Mathematical Methods of Engineering
Lecture Note 3
Nonlinear Equations in One Variable

3.1 Introduction

Many physical and engineering problems can be governed by nonlinear equation $f(x)=0$. These equations involve circular function, hyperbolic function, exponential and other transcendental functions and their combinations. The solution(values of *x)* of the nonlinear equation $f(x)=0$ is challenging. In fact, majority of equations cannot be solved analytically and so we have to solve them numerically. In this chapter, we shall consider some of the important approximate methods in finding the roots of the equations in one variable.

3.2 Number of Real Roots

3.2.1 Number of Real Roots by Graphical Method

A polynomial equation $a_0+a_1x+a_2x^2+\cdots+a_nx^n=0$ of degree *n* has exactly *n* roots. Some of them are real and others are complex. For a non-polynomial equation $f(x)=0$, there is no such rule of finding the number of roots. Geometrically, if the graph of $y=f(x)$ crosses the *x*-axis at $x=a$, then $x=a$ is a real root of $f(x)=0$. Now we shall consider graphically to find the number of real roots and its location.

Rewrite the equation $f(x)=0$ as $f_1(x)=f_2(x)$. At the point of intersection $x=x_1$ (say) of the graphs

$$y=f_1(x) \text{ and } y=f_2(x)$$

that is

$$f_1(x_1)=f_2(x_1)$$

and hence, $x=x_1$ is a root of the equation.

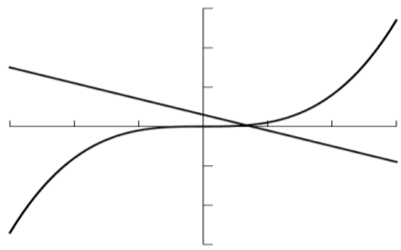
Thus the number of intersections of the two graphs will be the number of real roots.

Example 3.1

(a) Find the number of real roots of $x^3+4x-3=0$ by graphical method.
(b) Find the nmber of complex roots, if any.
(c) Use MATLAB function “roots” to find all the roots including complex roots.

(a)

$$x^3+4x-3=0$$
$$\Rightarrow x^3=3-4x$$
$$f_1(x)=x^3$$
$$f_2(x)=3-4x$$



There is one point of intersection here in the plot. So the number of real root is 1.

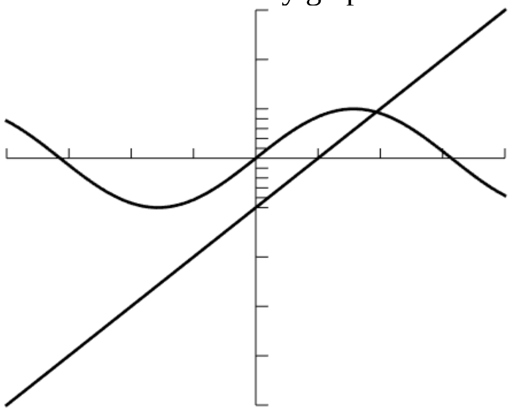
(b) It is a polynomial equation of degree three and hence the total number of roots is three.
So, number of complex roots = Total number of roots - Number of real roots

$$= 3 - 1 = 2.$$

(c)
```
>> p=[1 0 4 -3]        % entry of cubic polynomial
   p =    1     0     4    -3
>> Roots = roots(p)
   Roots =
    -0.3368 + 2.0833i
    -0.3368 - 2.0833i
     0.6736 + 0.0000i
```

Example 3.2: Find the number of real roots of $\sin x - x + 1 = 0$ by graphical method.

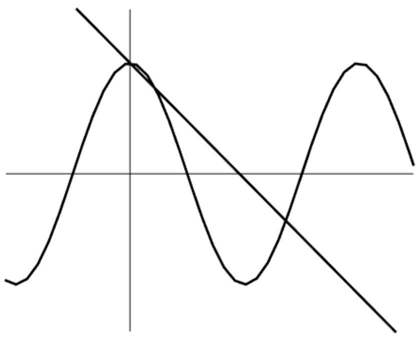
$$\sin x - x + 1 = 0$$
$$\Rightarrow \sin x = x - 1$$
$$f_1(x) = \sin x$$
$$f_2(x) = x - 1$$



There is one point of intersection here in the plot. So the number of real root is 1.

