Complex Differentiation and The Cauchy-Riemann Equation

Lecture-11

This Lecture Covers-

1. Definition of Analytic function

2. Necessary condition for a function to be analytic.

3. Cauchy-Riemann equations.

4. Some Examples and Exercises based on discussion in this lecture.

5. Sample Multiple Choice Questions (MCQs)

Euler's Formulae on complex number:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

And $$e^{-i\theta} = \cos\theta - i\sin\theta$$

Some Important Formulae:

$$\sin at = \frac{e^{iat} - e^{-iat}}{2i}$$

$$\cos at = \frac{e^{iat} + e^{-iat}}{2}$$

$$\sinh at = \frac{\mathrm{e}^{\mathrm{at}} - \mathrm{e}^{-\mathrm{at}}}{2}$$

$$\cosh at = \frac{\mathrm{e}^{\mathrm{at}} + \mathrm{e}^{-\mathrm{iat}}}{2}$$

$$\cos iay = \cosh ay;$$

$$-i\sin iay = \sinh ay$$

If a single valued function $f(z)$ is differentiable that is $f'(z)$ exists at every point of domain D except possibly at a finite number of exceptional points, then the function is said to be **analytic** in the domain D. These exceptional point at which $\mathrm{f}'(z)$ does not exist are called **singular points** or **singularities of the function.**

$\boldsymbol{f'(z) = u_x + iv_x}$

Necessary Condition for $\boldsymbol{f(z)}$ to be Analytic:

If $z = x + iy$ and
$f(z) = u(x, y) + iv(x, y)$
Satisfies the Cauchy-Riemann Equation (Rectangular & Polar form) , then $f(z)$ is said to be **analytic**

$\boldsymbol{f'(z) = e^{-i\theta}(u_r + iv_r)}$

Examples

For the function $f(z) = z^2$

(a) Separate real and imaginary parts
(b) Verify **C-R** equations.
(c) find $f'(z)$ using $u_x + i\, v_x$

Given, $f(z) = z^2$

$\Rightarrow u + iv = (x + iy)^2$
$\Rightarrow u + iv = x^2 + 2.x.iy + (iy)^2$
$\Rightarrow u + iv = x^2 + i.2xy + i^2y^2$
$\Rightarrow u + iv = x^2 + i.2xy - y^2 \quad [\because i^2 = -1]$
$\Rightarrow u + iv = (x^2 - y^2) + i.(2xy)$

Comparing both sides
Real part, $u = x^2 - y^2$
Imaginary part, $v = 2xy$

We must show that
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2 - y^2) = 2x$$
$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(2xy) = 2x$$
$$\text{So, } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2 - y^2) = -2y$$
$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(2xy) = 2y$$
$$\text{So, } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

We know,
$$f'(z) = u_x + iv_x$$

$$u_x = \frac{\partial u}{\partial x} = 2x$$
$$v_x = \frac{\partial v}{\partial x} = 2y$$
So, $f'(z) = u_x + iv_x$
$= 2x + i.2y$
$= 2(x + iy)$
$= 2z$

Examples

For the function $f(z) = z + e^{2z}$

(a) Separate real and imaginary parts
(b) Verify **C-R** equations.
(c) find $f'(z)$ using $u_x + i\, v_x$

Given, $f(z) = z + e^{2z}$

$\Rightarrow u + iv = (x + iy) + e^{2(x+iy)}$

$\Rightarrow u + iv = (x + iy) + e^{2x} e^{2iy}$

$\Rightarrow u + iv = (x + iy) + e^{2x}(\cos 2y + i \sin 2y)$

$\Rightarrow u + iv = (x + e^{2x} \cos 2y) + i(y + e^{2x} \sin 2y)$

Comparing both sides

Real part, $u = x + e^{2x} \cos 2y$

Imaginary part, $v = y + e^{2x} \sin 2y$

We must show that

$$u_x = v_y \text{ and } u_y = -v_x$$

$$u_x = \frac{\partial}{\partial x}(x + e^{2x} \cos 2y)$$
$$= 1 + 2e^{2x} \cos 2y$$
$$v_y = \frac{\partial}{\partial y}(y + e^{2x} \sin 2y)$$
$$= 1 + 2e^{2x} \cos 2y$$

So, $u_x = v_y$

$$u_y = \frac{\partial}{\partial y}(x + e^{2x} \cos 2y)$$
$$= -2e^{2x} \sin 2y$$
$$v_x = \frac{\partial}{\partial x}(y + e^{2x} \sin 2y)$$
$$= 2e^{2x} \sin 2y$$

So, $u_y = -v_x$.

We know,

$$f'(z) = u_x + iv_x$$
$$u_x = 1 + 2e^{2x} \cos 2y$$
$$v_x = 2e^{2x} \sin 2y$$

So, $f'(z) = u_x + iv_x$

$= 1 + 2e^{2x} \cos 2y + i2e^{2x} \sin 2y$

$= 1 + 2e^{2x}(\cos 2y + i \sin 2y)$

$= 1 + 2e^{2x} e^{i2y}$

$= 1 + 2e^{2x+i2y}$

$= 1 + 2e^{2(x+iy)}$

$= 1 + 2e^{2z}$

Examples

For the function $f(z) = \frac{1}{z^3}$

(a) Separate real and imaginary parts

(b) Verify **C-R** equations.

