

Lecture 05

Complex Variable, Laplace & Z- transformation

This Lecture Covers-

1. Inverse Laplace transformation using partial fraction with type unrepeated, repeated and complex or irrational factors.
2. Inverse Laplace transformation associated with unit step function.

Inverse Laplace Transformation Using Partial Fraction

Example on type Unrepeated Factors (Linear Factors)

Example: 01

$$\begin{aligned} & \mathcal{L}^{-1}\left\{\frac{1}{s^2 - 5s + 6}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{(s-3)(s-2)}\right\} \end{aligned}$$

$$\text{Let, } \frac{1}{(s-3)(s-2)} \equiv \frac{A}{s-3} + \frac{B}{s-2}$$

$$\Rightarrow 1 = A(s-2) + B(s-3)$$

$$\text{If } s = 2, B = -1 \text{ and if } s = 3, A = 1$$

Now,

$$\begin{aligned} &= \mathcal{L}^{-1}\left\{\frac{1}{(s-3)(s-2)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{(s-3)} - \frac{1}{(s-2)}\right\} \\ &= e^{3t} - e^{2t}. \end{aligned}$$

Short Technique for Linear factors

$$A = \frac{1}{s-2} \text{ at } s = 3; A = 1$$

$$B = \frac{1}{s-3} \text{ at } s = 2; B = -1$$

To Practice-

$$1. \quad F(s) = \frac{s+1}{s(s-2)(s+3)},$$

$$2. \quad F(s) = \frac{6}{(s+2)(s-4)},$$

$$3.. \quad F(s) = \frac{6s-17}{s^2-5s+6}.$$

Inverse Laplace Transformation Using Partial Fraction

Example on type Repeated Factors

Example: 01

$$\mathcal{L}^{-1} \left\{ \frac{4s + 5}{(s - 1)^2 (s + 2)} \right\}$$

$$\text{Let, } \frac{4s+5}{(s-1)^2(s+2)} \equiv \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2}$$

$$\Rightarrow 4s + 5$$

$$= A(s - 1)(s + 2) + B(s + 2) + C(s - 1)^2 \dots (1)$$

For $s = 1$, (1) implies

$$4 + 5 = 0 + B.3 + 0 \Rightarrow B = 3$$

For $s = -2$, (1) implies

$$-8 + 5 = 0 + 0 + C(-3)^2 \Rightarrow C = -\frac{1}{3}$$

For $s = 0$ (or any other value)

$$5 = A(-1)(2) + B(2) + C(1) \Rightarrow 5 = -2A + 6 - \frac{1}{3}$$
$$\Rightarrow A = \frac{1}{3}$$

Now,

$$\mathcal{L}^{-1} \left\{ \frac{4s + 5}{(s - 1)^2 (s + 2)} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{1/3}{s - 1} + \frac{3}{(s - 1)^2} - \frac{1/3}{s + 2} \right\}$$

$$= \frac{1}{3} e^t + 3t e^t - \frac{1}{3} e^{-2t}.$$

To Practice-

$$1. F(s) = \frac{s}{(s+1)^2},$$

$$2. F(s) = \frac{7s^2 + 14s - 9}{(s-1)^2 (s-2)}.$$

Or

Comparing both sides, we get

$$A + C = 0, A + B - 2C = 4$$

$$\text{and } -2A + 2B + C = 5$$

By solving, we get

$$A = \frac{1}{3}, B = 3 \text{ and } C = -\frac{1}{3}.$$

Inverse Laplace Transformation Using Partial Fraction

Example on type Factors with Complex or Irrational Roots

Example: 01

$$\mathcal{L}^{-1} \left\{ \frac{3s^2 + 13s + 26}{s(s^2 + 4s + 13)} \right\}$$

$$\text{Let, } \frac{3s^2 + 13s + 26}{s(s^2 + 4s + 13)} \equiv \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 13}$$

$$\Rightarrow 3s^2 + 13s + 26$$

$$= A(s^2 + 4s + 13) + (Bs + C)s$$

$$= As^2 + 4As + 13A + Bs^2 + Cs$$

$$\text{For } s = 0; \Rightarrow 26 = 13A \Rightarrow A = 2$$

Comparing both sides, we get

$$A + B = 3, \Rightarrow B = 1$$

$$4A + C = 13 \Rightarrow C = 5$$

So we get,

$$A = 2, B = 1 \text{ and } C = 5.$$

Now,

$$\mathcal{L}^{-1} \left\{ \frac{3s^2 + 13s + 26}{s(s^2 + 4s + 13)} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{2}{s} + \frac{s + 5}{(s^2 + 2 \cdot s \cdot 2 + 4) + 9} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{2}{s} + \frac{(s + 2) + 3}{(s + 2)^2 + 3^2} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{2}{s} + \frac{s + 2}{(s + 2)^2 + 3^2} + \frac{3}{(s + 2)^2 + 3^2} \right\}$$