Example 3.3: Find the number of real roots of $1 - x - \cos 3x = 0$ by graphical method.

$$1 - x - \cos 3x = 0$$
$$1 - x = \cos 3x$$
$$f_1(x) = 1 - x$$
$$f_2(x) = \cos 3x$$



There are three points of intersection here in the plot. So the number of real roots are 3.

3.2.2 Location of Roots

To locate the roots of $f(x) = 0$, first study the graph of $y = f(x)$ as shown below. If we can find two values of x, one for which $f(x)$ is positive, and one for which $f(x)$ is negative, then the curve must have crossed the x-axis and so must have passed through a root of the equation $f(x) = 0$.

In general, if $f(x)$ is continuous in $[a, b]$ and $f(a)$ and $f(b)$ are opposite in signs i.e., $f(a)f(b) < 0$, then there exists odd number of real roots (at least one root) of $f(x) = 0$ in (a, b).

But the only exception where it does not work is when curve touches the *x*-axis. For this case, the existence of a root can be determined by the sign of $f'(x)$ in the interval (a,b) containing the root and it will satisfy the condition $f'(a)f'(b) < 0$.

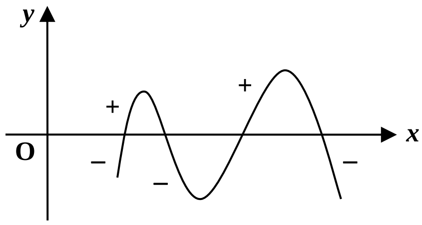


Fig 4.1

Example 3.4 The equation $(x-1)e^x = x$ has two real roots. For each root, find an interval where it lies.

Consider the values of function $f(x) = (x-1)e^x - x$ for different values of *x*:

x	*f*(*x*)
-2	1.59
-1	0.26
0	-1
1	-1
2	5.39

From the above table, we see that $f(-1)f(0) = -0.26 < 0$, a root lies in $(-1,\ 0)$,
and $f(1)f(2) = -5.39 < 0$, a root lies in $(1,2)$

3.3 Techniques to find real roots

3.3.1 Method of Bisection

Let $f(x)$ be continuous in $[a,b]$ and $f(a)f(b) < 0$, then there exists a real root of $f(x) = 0$ in (a,b). In this method we assume the mid-point $c = (a+b)/2$ is the approximation to the root.

If $f(c) = 0$, we conclude that c is a root of $f(x) = 0$. If $f(c) \neq 0$ and

(i) if $f(a)f(c) < 0$, the root is in (a, c) or

(ii) if $f(c)f(b) < 0$, the root is in (c, b).

By designating the new interval of root as $[a_1, b_1]$, we can calculate the next iterate x_1 by the formula

$$x_{n+1} = \frac{a_n + b_n}{2}, n = 1, 2, 3, \ \ldots$$

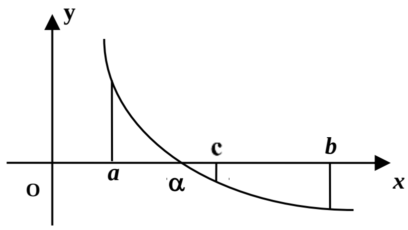


Fig 4.2 Bisection Method

Repeat the process until $|x_{n+1} - x_n| \leq \varepsilon$, where ε is the specified accuracy.

MATLAB Code:

```
function [a,b,c]=bisectionMethod()
   a=1;b=2;tol=0.01;

while abs((b-a))>tol
        c=(a+b)/2;

          if f(a)*f(c)<0
                    b=c;
          else
                    a=c;
          end
end
end
function f=f(x)
      f=x*x-3;
end
```

Example 3.5 Find the root of $f(x) = x^2 - 3$ using bisection method with accuracy $\varepsilon = 0.01$ in the interval [1, 2].

A	b	$f(a)$	$f(b)$	$c = (a + b)/2$	$f(c)$	$f(a)*f(c)$	Update	$b - a$
1.0	2.0	-2.0	1.0	1.5	-0.75	> 0	a = c	0.5
1.5	2.0	-0.75	1.0	1.75	0.062	< 0	b = c	0.25
1.5	1.75	-0.75	0.0625	1.625	-0.359	> 0	a = c	0.125
1.625	1.75	-0.3594	0.0625	1.6875	-0.1523	> 0	a = c	0.0625
1.6875	1.75	-0.1523	0.0625	1.7188	-0.0457	> 0	a = c	0.0313
1.7188	1.75	-0.0457	0.0625	1.7344	0.0081	< 0	b = c	0.0156
1.71988	1.7344	-0.0457	0.0081	1.7266	-0.0189	> 0	a = c	0.0078

3.3.2 Order of Convergence

There are different iterative methods in finding the roots of an equation. To compare the methods order of convergence is used.

