COMPLEX VARIABLE

LECTURE 9

OBJECTIVE:

Surveying the algebraic and geometric structure of the complex number system through

- Complex number
- Graphical representation
- Fundamental operations
- Conjugates
- Absolute value/modulus
- Power of imaginary unit
- Polar form and argument

Complex Numbers:

$z = a + ib$; where a and b are any real number and i is the imaginary unit.

- $i = \sqrt{-1}$ and $i^2 = -1$.
- If $a = 0$, the number $z = ib$ is called purely imaginary,
- If $b = 0$, the number $z = a$ is called real.
- Real part of z is: Re{z}= a
- Imaginary pat of z is: Im{z}= b.

Example: For $z = 2 - 4i$, Re{z}= 2 and Im{z}= -4.

Example: For $z = \frac{-1+2i}{3}$, Re{z}= $-\frac{1}{3}$ and Im{z]= $\frac{2}{3}$.

Conjugate:

- Conjugate of a complex number $z = a + ib$ is $\bar{z} = a - ib$.
- The geometric interpretation of a complex conjugate is the reflection along the real axis.

Example: If $z = 2 + 3i$ then conjugate of z will be $\bar{z} = 2 - 3i$.

Example: If $z = -2 - i$ then conjugate of z will be $\bar{z} = -2 + i$.

Example: If $z = -\frac{i}{3}$ then conjugate of z will be $\bar{z} = \frac{i}{3}$.

Example: If $z = 5$ then conjugate of z will be $\bar{z} = 5$.

Absolute value/Modulus:

- The distance from the origin to any complex number is the **absolute value** or **modulus.**
- Absolute value of a complex number $z = a + ib$ denoted by mod z or $|z|$

$$\text{mod } z = |z| = \sqrt{a^2 + b^2}$$

Example: If $z = 4 - 3i$ then

$$\text{mod } z = |z| = \sqrt{(4)^2 + (-3)^2} = 5.$$

Some properties of conjugate:

1. $\bar{\bar{z}} = z$

2. $\overline{z + w} = \bar{z} + \bar{w}$

3. $\overline{zw} = \bar{z}\bar{w}$

4. $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}, w \neq 0.$

Some properties of modulus:

