Name : Jannatul Ferdous Umama

ID : 20-42616-1

Sub : Assignment 3

[Slide 9, 10, 11, 12]

[Slide - 9]

Exercise

1. Express $\frac{(1+i)^2}{1-i}$ in terms of $a+ib$

$= \frac{1+2i+i^2}{1-i}$

$= \frac{2i}{1-i}$

$= \frac{2i(1+i)}{(1-i)(1+i)}$

$= \frac{2i+2i^2}{1-i^2}$

$= \frac{2i-2}{2}$

$= i-1$

$= -1+i$ (A1)

2. a) $\mathrm{Re}\left\{\dfrac{1+\sqrt{3}i}{1-i}\right\}$

$$\therefore \frac{1+\sqrt{3}i}{1-i} = \frac{(1+\sqrt{3}i)(1+i)}{(1+i)(1-i)}$$

$$= \frac{1+i+\sqrt{3}i+\sqrt{3}i^2}{1-i^2}$$

$$= \frac{1-\sqrt{3}+i(\sqrt{3}+1)}{2}$$

$$= \frac{1-\sqrt{3}}{2} + i\left(\frac{\sqrt{3}+1}{2}\right)$$

$$\therefore \mathrm{Re}\left\{\frac{1+\sqrt{3}i}{1-i}\right\} = \frac{1-\sqrt{3}}{2} \quad \text{(An)}$$

b) $\left|\dfrac{z}{\bar{z}}\right|$

We know, $z = a+ib$

$\bar{z} = a-ib$

$$\therefore \left|\frac{a+ib}{a-ib}\right| = \frac{|a+ib|}{|a-ib|}$$

$$= \frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}}$$

$$= 1 \quad \text{(An)}$$

e) $\operatorname{Im}\left\{\frac{z}{\bar{z}}\right\}$

We know, $z = a+ib$

$\bar{z} = a-ib$

$\therefore \frac{z}{\bar{z}} = \frac{a+ib}{a-ib} = \frac{(a+ib)(a+ib)}{(a+ib)(a-ib)}$

$= \frac{a^2 + aib + aib + i^2b^2}{a^2 - i^2b^2}$

$= \frac{a^2 - b^2 + 2iab}{a^2+b^2}$

$= \frac{a^2-b^2}{a^2+b^2} + i\frac{2ab}{a^2+b^2}$

$\therefore \operatorname{Im}\left\{\frac{z}{\bar{z}}\right\} = \frac{2ab}{a^2+b^2}$ (An)

3. a) $z = -1+i$

Here, $a=-1$, and $b=1$

We know that, $r = \sqrt{a^2+b^2}$

$= \sqrt{(-1)^2+1^2}$

$= \sqrt{2}$

and $\theta = \pi - \tan^{-1}\left|\frac{1}{-1}\right|$

$= \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ (An)

b) $z = -3 - \sqrt{3}i$

Here, $a = -3$ and $b = -\sqrt{3}$

We know that, $r = \sqrt{a^2+b^2}$

$$= \sqrt{(-3)^2 + (-\sqrt{3})^2}$$

$$= 2\sqrt{3}$$

and $\theta = \pi + \tan^{-1}\left|\frac{-\sqrt{3}}{-3}\right|$

$$= \pi + \frac{\pi}{6}$$

$$= \frac{7\pi}{6}$$

Hence, $z = re^{i\theta} = 2\sqrt{3}\, e^{i\frac{7\pi}{6}}$ (Ans)

c) $z = \frac{(1-i)^2}{1+i}$

$$= \frac{1-2i+i^2}{1+i}$$

$$= \frac{-2i}{1+i}$$

$$= \frac{-2i(1-i)}{(1+i)(1-i)}$$

$$= \frac{-2i+2i^2}{1-i^2}$$

$$= \frac{-2i-2}{2}$$
$$= -i-1$$
$$= -1-i$$

Here, $a=-1$, $b=-1$

We know, $r = \sqrt{a^2+b^2} = \sqrt{(-1)^2+(-1)^2}$
$$= \sqrt{2}$$

and $\theta = \pi + \tan^{-1}\left|\frac{-1}{-1}\right|$
$$= \pi + \frac{\pi}{4}$$
$$= \frac{5\pi}{4}$$

Hence, $z = re^{i\theta} = \sqrt{2}e^{i\frac{5\pi}{4}}$ (Ans)

4. a) $z = \sqrt{3}e^{i\pi/3}$

Here, $r=\sqrt{3}$ and $\theta = \frac{\pi}{3}$

We know, that, $a = r\cos\theta$
$$= \sqrt{3}\cos\frac{\pi}{3}$$
$$= \frac{\sqrt{3}}{2}$$

$b = r\sin\theta$
$$= \sqrt{3}\sin\frac{\pi}{3}$$
$$= \frac{3}{2}$$

Hence, $z = a + ib$

$$= \frac{\sqrt{3}}{2} + i\frac{3}{2}$$

$$= \frac{\sqrt{3} + 3i}{2}$$

$$= \frac{1}{2}(\sqrt{3} + 3i) \quad \text{(Ans)}$$

b) $z = 2e^{i\pi/4}$

Here, $r = 2$, $\theta = \frac{\pi}{4}$

We know that, $a = r\cos\theta$

$$= 2\cos\frac{\pi}{4}$$

$$= \sqrt{2}$$

$$b = r\sin\theta$$

$$= 2\sin\frac{\pi}{4}$$

$$= \sqrt{2}$$

Hence, $z = a + ib = \sqrt{2} + \sqrt{2}i$

$$= \sqrt{2}(1 + i) \quad \text{(Ans)}$$

5. a) $z = (-1-i)^4$

Here, $a = -1$ and $b = -1$

$r = \sqrt{a^2+b^2} = \sqrt{2}$

$\theta = \pi + \tan^{-1}\left|\frac{-1}{-1}\right|$

$= \pi + \frac{\pi}{4}$

$= \frac{5\pi}{4}$

Hence, $z = (-1-i)^4 = \left(\sqrt{2}e^{i\frac{5\pi}{4}}\right)^4$

$= \sqrt{2}e^{i5\pi}$

$\therefore \arg z = 5\pi$

$\therefore$ Principal argument $\operatorname{Arg} z = 5\pi - 4\pi$

$= \pi$ (Ans)

b) $z = (-2+2\sqrt{3}i)^3$

Here, $a = -2$ and $b = 2\sqrt{3}$

$r = \sqrt{(-2)^2 + (2\sqrt{3})^2}$

$= 4$

$\theta = \pi - \tan^{-1}\left|\frac{2\sqrt{3}}{-2}\right|$

$= \pi - \frac{\pi}{3}$

$= \frac{2\pi}{3}$

Hence, $z = (-2+2\sqrt{3}i)^3$

$= (4e^{i2\pi/3})^3$

$= 64e^{i2\pi}$

$\therefore \arg z = 2\pi$

$\therefore$ Principal argument : $\operatorname{Arg} z = 2\pi - 2\pi$

$= 0$

(Ans)

e) $z = \frac{(1+i)^3}{1-i}$

Let, $z_1 = (1+i)^3$ and $z_2 = 1-i$

For z_1 : $a = 1$ and $b = 1$

$r = \sqrt{a^2+b^2} = \sqrt{2}$

$\theta = \tan^{-1}\left|\frac{1}{1}\right| = \frac{\pi}{4}$

Hence, $z_1 = (1+i)^3 = (\sqrt{2}e^{i\pi/4})^3$

For z_2 : $a = 1$, $b = -1$

$r = \sqrt{a^2+b^2}$

$= \sqrt{2}$

θ

$$\theta = 2\pi - \tan^{-1}\left|\frac{-1}{1}\right|$$

$$= 2\pi - \frac{\pi}{4}$$

$$= \frac{7\pi}{4}$$

Hence, $z_2 = (1-i) = \sqrt{2}\, e^{i7\pi/4}$

So, $z = \frac{(1+i)^3}{1-i} = \frac{(\sqrt{2}e^{i\pi/4})^3}{(\sqrt{2}e^{i7\pi/4})}$

$$= \frac{2\sqrt{2}\, e^{i3\pi/4}}{\sqrt{2}\, e^{i7\pi/4}}$$

$$= \frac{2e^{i3\pi/4}}{e^{i7\pi/4}}$$

$$= 2e^{i3\pi/4 - i\frac{7\pi}{4}}$$

$$= 2e^{-i\pi}$$

$\therefore$ $\arg z = -\pi$

$\therefore$ Principal argument :

$$\operatorname{Arg} z = -\pi + 2\pi$$
$$= \pi \quad \text{(An)}$$

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[Slide - 10]

