Numerical Methods For Science and Engineering
Lecture Note 6
Interpolation & Curve Fitting

6.1 Introduction

In many occasions we are given only a few discrete set of values. To study the behavior of the function through those points a technique known as **interpolation** is introduced. Polynomial is a function which is easy to handle. The method of finding a polynomial that fits a selected set of points $(x, f(x))$ which behaves nearly the same way as the true function will be considered.

6.2 Polynomial Interpolation

Given the values of a function $f(x)$ at $(n+1)$ distinct points $x_0, x_1, x_2, \cdots, x_n$ we can construct a **unique** polynomial of degree less than equal to n which satisfies the conditions

$$p(x_i) = f(x_i), \quad i = 0, 1, 2, 3, \cdots, n$$

General Form : An nth degree polynomial can be taken as

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$$

To fit this polynomial to $(n+1)$ set of points we have to solve $(n+1)$ simultaneous equations and is very tedious.

Newton Interpolating Polynomial : A form which is convenient to use is suggested by Newton is

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

The unknown coefficients can be determined successively by substituting the set of values given. This form of representation is convenient in determining the unknown coefficients and plays an important role in the derivation of an interpolating polynomial.

Example 6.1 : Find the polynomial of least degree which takes the values

x	-1	1	2	5
$f(x)$	9	3	6	39

Solution: There are four set of values given. Let the approximated polynomial be

$$f(x) \approx a_0 + a_1(x+1) + a_2(x+1)(x-1) + a_3(x+1)(x-1)(x-2)$$

Using the values of x and $f(x)$ in turn, we get

From $x = -1,\ f(-1) = 9$, we get $\quad 9 = a_0$

From $x = 1,\ f(2) = 6$, we get $\quad 3 = 9 + a_1(2)$ or $a_1 = \frac{1}{2}93 - 9) = -3$

From $x = 2,\ f(2) = 3$, we get $\quad 3 = 9 - 3(3) + a_2(3)(1)$ or $a_2 = \frac{1}{3}(6) = 2$

From $x = 5,\ f(2) = 39$, we get $\quad 39 = 9 - 3(6) + 2(6)(4) + a_3(6)(4)(3)$ or

$$a_3 = \frac{1}{72}(39 - 39) = 0$$

Thus the polynomial is $f(x) = 9 - 3(x+1) + 2((x+1)(x-1)$

$$= 9 - 3x - 3 + 2x^2 - 2$$
$$= 2x^2 - 3x + 4$$

6.3 Divided Differences

Interpolating polynomials can be expressed in a variety of forms, and among these the Newton divided difference form is probably the convenient and efficient one.

Let the values of $f(x)$ corresponding to the arguments $x_0, x_1, \cdots, x_n$ be

$$f(x_0),\ f(x_1), \cdots, f(x_n).$$

The first divided difference for arguments x_0 and x_1 is defined by :

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

The second divided difference for arguments x_0, x_1 and x_2 is defined as :

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

Similarly higher divided differences are defined. The nth divided differences with (n+1) arguments is defined by $f[x_0, x_1, x_3, \cdots, x_n] = \dfrac{f[x_1, x_2, \cdots, x_n] - f[x_0, x_1, \cdots, x_{n-1}]}{x_n - x_0}$

Property 1: The divided differences are symmetric about their arguments i.e. does not depend on the order of the arguments.

Property 2: The nth divided differences of a polynomial of degree n is constant

6.4 Interpolation Formula using Divided Differences

6.4.1 Newton Divided Difference Interpolation

The interpolating polynomial $p(x)$ through the points $x_0, x_1, x_2, \cdots, x_n$ can be written in the Newton form as

$$f(x) \approx p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

Substituting $x = x_0, x_{1,} x_2, \cdots, x_n$, we have

$$f(x_0) = a_0$$

$$f(x_1) = f(x_0) + a_1(x_1 - x_0) \quad \text{or} \quad a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f[x_0, x_1]$$

$$f(x_2) = f(x_0) + f[x_0, x_1](x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

or $$(x_2 - x_1)a_2 = \frac{f(x_2) - f(x_0)}{(x_2 - x_0)} - f[x_0, x_1] = f[x_2, x_0] - f[x_0, x_1]$$

or $$a_2 = \frac{f[x_2, x_0] - f[x_0, x_1]}{(x_2 - x_1)} = \frac{f[x_1, x_2] - f[x_0, x_1]}{(x_2 - x_0)} = f[x_0, x_1, x_2]$$

Continuing the process it can be shown that $a_n = f[x_0, x_1, x_2, \cdots, x_n]$

Thus in terms of the divided differences interpolating polynomial can be written as

$$p(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots$$
$$+ f[x_0, x_1, x_2, \cdots, x_n](x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

This is known as **Newton's divided difference interpolation formula**.

If $f(x)$ is a polynomial through $(n+1)$ points $x_0, x_1, x_2, \cdots, x_n$, then the polynomial $g(x)$ through those points with an extra point $x = x_{n+1}$ is $g(x) = f(x) + b(x - x_0)(x - x_1)\cdots(x - x_n)$

The constant b can be calculated by substituting $x = x_{n+1}$.

Example 6.2

The table below gives the values of x and $f(x)$:

x :	-1	1	2	3	4
$f(x)$:	-7	-1	8	29	68

(i) Construct a divided-difference table for the above data.

(ii) Find the polynomial of least degree that incorporates the values in the table and find $f(5)$.

(iii) Find by linear interpolation a real root of $f(x) = 0$.

(iv) Find the polynomial $g(x)$ that takes the values of the above table and $g(5) = 203$.

Solution:

(i) The divided difference table for the given data is as follows:

x	$f(x)$	$f^1[\ ]$	$f^2[\ ]$	$f^3[\ ]$	$f^4[\ ]$
-1	$\boxed{-7}$				
1	-1	$\boxed{3}$			
2	8	9	$\boxed{2}$		
3	29	21	6	$\boxed{1}$	
4	68	39	9	1	0

(ii) The needed differences are enclosed by the double lined box.

By Newton's divided difference formula, we get

$$f(x) = -7 + 3(x+1) + 2(x+1)(x-1) + 1(x+1)(x-1)(x-2)$$

and
$$f(5) = -7 + 3(6) + 2(6)(4) + 1(6)(4)(3)$$
$$= -7 + 18 + 48 + 72 = 131$$

(iii) Here $\quad f(1)f(2) = (-1)(8) = -8 < 0$

Thus a root is in (1, 2).

From the table, we have

x	$f(x)$	1DD
1	-1	
2	8	9

Thus the root is the solution of

$$f(x) = -1 + 9(x-1) = 0$$

or
$$x = 1 + \tfrac{1}{9} \approx 1.111$$

(iv) The polynomial $g(x)$ can be written as

$$g(x) = f(x) + b(x+1)(x-1)(x-2)(x-3)(x-4)$$

where b is a constant.

Taking $x=5$, we have

$$g(5)=f(5)+b(6)(4)(3)(2)(1)$$

or $$203=131+144b$$

Hence $$b=\frac{203-131}{144}=\frac{72}{144}=\frac{1}{2}$$

The required polynomial is

$$g(x)=f(x)+\tfrac{1}{2}(x+1)(x-1)(x-2)(x-3)(x-4)$$

6.4.2 Newton Backward Divided Difference Formula

If the nodes are reordered as $x_n, x_{n-1}, \cdots, x_0$, the divided differences interpolating polynomial can be written as

$$\begin{aligned} p(x) = f(x_n) + f[x_n, x_{n-1}](x-x_n) + f[x_n, x_{n-1}, x_{n-2}](x-x_n)(x-x_{n-1}) + \cdots \\ + f[x_n, x_{n-1}, x_{n-2}, \cdots, x_0](x-x_n)(x-x_{n-1})\cdots(x-x_0) \end{aligned}$$

and is called the **Newton Backward Divided Difference** formula.