1. $|z_1 . z_2| = |z_1||z_2|$

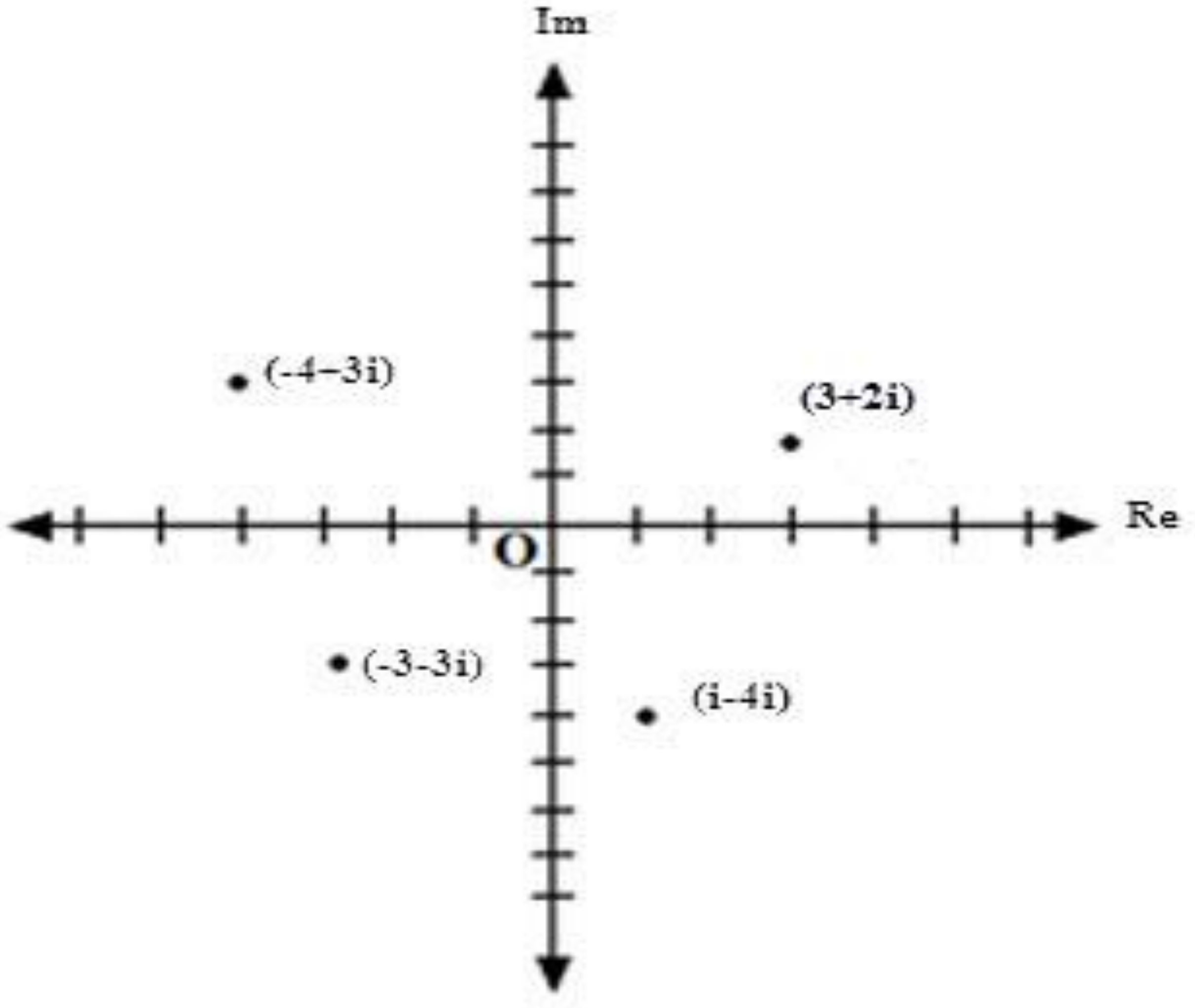
2. $|z|^2 = z . \bar{z}$

3. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$

Graphical Representation of Complex Number/ Argand Diagram:

- Mathematician Argand represented a complex number in a diagram known as **Argand diagram**.
- A complex number $z = a + ib$ can be represented as an ordered pair of real number (a, b).
- A complex number can be represented by points in a xy plane which is called **complex plane/ Argand diagram**.
- The horizontal axis is called the **real axis** and the vertical axis is called the **imaginary** axis.

Complex Numbers in complex plane:



Fundamental operations with complex number:

➢ Addition and Subtraction:

The **sum** and **difference** of complex numbers is defined by adding or subtracting their real components where $a, b \in \mathbb{R}$ i.e.:

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$
$$(a-bi)+(c-di)=(a-c)+(b-d)i$$

Example: Let , $z_1=(3+i)$ and $z_2=(1-7i)$

$$\therefore z_1+z_2=(3+1)+(1-7)\,i=4-6i$$
$$\text{And, } z_1-z_2=(3-1)+(1+7)\,i=2-8i.$$

➢ Product:

The commutative and distributive properties hold for the **product** of complex numbers:

$$(a+bi)(c+di)=a(c+di)+bi(c+di)$$
$$=ac+adi+bci+bdi^2=(ac-bd)+i\,(ad+bc).$$

Example: Let , $z_1=(3+i)$ and $z_2=(1-7i)$.

$$\therefore z_1 * z_2=(3+i)\,(1-7i)=3-21i+i-7i^2=3-20i+7=10-20i.$$

➢ Division:

For the division of two complex numbers to rationalize the denominator, multiply both the numerator and denominator by the conjugate of the denominator.

$$\frac{(a+bi)}{(c+di)} = \frac{(a+bi)}{(c+di)} \cdot \frac{(c-di)}{(c-di)} = \frac{(ac+bd)}{(c^2+d^2)} + i\,\frac{(bc-ad)}{(c^2+d^2)}.$$

Problem: Express $\frac{-3+i}{7-3i}$ in terms of $a+ib$.

Solution: $\frac{-3+i}{7-3i} = \frac{-3+i}{7-3i} * \frac{7+3i}{7+3i} = \frac{-21+7i-9i+3i^2}{7^2-(3i)^2} = \frac{-21-2i-3}{49+9}$

$$= \frac{-24-2i}{58} = -\frac{12}{29} - i\,\frac{1}{29}.$$

Problem: Find Re{z} and Im{z} where $z = \frac{3-2i}{1-2i}$.

