}$

$\Rightarrow 4s+5$

$= A(s-1)(s+2) + B(s+2) + C(s-1)^2 \ldots (1)$

$For\ s = 1, (1)\ implies$

$$4+5 = 0 + B.3 + 0 \Longrightarrow B = 3$$

$For\ s = -2, (1) implies$

$$-8+5 = 0+0+C(-3)^2 \Longrightarrow C = -\frac{1}{3}$$

$For\ s = 0\ (or\ any\ other\ value)$

$$5 = A(-1)(2) + B(2) + C(1) \Longrightarrow 5 = -2A + 6 - \frac{1}{3}$$

$$\Longrightarrow A = \frac{1}{3}$$

Now,

$$\mathcal{L}^{-1}\left\{\frac{4s+5}{(s-1)^2(s+2)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1/3}{s-1} + \frac{3}{(s-1)^2} - \frac{1/3}{s+2}\right\}$$

$$= \frac{1}{3}e^t + 3t\,e^t - \frac{1}{3}e^{-2t}.$$

**Or**

Comparing both sides, we get

$$A + C = 0, A + B - 2C = 4$$

$$and - 2A + 2B + C = 5$$

By solving, we get

$$A = \frac{1}{3}, B = 3 \text{ and } C = -\frac{1}{3}.$$

## To Practice-

1. $F(s) = \frac{s}{(s+1)^2}$ ,
2. $F(s) = \frac{7s^2+14s-9}{(s-1)^2(s-2)}$ .

# Inverse Laplace Transformation Using Partial Fraction

## Example on type  Factors with Complex or Irrational Roots

**Example: 01**

$$\mathcal{L}^{-1}\left\{\frac{3s^2+13s+26}{s(s^2+4s+13)}\right\}$$

Let, $\frac{3s^2+13s+26}{s(s^2+4s+13)} \equiv \frac{A}{s} + \frac{Bs+C}{s^2+4s+13}$

$\Rightarrow 3s^2+13s+26$

$= A(s^2+4s+13)+(Bs+C)s$

$= As^2+4As+13A+Bs^2+Cs$

For $s=0;\implies 26 = 13A \implies A=2$

Comparing both sides, we get

$A+B=3,\implies B=1$

$4A+C=13 \implies C=5$

So we get,

$A=2, B=1$ and $C=5$ .

Now,

$$\mathcal{L}^{-1}\left\{\frac{3s^2+13s+26}{s(s^2+4s+13)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2}{s}+\frac{s+5}{(s^2+2.s.2+4)+9}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2}{s}+\frac{(s+2)+3}{(s+2)^2+3^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2}{s}+\frac{s+2}{(s+2)^2+3^2}+\frac{3}{(s+2)^2+3^2}\right\}$$

$$= 2+e^{-2t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\}+e^{-2t}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$= 2+e^{-2t}\cos 3t+e^{-2t}\sin 3t$$

**To Practice-**

1. $F(s) = \frac{20}{(s^2+4s+1)\,(s+1)}$ ,

2. $F(s) = \frac{s}{(s^2+4)\,(s-1)}$ .

# Inverse Laplace Transformation Associated with Unit Step Function

## Formula

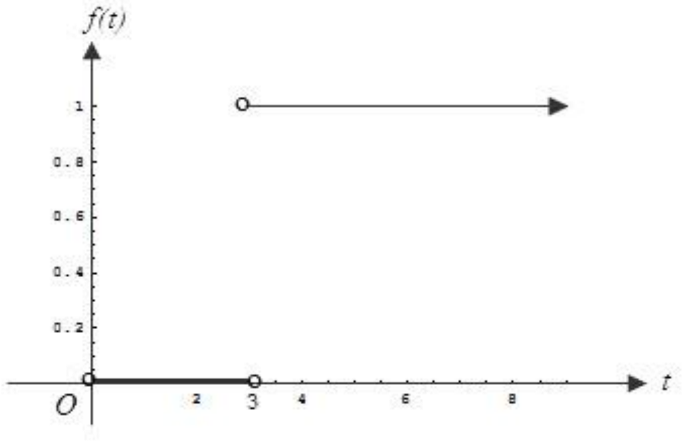
- $\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = u(t-a),$
- $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a).$

**Example 1:**
Find and sketch $f(t)$,
where $f(t) = \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\}.$

**Solution: Here,**
$F_1(s) = \frac{1}{s} \therefore f_1(t) = 1$

$$\therefore \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\}$$

$$= 1.u(t-3) = u_3(t) = \begin{cases} 0, & t < 3 \\ 1, & t \geq 3 \end{cases}.$$



# Inverse Laplace Transformation Associated with Unit Step Function

**Example 2:**

Find and sketch $f(t)$, where $f(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2}\right\}$.

