Chapter-1

Matrices

In mathematics, a matrix is a rectangular array of numbers, symbols, or expressions, arranged in rows and columns. Matrices play a huge role in graphics, any image is a matrix and each digit represents the intensity of a certain color at a certain grid point. Matrices are useful in any big data task, since a lot of the input data to problems is a collection of vectors (aka a matrix) relating multiple data points. Neural Networks rely heavily on matrices and matrix operations. Cryptography is the science of information security. Cryptography involves encrypting data so that a third party cannot intercept and read the data. In this cryptography matrix is a must which is very essential in engineering. In engineering, math reports are recorded using matrices. In architecture, matrices are used with computing. Software and hardware graphics processor uses matrices for performing operations such as scaling, translation, reflection and rotation. A person working in the field of AI who doesn't know matrix algebra is like a politician who doesn't know how to persuade. In machine learning, we often have to deal with structural data, which is generally represented as a table of rows and columns, or a matrix. A lot of problems in machine learning can be solved using matrix algebra. Transformation matrix plays an important role in robotics.

Matrix:

A matrix is a rectangular array of numbers of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & . & a_{mn} \end{bmatrix}$$

The numbers $a_{11}, a_{12}, ..., a_{mn}$ are called the elements or entries of the matrix.

The size of a matrix is described in terms of the number of rows (horizontal lines) and columns (vertical lines) it contains. To address the matrix size, it is called $m \times n$ or m by n size. In the size description, the 1st number always denotes the number of rows and 2nd denotes the number of column.

Matrices are generally denoted by capital letter A, B etc. Square brackets "[ ]" or curve brackets "( )" are used for the matrices notation.

The entry that occurs in row i and column j of a matrix will be denoted by a_{ij}. A general $m \times n$ matrix can be denoted as $[a_{ij}]_{m\times n}$ or $[a_{ij}]$.

Example: Some examples of matrices are

3×2 matrix (or 3 by 2 matrix): $\begin{bmatrix} 1 & 2 \\ -3 & 3 \\ 2 & 0 \end{bmatrix}$;

3×3 matrix (or 3 by 3 matrix): $\begin{bmatrix} 2 & 0 & 1 \\ -3 & 3 & 0 \\ 1 & -1 & 0 \end{bmatrix}$;

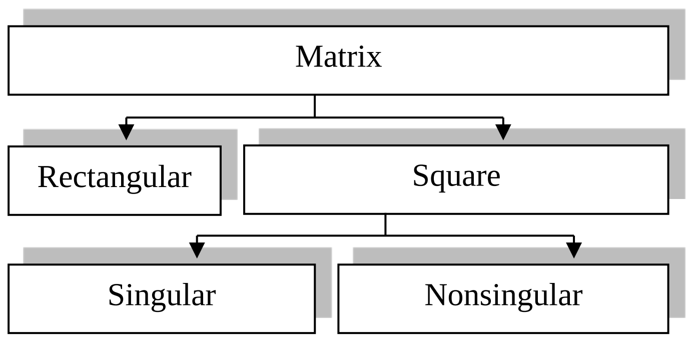
1×4 matrix (or 1 by 4 matrix): $\begin{bmatrix} 1 & 4 & 5 & 0 \end{bmatrix}$;

2×1 matrix (or 2 by 1 matrix): $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$;

1×1 matrix (or 1 by 1 matrix): $[5]$;

Classification of matrices:

A classification of matrices is, in a broad sense, as follows:



Rectangular matrix:

A matrix will be rectangular if it has m rows and n columns where $m \neq n$.

Example: $\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$

Square matrix:

A matrix in which the number of rows is equal to the number of columns is called a square matrix. Thus a $m \times n$ matrix will be a square matrix if $m = n$ and it will be referred as a square matrix of order n. The following matrix is square.

$$A = \begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & . & a_{nn} \end{bmatrix}$$

The entries $a_{11}, a_{22}, \ldots, a_{nn}$ are called the main diagonal entries of A.

Singular matrix:

The square matrix A is said to be singular if and only if (iff) its determinant is zero, i.e., $\det(A) = 0$.

Nonsingular matrix:

The square matrix A is said to be nonsingular if $\det(A) \neq 0$.

Row and Column matrices:

A row matrix is defined as a matrix having a single row and a column matrix is a matrix having a single column.

Example: $\begin{bmatrix} a_{11} & . & . & . & . & a_{1n} \end{bmatrix}$ is a row matrix.

Example: $\begin{bmatrix} a_{11} \\ a_{21} \\ . \\ . \\ . \\ a_{m1} \end{bmatrix}$ is a column matrix.