Solution: Here, $z = \frac{3-2i}{1-2i} = \frac{3-2i}{1-2i} * \frac{1+2i}{1+2i} = \frac{3+6i-2i+2}{1+4} = \frac{5+4i}{5} = 1 + \frac{4}{5}i.$

$\therefore \mathrm{Re}\{z\} = 1$ and $\mathrm{Im}\{z\} = \frac{4}{5}$.

Powers of imaginary unit i:

Power of imaginary unit i are:

$i^0 = 1, i^1 = i, i^2 = -1, i^3 = i^2.i = (-1).i,\ i^4 = i^3.i = (-i).i = 1$

$i^5 = i^4.i = i;\quad i^6 = i^5.i = -1;\quad i^7 = i^6.i = -i.$

∴By induction, for any positive integer n:

$i^{4n} = 1;\ i^{4n+1} = i;\ i^{4n+2} = -1;\ i^{4n+3} = -i.$

If n is a negative integer, then

$$i^{-n} = (i^{-1})^n = \left(\frac{1}{i}\right)^n = \left(\frac{i}{i.i}\right)^n = (-i)^n.$$

Problem: Evaluate $i^{105} + i^{23} + i^{20} - i^{34}$.

Solution:
$$i^{105} + i^{23} + i^{20} - i^{34}$$
$$= i^{4\cdot 26+1} + i^{4\cdot 5+3} + i^{4\cdot 5} - i^{4\cdot 8+2}$$
$$= i - i + 1 + 1 = 2.$$

Problem: If $z_1 = 1 - i$, $z_2 = -2 - 3i$ and $z_3 = 2i$, then evaluate $|3{z_1}^2 + z_1\overline{z_2} - 5z_3|$.

Solution: $|3{z_1}^2 + z_1\overline{z_2} - 5z_3| = |3(1-i)^2 + (1-i)(-2+3i) - 5 * 2i|$

$$= |3(1 - 2i - 1) + (-2 + 3i + 2i + 3) - 10i|$$

$$= |-6i + 1 + 5i - 10i| = |1 - 11i| = \sqrt{(1)^2 + (-11)^2} = \sqrt{122}\,.$$

Problem: Evaluate $\left|\frac{(1-i)^2}{1+i}\right|$.

Solution: $\left|\frac{(1-i)^2}{1+i}\right| = \left|\frac{1-2i-1}{1+i}\right| = \left|\frac{-2i}{1+i}\right| = \frac{\sqrt{(-2)^2}}{\sqrt{1+1}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$

Problem: Evaluate $\mathrm{Re}\left\{\frac{(1-i)^2}{1+i}\right\}$.

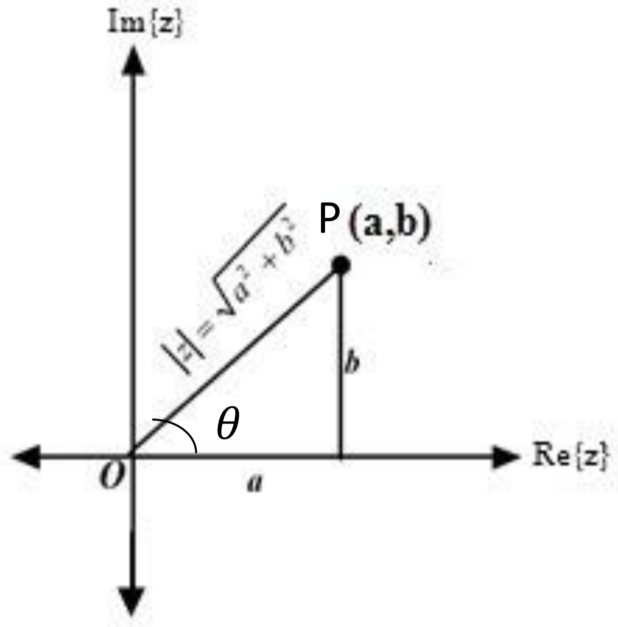
Solution: $\mathrm{Re}\left\{\frac{(1-i)^2}{1+i}\right\} = \mathrm{Re}\left\{\frac{-2i}{1+i}.\frac{1-i}{1+i}\right\} = \mathrm{Re}\left\{\frac{-2i+2i^2}{1^2-i^2}\right\}$

$$= \mathrm{Re}\left\{\frac{-2-2i}{2}\right\} = \mathrm{Re}\{-1-i\} = -1.$$

Polar form of Complex Number and Argument:

- If P a point in the complex plane corresponding to the complex number (a, b) or, $a + ib$ then,

  $a = rcos\theta$ and $b = rsin\theta$.



