Chapter 4

Solving a System of Linear Equations

Core Topics

Gauss elimination method (4.2).

Gauss elimination with pivoting (*4.3*).

Gauss–Jordan elimination method (4.4).

LU decomposition method (4.5).

Inverse of a matrix (4.6)

Iterative methods (Jacobi, Gauss–Seidel) (4.7).

Use of MATLAB's built-in functions for solving a system of linear equations (4.8).

Complementary Topics

Tridiagonal systems of equations (4.9).

Error, residual, norms, and condition number (4.10).

Ill-conditioned systems (4.11).

4.1 BACKGROUND

Systems of linear equations that have to be solved simultaneously arise in problems that include several (possibly many) variables that are dependent on each other. Such problems occur not only in engineering and science, which are the focus of this book, but in virtually any discipline (business, statistics, economics, etc.). A system of two (or three) equations with two (or three) unknowns can be solved manually by substitution or other mathematical methods (e.g., Cramer's rule, Section 2.4.6). Solving a system in this way is practically impossible as the number of equations (and unknowns) increases beyond three.

An example of a problem in electrical engineering that requires a solution of a system of equations is shown in Fig. 4-1. Using Kirchhoff's law, the currents i_1, i_2, i_3, and i_4 can be determined by solving the following system of four equations:

$$\begin{aligned} 9i_1 - 4i_2 - 2i_3 &= 24 \\ -4i_1 + 17i_2 - 6i_3 - 3i_4 &= -16 \\ -2i_1 - 6i_2 + 14i_3 - 6i_4 &= 0 \\ -3i_2 - 6i_3 + 11i_4 &= 18 \end{aligned} \tag{4.1}$$

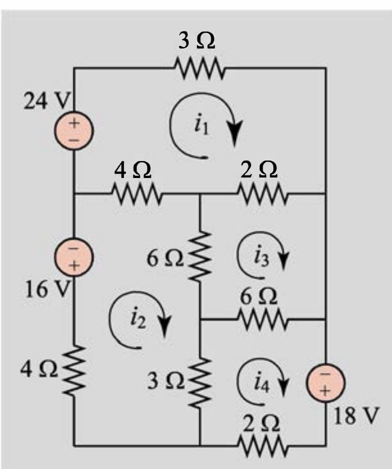


Figure 4-1: Electrical circuit.

Obviously, more complicated circuits may require the solution of a system with a larger number of equations. Another example that requires a solution of a system of equations is calculating the force in members of a truss. The forces in the eight members of the truss shown in Fig. 4-2 are determined from the solution of the following system of eight equations (equilibrium equations of pins A, B, C, and D):

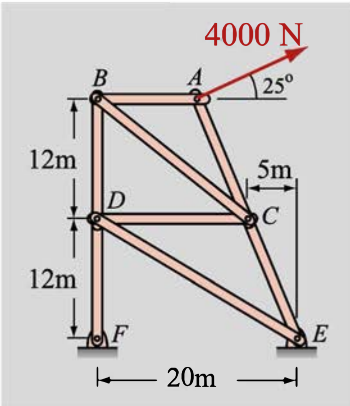


Figure 4-2: Eight-member truss.

$$
\begin{aligned}
&0.9231F_{AC} = 1690 && -F_{AB} - 0.3846F_{AC} = 3625 \\
&F_{AB} - 0.7809F_{BC} = 0 && 0.6247F_{BC} - F_{BD} = 0 \\
&F_{CD} + 0.8575F_{DE} = 0 && F_{BD} - 0.5145F_{DE} - F_{DF} = 0 \\
&0.3846F_{CE} - 0.3846F_{AC} - 0.7809F_{BC} - F_{CD} = 0 && \\
&0.9231F_{AC} + 0.6247F_{BC} - 0.9231F_{CE} = 0 &&
\end{aligned}
\tag{4.2}
$$

There are applications, for example, in finite element and finite difference analysis, where the system of equations that has to be solved contains thousands (or even millions) of simultaneous equations.

4.1.1 Overview of Numerical Methods for Solving a System of Linear Algebraic Equations

The general form of a system of n linear algebraic equations is:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad\quad \vdots \quad &\quad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n
\end{aligned}
\tag{4.3}
$$

$$
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix}
$$

Figure 4-3: A system of n linear algebraic equations.

The matrix form of the equations is shown in Fig. 4-3. Two types of numerical methods, ***direct*** and ***iterative***, are used for solving systems of linear algebraic equations. In direct methods, the solution is calculated by performing arithmetic operations with the equations. In iterative methods, an initial approximate solution is assumed and then used in an iterative process for obtaining successively more accurate solutions.

Direct methods

In direct methods, the system of equations that is initially given in the general form, Eqs. (4.3), is manipulated to an equivalent system of equations that can be easily solved. Three systems of equations that can be easily solved are the ***upper triangular***, ***lower triangular***, and ***diagonal*** forms.

The ***upper triangular*** form is shown in Eqs. (4.4), and is written in a matrix form for a system of four equations in Fig. 4-4.

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}
$$

Figure 4-4: A system of four equations in upper triangular form.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\
a_{33}x_3 + \ldots + a_{3n}x_n &= b_3 \\
\vdots \qquad\qquad \vdots \quad &\quad \vdots \\
a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n &= b_{n-1} \\
a_{nn}x_n &= b_n
\end{aligned}
\tag{4.4}
$$

The system in this form has all zero coefficients below the diagonal and is solved by a procedure called ***back substitution***. It starts with the last equation, which is solved for x_n. The value of x_n is then substituted in the next-to-the-last equation, which is solved for x_{n-1}. The process continues in the same manner all the way up to the first equation. In the case of four equations, the solution is given by:

$$x_4 = \frac{b_4}{a_{44}}, \quad x_3 = \frac{b_3 - a_{34}x_4}{a_{33}}, \quad x_2 = \frac{b_2 - (a_{23}x_3 + a_{24}x_4)}{a_{22}}, \quad \text{and}$$

$$x_1 = \frac{b_1 - (a_{12}x_2 + a_{13}x_3 + a_{14}x_4)}{a_{11}}$$

For a system of n equations in upper triangular form, a general formula for the solution using back substitution is:

$$\begin{aligned} x_n &= \frac{b_n}{a_{nn}} \\ x_i &= \frac{b_i - \sum_{j=i+1}^{j=n} a_{ij}x_j}{a_{ii}} \qquad i = n-1, n-2, \ldots, 1 \end{aligned} \tag{4.5}$$

In Section 4.2 the upper triangular form and back substitution are used in the Gauss elimination method.

The ***lower triangular*** form is shown in Eqs. (4.6), and is written in matrix form for a system of four equations in Fig. 4-5.

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Figure 4-5: A system of four equations in lower triangular form.

$$\begin{aligned} a_{11}x_1 \qquad\qquad\qquad\qquad &= b_1 \\ a_{21}x_1 + a_{22}x_2 \qquad\qquad\qquad &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \qquad\quad &= b_3 \\ \vdots \qquad\qquad \vdots \qquad\qquad\qquad &\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n &= b_n \end{aligned} \tag{4.6}$$

The system in this form has zero coefficients above the diagonal. A system in lower triangular form is solved in the same way as the upper triangular form but in an opposite order. The procedure is called ***forward substitution***. It starts with the first equation, which is solved for x_1. The value of x_1 is then substituted in the second equation, which is solved for x_2. The process continues in the same manner all the way down to the last equation. In the case of four equations, the solution is given by:

$$x_1 = \frac{b_1}{a_{11}}, \quad x_2 = \frac{b_2 - a_{21}x_1}{a_{22}}, \quad x_3 = \frac{b_3 - (a_{31}x_1 + a_{32}x_2)}{a_{33}}, \quad \text{and}$$

$$x_4 = \frac{b_4 - (a_{41}x_1 + a_{42}x_2 + a_{43}x_3)}{a_{44}} \tag{4.7}$$

For a system of n equations in lower triangular form, a general formula for the solution using forward substitution is:

$$x_1 = \frac{b_1}{a_{11}}$$
$$x_i = \frac{b_i - \sum_{j=1}^{j=i-1} a_{ij}x_j}{a_{ii}} \qquad i = 2, 3, \ldots, n \tag{4.8}$$

In Section 4.5 the lower triangular form is used together with the upper triangular form in the *LU* decomposition method for solving a system of equations.

The ***diagonal*** form of a system of linear equations is shown in Eqs. (4.9), and is written in matrix form for a system of four equations in Fig. 4-6.

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_2 \\ b_4 \end{bmatrix}$$

Figure 4-6: A system of four equations in diagonal form.

$$\begin{array}{lllll} a_{11}x_1 & & & & = b_1 \\ & a_{12}x_2 & & & = b_2 \\ & & a_{13}x_3 & & = b_3 \\ & & \vdots & & \vdots \\ & & & a_nx_n & = b_n \end{array} \tag{4.9}$$

A system in diagonal form has nonzero coefficients along the diagonal and zeros everywhere else. Obviously, a system in this form can be easily solved. A similar form is used in the Gauss–Jordan method, which is presented in Section 4.4.

From the three forms of simultaneous linear equations (upper triangular, lower triangular, diagonal) it might appear that changing a given system of equations to the diagonal form is the best choice because the diagonal system is the easiest to solve. In reality, however, the total number of operations required for solving a system is smaller when other methods are used.

Three direct methods for solving systems of equations—Gauss elimination (Sections 4.2 and 4.3), Gauss–Jordan (Section 4.4), and *LU* decomposition (Section 4.5)—and two indirect (iterative) methods—Jacobi and Gauss–Seidel (Section 4.7)—are described in this chapter.

4.2 GAUSS ELIMINATION METHOD

The Gauss elimination method is a procedure for solving a system of linear equations. In this procedure, a system of equations that is given in a general form is manipulated to be in ***upper triangular*** form, which is then solved by using back substitution (see Section 4.1.1). For a set of four equations with four unknowns the general form is given by:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= b_1 && (4.10a) \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= b_2 && (4.10b) \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 &= b_3 && (4.10c) \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 &= b_4 && (4.10d) \end{aligned} \tag{4.10}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Figure 4-7: Matrix form of a system of four equations.

The matrix form of the system is shown in Fig. 4-7. In the Gauss elimination method, the system of equations is manipulated into an equivalent system of equations that has the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= b_1 && (4.11a)\\ a'_{22}x_2 + a'_{23}x_3 + a'_{24}x_4 &= b'_2 && (4.11b)\\ a'_{33}x_3 + a'_{34}x_4 &= b'_3 && (4.11c)\\ a'_{44}x_4 &= b'_4 && (4.11d) \end{aligned} \qquad (4.11)$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a'_{33} & a'_{34} \\ 0 & 0 & 0 & a'_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ b'_4 \end{bmatrix}$$

Figure 4-8: Matrix form of the equivalent system.

The first equation in the equivalent system, (4.11*a*), is the same as (4.10*a*). In the second equation, (4.11*b*), the variable x_1 is eliminated. In the third equation, (4.11*c*), the variables x_1 and x_2 are eliminated. In the fourth equation, (4.11*d*), the variables x_1, x_2, and x_3 are eliminated. The matrix form of the equivalent system is shown in Fig. 4-8. The system of equations (4.11) is in upper triangular form, which can be easily solved by using back substitution.

In general, various mathematical manipulations can be used for converting a system of equations from the general form displayed in Eqs. (4.10) to the ***upper triangular*** form in Eqs. (4.11). One in particular, the Gauss elimination method, is described next. The procedure can be easily programmed in a computer code.

