Numerical Methods For Science and Engineering
Lecture Note 2
Solution of Linear System

2.1 Solution of Linear System by Elimination

Equivalent Systems: Two systems of equations are called equivalent if and only if they have the same solution set.

Elementary Transformations: A system of equations is transformed into an equivalent system if the following elementary operations are applied on the system:

(1) two equations are interchanged
(2) an equation is multiplied by a non-zero constant
(3) an equation is replaced by the sum of that equation and a multiple of any other equation.

Gaussian Elimination: The process which eliminates an unknown from succeeding equations using elementary operations is known as Gaussian elimination.
The equation which is used to eliminate an unknown from the succeeding equations is known as the **pivotal equation**. The coefficient of the eliminated variable in a pivotal equation is known as the **pivot**. If the pivot is zero, it cannot be used to eliminate the variable from the other equations. However, we can continue the elimination process by interchanging the equation with a nonzero pivot.

Solution of a Linear System

A systematic procedure for solving a linear system is to reduce a system that is easier to solve. One such system is the echelon form. The **back substitution** is then used to solve the system in reverse order.

A system is in ***echelon form*** or ***upper triangular form*** if

(i) all equations containing nonzero terms are above any equation with zeros only.
(ii) the first nonzero term in every equation occurs to the right of the first nonzero term term in the equation above it.

Solution of a Linear System

A systematic procedure for solving a linear system is to reduce a system that is easier to solve. One such system is the upper triangular form. The **back substitution** is then used to solve the system in reverse order.

2.2 Pivotal Elimination Method

Computers and calculators use fixed number of digits in its calculation and we may need to round the numbers. This introduces error in the calculations. Also, when two nearly equal numbers are subtracted, the accuracy in the calculation is lost. Consider the following example to study the difficulty faced in the elimination method.

Example 2.1 :

The linear system

$$0.003x + 71.08y = 71.11$$
$$4.23x - 8.16y = 34.14$$

has the exact solution $x = 10$ and $y = 1$.

Solve the above system using four-digit arithmetic i.e. keeping 4 s.f. at each value by

(a) Gaussian elimination without changing the order.

(b) Gaussian elimination by interchanging the order of the equations.

Solution

Using four-digit arithmetic the system can be written as

$$0.003000x + 71.08y = 71.11 \qquad (E_1)$$
$$4.230x - 8.160y = 34.14 \qquad (E_2)$$

(a) Here a multiplier is $m = 4.230 / 0.003 = 1410$.

Multiplying (E_1) by 1410 and retaining 4 digits with rounding, we have

$$4.230x + 100200 = 100300$$
$$4.230x - 8.160y = 34.14$$

Subtracting we get

$$100208.16y = 100265.86$$

To 4 digits this can be written as

$$1002 \times 10^2 y \approx 1003 \times 10^2$$

This gives

$$y = 1.000998 \approx 1.001$$

Substituting the value of *y* in (E_1), we get

$$x \approx \frac{71.11 - (71.08)(1.001)}{0.003}$$
$$\approx \frac{-0.04000}{0.003000} \approx -13.33$$

This produces a wrong result.

(b) Interchanging the rows, we can write

$$4.230x - 8.160y = 34.14 \qquad (E_3)$$
$$0.003000x + 71.08y = 71.11 \qquad (E_4)$$

For this system multiplier is

$$m_2 = \frac{0.003000}{4.230} \approx 0.0007092$$

The replacement of (E_4) by $(E_4 - m_2 \times E_3)$ reduces the system t0

$$4.230x - 8.160y = 34.14$$
$$71.09y \approx 71.09$$

The 4-digits arithmetic gives the solution

$$y = 1.000 \text{ and } x = 10.00$$

which is the exact solution of the system.

This example shows that the order in which we treat the equations for elimination affects the accuracy in the elimination process.

Note that $\max\{|0.003|, |4.23|\} = 4.23$. Thus in eliminating *x*, we should use the equation with maximum numerical coefficient of *x*.

2.2.1 Gaussian Elimination with Partial Pivoting

(a) Select pivotal equation for a variable for elimination (the equation with maximum numerical coefficient of that variable in the system).
(b) Eliminate the chosen variable from the remaining equations with respect to the pivotal equation.
(c) Repeat the process for the subsystem.
(d) Using back-substitution find solutions by using pivotal equations.

