Numerical Methods For Science and Engineering
Lecture Note 5
Spline Interpolation

5.1 Introduction

Spline interpolation function is a piecewise polynomial function joined together with certain conditions satisfied by them.
A function $f(x)$ of the form

$$f(x) = \begin{cases} f_1(x), & x_1 \le x < x_2 \\ f_2(x), & x_2 \le x < x_3 \\ \vdots & \\ f_{n-1}(x), & x_{n-1} \le x \le x_n \end{cases}$$

is called a **spline** of degree *m* if

(i) the domain of $f(x)$ is the interval $[x_1, x_n]$

(ii) $f(x), f'(x), f''(x) \ldots\ f^{(m-1)}$ are all continuous functions on $[x_1, x_n]$

(iii) $f(x)$ is a polynomial of degree less than equal to *m* on each subinterval $[x_k,\ x_{k+1}]$, $k = 1, 2, \cdots, n$.

5.1.1 Linear Spline Interpolation

For a linear spline through (x_k, y_k) we may take $f_k(x)$ is of the form

$$f_k(x) = a_k(x - x_k) + b_k, \qquad \text{for} \quad x_k \le x \le x_{k+1}'$$

Since the line passes through (x_k, y_k) and (x_{k+1}, y_{k+1}) we have

$$b_k = y_k$$

and
$$a_k = \frac{y_{k+1} - y_k}{x_{k+1} - x_k} = \frac{\Delta y_k}{h_k}$$

where $\Delta y_k = y_{k+1} - y_k$ and $h_k = x_{k+1} - x_k$.

The resulting linear spline curve $f_k(x)$ in $[x_k,\ x_{k+1}]$ can be written as

$$f_k(x) = y_k + \frac{\Delta y_k}{h_k}(x - x_k), \quad (k = 1, 2, \ldots, n - 1).$$

Example 5.1

Find the linear spline for the following data set

X	-1	1	2	5
Y	2.2	3.5	5.4	1.5

Hence estimate the value of $y(1.5)$.

Solution

Linear spline functions in different intervals are

$$f_1(x) = 2.2 + \frac{3.5 - 2.2}{1 - (-1)}(x+1) = 2.2 + 0.65(x+1), \qquad -1 \le x \le 1$$

$$f_2(x) = 3.5 + \frac{5.4 - 3.5}{2 - 1}(x-1) = 3.5 + 1.9(x-1), \qquad 1 \le x \le 2$$

$$f_3(x) = 5.4 + \frac{1.5 - 5.4}{5 - 2}(x-2) = 5.4 - 1.3(x-2), \qquad 2 \le x \le 5$$

Linear spline function is

$$f(x) = \begin{cases} 2.2 + 0.65(x+1), & -1 \le x \le 1 \\ 3.5 + 1.9(x-1), & 1 \le x \le 2 \\ 5.4 - 1.3(x-2) & 2 \le x \le 5 \end{cases}$$

The value $x = 1.5$ is in $1 \le x \le 2$. Thus

$$y(1.5) = 3.5 + 1.9(1.5 - 1) = 4.45.$$

5.2 Cubic Spline Interpolation

Cubic spline interpolation is used very often. It gives smoother curves than other types. To determine the cubic spline, we need to use cubic polynomial for each subintervals.

Consider the cubic polynomial $f_k(x)$ in each subinterval $[x_k, x_{k+1}]$, $k = 1, 2, \dots, n-1$ of the form

$$f_k(x) = a_k(x - x_{k-1})^3 + b_k(x - x_{k-1})^2 + c_k(x - x_{k-1}) + d_k, \quad (a_k \ne 0).$$

where a_k, b_k, c_k and d_k are to be determined.

Since the spline passes through (x_k, y_k) and $f_k(x)$,

$$f_1(x_0) = y_0, \quad \text{and } f_k(x_k) = y_k, \quad k = 1, 2, 3, \cdots, n.$$

$$f_k(x_k) = f_{k+1}(x_k), \qquad k = 1, 2, 3, \cdots, n-1$$

$$f_k'(x_k) = f_{k+1}'(x_k), \qquad k = 1, 2, 3, \cdots, n-1$$

$$f_k''(x_k) = f_{k+1}''(x_k), \qquad k = 1, 2, 3, \cdots, n-1$$

We can see that there are

$$1 + n + 3(n-1) = 4n - 2$$

conditions but we need to determine $4n$ constants. So we need to add two boundary conditions to get unique solution.

Normally we use three types of boundary conditions:

1. Second derivatives at end points are known

$$f_1''(x_0) = M_0 \text{ and } f_n''(x_n) = M_n.$$

   The special case

$$f_1''(x_0) = 0 \quad \text{and } f_n''(x_n) = 0$$

   give spline called natural **cubic spline**.

