5.3.3 Cubic Spline Interpolation

For the cubic spline through the points $(x_k, y_k), k = 0,1,2, \cdots, n$ we may take $f_k(x)$ is of the form

$$f_k(x) = a_k(x-x_k)^3 + b_k(x-x_k)^2 + c_k(x-x_k) + d_k \text{ in } [x_k, x_{k+1}]$$

Since the spline passes through (x_k, y_k) and $f_k(x)$, $f_k'(x)$ and $f_k''(x)$ are continuous at the interior points we get

$$d_k = y_k, \qquad k = 0,1,2, \cdots, n-1$$

Using the notation

$$M_k = f_k''(x_k) \text{ for } k = 0,1,2, \cdots, n-1 \text{ and } M_n = f_k''(x_n),$$

we have

$$b_k = \frac{M_k}{2}$$

$$a_k = \frac{M_{k+1} - M_k}{6h_k}$$

$$c_k = \frac{y_{k+1} - y_k}{h_k} - \frac{h_k}{6}(M_{k+1} + 2M_k)$$

.where M_k satisfy the recurrence relation

$$h_k M_k + 2(h_k + h_{k+1})M_{k+1} + h_{k+1}M_{k+2} = 6\left[\frac{\Delta y_{k+1}}{h_{k+1}} - \frac{\Delta y_k}{h_k}\right], \quad k = 0, 1, 2, \cdots, n\text{-}2$$

We still needs two more equations to determine M_k uniquely.

The two common end points constraints used in caluations are

Description of the strategy	Equations involving M_0 and M_n
Natural cubic spline "a relaxed curve": $S''(x_0) = 0$ and $S''(x_n) = 0$.	$M_0 = 0, \quad M_n = 0$
Clamped cubic spline: Specify $S'(x_0) = A$ and $S'(x_n) = B$.	$2M_0 + M_1 = \frac{6}{h_0}\left[\frac{\Delta y_0}{h_0} - A\right]$ $M_n + 2M_{n-1} = \frac{6}{h_{n-1}}\left[B - \frac{\Delta y_{n-1}}{h_{n-1}}\right]$
Curvature adjusted cubic Spline: Specify $S''(x_0) = A$ and $S''(x_n) = B$.	$M_0 = A, \quad M_n = B$

Example 5.6

A cubic spline for a function $f(x)$ is defined by

$$f(x) = \begin{cases} A(x+1)^3 + B(x+1)^2 + C(x+1) + 15, & -1 \le x \le 1 \\ a(x-1)^3 - 6(x-1)^2 - 13(x-1) + b, & 1 \le x \le 2 \end{cases}$$

Find the values of A, B, C, a, and b for

(i) the natural cubic spline.

(ii) the clamped cubic spline with the conditions $f'(-1) = -1$ and $f'(2) = 2$.

Solution

Let $\quad f_1(x) = A(x+1)^3 + B(x+1)^2 + C(x+1) + 15$

and $\quad f_2(x) = a(x-1)^3 - 6(x-1)^2 - 13(x-1) + b$

Then

$$f_1'(x) = 3A(x+1)^2 + 2B(x+1) + C$$

and
$$f_2'(x) = 3a(x-1)^2 - 12(x-1) - 13$$

$$f_1''(x) = 6A(x+1) + 2B$$
$$f_2''(x) = 6a(x-1) - 12$$

Conditions at the interior point $x = 1$ give

$$f_1(1) = f_2(1) \quad \Rightarrow \quad 8A + 4B + 2C + 15 = b \qquad (1)$$
$$f_1'(1) = f_2'(1) \quad \Rightarrow \quad 12A + 4B + C = -13 \qquad (2)$$
$$f_1''(1) = f_2''(1) \quad \Rightarrow \quad 12A + 2B = -12 \qquad (3)$$

(i) For natural cubic spline the boundary conditions give

$f_1''(-1) = 2B = 0$ or $B = 0$

$f_2''(2) = 6a - 12 = 0$ or $a = 2$

From (3), $12A = -12$ or $A = -1$

From (2), $-12 + C = -13$ or $C = -1$

From (1), $-8 - 2 + 15 = b$ or $b = 5$

The natural cubic spline function is

$$f(x) = \begin{cases} -1(x+1)^3 - 1(x+1) + 15, & -1 \le x \le 1 \\ 2(x-1)^3 - 6(x-1)^2 - 13(x-1) + 5, & 1 \le x \le 2 \end{cases}$$

