LECTURE 2

BOOK CHAPTER 4

Projectile Motion

Projectile Motion:

A particle moves in a vertical plane with some initial velocity $\vec{v}_0$ but its acceleration is always the freefall acceleration $\vec{g}$, which is downward. Such a particle is called a **projectile** (meaning that it is projected or launched), and its motion is called **projectile motion**.

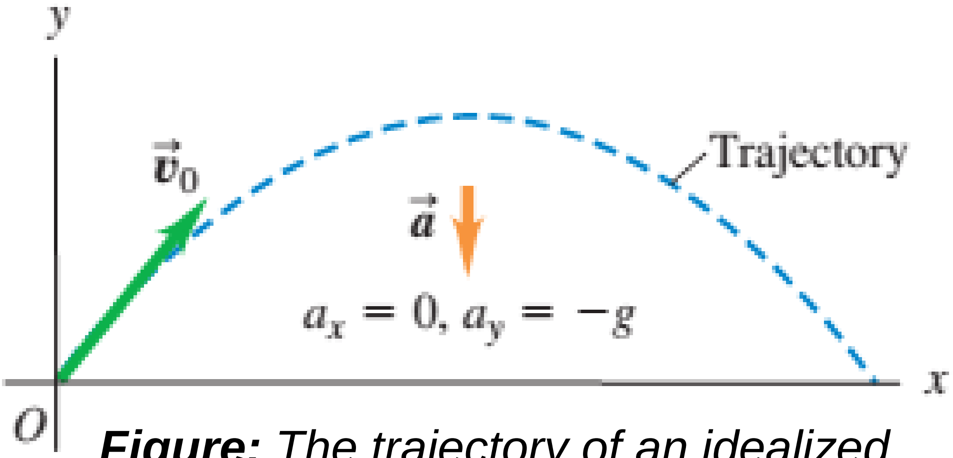
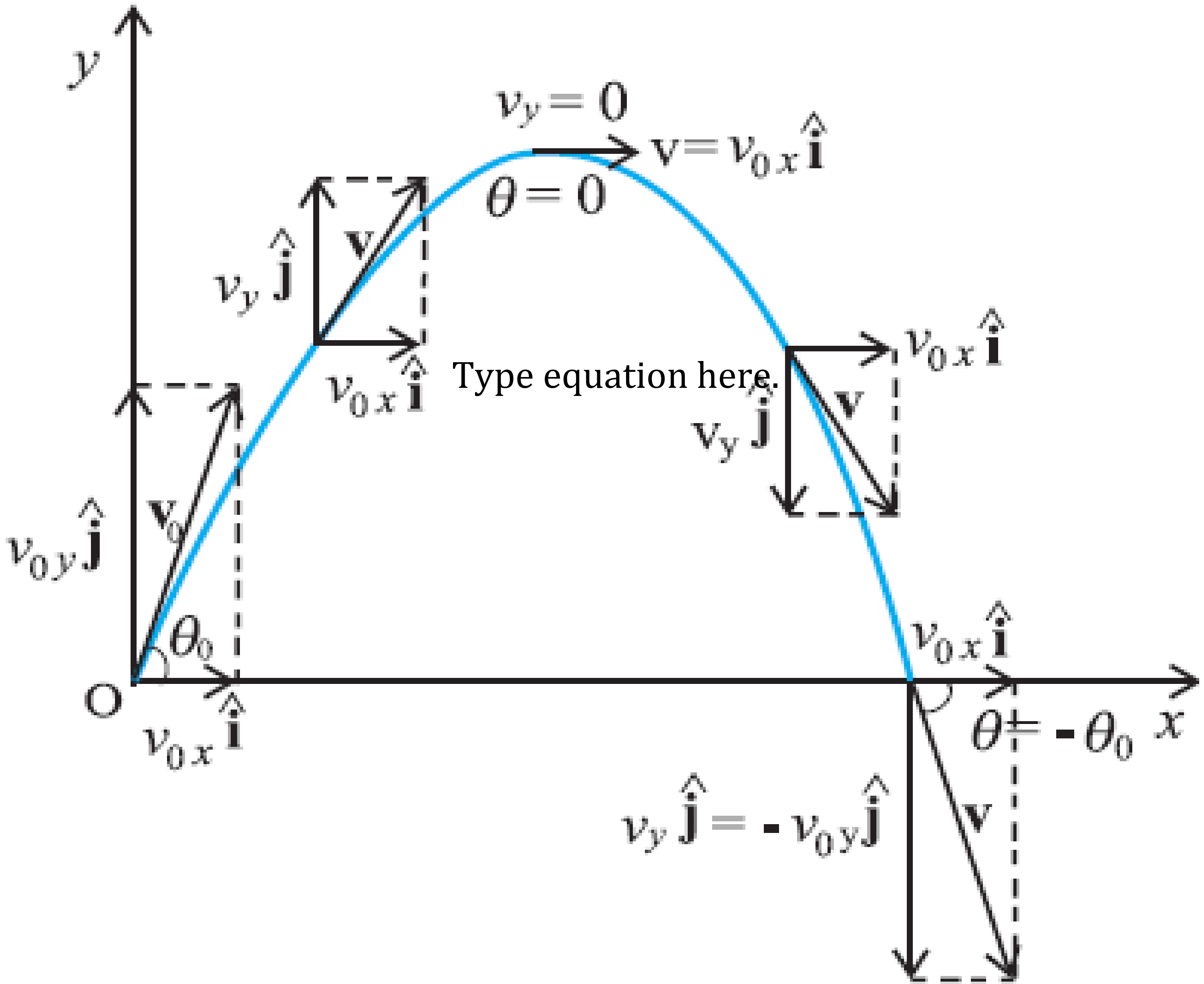