Where $r = |z| = \sqrt{a^2 + b^2}$ is the distance of $(a, b)$from the origin,

And θ is called the **amplitude or argument** of z which is the angle with real axis denoted by $arg\{z\}$.

Hence, we can write z in polar form as:

$$z = rcos\theta + irsin\theta$$

$$= r(cos\theta + isin\theta)$$

$$\therefore z = re^{i\theta}\text{[Euler Formulae].}$$

Principal argument: The **principal value** of the argument (sometimes called the **principal argument**) is the unique value of the argument that is in the range $0 \leq \theta < 2\pi$ or, $-\pi < \arg z \leq \pi$ and is denoted by Arg z.

$$\arg z = \theta = \operatorname{Arg} z + 2n\pi, \qquad (n = 0, \pm 1, \pm 2, \dots)$$

Some important properties of argument:

- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$
- $\arg(z^n) = n \arg(z)$.

Transformation of complex numbers:

Polar to rectangular: $z = re^{i\theta} \longrightarrow z = a + ib$

$$a = rcos\theta \text{ and } b = rsin\theta$$

Rectangular to Polar: $z = a + ib \longrightarrow z = re^{i\theta}$

$$r = \sqrt{a^2 + b^2} \text{ and}$$

$$\theta = \begin{cases} tan^{-1}\left(\dfrac{b}{a}\right); & if\ a > 0, b \geq 0\ (1st\ quadrant\ and\ on + ve\ real\ axis) \\ tan^{-1}\left(\dfrac{b}{a}\right) + \pi; & if\ a < 0\ (2nd, 3rd\ quadrant \text{ and } on - ve\ real\ axis) \\ tan^{-1}\left(\dfrac{b}{a}\right) + 2\pi; & if\ a > 0, b < 0\ (4th\ quadrant) \\ \dfrac{\pi}{2}; & if\ a = 0, b > 0\ (on + ve\ imaginary\ axis) \\ \dfrac{3\pi}{2}; & if\ a = 0, b < 0\ (on - ve\ imaginary\ axis) \\ Undefined; & if\ a = 0, b = 0 \end{cases} .$$

Problem: Find the rectangular form of $z = \sqrt{2}e^{i\frac{\pi}{4}}$.

Solution: Here $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$

We know that, $a = r\cos\theta = \sqrt{2}\cos\frac{\pi}{4} = 1$

And $b = r\sin\theta = \sqrt{2}\sin\frac{\pi}{4} = 1$

Hence $z = a + ib = 1 + i$.

Problem: Find the rectangular form of $z = 2e^{i\frac{5\pi}{6}}$.

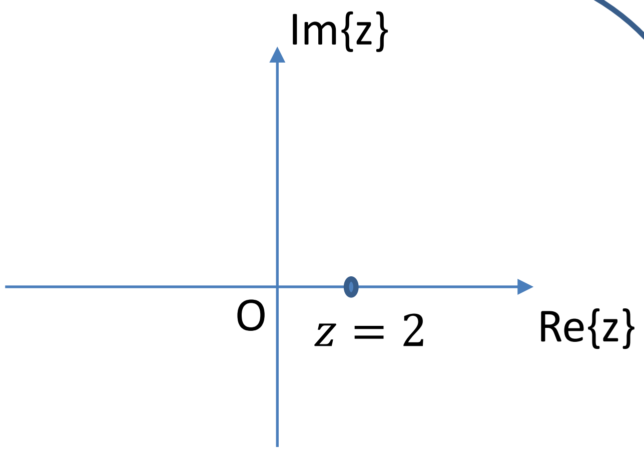
Solution: Here $r = 2$ and $\theta = \frac{5\pi}{6}$

We know that, $a = r\cos\theta = 2\cos(\frac{5\pi}{6}) = -2\frac{\sqrt{3}}{2} = -\sqrt{3}$

And $b = r\sin\theta = 2\sin(\frac{5\pi}{6}) = 2\frac{1}{2} = 1$

Hence $z = a + ib = -\sqrt{3} + i$.

Problem: Find the polar form of $z = 2$.