(c) find $f'(z)$ using $e^{-i\theta}(u_r + iv_r)$

Given, $f(z) = \frac{1}{z^3}$

$$\Rightarrow u + iv = \frac{1}{(re^{i\theta})^3}$$

$$\Rightarrow u + iv = \frac{1}{r^3 e^{i.3\theta}}$$

$$\Rightarrow u + iv = \frac{1}{r^3} e^{-i.3\theta}$$

$$\Rightarrow u + iv = \frac{1}{r^3}(\cos 3\theta - i \sin 3\theta)$$

$$\Rightarrow u + iv = \frac{1}{r^3}\cos 3\theta + i.\left(-\frac{1}{r^3}\sin 3\theta\right)$$

Comparing both sides

Real part, $u = \frac{1}{r^3}\cos 3\theta$

Imaginary part, $v = -\frac{1}{r^3}\sin 3\theta$

We must show that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial r} = \frac{\partial}{\partial r}\left(\frac{1}{r^3}\cos 3\theta\right) = -3\frac{1}{r^4}\cos 3\theta$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial}{\partial \theta}\left(-\frac{1}{r^3}\sin 3\theta\right) = -\frac{3}{r^3}\cos 3\theta$$

$$\text{So,} \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = \frac{\partial}{\partial r}\left(-\frac{1}{r^3}\sin 3\theta\right) = 3\frac{1}{r^4}\sin 3\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\frac{1}{r^3}\cos 3\theta\right) = -\frac{3}{r^3}\sin 3\theta$$

$$\text{So,} \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

We know,

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$

$$u_r = \frac{\partial u}{\partial r} = -\frac{3}{r^4}\cos 3\theta$$

$$v_r = \frac{\partial v}{\partial r} = \frac{3}{r^4}\sin 3\theta$$

So, $f'(z) = e^{-i\theta}(u_r + iv_r)$

$$= e^{-i\theta}\left(-\frac{3}{r^4}\right)(\cos 3\theta - i \sin 3\theta)$$

$$= -\frac{3}{r^4 e^{i4\theta}} = -3\frac{1}{(r\, e^{i\theta})^4} = -\frac{3}{z^4}$$

Exercise Set

For the following functions $f(z) =$

(a) $\bar{z}$,
(b) $2z^2 + 3\,e^z$,
(c) $2ze^z$,
(d) $3z^3$,
(e) $\frac{1}{z^9}$,
(f) z^5, and
(g) $z^{-\frac{2}{3}}$

I. Separate real and imaginary parts
II. Verify **C-R** equations
III. If analytic find $f'(z)$ using $u_x + iv_x$ or $e^{-i\theta}(u_r + iv_r)$

Sample MCQ

1. What is the real part of $f(z) = e^{z^2}$?

(a) $x^2 - y^2$ (b) $e^{x^2+y^2}(\cos 2xy)$ (c) $e^{x^2-y^2}(\cos 2xy)$ (d) $e^{x^2-y^2}(\sin 2xy)$

2. For $f(z) = \ln z$, which of the following is the real part? (Solution)

(a) $x^2 + y^2$ (b) $\frac{1}{2}(x^2 + y^2)$ (c) $\ln(x^2 + y^2)$ (d) $\frac{1}{2}\ln(x^2 + y^2)$

3. For $f(z) = \ln z$, which of the following is the imaginary part? (Solution)

(a) $\frac{y}{x}$ (b) $\tan^{-1}\left(\frac{y}{x}\right)$ (c) xy (d) $\sin^{-1}\left(\sqrt{x^2 + y^2}\right)$

4. For $u + iv = \sin z$, which of the following is u_x ? (Solution)

(a) $\cos x \cosh y$ (b) $\sin x \cosh y$ (c) $\sin x$ (d) $\sinh y$

5. For $u + iv = z^3$ which of the following is v_y?

(a) $x^2 + y^2$ (b) $3x^2 - 3y^2$ (c) $x^2 - y^2$ (d) $2x^2 - 2y^2$

Solution of MCQ no 2 and 3

$$\ln z$$
$$= \ln\left(re^{i\theta}\right)$$
$$= \ln r + \ln e^{i\theta}$$
$$= \ln r + i\theta \ln e$$
$$= \ln\left(\sqrt{x^2 + y^2}\right) + i\tan^{-1}\left(\frac{y}{x}\right)$$

So, Real part is

$$\ln\left(\sqrt{x^2 + y^2}\right)$$
$$= \ln(x^2 + y^2)^{\frac{1}{2}}$$
$$= \frac{1}{2}\ln(x^2 + y^2)$$

And imaginary part is

$$\tan^{-1}\left(\frac{y}{x}\right)$$

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Solution of MCQ no 4

$$u + iv = \sin z$$
$$\Rightarrow u + iv = \sin(x + iy)$$
$$\Rightarrow u + iv = \sin x \cos iy + \cos x \sin iy$$
$$\Rightarrow u + iv = \sin x \cosh y - i^2 \cos x \sin iy$$

[$\because \cos ix = \cosh x$] Reference

$$\Rightarrow u + iv = \sin x \cosh y + i \cos x\,(-i \sin iy)$$
$$\Rightarrow u + iv = \sin x \cosh y + i \cos x \sinh x$$

[$\because -i \sin ix = \sinh x$] Reference

$$\Rightarrow u = \sin x \cosh y$$
$$\Rightarrow u_x = \cos x \cosh y$$

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