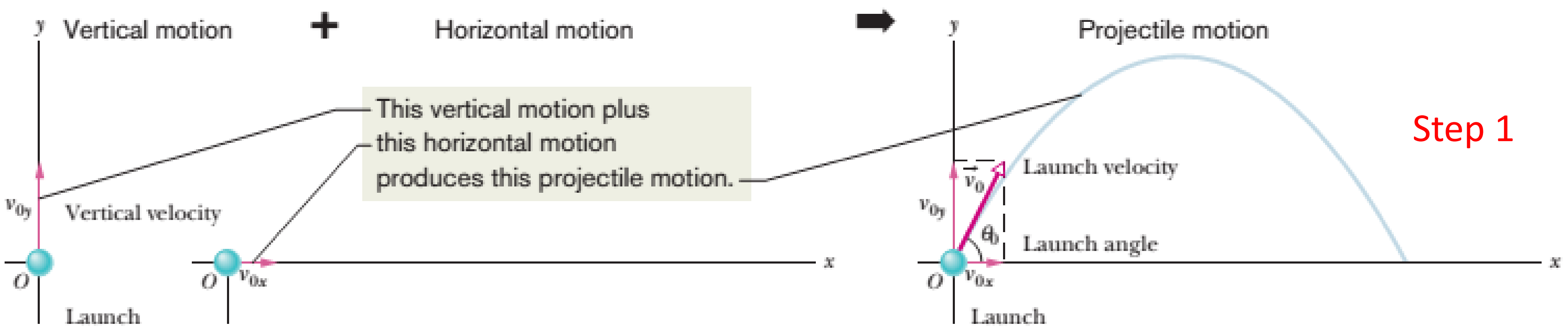
Figure: *The trajectory of an idealized projectile.*

Examples: A batted baseball, a thrown football, a package dropped from an airplane, and a bullet shot from a rifle are all projectiles.

Sketch of the path taken in projectile motion:



Sketch of the path taken in projectile motion (Step-by-Step):