Gauss elimination procedure (forward elimination)

The Gauss elimination procedure is first illustrated for a system of four equations with four unknowns. The starting point is the set of equations that is given by Eqs. (4.10). Converting the system of equations to the form given in Eqs. (4.11) is done in steps.

Step 1**:** In the first step, the first equation is unchanged, and the terms that include the variable x_1 in all the other equations are eliminated. This is done one equation at a time by using the first equation, which is called the ***pivot equation***. The coefficient a_{11} is called the ***pivot coefficient***, or the pivot element. To eliminate the term $a_{21}x_1$ in Eq. (4.10*b*), the pivot equation, Eq. (4.10*a*), is multiplied by $m_{21} = a_{21}/a_{11}$, and then the equation is subtracted from Eq. (4.10*b*):

$$\begin{array}{rl} & a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2 \\ - & m_{21}(a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4) = m_{21}b_1 \\ \hline & 0 + \underbrace{(a_{22} - m_{21}a_{12})}_{a'_{22}}x_2 + \underbrace{(a_{23} - m_{21}a_{13})}_{a'_{23}}x_3 + \underbrace{(a_{24} - m_{21}a_{14})}_{a'_{24}}x_4 = \underbrace{b_2 - m_{21}b_1}_{b'_2} \end{array}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Figure 4-9: Matrix form of the system after eliminating a_{21}.

It should be emphasized here that the pivot equation, Eq. (4.10*a*), itself is not changed. The matrix form of the equations after this operation is shown in Fig. 4-9.

Next, the term $a_{31}x_1$ in Eq. (4.10*c*) is eliminated. The pivot equation, Eq. (4.10*a*), is multiplied by $m_{31} = a_{31}/a_{11}$ and then is subtracted

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & a'_{32} & a'_{33} & a'_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ b_4 \end{bmatrix}$$

Figure 4-10: Matrix form of the system after eliminating a_{31}.

from Eq. (4.10*c*):

$$\begin{array}{rl} & a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = b_3 \\ - & m_{31}(a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4) = m_{31}b_1 \\ \hline & 0 + \underbrace{(a_{32} - m_{31}a_{12})}_{a'_{32}}x_2 + \underbrace{(a_{33} - m_{31}a_{13})}_{a'_{33}}x_3 + \underbrace{(a_{34} - m_{31}a_{14})}_{a'_{34}}x_4 = \underbrace{b_3 - m_{31}b_1}_{b'_3} \end{array}$$

The matrix form of the equations after this operation is shown in Fig. 4-10.

Next, the term $a_{41}x_1$ in Eq. (4.10*d*) is eliminated. The pivot equation, Eq. (4.10*a*), is multiplied by $m_{41} = a_{41}/a_{11}$ and then is subtracted from Eq. (4.10*d*):

$$\begin{array}{rl} & a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 = b_4 \\ - & m_{41}(a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4) = m_{41}b_1 \\ \hline & 0 + \underbrace{(a_{42} - m_{41}a_{12})}_{a'_{42}}x_2 + \underbrace{(a_{43} - m_{41}a_{13})}_{a'_{43}}x_3 + \underbrace{(a_{44} - m_{41}a_{14})}_{a'_{44}}x_4 = \underbrace{b_4 - m_{41}b_1}_{b'_4} \end{array}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & a'_{32} & a'_{33} & a'_{34} \\ 0 & a'_{42} & a'_{43} & a'_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ b'_4 \end{bmatrix}$$

Figure 4-11: Matrix form of the system after eliminating a_{41}.

This is the end of ***Step 1***. The system of equations now has the following form:

$$\left.\begin{array}{rcll} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 & = & b_1 & (4.12a) \\ 0 + a'_{22}x_2 + a'_{23}x_3 + a'_{24}x_4 & = & b'_2 & (4.12b) \\ 0 + a'_{32}x_2 + a'_{33}x_3 + a'_{34}x_4 & = & b'_3 & (4.12c) \\ 0 + a'_{42}x_2 + a'_{43}x_3 + a'_{44}x_4 & = & b'_4 & (4.12d) \end{array}\right\} \quad (4.12)$$

The matrix form of the equations after this operation is shown in Fig. 4-11. Note that the result of the elimination operation is to reduce the first column entries, except a_{11} (the pivot element), to zero.

Step 2: In this step, Eqs. (4.12*a*) and (4.12*b*) are not changed, and the terms that include the variable x_2 in Eqs. (4.12*c*) and (4.12*d*) are eliminated. In this step, Eq. (4.12*b*) is the pivot equation, and the coefficient a'_{22} is the pivot coefficient. To eliminate the term $a'_{32}x_2$ in Eq. (4.12*c*), the pivot equation, Eq. (4.12*b*), is multiplied by $m_{32} = a'_{32}/a'_{22}$ and then is subtracted from Eq. (4.12*c*):

$$\begin{array}{rl} & a'_{32}x_2 + a'_{33}x_3 + a'_{34}x_4 = b'_3 \\ - & m_{32}(a'_{22}x_2 + a'_{23}x_3 + a'_{24}x_4) = m_{32}b'_2 \\ \hline & 0 + \underbrace{(a'_{33} - m_{32}a'_{23})}_{a''_{33}}x_3 + \underbrace{(a'_{34} - m_{32}a'_{24})}_{a''_{34}}x_4 = \underbrace{b'_3 - m_{32}b'_2}_{b''_3} \end{array}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a''_{33} & a''_{34} \\ 0 & a'_{42} & a'_{43} & a'_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ b'_4 \end{bmatrix}$$

Figure 4-12: Matrix form of the system after eliminating a_{32}.

The matrix form of the equations after this operation is shown in Fig. 4-12.

Next, the term $a'_{42}x_2$ in Eq. (4.12*d*) is eliminated. The pivot equation, Eq. (4.12*b*), is multiplied by $m_{42} = a'_{42}/a'_{22}$ and then is subtracted from Eq. (4.12*d*):

$$\begin{array}{rrcl} & a'_{42}x_2 + a'_{43}x_3 + a'_{44}x_4 & = & b'_4 \\ - & m_{42}(a'_{22}x_2 + a'_{23}x_3 + a'_{24}x_4) & = & m_{42}b'_2 \\ \hline & 0 + \underbrace{(a'_{43} - m_{42}a'_{23})}_{a''_{43}}x_3 + \underbrace{(a'_{44} - m_{42}a'_{24})}_{a''_{44}}x_4 & = & \underbrace{b'_4 - m_{42}b'_2}_{b''_4} \end{array}$$

This is the end of ***Step 2***. The system of equations now has the following form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= b_1 && (4.13a) \\ 0 + a'_{22}x_2 + a'_{23}x_3 + a'_{24}x_4 &= b'_2 && (4.13b) \\ 0 + \quad 0 \quad + a''_{33}x_3 + a''_{34}x_4 &= b''_3 && (4.13c) \\ 0 + \quad 0 \quad + a''_{43}x_3 + a''_{44}x_4 &= b''_4 && (4.13d) \end{aligned} \qquad (4.13)$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a''_{33} & a''_{34} \\ 0 & 0 & a''_{43} & a''_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ b''_4 \end{bmatrix}$$

Figure 4-13: Matrix form of the system after eliminating a_{42}.

The matrix form of the equations at the end of ***Step 2*** is shown in Fig. 4-13.

***Step 3*:** In this step, Eqs. (4.13*a*), (4.13*b*), and (4.13*c*) are not changed, and the term that includes the variable x_3 in Eq. (4.13*d*) is eliminated. In this step, Eq. (4.13*c*) is the pivot equation, and the coefficient a''_{33} is the pivot coefficient. To eliminate the term $a''_{43}x_3$ in Eq. (4.13*d*), the pivot equation is multiplied by $m_{43} = a''_{43}/a''_{33}$ and then is subtracted from Eq. (4.13*d*):

$$\begin{array}{rrcl} & a''_{43}x_3 + a''_{44}x_4 & = & b''_4 \\ - & m_{43}(a''_{33}x_3 + a''_{34}x_4) & = & m_{43}b''_3 \\ \hline & \underbrace{(a''_{44} - m_{43}a''_{34})}_{a'''_{44}}x_4 & = & \underbrace{b''_4 - m_{43}b''_3}_{b'''_4} \end{array}$$

This is the end of ***Step 3***. The system of equations is now in an upper triangular form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= b_1 && (4.14a) \\ 0 + a'_{22}x_2 + a'_{23}x_3 + a'_{24}x_4 &= b'_2 && (4.14b) \\ 0 + \quad 0 \quad + a''_{33}x_3 + a''_{34}x_4 &= b''_3 && (4.14c) \\ 0 + \quad 0 \quad + \quad 0 \quad + a'''_{44}x_4 &= b'''_4 && (4.14d) \end{aligned} \qquad (4.14)$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a''_{33} & a''_{34} \\ 0 & 0 & 0 & a'''_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ b'''_4 \end{bmatrix}$$

Figure 4-14: Matrix form of the system after eliminating a_{43}.

The matrix form of the equations is shown in Fig. 4-14. Once transformed to upper triangular form, the equations can be easily solved by using back substitution. The three steps of the Gauss elimination process are illustrated together in Fig. 4-15.

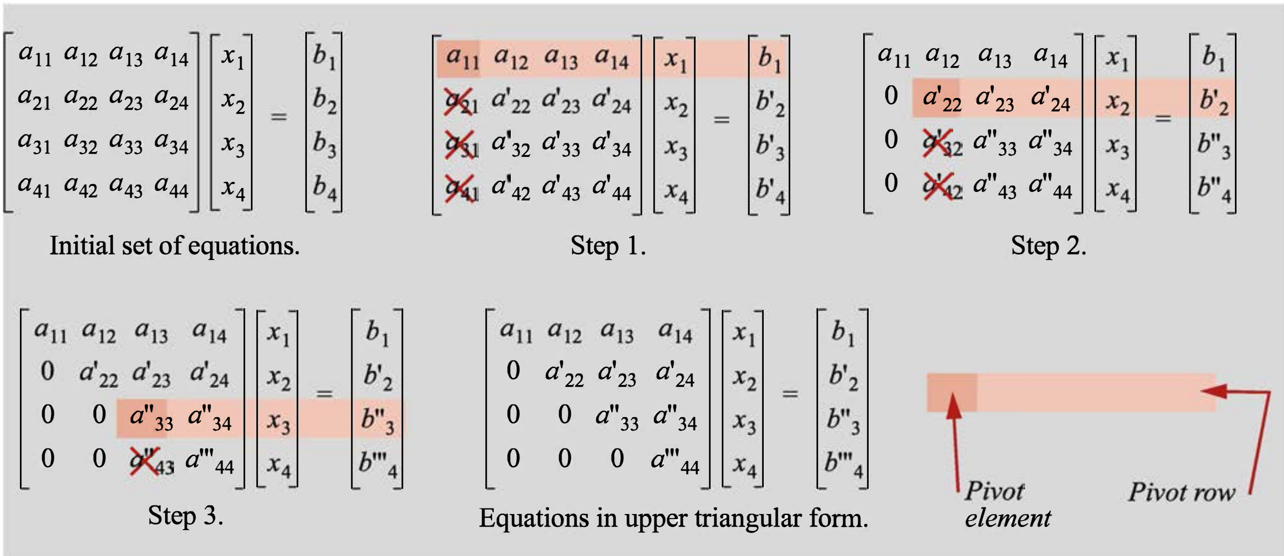


Figure 4-15: Gauss elimination procedure.

Example 4-1 shows a manual application of the Gauss elimination method for solving a system of four equations.

Example 4-1: Solving a set of four equations using Gauss elimination.

Solve the following system of four equations using the Gauss elimination method.

$$\begin{aligned} 4x_1 - 2x_2 - 3x_3 + 6x_4 &= 12 \\ -6x_1 + 7x_2 + 6.5x_3 - 6x_4 &= -6.5 \\ x_1 + 7.5x_2 + 6.25x_3 + 5.5x_4 &= 16 \\ -12x_1 + 22x_2 + 15.5x_3 - x_4 &= 17 \end{aligned}$$

SOLUTION

The solution follows the steps presented in the previous pages.

Step 1: The first equation is the pivot equation, and 4 is the pivot coefficient.

Multiply the pivot equation by $m_{21} = (-6)/4 = -1.5$ and subtract it from the second equation:

$$\begin{aligned} & -6x_1 + 7x_2 + 6.5x_3 - 6x_4 = -6.5 \\ - \; & (-1.5)(4x_1 - 2x_2 - 3x_3 + 6x_4) = (-6/4)\cdot 12 \\ \hline & 0x_1 + 4x_2 + 2x_3 + 3x_4 = 11.5 \end{aligned}$$

Multiply the pivot equation by $m_{31} = (1/4) = 0.25$ and subtract it from the third equation:

$$\begin{aligned} & x_1 + 7.5x_2 + 6.25x_3 + 5.5x_4 = 16 \\ - \; & (0.25)(4x_1 - 2x_2 - 3x_3 + 6x_4) = (1/4)\cdot 12 \\ \hline & 0x_1 + 8x_2 + 7x_3 + 4x_4 = 13 \end{aligned}$$

Multiply the pivot equation by $m_{41} = (-12)/4 = -3$ and subtract it from the fourth equation:

$$\begin{aligned} & -12x_1 + 22x_2 + 15.5x_3 - x_4 = 17 \\ - \; & (-3)(4x_1 - 2x_2 - 3x_3 + 6x_4) = -3\cdot 12 \\ \hline & 0x_1 + 16x_2 + 6.5x_3 + 17x_4 = 53 \end{aligned}$$

At the end of ***Step 1***, the four equations have the form:

$$\begin{aligned} 4x_1 - 2x_2 - 3x_3 + 6x_4 &= 12 \\ 4x_2 + 2x_3 + 3x_4 &= 11.5 \\ 8x_2 + 7x_3 + 4x_4 &= 13 \\ 16x_2 + 6.5x_3 + 17x_4 &= 53 \end{aligned}$$

Step 2: The second equation is the pivot equation, and 4 is the pivot coefficient.
Multiply the pivot equation by $m_{32} = 8/4 = 2$ and subtract it from the third equation:

$$\begin{array}{rrl} & 8x_2 + 7x_3 + 4x_4 &= 13 \\ - & 2(4x_2 + 2x_3 + 3x_4) &= 2 \cdot 11.5 \\ \hline & 0x_2 + 3x_3 - 2x_4 &= -10 \end{array}$$

Multiply the pivot equation by $m_{42} = 16/4 = 4$ and subtract it from the fourth equation:

$$\begin{array}{rrl} & 16x_2 + 6.5x_3 + 17x_4 &= 53 \\ - & 4(4x_2 + 2x_3 + 3x_4) &= 4 \cdot 11.5 \\ \hline & 0x_2 - 1.5x_3 + 5x_4 &= 7 \end{array}$$

At the end of ***Step 2***, the four equations have the form:

$$\begin{aligned} 4x_1 - 2x_2 - 3x_3 + 6x_4 &= 12 \\ 4x_2 + 2x_3 + 3x_4 &= 11.5 \\ 3x_3 - 2x_4 &= -10 \\ -1.5x_3 + 5x_4 &= 7 \end{aligned}$$

Step 3: The third equation is the pivot equation, and 3 is the pivot coefficient.
Multiply the pivot equation by $m_{43} = (-1.5)/3 = -0.5$ and subtract it from the fourth equation:

$$\begin{array}{rrl} & -1.5x_3 + 5x_4 &= 7 \\ - & -0.5(3x_3 - 2x_4) &= -0.5 \cdot -10 \\ \hline & 0x_3 + 4x_4 &= 2 \end{array}$$

At the end of ***Step 3***, the four equations have the form:

$$\begin{aligned} 4x_1 - 2x_2 - 3x_3 + 6x_4 &= 12 \\ 4x_2 + 2x_3 + 3x_4 &= 11.5 \\ 3x_3 - 2x_4 &= -10 \\ 4x_4 &= 2 \end{aligned}$$

Once the equations are in this form, the solution can be determined by back substitution. The value of x_4 is determined by solving the fourth equation:

$$x_4 = 2/4 = 0.5$$

Next, x_4 is substituted in the third equation, which is solved for x_3:

$$x_3 = \frac{-10 + 2x_4}{3} = \frac{-10 + 2 \cdot 0.5}{3} = -3$$

Next, x_4 and x_3 are substituted in the second equation, which is solved for x_2:

$$x_2 = \frac{11.5 - 2x_3 - 3x_4}{4} = \frac{11.5 - (2 \cdot -3) - (3 \cdot 0.5)}{4} = 4$$

Lastly, x_4, x_3 and x_2 are substituted in the first equation, which is solved for x_1:

$$x_1 = \frac{12 + 2x_2 + 3x_3 - 6x_4}{4} = \frac{12 + 2 \cdot 4 + 3 \cdot -3 - (6 \cdot 0.5)}{4} = 2$$

The extension of the Gauss elimination procedure to a system with n number of equations is straightforward. The elimination procedure starts with the first row as the pivot row and continues row after row down to one row before the last. At each step, the pivot row is used to eliminate the terms that are below the pivot element in all the rows that are below. Once the original system of equations is changed to upper triangular form, back substitution is used for determining the solution.

When the Gauss elimination method is programmed, it is convenient and more efficient to create one matrix that includes the matrix of coefficients $[a]$ and the right-hand-side vector $[b]$. This is done by appending the vector $[b]$ to the matrix $[a]$, as shown in Example 4-2, where the Gauss elimination method is programmed in MATLAB.

Example 4-2: MATLAB user-defined function for solving a system of equations using Gauss elimination.

Write a user-defined MATLAB function for solving a system of linear equations, $[a][x] = [b]$, using the Gauss elimination method. For function name and arguments, use `x = Gauss(a,b)`, where `a` is the matrix of coefficients, `b` is the right-hand-side column vector of constants, and `x` is a column vector of the solution.

Use the user-defined function `Gauss` to

(*a*) Solve the system of equations of Example 4-1.

(b) Solve the system of Eqs. (4.1).

SOLUTION

The following user-defined MATLAB function solves a system of linear equations. The program starts by appending the column vector $[b]$ to the matrix $[a]$. The new augmented matrix, named in the program `ab`, has the form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} & b_3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} & b_n \end{bmatrix}$$

Next, the Gauss elimination procedure is applied (forward elimination). The matrix is changed such that all the elements below the diagonal of `a` are zero:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ 0 & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\ 0 & 0 & a_{33} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{nn} & b_n \end{bmatrix}$$

At the end of the program, back substitution is used to solve for the unknowns, and the results are assigned to the column vector `x`.

Program 4-1: User-defined function. Gauss elimination.

```
function x = Gauss(a,b)
% The function solves a system of linear equations [a][x] = [b] using the Gauss
% elimination method.
% Input variables:
% a  The matrix of coefficients.
% b  Right-hand-side column vector of constants.
% Output variable:
% x  A column vector with the solution.

ab = [a,b];
[R, C] = size(ab);
for j = 1:R - 1
   for i = j + 1:R
      ab(i,j:C) = ab(i,j:C)- ab(i,j)/ab(j,j)*ab(j,j:C);
   end
end
x = zeros(R,1);
x(R) = ab(R,C)/ab(R,R);
for i = R - 1:-1:1
    x(i) = (ab(i,C) - ab(i,i + 1:R)*x(i + 1:R))/ab(i,i);
end
```

Append the column vector $[b]$ to the matrix $[a]$.

Pivot element.

The multiplier m_{ij}.

Pivot equation.

Gauss elimination procedure (forward elimination).

Back substitution.

The user-defined function `Gauss` is next used in the Command Window, first to solve the system of equations of Example 4-1, and then to solve the system of Eqs. (4.1).

```
>> A=[4 -2 -3 6; -6 7 6.5 -6; 1 7.5 6.25 5.5; -12 22 15.5 -1];
>> B = [12; -6.5; 16; 17];
>> sola = Gauss(A,B)
sola =
    2.0000
    4.0000
   -3.0000
    0.5000
>> C = [9 -4 -2 0; -4 17 -6 -3; -2 -6 14 -6; 0 -3 -6 11];
>> D = [24; -16; 0; 18];
>> solb = Gauss(C,D)
solb =
    4.0343
    1.6545
    2.8452
    3.6395
```

Solution for part (*a*).

Solution for part (*b*).

4.2.1 Potential Difficulties When Applying the Gauss Elimination Method

The pivot element is zero

Since the pivot row is divided by the pivot element, a problem will arise during the execution of the Gauss elimination procedure if the value of the pivot element is equal to zero. As shown in the next section, this situation can be corrected by changing the order of the rows. In a procedure called pivoting, the pivot row that has the zero pivot element is exchanged with another row that has a nonzero pivot element.

The pivot element is small relative to the other terms in the pivot row

Significant errors due to rounding can occur when the pivot element is small relative to other elements in the pivot row. This is illustrated by the following example.

Consider the following system of simultaneous equations for the unknowns x_1 and x_2:

$$\begin{aligned} 0.0003x_1 + 12.34x_2 &= 12.343 \\ 0.4321x_1 + x_2 &= 5.321 \end{aligned} \tag{4.15}$$

The exact solution of the system is $x_1 = 10$ and $x_2 = 1$.

The error due to rounding is illustrated by solving the system using Gaussian elimination on a machine with limited precision so that only four significant figures are retained with rounding. When the first equation of Eqs. (4.15) is entered, the constant on the right-hand side is rounded to 12.34.

The solution starts by using the first equation as the pivot equation and $a_{11} = 0.0003$ as the pivot coefficient. In the first step, the pivot equation is multiplied by $m_{21} = 0.4321/0.0003 = 1440$. With four significant figures and rounding, this operation gives:

$$(1440)(0.0003x_1 + 12.34x_2) = 1440 \cdot 12.34$$

or:

$$0.4320x_1 + 17770x_2 = 17770$$

The result is next subtracted from the second equation in Eqs. (4.15):

$$\begin{array}{rl} & 0.4321x_1 + x_2 = 5.321 \\ - & \underline{0.4320x_1 + 17770x_2 = 17770} \\ & 0.0001x_1 - 17770x_2 = -17760 \end{array}$$

After this operation, the system is:

$$\begin{aligned} 0.0003x_1 + 12.34x_2 &= 12.34 \\ 0.0001x_1 - 17770x_2 &= -17760 \end{aligned}$$

Note that the a_{21} element is not zero but a very small number. Next, the value of x_2 is calculated from the second equation:

$$x_2 = \frac{-17760}{-17770} = 0.9994$$

Then x_2 is substituted in the first equation, which is solved for x_1:

$$x_1 = \frac{12.34 - (12.34 \cdot 0.9994)}{0.0003} = \frac{12.34 - 12.33}{0.0003} = \frac{0.01}{0.0003} = 33.33$$

The solution that is obtained for x_1 is obviously incorrect. The incorrect value is obtained because the magnitude of a_{11} is small when compared to the magnitude of a_{12}. Consequently, a relatively small error (due to round-off arising from the finite precision of a computing machine) in the value of x_2 can lead to a large error in the value of x_1.