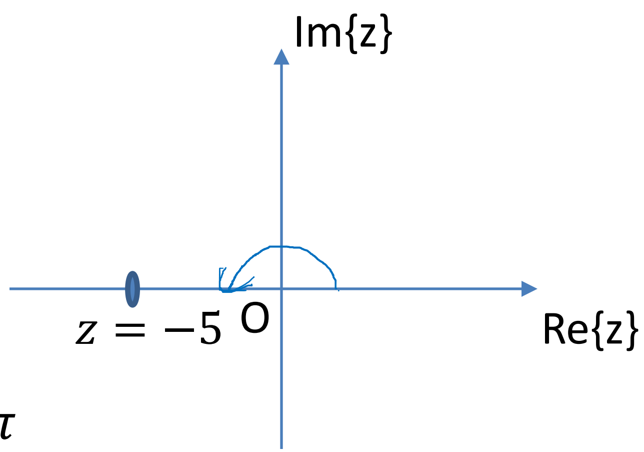
Solution: Here $a = 2$ and $b = 0$

We know that, $r = \sqrt{a^2 + b^2} = \sqrt{2^2} = 2$

And $\theta = tan^{-1}\left(\frac{b}{a}\right) = tan^{-1}\left(\frac{0}{2}\right) = 0$

Hence, $z = re^{i\theta} = 2$.

Problem: Find the polar form of $z = -5$.



Solution: Here $a = -5$ and $b = 0$

We know that, $r = \sqrt{a^2 + b^2} = \sqrt{(-5)^2} = 5$

And $\theta = tan^{-1}\left(\frac{b}{a}\right) + \pi = tan^{-1}\left(\frac{0}{-5}\right) + \pi = \pi$

Hence, $z = re^{i\theta} = 5e^{i\pi}$.

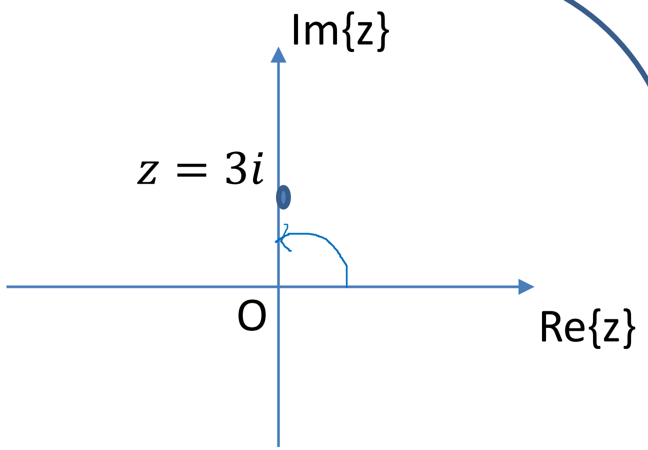
Problem: Find the polar form of $z = 3i$.

Solution: Here $a = 0$ and $b = 3$

We know that, $r = \sqrt{a^2 + b^2} = \sqrt{3^2} = 3$

And $\theta = \frac{\pi}{2}$

Hence, $z = re^{i\theta} = 3e^{i\frac{\pi}{2}}$.



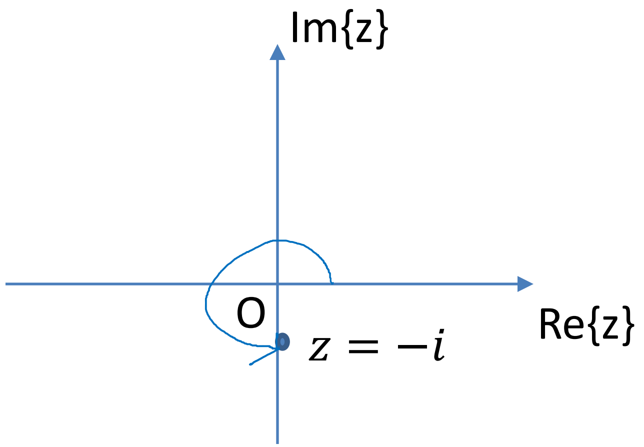
Problem: Find the polar form of $z = -i$.