Equal matrix:

Two matrices are said to be equal if they have the same order and their corresponding entries are equal.

Example: $A = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$. If $x = 2, y = 1, z = 3, t = 5$, that is corresponding entries are equal.

Diagonal matrix:

A **square matrix** is said to be diagonal if $a_{ij} = 0; i \neq j$ and $a_{ii} \neq 0$.

Unit matrix or identity matrix:

A **square matrix** is called a unit matrix or an identity matrix if only diagonal elements of the matrix are one. A unit matrix of order n is written as I_n.

Thus $I_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is unit matrix of order 2. So, $a_{ij} = 0,\ i \neq j$ and $a_{ij} = 1,\ i = j$

Transpose of a matrix:

The matrix obtained by interchanging rows and columns of a matrix A, is called the transpose of A, denoted by A^T.

If $A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \\ 7 & -5 \end{bmatrix}_{3\times 2}$ then $A^T = \begin{bmatrix} 3 & 4 & 7 \\ 2 & 1 & -5 \end{bmatrix}_{2\times 3}$

Symmetric matrix:

A symmetric matrix is a square matrix that is equal to the transpose of that matrix. Symmetric matrix is a special kind of square matrix where $a_{ij} = a_{ji}$ for all i and j.

Let, $A = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$, $A^T = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$ $\therefore A = A^T$

So, A is a symmetric matrix.

Skew-symmetric matrix:

A square matrix A is said to be skew symmetric if $A^T = -A$. That is, the transpose of a square matrix is equal to the negative of that matrix.

Example: $A = \begin{bmatrix} 0 & -6 \\ 6 & 0 \end{bmatrix}$, $A^T = \begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -6 \\ 6 & 0 \end{bmatrix} = -A$

So, the matrix A is a skew-symmetric matrix.

Matrix Algebra

Matrix addition and subtraction:

If A and B are matrices of the same size, then the sum $A + B$ is the matrix obtained by adding the entries of B to the corresponding entries of A and the difference $A - B$ is the matrix obtained by subtracting the entries of B from the corresponding entries of A. Matrices of different sizes cannot be added or subtracted.

$A = \begin{bmatrix} 4 & 8 \\ 9 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$,

$$A+B=\begin{bmatrix}5 & 10\\10 & 6\end{bmatrix}, \quad A-B=\begin{bmatrix}3 & 6\\8 & 0\end{bmatrix}.$$

Properties:

If the matrices A, B, C are comfortable for addition and if k is any scalar, then we can state that

1. $A + B = B + A$ (commutative Law)
2. $(A + B) + C = A + (B + C)$ (associative Law)
3. $A + 0 = 0 + A = A$ (where 0 is the zero matrix of the same order)
4. $k(A + B) = kA + kB = (A + B)k$ (where k is a scalar)

Scalar multiplication:

If A is any matrix and c is any scalar, then the product cA is the matrix obtained by multiplying each entry of the matrix A by c.

The matrix cA is said to be a scalar multiple of A.

Example: $A=\begin{bmatrix}2 & 3 & 4\\1 & 3 & 1\end{bmatrix} \quad \therefore 2A=\begin{bmatrix}4 & 6 & 8\\2 & 6 & 2\end{bmatrix}.$

Matrix multiplication:

If A is a $m \times n$ matrix and B is an $n \times r$ matrix, then the product AB is the $m \times r$ matrix whose entries are determined as follows:

To find the entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B multiply the corresponding entries from the row and column together and then add up the resulting products.

Let, $A=\begin{bmatrix}1 & 2 & 4\\2 & 3 & 0\end{bmatrix}_{2\times 3}$ and $B=\begin{bmatrix}4 & 1 & 4 & 3\\0 & -1 & 3 & 1\\2 & 7 & 5 & 2\end{bmatrix}_{3\times 4}$

$$\begin{aligned}[AB]_{2\times 4} &= \begin{bmatrix}1 & 2 & 4\\2 & 3 & 0\end{bmatrix}\begin{bmatrix}4 & 1 & 4 & 3\\0 & -1 & 3 & 1\\2 & 7 & 5 & 2\end{bmatrix}\\
&= \begin{bmatrix}1.4+2.0+4.2 & 1.1-2.1+4.7 & 1.4+2.3+4.5 & 1.3+2.1+4.2\\2.4+3.0+0.2 & 2.1-3.1+0.7 & 2.4+3.3+0.5 & 2.3+3.1+0.2\end{bmatrix}\\
&= \begin{bmatrix}12 & 27 & 30 & 13\\8 & -1 & 17 & 9\end{bmatrix}\end{aligned}$$

Multiplicative Properties:

If the matrices A, B, C are comfortable for the addition and multiplication, we have the following properties.

1. $(AB)C = A(BC)$ (commutative law)

2. $A(B+C) = AB + AC$ (distributivetive law)
3. $(A+B)C = AC + BC$ (distributivetive law)
4. $k(AB) = (kA)B = A(kB)$ ($where$ k is a scalar)

Theorem:

If A and B are matrices and A^T and B^T are the transpose of A and B respectively then,

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(AB)^T = B^T A^T$
4. $(\alpha A)^T = \alpha A^T$, where α is a scalar

Exercise 1.1

1. If $A = \begin{bmatrix} -5 & 6 & 1 \\ 2 & -2 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 8 & -4 \\ 1 & 2 & -9 \end{bmatrix}$, then find the matrices $7A, 3A + 2B,$ and $2A - 3B$.

2. If $A = \begin{bmatrix} -1 & 0 & 6 \\ 3 & 8 & 9 \\ -3 & 4 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 2 & -7 \\ -1 & -5 & 2 \\ 4 & 0 & 1 \end{bmatrix}$, then find the matrices $-2A + 4B,$ and $5A - 3B$.

3. If $A = \begin{bmatrix} -1 & 4 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 0 \\ -4 & 5 \end{bmatrix}$, compute AB and BA.

4. If $A = \begin{bmatrix} 6 & 0 \\ -4 & -2 \\ 9 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 5 \\ -2 & -3 \\ 0 & 3 \end{bmatrix}$, find AB and BA (if possible).

5. If $A = \begin{bmatrix} -1 & 4 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 0 \\ 6 & 5 \end{bmatrix}$, then prove that $(AB)^T = B^T A^T$.

6. If $A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & -1 \\ 0 & 5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & -2 & 8 \\ 0 & 6 & -2 \\ -1 & 0 & 9 \end{bmatrix}$ then prove that $(AB)^T = B^T A^T$.

Determinant:

Determinant is a value associated with a square matrix. It can be computed from the entries of the matrix by a specific arithmetic expression. The determinant of a matrix A is denoted by $\det(A)$, $\det A$, or $|A|$. The value of an $n \times n$ determinant can be evaluated by using either any one of the n rows or any one of the n columns. The determinant of the matrix

$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is $A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

and has the value $a(ei - fh) - b(di - fg) + c(dh - eg)$.

Example: Evaluate $\begin{vmatrix} 2 & 9 \\ -3 & 5 \end{vmatrix}$.

Solution: $\begin{vmatrix} 2 & 9 \\ -3 & 5 \end{vmatrix} = 2(5) - (-3)(9) = 37.$

Example: Evaluate $\begin{vmatrix} 1 & 1 & -4 \\ 2 & -4 & 0 \\ 0 & 2 & 1 \end{vmatrix}$.

Solution: $\begin{vmatrix} 1 & 1 & -4 \\ 2 & -4 & 0 \\ 0 & 2 & 1 \end{vmatrix} = 1\begin{vmatrix} -4 & 0 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} + (-4)\begin{vmatrix} 2 & -4 \\ 0 & 2 \end{vmatrix} = 1(-4) - 1\,(2) - 4(4) = -22$

The given determinant was evaluated with the help of **first row.** One can evaluate the determinant using any other row or column. In that case one must be careful about signs, i.e. putting ' $+$ ' & '$-$ ' . Two other methods are shown below:

When **column – I** is used:

$$\begin{vmatrix} 1 & 1 & -4 \\ 2 & -4 & 0 \\ 0 & 2 & 1 \end{vmatrix} = 1\begin{vmatrix} -4 & 0 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & -4 \\ 2 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & -4 \\ -4 & 0 \end{vmatrix} = 1(-4) - 2(1+8) + 0 = -22.$$

When **row – II** is used:

$$\begin{vmatrix} 1 & 1 & -4 \\ 2 & -4 & 0 \\ 0 & 2 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & -4 \\ 2 & 1 \end{vmatrix} + (-4)\begin{vmatrix} 1 & -4 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -2(1+8) - 4(1+0) - 0 = -22.$$

Now you should try to compute the same through other rows/columns. If you do not make any mistake, you will get -22 every time.