2. First derivatives at end points are known

$$f_1'(x_0) = d_0 \text{ and } f_n'(x_n) = d_n.$$

   give spline called **clampedcubic spline**.

3. Automatically adjusted boundary conditions known as **not-a-knot** cubic spline.
   This condition assumes that $f'''(x)$ are continuous at the second and last but one points.

$$f_1^{'''(x_1)} = f_2'''(x_1) \quad \text{and} \quad f_{n-1}'''(x_{n-1}) = f_n'''(x_{n-1}).$$

   Note that minimum number of data points is four for this condition to be used..

MATLAB Spline Interpolation Functions

(1) MATLAB function **spline**

yy=spline(x, Y, xx) for **not-a-knot** cubic spline

x, Y are inputs and xx expolant.

yy=spline(x, [dY0, Y, dYn], xx) for clamped cubic spline

` $dY0 = Y(x0)$and $dYn = Y'(xn)$

(2) **csape** spline interpolation with various end conditions

Syntax: sp=csape(X, Y, conds)

some of the conds are

'second'	adjusted second derivatives if not mentioned it uses $[0, 0]$
'clamped'	adjusted first derivatives
'not-a-knot'	uses not-a-knot condtion

Example 5.2

A natural cubic spline is defined by

$$f(x) = \begin{cases} A(x+1)^3 + B(x+1)^2 + C(x+1) + 1, & -1 \le x < 1 \\ D(x-1)^3 + 6(x-1)^2 + E(x-1) - 1, & 1 \le x \le 2 \end{cases}$$

(i) Use continuity and boundary conditions to estimate $A,\ B, C, D$ and E.

(ii) Find the value of $f(1.4)$ from the spline curve'

(iii) Use MATLB function "**sp=csape(x, y, 'conds')**" to construct natural cubic spline for the data set $(-1, 1), (1, -1)$ and $(2, 10)$. Find $f(1.4)$ using "**fnval(sp,x)**"' Plot the spline curve using "**fnplt(sp)**" along with the data points.

Solution

Let $f_1(x) = A(x+1)^3 + B(x+1)^2 + C(x+1) + 1,$

and $f_2(x) = D(x-1)^3 + 6(x-1)^2 + E(x-1) - 1,$

Then

$$f_1'(x) = 3A(x+1)^2 + 2B(x+1) + C$$
$$f_2'(x) = 3D(x-1)^2 + 12(x-1) + E$$

and

$$f_1''(x) = 6A(x+1) + 2B$$
$$f_2''(x) = 6D(x-1) + 12$$

(i) Conditions at the interior point $x = 1$ give

$$f_1(1) = f_2(1) \quad \Longrightarrow \quad 8A + 4B + 2C + 1 = -1 \qquad (1)$$
$$f_1'(1) = f_2'(1) \quad \Longrightarrow \quad 12\,A + 4B + C = E \qquad (2)$$
$$f_1''(1) = f_2''(1) \quad \Longrightarrow \quad 12A + 2B = 12 \qquad (3)$$

For natural cubic spline the boundary conditions give

$$f_1''(-1) = 2B = 0 \quad \text{or} \quad B = 0$$
$$f_2''(2) = 6D + 12 = 0 \quad \text{or} \quad D = -2$$

From (3), $12A = 12$ or $A = 1$

From (1), $8(1)+2C + 1 = -1$ or C$= -5$

From (2), $12(1) + (-5) = E$ or $E = 7$

The natural cubic spline function is

$$f(x) = \begin{cases} (x+1)^3 - 5(x+1) + 1 \quad , & -1 \le x < 1 \\ -2(x-1)^3 + 6(x-1)^2 + 7(x-1) - 1, & 1 \le x \le 2 \end{cases}$$

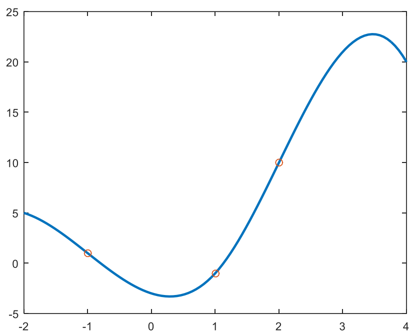
(ii) $f(1.4) = f_2(1.4) = -2(0.4)^3 + 6(0.4)^2 + 7(0.4) = 2.632.$