Solution: Here $a = 0$ and $b = -1$

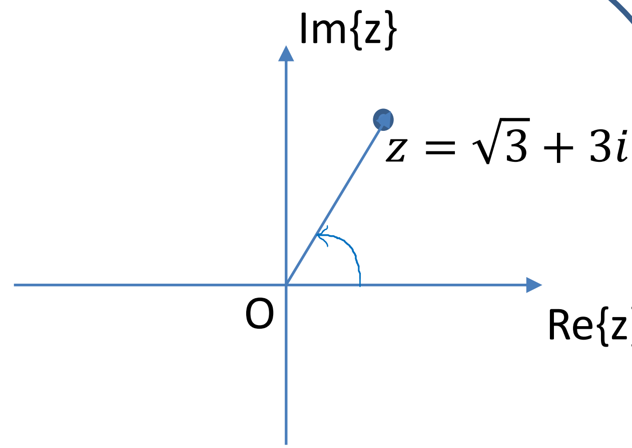
We know that, $r = \sqrt{a^2 + b^2} = \sqrt{(-1)^2} = 1$

And $\theta = \frac{3\pi}{2}$

Hence, $z = re^{i\theta} = e^{i\frac{3\pi}{2}}$.



Problem: Find the polar form of $z = \sqrt{3} + 3i$.



Solution: Here $a = \sqrt{3}$ and $b = 3$

We know that, $r = \sqrt{a^2 + b^2}$

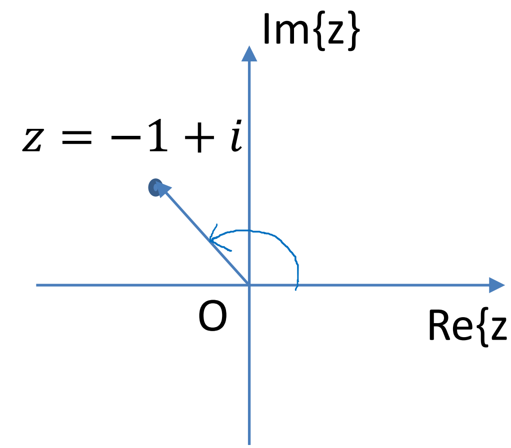
$$= \sqrt{\left(\sqrt{3}\right)^2 + 3^2} = 2\sqrt{3}$$

And $\theta = tan^{-1}\left(\frac{b}{a}\right) = tan^{-1}\left(\frac{3}{\sqrt{3}}\right)$

$$= tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$$

Hence, $z = re^{i\theta} = 2\sqrt{3}e^{i\frac{\pi}{3}}$.

Problem: Find the polar form of $z = -1 + i$.



Solution: Here $a = -$ and $b = 1$

We know that, $r = \sqrt{a^2 + b^2} = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

And $\theta = tan^{-1}\left(\frac{b}{a}\right) + \pi = tan^{-1}\left(\frac{1}{-1}\right) + \pi$

$$= -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$$

Hence, $z = re^{i\theta} = \sqrt{2}e^{i\frac{3\pi}{4}}$.

Problem: Find the polar form of $z = -2\sqrt{3} - 6i$.

Solution: Here $a = -2\sqrt{3}$ and $b = -6$

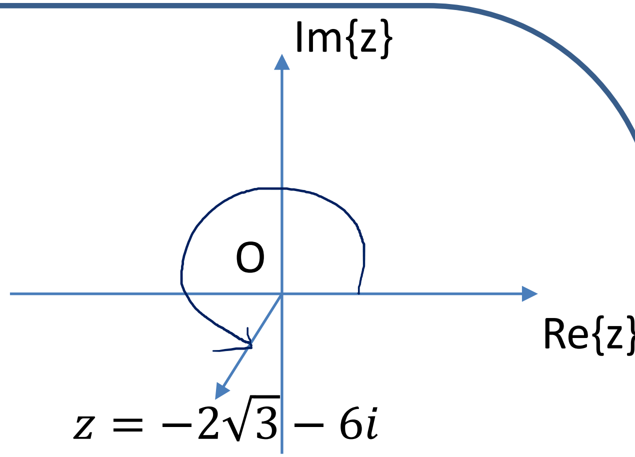
We know that, $r = \sqrt{a^2 + b^2}$

$$= \sqrt{\left(-2\sqrt{3}\right)^2 + (-6)^2} = 4\sqrt{3}$$

And $\theta = tan^{-1}\left(\frac{b}{a}\right) + \pi = tan^{-1}\left(\frac{-6}{-2\sqrt{3}}\right) + \pi$

$$= tan^{-1}\left(\sqrt{3}\right) + \pi = \frac{\pi}{3} + \pi = \frac{4\pi}{3}$$

Hence, $z = re^{i\theta} = 4\sqrt{3}e^{-i\frac{4\pi}{3}}$.



Problem: Find the polar form of $z = \sqrt{3} - i$.