The problem can be easily remedied by exchanging the order of the two equations in Eqs. (4.15):

$$\begin{aligned} 0.4321x_1 + x_2 &= 5.321 \\ 0.0003x_1 + 12.34x_2 &= 12.343 \end{aligned} \tag{4.16}$$

Now, as the first equation is used as the pivot equation, the pivot coefficient is $a_{11} = 0.4321$. In the first step, the pivot equation is multiplied by $m_{21} = 0.0003/0.4321 = 0.0006943$. With four significant figures and rounding this operation gives:

$$(0.0006943)(0.4321x_1 + x_2) = 0.0006943 \cdot 5.321$$

or:

$$0.0003x_1 + 0.0006943x_2 = 0.003694$$

The result is next subtracted from the second equation in Eqs. (4.16):

$$\begin{array}{rrcl} - & 0.0003x_1 + 12.34x_2 & = & 12.34 \\ & 0.0003x_1 + 0.0006943x_2 & = & 0.003694 \\ \hline & 12.34x_2 & = & 12.34 \end{array}$$

After this operation, the system is:

$$\begin{aligned} 0.4321x_1 + x_2 &= 5.321 \\ 0x_1 + 12.34x_2 &= 12.34 \end{aligned}$$

Next, the value of x_2 is calculated from the second equation:

$$x_2 = \frac{12.34}{12.34} = 1$$

Then x_2 is substituted in the first equation that is solved for x_1:

$$x_1 = \frac{5.321 - 1}{0.4321} = 10$$

The solution that is obtained now is the exact solution.

In general, a more accurate solution is obtained when the equations are arranged (and rearranged every time a new pivot equation is used) such that the pivot equation has the largest possible pivot element. This is explained in more detail in the next section.

Round-off errors can also be significant when solving large systems of equations even when all the coefficients in the pivot row are of the same order of magnitude. This can be caused by a large number of operations (multiplication, division, addition, and subtraction) associated with large systems.

4.3 GAUSS ELIMINATION WITH PIVOTING

In the Gauss elimination procedure, the pivot equation is divided by the pivot coefficient. This, however, cannot be done if the pivot coefficient is zero. For example, for the following system of three equations:

$$\begin{aligned} 0x_1 + 2x_2 + 3x_3 &= 46 \\ 4x_1 - 3x_2 + 2x_3 &= 16 \\ 2x_1 + 4x_2 - 3x_3 &= 12 \end{aligned}$$

the procedure starts by taking the first equation as the pivot equation and the coefficient of x_1, which is 0, as the pivot coefficient. To eliminate the term $4x_1$ in the second equation, the pivot equation is supposed to be multiplied by $4/0$ and then subtracted from the second equation. Obviously, this is not possible when the pivot element is equal to zero. The division by zero can be avoided if the order in which the equations are written is changed such that in the first equation the first coefficient is not zero. For example, in the system above, this can be done by exchanging the first two equations.

In the general Gauss elimination procedure, an equation (or a row) can be used as the pivot equation (pivot row) only if the pivot coefficient (pivot element) is not zero. If the pivot element is zero, the equation (i.e., the row) is exchanged with one of the equations (rows) that are below, which has a nonzero pivot coefficient. This exchange of rows, illustrated in Fig. 4-16, is called ***pivoting***.

After the first step, the second equation has a pivot element that is equal to zero.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & a'_{23} & a'_{24} \\ 0 & a'_{32} & a'_{33} & a'_{34} \\ 0 & a'_{42} & a'_{43} & a'_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ b'_4 \end{bmatrix}$$

Using pivoting, the second equation is exchanged with the third equation.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{32} & a'_{33} & a'_{34} \\ 0 & 0 & a'_{23} & a'_{24} \\ 0 & a'_{42} & a'_{43} & a'_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_3 \\ b'_2 \\ b'_4 \end{bmatrix}$$

Figure 4-16: Illustration of pivoting.

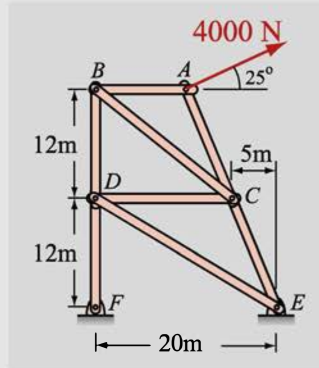
Additional comments about pivoting

- If during the Gauss elimination procedure a pivot equation has a pivot element that is equal to zero, then if the system of equations that is being solved has a solution, an equation with a nonzero element in the pivot position can always be found.
- The numerical calculations are less prone to error and will have fewer round-off errors (see Section 4.2.1) if the pivot element has a larger numerical absolute value compared to the other elements in the same row. Consequently, among all the equations that can be exchanged to be the pivot equation, it is better to select the equation whose pivot element has the largest absolute numerical value. Moreover, it is good to employ pivoting for the purpose of having a pivot equation with the pivot element that has a largest absolute numerical value at all times (even when pivoting is not necessary).

The addition of pivoting to the programming of the Gauss elimination method is shown in the next example. The addition of pivoting every time a new pivot equation is used, such that the pivot row will have the largest absolute pivot element, is assigned as an exercise in Problem 4.21.

Example 4-3: MATLAB user-defined function for solving a system of equations using Gauss elimination with pivoting.

Write a user-defined MATLAB function for solving a system of linear equations $[a][x] = [b]$ using the Gauss elimination method with pivoting. Name the function `x = GaussPivot(a,b)`, where `a` is the matrix of coefficients, `b` is the right-hand-side column vector of constants, and `x` is a column vector of the solution. Use the function to determine the forces in the loaded eight-member truss that is shown in the figure (same as in Fig. 4-2).



SOLUTION

The forces in the eight truss members are determined from the set of eight equations, Eqs. (4.2). The equations are derived by drawing free body diagrams of pins A, B, C, and D and applying equations of equilibrium. The equations are rewritten here in a matrix form (intentionally, the equations are written in an order that requires pivoting):

$$\begin{bmatrix} 0 & 0.9231 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -0.3846 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0.8575 & 0 \\ 1 & 0 & -0.7809 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.3846 & -0.7809 & 0 & -1 & 0.3846 & 0 & 0 \\ 0 & 0.9231 & 0.6247 & 0 & 0 & -0.9231 & 0 & 0 \\ 0 & 0 & 0.6247 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -0.5145 & -1 \end{bmatrix} \begin{bmatrix} F_{AB} \\ F_{AC} \\ F_{BC} \\ F_{BD} \\ F_{CD} \\ F_{CE} \\ F_{DE} \\ F_{DF} \end{bmatrix} = \begin{bmatrix} 1690 \\ 3625 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.17}$$

The function `GaussPivot` is created by modifying the function `Gauss` listed in the solution of Example 4-2.

Program 4-2: User-defined function. Gauss elimination with pivoting.

```
function x = GaussPivot(a,b)
% The function solves a system of linear equations ax = b using the Gauss
% elimination method with pivoting.
% Input variables:
% a   The matrix of coefficients.
% b   Right-hand-side column vector of constants.
% Output variable:
% x   A column vector with the solution.

ab = [a,b];
[R, C] = size(ab);
for j = 1:R - 1
% Pivoting section starts
   if ab(j,j) = = 0
```

Check if the pivot element is zero.

```
      for k = j + 1:R
          if ab(k,j) ~ = 0
              abTemp = ab(j,:);
              ab(j,:) = ab(k,:);
              ab(k,:) = abTemp;
              break
          end
      end
    end
% Pivoting section ends
    for i = j + 1:R
        ab(i,j:C) = ab(i,j:C) - ab(i,j)/ab(j,j)*ab(j,j:C);
    end
end
x = zeros(R,1);
x(R) = ab(R,C)/ab(R,R);
for i = R - 1:-1:1
    x(i) = (ab(i,C) - ab(i,i + 1:R)*x(i + 1:R))/ab(i,i);
end
```

If pivoting is required, search in the rows below for a row with nonzero pivot element.

Switch the row that has a zero pivot element with the row that has a nonzero pivot element.

Stop searching for a row with a nonzero pivot element.

The user-defined function `GaussPivot` is next used in a script file program to solve the system of equations Eq. (4.17).

```
% Example 4-3
a=[0 0.9231 0 0 0 0 0 0; -1 -0.3846 0 0 0 0 0 0; 0 0 0 0 1 0 0.8575 0; 1 0 -0.7809 0 0 0 0 0
    0 -0.3846 -0.7809 0 -1 0.3846 0 0; 0 0.9231 0.6247 0 0 -0.9231 0 0
    0 0 0.6247 -1 0 0 0 0; 0 0 0 1 0 0 -0.5145 -1];
b = [1690;3625;0;0;0;0;0;0];
Forces = GaussPivot(a,b)
```

When the script file is executed, the following solution is displayed in the Command Window.

```
Forces =
  -4.3291e+003
   1.8308e+003
  -5.5438e+003
  -3.4632e+003
   2.8862e+003
  -1.9209e+003
  -3.3659e+003
  -1.7315e+003
>>
```

$$\begin{bmatrix} F_{AB} \\ F_{AC} \\ F_{BC} \\ F_{BD} \\ F_{CD} \\ F_{CE} \\ F_{DE} \\ F_{DF} \end{bmatrix}$$

4.4 GAUSS–JORDAN ELIMINATION METHOD

The Gauss–Jordan elimination method is a procedure for solving a system of linear equations, $[a][x]=[b]$. In this procedure, a system of equations that is given in a general form is manipulated into an equivalent system of equations in ***diagonal*** form (see Section 4.1.1) with normalized elements along the diagonal. This means that when the diagonal form of the matrix of the coefficients, $[a]$, is reduced to the identity matrix, the new vector $[b']$ is the solution. The starting point of the procedure is a system of equations given in a general form (the illustration that follows is for a system of four equations):

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= b_1 \quad (4.18a)\\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= b_2 \quad (4.18b)\\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 &= b_3 \quad (4.18c)\\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 &= b_4 \quad (4.18d) \end{aligned} \tag{4.18}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{bmatrix} = \begin{bmatrix} b_1\\ b_2\\ b_3\\ b_4 \end{bmatrix}$$

Figure 4-17: Matrix form of a system of four equations.

The matrix form of the system is shown in Fig. 4-17. In the Gauss–Jordan elimination method, the system of equations is manipulated to have the following diagonal form:

$$\begin{aligned} x_1 + 0 + 0 + 0 &= b'_1 \quad (4.19a)\\ 0 + x_2 + 0 + 0 &= b'_2 \quad (4.19b)\\ 0 + 0 + x_3 + 0 &= b'_3 \quad (4.19c)\\ 0 + 0 + 0 + x_4 &= b'_4 \quad (4.19d) \end{aligned} \tag{4.19}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{bmatrix} = \begin{bmatrix} b'_1\\ b'_2\\ b'_3\\ b'_4 \end{bmatrix}$$

Figure 4-18: Matrix form of the equivalent system after applying the Gauss–Jordan method.

The matrix form of the equivalent system is shown in Fig. 4-18. The terms on the right-hand side of the equations (column $[b']$) are the solution. In matrix form, the matrix of the coefficients is transformed into an identity matrix.

Gauss–Jordan elimination procedure

The Gauss–Jordan elimination procedure for transforming the system of equations from the form in Eqs. (4.18) to the form in Eqs. (4.19) is the same as the Gauss elimination procedure (see Section 4.2), except for the following two differences:

- The pivot equation is normalized by dividing all the terms in the equation by the pivot coefficient. This makes the pivot coefficient equal to 1.
- The pivot equation is used to eliminate the off-diagonal terms in **ALL** the other equations. This means that the elimination process is applied to the equations (rows) that are above and below the pivot equation. (In the Gaussian elimination method, only elements that are below the pivot element are eliminated.)

When the Gauss–Jordan procedure is programmed, it is convenient and more efficient to create a single matrix that includes the matrix of coefficients $[a]$ and the vector $[b]$. This is done by appending the vector $[b]$ to the matrix $[a]$. The augmented matrix at the starting point of the procedure is shown (for a system of four equations) in Fig. 4-19*a*. At the end of the procedure, shown in Fig. 4-19*b*, the elements of $[a]$ are replaced by an identity matrix, and the column $[b']$ is the solution.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & b_4 \end{bmatrix} \xrightarrow{\text{Gauss–Jordan procedure}} \begin{bmatrix} 1 & 0 & 0 & 0 & b'_1 \\ 0 & 1 & 0 & 0 & b'_2 \\ 0 & 0 & 1 & 0 & b'_3 \\ 0 & 0 & 0 & 1 & b'_4 \end{bmatrix}$$

(*a*) (*b*)

Figure 4-19: Schematic illustration of the Gauss–Jordan method.

The Gauss–Jordan method can also be used for solving several systems of equations $[a][x] = [b]$ that have the same coefficients $[a]$ but different right-hand-side vectors $[b]$. This is done by augmenting the matrix $[a]$ to include all of the vectors $[b]$. In Section 4.6.2 the method is used in this way for calculating the inverse of a matrix.

The Gauss–Jordan elimination method is demonstrated in Example 4-4 where it is used to solve the set of equations solved in Example 4-1.

Example 4-4: Solving a set of four equations using Gauss–Jordan elimination.

Solve the following set of four equations using the Gauss–Jordan elimination method.

$$\begin{aligned} 4x_1 - 2x_2 - 3x_3 + 6x_4 &= 12 \\ -6x_1 + 7x_2 + 6.5x_3 - 6x_4 &= -6.5 \\ x_1 + 7.5x_2 + 6.25x_3 + 5.5x_4 &= 16 \\ -12x_1 + 22x_2 + 15.5x_3 - x_4 &= 17 \end{aligned}$$

SOLUTION

The solution is carried out by using the matrix form of the equations. In matrix form, the system is:

$$\begin{bmatrix} 4 & -2 & -3 & 6 \\ -6 & 7 & 6.5 & -6 \\ 1 & 7.5 & 6.25 & 5.5 \\ -12 & 22 & 15.5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 12 \\ -6.5 \\ 16 \\ 17 \end{bmatrix}$$

For the numerical procedure, a new matrix is created by augmenting the coefficient matrix to include the right-hand side of the equation:

$$\begin{bmatrix} 4 & -2 & -3 & 6 & 12 \\ -6 & 7 & 6.5 & -6 & -6.5 \\ 1 & 7.5 & 6.25 & 5.5 & 16 \\ -12 & 22 & 15.5 & -1 & 17 \end{bmatrix}$$

The first pivoting row is the first row, and the first element in this row is the pivot element. The row is normalized by dividing it by the pivot element:

$$\begin{bmatrix} \frac{4}{4} & \frac{-2}{4} & \frac{-3}{4} & \frac{6}{4} & \frac{12}{4} \\ -6 & 7 & 6.5 & -6 & -6.5 \\ 1 & 7.5 & 6.25 & 5.5 & 16 \\ -12 & 22 & 15.5 & -1 & 17 \end{bmatrix} = \begin{bmatrix} 1 & -0.5 & -0.75 & 1.5 & 3 \\ -6 & 7 & 6.5 & -6 & -6.5 \\ 1 & 7.5 & 6.25 & 5.5 & 16 \\ -12 & 22 & 15.5 & -1 & 17 \end{bmatrix}$$

Next, all the first elements in rows 2, 3, and 4 are eliminated:

$$\begin{bmatrix} 1 & -0.5 & -0.75 & 1.5 & 3 \\ -6 & 7 & 6.5 & -6 & -6.5 \\ 1 & 7.5 & 6.25 & 5.5 & 16 \\ -12 & 22 & 15.5 & -1 & 17 \end{bmatrix} \begin{matrix} \\ \leftarrow -(-6)[1\ -0.5\ -0.75\ 1.5\ 3] \\ \leftarrow -(1)[1\ -0.5\ -0.75\ 1.5\ 3] \\ \leftarrow -(-12)[1\ -0.5\ -0.75\ 1.5\ 3] \end{matrix} = \begin{bmatrix} 1 & -0.5 & -0.75 & 1.5 & 3 \\ 0 & 4 & 2 & 3 & 11.5 \\ 0 & 8 & 7 & 4 & 13 \\ 0 & 16 & 6.5 & 17 & 53 \end{bmatrix}$$

The next pivot row is the second row, with the second element as the pivot element. The row is normalized by dividing it by the pivot element:

$$\begin{bmatrix} 1 & -0.5 & -0.75 & 1.5 & 3 \\ 0 & \frac{4}{4} & \frac{2}{4} & \frac{3}{4} & \frac{11.5}{4} \\ 0 & 8 & 7 & 4 & 13 \\ 0 & 16 & 6.5 & 17 & 53 \end{bmatrix} = \begin{bmatrix} 1 & -0.5 & -0.75 & 1.5 & 3 \\ 0 & 1 & 0.5 & 0.75 & 2.875 \\ 0 & 8 & 7 & 4 & 13 \\ 0 & 16 & 6.5 & 17 & 53 \end{bmatrix}$$

Next, all the second elements in rows 1, 3, and 4 are eliminated:

$$\begin{bmatrix} 1 & -0.5 & -0.75 & 1.5 & 3 \\ 0 & 1 & 0.5 & 0.75 & 2.875 \\ 0 & 8 & 7 & 4 & 13 \\ 0 & 16 & 6.5 & 17 & 53 \end{bmatrix} \begin{matrix} \leftarrow -(-0.5)[0\ 1\ 0.5\ 0.75\ 2.875] \\ \\ \leftarrow -(8)[0\ 1\ 0.5\ 0.75\ 2.875] \\ \leftarrow -(16)[0\ 1\ 0.5\ 0.75\ 2.875] \end{matrix} = \begin{bmatrix} 1 & 0 & -0.5 & 1.875 & 4.4375 \\ 0 & 1 & 0.5 & 0.75 & 2.875 \\ 0 & 0 & 3 & -2 & -10 \\ 0 & 0 & -1.5 & 5 & 7 \end{bmatrix}$$

The next pivot row is the third row, with the third element as the pivot element. The row is normalized by dividing it by the pivot element:

$$\begin{bmatrix} 1 & 0 & -0.5 & 1.875 & 4.4375 \\ 0 & 1 & 0.5 & 0.75 & 2.875 \\ 0 & 0 & \frac{3}{3} & \frac{-2}{3} & \frac{-10}{3} \\ 0 & 0 & -1.5 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -0.5 & 1.875 & 4.4375 \\ 0 & 1 & 0.5 & 0.75 & 2.875 \\ 0 & 0 & 1 & -0.667 & -3.333 \\ 0 & 0 & -1.5 & 5 & 7 \end{bmatrix}$$

Next, all the third elements in rows 1, 2, and 4 are eliminated:

$$\begin{bmatrix} 1 & 0 & -0.5 & 1.875 & 4.4375 \\ 0 & 1 & 0.5 & 0.75 & 2.875 \\ 0 & 0 & 1 & -0.667 & -3.333 \\ 0 & 0 & -1.5 & 5 & 7 \end{bmatrix} \begin{matrix} \leftarrow -(-0.5)[0\ 0\ 1\ -0.667\ -3.333] \\ \leftarrow -(0.5)[0\ 0\ 1\ -0.667\ -3.333] \\ \\ \leftarrow -(-1.5)[0\ 0\ 1\ -0.667\ -3.333] \end{matrix} = \begin{bmatrix} 1 & 0 & 0 & 1.5417 & 2.7708 \\ 0 & 1 & 0 & 1.0833 & 4.5417 \\ 0 & 0 & 1 & -0.667 & -3.333 \\ 0 & 0 & 0 & 4 & 2 \end{bmatrix}$$

The next pivot row is the fourth row, with the fourth element as the pivot element. The row is normalized by dividing it by the pivot element:

$$\begin{bmatrix} 1 & 0 & 0 & 1.5417 & 2.7708 \\ 0 & 1 & 0 & 1.0833 & 4.5417 \\ 0 & 0 & 1 & -0.667 & -3.333 \\ 0 & 0 & 0 & \frac{4}{4} & \frac{2}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1.5417 & 2.7708 \\ 0 & 1 & 0 & 1.0833 & 4.5417 \\ 0 & 0 & 1 & -0.667 & -3.333 \\ 0 & 0 & 0 & 1 & 0.5 \end{bmatrix}$$

Next, all the fourth elements in rows 1, 2, and 3 are eliminated:

$$\begin{bmatrix} 1 & 0 & 0 & 1.5417 & 2.7708 \\ 0 & 1 & 0 & 1.0833 & 4.5417 \\ 0 & 0 & 1 & -0.667 & -3.333 \\ 0 & 0 & 0 & 1 & 0.5 \end{bmatrix} \begin{matrix} \longleftarrow & -(1.5417)[0\ 0\ 0\ 1\ 0.5] \\ \longleftarrow & -(1.0833)[0\ 0\ 0\ 1\ 0.5] \\ \longleftarrow & -(-0.667)[0\ 0\ 0\ 1\ 0.5] \\ & \end{matrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \boxed{2} \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0.5 \end{bmatrix}$$

The solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -3 \\ 0.5 \end{bmatrix}$$

The Gauss–Jordan elimination method with pivoting

It is possible that the equations are written in such an order that during the elimination procedure a pivot equation has a pivot element that is equal to zero. Obviously, in this case it is impossible to normalize the pivot row (divide by the pivot element). As with the Gauss elimination method, the problem can be corrected by using pivoting. This is left as an exercise in Problem 4.22.

4.5 *LU* DECOMPOSITION METHOD

Background

The Gauss elimination method consists of two parts. The first part is the elimination procedure in which a system of linear equations that is given in a general form, $[a][x]=[b]$, is transformed into an equivalent system of equations $[a'][x]=[b']$ in which the matrix of coefficients $[a']$ is upper triangular. In the second part, the equivalent system is solved by using back substitution. The elimination procedure requires many mathematical operations and significantly more computing time than the back substitution calculations. During the elimination procedure, the matrix of coefficients $[a]$ and the vector $[b]$ are both changed. This means that if there is a need to solve systems of equations that have the same left-hand-side terms (same coefficient matrix $[a]$) but different right-hand-side constants (different vectors $[b]$), the elimination procedure has to be carried out for each $[b]$ again. Ideally, it would be better if the operations on the matrix of coefficients $[a]$ were dissociated from those on the vector of constants $[b]$. In this way, the elimination procedure with $[a]$ is done only once and then is used for solving systems of equations with different vectors $[b]$.

One option for solving various systems of equations $[a][x] = [b]$ that have the same coefficient matrices $[a]$ but different constant vectors $[b]$ is to first calculate the inverse of the matrix $[a]$. Once the inverse matrix $[a]^{-1}$ is known, the solution can be calculated by:

$$[x] = [a]^{-1}[b]$$

Calculating the inverse of a matrix, however, requires many mathematical operations, and is computationally inefficient. A more efficient method of solution for this case is the *LU* decomposition method.

In the *LU* decomposition method, the operations with the matrix $[a]$ are done without using, or changing, the vector $[b]$, which is used only in the substitution part of the solution. The *LU* decomposition method can be used for solving a single system of linear equations, but it is especially advantageous for solving systems that have the same coefficient matrices $[a]$ but different constant vectors $[b]$.

The LU decomposition method

The *LU* decomposition method is a method for solving a system of linear equations $[a][x] = [b]$. In this method the matrix of coefficients $[a]$ is decomposed (factored) into a product of two matrices $[L]$ and $[U]$:

$$[a] = [L][U] \tag{4.20}$$

where the matrix $[L]$ is a lower triangular matrix and $[U]$ is an upper triangular matrix. With this decomposition, the system of equations to be solved has the form:

$$[L][U][x] = [b] \tag{4.21}$$

To solve this equation, the product $[U][x]$ is defined as:

$$[U][x] = [y] \tag{4.22}$$

and is substituted in Eq. (4.21) to give:

$$[L][y] = [b] \tag{4.23}$$

Now, the solution $[x]$ is obtained in two steps. First, Eq. (4.23) is solved for $[y]$. Then, the solution $[y]$ is substituted in Eq. (4.22), and that equation is solved for $[x]$.

Since the matrix $[L]$ is a lower triangular matrix, the solution $[y]$ in Eq. (4.23) is obtained by using the forward substitution method. Once $[y]$ is known and is substituted in Eq. (4.22), this equation is solved by using back substitution, since $[U]$ is an upper triangular matrix.

For a given matrix $[a]$ several methods can be used to determine the corresponding $[L]$ and $[U]$. Two of the methods, one related to the Gauss elimination method and another called Crout's method, are described next.

4.5.1 LU Decomposition Using the Gauss Elimination Procedure

When the Gauss elimination procedure is applied to a matrix $[a]$, the elements of the matrices $[L]$ and $[U]$ are actually calculated. The upper triangular matrix $[U]$ is the matrix of coefficients $[a]$ that is obtained at the end of the procedure, as shown in Figs. 4-8 and 4-14. The lower triangular matrix $[L]$ is not written explicitly during the procedure, but the elements that make up the matrix are actually calculated along the way. The elements of $[L]$ on the diagonal are all 1, and the elements below the diagonal are the multipliers m_{ij} that multiply the pivot equation when it is used to eliminate the elements below the pivot coefficient (see the ***Gauss elimination procedure*** in Section 4.2). For the case of a system of four equations, the matrix of coefficients $[a]$ is (4×4), and the decomposition has the form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ m_{21} & 1 & 0 & 0 \\ m_{31} & m_{32} & 1 & 0 \\ m_{41} & m_{42} & m_{43} & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a''_{33} & a''_{34} \\ 0 & 0 & 0 & a'''_{44} \end{bmatrix} \tag{4.24}$$

A numerical example illustrating LU decomposition is given next. It uses the information in the solution of Example 4-1, where a system of four equations is solved by using the Gauss elimination method. The matrix $[a]$ can be written from the given set of equations in the problem statement, and the matrix $[U]$ can be written from the set of equations at the end of ***step 3*** (page 107). The matrix $[L]$ can be written by using the multipliers that are calculated in the solution. The decomposition has the form:

$$\begin{bmatrix} 4 & -2 & -3 & 6 \\ -6 & 7 & 6.5 & -6 \\ 1 & 7.5 & 6.25 & 5.5 \\ -12 & 22 & 15.5 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1.5 & 1 & 0 & 0 \\ 0.25 & 2 & 1 & 0 \\ -3 & 4 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 & -3 & 6 \\ 0 & 4 & 2 & 3 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 4 \end{bmatrix} \tag{4.25}$$

The decomposition in Eq. (4.25) can be verified by using MATLAB:

```
>> L = [1,0,0,0;-1.5,1,0,0;0.25,2,1,0;-3,4,-0.5,1]
L =
    1.0000         0         0         0
   -1.5000    1.0000         0         0
    0.2500    2.0000    1.0000         0
   -3.0000    4.0000   -0.5000    1.0000
>> U = [4,-2,-3,6;0,4,2,3;0,0,3,-2;0,0,0,4]
```

```
U =
     4    -2    -3     6
     0     4     2     3
     0     0     3    -2
     0     0     0     4
>> L*U
ans =
    4.0000   -2.0000   -3.0000    6.0000
   -6.0000    7.0000    6.5000   -6.0000
    1.0000    7.5000    6.2500    5.5000
  -12.0000   22.0000   15.5000   -1.0000
```

Multiplication of the matrices L and U verifies that the answer is the matrix $[a]$.

4.5.2 LU Decomposition Using Crout's Method

In this method the matrix $[a]$ is decomposed into the product $[L][U]$, where the diagonal elements of the matrix $[U]$ are all 1s. It turns out that in this case, the elements of both matrices can be determined using formulas that can be easily programmed. This is illustrated for a system of four equations. In Crout's method, the LU decomposition has the form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{bmatrix} \begin{bmatrix} 1 & U_{12} & U_{13} & U_{14} \\ 0 & 1 & U_{23} & U_{24} \\ 0 & 0 & 1 & U_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.26}$$

Executing the matrix multiplication on the right-hand side of the equation gives:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} L_{11} & (L_{11}U_{12}) & (L_{11}U_{13}) & (L_{11}U_{14}) \\ L_{21} & (L_{21}U_{12}+L_{22}) & (L_{21}U_{13}+L_{22}U_{23}) & (L_{21}U_{14}+L_{22}U_{24}) \\ L_{31} & (L_{31}U_{12}+L_{32}) & (L_{31}U_{13}+L_{32}U_{23}+L_{33}) & (L_{31}U_{14}+L_{32}U_{24}+L_{33}U_{34}) \\ L_{41} & (L_{41}U_{12}+L_{42}) & (L_{41}U_{13}+L_{42}U_{23}+L_{43}) & (L_{41}U_{14}+L_{42}U_{24}+L_{43}U_{34}+L_{44}) \end{bmatrix} \tag{4.27}$$

The elements of the matrices $[L]$ and $[U]$ can be determined by solving Eq. (4.27). The solution is obtained by equating the corresponding elements of the matrices on both sides of the equation. Looking at Eq. (4.27), one can observe that the elements of the matrices $[L]$ and $[U]$ can be easily determined row after row from the known elements of $[a]$ and the elements of $[L]$ and $[U]$ that are already calculated. Starting with the first row, the value of L_{11} is calculated from $L_{11} = a_{11}$. Once L_{11} is known, the values of U_{12}, U_{13}, and U_{14} are calculated by:

$$U_{12} = \frac{a_{12}}{L_{11}} \quad U_{13} = \frac{a_{13}}{L_{11}} \quad \text{and} \quad U_{14} = \frac{a_{14}}{L_{11}} \tag{4.28}$$

Moving on to the second row, the value of L_{21} is calculated from $L_{21} = a_{21}$ and the value of L_{22} is calculated from:

$$L_{22} = a_{22} - L_{21}U_{12} \tag{4.29}$$

With the values of L_{21} and L_{22} known, the values of U_{23} and U_{24} are determined from:

$$U_{23} = \frac{a_{23} - L_{21}U_{13}}{L_{22}} \quad \text{and} \quad U_{24} = \frac{a_{24} - L_{21}U_{14}}{L_{22}} \tag{4.30}$$

In the third row:

$$L_{31} = a_{31}, \quad L_{32} = a_{32} - L_{31}U_{12}, \quad \text{and} \quad L_{33} = a_{33} - L_{31}U_{13} - L_{32}U_{23} \tag{4.31}$$

Once the values of L_{31}, L_{32}, and L_{33} are known, the value of U_{34} is calculated by:

$$U_{34} = \frac{a_{34} - L_{31}U_{14} - L_{32}U_{24}}{L_{33}} \tag{4.32}$$

In the fourth row, the values of L_{41}, L_{42}, L_{43}, and L_{44} are calculated by:

$$L_{41} = a_{41}, \quad L_{42} = a_{42} - L_{41}U_{12}, \quad L_{43} = a_{43} - L_{41}U_{13} - L_{42}U_{23}, \quad \text{and}$$
$$L_{44} = a_{44} - L_{41}U_{14} - L_{42}U_{24} - L_{43}U_{34} \tag{4.33}$$

A procedure for determining the elements of the matrices $[L]$ and $[U]$ can be written by following the calculations in Eqs. (4.28) through (4.33). If $[a]$ is an $(n \times n)$ matrix, the elements of $[L]$ and $[U]$ are given by:

Step 1: Calculating the first column of $[L]$:

$$\text{for} \quad i = 1, 2, \ldots, n \qquad L_{i1} = a_{i1} \tag{4.34}$$

Step 2: Substituting 1s in the diagonal of $[U]$:

$$\text{for} \quad i = 1, 2, \ldots, n \qquad U_{ii} = 1 \tag{4.35}$$

Step 3: Calculating the elements in the first row of $[U]$ (except U_{11} which was already calculated):

$$\text{for} \quad j = 2, 3, \ldots, n \qquad U_{1j} = \frac{a_{1j}}{L_{11}} \tag{4.36}$$

Step 4: Calculating the rest of the elements row after row (i is the row number and j is the column number). The elements of $[L]$ are calculated first because they are used for calculating the elements of $[U]$:

for $i = 2, 3, \ldots, n$

$$\text{for} \quad j = 2, 3, \ldots, i \qquad L_{ij} = a_{ij} - \sum_{k=1}^{k=j-1} L_{ik}U_{kj} \tag{4.37}$$

$$\text{for} \quad j = (i+1), (i+2), \ldots, n \qquad U_{ij} = \frac{a_{ij} - \sum_{k=1}^{k=j-1} L_{ik}U_{kj}}{L_{ii}} \tag{4.38}$$

Examples 4-5 and 4-6 show how the LU decomposition with Crout's method is used for solving systems of equations. In Example 4-5 the calculations are done manually, and in Example 4-6 the decomposition is done with a user-defined MATLAB program.

Example 4-5: Solving a set of four equations using LU decomposition with Crout's method.