Step 1

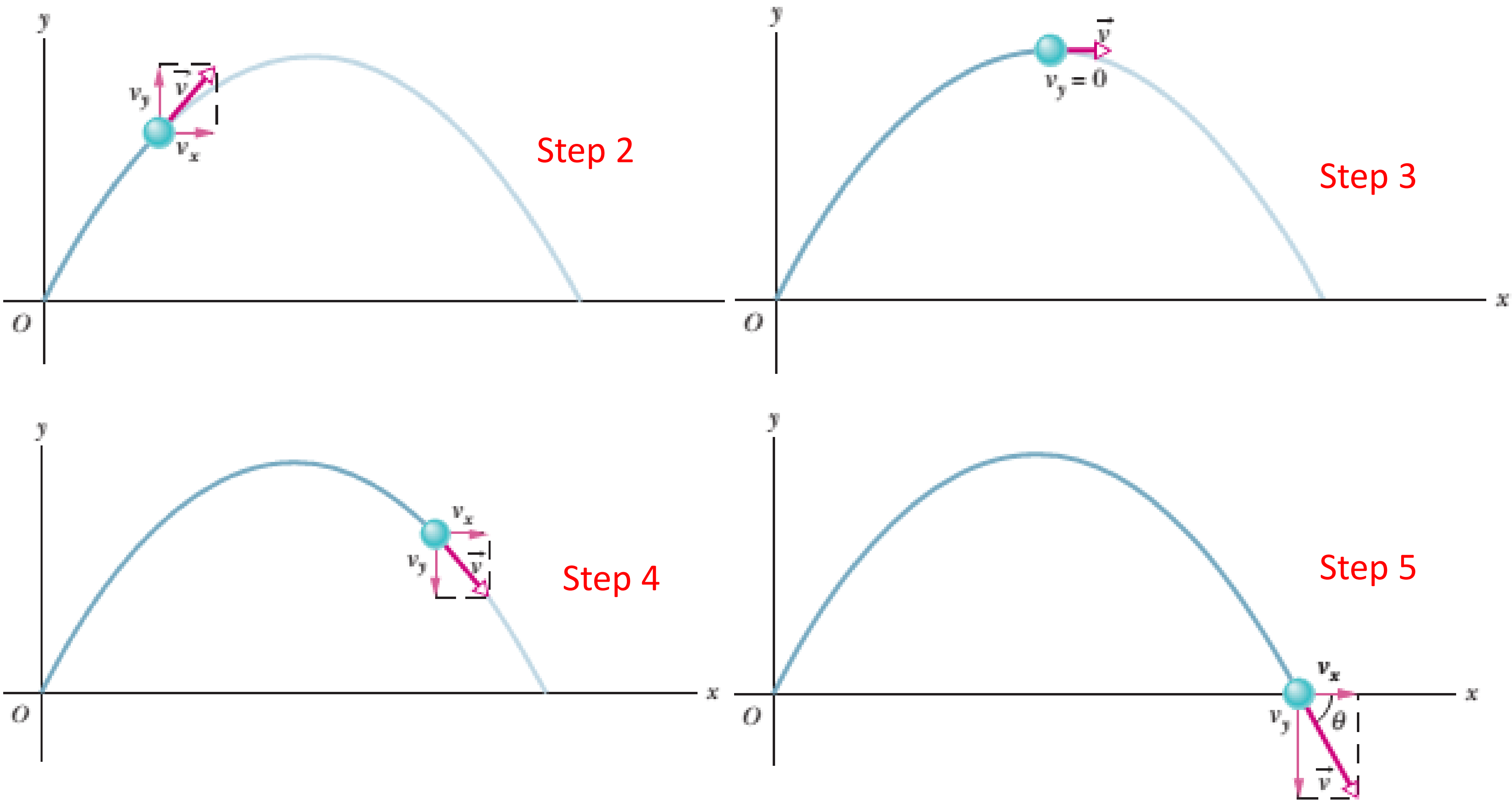
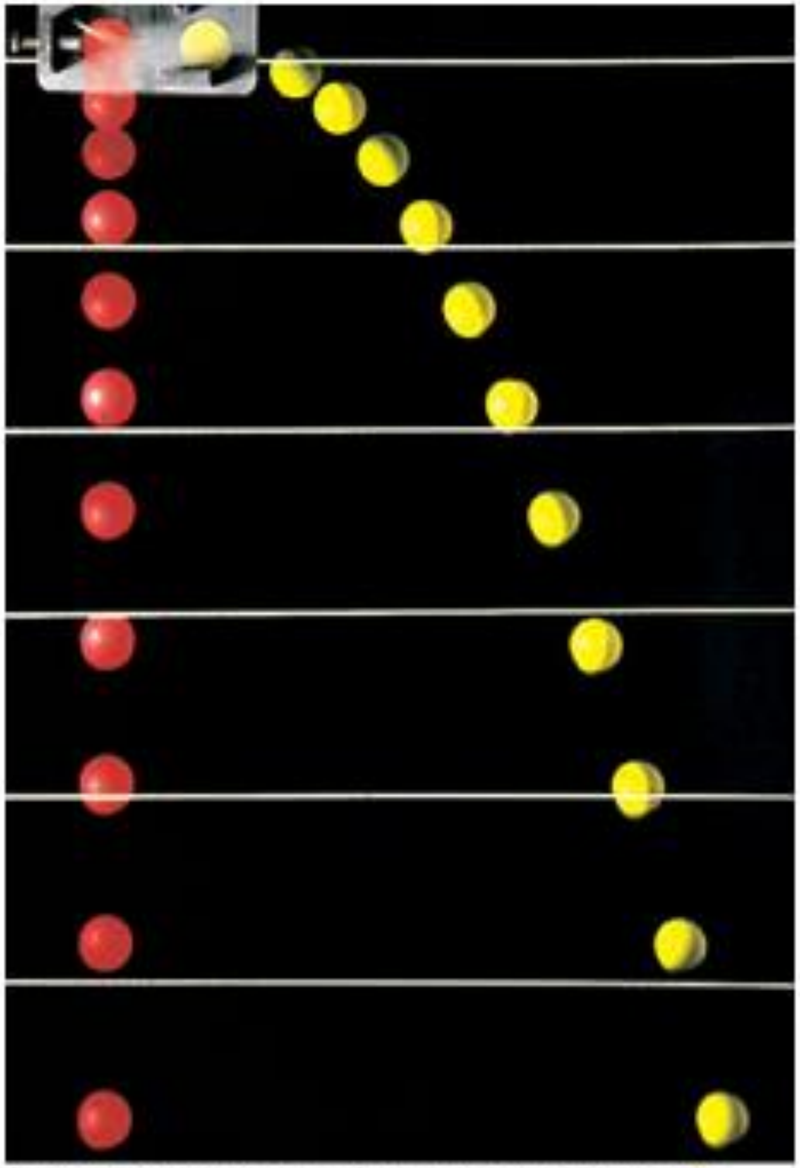