Let ε_n be the error in the nth iteration for a root α of $f(x) = 0$, then

$$\varepsilon_n = |x_n - \alpha|$$

If

$$\lim_{n\to\infty} \frac{\varepsilon_{n-1}}{\varepsilon_n^R} = A \quad \text{(const}\}$$

then the order of convergence of the sequence $\{x_n\}$ is R.

In special case,

If $R = 1$, the convergence is called linear.
If $R = 2$, the convergence is called quadratic.
If $1 < R < 2$, the convergence is superlinear.

3.3.3 The Secant Method

In **Secant** method two values of x near the root is used and the root is approximated by the x-intercept of the secant line (chord) joining the two points. The straight line through the two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is $y - f(x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_2)$

On the x-axis $y = 0$ and let $x = x_3$, then $-f(x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x_3 - x_2)$

Solving for x_3, $\quad x_3 = x_2 - \frac{(x_2 - x_1)}{f(x_2) - f(x_1)} f(x_2)$

The estimated value will be closer to the root than either of the two initial points. We continue the process to get better approximation of the root by using the last two computed points using the iteration formula

$$x_{n+2} = x_{n+1} - \frac{(x_{n+1} - x_n)}{f(x_{n+1}) - f(x_n)} f(x_{n+1}), \; n \geq 1$$

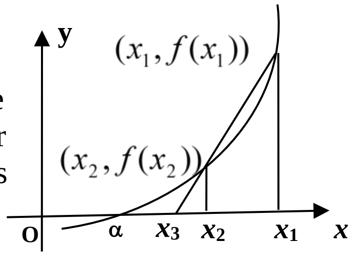


Fig.4.3 The Secant Method

Here it is not necessary that the interval $[x_{n+1}, x_n]$ should contain the root i.e. $f(x_{n+1})f(x_n) < 0$. In selecting, x_1 and x_2 care should be taken so that x_2 is closer to the root than x_1 to get rapid convergence. This can be achieved by selecting x_1 and x_2 such that $|f(x_2)| < |f(x_1)|$

Secant Method in MATLAB

```
a=input('Enter function:','s');
f=inline(a)
x(1)=input('Enter first point of guess interval: ');
x(2)=input('Enter second point of guess interval: ');
n=input('Enter allowed Error in calculation: ');
iteration=0;
fori=3:100
  x(i) = x(i-1) - (f(x(i-1)))*((x(i-1) - x(i-2))/(f(x(i-1)) - f(x(i-2))));
   iteration=iteration+1;
if abs((x(i)-x(i-1))/x(i))*100<n
        root=x(i)
        iteration=iteration
break
end
end
```

Example 3.6: Find the root of $f(x) = \cos x + 2\sin x + x^2$ using secant method initiating with $x_1 = 0$ and $x_2 = -0.1$ with tolerancy/accuracy $\varepsilon = 0.001$

n	x_{n+2}	$f(x_{n+2})$	$\lvert x_{n+2} - x_{n+1} \rvert$
1	-0.5136	0.1522	0.4136
2	-0.6100	0.0457	0.0964
3	-0.6514	0.0065	0.0414
4	-0.6582	0.0013	0.0068
5	-0.6598	0.0006	0.0016
6	0.6595	0.0002	0.0003

3.3.4 Newton-Raphson Method

The procedure known as **Newton's method** is also called **Newton-Raphson method**. In this method, the root of the equation $f(x)=0$ is approximated by the x-intercept of the tangent line through a guess value x_0 near the root.

The equation of the tangent through $(x_0, f(x_0))$ is

$$y - f(x_0) = f'(x_0)(x - x_0)$$

On the x-axis $y=0$ and let $x = x_1$, then

$$-f(x_0) = f'(x_0)(x_1 - x_0)$$

Solving for x_1, $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

The process can be repeated with the new estimate of x until we reach the required degree of accuracy.

In general, the iterative formula for the process can be expressed as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, 3, \cdots$$

Or for a simple root the convergence of the Newton-Raphson method is of order two.

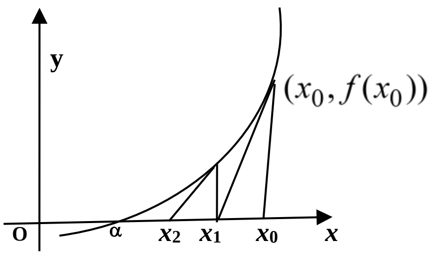


Fig.4.4 Newton-Raphson Method

```
syms x;
fprintf('Newton Raphson Method\n');
Fun=input('\nType a function \n');
xi=input('\nType initial value\n');
F = sym(Fun);   % reduce function to symbolic form
dF = diff(F);   % evaluate derivative in symbolic form
tol=1.0e-5;

for k=1:10
Fxi = subs(F, 'x', xi);     % caalculate value of the function
dFxi = subs(dF, 'x', xi);   % calculaate value of derivative
xi2=xi-Fxi/dFxi;   % iteration using Newton-Raphson formula
xn=vpa(xi2,3);   % reduce vaues to 7 d.p.
IX=[k xn];
disp(IX)
tol2=abs(xn-xi);
if(tol2<tol)
break;
end
xi=xn;
end
```

Example 3.7: Consider the function $f(x) = \sin x + 3x - 1$.

i. Perform one iteration using Newton-Raphson formula for finding its root near $x = 0$.
ii. Write MATLAB syntax for finding the root in $[0, 1]$ using MATLAB function "**fzero**".

Solution: i.

$$f(x) = \sin x + 3x - 1$$
$$f'(x) = \cos x + 3$$
$$f(0) = -1, \quad f'(0) = 4$$
$$x_1 = 0 - \frac{-1}{4} = 0.25$$

iii. MATLAB code:

```
>> F=@(x) sin(x)+3*x-1;              % entered as function handle
>> Sol = fzero(F, [0,1])

Sol =
   0.2507
```

3.3.5 Starting Value for Iteration

If the starting value is reasonably close to the root, the number of iterations needed will be less and calculation time will be saved. Use specified starting value, if stated. Otherwise, first find an interval in which a root lies and then choose as a starting value, x_0, either

(i) one of the end-points of the interval where the magnitudes of the value of the function is small., or
(ii) guess an internal point of the interval closer to the root.

Example 3.8: Suppose that the displacement of a body at time t under a dumping oscillation is given by

$$x(t) = e^{-t}(\cos\sqrt{3}\,t + 2\sqrt{3}\sin\sqrt{3}\,t)$$

Determine the time required for the body to come to rest i.e. $x(t) = 0$.

i. Apply bisection method **four** times in the interval $[1, 2]$ to find the new smaller interval where the time t lies.
ii. Perform **four** iteration using Newton-Raphson formula for finding its time t near $t = 1$.
iii. Perform **four** iteration using secant method for finding its time t initiating with $t_0 = 1$ and $t_1 = 2$.

Solution:
i.

a	*b*	*f(a)*	*f(b)*	*c = (a + b)/2*	*f(c)*	*f(a)*f(c)*	*Update*
1.0	2.0	1.1988	-0.2769	1.5	0.2088	> 0	a = c
1.5	2.0	0.2088	-0.2769	1.75	-0.1063	< 0	b = c
1.5	1.75	0.2088	-0.1063	1.625	0.0326	> 0	a = c
1.625	1.75	0.0326	-0.1063	1.6875	-0.0415	< 0	b = c

ii. $x(t) = e^{-t}(\cos\sqrt{3}\,t + 2\sqrt{3}\sin\sqrt{3}\,t)$

$$x'(t) = e^{-t}\left(-\sqrt{3}\sin\sqrt{3}\,t + 6\cos\sqrt{3}\,t\right) - e^{-t}(\cos\sqrt{3}\,t + 2\sqrt{3}\sin\sqrt{3}\,t)$$

Using the formula $t_{n+1} = t_n - \frac{x(t_n)}{x'(t_n)}$ for $n = 0, 1, 2, 3$ we get
$t_0 = 1, t_1 = 1.5494, t_2 = 1.6429, t_3 = 1.6515, t_4 = 1.6515.$

iii. Using the formula $t_{n+2} = t_{n+1} - \frac{t_{n+1} - t_n}{x(t_{n+1}) - x(t_n)}$ for $n = 0, 1, 2, 3$ we get
$t_0 = 1, t_1 = 2, t_2 = 1.6429, t_3 = 1.6515, t_4 = 1.6515$

3.3.6 Multiple Roots

Equal (repeated) roots are known as the multiple roots. If the root α of $f(x) = 0$ is a repeated root, then we may write

$$f(x) = (x - \alpha)^m g(x) = 0$$

where $g(x)$ is bounded and $g(\alpha) \neq 0$.