Solve the following set of four equations (the same as in Example 4-1) using LU decomposition with Crout's method.

$$\begin{aligned} 4x_1 - 2x_2 - 3x_3 + 6x_4 &= 12 \\ -6x_1 + 7x_2 + 6.5x_3 - 6x_4 &= -6.5 \\ x_1 + 7.5x_2 + 6.25x_3 + 5.5x_4 &= 16 \\ -12x_1 + 22x_2 + 15.5x_3 - x_4 &= 17 \end{aligned}$$

SOLUTION

First, the equations are written in matrix form:

$$\begin{bmatrix} 4 & -2 & -3 & 6 \\ -6 & 7 & 6.5 & -6 \\ 1 & 7.5 & 6.25 & 5.5 \\ -12 & 22 & 15.5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 12 \\ -6.5 \\ 16 \\ 17 \end{bmatrix} \tag{4.39}$$

Next, the matrix of coefficients $[a]$ is decomposed into the product $[L][U]$, as shown in Eq. (4.27). The decomposition is done by following the steps listed on the previous page:

Step 1: Calculating the first column of $[L]$:

for $i = 1, 2, 3, 4$ $\quad L_{i1} = a_{i1}$: $\; L_{11} = 4, \quad L_{21} = -6, \quad L_{31} = 1, \quad L_{41} = -12$

Step 2: Substituting 1s in the diagonal of $[U]$:

for $i = 1, 2, 3, 4$ $\quad U_{ii} = 1$: $\; U_{11} = 1, \quad U_{22} = 1, \quad U_{33} = 1, \quad U_{44} = 1.$

Step 3: Calculating the elements in the first row of $[U]$ (except U_{11} which was already calculated):

for $j = 2, 3, 4$ $\quad U_{1j} = \dfrac{a_{1j}}{L_{11}}$: $\; U_{12} = \dfrac{a_{12}}{L_{11}} = \dfrac{-2}{4} = -0.5, \quad U_{13} = \dfrac{a_{13}}{L_{11}} = \dfrac{-3}{4} = -0.75,$

$$U_{14} = \frac{a_{14}}{L_{11}} = \frac{6}{4} = 1.5$$

Step 4: Calculating the rest of the elements row after row, starting with the second row (i is the row number and j is the column number). In the present problem there are four rows, so i starts at 2 and ends with 4. For each value of i (each row), the elements of L are calculated first, and the elements of U are calculated subsequently. The general form of the equations is (Eqs. (4.37) and (4.38)):

for $i = 2, 3, 4,$

$$\text{for } j = 2, 3, \ldots, i \qquad L_{ij} = a_{ij} - \sum_{k=1}^{k=j-1} L_{ik} U_{kj} \tag{4.40}$$

$$\text{for } j = (i+1), (i+2), \ldots, n \qquad U_{ij} = \frac{a_{ij} - \sum_{k=1}^{k=i-1} L_{ik} U_{kj}}{L_{ii}} \tag{4.41}$$

Starting with the second row, $i = 2$,

for $j = 2$: $L_{22} = a_{22} - \sum_{k=1}^{k=1} L_{2k}U_{k2} = a_{22} - L_{21}U_{12} = 7 - (-6 \cdot -0.5) = 4$

for $j = 3, 4$: $$U_{23} = \frac{a_{23} - \sum_{k=1}^{k=1} L_{2k}U_{k3}}{L_{22}} = \frac{a_{23} - (L_{21}U_{13})}{L_{22}} = \frac{6.5 - (-6 \cdot -0.75)}{4} = 0.5$$

$$U_{24} = \frac{a_{24} - \sum_{k=1}^{k=1} L_{2k}U_{k4}}{L_{22}} = \frac{a_{24} - (L_{21}U_{14})}{L_{22}} = \frac{-6 - (-6 \cdot 1.5)}{4} = 0.75$$

Next, for the third row, $i = 3$,

for $j = 2, 3$: $L_{32} = a_{32} - \sum_{k=1}^{k=1} L_{3k}U_{k2} = a_{32} - L_{31}U_{12} = 7.5 - (1 \cdot -0.5) = 8$

$$L_{33} = a_{33} - \sum_{k=1}^{k=2} L_{3k}U_{k3} = a_{33} - (L_{31}U_{13} + L_{32}U_{23}) = 6.25 - (1 \cdot -0.75 + 8 \cdot 0.5) = 3$$

for $j = 4$: $$U_{34} = \frac{a_{34} - \sum_{k=1}^{k=2} L_{3k}U_{k4}}{L_{33}} = \frac{a_{34} - (L_{31}U_{14} + L_{32}U_{24})}{L_{33}} = \frac{5.5 - (1 \cdot 1.5 + 8 \cdot 0.75)}{3} = -0.6667$$

For the last row, $i = 4$,

for $j = 2, 3, 4$: $L_{42} = a_{42} - \sum_{k=1}^{k=1} L_{4k}U_{k2} = a_{42} - L_{41}U_{12} = 22 - (-12 \cdot -0.5) = 16$

$$L_{43} = a_{43} - \sum_{k=1}^{k=2} L_{4k}U_{k3} = a_{43} - (L_{41}U_{13} + L_{42}U_{23}) = 15.5 - (-12 \cdot -0.75 + 16 \cdot 0.5) = -1.5$$

$$L_{44} = a_{44} - \sum_{k=1}^{k=4} L_{4k}U_{k4} = a_{44} - (L_{41}U_{14} + L_{42}U_{24} + L_{43}U_{34}) = -1 - (-12 \cdot 1.5 + 16 \cdot 0.75 + -1.5 \cdot -0.6667) = 4$$

Writing the matrices $[L]$ and $[U]$ in a matrix form,

$$L = \begin{bmatrix} 4 & 0 & 0 & 0 \\ -6 & 4 & 0 & 0 \\ 1 & 8 & 3 & 0 \\ -12 & 16 & -1.5 & 4 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & -0.5 & -0.75 & 1.5 \\ 0 & 1 & 0.5 & 0.75 \\ 0 & 0 & 1 & -0.6667 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To verify that the two matrices are correct, they are multiplied by using MATLAB:

```
>> L = [4 0 0 0; -6, 4 0 0; 1 8 3 0; -12 16 -1.5 4];
>> U = [1 -0.5 -0.75 1.5; 0 1 0.5 0.75; 0 0 1 -0.6667; 0 0 0 1];
>> L*U
ans =
   4.0000   -2.0000   -3.0000    6.0000
  -6.0000    7.0000    6.5000   -6.0000
   1.0000    7.5000    6.2500    5.4999
 -12.0000   22.0000   15.5000   -1.0000
```

This matrix is the same as the matrix of coefficients in Eq. (4.39) (except for round-off errors).

Once the decomposition is complete, a solution is obtained by using Eqs. (4.22) and (4.23). First, the matrix $[L]$ and the vector $[b]$ are used in Eq. (4.23), $[L][y] = [b]$, to solve for $[y]$:

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ -6 & 4 & 0 & 0 \\ 1 & 8 & 3 & 0 \\ -12 & 16 & -1.5 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 12 \\ -6.5 \\ 16 \\ 17 \end{bmatrix} \tag{4.42}$$

Using forward substitution, the solution is:

$y_1 = \dfrac{12}{4} = 3$, $\quad y_2 = \dfrac{-6.5 + 6y_1}{4} = 2.875$, $\quad y_3 = \dfrac{16 - y_1 - 8y_2}{3} = -3.333$, and

$y_4 = \dfrac{17 + 12y_1 - 16y_2 + 1.5y_3}{4} = 0.5$

Next, the matrix $[U]$ and the vector $[y]$ are used in Eq. (4.22), $[U][x] = [y]$, to solve for $[x]$:

$$\begin{bmatrix} 1 & -0.5 & -0.75 & 1.5 \\ 0 & 1 & 0.5 & 0.75 \\ 0 & 0 & 1 & -0.6667 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2.875 \\ -3.333 \\ 0.5 \end{bmatrix} \tag{4.43}$$

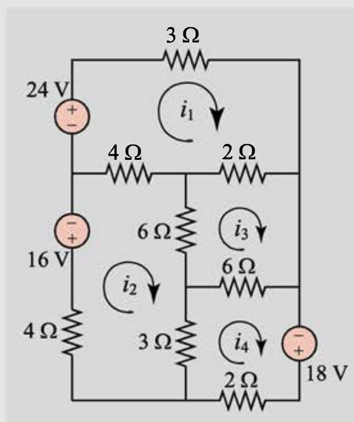
Using back substitution, the solution is:

$x_4 = \dfrac{0.5}{1} = 0.5$, $\quad x_3 = -3.333 + 0.6667x_4 = -3$, $\quad x_2 = 2.875 - 0.5x_3 - 0.75x_4 = 4$, and

$x_1 = 3 + 0.5x_2 + 0.75x_3 - 1.5x_4 = 2$

Example 4-6: MATLAB user-defined function for solving a system of equations using *LU* decomposition with Crout's method.

Determine the currents i_1, i_2, i_3, and i_4 in the circuit shown in the figure (same as in Fig. 4-1). Write the system of equations that has to be solved in the form $[a][i] = [b]$. Solve the system by using the LU decomposition method, and use Crout's method for doing the decomposition.



SOLUTION

The currents are determined from the set of four equations, Eq. (4.1). The equations are derived by using Kirchhoff's law. In matrix form, $[a][i] = [b]$, the equations are:

$$\begin{bmatrix} 9 & -4 & -2 & 0 \\ -4 & 17 & -6 & -3 \\ -2 & -6 & 14 & -6 \\ 0 & -3 & -6 & 11 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} = \begin{bmatrix} 24 \\ -16 \\ 0 \\ 18 \end{bmatrix} \tag{4.44}$$

To solve the system of equations, three user-defined functions are created. The functions are as follows:

`[L U] = LUdecompCrout(A)` This function decomposes the matrix `A` into lower triangular and upper triangular matrices `L` and `U`, respectively.

`y = ForwardSub(L,b)` This function solves a system of equations that is given in lower triangular form.

`x = BackwardSub(L,b)` This function solves a system of equations that is given in upper triangular form.

Listing of the user-defined function `LUdecompCrout`:

Program 4-3: User-defined function. *LU* decomposition using Crout's method.

```
function [L, U] = LUdecompCrout(A)
% The function decomposes the matrix A into a lower triangular matrix L
% and an upper triangular matrix U, using Crout's method, such that A = LU.
% Input variables:
% A  The matrix of coefficients.
% b  Right-hand-side column vector of constants.
% Output variable:
% L  Lower triangular matrix.
% U  Upper triangular matrix.

[R, C]=size(A);
for i=1:R
   L(i,1)=A(i,1);                    Eq. (4.34).
   U(i,i)=1;                         Eq. (4.35).      Steps 1 and 2 (page 122).
end
for j=2:R
   U(1,j)=A(1,j)/L(1,1);             Eq. (4.36).      Step 3 (page 122).
end
for i=2:R
   for j=2:i
     L(i,j)=A(i,j)-L(i,1:j-1)*U(1:j-1,j);         Eq. (4.37).
   end                                                     Step 4 (page 122).
   for j=i + 1:R
     U(i,j)=(A(i,j)-L(i,1:i-1)*U(1:i-1,j))/L(i,i);  Eq. (4.38).
   end
end
```

Listing of the user-defined function `ForwardSub`:

Program 4-4: User-defined function. Forward substitution.

```
function y = ForwardSub(a,b)
% The function solves a system of linear equations ax = b
% where a is lower triangular matrix by using forward substitution.
% Input variables:
% a  The matrix of coefficients.
% b  A column vector of constants.
% Output variable:
```

```
% y  A column vector with the solution.

n=length(b);
y(1,1)=b(1)/a(1,1);
for i=2:n
    y(i,1)=(b(i)-a(i,1:i-1)*y(1:i-1,1))./a(i,i);
end
```

Eq. (4.8).

Listing of the user-defined function `BackwardSub`:

Program 4-5: User-defined function. Back substitution.

```
function y = BackwardSub(a,b)
% The function solves a system of linear equations ax = b
% where a is an upper triangular matrix by using back substitution.
% Input variables:
% a  The matrix of coefficients.
% b  A column vector of constants.
% Output variable:
% y  A column vector with the solution.

n=length(b);
y(n,1)=b(n)/a(n,n);
for i=n-1:-1:1
   y(i,1)=(b(i)-a(i,i+1:n)*y(i+1:n,1))./a(i,i);
end
```

Eq. (4.5).