Figure: The *projectile motion* of an object launched into the air at the origin of a coordinate system and with launch velocity $\vec{v}_0$ at angle θ_0. The motion is a combination of vertical motion (constant acceleration) and horizontal motion (constant velocity), as shown by the velocity components.

The adjacent Figure shows two balls with different *x*-motion but identical *y*-motion; one is dropped from rest and the other is projected horizontally, but both balls fall the same distance in the same time.



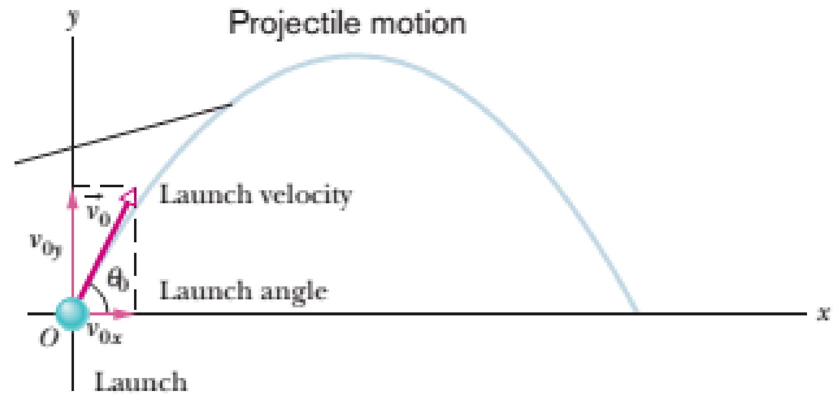
Richard Megna/Fundamental Photographs

The Horizontal Motion:

Type equation here.