The root α is called a *multiple root of multiplicity m*. We obtain from the above equation

$$f(\alpha) = f'(\alpha) = \cdots = f^{(m-1)}(\alpha) = 0, \quad f^{(m)}(\alpha) \neq 0$$

For a multiple root, the order of convergence is reduced, but the order may be increased by modifying the methods discussed. If the multiplicity *m* of the root is known in advance, then some of the methods can be modified so that they have the same rate of convergence as that for simple roots.

3.3.7 Modified Newton-Raphson Method

For a multiple root the order of convergence of the Newton-Raphson formula is linear. The order can be increased by the modified formula

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, 3, \cdots$$

where *m* is the multiplicity of the root.

The order of convergence of the above is two as that of the simple root.

When the multiplicity of the root is not known in advance, we may proceed as follows;

The function

$$u(x) = \frac{f(x)}{f'(x)}$$

has a simple root α regardless of the multiplicity of the root of $f(x) = 0$.

When the Newton-Raphson method is applied to the simple root αof $u(x) = 0$, we have

$$x_{n+1} = x_n - \frac{u(x_n)}{u'(x_n)}$$

or

$$x_{n+1} = x_n - \frac{f(x_n) f'(x_n)}{[f'(x_n)]^2 - f(x_n) f''(x_n)} .$$

Example 3.9: The equation $686x^3 - 735x^2 + 125 = 0$ has a double root near $x = 0.5$. Compute the iterative results using modified Newton-Raphson formula.
Here we may take

$$f(x) = 686x^3 - 735x^2 + 125$$

and

$$f'(x) = 2058x^2 - 1470x.$$

Newton-Raphson iterative formula for double root is

$$x_{n+1} = x_n - 2\,\frac{f(x_n)}{f'(x_n)}$$

$$= x_n + \frac{2(686x_n^3 - 735x_n^2 + 125)}{2058x_n^2 - 1470x_n}$$

N	xn	f(xn)	f'(xn)
0	0.5	27	-220.5
1	0.7449	0.70855	46.93183
2	0.7147	0.00013	0.609353
3	0.7143	1.5E-07	0.021
4	0.7143	1.5E-07	0.021

3.3.8 Fixed Point Iteration Method

A **fixed point** of a function $g(x)$ is a real number α such that $\alpha = g(\alpha)$. This means α is a root of the equation $x = g(x)$. To find a root of the equation $f(x) = 0$ by an iterative method, first rearrange the equation into a form $x = g(x)$. The function $g(x)$ is called the iteration function. Note that there is no unique form $x = g(x)$ into which the equation can be rearranged.

An iteration formula is then $x_{n+1} = g(x_n), \quad n = 0, 1, 2, 3, \cdots$

If $x_0$is an approximation close to a root of $f(x) = 0$ and $x_{n+1} = g(x_n)$ is an iterative formula used to find the root of the equation near x_0, then

(i) if $|g'(x_1)| < 1$, the sequence $x_2, x_3, x_4, \cdots$ will converge to the root. In particular,

(a)if $-1 < g'(x_1) < 0$, the sequence will oscillate and converge to the root

(b)if $0 < g'(x_1) < 1$, the sequence will converge to the root without oscillating.

(ii) if $|g'(x_1)| \geq 1$, the sequence $x_2, x_3, x_4, \cdots$ will diverge.

Example 3.10 : The equation $x^3 + 2x - 5 = 0$ has a root near $x = 1.4$. Using following iteration formulae, perform few iterations and comment on the results.

(a) $x_{n+1} = \frac{1}{2}(5 - x_n^3)$ (b) $x_{n+1} = (5 - 2x_n)^{1/3}$ (c) $x_{n+1} = \dfrac{2x_n^3 + 5}{3x_n^2 + 2}$

The calculation using the three iteration formulae are as follows:

n	Formula (a)	Formula (b)	Formula (c)
1	1.4	1.4	1.4
2	1.128	1.300591	1.33096
3	1.782375	1.338646	1.32827
4	-0.331181	1.324336	1.32827
5	2.518162	1.329753	1.32827
6	-5.48401	1.327708	1.32827
7	84.96401	1.328481	1.32827
8	-306670	1.328189	1.32827
9	1.44E+16	1.328299	1.32827
10	-1.5E+48	1.328257	1.32827

The iteration function in (a) is $g_a(x) = \frac{1}{2}(5 - x^3)$ and $g'_a(x) = -\frac{3}{2}x^2$

With $x_1 = 1.4$, $g'_a(1.4) = -2.94$.