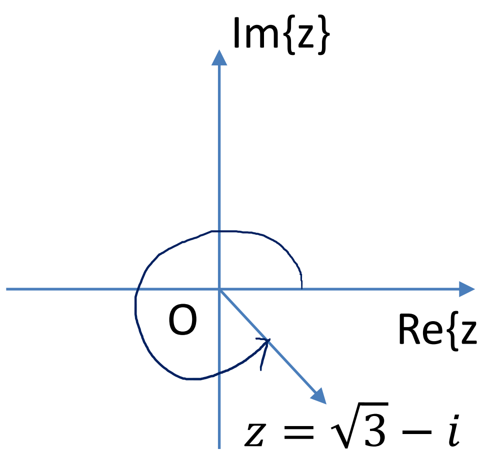
Solution: Here $a = -$ and $b = 1$

We know that, $r = \sqrt{a^2 + b^2} = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$

And $\theta = tan^{-1}\left(\frac{b}{a}\right) + 2\pi = tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) + 2\pi$

$$= -\frac{\pi}{6} + 2\pi = \frac{11\pi}{6}.$$

Hence, $z = re^{i\theta} = 2e^{-i\frac{\pi}{6}}$.



Problem: Find the polar form of $z = \left(\frac{1-i}{1+i}\right)^{18}$.

Solution: Let $z_1 = 1 - i$ and $z_2 = 1 + i$.

For z_1: $a = 1$ and $b = -1$

$\therefore r = \sqrt{(1)^2+(-1)^2} = \sqrt{2}$ and $\theta = tan^{-1}\left(\frac{-1}{1}\right) + 2\,\pi = \frac{7\pi}{4}$

Hence, $z_1 = \sqrt{2}e^{i\frac{7\pi}{4}}$.

Similarly,

For z_2: $a = 1$ and $b = 1$

$\therefore r = \sqrt{(1)^2+(1)^2} = \sqrt{2}$ and $\theta = tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

Hence, $z_2 = \sqrt{2}e^{i\frac{\pi}{4}}$.

So, $z = \left(\frac{1-i}{1+i}\right)^{18} = \left(\frac{\sqrt{2}e^{i\frac{7\pi}{4}}}{\sqrt{2}e^{i\frac{\pi}{4}}}\right)^{18}$

$= \frac{e^{i\frac{63\pi}{2}}}{e^{i\frac{9\pi}{2}}} = e^{i\frac{63\pi}{2}-i\frac{9\pi}{2}} = e^{i\frac{54\pi}{2}} = e^{i27\pi}$.

Finding principal argument (Arg z) from general argument (arg z):

- If arg{z} is in $0 \leq \theta < 2\pi$ then $\arg z = \operatorname{Arg} z$.

Example: For $z = 3e^{i\frac{\pi}{6}}$, $\arg z = \frac{\pi}{6} = \operatorname{Arg}$ z.

- If $\arg z \geq 2\pi$ then subtract maximum no. of $2n\pi$

Example: For z $= 15e^{i\frac{15\pi}{2}}$, $\arg z = \frac{15\pi}{2} = 7.5\pi$

So, $\operatorname{Arg} z = 7.5\pi = 6\pi = 1.5\pi = \frac{3\pi}{2}$.

- If $\arg z < 0$ then add minimum no. of $2n\pi$

Example: For z $= 2e^{-i5\pi}$, $\arg z = -5\pi$

So, $\operatorname{Arg} z = -5\pi + 6\pi = \pi$.

Problem: Find the principal argument of $z = (2+2i)^4$.

Solution: Let $z_1 = 2+2i$; $a = 2, b = 2$;

$\therefore r = \sqrt{4+4} = 2\sqrt{2}$ and $\theta = tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}$

Hence, $z = (2+2i)^4 = \left(2\sqrt{2}e^{i\frac{\pi}{4}}\right)^4 = 2\sqrt{2}\, e^{i\pi}$

$\therefore \arg z = \pi$; $\quad\therefore$Principal argument:$Arg\ z = \pi$.

Problem: Find the principal argument of $z = \left(\frac{2\sqrt{3}+2i}{1-\sqrt{3}i}\right)^6$.

Solution: Let $z_1 = 1+\sqrt{3}i$ and $z_2 = 1-\sqrt{3}i$.

For z_1: $a = 1$ and $b = \sqrt{3}$

$\therefore r = \sqrt{(2\sqrt{3})^2+(2)^2} = 4$ and $\theta = tan^{-1}\left(\frac{2}{2\sqrt{3}}\right) = \frac{\pi}{6} \therefore z_1 = 4e^{i\frac{\pi}{6}}$.