At any time *t*, the projectile's horizontal displacement $x - x_0$ from an initial position x_0 is given by

$$x - x_0 = v_{0x}\, t + \frac{1}{2} a_x t^2$$



Where $acceleration\ along\ x - axis, a_x = 0$

Using $v_{0x} = v_0 \cos\theta_0$ we can write

$$x - x_0 = (v_0 \cos\theta_0)\ t \qquad \text{.......... (1)}$$

At any time *t*, the projectile's horizontal velocity $v_{0x} = v_x$

The Vertical Motion:

At any time *t*, the projectile's vertical displacement $y - y_0$ from an initial position y_0 is given by

$$y - y_0 = v_{0y}\, t - \frac{1}{2} g t^2 \qquad [\text{ where, } a_y = -g]$$

$$y - y_0 = (v_0 \sin\theta_0)\ t - \frac{1}{2} g t^2 \qquad [\text{ where, } v_{0y} = v_0 \sin\theta_0]$$

.... (2)

At any time *t*, the projectile's vertical velocity

$$v_y = v_0 \sin\theta_0 - gt \qquad [\text{v} = u + at]$$

And also we can express v_y as

$$v_y^2 = (v_0 \sin\theta_0)^2 - 2\,g(y - y_0) \qquad [\text{v}^2 = u^2 + 2as]$$

- Show that the path of a projectile is a parabola.

From equation (1) we can write

$$t = \frac{x - x_0}{v_0 \cos\theta_0}$$

Using the value of t in equation (2), we get

$$y - y_0 = v_0 \sin\theta_0 \frac{x - x_0}{v_0 \cos\theta_0} - \frac{1}{2} g \left(\frac{x - x_0}{v_0 \cos\theta_0}\right)^2$$

For simplicity, we let $x_0 = 0 \; and \; y_0 = 0$.

Therefore, the equation becomes

$$y = (\tan\theta_0)x + \frac{1}{2}g\left(-\frac{1}{v_0\cos\theta_0}\right)^2 \mathrm{x}^2 \qquad \text{..................... (3)}$$

$$\mathrm{a} = \tan\theta_0 \qquad\qquad \mathrm{b} = \frac{1}{2}g\left(-\frac{1}{v_0\cos\theta_0}\right)^2$$

Where $\theta_0, g \; and \; v_0 \; are \; constants$.

Equation (3) is of the form $y = ax + bx^2 \;, where \; a \; and \; b \; are \; constants$.

This is the equation of a parabola, so the path is *parabolic*.

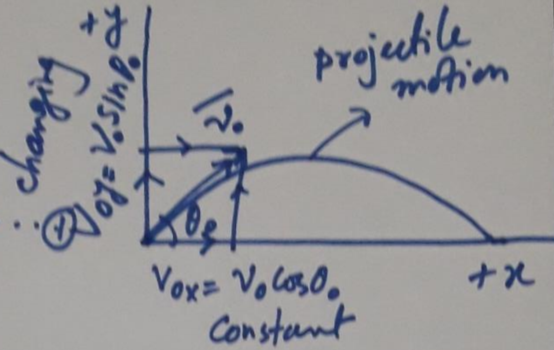
Equation of the projectile's path: $y = ax + bx^2$

$\overline{V}_0 = V_{0x}\,\hat{i} + V_{0y}\,\hat{j}$

$\overline{V}_0 = V_0 \cos\theta_0\,\hat{i} + V_0 \sin\theta_0\,\hat{j}$

$V_{0x} = V_0 \cos\theta_0$

Verticle motion: $V_{0y} = V_0 \sin\theta_0$ . . . ①



Equation of motion along y-axis,

$$y - y_0 = V_{0y} t - \frac{1}{2} g t^2 \quad \ldots \text{②}$$

$$= (V_0 \sin\theta_0) t - \frac{1}{2} g t^2$$

Equation of motion along x-axis,

$$x - x_0 = V_{0x} t$$

$$t = \frac{x - x_0}{V_{0x}} = \frac{x - x_0}{V_0 \cos\theta_0}$$

$$\therefore y - y_0 = (V_0 \sin\theta_0)\left(\frac{x - x_0}{V_0 \cos\theta_0}\right) - \frac{1}{2} g \left(\frac{x - x_0}{V_0 \cos\theta_0}\right)^2$$