For convenience of calculations we made the coefficients of eliminated variable to unity before elimination. This is illustrated through an example below.

Example 2.2 : The currents running through an electrical system are given by the following system of equations. The three currents, I1, I2, and I3, are measured in amps.

$$8I_1 + 3I_2 - 5I_3 = 3$$
$$10I_1 + 7I_2 + 2I_3 = 4\ .$$
$$6I_1 + 4I_2 + 7I_3 = 8$$

a. Solve the above system to find the currents in this circuit using Gaussian elimination with partial pivoting.
b. Justify your result by direct substitution in the original equation.
c. Write MATLAB codes to solve by left division (backslash) operator.

Solution : a.

Oper	I_1	I_2	I_3	Constant	Eqn #	
	8	3	-5	3	E1	
	10	7	2	4	E2	
	6	4	7	8	E3	
E2/10	1.000	0.700	0.200	0.400	E4	1st Piv eqn
E1/8	1.000	0.375	-0.625	0.375	E5	
E3/6	1.000	0.667	1.167	1.333	E6	
E5-E4	0	-0.325	-0.825	-0.025	E7	
E6-E4	0	-0.033	0.967	0.933	E8	
E7/(-0.325)	0	1	2.538	0.0769	E9	2nd Piv eqn
E8/(-0.03)	0	1	-32.233	-31.1	E10	
E9-E10	0	0	34.771	31.177		
E10/(-32.233)	0	0	1	0.897		3rd Piv eqn

The triangular form of the system can be summarized as

$$I_1 + 0.7I_2 + 0.2I_3 = 0.4$$
$$I_2 + 2.538I_3 = 0.077$$
$$I_3 = 0.8907$$

Solving by back substitution, we have, $I_3 = 0.8907$

$$I_2 = 0.077 - 2.538(0.890) = -2.182$$
$$I_1 = 0.4 - 0.7(-2.182) - 0.2(0.8907) = 1.759$$

b. Justification: $E1$: $8(1.759) + 3(-2.182) - 5(0.8907) = 3.01 \cong 3$

$E2$: $10(1.759) + 7(-2.182) + 2(0.8907) = 4.02 \approx 4$

$E3$: $6(1.759) + 4(-2.182) + 7(0.8907) = 8.01 \approx 8$

c.
```
>> A=[8 3 -5; 10 7 2; 6 4 7];
>> b=[3; 4; 8];
>> Solution = A\b
Solution =    1.7500, -2.1829, 0.8902
```

Example 2.3 : Cory, Josh and Dan went shopping for Halloween treats. Cory bought 3 chocolate pumpkins, 4 masks and 8 candy witches. He spent \$36.65. Josh bought 5 chocolate pumpkins, 3 masks and 10 candy witches. He spent \$37.50. Dan bought 4 chocolate pumpkins, 5 masks and 6 candy witches. He spent \$43.45.
i. Write a system of equations to represent this problem
ii. Calculate the unit price of each item purchased using Gaussian elimination with pivoting.

Solution:
i. Let $x =$ price of a chocolate pumpkin
$y =$ price of a mask
$z =$ price of a candy witche
Now we can setup our system of equations as:

$$3x + 4y + 8z = 36.65$$
$$5x + 3y + 10z = 37.50$$
$$4x + 5y + 6z = 43.45$$

ii.

Oper	x	y	z	Constant	Eqn #	
	3	4	8	36.65	E1	
	5	3	10	37.50	E2	
	4	5	6	43.45	E3	
E2/5	1.000	0.600	2.000	7.500	E4	1^{st} Piv eq^n
E1/3	1.000	1.333	2.667	12.217	E5	
E3/4	1.000	1.250	1.500	10.863	E6	
E4-E5	0	-0.733	-0.667	-4.717	E7	
E4-E6	0	-0.650	0.500	-3.363	E8	
E7/(-0.733)	0	1.000	0.910	6.435	E9	2^{nd} Piv eq^n
E8/(-0.650)	0	1.000	-0.769	5.174	E10	
E9-E10	0	0	1.679	1.261	E11	
E10/(1.679)	0	0	1.000	0.751	E12	3^{rd} Piv eq^n