**Solution:**

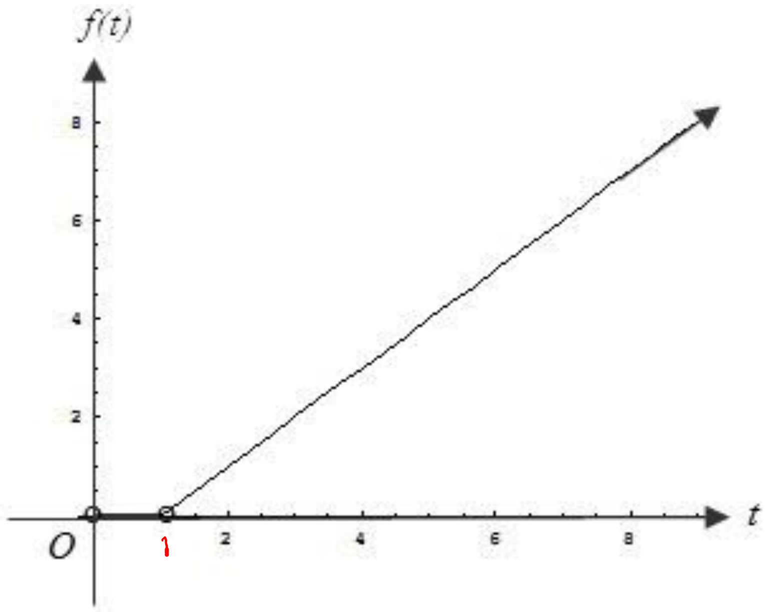
Let, $F(s) = \frac{1}{s^2}$ and $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t = f(t)$.

We know that,

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u_a(t)$$

So, $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2}\right\} = (t-1)u(t-1)$

$$= \begin{cases} 0, & t < 1 \\ t-1, & t \geq 1 \end{cases}.$$



# Inverse Laplace Transformation Associated with Unit Step Function

**Example 3:**

Find and sketch $f(t)$, where $f(t) = \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}$.

**Solution:**

Let, $F(s) = \frac{1}{s^2+1}$ and $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t = f(t)$.

We know that,

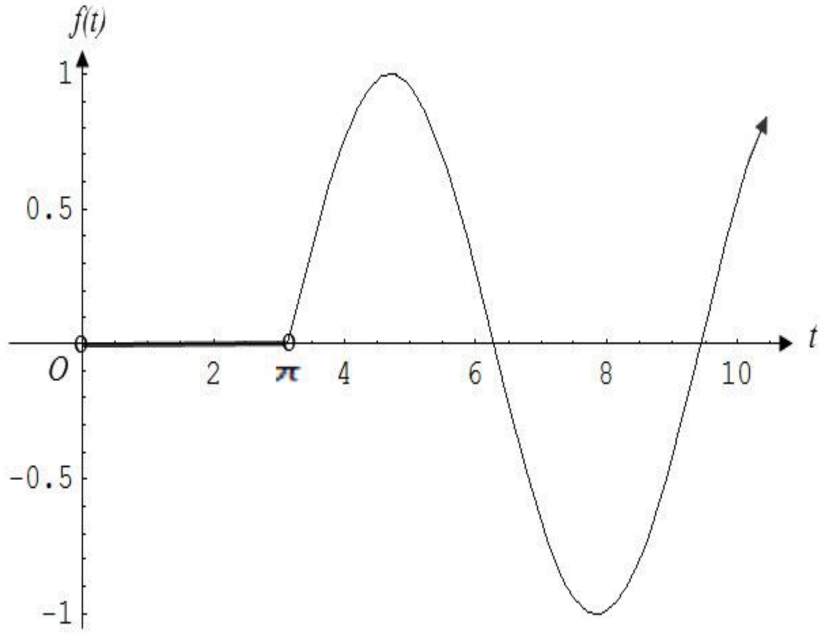
$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$$

So, $\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}$

$= \sin(t-\pi)\,u(t-\pi) = -\sin(\pi - t)\,u(t-\pi)$

$= -\sin t\, u(t-\pi)$

$$= \begin{cases} 0, & t < \pi \\ -\sin t, & t \geq \pi \end{cases}.$$



# Inverse Laplace Transformation Associated with Unit Step Function

**To Practice**

1. $F(s) = 3\left(\frac{e^{-5s}}{s}\right)$

2. $F(s) = 4\left(\frac{e^{-3s}}{s^2}\right)$

3. $F(s) = \frac{se^{-\pi s}}{s^2+25}$

4. $F(s) = \frac{2\left(e^{-3s}-3e^{-4s}\right)}{s}$

5. $F(s) = \frac{5\left(e^{-\pi s}+e^{-2\pi s}\right)}{s^2+25}$

# Learning Outcomes

After completing this lecture you will know how to evaluate inverse Laplace transformation using partial fraction and also inverse Laplace transformation associated with unit step function.

# Sample MCQ

1. $\mathcal{L}^{-1}\left\{\frac{6}{(s+2)\,(s-4)}\right\} = ?$

(a) $e^{t} - e^{-2t}$ (b) $e^{4t} - e^{-2t}$ (c) $e^{4t} - e^{2t}$ (d) $e^{3t} - e^{-2t}$

2. $\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2}\right\} = ?$

(a) $e^{-t} - t\,e^{-t}$ (b) $e^{t} - t\,e^{-t}$ (c) $e^{-t} - e^{-t}$ (d) $e^{-t} - t\,e^{t}$

3. $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)\,(s-1)}\right\} = ?$

(a) $\frac{2}{5}\sin 2t - \frac{1}{5}\cos 2t + \frac{1}{5}\,e^{t}$ (b) $\frac{2}{3}\sin 2t - \frac{1}{5}\cos 2t + \frac{1}{5}\,e^{t}$

(c) $\sin 2t - \frac{1}{5}\cos 2t + \frac{1}{5}\,e^{t}$ (d) $\frac{2}{5}\sin 2t + \frac{1}{5}\cos 2t + \frac{1}{5}\,e^{t}$