The functions are then used in a MATLAB computer program (script file) that is used for solving the problem by following these steps:

- The matrix of coefficients $[a]$ is decomposed into upper $[U]$ and lower $[L]$ triangular matrices (using the `LUdecompCrout` function).
- The matrix $[L]$ and the vector $[b]$ are used in Eq. (4.23), $[L][y] = [b]$, to solve for $[y]$, (using the `ForwardSub` function).
- The solution $[y]$ and the matrix $[U]$ are used in Eq. (4.22), $[U][i] = [y]$, to solve for $[i]$ (using the `BackwardSub` function).

Script file:

Program 4-6: Script file. Solving a system with Crout's *LU* decomposition.

```
% This script file solves a system of equations by using
% the Crout's LU decomposition method.
a = [9 -4 -2 0; -4 17 -6 -3; -2 -6 14 -6; 0 -3 -6 11];
b = [24; -16; 0; 18];
[L, U]=LUdecompCrout(a);
y=ForwardSub(L,b);
i=BackwardSub(U,y)
```

When the script file is executed, the following solution is displayed in the Command Window:

```
i =
    4.0343
    1.6545
    2.8452
    3.6395
```

The script file can be easily modified for solving the systems of equations $[a][i] = [b]$ for the same matrix $[a]$, but different values of $[b]$. The LU decomposition is done once, and only the last two steps have to be executed for each $[b]$.

4.5.3 LU Decomposition with Pivoting

Decomposition of a matrix $[a]$ into the matrices $[L]$ and $[U]$ means that $[a] = [L][U]$. In the presentation of Gauss and Crout's decomposition methods in the previous two subsections, it is assumed that it is possible to carry out all the calculations without pivoting. In reality, as was discussed in Section 4.3, pivoting may be required for a successful execution of the Gauss elimination procedure. Pivoting might also be needed with Crout's method. If pivoting is used, then the matrices $[L]$ and $[U]$ that are obtained are not the decomposition of the original matrix $[a]$. The product $[L][U]$ gives a matrix with rows that have the same elements as $[a]$, but due to the pivoting, the rows are in a different order. When pivoting is used in the decomposition procedure, the changes that are made have to be recorded and stored. This is done by creating a matrix $[P]$, called a permutation matrix, such that:

$$[P][a] = [L][U] \tag{4.45}$$

If the matrices $[L]$ and $[U]$ are used for solving a system of equations $[a][x] = [b]$ (by using Eqs. (4.23) and (4.22)), then the order of the rows of $[b]$ have to be changed such that it is consistent with the pivoting. This is done by multiplying $[b]$ by the permutation matrix, $[P][b]$. Use of the permutation matrix is shown in Section 4.8.3, where the decomposition is done with MATLAB's built-in function.

4.6 INVERSE OF A MATRIX

The inverse of a square matrix $[a]$ is the matrix $[a]^{-1}$ such that the product of the two matrices gives the identity matrix $[I]$.

$$[a][a]^{-1} = [a]^{-1}[a] = [I] \tag{4.46}$$

The process of calculating the inverse of a matrix is essentially the same as the process of solving a system of linear equations. This is illustrated for the case of a (4×4) matrix. If $[a]$ is a given matrix and $[x]$ is the unknown inverse of $[a]$, then:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.47}$$

Equation (4.47) can be rewritten as four separate systems of equations, where in each system one column of the matrix $[x]$ is the unknown:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{41} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \\ x_{42} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \\ x_{43} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_{14} \\ x_{24} \\ x_{34} \\ x_{44} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{4.48}$$

Solving the four systems of equations in Eqs. (4.48) gives the four columns of the inverse of $[a]$. The systems of equations can be solved by using any of the methods that have been introduced earlier in this chapter (or other methods). Two of the methods, the *LU* decomposition method and the Gauss–Jordan elimination method, are described in more detail next.

4.6.1 Calculating the Inverse with the LU Decomposition Method

The *LU* decomposition method is especially suitable for calculating the inverse of a matrix. As shown in Eqs. (4.48), the matrix of coefficients in all four systems of equations is the same. Consequently, the *LU* decomposition of the matrix $[A]$ is calculated only once. Then, each of the systems is solved by first using Eq. (4.23) (forward substitution) and then Eq. (4.22) (back substitution). This is illustrated, by using MATLAB, in Example 4-7.

Example 4-7: Determining the inverse of a matrix using the *LU* decomposition method.

Determine the inverse of the matrix $[a]$ by using the LU decomposition method.

$$[a] = \begin{bmatrix} 0.2 & -5 & 3 & 0.4 & 0 \\ -0.5 & 1 & 7 & -2 & 0.3 \\ 0.6 & 2 & -4 & 3 & 0.1 \\ 3 & 0.8 & 2 & -0.4 & 3 \\ 0.5 & 3 & 2 & 0.4 & 1 \end{bmatrix} \tag{4.49}$$

Do the calculations by writing a MATLAB user-defined function. Name the function `invA = InverseLU(A)`, where `A` is the matrix to be inverted, and `invA` is the inverse. In the function, use the functions `LUdecompCrout`, `ForwardSub`, and `BackwardSub` that were written in Example 4-6.

SOLUTION

If the inverse of $[a]$ is $[x]$ ($[x]=[a]^{-1}$), then $[a][x]=[I]$, which are the following five sets of five systems of equations that have to be solved. In each set of equations, one column of the inverse is calculated.

$$\begin{bmatrix} 0.2 & -5 & 3 & 0.4 & 0 \\ -0.5 & 1 & 7 & -2 & 0.3 \\ 0.6 & 2 & -4 & 3 & 0.1 \\ 3 & 0.8 & 2 & -0.4 & 3 \\ 0.5 & 3 & 2 & 0.4 & 1 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{41} \\ x_{51} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0.2 & -5 & 3 & 0.4 & 0 \\ -0.5 & 1 & 7 & -2 & 0.3 \\ 0.6 & 2 & -4 & 3 & 0.1 \\ 3 & 0.8 & 2 & -0.4 & 3 \\ 0.5 & 3 & 2 & 0.4 & 1 \end{bmatrix}\begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \\ x_{42} \\ x_{52} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0.2 & -5 & 3 & 0.4 & 0 \\ -0.5 & 1 & 7 & -2 & 0.3 \\ 0.6 & 2 & -4 & 3 & 0.1 \\ 3 & 0.8 & 2 & -0.4 & 3 \\ 0.5 & 3 & 2 & 0.4 & 1 \end{bmatrix}\begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \\ x_{43} \\ x_{53} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0.2 & -5 & 3 & 0.4 & 0 \\ -0.5 & 1 & 7 & -2 & 0.3 \\ 0.6 & 2 & -4 & 3 & 0.1 \\ 3 & 0.8 & 2 & -0.4 & 3 \\ 0.5 & 3 & 2 & 0.4 & 1 \end{bmatrix}\begin{bmatrix} x_{14} \\ x_{24} \\ x_{34} \\ x_{44} \\ x_{54} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0.2 & -5 & 3 & 0.4 & 0 \\ -0.5 & 1 & 7 & -2 & 0.3 \\ 0.6 & 2 & -4 & 3 & 0.1 \\ 3 & 0.8 & 2 & -0.4 & 3 \\ 0.5 & 3 & 2 & 0.4 & 1 \end{bmatrix}\begin{bmatrix} x_{15} \\ x_{25} \\ x_{35} \\ x_{45} \\ x_{55} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{4.50}$$

The solution is obtained with the user-defined function `InverseLU` that is listed below. The function can be used for calculating the inverse of any sized square matrix.

The function executes the following operations:

- The matrix $[a]$ is decomposed into matrices $[L]$ and $[U]$ by applying Crout's method. This is done by using the function `LUdecompCrout` that was written in Example 4-6.
- Each system of equations in Eqs. (4.50) is solved by using Eqs. (4.23) and (4.22). This is done by first using the function `ForwardSub` and subsequently the function `BackwardSub` (see Example 4-6).

Program 4-7: User-defined function. Inverse of a matrix.

```
function invA = InverseLU(A)
% The function calculates the inverse of a matrix
% Input variables:
% A  The matrix to be inverted.
% Output variable:
% invA   The inverse of A.
```

```
[nR nC] = size(A);
I=eye(nR);
[L U]= LUdecompCrout(A);
for j=1:nC
    y=ForwardSub(L,I(:,j));
    invA(:,j)=BackwardSub(U,y);
end
```

Create an identity matrix of the same size as $[A]$.

Decomposition of $[A]$ into $[L]$ and $[U]$.

In each pass of the loop, one set of the equations in Eqs. (4.50) is solved. Each solution is one column in the inverse of the matrix.

The function is then used in the Command Window for solving the problem.

```
>> F=[0.2 -5 3 0.4 0; -0.5 1 7 -2 0.3; 0.6 2 -4 3 0.1; 3 0.8 2 -0.4 3; 0.5 3 2 0.4 1];
>> invF = InverseLU(F)
invF =
   -0.7079    2.5314    2.4312    0.9666   -3.9023
   -0.1934    0.3101    0.2795    0.0577   -0.2941
    0.0217    0.3655    0.2861    0.0506   -0.2899
    0.2734   -0.1299    0.1316   -0.1410    0.4489
    0.7815   -2.8751   -2.6789   -0.7011    4.2338
>> invF*F
ans =
    1.0000   -0.0000    0.0000   -0.0000   -0.0000
    0.0000    1.0000    0.0000   -0.0               0
         0   -0.0000    1.0000   -0.0000   -0.0000
   -0.0000    0.0000   -0.0000    1.0000   -0.0000
   -0.0000    0.0000   -0.0000   -0.0000    1.0000
```

The solution $[F]^{-1}$.

Check if $[F][F]^{-1} = [I]$.

4.6.2 Calculating the Inverse Using the Gauss–Jordan Method

The Gauss–Jordan method is easily adapted for calculating the inverse of a square $(n \times n)$ matrix $[a]$. This is done by first appending an identity matrix $[I]$ of the same size as the matrix $[a]$ to $[a]$ itself. This is shown schematically for a (4×4) matrix in Fig. 4-20*a*. Then, the Gauss–Jordan procedure is applied such that the elements of the matrix $[a]$ (the left half of the augmented matrix) are converted to 1s along the diagonal and 0s elsewhere. During this process, the terms of the identity matrix in Fig. 4-20*a* (the right half of the augmented matrix) are changed and become the elements $[a']$ in Fig. 4-20*b*, which constitute the inverse of $[a]$.

$$\underset{(a)}{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 1 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 & 1 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 & 0 & 1 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 & 0 & 0 & 1 \end{bmatrix}} \xrightarrow{\text{Gauss–Jordan procedure}} \underset{(b)}{\begin{bmatrix} 1 & 0 & 0 & 0 & a'_{11} & a'_{12} & a'_{13} & a'_{14} \\ 0 & 1 & 0 & 0 & a'_{21} & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & 1 & 0 & a'_{31} & a'_{32} & a'_{33} & a'_{34} \\ 0 & 0 & 0 & 1 & a'_{41} & a'_{42} & a'_{43} & a'_{44} \end{bmatrix}}$$

Figure 4-20: Calculating the inverse with the Gauss–Jordan method.