(ii)

```
>> clear
>> x=[-1 1 2];
>> y=[1 -1 10];
>> sp=csape(x,y,'second');
>> y1=fnval(sp,1.4)

y1 =
    2.6320

>> fnplt(sp, 2, [-2, 4])
>> hold on                    % used to plot in the same figure
>> plot(x, y,'O')
>> hold off
```



Example:5.3

A cubic spline $f(x)$ which interpolates the data $(1,-10),(2,-6),(4,2),(5,18)$ is defined by
$$f(x)=\begin{cases}(x-1)^3+b_1(x-1)^2+c_1(x-1)-10, & 1\le x<2\\ a_2(x-2)^3-(x-2)^2+c_2(x-2)-6, & 2\le x<4\\ a_3(x-4)^3+b_3(x-4)^2+10(x-4)+2, & 4\le x\le 5\end{cases}$$

(i) If the spline satisfies the not-a-knot boundary conditions, find a_2 and a_3.

(ii) Using continuity of $f''(x)$ at $x=2$ and 4, find b_1 and b_3.

(iii) Using continuity of $f'(x)$ at $x=2$ and 4, find c_1 and c_2.

(iv) Show that $f(x)$ passes through $(5,18)$.

(v) Use MATLB function "**sp=spline(x, y)**" to construct the spline curve and find coefficients by **"sp.coefs**".

(vi) Write down MATLAB codes using "**fnval(sp,x)**" to estimate the values of $f(x)$ for $x = 1.4, 2.5$ and 4.8 from the spline curve.

(vii) Write down MATLAB codes using "**fnplt(sp)**" to plot the spline curve and the given data points.

Solution : Let the cubic spline be $f(x)=\begin{cases}f_1(x), & 1\le x<2\\ f_2(x), & 2\le x<4\\ f_3(x), & 4\le x\le 5\end{cases}$

where
$$f_1(x)=(x-1)^3+b_1(x-1)^2+c_1(x-1)-10,$$
$$f_2(x)=a_2(x-2)^3-(x-2)^2+c_2(x-2)-6,$$
$$f_3(x)=a_3(x-4)^3+b_3(x-4)^2+10(x-4)+2.$$

Then
$$f_1{}'(x)=3(x-1)^2+2b_1(x-1)+c_1,$$
$$f_2{}'(x)=3a_2(x-2)^2-2(x-2)+c_2,$$
$$f_3{}'(x)=3a_3(x-4)^2+2b_3(x-4)+10.$$

And
$$f_1''(x)=6(x-1)+2b_1,$$
$$f_2''(x)=6a_2(x-2)-2,$$
$$f_3''(x)=6a_3(x-4)+2b_3.$$

Also
$$f_1'''(x)=6,\quad f_2'''(x)=6a_2,\quad f_3'''(x)=6a_3.$$

i. Spline curve satisfies not-a-knot boundary conditions. Thus
$$f_1'''(2)=f_2'''(2)\quad\Rightarrow\quad 6=6a_2\quad \text{or}\ a_2=1.$$
$$f_3'''(4)=f_2'''(4)\quad or\quad 6a_3=6a_2=6\quad or\quad a_3=1.$$

ii. Continuity of $f''(x)$ at $x=2$ and 4 gives
$$f_1''(2)=f_2''(2)\quad\Rightarrow\quad 6(1)+2b_1=-2\quad \text{or}\quad b_1=-4.$$

And $\quad f_2''(4)=f_3''(4)\quad\Rightarrow\quad 6a_2(2)-2=2b_3\quad \text{or}\quad b_3=\frac{1}{2}[6(1)(2)-2]=5.$

iii. Continuity of $f'(x)$ at $x = 2$ and 4 gives

$$f_1'(2) = f_2'(2) \quad \Rightarrow \quad 3 + 2b_1 + c_1 = c_2.$$

And $f_2'(4) = f_3'(4) \quad \Rightarrow \quad 3a_2(4) - 2(2) + c_2 = 10$

$$c_2 = 10 - 3(1)(4) + 4 = 2$$

and $c_1 = 2 - 3 - 2(-4) = 7.$

iv. $f(5) = f_3(5) = 1(1) + 5(1) + 10(1) + 2 = 18.$

v.