$$= \left(\frac{\sin\theta_0}{\cos\theta_0}\right)(x - x_0) + \left\{\frac{-g(x - x_0)^2}{2V_0^2 \cos^2\theta_0}\right\}$$

$$y = (\tan\theta_0)x + \left(\frac{-g}{2V_0^2 \cos^2\theta}\right)x^2 \quad \Big| \quad x_0 = 0,\; y_0 = 0$$

$$y = ax + bx^2 \quad \Big| \quad a = \tan\theta_0, \quad b = \frac{-g}{2V_0^2 \cos^2\theta_0}$$

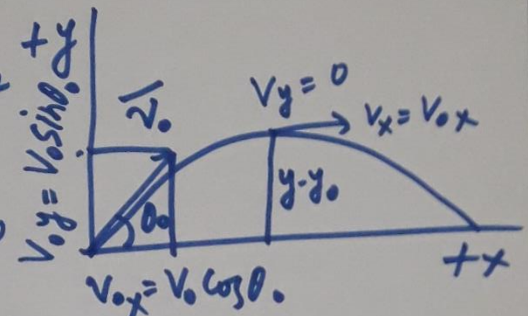
parabola

Type equation

Maximum height: $y = \dfrac{v_0^2 \sin^2\theta_0}{2g}$

y-axis:

$y - y_0 = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2$



At maximum height, $V_y = 0$

$V_y = V_{0y} - gt = V_0 \sin\theta_0 - gt$

$0 = V_0 \sin\theta_0 - gt$

$gt = V_0 \sin\theta_0$

$\therefore t = \dfrac{V_0 \sin\theta_0}{g}$

$$y - y_0 = V_0 \sin\theta_0 \left(\frac{V_0 \sin\theta_0}{g}\right) - \frac{1}{2}g\left(\frac{V_0 \sin\theta_0}{g}\right)^2$$

$$= \frac{v_0^2 \sin^2\theta_0}{g} - \frac{1}{2}g\left(\frac{v_0^2 \sin^2\theta_0}{g^2}\right)$$

$$= \left(\frac{v_0^2 \sin^2\theta_0}{g}\right) - \frac{1}{2}\left(\frac{v_0^2 \sin^2\theta_0}{g}\right)$$

$$= \left(1 - \frac{1}{2}\right)\frac{v_0^2 \sin^2\theta_0}{g}$$

$y - y_0 = \dfrac{v_0^2 \sin^2\theta_0}{2g}$ $\quad \therefore y = \dfrac{v_0^2 \sin^2\theta_0}{2g}$ / $y_0 = 0$

❑ Equations for the horizontal range and the maximum horizontal range of a projectile:

The ***horizontal range R*** of the projectile is the *horizontal* distance the projectile has traveled when it returns to its initial height (the height at which it is launched). That is $x - x_0 = R \; when \; y - y_0 = 0$.

Using $x - x_0 = R \; in \; equation \; (1) and \; y - y_0 = 0 \; in \; equation \; (2)$, we get

$$R = (v_0 \cos\theta_0)\, t$$ [From equation (1)]

And $0 = (v_0 \sin\theta_0)\; t - \frac{1}{2} g t^2$ [From equation (2)]

$$or \quad (v_0 \sin\theta_0)\; t = \frac{1}{2} g t^2 \qquad or \quad t = \frac{2 v_0 \sin\theta_0}{g}$$

Therefore, $$R = (v_0 \cos\theta_0)\, \frac{2 v_0 \sin\theta_0}{g} = \frac{v_0^2\,(2\sin\theta_0\cos\theta_0)}{g}$$

$$\boxed{R = \frac{v_0^2 \sin 2\theta_0}{g}} \quad \ldots\ldots(3)$$

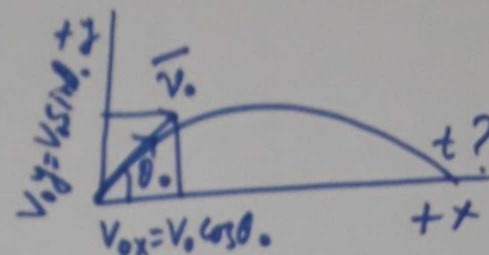
***Caution:* This equation does not give the horizontal distance traveled by a projectile when the final height is not the launch height.**