1. a) $z^2+9=0$

Here the numbers of roots are 2

$z^2=-9$

$\Rightarrow z^2 = 9e^{i\pi}$

$\Rightarrow z = (9e^{i\pi})^{1/2}$

$\Rightarrow z = \left(9e^{i(\pi+2n\pi)}\right)^{1/2}$

$\Rightarrow z = 9^{1/2} e^{i\left(\frac{\pi+2n\pi}{2}\right)}$

$= 3e^{i\left(\frac{\pi+2n\pi}{2}\right)}$

$r = \sqrt{(-9)^2} = 9$

$\theta = \pi - \tan^{-1}\left|\frac{0}{-9}\right|$

$= \pi - 0$

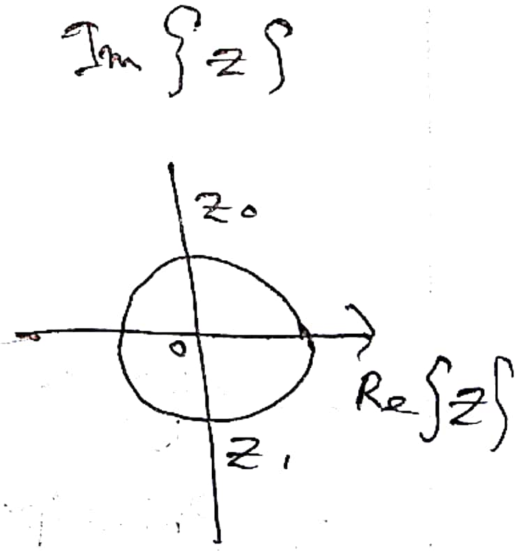
$= \pi$

$\therefore \theta = \theta + 2n\pi$

$\therefore n = 0, 1$

When $n=0$, $z_0 = 3e^{i\pi/2}$

When $n=1$, $z_1 = 3e^{i3\pi/2}$



The distance of each root from the origin is same as 3 and the angular distance of two consecutive roots are same.

b) $z^3 - \sqrt{3} - i = 0$

Here the numbers of roots are 3

$z^3 - \sqrt{3} - i = 0$

$\Rightarrow z = (\sqrt{3} + i)^{1/3}$

$\Rightarrow z = (2e^{i\pi/6})^{1/3}$

$\Rightarrow z = \left(2e^{i\left(\frac{\pi}{6} + 2n\pi\right)}\right)^{1/3}$

$\Rightarrow z_n = 2^{1/3} e^{i\left(\frac{\pi + 12n\pi}{18}\right)}$

$r = \sqrt{(\sqrt{3})^2 + 1^2}$
$= 2$

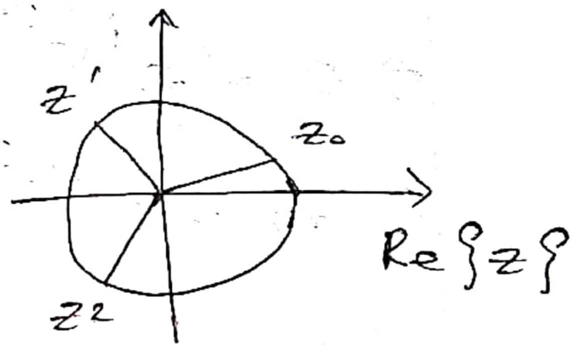
$\theta = \tan^{-1}\left|\frac{1}{\sqrt{3}}\right|$
$= \frac{\pi}{6}$

$n = 0, 1, 2$

When $n = 0$, $z_0 = 2^{1/3} e^{i\pi/18}$

When $n = 1$, $z_1 = 2^{1/3} e^{i\frac{13\pi}{18}}$

When $n = 2$, $z_1 = 2^{1/3} e^{i\frac{25\pi}{18}}$



The distance of each root from the origin is same as $2^{1/3}$ and the angular distance $2\pi/3$ of two consecutive roots are same.

e) $z^3 = -i$

there the numbers of roots are 3

$z^3 = -i$

$\Rightarrow z^3 = 1 \cdot e^{i\,3\pi/2}$

$\Rightarrow z = \left(e^{i\,3\pi/2}\right)^{1/3}$

$\Rightarrow z = \left(e^{i\left(\frac{3\pi}{2} + 2n\pi\right)}\right)^{1/3}$

$\Rightarrow z = \left(e^{i\left(\frac{3\pi + 4n\pi}{2}\right)}\right)^{1/3}$

$\Rightarrow z_n = e^{i\left(\frac{3\pi + 4n\pi}{6}\right)};$

$r = \sqrt{(-1)^2} = 1$

$\theta = 2\pi - \tan^{-1}\left|\frac{-1}{0}\right|$

$= 2\pi - \frac{\pi}{2}$

$= \frac{3\pi}{2}$

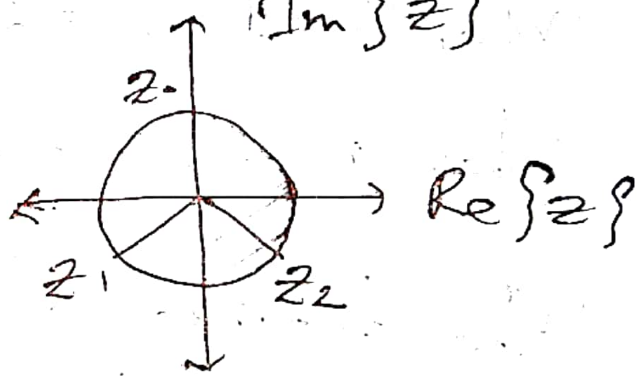
$[\because \theta = \theta + 2n\pi]$

$n = 0, 1, 2$

When, $n=0$, $z_0 = e^{i\pi/2}$

When, $n=1$, $z_1 = e^{i\,7\pi/6}$

When, $n=2$, $z_2 = e^{i\,11\pi/6}$



The distance of each root from the origin is same as 1 and the angular distance $\frac{2\pi}{3}$ of two consecutive roots are same.

d) $z^4 - 1 = 0$

Here the numbers of roots are 4.

$z^4 - 1 = 0$

$\Rightarrow z^4 = 1$

$\Rightarrow z = (1)^{1/4}$

$\Rightarrow z = (e^{i0})^{1/4}$

$\Rightarrow z = \left(e^{i(0+2n\pi)}\right)^{1/4}$ $\quad [\because \theta = \theta + 2n\pi]$

$\Rightarrow z_n = e^{i\frac{2n\pi}{4}}$; $\quad n = 0, 1, 2, 3$

$r = \sqrt{1^2} = 1$

$\theta = \tan^{-1}\left|\frac{0}{1}\right|$

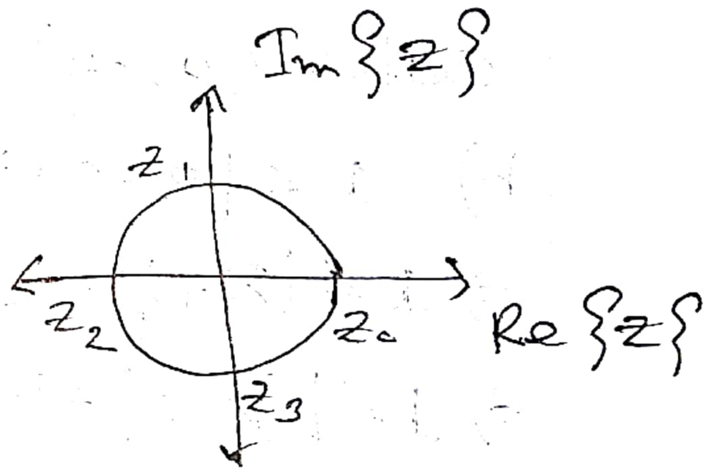
$= 0$

When $n = 0$, $z_0 = e^{i0} = 1$

When $n = 1$, $z_1 = e^{i\pi/2}$

When $n = 2$, $z_2 = e^{i\pi}$

When $n = 3$, $z_3 = e^{i3\pi/2}$



The distance of each root from the origin is same as 1 and the angular distance $\frac{\pi}{2}$ of two consecutive roots are same.