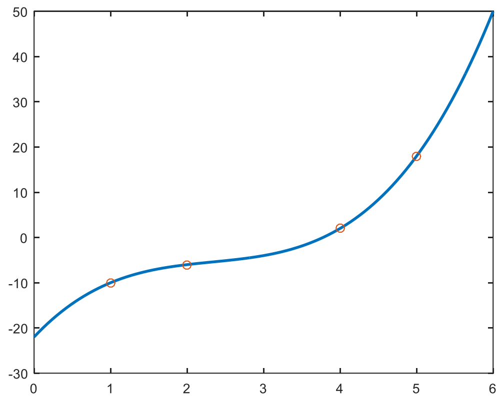
```
>> x=[1 2 4 5];
>> y=[-10 -6 2 18];
>> sp=spline(x,y);
>> format short g
>> Coefficients = sp.coefs

Coefficients =
     1    -4     7   -10
     1    -1     2    -6
     1     5    10     2
```

vi.

```
>> x1=[1.4, 2.5, 4.8];
>> y1=fnval(sp,x1);
>> xy_value =[x1', y1']

xy_value =
   1.4    -7.776
   2.5    -5.125
   4.8    13.712
```



vii.

```
>> fnplt(sp, [0, 6])
>> hold on
>> plot(x,y,'o')
```

Exercise 5

1. In a chemical reaction the concentration level y of the product at time ***t*** (minute) was measured every half hour. The following results were found:

t	1.0	1.5	2.0	2.5
y	0.24	0.27	0.31	0.36

Construct a linear spline interpolation to estimate the concentration level at 2.25 minute.

2. Use the portion of the given steam table for superheated H_2O at 200 MPa to find the corresponding entropy, s, for a specific volume, v, of 0.118 m^3/kg with linear spline.

V (m^3/kg)	0.2037	0.2114	0.32547	0.33213
S (kJ/kg K)	6.5147	6.6453	6.8664	6.9513

3. Given the following set of values of x and $f(x)$:

$$(-1, 6), (1, 2) \text{ and } (2, 12).$$

Find the linear spline passing through the above points.

A natural cubic spline through the above points are defined by

$$f(x) = \begin{cases} A(x+1)^3 + B(x+1)^2 + C(x+1) + 6, & -1 \le x < 1 \\ D(x-1)^3 + 6(x-1)^2 + E(x-1) + 2, & 1 \le x \le 2 \end{cases}$$

Use continuity and boundary conditions to find equations satisfied by $A,\ B, C, D$ and E.

Solve for $A,\ B, C, D$ and E.

Find the value of $f(0.4)$ from linear spline and $f(1.4)$ from cubic curve.

Write MATLAB codes using "**sp=csape(x, y, 'second')**" to construct natural cubic spline for the above data set. Plot the spline curve using "**fnplt(sp)**" along with the data points.

4. A natural cubic spline $f(x)$ is defined by

$$f(x) = \begin{cases} A(x+1)^3 + B(x+1)^2 + C(x+1) + 7, & -1 \le x \le 2 \\ D(x-2)^3 + 5(x-2)^2 + E(x-2) + 2, & 2 \le x \le 4 \end{cases}$$

i. Use continuity and boundary conditions to find the values of *A*, *B*, *C*, *D*, and *E*.

ii. Estimate the value of $f(3.5)$.

Iii. Write a MATLB code using function "**spline(x, y)**" to construct the spline curve and "**fnval(function,x)**" to estimate the values of $f(x)$ for $x = -0.5,$ and 3.9.

5. A natural cubic spline $f(x)$ is defined by

$$f(x) = \begin{cases} A(x+1)^3 + B(x+1)^2 + C(x+1) - 6, & -1 \le x \le 1 \\ D(x-1)^3 + 12(x-1)^2 + E(x-1) + 2, & 1 \le x \le 3 \end{cases}$$

i. Use continuity and boundary conditions to find the values of *A*, *B*, *C*, *D* and *E*.

ii. Estimate $f(2)$from the spline curve.

iii. Use MATLAB function "**sp=spline(x, y)**" and "**fnval(sp,x)**" to construct the spline curve from the data $(1, -10), (2, -6), (4, 2), (6, 15)$ and find the values of $f(x)$ for $x = 1.4, 2.5$ and 4.8 from the spline curve.

6. A clamped cubic spline function $f(x)$ through $(1,2), (2,3), (3,5)$ is defined by

$$f(x) = \begin{cases} 1.5(x-1)^3 + A(x-1)^2 + B(x-1) + 2, & 1 \le x \le 2 \\ C(x-2)^3 + 2(x-2)^2 + D(x-2) + 3, & 2 \le x \le 3 \end{cases}$$

Given that $f'(1) = 2$ and $\mathrm{f}'(3) = 1$.

i. Use continuity and boundary conditions to find the values of *A*, *B*, *C* and *D*.
ii. Estimate $f(2.5)$from the spline curve.
iii. Use MATLB function "**sp=spline(x, y)**" and "**fnval(sp,x)**" to construct the spline curve from the data $(1,2), (2,3), (3,5)$ and find the values of $f(x)$ for $x = 1.4$ and 2.5 from the spline curve.