Horizontal range: $R = \frac{v_0^2 \sin 2\theta_0}{g}$ ①

x-axis:

$x - x_0 = V_{0x} t$

$x - x_0 = (V_0 \cos\theta_0) t$



$R = (V_0 \cos\theta_0) t$ - - ①

$x - x_0 = R$

y-axis:

$y - y_0 = V_{0y} t - \frac{1}{2} g t^2$ | $y - y_0 = 0$

$0 = (V_0 \sin\theta_0) t - \frac{1}{2} g t^2$

$t (V_0 \sin\theta_0 - \frac{1}{2} g t) = 0$

$\therefore t = 0$ or $V_0 \sin\theta_0 - \frac{1}{2} g t = 0$

$V_0 \sin\theta_0 = \frac{1}{2} g t$

$\therefore t = \frac{2 V_0 \sin\theta_0}{g}$

$R = V_0 \cos\theta_0 \left(\frac{2 V_0 \sin\theta_0}{g}\right) = \frac{V_0^2 (2 \sin\theta_0 \cos\theta_0)}{g}$

$R = \frac{V_0^2 \sin 2\theta_0}{g}$

$$R_{max} = \frac{v_0^2}{g}(1) = \frac{v_0^2}{g}$$ Maximum range: 2

R has its maximum value when

$\sin 2\theta_0 = 1$

$\sin 2\theta_0 = \sin 90^\circ$

$2\theta_0 = 90^\circ$

$\therefore \theta_0 = 45^\circ$

The horizontal range R is maximum for a launch angle of 45°.

The value of R is maximum in equation (3) when $\sin 2\theta_0 = 1$

$$or\ \ Sin\,2\theta_0 = Sin\,90$$

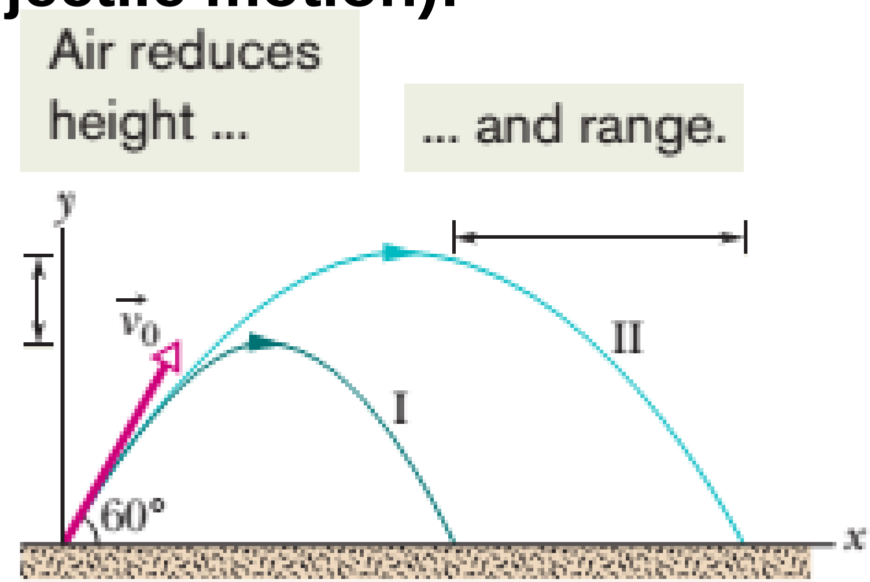
$$or\ 2\theta_0 = 90^0$$

$$\boxed{\theta_0 = 45^0}$$

$$\boxed{Maximum\ horizontal\ range, R = \frac{v_0^2}{g}}$$

The Effects of the Air (in the projectile motion):

	Path I (Air)	Path II (Vacuum)
Range	98.5 m	177 m
Maximum height	53.0 m	76.8 m
Time of flight	6.6 s	7.9 s



The launch angle is 60° and the launch speed is 44.7 m/s.

Projectile motion problem solving: