COMPLEX VARIABLE

Lecture 10

OBJECTIVE

- Obtaining complex roots of *polynomial equations* using De Moivre's theorem
- Discussion and sketch of complex inequality.

De`Moivre's Theorem:

If $z = r(\cos\theta + i\sin\theta)$ and n are positive integers, then

$$z^n = r^n(\cos n\theta + i \sin n\theta) = \left(re^{i\theta}\right)^n = r^n e^{in\theta}$$

- To find the nth power of a complex number, take the nth power of the absolute value or length and multiply the argument by n.

Problem: Find all values of z for which $z^3+2-i2\sqrt{3}=0$ and also locate these values in the complex plane.

Solution: Given, $z^3+2-i2\sqrt{3}=0$.

$$r=\sqrt{(-2)^2+\left(2\sqrt{3}\right)^2}=4$$

$$\theta=tan^{-1}\left(\frac{2\sqrt{3}}{-2}\right)+\pi$$

$$=-\frac{\pi}{3}+\pi=\frac{2\pi}{3}$$

Here the numbers of roots are 3.

$$z^3+2-i2\sqrt{3}=0$$

$$\Rightarrow z=\left(-2+i2\sqrt{3}\right)^{\frac{1}{3}}$$

$$\Rightarrow z=\left(4\ e^{i\frac{2\pi}{3}}\right)^{\frac{1}{3}}$$

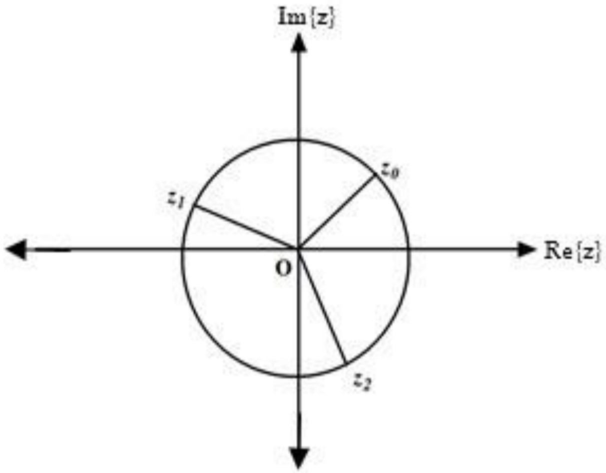
$$\Rightarrow\ z=\left(2^2\ e^{\ i\left(\frac{2\pi}{3}+2n\pi\right)}\right)^{\frac{1}{3}}\quad [\because\ \theta=\theta+2n\pi]$$

$$\Rightarrow\ z_n=2^{\frac{2}{3}}\ e^{\ i\left(\frac{2\pi+6n\pi}{9}\right)}\ ;n=0,1,2$$

when $n=0,\ z_0=2^{\frac{2}{3}}\ e^{\ i\left(\frac{2\pi}{9}\right)}$

when $n=1,\ z_1=2^{\frac{2}{3}}\ e^{\ i\left(\frac{8\pi}{9}\right)}$

when $n=2,\ z_2=2^{\frac{2}{3}}\ e^{\ i\left(\frac{14\pi}{9}\right)}$



The distance of each root from the origin is same as $2^{\frac{2}{3}}$ and the angular distance $\frac{2\pi}{3}$ of two consecutive roots are same.

Problem: Find all values of z for which $z^4 - 81 = 0$ and also locate these values in the complex plane.

Solution: Given, $z^4 - 81 = 0$.

Here the numbers of roots are 3.

$$z^4 - 81 = 0$$

$$\Rightarrow z = (81)^{\frac{1}{4}}$$

$$\Rightarrow z = \left(81\ e^{i\,0}\right)^{\frac{1}{4}}$$

$$\Rightarrow z = \left(3^4\ e^{\ i\,(0+2n\pi)}\right)^{\frac{1}{4}}\ [\because\ \theta = \theta + 2n\pi]$$

$$\Rightarrow z_n = 3\ e^{\ i\left(\frac{2n\pi}{4}\right)}\ ; n = 0,1,2,3$$

$$r = \sqrt{(81)^2} = 81$$

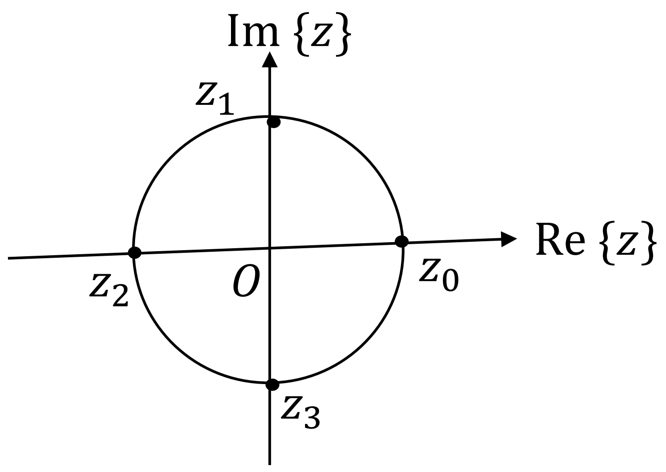
$$\theta = tan^{-1}\left(\frac{0}{81}\right) = 0$$