$$= 2 + e^{-2t} \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 3^2} \right\} + e^{-2t} \mathcal{L}^{-1} \left\{ \frac{3}{s^2 + 3^2} \right\}$$

$$= 2 + e^{-2t} \cos 3t + e^{-2t} \sin 3t$$

To Practice-

$$1. F(s) = \frac{20}{(s^2 + 4s + 1)(s + 1)},$$

$$2. F(s) = \frac{s}{(s^2 + 4)(s - 1)}.$$

Inverse Laplace Transformation Associated with Unit Step Function

Formula

- $\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = u(t - a),$
- $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t - a)u(t - a).$

Example 1:

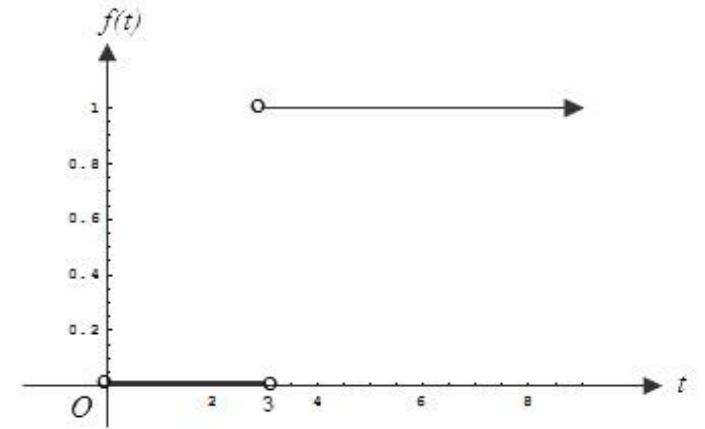
Find and sketch $f(t)$,
where $f(t) = \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\}.$

Solution: Here,

$$F_1(s) = \frac{1}{s} \therefore f_1(t) = 1$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\}$$

$$= 1 \cdot u(t - 3) = u_3(t) = \begin{cases} 0, & t < 3 \\ 1, & t \geq 3 \end{cases}$$



Inverse Laplace Transformation Associated with Unit Step Function

Example 2:

Find and sketch $f(t)$, where $f(t) = \mathcal{L}^{-1} \left\{ \frac{e^{-s}}{s^2} \right\}$.

Solution:

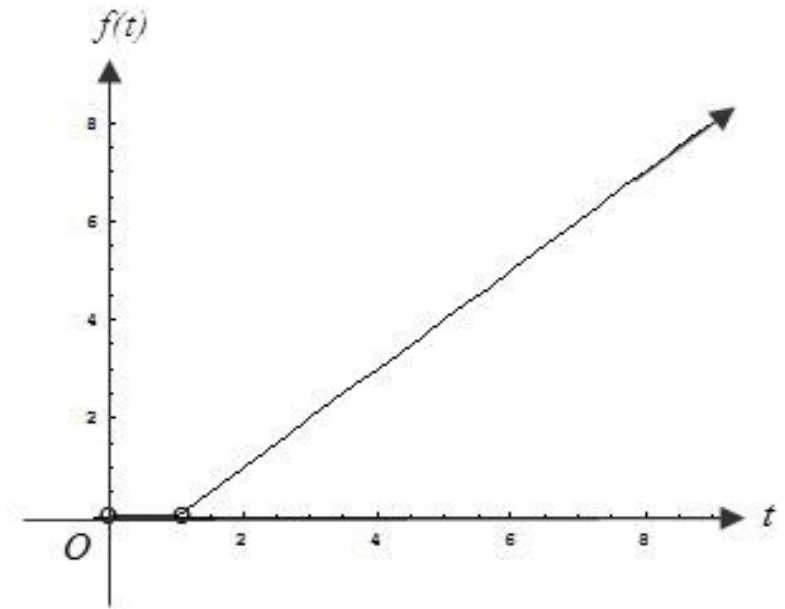
Let, $F(s) = \frac{1}{s^2}$ and $\mathcal{L}^{-1} \left\{ \frac{1}{s^2} \right\} = t = f(t)$.

We know that,

$$\mathcal{L}^{-1} \{ e^{-as} F(s) \} = f(t - a) u_a(t)$$

So, $\mathcal{L}^{-1} \left\{ \frac{e^{-s}}{s^2} \right\} = (t - 1) u(t - 1)$

$$= \begin{cases} 0, & t < 1 \\ t - 1, & t \geq 1 \end{cases}$$



Inverse Laplace Transformation Associated with Unit Step Function

Example 3:

Find and sketch $f(t)$, where $f(t) = \mathcal{L}^{-1} \left\{ \frac{e^{-\pi s}}{s^2+1} \right\}$.

