Problem: If $z_1 = 1 - i\,, z_2 = -2 - 3i$ and $z_3 = 2i$, then evaluate $|3{z_1}^2 + z_1\overline{z_2} - 5z_3|$.

Solution: $|3{z_1}^2 + z_1\overline{z_2} - 5z_3| = |3(1-i)^2 + (1-i)(-2+3i) - 5*2i|$

$$= |3(1-2i-1) + (-2+3i+2i+3) - 10i|$$

$$= |-6i + 1 + 5i - 10i| = |1 - 11i| = \sqrt{(1)^2 + (-11)^2} = \sqrt{122}\,.$$

Problem: Evaluate $\left|\frac{(1-i)^2}{1+i}\right|$.

Solution: $\left|\frac{(1-i)^2}{1+i}\right| = \left|\frac{1-2i-1}{1+i}\right| = \left|\frac{-2i}{1+i}\right| = \frac{\sqrt{(-2)^2}}{\sqrt{1+1}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$

Problem: Evaluate $\mathrm{Re}\left\{\frac{(1-i)^2}{1+i}\right\}$.

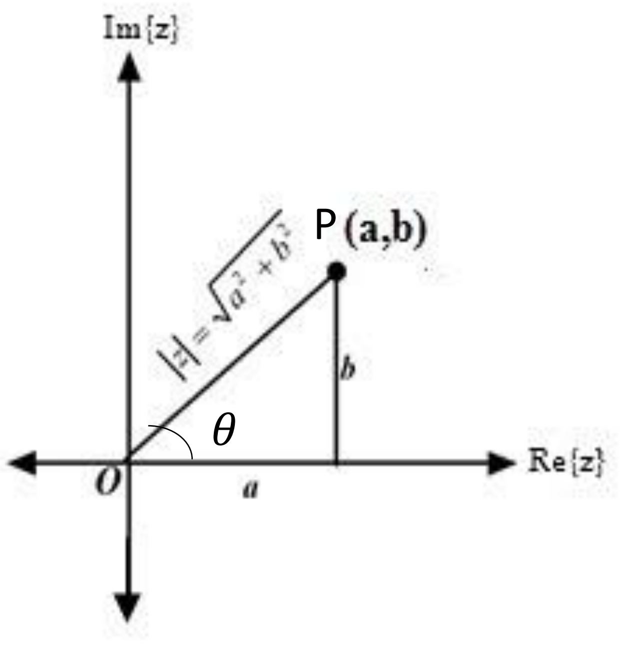
Solution: $\mathrm{Re}\left\{\frac{(1-i)^2}{1+i}\right\} = \mathrm{Re}\left\{\frac{-2i}{1+i}.\frac{1-i}{1-i}\right\} = \mathrm{Re}\left\{\frac{-2i+2i^2}{1^2+1^2}\right\}$

$$= \mathrm{Re}\left\{\frac{-2-2i}{2}\right\} = \mathrm{Re}\{-1-i\} = -1.$$

Polar form of Complex Number and Argument:

- If P a point in the complex plane corresponding to the complex number (a, b) or, $a + ib$ then,

  $a = r cos\theta$ and $b = r sin\theta$.



Where $r = |z| = \sqrt{a^2 + b^2}$ is the distance of $(a, b)$from the origin,

And θ is called the **amplitude or argument** of z which is the angle with real axis denoted by $arg\{z\}$.

Hence, we can write z in polar form as:

$$z = rcos\theta + irsin\theta$$

$$= r(cos\theta + isin\theta)$$

$$\therefore z = re^{i\theta} \text{[Euler Formulae]}.$$

Principal argument: The **principal value** of the argument (sometimes called the **principal argument**) is the unique value of the argument that is in the range $0 \leq \theta < 2\pi$ or, $-\pi < \theta \leq \pi$ and is denoted by Arg z.

$$\arg z = \operatorname{Arg} z + 2n\pi, \qquad (n = 0, \pm 1, \pm 2, \dots)$$

Some important properties of argument:

- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$
- $\arg(z^n) = n \arg(z)$.