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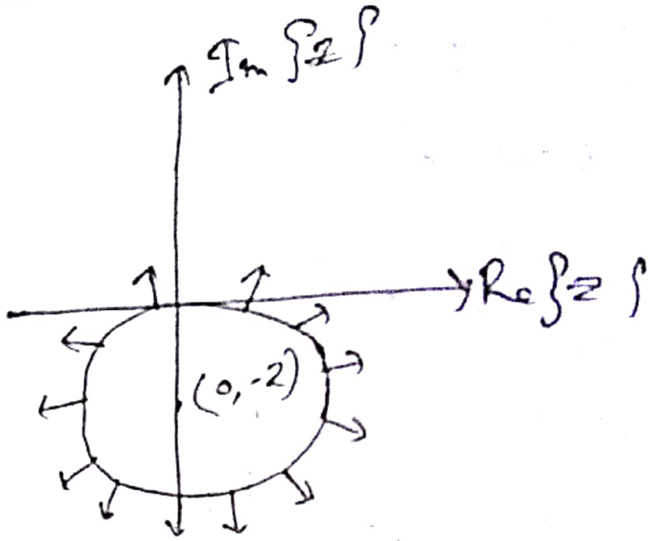
2. a) $|z+2i|>4$

$\Rightarrow |x+iy+2i|>4$

$\Rightarrow |x+i(y+2)|>4$

$\Rightarrow \sqrt{x^2+(y+2)^2}>4$

$\Rightarrow x^2+(y+2)^2>4^2$



∴ Given inequality represents the region outside the circle of radius 4 with center (0, -2)

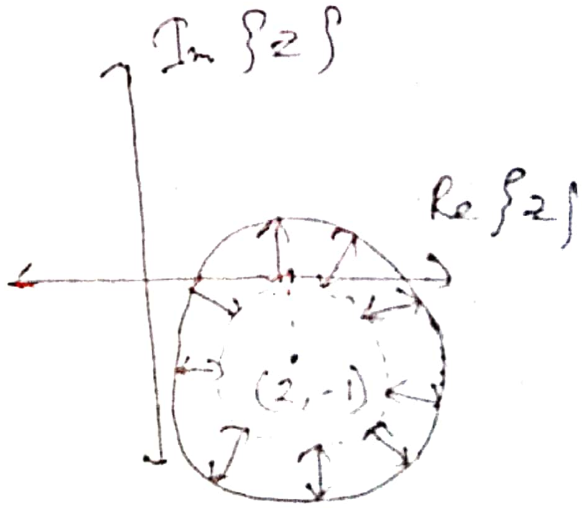
b) $1<|z-2+i|\leq 3$

$\Rightarrow 1<|x+iy-2+i|\leq 3$

$\Rightarrow 1<|(x-2)+i(y+1)|\leq 3$

$\Rightarrow 1<\sqrt{(x-2)^2+(y+1)^2}\leq 3$

$\Rightarrow 1<(x-2)^2+(y+1)^2\leq 3^2$



∴ Given inequality represents the regin between two concentric circles of radious 1 and 3 with center (2, -1)

e) $Im\{z^2\} = 9$

$\Rightarrow Im\{(x+iy)^2\} = 9$

$\Rightarrow Im\{x^2 + 2xiy + i^2y^2\} = 9$

$\Rightarrow Im\{x^2 - y^2 + i2xy\} = 9$

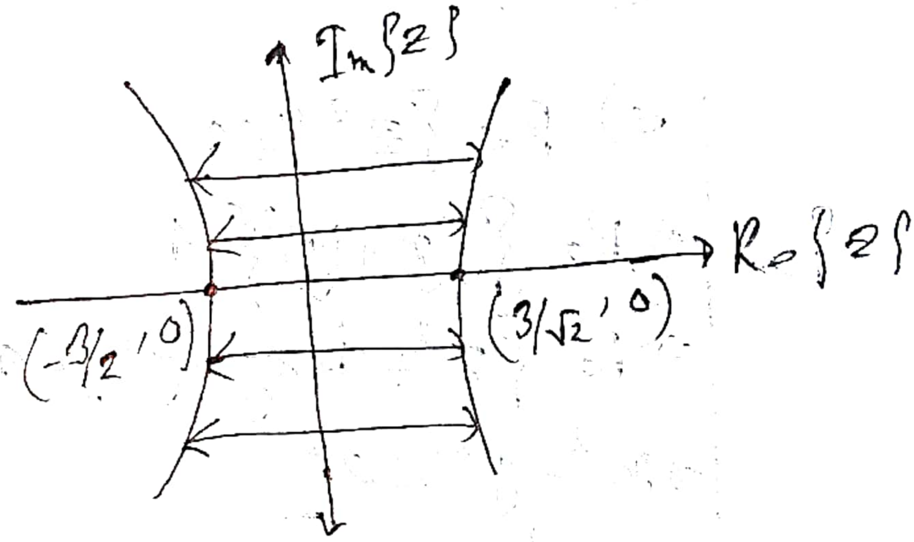
$\Rightarrow 2xy = 9$

$\Rightarrow xy = \frac{9}{2}$

$\Rightarrow xy = \left(\frac{3}{\sqrt{2}}\right)^2$

$\left[\because x = \frac{u+v}{(3/\sqrt{2})}, y = \frac{u-v}{(3/\sqrt{2})}\right]$

$\Rightarrow \frac{u^2}{(3/\sqrt{2})^2} - \frac{v^2}{(3/\sqrt{2})^2} = 1$



Given inegulity represents the region inside the hyperbola with verties $\left(\pm\frac{3}{\sqrt{2}}, 0\right)$ and center $(0, 0)$

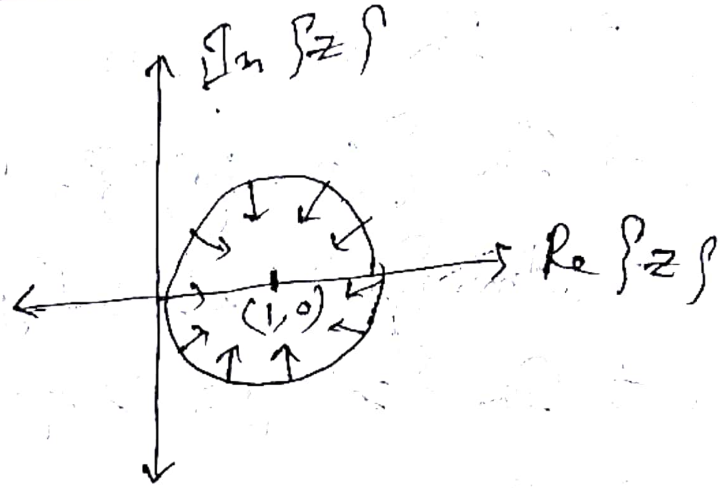
d) $|z-1| \leq 1$

$\Rightarrow |x+iy-1| \leq 1$

$\Rightarrow |(x-1)+iy| \leq 1$

$\Rightarrow \sqrt{(x-1)^2+y^2} \leq 1$

$\Rightarrow (x-1)^2+y^2 \leq 1$



Given inequality represents the region inside the circle of radius 1 with center (1,0)