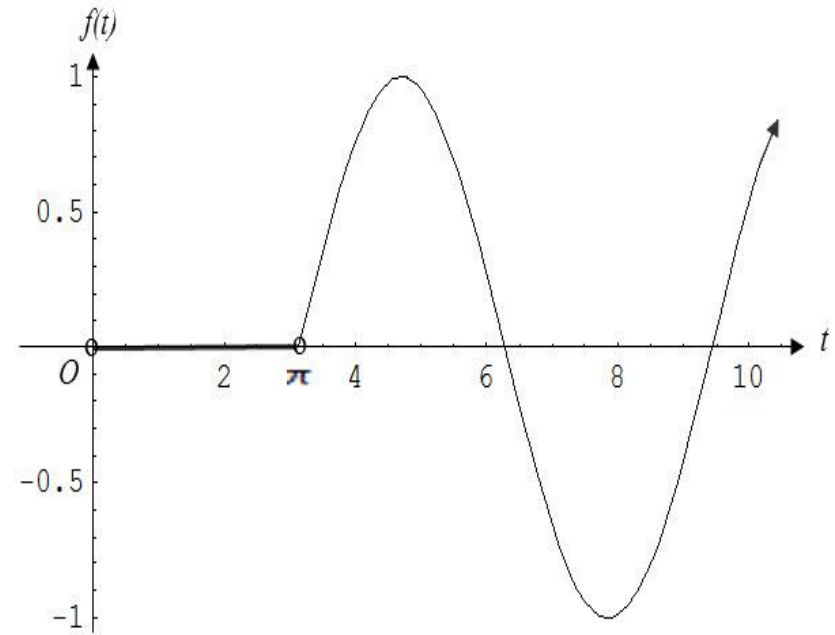
Solution:

Let, $F(s) = \frac{1}{s^2+1}$ and $\mathcal{L}^{-1} \left\{ \frac{1}{s^2+1} \right\} = \sin t = f(t)$.

We know that,

$$\mathcal{L}^{-1} \{ e^{-as} F(s) \} = f(t-a) u(t-a)$$

$$\begin{aligned} \text{So, } \mathcal{L}^{-1} \left\{ \frac{e^{-\pi s}}{s^2+1} \right\} &= \sin(t-\pi) u(t-\pi) = -\sin(\pi-t) u(t-\pi) \\ &= -\sin t u(t-\pi) \\ &= \begin{cases} 0, & t < \pi \\ -\sin t, & t \geq \pi \end{cases} \end{aligned}$$



Inverse Laplace Transformation Associated with Unit Step Function

To Practice

$$1. F(s) = 3 \left(\frac{e^{-5s}}{s} \right)$$

$$2. F(s) = 4 \left(\frac{e^{-3s}}{s^2} \right)$$

$$3. F(s) = \frac{se^{-\pi s}}{s^2+25}$$

$$4. F(s) = \frac{2(e^{-3s}-3e^{-4s})}{s}$$

$$5. F(s) = \frac{5(e^{-\pi s}+e^{-2\pi s})}{s^2+25}$$

Learning Outcomes

After completing this lecture you will know how to evaluate inverse Laplace transformation using partial fraction and also inverse Laplace transformation associated with unit step function.

Sample MCQ

1. $\mathcal{L}^{-1} \left\{ \frac{6}{(s+2)(s-4)} \right\} = ?$

(a) $e^t - e^{-2t}$

(b) $e^{4t} - e^{-2t}$

(c) $e^{4t} - e^{2t}$

(d) $e^{3t} - e^{-2t}$

2. $\mathcal{L}^{-1} \left\{ \frac{s}{(s+1)^2} \right\} = ?$

(a) $e^{-t} - t e^{-t}$

(b) $e^t - t e^{-t}$

(c) $e^{-t} - e^{-t}$

(d) $e^{-t} - t e^t$

3. $\mathcal{L}^{-1} \left\{ \frac{s}{(s^2+4)(s-1)} \right\} = ?$

(a) $\frac{2}{5} \sin 2t - \frac{1}{5} \cos 2t + \frac{1}{5} e^t$

(b) $\frac{2}{3} \sin 2t - \frac{1}{5} \cos 2t + \frac{1}{5} e^t$

(c) $\sin 2t - \frac{1}{5} \cos 2t + \frac{1}{5} e^t$

(d) $\frac{2}{5} \sin 2t + \frac{1}{5} \cos 2t + \frac{1}{5} e^t$