Similarly, For z_2: $a = 1$ and $b = -\sqrt{3}$

$\therefore r = \sqrt{(1)^2+(-\sqrt{3})^2} = 2$ and $\theta = tan^{-1}\left(-\frac{\sqrt{3}}{1}\right) + 2\pi = \frac{5\pi}{3} \therefore z_2 = 2e^{i\frac{5\pi}{3}}$

So,$z = \left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^6 = \left(\frac{4e^{i\frac{\pi}{6}}}{2e^{i\frac{5\pi}{3}}}\right)^6 = 2^6\frac{e^{i\pi}}{e^{i10\pi}} = 2^6e^{i\pi-i10\pi} = 2^6e^{-i9\pi}$.

$\therefore \arg z = -9\pi$; $\quad\therefore$Principal argument:$Arg\ z = -9\pi+10\pi = \pi$.

Problem: Find the principal argument of $z = \left(\frac{1-i}{1+i}\right)^{18}$.

Solution: Let $z_1 = 1 - i$ and $z_2 = 1 + i$.

For z_1: $a = 1$ and $b = -1$

$\therefore r = \sqrt{(1)^2+(-1)^2} = \sqrt{2}$ and $\theta = tan^{-1}\left(\frac{-1}{1}\right) + 2\,\pi = \frac{7\pi}{4}$

Hence, $z_1 = \sqrt{2}e^{i\frac{7\pi}{4}}$.

Similarly,

For z_2: $a = 1$ and $b = 1$

$\therefore r = \sqrt{(1)^2+(1)^2} = \sqrt{2}$ and $\theta = tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

Hence, $z_2 = \sqrt{2}e^{i\frac{\pi}{4}}$.

So, $z = \left(\frac{1-i}{1+i}\right)^{18} = \left(\frac{\sqrt{2}e^{i\frac{7\pi}{4}}}{\sqrt{2}e^{i\frac{\pi}{4}}}\right)^{18}$

$= \frac{e^{i\frac{63\pi}{2}}}{e^{i\frac{9\pi}{2}}} = e^{i\frac{63\pi}{2}-i\frac{9\pi}{2}} = e^{i\frac{54\pi}{2}} = e^{i27\pi}$.

$\therefore \arg z = 27\pi$; $\quad \therefore$Principal argument:$Arg\ z = 27\pi - 26\pi = \pi$.

Exercise:

1. Express $\frac{(1+i)^2}{1-i}$ in terms of $a+ib$.
2. Evaluate each of the followings:

(a) Re$\left\{\frac{1+\sqrt{3}\,i}{1-i}\right\}$, (b) $\left|\frac{z}{\bar{z}}\right|$, (c)Im$\left\{\frac{z}{\bar{z}}\right\}$.

3. Convert the following numbers into polar form:

(a) $z=-1+i$, (b) $z=-3-\sqrt{3}i$, (c) $z=\frac{(1-i)^2}{1+i}$.

4. Convert the following numbers into rectangular form:

$z=\sqrt{3}\,e^{i\,\frac{\pi}{3}}$ and $z=2e^{i\,\frac{\pi}{4}}$.

5. Find the principle argument of the followings:

(a) $z=(-1-i)^4$, (b) $z=\left(-2+2\sqrt{3}\,i\right)^3$, $z=\frac{(1+i)^3}{(1-i)}$.

MULTIPLE CHOICE QUESTION

1. The standard form of the complex number $\dfrac{(1+i)^2}{1-i}$ is

   (a) $-1-i$ (b) $-1+i$ (c) $1+i$ (d) None

2. The polar form of the complex number $-1+i$ is

   (a) $\sqrt{2}e^{i\frac{5\pi}{4}}$ (b) $-\sqrt{2}e^{i\frac{5\pi}{4}}$ (c) $\sqrt{2}e^{i\frac{3\pi}{4}}$ (d) None

3. The standard rectangular form of the complex number $z=\sqrt{3}e^{i\frac{\pi}{2}}$ is

   (a) $-\sqrt{3}+i$ (b) $-\sqrt{3}+\sqrt{3}i$ (c) $\sqrt{3}i$ (d) None

4. The principle argument of the complex number $\left(-1+i\right)^4$ is

   (a) π (b) 2π (c) 3π (d) None