(ii) For clamped cubic spline the boundary conditions give

$f_1'(-1) = C = -1$ or $C = -1$

$f_2'(2) = 3a - 12 - 13 = 2$ or $a = 9$

From (2 & 3), $12A + 4B = -12$ and $12A + 2B = -12$

Solving we have $B = 0$ and $A = -1$

From (1), $8(-1) + 4(0) + 2(-1) + 15 = b$ or $b = 5$

The clamped cubic spline function is

$$f(x) = \begin{cases} -(x+1)^3 - 1(x+1) + 15, & -1 \le x \le 1 \\ 9(x-1)^3 - 6(x-1)^2 - 13(x-1) + 5, & 1 \le x \le 2 \end{cases}$$

Example 5.6

A cubic spline $f(x)$ which passes through the following data npoints:

$$(-1, 6),\ (1, 2) \text{ and } (2, 12).$$

Find

(a) the natural cubic spline.

(b) the clamped cubic spline with conditions $f'(-1) = 4$ and $f'(2) = 1$.

(c) the curvature adjusted cubic spline with the second derivative boundary conditions $f''(0) = 4$ and $f''(2) = 4$.

Solution

Let the cubic spline be

$$f(x) = \begin{cases} f_1(x), & -1 \le x \le 1 \\ f_2(x), & 1 \le x \le 2 \end{cases}$$

where

$$f_1(x) = a_1(x+1)^3 + b_1(x+1)^2 + c_1(x+1) + d_1$$

$$f_2(x) = a_2(x-1)^3 + b_2(x-1)^2 + c_2(x-1) + d_2$$

Then

$$f_1'(x) = 3a_1(x+1)^2 + 2b_1(x+1) + c_1$$
$$f_2'(x) = 3a_2(x-1)^2 + 2b_2(x-1) + c_2$$

and

$$f_1''(x) = 6a_1(x+1) + 2b_1$$
$$f_2''(x) = 6a_2(x-1) + 2b_2$$

The curve passes through $(-1, 6)$, $(1, 2)$ and $(2, 12)$, and we have

$$f_1(-1) = d_1 = 6 \qquad (1)$$
$$f_2(1) = d_2 = 2 \qquad (2)$$
$$f_2(2) = a_2 + b_2 + c_2 + d_2 = 12 \qquad (3)$$

Conditions at the interior point $x = 1$ give

$$f_1(1) = f_2(1) \quad \Rightarrow \quad 8a_1 + 4b_1 + 2c_1 + d_1 = d_2 \qquad (4)$$
$$f_1'(1) = f_2'(1) \quad \Rightarrow \quad 12a_1 + 4b_1 + c_1 = c_2 \qquad (5)$$
$$f_1''(1) = f_2''(1) \quad \Rightarrow \quad 12a_1 + 2b_1 = 2b_2 \qquad (6)$$

(a) For natural cubic spline the boundary conditions give

$$f_1''(-1) = 2b_1 = 0 \qquad \text{or} \qquad b_1 = 0 \qquad (7)$$
$$f_2''(2) = 6a_2 + 2b_2 = 0 \qquad (8)$$

From (1), (2) and (7), we have

$$b_1 = 0, \quad d_1 = 6 \quad \text{and} \quad d_2 = 2.$$

From (6), $\quad b_2 = 6a_1$

From (8), $\quad 6a_2 = -2b_2 = -12a_1 \quad$ or $\quad a_2 = -2a_1$

From (4), $\quad 8a_1 + 2c_1 + 6 = 2 \quad$ or $\quad c_2 = -2 - 4a_1$

From (5), $\quad c_2 = 12a_1 - 2 - 4a_1 = -2 + 8a_1$

Substituting in (3), we have

$$-2a_1 + 6a_1 - 2 + 8a_1 + 2 = 12 \quad \text{or} \quad 12a_1 = 12$$

Therefore $\quad a_1 = 1$

Thus the values of the coefficients are

$$a_1 = 1, \; b_1 = 0, \; c_1 = -6, \; d_1 = 6$$

and $\quad a_2 = -2, \; b_2 = 6, \; c_2 = 6, \; d_2 = 2$

The natural cubic spline function is

$$f(x) = \begin{cases} (x+1)^3 - 6(x+1) + 6, & -1 \le x \le 1 \\ -2(x-1)^3 + 6(x-1)^2 + 6(x-1) + 2, & 1 \le x \le 2 \end{cases}$$

(b) For clamped cubic spline the boundary conditions give

$$f_1'(-1) = c_1 = 4 \qquad (9)$$
$$f_2'(2) = 3a_2 + 2b_2 + c_2 = 1 \qquad (10)$$

