Mathematical Methods For Science and Engineering
Lecture Note 4
Nonlinear System of Equations

4.1 Newton-Raphson Method

Recall that the Newton-Raphson formula to find the root of the equation $f(x) = 0$ can be written as

$$x_{n+1} = x_n - [f'(x_n)]^{-1} f(x_n).$$

For a system of nonlinear equations, the Newton-Raphson formula in matrix form can be expressed as

$$X_{n+1} = X_n - [J(X_n)]^{-1} F(X_n).$$

where *X* is a variables matrix, *F*(*X*) is a function matrix and *J(X)* is a Jacobian matrix.

The variable, function and Jacobian matrices for a system of two variables can be written as

$$X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad F = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}, \quad J(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}.$$

This method can be extended for equations with more variables.

Example 4.1 Consider the nonlinear system $\begin{matrix} x^2 + y - 1 = 0 \\ y^3 - 2x + 4 = 0 \end{matrix}$.

(a) Find the Jacobian matrix $J(x,y)$ for the above system.
(b) Evaluate the inverse of the Jacobian matrix at $(1.4, -1)$.
(c) Write down the iterative formula for the above system based on the Newton-Raphson method.
(d) Estimate the root to 2 d.p. using the using the above iterative formula once starting with $(x_0, y_0) = (1.4, -1)$.
(e) Write down MATLAB commands to execute the iteration four times.
(f) Use MATLAB function "**fsolve(fun,x0)**" to find the above root.

Solution:

(a) The function matrix and Jacobian matrix are

$$F(x,y) = \begin{pmatrix} x^2 + y - 1 \\ y^3 - 2x + 4 \end{pmatrix}, \qquad J(x,y) = \begin{pmatrix} 2x & 1 \\ -2 & 3y^2 \end{pmatrix}$$

(b)

Note: The inverse of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be calculated by

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

At the point $(1.4, -1)$. we have

$$J(1.4,-1) = \begin{pmatrix} 2.8 & 1 \\ -2 & 3 \end{pmatrix}$$

and

$$[J(1.4,-1)]^{-1} = \begin{pmatrix} 0.2885 & -0.0962 \\ 0.1923 & 0.2692 \end{pmatrix}$$

(c) The iterative formula is

$$X_{n+1} = X_n - [J(X_n)]^{-1}\, F(X_n).$$

where

$$X_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}.$$

(d) Using $n = 0$, we have

$$X1 = X0 - [J(X0)]^{-1} F(X0)$$

where

$$X_0 = \begin{pmatrix} 1.4 \\ -1 \end{pmatrix}, \quad F(X_0) = F(1.4,-1) = \begin{pmatrix} -0.04 \\ 0.2 \end{pmatrix},$$

and

$$[J(X_0)]^{-1} = [J(1.4,-1)]^{-1} = \begin{pmatrix} 0.2885 & -0.0962 \\ 0.1923 & 0.2692 \end{pmatrix}$$

Thus

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1.4 \\ -1 \end{pmatrix} - \begin{pmatrix} 0.2885 & -0.0962 \\ 0.1923 & 0.2692 \end{pmatrix}\begin{pmatrix} -0.04 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 1.4308 \\ -1.0462 \end{pmatrix}$$

root to 2 d.p. is

$$x_1 = 1.43 \quad \text{and} \quad y_1 = -1.05.$$

Further iterations can be done by updating $X0$ by $X1$.

(e) Define the functions as $F = \begin{bmatrix} x^2 + y - 1 \\ y^3 - 2x + 4 \end{bmatrix}$

we have the Jacobian $J = \begin{bmatrix} 2x & 1 \\ -2 & 3y^2 \end{bmatrix}$.

```
>> clear
>> F=@(x,y) [x.^2+y-1; y.^3-2*x+4];
>> J=@(x,y)[2*x,  1;  -2,  3*y.^2];
>> x(1)=1.4;
>> y(1)=-1;
>> for n=1:50
     Xn=[x(n); y(n)];
     Fn=F(x(n), y(n));
     Jn=J(x(n), y(n));
     Xn1=Xn-Jn\Fn;
   x(n+1)=Xn1(1);
  y(n+1)=Xn1(2);
end

>> Roots=[x',y']

Roots =
    1.4000   -1.0000
    1.4308   -1.0462
    1.4299   -1.0447
    1.4299   -1.0447
    1.4299   -1.0447
```

```
(f)  >> clear
% Left_hand side of equations as vector
>> F=@(x) [x(1)^2+x(2)-1; x(2)^3-2*x(1)+4];
% Guess solution
>> x0=[1; -1];
% Solve the system
>> x=fsolve(F, x0)

x =
    1.4299
   -1.0447
```