when $n = 0$, $z_0 = 3\ e^{\ i\,0} = 3$

when $n = 1$, $z_1 = 3\ e^{\ i\frac{\pi}{2}}$

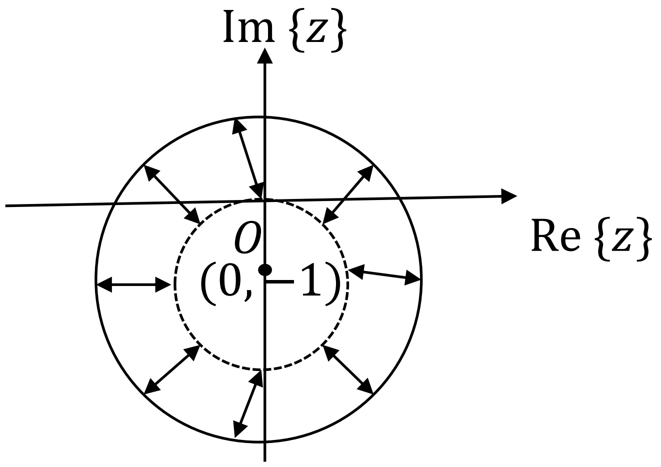
when $n = 2$, $z_2 = 3\ e^{\ i\pi}$

when $n = 3$, $z_3 = 3\ e^{\ i\frac{3\pi}{2}}$



The distance of each root from the origin is same as 3 and the angular distance $\frac{\pi}{2}$ of two consecutive roots are same.

Problem: Describe and graph the locus represented by $1 < |z+i| \leq 2$.



Solution: Given, $1 < |z+i| \leq 2$.

$\Longrightarrow 1 < |x+iy+i| \leq 2$

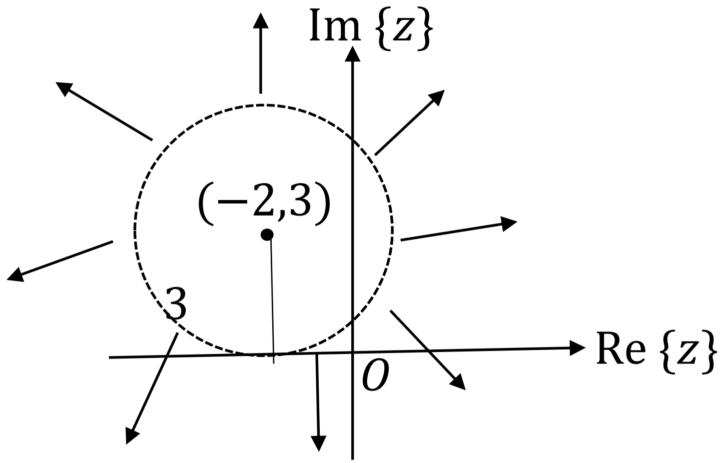
$\Longrightarrow 1 < |x+i(y+1)| \leq 2$

$\Longrightarrow 1 < \sqrt{x^2+(y+1)^2} \leq 2$

$\Longrightarrow 1 < (x-0)^2+(y-(-1))^2 \leq 2^2$

$\therefore$ Given inequality represents the region between two concentric circles of radii 1 and 2 with center $(0,-1)$.

Problem: Describe and graph the locus represented by $|z+2-3i| > 3$.



Solution: Given, $|z+2-3i| > 3$.

$\Longrightarrow |x+iy+2-3i| > 3$

$\Longrightarrow |(x+2)+i(y-3)| > 3$

$\Longrightarrow \sqrt{(x+2)^2+(y-3)^2} > 3$

$\Longrightarrow (x+2)^2+(y-3)^2 > 3$

$\therefore$ Given inequality represents the region outside the circle of radius 3 with center $(-2,3)$.

Problem: Describe and graph the locus represented by $\mathrm{Re}\{z^2\} \leq 4$

Solution: Given, $\mathrm{Re}\{z^2\} \leq 4$

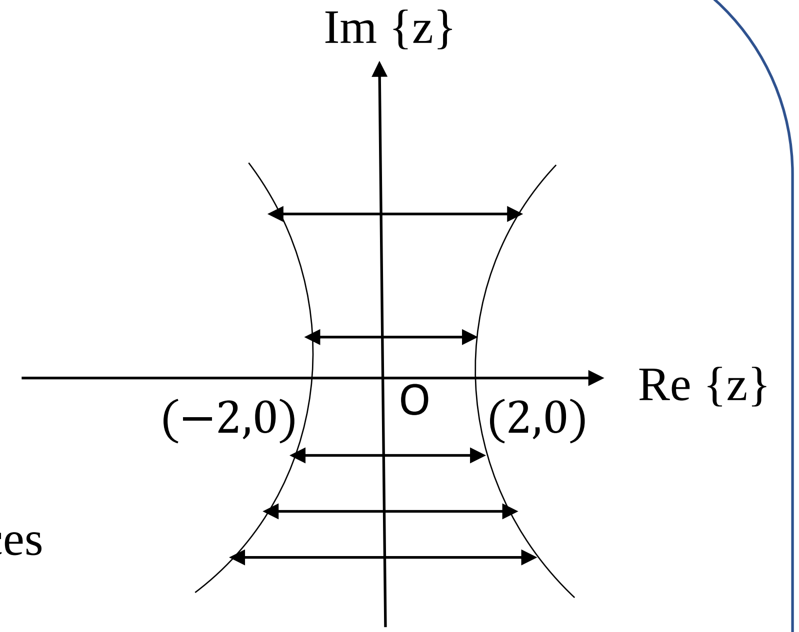
$\Longrightarrow \mathrm{Re}\{(x+iy)^2\} \leq 4$

$\Longrightarrow \mathrm{Re}\{x^2 + i\,2xy - y^2\} \leq 4$

$\Longrightarrow x^2 - y^2 \leq 2^2$

$\Longrightarrow \frac{x^2}{2^2} - \frac{y^2}{2^2} \leq 1$

$\therefore$ Given inequality represents the region inside the hyperbola with vertices $(\pm 2, 0)$ and center $(0,0)$.



Problem: Describe and graph the locus represented by $\frac{\pi}{6} \leq \arg z \leq \frac{\pi}{4}$.

Solution: Given, $\frac{\pi}{6} \leq \arg z \leq \frac{\pi}{4}$.

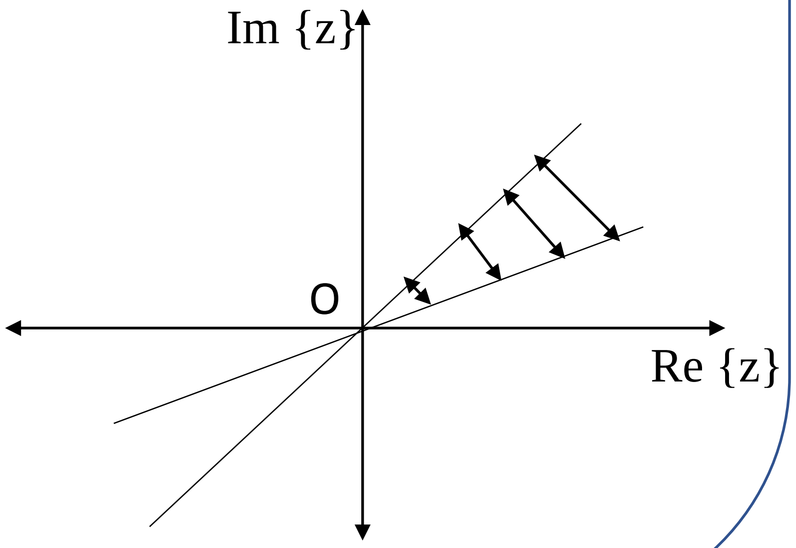
$\Longrightarrow \frac{\pi}{6} \leq tan^{-1}\left(\frac{y}{x}\right) \leq \frac{\pi}{4}$

$\Longrightarrow tan\frac{\pi}{6} \leq \frac{y}{x} \leq tan\frac{\pi}{4}$

$\Longrightarrow \frac{1}{\sqrt{3}} \leq \frac{y}{x} \leq 1$

$\Longrightarrow \frac{1}{\sqrt{3}}x \leq y \leq x$

$\therefore$ Given inequality represents the region between the lines $y = \frac{1}{\sqrt{3}}x$ and $y = x$ in 1st quadrant.



Exercise

1. Find all values of z for the following equations and also locate these values in the complex plane:
    (a) $z^2 + 9 = 0$.
    (b) $z^3 - \sqrt{3} - i = 0$.
    (c) $z^3 = -i$.
    (d) $z^4 - 1 = 0$.
2. Describe and graph the locus represented by each of the followings:
    (a) $|z + 2i| > 4$.
    (b) $1 < |z - 2 + i| \leq 3$.
    (c) $\mathrm{Im}\{z^2\} = 9$.
    (d) $|z - 1| \leq 1$.
    (e) $\mathrm{Re}\{z^2\} < 4$.
    (f) $\frac{\pi}{6} \leq \arg z \leq \frac{\pi}{3}$.

MULTIPLE CHOICE QUESTION

1. The complex equation $z^3 - i + 1 = 0$ has how many roots?

   (a) 2 (b) 3 (c) 1 (d) None

2. The angular distance of each individual root from the next root is

   (a) same (b) different (c) both (d) None

3. The roots of the complex equation $z^2 - i + 1 = 0$ are

   (a) $\sqrt{2}e^{i\frac{3\pi}{4}}, \sqrt{2}e^{i\frac{11\pi}{4}}$ (b) $\sqrt{2}e^{i\frac{\pi}{4}}, \sqrt{2}e^{i\frac{3\pi}{4}}$ (c) $\sqrt{2}e^{i\frac{3\pi}{4}}, \sqrt{2}e^{i\frac{7\pi}{4}}$ (d) None