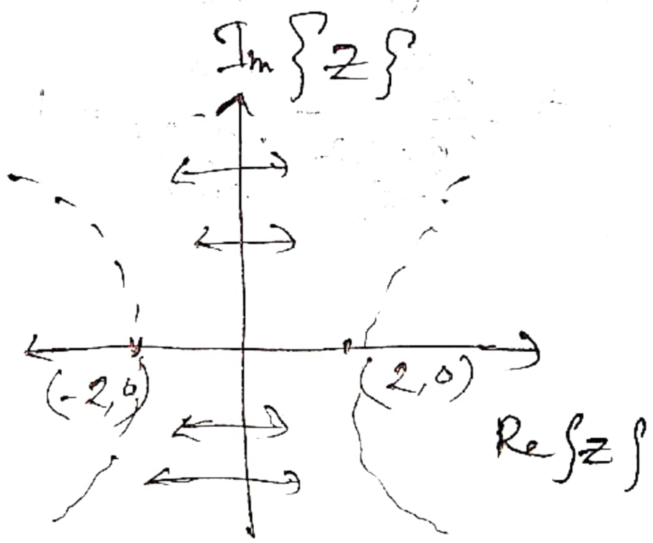
e) $Re\{z^2\} \leq 4$

$\Rightarrow Re\{(x+iy)^2\} \leq 4$

$\Rightarrow Re\{x^2+i2xy-y^2\} < 4$

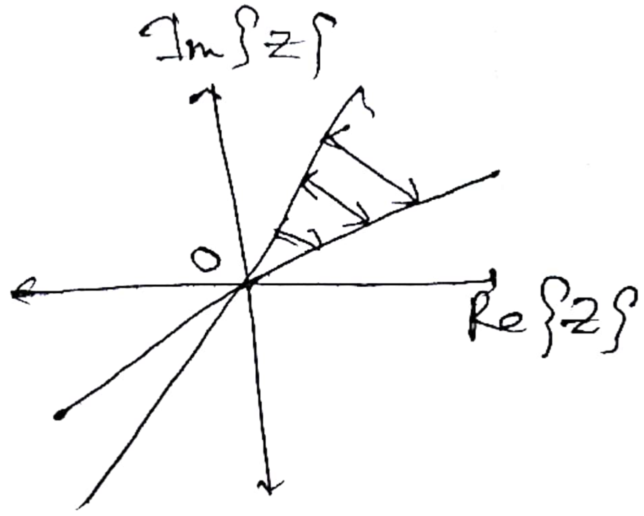
$\Rightarrow x^2-y^2 < 2^2$

$\Rightarrow \frac{x^2}{2^2} - \frac{y^2}{2^2} < 1$



Given inequality represents the region inside the hyperbola with vertices $(\pm 2, 0)$ and center $(0,0)$

f) $\frac{\pi}{6} \le \arg z \le \frac{\pi}{3}$



$\Rightarrow \frac{\pi}{6} \le \tan^{-1}\left(\frac{y}{x}\right) \le \frac{\pi}{3}$

$\Rightarrow \tan\frac{\pi}{6} \le \frac{y}{x} \le \tan\frac{\pi}{3}$

$\Rightarrow \frac{1}{\sqrt{3}} \le \frac{y}{x} \le \sqrt{3}$

$\Rightarrow \frac{x}{\sqrt{3}} \le y \le \sqrt{3}x$

$\therefore$ Given inequality represents the region between lines $y = \frac{1}{\sqrt{3}}x$ and $y = \sqrt{3}x$ in 1s quadrant.

[Slide - 11]

a) $f(z) = \bar{z}$

(i) Given $f(z) = \bar{z}$

$\Rightarrow u + iv = x - iy$

Comparing both sides,

Real Part, $u = x$

Imaginary Part $v = -y$ (Ans)

(ii) We must show that,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\delta x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x) = 1$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(-y) = -1$$

So, $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x) = 0$$

$$-\frac{\partial v}{\partial x} = -\frac{\partial}{\partial x}(-y) = 0$$

So, $\frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x}$

Since, $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ the cauchy - Rieman equation do not hold.

$\therefore$ f (z) is no where differentiable.

(Ans)

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b) $f(z) = 2z^2 + 3e^z$

(1) Given $f(z) = 2z^2 + 3e^z$

$\Rightarrow u + iv = 2(x+iy)^2 + 3e^{(x+iy)}$

$\Rightarrow u + iv = 2(x^2 - y^2 + i2xy) + 3e^{x}e^{iy}$

$\Rightarrow u + iv = 2(x^2 - y^2 + i2xy) + 3e^x(\cos y + i\sin y)$

$\Rightarrow u + iv = (2x^2 - 2y^2 + 3e^x\cos y)$
$\quad + i(4xy + 3e^x\sin y)$

Comparing both sides

Real part $u = 2x^2 - 2y^2 + 3e^x\cos y$

Imaginary part $v = 4xy + 3e^x\sin y$

(11) We must show that

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial x}$

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(2x^2 - 2y^2 + 3e^x\cos y)$

$= 4x + 3\cos y\, e^x$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(4xy + 3^{x}\sin y) = 4x + 3e^{x}\cos y$$

$$\text{So, } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(2x^2 - 2y^2 + 3e^{x}\cos y\right) = -4y - 3e^{x}\sin y$$

$$= -(4y + 3e^{x}\sin y)$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(4xy + 3e^{x}\sin y\right) = 4y + 3e^{x}\sin y$$

$$\text{So, } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{(An)}$$

(iii) We know,

$$f'(z) = u_x + iv_x$$

$$u_x = 4x + 3\cos y\, e^{x}$$

$$v_x = 4y + 3e^{x}\sin y$$

So, $f'(z) = u_x + iv_x$

$$= 4x + 3\cos y\, e^{x} + i\left(4y + 3e^{x}\sin y\right)$$

$$= 4(x + iy) + 3e^{x}(\cos y + i\sin y)$$

$$= 4(x + iy) + 3e^{x}e^{iy}$$

$$= 4(x + iy) + 3e^{x+iy}$$

$$= 4z + 3e^{z} \qquad \text{(Ans)}$$

e) $f(z) = 2ze^{z}$

(1) Given, $f(z) = 2ze^{z}$

$\Rightarrow u+iv = 2(x+iy)e^{x+iy}$

$\Rightarrow u+iv = 2e^{x}(x+iy)(\cos y + i\sin y)$

$\Rightarrow u+iv = 2e^{x}x\cos y + 2e^{x}i^{2}y\sin y$

$+ i(2e^{x}y\cos y) + (2e^{x}x\sin y)$

$\Rightarrow u+iv = 2e^{x}x\cos y - 2ye^{x}\sin y$

$+ i(2e^{x}y\cos y + 2xe^{x}\sin y)$

Comparing both sides,

Real Part, $u = 2e^{x}x\cos y - 2ye^{x}\sin y$

Imaginary Part, $V = 2e^{x}y\cos y + 2xe^{x}\sin y$

(11) We must show that

$u_x = V_y$ and $u_y = -V_x$

$v_x = \frac{\partial}{\partial x}(2e^{x}x\cos y - 2e^{x}y\sin y)$

$= 2\cos y(xe^{x} + e^{x}) - 2y\sin y\, e^{x}$

$$V_y = \frac{\partial}{\partial y}\left(2e^x y\cos y + 2xe^x \sin y\right)$$

$$= 2e^x(-y\sin y + \cos y) + 2xe^x\cos y$$

$$= 2\cos y\,(e^x + xe^x) - 2y\sin y\, e^x$$

So, $U_x = V_y$

$$V_y = \frac{\partial}{\partial y}\left(2e^x x\cos y - 2ye^x \sin y\right)$$

$$= -2e^x x\sin y - 2e^x(y\cos y + \sin y)$$

$$= -(2e^x x\sin y + 2e^x y\cos y + 2e^x \sin y)$$

$$V_x = \frac{\partial}{\partial x}\left(2e^x y\cos y + 2xe^x \sin y\right)$$

$$= 2e^x y\cos y + 2\sin y\, e^x + 2xe^x \sin y$$

So, $U_y = -V_x$

(iii) We know,

$$f'(z) = U_x + iV_x$$

$$U_x = 2xe^x\cos y + 2e^x\cos y - 2ye^x\sin y$$

$$V_x = 2ye^x\cos y + 2e^x\sin y + 2xe^x\sin y$$

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So, $f'(z) = U_x + iV_x$

$$= 2xe^x\cos y + 2e^x\cos y - 2ye^x\sin y + i(2ye^x\cos y + 2e^x\sin y + 2xe^x\sin y)$$

$$= 2xe^x\cos y + 2i^2ye^x\sin y + i2ye^x\cos y + i2xe^x\sin y + 2e^x(\cos y + i\sin y)$$

$$= 2xe^x(\cos y + i\sin y) + i2ye^x(\cos y + i\sin y) + 2e^x(\cos y + i\sin y)$$

$$= 2xe^xe^{iy} + i2ye^xe^{iy} + 2e^xe^{iy}$$

$$= 2xe^{x+iy} + i2ye^{x+iy} + 2e^z$$

$$= 2e^z(x+iy) + 2e^z$$

$$= 2e^z z + 2e^z$$

$$= 2(ze^z + e^z)$$

(An)