So $|g'_a(1.4)| > 1$ and the sequence will not converge to the root as shown in the above table.

The iteration function in (b) is $g_b(x) = (5 - 2x)^{1/3}$ and $g'_b(x) = -\frac{2}{3}\frac{1}{(5-2x)^{2/3}}$

With $x_1 = 1.4$, $g'_b(1.4) = -0.394$.

Since $g'_b(1.4)$ is negative and $|g'_b(1.4)| < 1$, the sequence will converge to the root with oscillation as shown in the above table.

The iteration function in (c) is $g_c(x) = \frac{2x^3 + 5}{3x^2 + 2}$ and $g'_c(x) = \frac{6x(x^3 + 2x - 5)}{(3x^2 + 2)^2}$

With $x_1 = 1.4$, $g'_c(1.4) = 0.0736$.

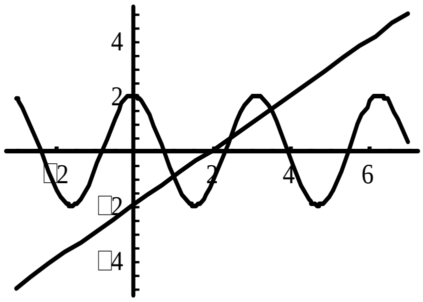
Since $g'_c(1.4)$ is positive and $|g'_c(1.4)| < 1$ with small value, the sequence will converge rapidly to the root without oscillation as shown in the above table.

Example 3.11 : Given that $f(x) = 2\cos 2x + 2 - x$.

a. i. Find the number of real roots of the equation $f(x) = 0$.
   ii. For each root, find an interval where it lies.
b. i. Show that the equation $f(x) = 0$ has a root in $[0.8, 1.6]$.
   ii. Use Bisection method twice to find the new smaller interval where the root lies.
c. Apply Secant method once to find the root to 2 d.p in the last interval obtained by using Bisection method.
d. i. Write down an iteration formula based on Newton-Raphson formula.
   ii. Perform one iteration starting with the value obtained in (c).
   iii. Write down MATLAB commands to execute the iteration four times.
e. An iterative formula $x_{n+1} = \frac{1}{4}(2 + 3x_n + 2\cos 2x_n)$ can be used to estimate the root of $f(x) = 0$
   i. State with reason whether the iterative formula will converge to the root near $x_0 = 3.5$.
   ii. If the iterative formula converges to the root do the iteration two times to estimate the root to 3 decimal places
   iii. Write down MATLAB commands to execute the iterations five times.

a. i. The equation $f(x)=0$ i.e. $2\cos 2x+2-x=0$ can be written as $2\cos 2x=x-2$

Consider the graphs of $y=2\cos 2x$ and $y=x-2$. The two curves intersect at three points and hence the number of real roots is 3.



ii. Consider the values of $f(x)$ for different values of x:

x	*f(x)*
0	4
1	0.2
2	-1.3
3	0.9
4	-2.3

From the above table we see that

$f(1)f(2)$ = - 0.26 <0, a root lies in (1, 2)
$f(2)f(3)$ = - 1.17 <0, a root lies in (2, 3)
$f(3)f(4)$ = - 2.27 <0, a root lies in (3, 4)

b. i. It can be seen that $f(0.8)=1.142$ and $f(1.6)=-1.596$

Here $f(0.8)f(1.6)<0$. Thus, a root lies in $(0.8,\ 1.6)$.

ii. Applying Bisection method on $(0.8,\ 1.6)$, we have

A	*b*	*f(a)*	*f(b)*	*c = (a + b)/2*	*f(c)*	*f(a)*f(c)*	*Update*	*b − a*
0.8	1.6	1.142	-1.596	1.2	-0.675	< 0	b = c	0.8
0.8	1.2	1.142	-0.675	1.0	0.16771	> 0	a = c	0.4
1.0	1.2	0.16771	-0.675	1.1				

c. Given starting values for Secant method is $x_1 = 1$ and $x_2 = 1.2$

n	x_n	$f(x_n)$
1	1	0.1677
2	1.2	-0.6748
3	1.0398	

Root to 2 d.p. is $x_3 = 1.04$.

d. i. When Newton-Raphson method applied to the equation, we have

$$f(x)=2\cos 2x+2-x$$

$$f'(x)=-4\sin 2x-1.$$

Newton-Raphson iterative formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{2\cos(2x_n)+2-x_n}{-4\sin(2x_n)-1}.$$

ii. Using the starting value $x_0 = 1.04$ from (c), we may proceed as follows:

n	x_n	$f(x_n)$	$f'(x_n)$
0	1.04	-0.01496	-4.49253
2	1.03667		

iii.
```
f=@(x) 2*cos(2*x)+2-x;
fd=@(x) -4*sin(2*x)-1;
x(1)=1.04;
for n=1:4
x(n+1)=x(n)-f(x(n))/fd(x(n));
end
```
Output: >> Solution = x

1.0400   1.0367   1.0367   1.0367   1.0367

e. i. In this case the iterative function $g(x)$ is given by

$$g(x) = \frac{1}{4}(2 + 3x + 2\cos 2x)$$

$$g'(x) = \frac{1}{4}(3 - 4\sin 2x)$$

and $$g'(3.5) = 0.093$$

The iterative formula will converge to the root near 3.5 since $|g'(3.5)| = 0.093 < 1$

ii. With $x_1 = 3.5$, the successive iterates are given below:

n	x_n	$\lvert x_{n+1} - x_n \rvert$
1	3.5	
2	3.501951	0.01951
3	3.502130	0.000179

iii.
```
f=@(x) (2+3*x+2*cos(2*x))/4;
x(1)=3.5;
for n=1:5
x(n+1)=f(x(n));
end
```
Output: Solution = x

3.5000   3.5020   3.5021   3.5021   3.5021   3.5021

Exercise 3

1. Given the following polynomial equations and an interval.

a. $x^3 - 5x + 1 = 0$; $[2, 3]$,   c. $x^4 - 2x - 5 = 0$; $[0, 2]$,

b. $x^3 + x^2 - 2x - 5 = 0$; $[1,2]$,   d. $x^4 + x^2 - 80 = 0$; $[2.90, 2.92]$.

i. Find the number of real roots of the equation by graphical method. Find also the number of complex roots, if any.

ii. Apply bisection method two times in the given interval to find the new smaller interval of this root.

iii. Apply secant method to estimate the root correct to 2 d.p. in the last interval acquired by using bisection method.

iv. Write down an iteration formula based on Newton-Raphson method.

v. Perform one iteration starting using the above formula (iv.) with a suitable value in the given interval to estimate the root to 2 d.p.

vi. Write down MATLAB codes to execute the iteration four times.

vii. Write MATLAB commands "**roots**" to find all the roots including complex roots.

2. Given the following nonlinear equations.

a. $2\sin\left(\frac{x}{2}\right) - x^2 + 1 = 0,$

b. $\cos 2x + 4 - x = 0,$

c. $\frac{\sin x}{x} + e^{-x} = 0$

d. $x\,(\cos x + x - x^2) = 0,$

e. $2e^x - x^2 - 7 = 0,$

f. $x^2 + \ln x - 2 = 0.$

g. $\ln(x) - 2x + 7 = 0,$

h. $\cosh(x) - x^2 = 0,$

i. $\sin^{-1} x + x^2 - 1 = 0.$

i. For each functions, find a suitable interval for which the root lies.

ii. Use the bisection method two times to find the new smaller interval in one of the interval obtained in(ii).

iii. Use secant method in the interval obtained in (iii) to find the root of the equation correct to 2 decimal places.

iv. Write an iterative formula based on Newton-Raphson method and iterate 2 times with a suitable value.

v. Write MATLAB codes to execute the iterative formula five times in part (v).

vi. Use MATLAB builtin function "**fzero"** to find all the roots of each equation using an interval or a guess value.

3. Given the equation $5x + \text{sinhx} - e^x - 6 = 0.$

i. Apply Newton-Raphson method in the interval $[1,2]$ to estimate the root correct to 3 d.p.

ii. The following iterative formulae are suggested to estimate the root of the above equation.

a. $x_{n+1} = \frac{1}{5}(e^x - \text{sinhx} + 6)$ b. $x_{n+1} = \frac{1}{6}(e^x - \text{sinhx} + 6)$)

State with reason which iterative formula will converge faster to the root near $x = 1.$

iii. Use the suitable iterative formula from the above two (a) and (b) to find the root correct to 2 decimal places.

iv. Write MATLAB codes to execute the iterative formula in (iii) five times.

4. Given the equation $4\cos\left(\frac{x}{2}\right) - 3x = 0$

i. Apply bisection method **two** times in the interval $[1.0, 1.2]$ to find the new smaller interval where the root lies.

ii. The following iterative formulae are suggested to estimate the root of the above equation

(i) $x_{n+1} = \frac{1}{4}\left[x_n + 4\cos\left(\frac{x_n}{2}\right)\right],$ (ii) $x_{n+1} = \frac{4}{3}\cos\left(\frac{x_n}{2}\right).$

State with reason which iterative formula will converge faster to the root near $x = 1.1.$

iii. Use the suitable iterative formula from the above two ((i) and (ii)) to find the root correct to 2 decimal places.

iv. Use MATLAB built-in function "**fzero(fun,x0)**" to find all the roots of the given equation in the interval $[0, 2]$.

5.

Given the equation $xe^x - x - 3 = 0$.

i. The following iterative formulae are suggested to estimate the root of the above equation.

$$\text{a. } x_{n+1} = \frac{1}{8}[9x_n - x_n e^{x_n} + 3], \qquad \text{b. } x_{n+1} = \frac{1}{10}[11x_n - x_n e^{x_n} + 3].$$

State with reason which iterative formula will converge faster to the root near $x = 1.2$.

ii. Use the suitable iterative formula from the above two (a) and (b) to find the root correct to 2 decimal places.

iii. Write MATLAB codes to execute the iterative formula used in (ii) five times.

6. Consider the polynomial equation $x^4 + x^2 - 2x - 5 = 0$.

a. Find the number of real roots of the equation by graphical method. Find also the number of complex roots, if any.

b. Show that the equation has a root in $[1, 2]$.

c. An iterative formula $x_{n+1} = (5 + 2x_n - x_n^2)^{1/4}$ may be used to estimate the root. Verify whether the iterative formula will converge to the root near $x_0 = 1.7$.

d. If the iterative formula converges to the root do the iteration two times to estimate the root to 3 decimal places.

e. Write MATLAB commands "**roots(p)**" to find all the roots including complex roots.

7. A quarterback throws a pass to his wide receiver running a route. The quarterback releases the ball at a height of h_Q. The wide receiver is supposed to catch the ball straight down the field 60 ft away at a height of h_R. The equation that describes the motion of the football is the familiar equation of projectile motion from physics:

$$y = x\tan(\theta) - \frac{1}{2}\frac{x^2 g}{v_0^2}\frac{1}{\cos^2(\theta)} + h_Q$$

where x and y are the horizontal and vertical distance, respectively, $g = 32.2\ \text{ft/s}^2$ is the acceleration due to gravity, v_0 is the initial velocity of the football as it leaves the quarterback's throwing hand. For $v_0 = 50$ ft/s, $x = 60$ ft, $h_Q = 6.5$ ft, and $h_R = 7$ ft, find the angle θ at which the quarterback must launch the ball.

a. Apply bisection method **three** times in the interval $[0, 0.5]$ to find the new smaller interval where the angle θ lies.

b. Perform **three** iteration using Newton-Raphson formula for finding its angle θ near $\theta = 0$

c. Perform **three** iteration using secant method for finding its angle θ initiating with $\theta_1 = 0$ and $\theta_2 = 1$.

8. You are designing a spherical tank (Fig. P5.17) to hold water for a small village in a developing country. The volume of liquid it can hold can be computed as

$$V = \pi h^2 \frac{[3R - h]}{3}$$

where V = volume [m^3], h = depth of water in tank [m], and R = the tank radius [m].

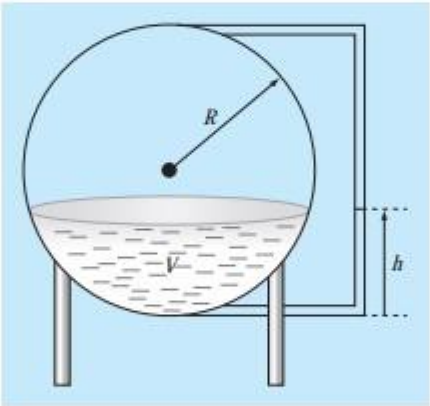


Figure P5.17

If $R = 3$ m, to what depth must the tank be filled so that it holds 30 m^3?

Use N-R method to calculate the depth.