4.2 Fixed Point Iteration Method for Nonlinear System

Consider the nonlinear system of two variables of the form

$$f_1(x, y) = 0 \quad \text{and} f_2(x, y) = 0 \tag{1}$$

By rearrangements the system can be expressed as

$$x = g_1(x, y) \text{and} \quad y = g_2(x, y) \tag{2}$$

If there is a point (p, q) such that

$$p = g_1(p, q) \text{and} \quad q = g_2(p, q)$$

then the point (p, q) is a **fixed point** of the system and it is a root of the system.

With the arrangement (2), we may assume an iterative formula

$$x_{n+1} = g_{1(x_n,y_n)} \text{ and } y_{n+1} = g_2(x_n, y_n) \tag{3}$$

where$n = 0, 1, 2, 3, \cdots$.

Without further details it may be mentioned that if (x_0, y_0) is close to the fixed point (p, q) and if

$$\left|\frac{\partial}{\partial x} g_1(x_0, y_0)\right| + \left|\frac{\partial}{\partial y} g_1(x_0, y_0)\right| < 1 \tag{4a}$$

and

$$\left|\frac{\partial}{\partial x} g_2(x_0, y_0)\right| + \left|\frac{\partial}{\partial y} g_2(x_0, y_0)\right| < 1 \tag{4b}$$

then the iterative formula (3) will converge to the fixed point (p, q).

If the conditions (4) are not satisfied, the iterative process might diverge.

The above method can be extended for more than two variables.

Seidel Iteration

An improvement of the iteration process is the Seidel iteration similar to Gauss-Seidel iteration for linear system. In Seidel iteration process the iterative formulas are

$$x_{n+1} = g_1(x_n, y_n)$$
$$y_{n+1} = g_2(x_{n+1}, y_n)$$

Example 4.2

Consider the system of equations $\quad y = x^2 - 5x + 3$

$$x^2 + 4y^2 = 4$$

(a) Plot the graphs of the above system using MATLAB function "**ezplot(fun)**". From your plot find the number of real roots and estimate the roots.

(b) A fixed point iteration formula is suggested to estimate root at (x_0, y_0)

$$x_{n+1} = \frac{1}{5}(x_n^2 - y_n + 3)$$
$$y_{n+1} = \frac{1}{10}(4 - x_n^2 - 4y_n^2 + 10y_n)$$

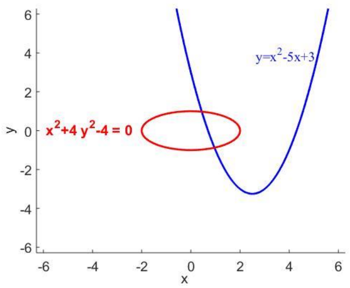
(i) Verify whether the above iterative formula will converge to the root near $(x_0, y_0) = (0.4, 1.0)$. If converges, perform one iteration otherwise suggest another fixed point iterative formula which converge to root.

(ii) Verify whether the above iterative formula will converge to the root near $(x_0, y_0) = (0.9, -1)$. If converges, perform one iteration otherwise suggest another fixed point iterative formula which converge to the root.

(c) Write MATLAB commands to execute the iterative formula in b(ii) five times.

(a) MATAB codes for ezplot

```
>> clear
>> figure
>> hold on
>> ezplot('y-x.^2+5*x-3')
>> ezplot('x.^2+4*y.^2-4')
>> hold off
```



From graph it can be seen that the system has two real roots near $(0.4, 1.0)$and $(0.9, -0.9)$.