6.5 Lagrange Interpolating Polynomial

Lagrange polynomial of degree one passing through two points (x_0, y_0) and (x_1, y_1) is written as

$$L_1(x) = \frac{x-x_1}{x_0-x_1}y_0 + \frac{x-x_0}{x_1-x_0}y_1$$

Lagrange polynomial of degree two passing through three points (x_0, y_0), (x_1, y_1) and (x_2, y_2) is written as

$$L_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

Lagrange polynomial of degree three passing through four points (x_0, y_0), (x_1, y_1), (x_2, y_2) and (x_3, y_3) is written as

$$\begin{aligned} L_3(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 \\ &+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3 \end{aligned}$$

In general, the Lagrange polynomial of degree n passing through $(n+1)$ points (x_0, y_0), (x_1, y_1), $\cdots$, (x_n, y_n) is written as

$$\begin{aligned} L_n(x) &= \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}y_1 \\ &+ \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}y_n \end{aligned}$$

Example 6.3

The following table gives the values of an empirical function $f(x)$ for certain values of x.

x	0	1	2	3
$f(x)$	–4	–1	8	29

Use the Lagrange interpolation formula to estimate

(i) the value of $f(2.5)$

(ii) the root of the equation $f(x) = 0$ to 3 decimal places.

(i) Applying Lagrange's formula, we have

$$f(x) = -4\frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} - 1\frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)}$$
$$+8\frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)} + 29\frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)}$$

and

$$f(2.5) = -4\frac{(1.5)(0.5)(-0.5)}{(-1)(-2)(-3)} - 1\frac{(2.5)(0.5)(-0.5)}{(1)(-1)(-2)}$$
$$+8\frac{(2.5)(1.5)(-0.5)}{(2)(1)(-1)} + 29\frac{(2.5)(1.5)(0.5)}{(3)(2)(1)}$$
$$= -0.25 + 0.3125 + 7.5 + 9.0625$$
$$= 16.625$$

(ii) Let $y = f(x)$. Then the root of $f(x) = 0$ corresponds to $y = 0$. To find the root let us use the Lagrange formula in reverse order i.e. consider the polynomial in terms of y.

y	–4	–1	8	29
x	0	1	2	3

Then

$$x = 0 + 1\frac{(y+4)(y-8)(y-29)}{(-1+4)(-1-8)(-1-29)} + 2\frac{(y+4)(y+1)(y-29)}{(8+4)(8+1)(8-29)} + 3\frac{(y+4)(y+1)(y-8)}{(29+4)(29+1)(29-8)}$$

When $y = 0$, then

$$x = 0 + 1\frac{(4)(-8)(-29)}{(3)(-9)(-30)} + 2\frac{(4)(1)(-29)}{(12)(9)(-21)} + 3\frac{(4)(1)(-8)}{(33)(30)(21)}$$
$$= 1.1457 + 0.1023 - 0.0046$$
$$= 1.2434$$

Exercise 6.4 The upward velocity of a rocket is given as a function of time below:

t (s)	10	15	20	22.5	30
$v(t)$ (m/s)	227	363	517	603	903

i. Construct a divided-difference table for the above data.
ii. Determine the value of the velocity at $t = 17$ seconds using two suitable points.
iii. Determine the value of the velocity at $t = 17$ seconds using three suitable points.
iv. Find the polynomial which passes through all the points and find $v(35)$.
v. Use Lagrange interpolating polynomial to estimate
   a. the value of t for $v(t) = 400$ using two suitable points.
   b. the value of t for $v(t) = 400$ using three suitable points.
vi. Write down MATLAB codes using "**polyfit(x, y, n)**" and "**polyval(p, x)**" for the following.
   a. Find the polynomial of least degree that incorporates all the values in the table. and estimate the velocities corresponding to $.t = 17,\ 25$ and 35 seconds.
   b. Draw the figure showing fitted polynomial and the given points.

Solution:

i.

t	$v(t)$	$v^1[\ ]$	$v^2[\ ]$	$v^3[\ ]$	$v^4[\ ]$
10	227				
15	363	27.2			
20	517	30.8	0.36		
22.5	603	34.4	0.48	0.0096	
30	903	40.0	0.56	0/0043	0.0002

ii. Note that $15 < 17 < 20$ and using the relevant part of the table

t	$v(t)$	
15	363	
20	517	30.8

we have the linear polynomial $v(t) = 363 + 30.8\,(t - 15)$.

And $$v(17) \approx 363 + 30.8(2) = 424.6.$$

iii. Note that $17 - 10 = 7$ and $22.5 - 17 = 5.5$. Thus we may choose points corresponding to $t = 15, 20$ and 22.5. Collecting the relevant part of the table

t	$v(t)$	$v^1[\ ]$	$v^2[\ ]$
15	363		
20	517	30.8	
22.5	603	34.4	0.48

The polynomial with 3 points is $v(t) = 363 + 30.8(t - 15) + 0.48(t - 15)(t - 20)$.

And $$v(17) = 363 + 30.8(2) + 0.48(2)(-3) = 421.72.$$

iv. Polynomial passing through all the points is
$$v(t) = 227 + 27.2(t-10) + 0.36(t-10)(t-15) + 0.0096(t-10)(t-15)(t-20)$$
$$-0.0002(t-10)(t-15)(t-20)(t-22.5)$$
And $v(35) = 227 + 27.2(25) + 0.36(25)(20) + +0.0096(25)(20)(15)$
$$-0.0002(25)(20)(25)(12.5)$$
$$= 1140.25.$$

v. For a given v we need to calculate the value of t, so consider the Lagrange polynomial in reverse order.

a. With two points consider

v	363	517
t	15	20

and the Lagrange polynomial is $t = 15\frac{(v-517)}{(363-517)} + 20\frac{(v-363)}{(517-363)}$

For $v = 400$,
$$t = \frac{15(-117)}{-154} + \frac{20(37)}{154} = \frac{2495}{154} = 16.2013.$$

b. With three points consider

v	227	363	517
t	10	15	20

and the Lagrange polynomial is
$$t = 10\frac{(v-363)(v-517)}{(227-363)(227-517)} + 15\frac{(v-227)(v-517)}{(363-227)(363-517)}$$
$$+ 20\frac{(v-227)(v-363)}{(517-227)(517-363)}.$$
For $v = 400$,
$$t = \frac{10(37)(-117)}{(-136)(-290)} + \frac{15(173)(-117)}{(136)(-154)} + \frac{20(173)(37)}{(290)(154)}$$
$$= -1.0976 + 14.4965 + 2.8665 = 16.2654.$$

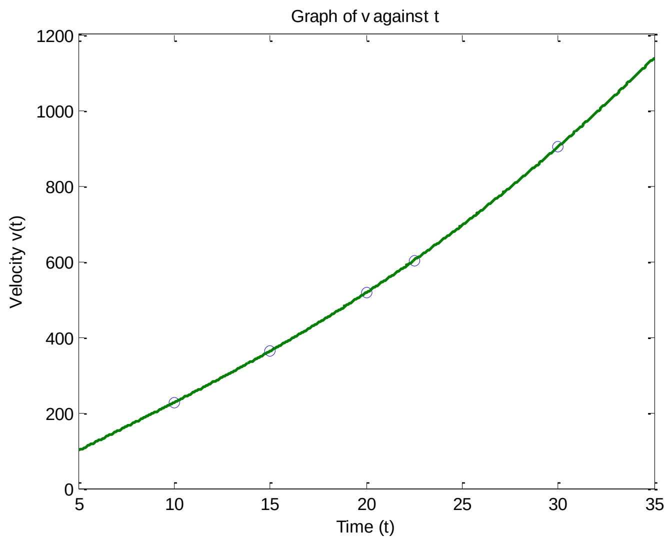
vi. MATLAB CODES

a.
```
>> t=[10  15  20  22.5  30];
>> v=[227  363  517  603  903];
>> pt=polyfit(t,v,4)
pt =   -0.0002    0.0240   -0.4267   28.2000  -34.2000

>> t1=[17  25  30];
>> v1=polyval(pt,t1);
>> % Output value of v for t
```

```
        >> t_v =[t1',v1']
t_v =    17.0000     421.9875
         25.0000     695.8000
         30.0000     903.0000
```

b.
```
>>  t=[10  15  20  22.5  30];
>>  v=[227  363  517  603  903];
>> pt=polyfit(t,v,4);
>> t1=linspace(5,35,500);         % generates 500 values
>>  v1=polyval(pt,t1);              % calculates values of v
>> plot(t, v,'o',t1,v1);
>> title('Graph of v against t');
>> xlabel('Time (t)');
>> ylabel('Velocity v(t)');
```



Exercise 6.1

1. The table below gives the velocity v at time t

t(s)	1	3	4	7
v(m/s)	3	5	21	201

i. Construct a divided-difference table for the above data.
ii. Find the polynomial of least degree that incorporates the values in the table.
iii. Find the acceleration at time $t = 6s$.
iv. Find the distance function when $S(0) = 2$.

2. . The table below gives the values of x and $f(x)$

x	-2	0	3	6	7
f(*x*)	2	-4	-58	842	1802

i. Construct a divided-difference table for the above data.
ii. Find the polynomial which passes through all the points of the table and find $f(5)$.
iii. Find the polynomial $g(x)$ that takes the values of the above table and $g(5) = 549$.
iv. Use Lagrange interpolating polynomial to estimate
   a. the value of $f(4)$ using two suitable points.
   b. the value of *x* for $f(x) = 0$ using three suitable points.
v. Write down MATLAB codes using "**polyfit(x, y, n)**" and "**polyval(p, x)**" for the following. Find the polynomial of least degree that incorporates all the values in the table and estimate the values corresponding to $x = 1,\ 5$ and 9.

3. The table below gives the values of x and $f(x)$

x	4	5	7	9	11
$f(x)$	62	95	185	307	461

i. Construct a divided-difference table for the above data.
ii. Find the polynomial which passes through all the points of the table and find $f(12)$.
iii. Find the polynomial $g(x)$ that takes the values of the above table and $g(12) = 1280$.
iv. Use Lagrange interpolating polynomial to estimate
   a. the value of $f(8)$ using two points.
   b. the value of *x* for $f(x) = 380$ using three points.
v. Write down MATLAB codes using "**polyfit(x, y, n)**" and "**polyval(p, x)**" for the following. Find the polynomial of least degree that incorporates all the values in the table and estimate the values corresponding to $x = 1,\ 3$ and 5.

4. The table below gives the values of *x* and *f*(*x*):

x	-2	-1	0	3
f(*x*)	12	14	10	22

i.

Construct a divided-difference table for the above data.

ii. Find the polynomial of least degree that incorporates the values in the table and find $f(8)$.

iii. Given $g(8) = 1202$, find the polynomial $g(x)$ that also takes the values of the above table.

iv. Use Lagrange interpolation formula to find

  a. a real root of $f(x) = 0$ using linear approximation.

  b. a real root of $f(x) = 0$ using all the points.

v. Write down MATLAB codes using “**polyfit(x, y, n)**” and “**polyval(p, x)**” to plot the figure showing fitted polynomial and the given points.

Curve Fitting

6.6 Introduction

The purpose of curve fitting is to find the parameters values of the model function that closely match the data's. The fitted curves can be used to estimate the values of one variable corresponding to the specified values of the other variable. The method of least squares may be one of the most systematic procedure to fit a curve through the given data points. In polynomial interpolation we have considered the problem of finding polynomial of least degree which agree with the tabulated data's. Spline interpolation is a form of polynomial interpolation where the interpolant is a piecewise polynomial called spline. This means that between two points there is a piecewise polynomial curve which joined smoothly to the neighboring curves. Cubic spline has different important applications. One of the important applications is in Computer graphics.

6.7 Curve Fitting by Least Squares Method

The method of least squares may be one of the most systematic procedures to fit a curve through given data points.
Consider the problem of fitting a set of n data points

$$(x_r, y_r), \quad r = 1.2.3, \cdots, n$$

to a curve $Y = f(x)$ whose values depends on m parameters $c_1, c_2, c_3, \cdots, c_m$. The values of the function at a point depends on the values of the parameter involved. In least square method we determine a set of values of the parameter $c_1, c_2, c_3, \cdots, c_m$ such that the sum of the squares of the error

$$E(c_1, c_2, \cdots, c_m) = \sum_{ii=1}^{n} \left[f(x_i, c_1, c_2, \cdots, c_m) - y_i \right]^2$$

is minimum.
The necessary conditions for E to have a minimum is that

$$\frac{\partial E}{\partial c_r} = 0, \qquad r = 1,2,3, \cdots, m$$

This condition gives a system of m equations, ***called normal equations***, in m unknowns $c_1, c_2, c_3, \cdots, c_m$.
If the parameters appear in the function in non-linear form, the normal equations become non-linear and are difficult to solve. This difficulty may be avoided if $f(x)$ is transformed to a form which is linear in parameters.

Note that $\sum_{i=1}^{n} 1 = n$.

Example 6.5
Given the following set of values of x and y:

X	1	2	3	4	5	6
Y	1.553	1.638	0.685	–0.428	–0.679	0.164

A physicist wants to approximate the data using a periodic curve $y = a + b\sin x$.
Estimate the parameters a and b to 2 decimal places using least squares method.

Solution

Sum of the square deviation is

$$E(a,b) = \sum_{i=1}^{6} (a + b\sin x_i - y_i)^2$$

At minimum,

$$\frac{\partial E}{\partial a} = 0 \text{ and } \quad \frac{\partial E}{\partial b} = 0$$

These conditions gives

$$\sum_{i=1}^{6} 2(a + b\sin x_i - y_i)1 = 0$$

$$\sum_{i=1}^{6} 2(a + b\sin x_i - y_i)\sin x_i = 0$$

which can be rearranged as

$$a\sum 1 + b\sum \sin x_i = \sum y_i$$

$$a\sum \sin x_i + b\sum \sin^2 x_i = \sum y_i \sin x_i$$

The sum can be calculated as follows

X	***y***	**sin *x***	**sin²*x***	***y* sin *x***
1	1.553	0.8415	0.7081	1.3068
2	1.638	0.9093	0.8268	1.4894
3	0.685	0.1411	0.0199	0.0967
4	-0.428	-0.7568	0.5727	0.3239
5	-0.679	-0.9589	0.9195	0.6511
6	0.164	-0.2794	0.0781	-0.0458

Sum	2.933	-0.1032	3.1251	3.8221

The normal equations are

$$6\,a - 0.1032\,b = 2.933$$
$$-0.1032\,a + 3.1251\,b = 3.8221$$

By dividing each equation by the coefficient of a, we have

$$a - 0.0172\,b = -0.4888$$
$$a - 30.282\,b = -37.0359$$

Subtracting the equations

$$30.2648\,b = 37.5247$$

Solving we have $b = 1.2399 \approx 1.24$

$$a = 0.5108 \approx 0.51$$

Example 6.6

The height of a child is measured at different ages and listed below:

t (yrs)	3	6	9	12	15
H (ft)	2.87	3.60	4.28	4.88	5.35

It is believed that height follows saturation growth model

$$H = \frac{6.45}{1 + a_2 \exp(-a_3 t)}.$$

i. Use a suitable substitution to reduce the above relation to a linearized form in parameters.
ii. Use least square method to find the normal equation of the above data
iii. Estimate, to 2 decimal places, the values of a_2 and a_3
iv. Estimate the height when the child becomes 20 years old.
v. Use MATLAB function $\mathrm{a} = \mathrm{lsqcurvefit}(\mathrm{fun}, \mathrm{a_0}, \mathrm{xdata}, \mathrm{ydata})$ to fit the general form like $H = \frac{a_1}{1+a_2 \exp(-a_3 t)}$.

Solution

i. The curve $H = \frac{6.45}{1+a_2 \exp(-a_3 t)}$ is to be fitted to the given data.

The equation of the curve can be rewritten as $\frac{6.45}{H} - 1 = a_2 e^{-a_3 t}$

Taking logarithm of both sides, we get $\ln\left(\frac{6.45}{H} - 1\right) = \ln a_2 - a_3\mathrm{t}$,

which can be written in the form

$$Y = A + BX$$

where $Y = \ln\left(\frac{6.45}{H} - 1\right)$, $H = t$, $A = \ln a_2$ and $B = -a_3$.

ii. Sum of the square deviation is

$$E(A,B)=\sum_{i=1}^{5}\left(A+BX_i-Y_i\right)^2$$

At minimum, $\frac{\partial E}{\partial A}=0$ and $\frac{\partial E}{\partial B}=0$

These conditions give

$$\sum 2(A+BX_i-Y_i)1=0$$
$$\sum 2(A+BX_i-Y_i)X_i=0$$

which can be rearranged as

$$A\sum 1+B\sum X_i=\sum Y_i$$
$$A\sum X_i+B\sum X_i^2=\sum X_iY_i$$

The sum can be calculated in a tabular form as shown below:

N	T	H	X	Y	XY	X^2
1	3	2.87	3	0.221	0.663	9
2	6	3.60	6	-0.234	-1.402	36
3	9	4.28	9	-0.679	-6.113	81
4	12	4.88	12	-1.134	-13.609	144
5	15	5.35	15	-1.582	-23.727	225
Sum			45	-3.408	-44.187	495

Normal Equations

$$5A+45B=-3.408$$
$$45A+495B=-44.187$$

iii. Solutions:

$$A+9B=-0.682$$
$$A+11B=-0.982$$
$$-2B=0.3$$

$B=-0.150$ $\quad a_3=0.15$

$A=0.668$ $\quad a_2=1.95$

iv. The fitting curve is $H=\frac{6.45}{1+1.45\exp(-0.15t)}$.

From the equation of the curve, we get

when $t=20$ then $H=5.88$.

v. `>> xd=[3 6 9 12 15];` % state x-values

```
>> yd=[2.87 3.60 4.28 4.88 5.35];        % staet y-values
 Define fitting curve in terms of parameters as vector a
 >> fun=@(a,xd) a(1)./(1+a(2).*exp(-a(3).*xd));
 >>  a0=[6,2,0.2];                        % guess parameter values
% To fit the curve use MATLAB function lsqcurvefit with following syntax
 >> a=lsqcurvefit(Fd,a0,xd,yd)
```

Exercise 6.2

1. Find the least square line $v = b + 2at$ to the following data (where b, a are constants)

t	0	2	4	5	7
v	6.7	9.2	11.5	15.6	19.2

2. Average price, P, of a certain type of second-hand car is believed to be related to its age,t years, by an equation of the form

$$P = \frac{50}{a+be^{\frac{t}{4}}}$$

where$a$ and b are constants. Data from a recent newspaper give the following average price (in Taka) for used car of this type,

t (yrs)	2	4	6	8
P (lac)	20.50	17.25	14.50	11.75

(i) Estimate the values of a and b rounded to 3 significant figures.
(ii) Estimate the values of a car of this type that is 10 years old and the original new price.

3. A bowl of hot water is kept in a room of constant temperature 25^0C. At 5 minutes interval temperature of the water is recorded and listed as given below.

t in minute	5	10	15	20	25
T in ^{0}C	76.8	70.4	64.2	58.8	54.1

The law of cooling can be assumed to be of the form $T = 27 + ae^{-kt}$.
(i) Find, to 2 significant figures, the best values of a and k.
(ii) Estimate the initial temperature.
(iii) Estimate the time, to the nearest minute, when the temperature of the water in the bowl will be 50^0C.

4. The equation $v = 70 - ce^{-kt}$ can be used for calculating the speed of a moving car, where c and k are constants.

The table below shows the speed of the car at various times

t	4	8	12	16	20

v	23.21	28.52	33.07	36.96	40.29

(a) Estimate the values of c and k rounded to 2 significant figures.
(b) Find the time, to the nearest second, when the speed is 45 ms^{-1}.