The triangular form of the system can be summarized as

$$x + 0.600y + 2.000z = 7.500$$
$$y + 0.910z = 6.435$$
$$z = 0.751$$

Solving by back substitution, we have, $z = 0.751$

$$y = 6.435 - 0.91(0.751) = 5.752$$
$$x = 7.5 - 0.6(5.752) - 2(0.751) = 2.547$$

Partial pivoting is adequate for most of the simultaneous equations which arise in practice. But we may encounter sets of equations where wrong or incorrect solutions may results. To improve the calculation in such cases **total pivoting** is used. In total pivoting, maximum magnitude of the coefficients of the variables is used for the pivot in each elimination.

2.3 Solution of Linear System by Iterative Method

Iterative method for linear system is similar as the method of fixed-point iteration for an equation in one variable. To solve a linear system by iteration, we solve each equation for one of the variables, in turn, in terms of the other variables. Starting from an approximation to the solution, if convergent, derive a new sequence of approximations. Repeat the calculations till the required accuracy is obtained.

An iterative method converges, for any choice of the first approximation, if every equation satisfies the condition that the magnitude of the coefficient of solving variable is greater than the sum of the absolute values of the coefficients of the other variables. A system satisfying this condition is called diagonally dominant. A linear system can always be reduced to diagonally dominant by elementary operations.

For example, in the system

$$x+2y+10z=10 \quad (E_1) \qquad |1|<|2|+|10|$$
$$x-10y-z=24 \quad (E_2) \qquad |-10|>|1|+|-1|$$
$$11x+5y+8z=31 \quad (E_3) \qquad |8|<|11|+|5|$$

is not diagonally dominant. Rearranging as $(E_3)-(E_1)$, (E_2), (E_1), we have

$$10x+3y-2z=21 \qquad |10|>|3|+|-2|$$
$$x-10y-z=24 \qquad |-10|>|1|+|-1|$$
$$x+2y+10z=10 \qquad |10|>|1|+|2|$$

The system reduces to diagonally dominant form.

2.3.1. Jacobi Iterative Method:

In this method, a fixed set of values is used to calculate all the variables and then repeated for the next iteration with the values obtained previously. The iterative formulas of the above system are

$$x_{n+1}=\frac{1}{10}(21-3y_n+2z_n)$$
$$y_{n+1}=-\frac{1}{10}(24-x_n+z_n)$$
$$z_{n+1}=\frac{1}{10}(10-x_n-2y_n)$$

Gauss-Seidel iterative method is more efficient than Jacobi's iterative method and explained through an example.

2.3.2. Gauss-Seidel Iterative Method

In this method, the values of each variable is calculated using the most recent approximations to the values of the other variables. The Gauss-Seidel iterative formulas of the above system are

$$x_{n+1}=\frac{1}{10}(21-3y_n+2z_n)$$
$$y_{n+1}=-\frac{1}{10}(24-x_{n+1}+z_n)$$
$$z_{n+1}=\frac{1}{10}(10-x_{n+1}-2y_{n+1})$$

If initial values are not supplied we can start with initial values

$$x_0=0, \quad y_0=0, \quad z_0=0.$$

and perform the iterations until required accuracy is achieved.

Example 2.4: Given the linear system

$$6x+5y+3z=7 \qquad \text{Eq}(1)$$
$$8x-3y+2z=16 \qquad \text{Eq}(2)\ .$$
$$10x-7y-8z=15 \qquad \text{Eq}(3)$$

a. Reduce the above system to diagonally dominant form.
b. Write the corresponding Gauss-Seidel iteration formula.

c. Compute two iterations to estimate the roots to 2 d.p. with $x_0 = 1.5$, $y_0 = -1$ and $z_0 = 1$.

d. Write MATLAB code to iterate the above formula four times.