(b) (i) From the iterative formula let us define

$$g_1(x,y) = \frac{1}{5}(x^2 - y + 3)$$

and

$$g_2(x,y) = \frac{1}{10}(4 - x^2 - 4y^2 + 10y)$$

Here

$$\frac{\partial}{\partial x} g_1(x,y) = \frac{2x}{5}, \qquad \frac{\partial}{\partial y} g_1(x,y) = -\frac{1}{5}$$

and

$$\frac{\partial}{\partial x} g_2(x,y) = -\frac{2x}{10}, \qquad \frac{\partial}{\partial y} g_2(x,y) = \frac{1}{10}(-8y + 10)$$

Near $(0.4, 1.)$, we have

$$\left|\frac{\partial g_1}{\partial x}\right| + \left|\frac{\partial g_1}{\partial y}\right| = |0.16| + |-0.2| = 0.36 < 1$$

and

$$\left|\frac{\partial g_2}{\partial x}\right| + \left|\frac{\partial g_2}{\partial y}\right| = |-0.08| + \left|\frac{-8(1) + 10}{10}\right| = 0.28 < 1$$

The partial derivative tests are satisfied and hence the iterations converges to the root near $(0.4, 1.0)$.
We can use iterative formulas for the root near $(0.4, 1.0)$ as

$$x_{n+1} = \frac{1}{5}(x_n^2 - y_n + 3)$$

$$y_{n+1} = \frac{1}{10}(4 - x_n^2 - 4y_n^2 + 10y_n)$$

In Seidel iteration the second iterative equation will be of the form

$$y_{n+1} = \frac{1}{10}(4 - x_{n+1}^2 - 4y_n^2 + 10y_n)$$

Calculations with the above iterative formulas with $x_0 = 0.4$ and $y_0 = 1.0$ are given below:

Jacobi iteration		
n	x_n	y_n
0	0.4	1.0
1	0.432	0.984

Seidel iteration		
n	x_n	y_n
0	0.4	1.0
1	0.432	0.9813

(ii) Near (0.9, -1.0) we have

$$\left|\frac{\partial g_1}{\partial x}\right| + \left|\frac{\partial g_1}{\partial y}\right| = |0.36| + |-0.2| = 0.56 < 1$$

$g_1 x, y)$ satisfy the partial derivative test.

But $$\left|\frac{\partial g_2}{\partial x}\right| + \left|\frac{\partial g_2}{\partial y}\right| = |-0.18| + \left|\frac{-8(-1)+10}{10}\right| = 1.98 > 1$$

Test fails and the convergence is not guaranteed.

Rearranging the second equation by

$$x^2 + 4y^2 - 4 + 10y = 10y$$

we have

$$y = \frac{1}{10}(x^2 + 4y^2 - 4 + 10y)\,.$$

Corresponding iterative function can be of the form

$$g_3(x,y) = \frac{1}{10}(x^2 + 4y^2 - 4 + 10y)\,.$$

and

$$\frac{\partial}{\partial x} g_3(x,y) = \frac{2x}{10}\,, \qquad \frac{\partial}{\partial y} g_3(x,y) = \frac{1}{10}(8y + 10)$$

At $(0.9, -1)$,

$$\left|\frac{\partial g_3}{\partial x}\right| + \left|\frac{\partial g_3}{\partial y}\right| = \left|\frac{1.8}{10}\right| + \left|\frac{8(-1) + 10}{10}\right| = 0.38 < 1$$

$g_1 x, y)$ satisfies the partial derivative test.

Thus a fixed point iterative formula which converge to root near $(0.9, -1)$ is

$$x_{n+1} = \frac{1}{5}(x_n^2 - y_n + 3)$$

$$y_{n+1} = \frac{1}{10}(x_n^2 + 4y_n^2 - 4 + 10y_n)$$

(e)
```
>> clear
>> x(1)=0.9;
>> y(1)=-1;
>> for n=1:5
    x(n+1)=(x(n)^2-y(n)+3)/5;
    y(n+1)=(x(n)^2+4*y(n)^2-4+10*y(n))/10;
    end
>> Iterative_Roots =[x',y']

Iterative_Roots =

   0.9000   -1.0000
   0.9620   -0.9190
   0.9689   -0.8886
   0.9655   -0.8789
   0.9622   -0.8767
   0.9605   -0.8767
```

Exercise 4

1. Consider the following nonlinear systems:

    (a) $x^3 + y^2 = 2, x^2 - y = 1$ starting with $(x_0, y_0) = (1.2, -0.5)$

    (b) $3x^2 - y^2 = 0,\ \ 3xy^2 - x^3 - 1 = 0$ starting with $(x_0, y_0) = (1, 1)$.

    (c) $\ln(x^2 + y^2) + y = 1, \sqrt{x} + xy^2 = 0$ starting with $(x_0, y_0) = (2.4, -0.6)$.