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d) $f(z) = 3z^3$

(i) Given, $f(z) = 3z^3$

$$\Rightarrow u + iv = 3(re^{i\theta})^3$$

$$\Rightarrow u + iv = 3r^3 e^{i3\theta}$$

$$\Rightarrow u + iv = 3r^3(\cos 3\theta + i \sin 3\theta)$$

$$\Rightarrow u + iv = 3r^3 \cos 3\theta + i(3r^3 \sin 3\theta)$$

Comparing both sides,

Real part, $u = 3r^3 \cos 3\theta$

Imaginary part $v = 3r^3 \sin 3\theta$

(ii) We must show that,

$$\frac{\partial v}{\partial r} = \frac{1}{r} \cdot \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \cdot \frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial r} = \frac{\partial}{\partial r}(3r^3 \cos 3\theta) = 9r^2 \cos 3\theta$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial}{\partial \theta}(3r^3 \sin 3\theta) = 9r^3 \cos 3\theta$$

$$\text{So, } \frac{\partial u}{\partial r} = \frac{1}{r} \cdot \frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = \frac{\partial}{\partial r}(3r^3 \sin 3\theta) = 9r^2 \sin 3\theta$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial}{\partial \theta}(3r^3 \cos 3\theta) = -9r^3 \sin 3\theta$$

$$\text{So, } \frac{\partial v}{\partial r} = -\frac{1}{r} \cdot \frac{\partial u}{\partial \theta}$$

(iii) We know,

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$

$$u_r = 9r^2 \cos 3\theta$$

$$v_r = 9r^2 \sin 3\theta$$

So, $f'(z) = e^{-i\theta}(9r^2\cos 3\theta + i\, 9r^2 \sin 3\theta)$

$$= e^{-i\theta}\, 9r^2(\cos 3\theta + \sin 3\theta)$$

$$= 9r^2 e^{-i\theta} e^{i3\theta}$$

$$= 9r^2 e^{i2\theta}$$

$$= 9(re^{i\theta})^2$$

$$= 9z^2 \quad \text{(Ans)}$$

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e) $f(z) = \frac{1}{z^9}$

(i) Given, $f(z) = \frac{1}{z^9}$

$\Rightarrow u + iv = \frac{1}{(re^{i\theta})^9}$

$\Rightarrow u + iv = \frac{1}{r^9} e^{-i9\theta}$

$\Rightarrow u + iv = \frac{1}{r^9} (\cos 9\theta - i\sin 9\theta)$

$\Rightarrow u + iv = \frac{1}{r^9} \cos 9\theta + (-i \frac{1}{r^9} \cdot \sin 9\theta)$

Comparing both sides,

Real part, $u = \frac{1}{r^9} \cos 9\theta$

Imaginary part, $v = -\frac{1}{r^9} \sin 9\theta$

(ii) We must show that,

$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$

$\frac{\partial v}{\partial r} = \frac{\partial}{\partial r} \left(\frac{1}{r^9} \cos 9\theta \right) = -9 \frac{1}{r^{10}} \cos 9\theta$

$\frac{\partial v}{\partial \theta} = \frac{\partial}{\partial \theta} \left(-\frac{1}{r^9} \sin 9\theta \right) = -9 \frac{1}{r^9} \cos 9\theta$

So, $\frac{\partial v}{\partial r} = \frac{\partial v}{\partial \theta} \frac{1}{r}$

$$\frac{\partial v}{\partial r} = \frac{\partial}{\partial r}\left(-\frac{1}{r^9} \sin 9\theta\right)$$

$$= 9 \frac{1}{r^{10}} \sin 9\theta$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\frac{1}{r^9} \cos 9\theta\right) = -9 \frac{1}{r^9} \sin 9\theta$$

So, $\dfrac{\partial v}{\partial r} = \dfrac{-1}{r} \cdot \dfrac{\partial v}{\partial \theta}$

(iii) We know,

$$f'(z) = e^{-i\theta}(u_r + i v_r)$$

$$u_r = -9 \frac{1}{r^{10}} \cos 9\theta$$

$$v_r = 9 \cdot \frac{1}{r^{10}} \sin 9\theta$$

So, $f'(z) = e^{-i\theta}(u_r + i v_r)$

$$= e^{-i\theta}\left(-9\frac{1}{r^{10}} \cos 9\theta + i \frac{9}{r^{10}} \sin 9\theta\right)$$

$$= e^{-i\theta} \cdot -\frac{9}{r^{10}} (\cos 9\theta - \sin 9\theta)$$

$$= e^{-i\theta} \cdot -\frac{9}{r^{10}} e^{-i9\theta}$$

$$= -\frac{9}{r^{10}} \cdot e^{-i10\theta}$$

$$= -9 \cdot \frac{1}{(re^{i\theta})^{10}} = -\frac{9}{z^{10}} \quad \text{(Ans)}$$

f) $f(z) = z^5$

(i) Given, $f(z) = z^5$

$\Rightarrow u + iv = (re^{i\theta})^5$

$\Rightarrow u + iv = r^5 e^{i5\theta}$

$\Rightarrow u + iv = r^5 (\cos 5\theta + i\sin 5\theta)$

$\Rightarrow u + iv = r^5 \cos 5\theta + i r^5 \sin 5\theta$

Comparing both sides,

Real part, $u = r^5 \cos 5\theta$

Imaginary part $v = r^5 \sin 5\theta$

(ii) We must show that,

$$\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \cdot \frac{1}{r} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial r} = \frac{\partial}{\partial r}(r^5 \cos 5\theta) = 5\cancel{\theta} r^5 \cos 5\theta$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial}{\partial \theta}(r^5 \sin 5\theta) = 5 r^5 \cos 5\theta$$

So, $\dfrac{\partial u}{\partial r} = \dfrac{1}{r} \cdot \dfrac{\partial v}{\partial \theta}$

$$\frac{\partial v}{\partial r} = \frac{\partial}{\partial r}(r^5 \sin 5\theta) = 5 r^4 \sin 5\theta$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial}{\partial \theta}(r^5 \cos 5\theta) = -5r^5 \sin 5\theta$$

$$So, \frac{\partial v}{\partial r} = -\frac{1}{r} \cdot \frac{\partial u}{\partial \theta}$$

(iii) We know,

$$f'(z) = e^{i\theta}(u_r + iv_r)$$

$$u_r = 5r^4 \cos 5\theta$$

$$v_r = 5r^4 \sin 5\theta$$

$$\text{So, } f'(z) = e^{-i\theta}(u_r + iv_r)$$

$$= e^{-i\theta}(5r^4 \cos 5\theta + 5ir^4 \sin 5\theta)$$

$$= e^{-i\theta} 5r^4 (\cos 5\theta + i \sin 5\theta)$$

$$= 5r^4 e^{i4\theta}$$

$$= 5(re^{i\theta})^4$$

$$= 5z^4 \text{ (Ans)}$$

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9. $f(z) = z^{-2/3}$

(I) Given, $f(z) = z^{-2/3}$

$\Rightarrow u + iv = (re^{i\theta})^{-2/3}$

$\Rightarrow u + iv = r^{-2/3} e^{i(-\frac{2}{3}\theta)}$

$\Rightarrow u + iv = r^{-2/3} (\cos 2/3\,\theta - i \sin \frac{2}{3}\theta)$

$\Rightarrow u + iv = r^{-2/3} \cos \frac{2}{3}\theta + i(-r^{-2/3} \sin \frac{2}{3}\theta)$

Comparing both sides,

Real part, $u = r^{-2/3} \cos \frac{2}{3}\theta$

Imaginary part, $v = -r^{-2/3} \sin \frac{2}{3}\theta$

(II) We must show that,

$\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \cdot \frac{1}{r}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r} \cdot \frac{\partial v}{\partial \theta}$

$\frac{\partial v}{\partial r} = \frac{\partial}{\partial r}\left(r^{-\frac{2}{3}} \cos \frac{2}{3}\theta\right) = -\frac{2}{3} r^{-5/3} \cos 2/3\,\theta$

$\frac{\partial v}{\partial \theta} = \frac{\partial}{\partial \theta}\left(-r^{-2/3} \sin \frac{2}{3}\theta\right) = -\frac{2}{3} r^{-2/3} \cos \frac{2}{3}\theta$

