

Complex Differentiation and The Cauchy-Riemann Equation

Lecture-11

This Lecture Covers-

1. Definition of Analytic function
2. Necessary condition for a function to be analytic.
3. Cauchy-Riemann equations.
4. Some Examples and Exercises based on discussion in this lecture.
5. Sample Multiple Choice Questions (MCQs)

Euler's Formulae on complex number:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

And

$$e^{-i\theta} = \cos \theta - i \sin \theta$$

Some Important Formulae:

$$\sin at = \frac{e^{iat} - e^{-iat}}{2i}$$

$$\cos at = \frac{e^{iat} + e^{-iat}}{2}$$

$$\sinh at = \frac{e^{at} - e^{-at}}{2}$$

$$\cosh at = \frac{e^{at} + e^{-at}}{2}$$

$$\cos iay = \cosh ay;$$

$$-i \sin iay = \sinh ay$$

If a single valued function $f(z)$ is differentiable that is $f'(z)$ exists at every point of domain D except possibly at a finite number of exceptional points, then the function is said to be **analytic** in the domain D . These exceptional point at which $f'(z)$ does not exist are called **singular points** or **singularities of the function**.

$$f'(z) = u_x + iv_x$$

Necessary Condition for $f(z)$ to be Analytic:

If $z = x + iy$ and
 $f(z) = u(x, y) + iv(x, y)$
Satisfies the Cauchy-Riemann Equation (Rectangular & Polar form) , then $f(z)$ is said to be **analytic**

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$

Examples

For the function $f(z) = z^2$

- (a) Separate real and imaginary parts
- (b) Verify **C-R** equations.
- (c) find $f'(z)$ using $u_x + i v_x$

Given, $f(z) = z^2$

$$\begin{aligned}\Rightarrow u + iv &= (x + iy)^2 \\ \Rightarrow u + iv &= x^2 + 2 \cdot x \cdot iy + (iy)^2 \\ \Rightarrow u + iv &= x^2 + i \cdot 2xy + i^2 y^2 \\ \Rightarrow u + iv &= x^2 + i \cdot 2xy - y^2 \quad [\because i^2 = -1] \\ \Rightarrow u + iv &= (x^2 - y^2) + i \cdot (2xy)\end{aligned}$$

Comparing both sides

Real part, $u = x^2 - y^2$

Imaginary part, $v = 2xy$

We must show that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} (x^2 - y^2) = 2x$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y} (2xy) = 2x$$

$$\text{So, } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} (x^2 - y^2) = -2y$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x} (2xy) = 2y$$

$$\text{So, } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

We know,

$$f'(z) = u_x + i v_x$$

$$u_x = \frac{\partial u}{\partial x} = 2x$$

$$v_x = \frac{\partial v}{\partial x} = 2y$$

$$\begin{aligned}\text{So, } f'(z) &= u_x + i v_x \\ &= 2x + i \cdot 2y \\ &= 2(x + iy) \\ &= 2z\end{aligned}$$

Exercise Set

For the following functions $f(z) =$

(a) \bar{z} ,

(b) $2z^2 + 3e^z$,

(c) $2ze^z$,

(d) $3z^3$,

(e) $\frac{1}{z^9}$,

(f) z^5 , and

(g) $z^{-\frac{2}{3}}$


I. Separate real and imaginary parts


II. Verify **C-R** equations


III. If analytic find $f'(z)$ using $u_x + iv_x$ or $e^{-i\theta}(u_r + iv_r)$


Sample MCQ

1. What is the real part of $f(z) = e^{z^2}$?


 (a) $x^2 - y^2$


 (b) $e^{x^2+y^2}(\cos 2xy)$


 (c) $e^{x^2-y^2}(\cos 2xy)$


 (d) $e^{x^2-y^2}(\sin 2xy)$

2. For $f(z) = \ln z$, which of the following is the real part? ([Solution](#))


 (a) $x^2 + y^2$


 (b) $\frac{1}{2}(x^2 + y^2)$


 (c) $\ln(x^2 + y^2)$


 (d) $\frac{1}{2}\ln(x^2 + y^2)$

3. For $f(z) = \ln z$, which of the following is the imaginary part? ([Solution](#))

 (a) $\frac{y}{x}$


 (b) $\tan^{-1}\left(\frac{y}{x}\right)$


 (c) xy


 (d) $\sin^{-1}\left(\sqrt{x^2 + y^2}\right)$

4. For $u + iv = \sin z$, which of the following is u_x ? ([Solution](#))


 (a) $\cos x \cosh y$


 (b) $\sin x \cosh y$


 (c) $\sin x$


 (d) $\sinh y$

5. For $u + iv = z^3$ which of the following is v_y ?

 (a) $x^2 + y^2$

 (b) $3x^2 - 3y^2$

 (c) $x^2 - y^2$

 (d) $2x^2 - 2y^2$

Solution of MCQ no 2 and 3

$$\begin{aligned}\ln z &= \ln(re^{i\theta}) \\ &= \ln r + \ln e^{i\theta} \\ &= \ln r + i\theta \ln e \\ &= \ln\left(\sqrt{x^2 + y^2}\right) + i \tan^{-1}\left(\frac{y}{x}\right)\end{aligned}$$

So, Real part is

$$\begin{aligned}\ln\left(\sqrt{x^2 + y^2}\right) \\ &= \ln(x^2 + y^2)^{\frac{1}{2}} \\ &= \frac{1}{2} \ln(x^2 + y^2)\end{aligned}$$

And imaginary part is

$$\tan^{-1}\left(\frac{y}{x}\right)$$

[\[Back to Current Slide\]](#)

Solution of MCQ no 4

$$u + iv = \sin z$$

$$\Rightarrow u + iv = \sin(x + iy)$$

$$\Rightarrow u + iv = \sin x \cos iy + \cos x \sin iy$$

$$\Rightarrow u + iv = \sin x \cosh y - i^2 \cos x \sin iy$$

[$\because \cos ix = \cosh x$] [Reference](#)

$$\Rightarrow u + iv = \sin x \cosh y + i \cos x (-i \sin iy)$$

$$\Rightarrow u + iv = \sin x \cosh y + i \cos x \sinh x$$

[$\because -i \sin ix = \sinh x$] [Reference](#)

$$\Rightarrow u = \sin x \cosh y$$

$$\Rightarrow u_x = \cos x \cosh y$$

[\[Back to Current Slide\]](#)