Complex Variable, Laplace & Z- transformation

Lecture 07

This Lecture Covers-

1. Solving Simultaneous Ordinary Differential Equations by Laplace Transform

2. Some examples & exercises on solving simultaneous ODE.

Solve the following system of differential equations where $x(t) \equiv x$, $y(t) \equiv y$, $\dot{y} \equiv \frac{dy(t)}{dt}$ and $\dot{x} \equiv \frac{dx(t)}{dt}$, using Laplace transformation.

$$\begin{cases} \dfrac{dx(t)}{dt} = 2x(t) - 3y(t) \dots (i) \\ \dfrac{dy(t)}{dt} = y(t) - 2x(t) \dots (ii) \end{cases} \text{subject to } x(0) = 8, y(0) = 3.$$

Solution:

Taking the Laplace transforms of both equations

$$\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = 2\mathcal{L}\{x(t)\} - 3\mathcal{L}\{y(t)\}$$

$$\Rightarrow sX(s) - x(0) = 2X(s) - 3Y(s)$$

$$\Rightarrow (s-2)X(s) + 3Y(s) = 8\ldots\ldots\text{(iii)}$$

$$\mathcal{L}\left\{\frac{dy(t)}{dt}\right\} = \mathcal{L}\{y(t)\} - 2\mathcal{L}\{x(t)\}$$

$$\Rightarrow sY(s) - y(0) = Y(s) - 2X(s)$$

$$\Rightarrow 2X(s) + (s-1)Y(s) = 3\ldots\ldots\text{(iv)}$$

Now solving equation (iii) & (iv) simultaneously using **Cramer's rule** and partial fraction we get,

$$X(s) = \frac{\begin{vmatrix} 8 & 3 \\ 3 & s-1 \end{vmatrix}}{\begin{vmatrix} s-2 & 3 \\ 2 & s-1 \end{vmatrix}} = \frac{8s-17}{s^2-3s-4}$$

$$= \frac{8s-17}{(s+1)(s-4)} = \frac{5}{s+1} + \frac{3}{s-4}$$

$$\boxed{A = \frac{-8-17}{-5} = 5 \quad B = \frac{32-17}{5} = 3}$$

$$\text{And, } Y(s) = \frac{\begin{vmatrix} s-2 & 8 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} s-2 & 3 \\ 2 & s-1 \end{vmatrix}} = \frac{3s-22}{s^2-3s-4}$$

$$= \frac{3s-22}{(s+1)(s-4)} = \frac{5}{s+1} - \frac{2}{s-4}$$

$$\boxed{A = \frac{-3-22}{-5} = 5 \quad B = \frac{12-22}{5} = -2}$$

Now taking inverse Laplace transform we get,

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{5}{s+1} + \frac{3}{s-4}\right\}$$

$$\Rightarrow x(t) = 5e^{-t} + 3e^{4t}$$

$$\text{And,} \quad \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{5}{s+1} - \frac{2}{s-4}\right\}$$

$$y(t) = 5e^{-t} - 2e^{4t}$$

Solve the following system of differential equations where $x(t) \equiv x$, $y(t) \equiv y$, $\dot{y} \equiv \frac{dy(t)}{dt}$ and $\dot{x} \equiv \frac{dx(t)}{dt}$, using Laplace transformation.

$$\begin{cases} \dot{x} = x + y \\ \dot{y} = 4x + y; \end{cases}$$ subject to $x(0) = 1, y(0) = 2$.

Solution:

Taking the Laplace transforms of both equations

$$\mathcal{L}\{\dot{x}\} = \mathcal{L}\{x(t)\} + \mathcal{L}\{y(t)\}$$

$$\Rightarrow sX(s) - x(0) = X(s) + Y(s)$$

$$\Rightarrow (s-1)X(s) - Y(s) = 1 \ldots\ldots \text{(iii)}$$

$$\mathcal{L}\{\dot{y}\} = 4\mathcal{L}\{x(t)\} + \mathcal{L}\{y(t)\}$$

$$\Rightarrow sY(s) - y(0) = 4X(s) + Y(s)$$

$$\Rightarrow -4X(s) + (s-1)Y(s) = 2 \ldots\ldots \text{(iv)}$$

Now solving equation (iii) & (iv) simultaneously using **Cramer's rule** and partial fraction we get,

$$X(s) = \frac{\begin{vmatrix} 1 & -1 \\ 2 & s-1 \end{vmatrix}}{\begin{vmatrix} s-1 & -1 \\ -4 & s-1 \end{vmatrix}} = \frac{(s-1)+2}{(s-1)^2-4}$$

$$= \frac{(s-1)}{(s-1)^2-4} + \frac{2}{(s-1)^2-4}$$

And, $$Y(s) = \frac{\begin{vmatrix} s-1 & 1 \\ -4 & 2 \end{vmatrix}}{\begin{vmatrix} s-1 & -1 \\ -4 & s-1 \end{vmatrix}} = \frac{2(s-1)+4}{(s-1)^2-4}$$

$$= \frac{2(s-1)}{(s-1)^2-4} + \frac{4}{(s-1)^2-4}$$

Now taking inverse Laplace transform we get,

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{(s-1)}{(s-1)^2-4} + \frac{2}{(s-1)^2-4}\right\}$$

$$\Rightarrow x(t) = e^t(\cosh 2t + \sinh 2t)$$

And, $$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2(s-1)}{(s-1)^2-4} + \frac{4}{(s-1)^2-4}\right\}$$

$$y(t) = e^t(2\cosh 2t + 2\sinh 2t)$$

Exercises

Solve the following system of differential equations where $x(t) \equiv x$, $y(t) \equiv y$, $\dot{y} \equiv \frac{dy(t)}{dt}$ and $\dot{x} \equiv \frac{dx(t)}{dt}$, using Laplace transformation. Also justify your answers.

1. $\dot{x} = y$
   $\dot{y} = 16x;\;\; x(0) = 0, y(0) = 4.$

2. $\dot{x} = -4y$
   $\dot{y} = x;\;\; x(0) = 2, y(0) = 0.$

3. $\dot{x} = 2x + y$
   $\dot{y} = 4x + 2y;\;\; x(0) = 1, y(0) = 6.$

4. $\dot{x} = 3x + y$
   $\dot{y} = 4x + 3y;\;\; x(0) = 3, y(0) = 2.$

Learning Outcomes

After completing this lecture one can easily solve differential equation and also system of differential equation using Laplace transformation.

Sample MCQ

For

$\dot{x} = y$

$\dot{y} = 16x;\;\; x(0) = 0, y(0) = 4$; answer the following questions: (1-3)

1. Which one is the corresponding system of equations in $X(s)$ and $Y(s)$?

(a) $sX(s) - Y(s) = 0$
$16\,X(s) + sY(s) = 4$

(b) $sX(s) - Y(s) = 0$
$-16\,X(s) + sY(s) = 4$

(c) $sX(s) + Y(s) = 0$
$-16\,X(s) + sY(s) = 4$

(d) $sX(s) - Y(s) = 0$
$-16\,X(s) - sY(s) = 4$

2. If we solve the system of equations obtained in 1, then which one is correct for $X(s)$?

(a) $X(s) = \frac{2s+4}{s^2+16}$ (b) $X(s) = \frac{2s-4}{s^2-16}$ (c) $X(s) = \frac{2s+4}{s^2-16}$ (d) $X(s) = \frac{s+4}{s^2-16}$

3. Evaluate $y(t)$ for the given system of differential equations:

(a) $4\cosh 4t + 8\sinh 4t$ (b) $\cosh 4t + 8\sinh 4t$ (c) $4\cosh 4t - 8\sinh 4t$ (d) $4\cosh 4t + 8\sinh t$