So, $\frac{\partial u}{\partial r} = \frac{1}{r} \cdot \frac{\partial v}{\partial \theta}$

$$\frac{\partial v}{\partial r} = \frac{\partial}{\partial r}\left(-\frac{1}{r^{2/3}} \sin \frac{2}{3}\theta\right)$$
$$= \frac{2}{3} \frac{1}{r^{5/3}} \sin \frac{2}{3}\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial}{\partial \theta}\left(r^{-2/3} \cos \frac{2}{3}\theta\right) = -\frac{2}{3} \cdot \frac{1}{r^{2/3}} \sin \frac{2}{3}\theta$$

So, $\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$

(iii) We know,

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$

$$u_r = -\frac{2}{3} r^{-5/3} \cos 2/3\,\theta$$

$$v_r = \frac{2}{3} r^{-5/3} \sin \frac{2}{3}\theta$$

So, $f'(z) = e^{-i\theta}(u_r + iv_r)$

$$= e^{-i\theta}\left(-\frac{2}{3} r^{-5/3} \cos \frac{2}{3}\theta + i \frac{2}{3} r^{-5/3} \sin \frac{2}{3}\theta\right)$$

$$= e^{-i\theta}\left(-\frac{2}{3} r^{-5/3}\right)\left(\cos 2/3\,\theta - \sin \frac{2}{3}\theta\right)$$

$$= e^{-i\theta}\left(-\frac{2}{3}r^{-5/3}\right) e^{-i\,2/3\,\theta}$$

$$= -\frac{2}{3}r^{-5/3}\,e^{-5/3\,\theta}$$

$$= -\frac{2}{3}\cdot\frac{1}{(re^{i\theta})^{5/3}}$$

$$= -\frac{2}{3}\cdot\frac{1}{z^{5/3}} \quad \text{(An)}$$

[Slide - 12]

1. (1) $w = 2z - (2+3i)$; $x=2, y=0, x=5$ and $y=4$

Given, $w = 2z - (2+3i)$

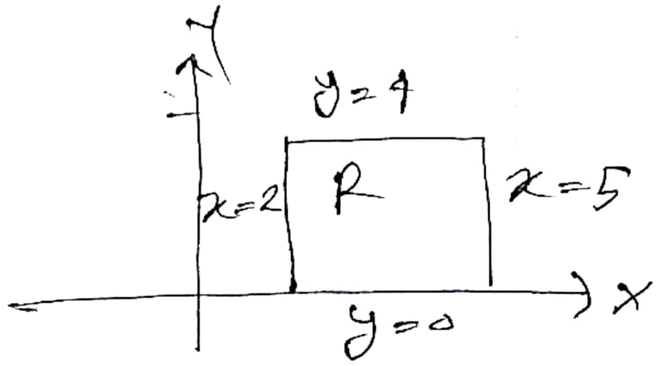
$\Rightarrow u + iv = 2(x+iy) - 2 - 3i$

$\Rightarrow u + iv = (2x-2) + i(2y-3)$

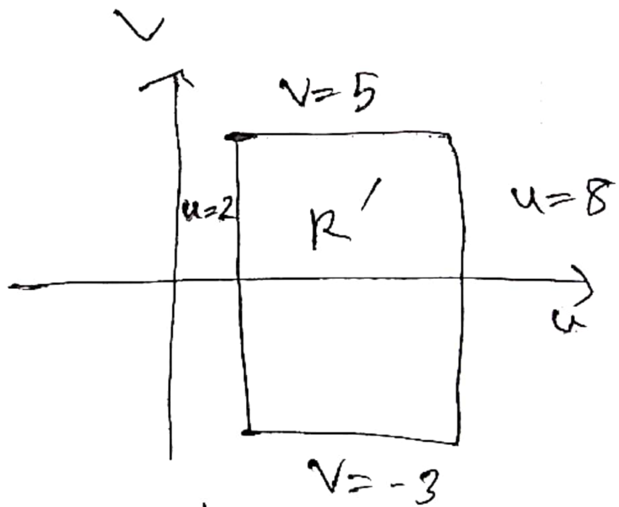
Hence, $u = 2x - 2$

and

$v = 2y - 3$



z plane



w plane.

when $x = 2 \Rightarrow u = 2$

$y = 0 \Rightarrow v = -3$

$x = 5 \Rightarrow u = 8$

$y = 4 \Rightarrow v = 5$

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(11) $W = \frac{1}{2} e^{\pi/2 \, iz} + 2i$

$f(z) = \frac{1}{2} e^{\pi i/2} z + 2i$

$\Rightarrow u + iv = \frac{1}{2} \left(\cos\frac{\pi}{2} + i\sin \pi/2\right) (x + iy) + 2i$

$\Rightarrow u + iv = \frac{1}{2} i(x + iy) + 2i$

$\Rightarrow u + iv = \frac{xi}{2} + \frac{i^2 y}{2} + 2i$

$\Rightarrow u + iv = -\frac{y}{2} + i\left(\frac{x}{2} + 2\right)$

So, $u = -\frac{y}{2}$

$v = \frac{x}{2} + 2$

When, $x = 2$, $y = 0$

(1) $u = -\frac{y}{2}$
$= 0$

$v = \frac{x}{2} + 2$
$= \frac{2}{2} + 2$
$= 3$

$(0, 3)$

(2) When,
$x = 4$, $y = 4$

$u = -\frac{y}{2}$
$= -\frac{4}{2}$
$= -2$

$v = \frac{x}{2} + 2$
$= \frac{2}{2} + 2$
$= 3$

$(-2, 3)$

(3) When,
$x = 4$, $y = 0$

$u = -\frac{y}{2}$
$= 0$

$v = \frac{x}{2} + 2$
$= \frac{4}{2} + 2$
$= 4$

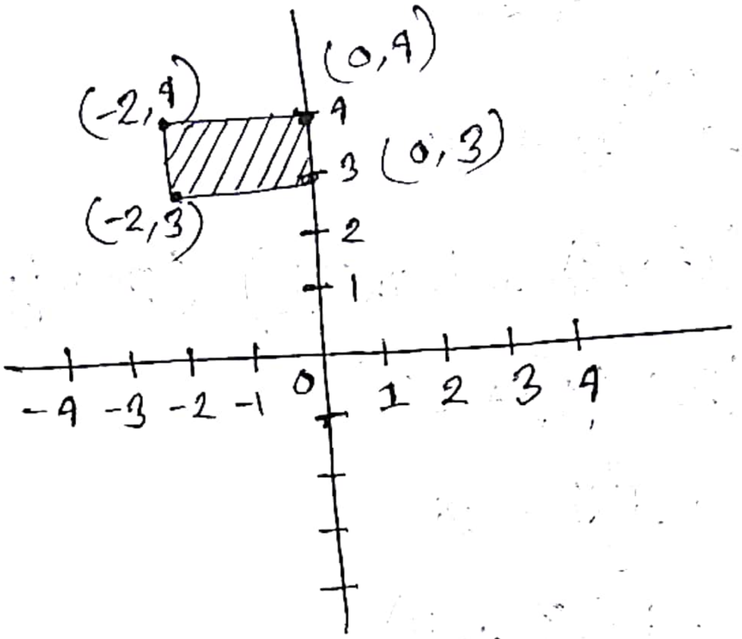
$(0, 4)$

(4) When,
$x = 4$, $y = 4$

$u = -\frac{y}{2}$
$= -2$

$v = \frac{x}{2} + 2$
$= 4$

$(-2, 4)$



(iii) $W = \sqrt{2} e^{\pi i/4} z - (1-i)$

$f(z) = \sqrt{2} e^{\pi i/4} z - (1-i)$

$\Rightarrow u + iv = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)(x+iy) - (1-i)$

$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\cdot\frac{1}{\sqrt{2}}\right)(x+iy) - (1-i)$