From (1) and (2), we have

$$d_1 = 6 \quad \text{and} \quad d_2 = 2.$$

From (4), $\quad 8a_1 + 4b_1 = 2 - 6 - 2(4) = -12$

or $\quad 2a_1 + b_1 = -3$

From (5), $\quad c_2 = 4a_1 + 8a_1 + 4b_1 + c_1 = 4a_1 - 8$

From (6), $b_2 = 4a_1 + 2a_1 + b_1 = 4a_1 - 3$

From (3), $a_2 = 12 - (4a_1 - 3) - (4a_1 - 8) - 2 = 21 - 8a_1$

Substituting in (10), we have

$$63 - 24a_1 + 8a_1 - 6 + 4a_1 - 8 = 1 \quad \text{or} \quad 12a_1 = 48$$

Therefore
$$a_1 = 4$$

Thus the values of the coefficients are

$$a_1 = 4,\ b_1 = -11,\ c_1 = 4,\ d_1 = 6$$

and
$$a_2 = -11,\ b_2 = 13,\ c_2 = 8,\ d_2 = 2$$

The clamped cubic spline function is

$$f(x) = \begin{cases} 4(x+1)^3 - 11(x+1)^2 + 4(x+1) + 6, & -1 \le x \le 1 \\ -11(x-1)^3 + 13(x-1)^2 + 8(x-1) + 2, & 1 \le x \le 2 \end{cases}$$

(c) For curvature adjusted cubic spline the boundary conditions give

$$f_1''(-1) = 2b_1 = 4 \quad \text{or} \quad b_1 = 2 \qquad (11)$$
$$f_2''(2) = 6a_2 + 2b_2 = 4 \quad \text{or} \quad 3a_2 + b_2 = 2 \qquad (12)$$

From (1) and (2), we have

$$d_1 = 6 \quad \text{and} \quad d_2 = 2.$$

From (4), $8a_1 + 2c_1 = 2 - 6 - 4(2) = -12$

or $c_1 = -6 - 4a_1$

From (5), $c_2 = 12a_1 + 4(2) - 6 - 4a_1 = 8a_1 + 2$

From (6), $b_2 = 6a_1 + 2$

From (3), $a_2 = 12 - (6a_1 + 2) - (8a_1 + 2) - 2 = 6 - 14a_1$

Substituting in (12), we have

$$18 - 42a_1 + 6a_1 + 2 = 2 \quad \text{or} \quad 36a_1 = 18$$

Therefore
$$a_1 = 1/2$$

Thus the values of the coefficients are

$$a_1 = 1/2,\ b_1 = 2,\ c_1 = -8,\ d_1 = 6$$

and
$$a_2 = -1,\ b_2 = 5,\ c_2 = 6,\ d_2 = 2$$

The curvature adjusted cubic spline function is

$$f(x) = \begin{cases} \frac{1}{2}(x+1)^3 + 2(x+1)^2 - 8(x+1) + 6, & -1 \le x \le 1 \\ -(x-1)^3 + 5(x-1)^2 + 6(x-1) + 2, & 1 \le x \le 2 \end{cases}$$

EXERCISES 5.2

1. Determine whether this function is a first degree spline:

$$f(x) = \begin{cases} x & -1 \le x \le 1 \\ 1 - 2(x-1), & 1 \le x \le 2 \\ -1 + 3(x-2), & 2 \le x \le 3 \end{cases}$$

2. Is $f(x) = |x|$ a first degree spline? Why or why not?

3. Find linear spline for the following data:

x	-1	0	1/2	1	2
y	2	1	4	2	3

Hence estimate the values of $y(-0.5)$ and $y(1.5)$.

4. Are these functions quadratic splines? Explain why or why not.

(a) $$f(x)=\begin{cases}0.1x^2, & 0\le x\le 1\\ 9.3x^2-18.4x+9.2, & 1\le x\le 1.3\end{cases}$$

(b) $$f(x)=\begin{cases}-(x+1)^2+2, & -1\le x\le 0\\ x^2-2x+1, & 0\le x\le 1\\ x^2-1, & x>1\end{cases}$$

5. A quadratic spline $S(x)$ is defined by

$$S(x)=\begin{cases}a(x+2)^2+x+3, & -2\le x\le 0\\ -2x^2+bx+5, & 0\le x\le 1\\ (x-1)^2+c(x-1)+d, & 1\le x\le 2\end{cases}$$

Find a, b, c and d.