    (d) $4x^2 - 20x + \frac{1}{4}y^2 + 8 = 0, \frac{1}{2}xy^2 + 2x - 5y + 8 = 0$ starting with $(x_0, y_0) = (0.5, 1.9)$.

    (e) $y\sin(xy) + 2 = 0,\ \ \cos(xy) + x - y = 0$ starting with $(x_0, y_0) = (1, 2)$.

    (f) $\tan^{-1} x + e^{y^2} = 5, \cos x^2 + xy = 10$ starting with $(x_0, y_0) = (-1.6, -1.3)$.

Now using Newton-Raphson method answer the following questions. For each of the above nonlinear system of equations:

(i) Find the Jacobian matrix $J(x, y)$ for the above system.

(ii)Evaluate the inverse of the Jacobian matrix at the given point.

(iii) Write down the iterative formula for Newton-Raphson method.

(iv) Estimate the root to 2 d.p. using the using the above iterative formula **once** starting with the given point.

(v)Write down MATLAB commands to execute the iteration four times.

(vi) Use MATLAB function “**fsolve (fun,x0)**” to find the root.

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2. Sketch the curves represented by the following non-linear equations,

$$2e^x + y = 0,$$
$$3x^2 + 4y^2 = 8.$$

a) Solve using Newton’s method with initial estimate $(-1, -2)$.

b) Repeat (a) but with an initial estimate $(-2,0)$.

c) Use graphical approach to validate the results in (a) and (b).

3. Sketch the curves represented by the following non-linear equations,

$$x^2 - y^2 = 4,$$
$$x^2 + y^2 = 16.$$

a) Solve using Newton’s method with initial estimate $(3.0,2.5)$.

b) Repeat (a) but with an initial estimate $(-3, -2.5)$.

c) Use graphical approach to validate the results in (a) and (b).

4. The system of non-linear equations

$$4x + \ln(x^2 + y) = 1 \quad \text{and} \quad 8y - \ln(x + y^2) = 1$$

has a solution near $(x_0, y_0) = (1,\ 2)$.

The following iterative formula is suggested to estimate the roots.

$$x_{n+1} = \frac{1}{4}(1 - \ln(x_n^2 + y_n))$$

$$y_{n+1} = \frac{1}{8}(1 + \ln\,(x_n + y_n^2))$$

i. Verify whether the above iterative formula will converge to the root near $(x_0, y_0) = (1, 2)$. If converges, perform ONE iteration otherwise write 'Not Convergent'.

ii. Write MATLAB commands to execute the iterative formula five times.

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5. Determine the roots of the following simultaneous non linear equations using (a) Newton-Raphson method, (b) fixed-point iteration method.

$$x = y + x^2 - 0.5,$$

$$y = x^2 - 5xy$$

Employ initial guesses of $x_0 = y_0 = 1.0$ and discuss the results.

6. The following system has a root near $(x_0, y_0) = (0.2,\ 0.3)$.

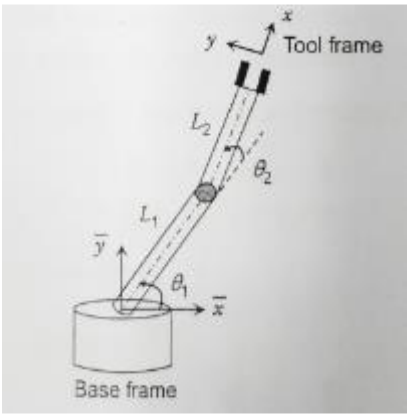
$$2x^2 + y^2 - 15x + 2 = 0,$$
$$xy^2 + x - 10y + 5 = 0.$$

Estimate the root correct to 3 decimal places using fixed point iterative method.

7. A planar, two-link robot arm is shown in the figure. The coordinate system xy is the tool frame and attached to the end-effector. The coordinates of the end-effector relative to the base frame are expressed as

$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2)$$
$$y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2)$$

Suppose the lengths, in consistent physical units, of the two links are $L_1 = 1$ and $L_2 = 2$, and that $x = 2.5, y = 1.4$. Find the joint angles θ_1 and θ_2 (in radians) using Newton's method with initial estimate of (0.8,0.9)


x
y
Tool frame
L_2
θ_2
L_1
$\bar{y}$
θ_1
$\bar{x}$
Base frame