Transformation of complex numbers:

Polar to rectangular: $z = re^{i\theta} \longrightarrow z = a + ib$

$$a = r cos\theta \text{ and } b = r sin\theta$$

Rectangular to Polar: $z = a + ib \longrightarrow z = re^{i\theta}$

$$r = \sqrt{a^2 + b^2} \text{ and}$$

$$\theta = \begin{cases} tan^{-1}\left(\frac{b}{a}\right); & if\ a > 0, b \geq 0\ (1st\ quadrant\ and\ on + ve\ real\ axis) \\ tan^{-1}\left(\frac{b}{a}\right) + \pi; & if\ a < 0\ (2nd, 3rd\ quadrant \text{ and } on - ve\ real\ axis) \\ tan^{-1}\left(\frac{b}{a}\right) + 2\pi; & if\ a > 0, b < 0\ (4th\ quadrant) \\ \frac{\pi}{2}; & if\ a = 0, b > 0\ (on + ve\ imaginary\ axis) \\ \frac{3\pi}{2}; & if\ a = 0, b < 0\ (on - ve\ imaginary\ axis) \\ Undefined; & if\ a = 0, b = 0 \end{cases}.$$

Problem: Find the rectangular form of $z = \sqrt{2}e^{i\frac{\pi}{4}}$.

Solution: Here $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$

We know that, $a = r\cos\theta = \sqrt{2}\cos\frac{\pi}{4} = 1$

And $b = r\sin\theta = \sqrt{2}\sin\frac{\pi}{4} = 1$

Hence $z = a + ib = 1 + i$.

Problem: Find the rectangular form of $z = 2e^{i\frac{5\pi}{6}}$.

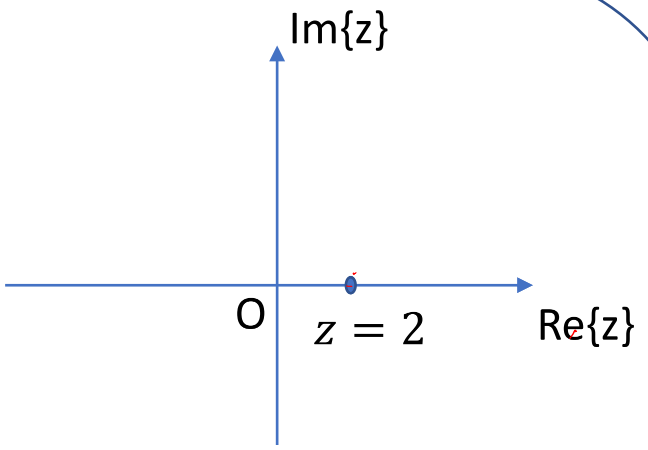
Solution: Here $r = 2$ and $\theta = \frac{5\pi}{6}$

We know that, $a = r\cos\theta = 2\cos(\frac{5\pi}{6}) = -2\frac{\sqrt{3}}{2} = -\sqrt{3}$

And $b = r\sin\theta = 2\sin(\frac{5\pi}{6}) = 2\frac{1}{2} = 1$

Hence $z = a + ib = -\sqrt{3} + i$.

Problem: Find the polar form of $z = 2$.



Solution: Here $a = 2$ and $b = 0$

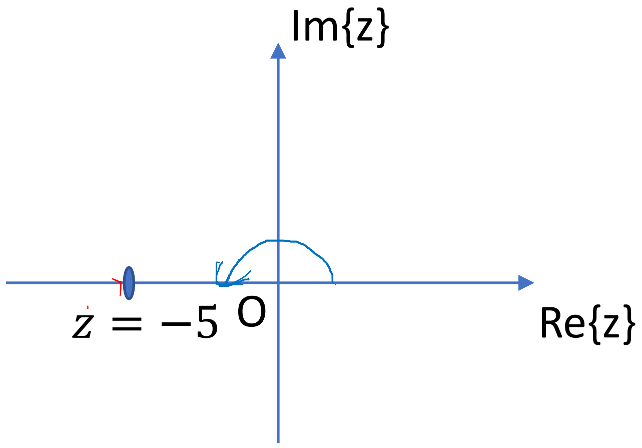
We know that, $r = \sqrt{a^2 + b^2} = \sqrt{2^2} = 2$

And $\theta = 0$

Hence, $z = re^{i\theta} = 2$.

Exercise: $z = 3, \frac{5}{2}, \frac{1}{3}$.

Problem: Find the polar form of $z = -5$.



Solution: Here $a = -5$ and $b = 0$

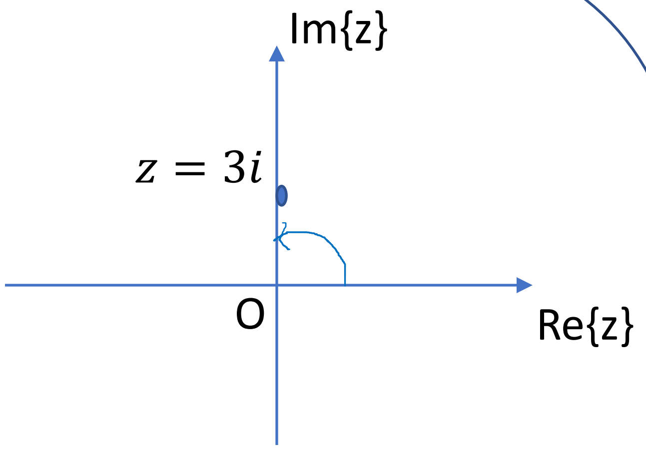
We know that, $r = \sqrt{a^2 + b^2} = \sqrt{(-5)^2} = 5$

And $\theta = \pi$

Hence, $z = re^{i\theta} = 5e^{i\pi}$.

Exercise: $z = -2, -\frac{3}{2}, -\frac{1}{4}$.

Problem: Find the polar form of $z = 3i$.



Solution: Here $a = 0$ and $b = 3$

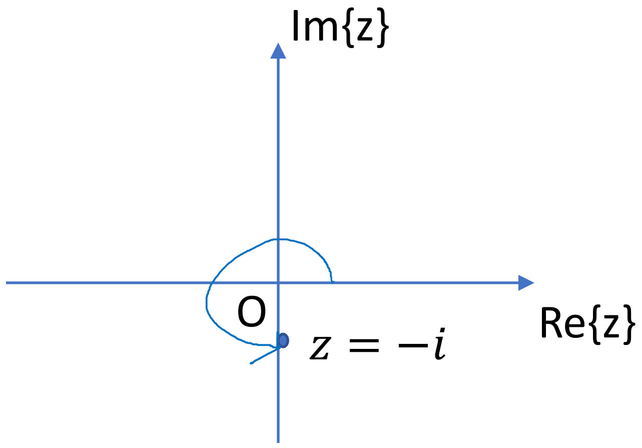
We know that, $r = \sqrt{a^2 + b^2} = \sqrt{3^2} = 3$

And $\theta = \frac{\pi}{2}$

Hence, $z = re^{i\theta} = 3e^{i\frac{\pi}{2}}$.

Exercise: $z = i, \frac{5}{2}i, \frac{2}{3}i$.

Problem: Find the polar form of $z = -i$.



Solution: Here $a = 0$ and $b = -1$

We know that, $r = \sqrt{a^2 + b^2} = \sqrt{(-1)^2} = 1$

And $\theta = \frac{3\pi}{2}$

Hence, $z = re^{i\theta} = e^{i\frac{3\pi}{2}}$.

Exercise: $z = -3i, -\frac{3}{2}i$.

Problem: Find the polar form of $z = \sqrt{3} + 3i$.

Solution: Here $a = \sqrt{3}$ and $b = 3$

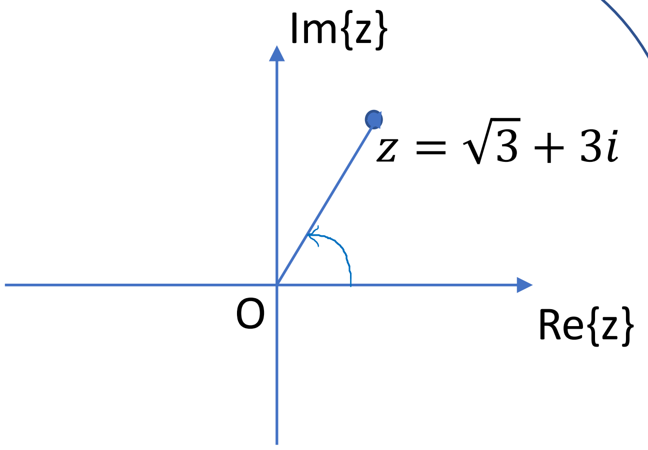
We know that, $r = \sqrt{a^2 + b^2}$

$$= \sqrt{\left(\sqrt{3}\right)^2 + 3^2} = 2\sqrt{3}$$

And $\theta = tan^{-1}\left(\frac{b}{a}\right) = tan^{-1}\left(\frac{3}{\sqrt{3}}\right)$

$$= tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$$

Hence, $z = re^{i\theta} = 2\sqrt{3}e^{i\frac{\pi}{3}}$.



Exercise: $z = 6 + 2\sqrt{3}i, 1 + i$.

Problem: Find the polar form of $z = -1 + i$.

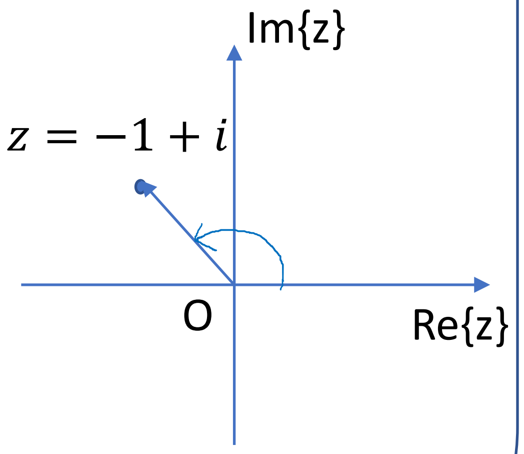
Solution: Here $a = -1$ and $b = 1$

We know that, $r = \sqrt{a^2 + b^2} = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

And $\theta = tan^{-1}\left(\frac{b}{a}\right) + \pi = tan^{-1}\left(\frac{1}{-1}\right) + \pi$

$$= -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$$

Hence, $z = re^{i\theta} = \sqrt{2}e^{i\frac{3\pi}{4}}$.



Exercise: $z = -1 + \sqrt{3}i$, $z = -2 + 2\sqrt{3}i$.

Problem: Find the polar form of $z = -2\sqrt{3} - 6i$.

Solution: Here $a = -2\sqrt{3}$ and $b = -6$

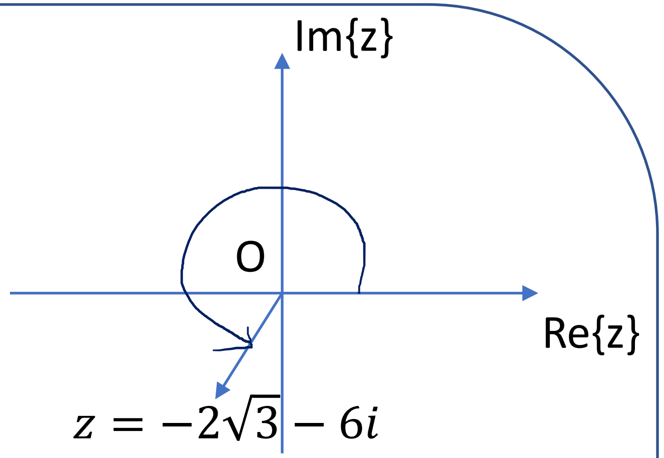
We know that, $r = \sqrt{a^2 + b^2}$

$$= \sqrt{\left(-2\sqrt{3}\right)^2 + (-6)^2} = 4\sqrt{3}$$

And $\theta = tan^{-1}\left(\frac{b}{a}\right) + \pi = tan^{-1}\left(\frac{-6}{-2\sqrt{3}}\right) + \pi$

$$= tan^{-1}\left(\sqrt{3}\right) + \pi = \frac{\pi}{3} + \pi = \frac{4\pi}{3}$$

Hence, $z = re^{i\theta} = 4\sqrt{3}e^{i\frac{4\pi}{3}}$.



Exercise: $z = -1 - i$, $z = -3 - \sqrt{3}i$.

Problem: Find the polar form of $z = \sqrt{3} - i$.

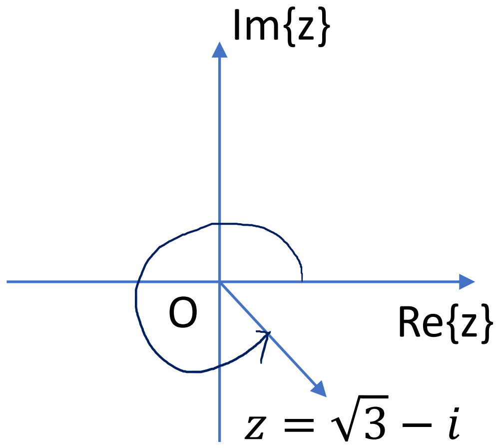
Solution: Here $a = \sqrt{3}$ and $b = -1$

We know that, $r = \sqrt{a^2 + b^2} = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$

And $\theta = tan^{-1}\left(\frac{b}{a}\right) + 2\pi = tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) + 2\pi$

$$= -\frac{\pi}{6} + 2\pi = \frac{11\pi}{6}.$$

Hence, $z = re^{i\theta} = 2e^{i\frac{11\pi}{6}}$.



Exercise: $z = 1 - \sqrt{3}i$, $z = 2 - 2\sqrt{3}i$.

Problem: Find the polar form of $z = \left(\frac{1-i}{1+i}\right)^{18}$.

Solution: Let $z_1 = 1 - i$ and $z_2 = 1 + i$.

For z_1: $a = 1$ and $b = -1$

$\therefore r = \sqrt{(1)^2+(-1)^2} = \sqrt{2}$

and $\theta = tan^{-1}\left(\frac{-1}{1}\right) + 2\,\pi = -\frac{\pi}{4} + 2\pi = \frac{7\pi}{4}$

Hence, $z_1 = \sqrt{2}e^{i\frac{7\pi}{4}}$.

Similarly, For z_2: $a = 1$ and $b = 1$

$\therefore r = \sqrt{(1)^2+(1)^2} = \sqrt{2}$

and $\theta = tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

Hence, $z_2 = \sqrt{2}e^{i\frac{\pi}{4}}$.

So, $z = \left(\frac{1-i}{1+i}\right)^{18} = \left(\frac{\sqrt{2}e^{i\frac{7\pi}{4}}}{\sqrt{2}e^{i\frac{\pi}{4}}}\right)^{18}$

$= \frac{e^{i\frac{63\pi}{2}}}{e^{i\frac{9\pi}{2}}} = e^{i\frac{63\pi}{2} - i\frac{9\pi}{2}} = e^{i\frac{54\pi}{2}} = e^{i27\pi}$.

Problem: Find the polar form of $z = \left(\sqrt{3} - i\right)^2\left(-1 + \sqrt{3}i\right)^5$.

Solution: Let $z_1 = \sqrt{3} - i$ and $z_2 = -1 + \sqrt{3}i$.

For z_1: $a = \sqrt{3}$ and $b = -1$

$\therefore r = \sqrt{(\sqrt{3})^2+(-1)^2} = 2$

and $\theta = tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) + 2\,\pi = -\frac{\pi}{6} + 2\pi = \frac{11\pi}{6}$

Hence, $z_1 = 2e^{i\frac{11\pi}{6}}$.

Similarly, For z_2: $a = -1$ and $b = \sqrt{3}$

$\therefore r = \sqrt{(-1)^2+(\sqrt{3})^2} = 2$

and $\theta = tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) + \pi = -\frac{\pi}{3} + \pi = \frac{2\pi}{3}$

Hence, $z_2 = 2e^{i\frac{2\pi}{3}}$.

So, $z = (1 - i)^2(1 + i)^5 = \left(2e^{i\frac{11\pi}{6}}\right)^2\left(2e^{i\frac{2\pi}{3}}\right)^5$

$= 2^7\, e^{i\frac{11\pi}{3}}e^{i\frac{10\pi}{3}} = 2^7 e^{i\left(\frac{11\pi}{3}+\frac{10\pi}{3}\right)} = 128e^{i7\pi}$.

Finding principal argument (Arg z) from general argument ($\arg z$):

- If $\arg z$ is in $0 \le \theta < 2\pi$ then $\arg z = \operatorname{Arg} z$.

Example: For $z = 3e^{i\frac{\pi}{6}}$, $\arg z = \frac{\pi}{6} = \operatorname{Arg}$ z.

- If $\arg z \ge 2\pi$ then subtract maximum no. of $2n\pi$

Example: For z $= 15e^{i\frac{15\pi}{2}}$, $\arg z = \frac{15\pi}{2} = 7.5\pi$

So, $\operatorname{Arg} z = 7.5\pi - 6\pi = 1.5\pi = \frac{3\pi}{2}$.

- If $\arg z < 0$ then add minimum no. of $2n\pi$

Example: For z $= 2e^{-i5\pi}$, $\arg z = -5\pi$

So, $\operatorname{Arg} z = -5\pi + 6\pi = \pi$.

Problem: Find the principal argument of $z = (2+2i)^4$.

Solution: Let $z_1 = 2+2i$; $a = 2, b = 2$;

$\therefore r = \sqrt{4+4} = 2\sqrt{2}$ and $\theta = tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}$

Hence, $z = (2+2i)^4 = \left(2\sqrt{2}e^{i\frac{\pi}{4}}\right)^4 = 64\, e^{i\pi}$.

$\therefore \arg z = \pi$; $\quad \therefore$Principal argument:$Arg\ z = \pi$.

Problem: Find the principal argument of $z = \left(\frac{2\sqrt{3}+2i}{1-\sqrt{3}i}\right)^6$.

Solution: Let $z_1 = 2\sqrt{3}+2i$ and $z_2 = 1-\sqrt{3}i$.

For z_1: $a = 2\sqrt{3}$ and $b = 2$

$\therefore r = \sqrt{(2\sqrt{3})^2+(2)^2} = 4$ and $\theta = tan^{-1}\left(\frac{2}{2\sqrt{3}}\right) = \frac{\pi}{6} \therefore z_1 = 4e^{i\frac{\pi}{6}}$.

Similarly, For z_2: $a = 1$ and $b = -\sqrt{3}$

$\therefore r = \sqrt{(1)^2+(-\sqrt{3})^2} = 2$ and $\theta = tan^{-1}\left(-\frac{\sqrt{3}}{1}\right) + 2\pi = \frac{5\pi}{3} \therefore z_2 = 2e^{i\frac{5\pi}{3}}$

So,$z = \left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^6 = \left(\frac{4e^{i\frac{\pi}{6}}}{2e^{i\frac{5\pi}{3}}}\right)^6 = 2^6\frac{e^{i\pi}}{e^{i10\pi}} = 2^6 e^{i\pi-i10\pi} = 2^6e^{-i9\pi}$.

$\therefore \arg z = -9\pi$; $\quad \therefore$Principal argument:$Arg\ z = -9\pi+10\pi = \pi$.

Problem: Find the principal argument of $z = \left(\frac{1-i}{1+i}\right)^{18}$.

Solution: Let $z_1 = 1 - i$ and $z_2 = 1 + i$.

For z_1: $a = 1$ and $b = -1$

$\therefore r = \sqrt{(1)^2+(-1)^2} = \sqrt{2}$ and $\theta = tan^{-1}\left(\frac{-1}{1}\right) + 2\,\pi = \frac{7\pi}{4}$

Hence, $z_1 = \sqrt{2}e^{i\frac{7\pi}{4}}$.

Similarly,

For z_2: $a = 1$ and $b = 1$

$\therefore r = \sqrt{(1)^2+(1)^2} = \sqrt{2}$ and $\theta = tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

Hence, $z_2 = \sqrt{2}e^{i\frac{\pi}{4}}$.

So, $z = \left(\frac{1-i}{1+i}\right)^{18} = \left(\frac{\sqrt{2}e^{i\frac{7\pi}{4}}}{\sqrt{2}e^{i\frac{\pi}{4}}}\right)^{18} = \frac{e^{i\frac{63\pi}{2}}}{e^{i\frac{9\pi}{2}}}$

$= e^{i\frac{63\pi}{2} - i\frac{9\pi}{2}} = e^{i\frac{54\pi}{2}} = e^{i27\pi}$.

$\therefore \arg z = 27\pi$; $\quad \therefore$Principal argument: $Arg\ z = 27\pi - 26\pi = \pi$.

Exercise:

1. Express $\frac{(1+i)^2}{1-i}$ in terms of $a + ib$.
2. Evaluate each of the followings:

(a) $\mathrm{Re}\left\{\frac{1+\sqrt{3}\,i}{1-i}\right\}$, (b) $\left|\frac{z}{\bar{z}}\right|$, (c)$\mathrm{Im}\left\{\frac{z}{\bar{z}}\right\}$.

Hints for (b) and (c): Let $z = a + ib$.

3. Convert the following numbers into polar form:

(a) $z = -1 + i$, (b) $z = -3 - \sqrt{3}i$, (c) $z = \frac{(1-i)^2}{1+i}$.

4. Convert the following numbers into rectangular form:

$z = \sqrt{3}\, e^{i\,\frac{\pi}{3}}$ and $z = 2e^{i\,\frac{\pi}{4}}$.

5. Find the principal argument of the followings:

(a) $z = (-1 - i)^4$, (b) $z = \left(-2 + 2\sqrt{3}\, i\right)^3$, $z = \frac{(1+i)^3}{(1-i)}$.

MULTIPLE CHOICE QUESTION

1. The standard form of the complex number $\dfrac{(1+i)^2}{1-i}$ is

   (a) $-1-i$ (b) $-1+i$ (c) $1+i$ (d) None

2. The polar form of the complex number $-1+i$ is

   (a) $\sqrt{2}e^{i\frac{5\pi}{4}}$ (b) $-\sqrt{2}e^{i\frac{5\pi}{4}}$ (c) $\sqrt{2}e^{i\frac{3\pi}{4}}$ (d) None

3. The standard rectangular form of the complex number $z=\sqrt{3}e^{i\frac{\pi}{2}}$ is

   (a) $-\sqrt{3}+i$ (b) $-\sqrt{3}+\sqrt{3}i$ (c) $\sqrt{3}i$ (d) None

4. The principal argument of the complex number $\left(-1+i\right)^4$ is

   (a) π (b) 2π (c) 3π (d) None