$= (1+i)(x+iy) - (1-i)$

$= x + ix + iy + i^2 y - 1 + i$

$= (x-y-1) + i(x+y+1)$

So, $u = x - y - 1$

$v = x + y + 1$

(1) when,

$x=2,\ y=0$

$u=x-y-1 = 2-0-1 = 1$ | $v=x+y+1 = 2+0+1 = 3$

$(1,3)$

(2) when,

$x=2,\ y=4$

$u=x-y-1 = 2-4-1 = -3$ | $v=x+y+1 = 2+4+1 = 7$

$(-3,7)$

(3) When,

$x=4,\ y=0$

$u=x-y-1 = 4-0-1 = 3$ | $v=x+y+1 = 4+0+1 = 5$

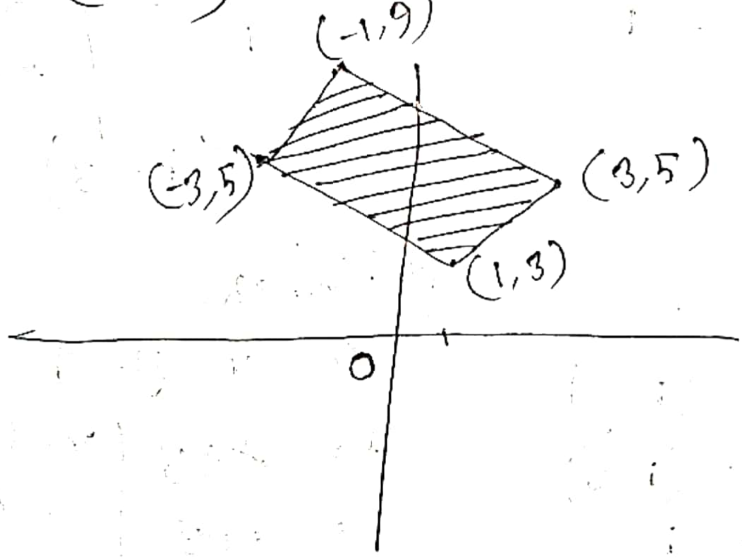
$(3,5)$

(4) when,

$x=4,\ y=4$

$u=x-y-1 = 4-4-1 = -1$ | $v=x+y+1 = 4+4+1 = 9$

$(-1,9)$



(iv) $W = e^{i\pi}z + 3 + i$

$f(z) = (\cos\pi + i\sin\pi)(x+iy) + 3 + i$

$\Rightarrow u + iv = (-1)(x+iy) + 3 + i$

$= -x - iy + 3 + i$

$= (-x+3) + i(1+y)$

So, $u = (-x+3)$

$v = (1-y)$

(1) When,

$x = 2,\ y = 0$

$u = -x+3$	$v = (1-y)$
$= -2+3$	$= (1-0)$
$= 1$	$= 1$

$(1,1)$

(2) When

$x = 2,\ y = 4$

$u = -x+3$	$v = (1-y)$
$= -2+3$	$= (1-4)$
$= 1$	$= -3$

$(1,-3)$

(3) When,

$x = 4,\ y = 0$

$u = -x+3$	$v = (1-y)$
$= -4+3$	$= (1-0)$
$= -1$	$= 1$

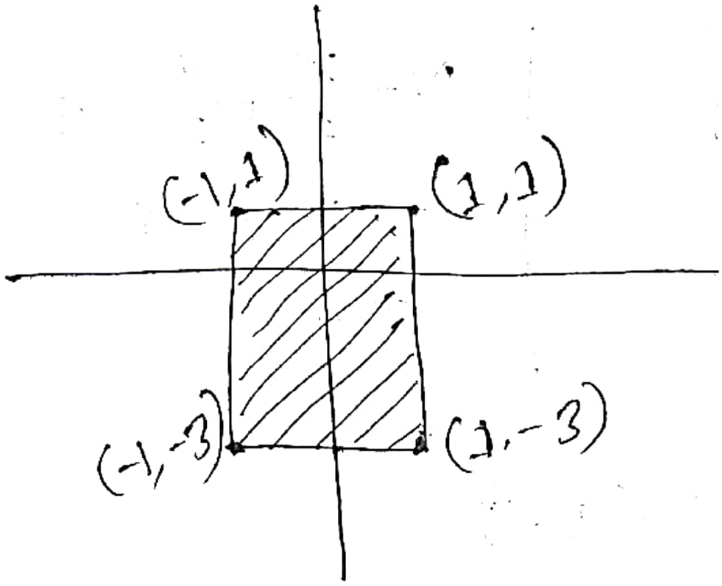
$(-1,1)$

(4) When,

$x = 4,\ y = 4$

$u = -x+3$	$v = (1-y)$
$= -4+3$	$= (1-4)$
$= -1$	$= 3$

$(-1,3)$



(v) $W = \frac{1}{\sqrt{2}} e^{\frac{\pi i}{4}} z + 1 - 3i$

$f(z) = \frac{1}{\sqrt{2}} (\cos \pi/4 + i \sin \pi/4)(x+iy) + 1 - 3i$

$u + iv = \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right)(x+iy) + 1 - 3i$

$= \frac{1}{2}(1+i)(x+iy) + 1 - 3i$

$= \frac{1}{2}(x + ix + iy - y) + 1 - 3i$

$= \left(x/2 - \frac{y}{2} + 1 \right) + i \left(\frac{x}{2} + \frac{y}{2} - 3 \right)$