Solution:

a. Eq(2) $8x - 3y + 2z = 16,$ $\lfloor 8 \rfloor \geq \lfloor -3 \rfloor + \lfloor 2 \rfloor$

Eq(1)-Eq(2) $-2x + 8y + z = -9,$ $\lfloor 8 \rfloor \geq \lfloor -2 \rfloor + \lfloor 1 \rfloor$

Eq(2)-Eq(3) $-2x + 4y + 10z = 1,$ $\lfloor 10 \rfloor \geq \lfloor -2 \rfloor + \lfloor 4 \rfloor$

b. Gauss-Seidel formula

$$x_{n+1} = \frac{1}{8}[16 + 3y_n - 2z_n]$$
$$y_{n+1} = \frac{1}{8}[-9 + 2x_{n+1} - z_n]$$
$$z_{n+1} = \frac{1}{10}[1 + 2x_{n+1} - 4y_{n+1}]$$

c. Starting with initial values $x_0 = 1.5$, $y_0 = -1$, $z_0 = 1$

When $n = 0$, we have

$$x_1 = \frac{1}{8}[16 + 3(-1) - 2(1)] = 1.375$$
$$y_1 = \frac{1}{8}[-9 + 2(1.375) - 1] = -0.906$$
$$z_1 = \frac{1}{10}[1 + 2(1.375) - 4(-0.906)] = 0.737$$

For $n = 1$, we have

$$x_2 = \frac{1}{8}[16 + 3(-0.906) - 2(0.737)] = 1.476$$
$$y_2 = \frac{1}{8}[-9 + 2(1.476) - -0.906] = -0.848$$
$$z_2 = \frac{1}{10}[1 + 2(1.476) - 4(-0.848)] = 0.734$$

Solution to 2 d.p. is $x = 1.48$, $y = -0.85$, $z = 0.73$.

d.

```
>> x(1)=1.5;
>> y(1)=-1;
>> z(1)=1;
>> iter(1)=0;
>> for n=1:4
 iter(n+1)=n;
 x(n+1)=(16+3*y(n)-2*z(n))/8;
 y(n+1)=(-9+2*x(n+1)-z(n))/8;
 z(n+1)=(1+2*x(n+1)-4*y(n+1))/10;
end
>> Solution = [iter',x',y',z']
Solution =
        0    1.5000   -1.0000    1.0000
   1.0000    1.3750   -0.9063    0.7375
   2.0000    1.4758   -0.8482    0.7345
   3.0000    1.4983   -0.8422    0.7366
   4.0000    1.5000   -0.8421    0.7368
```

Exercise-2

1. A chemist has 3 different acid solutions. The first acid solution contains 10% acid, the second contains 40% and the third contains 60% acid. The chemist wants to use all 3 solutions to obtain a mixture of 100 liters containing 45% acid. The supply of 40% acid is low. Consequently, twice as much 10% solution as 40% solution must be used. How many liters of each solution should be used?

2. A movie theater is filled the capacity of 350. The theater charges 40 taka for children, 70 for students and 120 for adults. There are half as many adults as there are students. If the total ticket sales were 24150 taka, how many children, students and adults attended?

3.
a. $5x - 4y + 3z = 21, \quad 2x + 3y + 2z = 20, \quad 8x + 2y + z = 13.$
b. $3x - 4y + 6z = 12, \quad 7x + y - 5z = 18, \quad 2x + 9y + 4z = 14.$
c. $2x + y - 3z = 17, \quad 5x - 2y + 3z = 6, \quad x - 8y + z = 11.$

i. Solve the above linear system using Gaussian elimination with pivoting (partial/total).

ii. Justify your result by direct substitution in the original equations.

iii. Write MATLAB codes to solve by left division (backslash) operator.

4.
a. $x + 8y + 3z = 10, \; 3x - 5y + 7z = 4, \quad 3x - y - z = 1.$
using $x_0 = 0.85, \; y_0 = 0.8$ and $z_0 = 0.75$
b. $2x + 10y - 7z = 20, \; 3x - 7y - 5z = 18, \; 8x - 5y - 2z = 12.$
using $x_0 = 0.6, \; y_0 = -0.1$ and $z_0 = -3.$
c. $5x + 9y + 12z = 9, \quad 8x - 4y - 11z = 14, \quad -2x + 5y + z = 10.$
using $x_0 = 0.75, \; y_0 = 2.5$ and $z_0 = -1.5.$

i. Reduce the above system to diagonally dominant form.

ii. Write the corresponding Gauss-Seidel and Jacobi iteration formula.

iii. Compute two iterations to estimate the roots to 3 d.p. with the given initial values.

iv. Justify your result by direct substitution in the original equations.

v. Write MATLAB codes to solve by left division (backslash) operator.

5. Consider the linear system: $4x + 2y + z = 7, \; 4x + 5y + 3z = 4, 4x + 5y + 7z = 3$

i. Reduce the above system to diagonally dominant form.

ii. Write the corresponding Gauss-Seidel iteration formula.

iii. Compute two iterations to estimate the roots to 2 d.p with the following initial values x= 2, y=-0.75, z=-0.2.

iv. Justify your result by direct substitution in the original equations.

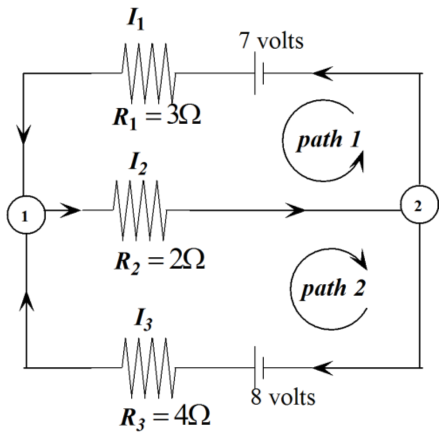
v. Write MATLAB codes to iterate the above formula four times.

6. Determine the loop currents I_1, I_2 and I_3 of the given circuit by solving the linear system using Gaussian elimination with pivoting.

For loop 1: $I_1 - I_2 + I_3 = 0$

For loop 2: $3I_1 + 2I_2 = 7$

For loop 3: $2I_2 + 4I_3 = 8$



7. Given the linear system: $10x + 5y + 3z = 21, 6x + 3y - 7z = 22, 3x + 16y + 4z = 14.$

i. Use Gaussian elimination with partial pivoting to solve the system giving results to 2 d.p.

ii. Reduce the above system to diagonally dominant form and write the corresponding Gauss- Seidel iteration formula.

iii. Compute **one** iterations to estimate the solutions to 2 d.p. with $x_0 = 2, y_0 = 0.8, z_0 = -1.$

iv. Write MATLAB codes to solve by left division (backslash) operator.

8. Given the linear system: $6x + 5y - 8z = 24, 10x + 3y + 4z = 11, 8y + 3z = 10.$

i. Use Gaussian elimination with partial pivoting to solve the system giving results to 2 d.p.

ii. Reduce the above system to diagonally dominant form and write the corresponding

iii. Gauss-Seidel iteration formula.

iv. Compute one iterations to estimate the solutions to 2 d.p. with $x_0 = 1, y_0 = 1.5,$

9. Tracy, Danielle and Sherri bought snacks for a birthday party. They each bought the items shown in the following table at the local convenience store:

Number of bags of potato chips	Number of litres of pop	Number of chocolate bars	Cost ($)
2	9	5	21.00
3	2	10	20.88
8	3	4	13.17

i. Construct the system of linear equation from the above problem.

ii. Reduce the system obtain in (i) to diagonally dominant form.

iii. Write the corresponding Gauss-Seidel iteration formula.

iv. Compute two iterations to estimate the roots to 2 d.p using the following initial values $x_0 = 0.2,\ y_0 = 1.25$ and $z_0 = 165.$

v. Justify your result by direct substitution in the original equations.

vi. Write MATLAB codes to iterate the above formula four times.

10. Jesse, Maria and Charles went to the local craft store to purchase supplies for making decorations for the upcoming dance at the high school. Jesse purchased three sheets of craft paper, four boxes of markers and five glue sticks. His bill, before taxes was $24.40. Maria spent $30.40 when she bought six sheets of craft paper, five boxes of markers and two glue sticks. Charles, purchases totaled $13.40 when he bought three sheets of craft paper, two boxes of markers and one glue stick. Determine the unit cost of each item.

i. Construct the system of linear equation from the above problem.

ii. Solve the system of linear equation using Gaussian elimination with partial pivoting.

iii. Write MATLAB codes to solve by left division (backslash) operator.

11. A local computer company sells three types of laptop computers to three nearby stores. The number of laptops ordered by each store and the amount owing to the company for the order is shown in the following table:

Store	Laptop A	Laptop B	Laptop C	Amount Owing($)
Wal-Mart	10	8	6	21 200
Sears	7	9	5	18 700
Target	8	4	3	13 000

i. Write a system of equations to represent the above information.

ii. Solve the system of linear equation using Gaussian elimination with partial pivoting.

iii. Write MATLAB codes to solve by left division (backslash) operator.

12. Consider the linear system: $5x + 2y + z = 7, 2x - 4y + 3z = 6, 3x + 5y + 7z = 6$

i. Reduce the above system to diagonally dominant form.

ii. Write the corresponding Gauss-Seidel iteration formula.

iii. Compute two iterations to estimate the roots to 3 d.p.

iv. Justify your result by direct substitution in the original equations.

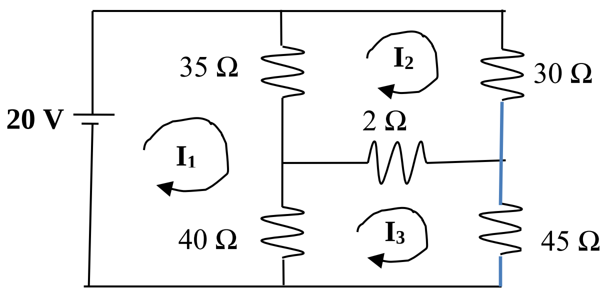
v. Write MATLAB codes to iterate the above formula four times.
Answer: $x = 1.453, y = -0.389, z = 0.512$

13. The following system of equations was generated by applying the mesh current law in the circuit shown in the adjacent figure.

$$35(I_1 - I_2) + 40(I_1 - I_3) = 20$$

$$35(I_2 - I_1) + 30I_2 + 2(I_2 - I_3) = 0$$

$$40(I_3 - I_1) + 2(I_3 - I_2) + 45I_3 = 0$$



i. Write down the system in simplified form.

ii. Solve the system of equations using Gauss-Seidel iteration correct to 2 decimal places.

14. Given the linear system: $9x + 5y + 3z = 20, 5x + 4y - 7z = 21, 3x + 9y + 4z = 16.$

i. Use Gaussian elimination with partial pivoting to solve the system giving results to 2 d.p.

ii. Reduce the above system to diagonally dominant form and write the corresponding Gauss- Seidel iteration formula.

iii. Compute **one** iterations to estimate the solutions to 2 d.p. with $x_0 = 1.5, y_0 = 1.6, z_0 = -1.$

iv. Write MATLAB codes to solve by left division (backslash) operator.

15. Given the linear system: $6x + 5y - 8z = 14, 8x + 3y + 4z = 16, 2x + 7y + 3z = 12.$

i. Use Gaussian elimination with partial pivoting to solve the system giving results to 2 d.p.

ii. Reduce the above system to diagonally dominant form and write the corresponding Gauss-Seidel iteration formula.

iii. Compute one iterations to estimate the solutions to 2 d.p. with $x_0 = 1.4, y_0 = 1.2,$ and $z_0 = 0.2.$

iv. Write MATLAB codes to solve by left division (backslash) operator.

Answer: $x = 1.47, y = 1.24, z = 0.1$

18. An electronics company produces transistors, resistors, and computer chips. Each transistor requires four units of copper, one unit of zinc, and two units of glass. Each resistor requires three, three, and one units of the three materials, respectively, and each computer chip requires two, one, and three units of these materials, respectively. Putting this information into table form, we get:

Component	Copper	Zinc	Glass
Transistors	4	1	2
Resistors	3	3	1
Computer chips	2	1	3

Supplies of these materials vary from week to week, so the company needs to determine a different production run each week. For example, one week the total amounts of materials available are 960 units of copper, 510 units of zinc, and 610 units of glass. Set up the system of equations modeling the production run, and use Excel, MATLAB, or Mathcad, to solve for the number of transistors, resistors, and computer chips to be manufactured this week.