6. Find the quadratic spline through the given points with the given conditions.

(a)

x	-1	1	4
y	1	-1	11

with $y''(-1)=1$.

(b)

x	1	2	4
y	2	5	1

with $y''(1)=-1$.

7. Find quadratic splines satisfying the following data points:
   (a) $(0,-1)$, $(1,1)$, $(3,-3)$ and $(6,65)$
   (b) $(-1,1)$, $(1,5)$, $(3,-3)$ and $(5,9)$
   (c) $(3.0, 2.5)$, $(4.5, 1.0)$, $(7.0, 2.5)$ and $(9.0, 0.5)$

8. Prove that the derivative of a quadratic spline is a first degree spline.

9. Show that the indefinite integral of a first-degree spline is a second-degree spline.

10. Determine whether $f(x)$ is a cubic spline with knots -1, 0, 1 and 2:

$$f(x)=\begin{cases}1+2(x+1)+(x+1)^3, & -1\le x\le 0\\ 4+5x+3x^3, & 0\le x\le 1\\ 11+1(x-1)+3(x-1)^2+(x-1)^3, & 1\le x\le 2\end{cases}$$

11. A natural cubic spline for a function $f(x)$ is defined by

$$f(x)=\begin{cases}a(x+1)^3+b(x+1)^2-8(x+1)+c, & -1\le x\le 1\\ -(x-1)^3+A(x-1)^2+B(x-1)-6, & 1\le x\le 3\end{cases}$$

Find the values of a, b, c, A and B.
Hence estimate $f(1.5)$.

12. A natural cubic spline for a function $f(x)$ is defined by

$$f(x)=\begin{cases}A(x+1)^3+B(x+1)^2-5(x+1)+5, & -1\le x\le 0\\ x^3+3x^2-2x+1, & 0\le x\le 1\\ a(x-1)^3+b(x-1)^2+c(x-1)+d, & 1\le x\le 2\end{cases}$$

Find the values of A, B, a, b, c and d.
Hence estimate the values of $f(-0.5)$ and $f(1.5)$.

13. A clamped cubic spline $f(x)$ is defined by

$$f(x)=\begin{cases}(x-1)^3+a(x-1)^2+b(x-1)+c, & 1\le x\le 2\\ A(x-2)^3+5(x-2)^2+B(x-2)+6, & 2\le x\le 4\end{cases}$$

Given that $f'(1)=0$ and $f'(4)=3$ find a, b, c, A amd B.

14. A clamped cubic spline for a function $f(x)$ is defined by

$$f(x)=\begin{cases}3(x-1)+2(x-1)^2-(x-1)^3, & 1\le x\le 2\\ a+b(x-2)+c(x-2)^2+d(x-2)^3, & 2\le x\le 3\end{cases}$$

Given $f'(1)=f'(3)$, find a, b, c, and d.

15. A curvature adjusted cubic spline $f(x)$ is defined by

$$f(x)=\begin{cases}a(x+1)^3+b(x+1)^2-10(x+1)+c, & -1\le x\le 1\\ A(x-1)^3+11(x-1)^2+B(x-1)-7, & 1\le x\le 3\end{cases}$$

Given that $f''(-1)=-2$ and $f''(3)=4$ find a, b, c, A amd B.
Hence find $f(2)$.

16. Find the natural cubic splines satisfying the following data points:
(a) $(0,1)$, $(1,1)$ and $(2,5)$ (b) $(-1,1)$, $(0,2)$ and $(1,-1)$
(c) $(1,4)$, $(2,6)$ and $(3,24)$ (d) $(-1,4)$, $(2,10)$ and $(3,36)$

17. Consider the points $(-1,1)$, $(1,-3)$ and $(3,57)$. Find
(a) the natural cubic spline.
(b) the clamped cubic spline with conditions $f'(-1)=-2$ and $f'(3)=-2$.
(c) the curvature adjusted cubic spline with the second derivative boundary conditions $f''(-1)=-1$ and $f''(3)=1$.

18. Find the natural cubic spline which fits the following data:

x	1	2	3	4
y	1	5	11	8

and hence find the values of $y(1.5)$.

19. Find the natural cubic spline which fits the following data:

x	1	2	3	4	5
$f(x)$	6	−3	6	2	−6

Find $f(x)$ at $x = 1.3$.

20. Consider the points

x	−1	0	3	4
y	9	26	56	29

(a) Find the natural cubic spline which fits this data and hence estimate the value of $y(1)$.
(b) Find the clamped cubic spline with conditions $S'(-1)=1$ and $S'(4)=-1$.