So, $u = \frac{x}{2} - \frac{y}{2} + 1$

$v = \frac{x}{2} + \frac{y}{2} - 3$

(1) When, $x = 2, y = 0$

$u = \frac{x}{2} - \frac{y}{2} + 1$
$= 1 - 0 + 1$
$= 2$

$v = \frac{x}{2} + \frac{y}{2} - 3$
$= 1 + 0 - 3$
$= -2$

$(2, -2)$

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(2) When,

$x=2,\ y=4$

$u=\frac{x}{2}-\frac{y}{2}+1$

$=1-2+1$

$=0$

$v=\frac{x}{2}+\frac{y}{2}-3$

$=1+2-3$

$=0$

$(0,0)$

(3) When,

$x=4,\ y=0$

$u=\frac{x}{2}-\frac{y}{2}+1$

$=2-0+1$

$=3$

$v=\frac{x}{2}+\frac{y}{2}-3$

$=2+0-3$

$=-1$

$(3,-1)$

(4) When,

$x=4,\ y=4$

$u=\frac{x}{2}-\frac{y}{2}+1$

$=2-2+1$

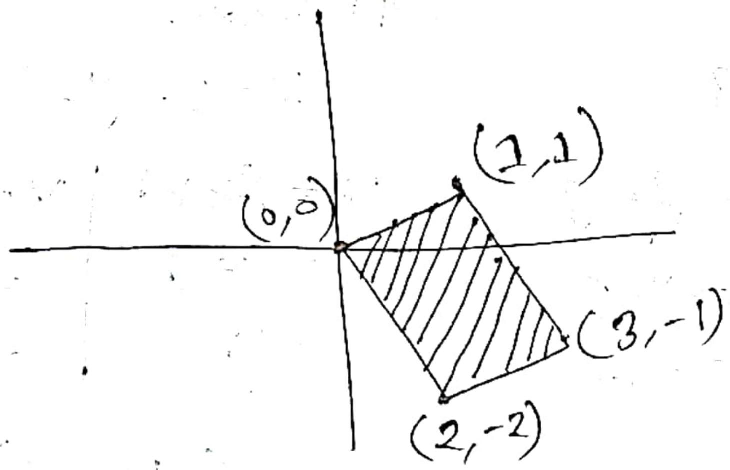
$=1$

$v=\frac{x}{2}+\frac{y}{2}-3$

$=2+2-3$

$=1$

$(1,1)$



2. ① $W = 3z + 1 - 3i$

$f(z) = 3z + 1 - 3i$

$\Rightarrow u + iv = 3(x+iy) + 1 - 3i$

$= 3x + 3iy + 1 - 3i$

$= (3x+1) + i(3y-3)$

$u = 3x+1$

$v = 3y-3$

① When,

$x=1, y=0$

$u = 3x+1 = 3+1 = 4$

$v = 3y-3 = 0-3 = -3$

$(4, -3)$

② When

$x_1 = 1, y = 0$

$u = 3x+1 = 3+1 = 4$

$v = 3y-3 = 0-3 = -3$

$(4, -3)$

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(3) When,

$x = 1, y = -3$

$u = 3x + 1$
$= 3 + 1$
$= 4$

$v = 3y - 3$
$= -9 - 3$
$= -12$

$(4, -12)$

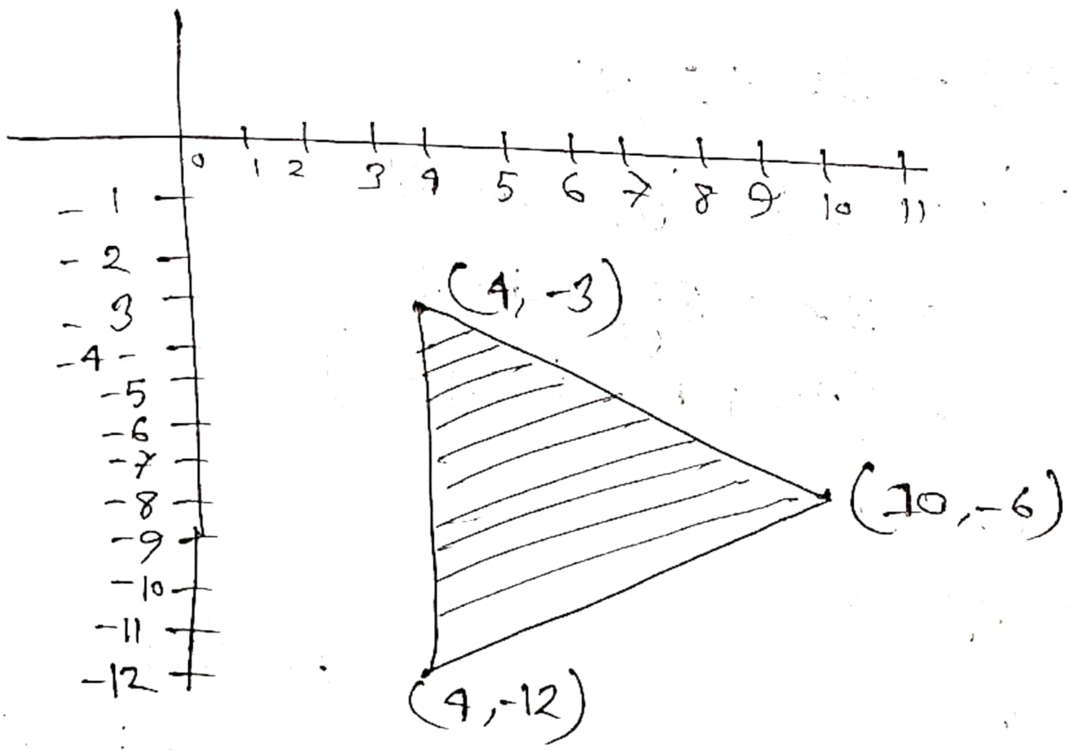
(4) When,

$x = 3, y = -1$

$u = 3x + 1$
$= 9 + 1$
$= 10$

$v = 3y - 3$
$= -3 - 3$
$= -6$

$(10, -6)$



(11) $W = iz + 3 + 2i$

$f(z) = i(x+iy) + 3 + 2i$

$\Rightarrow u + iv = ix + i^2 y + 3 + 2i$

$= ix - y + 3 + 2i$

$= (-y+3) + i(2+x)$

So, $u = -y+3$

$v = 2+x$

(1) When,

$x = 1, y = 0,$

$u = -y + 3$
$= 0+3$
$= 3$

$v = 2+x$
$= 2+1$
$= 3$

$(3,3)$

(2) When,

$x = 1, y = -3$

$u = -y+3$
$= 3+3$
$= 6$

$v = 2+x$
$= 2+1$
$= 3$

$(6,3)$

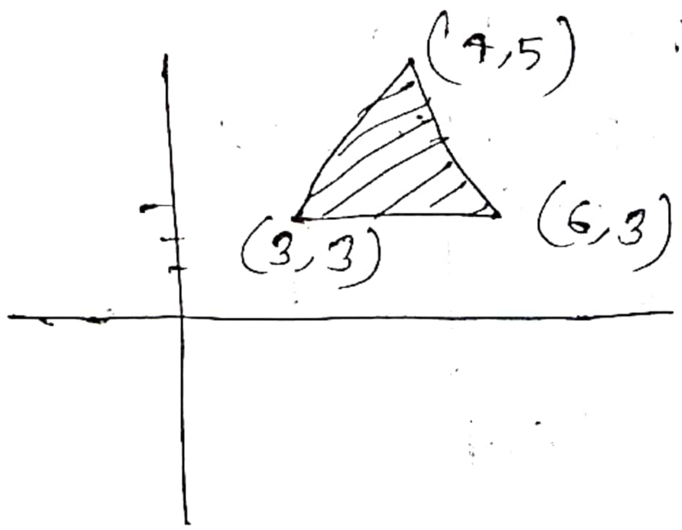
(3) When,

$x = 3, y = -1$

$u = -y+3$
$= 1+3$
$= 4$

$v = 2+x$
$= 2+x$
$= 2+3$
$= 5$

$(4,5)$



(iii) $W = (1+2i)z - i$

$f(z) = (1+2i)(x+iy) - i$

$\Rightarrow u + iv = x + 2ix + iy + 2i^2y - i$

$= x + 2ix + iy - 2y - i$

$= (x-2y) + i(2x + y + y - 1)$

So, $u = x - 2y$, $v = 2x + y - 1$

When,

$x = 1, y = 0$

$u = x - 2y$
$= 1 - 0$
$= 1$

$v = 2x + y - 1$
$= 2 + 0 - 1$
$= 1$

$(1, 1)$

When,

$x = 1, y = 3$

$u = x - 2y$
$= 1 + 6$
$= 7$

$v = 2x + y$
$= 2 - 6 - 1$
$= -5$

$(7, -5)$

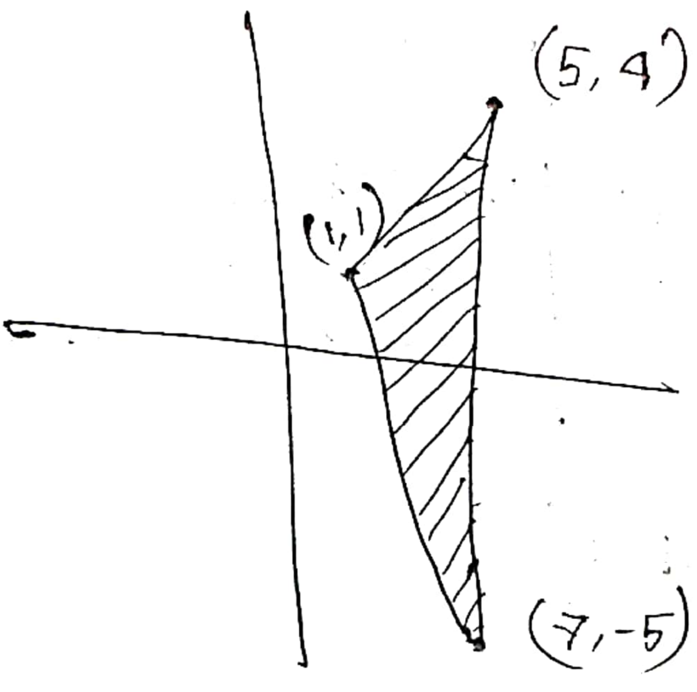
When,

$x = 3, y = -1$

$u = x - 2y$
$= 3 + 2$
$= 5$

$v = 2x + y - 1$
$= 6 - 1 - 1$
$= 4$

$(5, 4)$



(iv) $W = \frac{1}{2} e^{\pi i/2} z - 4$

$f(z) = \frac{1}{2} e^{\pi i/2} z - 4$

$\Rightarrow u + iv = \frac{1}{2} \left(\cos\frac{\pi}{2} + i \sin \pi/2\right)(x+iy) - 4$

$= \frac{1}{2}(i)(x+iy) - 4$

$= \frac{1}{2}(ix - i^2 y) - 4$

$= \frac{1}{2}(ix - y) - 4$

$= \frac{ix}{2} - \frac{y}{2} - 4$

So, $u = \left(-\frac{y}{2} - 4\right)$, $v = \frac{x}{2}$

when,

$x = 1, y = 0$

$u = \left(-\frac{y}{2} - 4\right)$ | $v = \frac{x}{2}$

$= -4$ | $= \frac{1}{2}$

$\left(-4, \frac{1}{2}\right)$

When, $x = 1$, $y = 3$

$u = \left(-\frac{y}{2} - 4\right)$ | $v = \frac{x}{2}$

$= (3/2 - 4)$ | $= \frac{1}{2}$

$= \frac{5}{2}$

$\left(5/2, \frac{1}{2}\right)$

when,

$x = 3, y = -1$

$u = -\frac{y}{2} - 4$ | $v = \frac{x}{2}$

$= \frac{1}{2} - 4$ | $= \frac{3}{2}$

$= -\frac{7}{2}$

$\left(-\frac{7}{2}, \frac{3}{2}\right)$