Example: Find the value of the determinant $\begin{vmatrix} 2 & -2 & 0 & 3 \\ 1 & 0 & 1 & -1 \\ 2 & 1 & -4 & 0 \\ 0 & -1 & 2 & 1 \end{vmatrix}$

Solution:

$$\begin{vmatrix} 2 & -2 & 0 & 3 \\ 1 & 0 & 1 & -1 \\ 2 & 1 & -4 & 0 \\ 0 & -1 & 2 & 1 \end{vmatrix} = 2\begin{vmatrix} 0 & 1 & -1 \\ 1 & -4 & 0 \\ -1 & 2 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 1 & -1 \\ 2 & -4 & 0 \\ 0 & 2 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} - 3\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & -4 \\ 0 & -1 & 2 \end{vmatrix}$$

$$= 2(1) + 2(-10) + 0 - 3(-4) = -6.$$

Exercise 1.2

1. Find the value of the determinant $\begin{vmatrix} 7 & -3 & 0 \\ 3 & 4 & 3 \\ -5 & 0 & -5 \end{vmatrix}$.

2. Find the values of the constant α so that the determinant of the following matrix becomes zero
$$A = \begin{bmatrix} -10 & 4 & 4\alpha \\ -1 & 2 & 0 \\ 1 & \alpha & -2 \end{bmatrix}.$$

3. Find the value of the determinant $\begin{vmatrix} -1 & 0 & 2 & -3 \\ 2 & 1 & -1 & 1 \\ -2 & -1 & 3 & 0 \\ 0 & 1 & -2 & -1 \end{vmatrix}$.

4. Find the value of the determinant $\begin{vmatrix} 3 & -1 & 5 \\ -3 & 4 & -3 \\ 5 & -1 & 0 \end{vmatrix}$.

5. Find the value of the determinant $\begin{vmatrix} -1 & 0 & 3 \\ 6 & -4 & 6 \\ 1 & -1 & 0 \end{vmatrix}$.

6. For the matrix $A = \begin{bmatrix} 10 & -4 & 4\alpha \\ -1 & 2 & 0 \\ 1 & \alpha & -2 \end{bmatrix}$, determine the value(s) of α for which A will be singular.

Matrix Inverse

Inverse of a matrix:

The inverse of a square matrix A is denoted by A^{-1} such that $AA^{-1} = A^{-1}A = I_n$,
where I_n is the identity matrix of order n.

Two commonly used methods for finding the inverse are:
i) Using adjoint matrix / cofactor method and
ii) Using elementary row operation.

Minor of an element of a matrix:

Deleting ith row and jth column, we get a new determinant of order (n-1) is called minor of a_{ij} is represented by M_{ij}.

Consider a matrix, $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. Minor of a_{11} is represented by M_{11} and is defined $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$.

Cofactor of an element of a matrix:

If we multiply the minor of the element in the i-th row and j-th column of the determinant of the matrix by $(-1)^{i+j}$, the product is called the cofactor of the element.

Cofactor of the element a_{ij} is represented by A_{ij} and is defined by

$$A_{ij} = (-1)^{i+j} \times M_{ij}$$

Example: If $A=\begin{bmatrix}3 & 4 & 7\\ -2 & 5 & 6\\ 7 & 3 & -9\end{bmatrix}$, find the cofactor of 6.

Solution: The cofactor of 6 is

$$\begin{aligned}A_{23} &= (-1)^{2+3}\begin{vmatrix}3 & 4\\ 7 & 3\end{vmatrix}\\ &= -(9-28)\\ &= 19.\end{aligned}$$

Adjoint of a square matrix:

Let $A=[a_{ij}]_{n\times n}$ be a square matrix of order n, then the adjoint of A is defined to be the transpose of matrix $[A_{ij}]_{n\times n}$, where A_{ij} is cofactor of a_{ij} in A.

In other words, let

$$A=\begin{pmatrix}a_{11} & a_{12} & . & . & . & a_{1n}\\ a_{21} & a_{22} & . & . & . & a_{2n}\\ . & . & . & . & . & .\\ . & . & . & . & . & .\\ . & . & . & . & . & .\\ a_{n1} & a_{n2} & . & . & . & a_{nn}\end{pmatrix}_{n\times n}$$

Adj (A) $=\begin{pmatrix}A_{11} & A_{12} & . & . & . & A_{1n}\\ A_{21} & A_{22} & . & . & . & A_{2n}\\ . & . & . & . & . & .\\ . & . & . & . & . & .\\ . & . & . & . & . & .\\ A_{n1} & A_{n2} & . & . & . & A_{nn}\end{pmatrix}^T$ where A_{ij} is the Co factor of a_{ij}.

Example: Find the adjoint of the matrix $A=\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{pmatrix}$.

Solution: adj $(A)=\begin{pmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{pmatrix}^T$ where,

$A_{11}=(-1)^{1+1}\begin{vmatrix}2 & -3\\ -1 & 3\end{vmatrix}=3$, $A_{12}=(-1)^{1+2}\begin{vmatrix}1 & -3\\ 2 & 3\end{vmatrix}=-9$, $A_{13}=(-1)^{1+3}\begin{vmatrix}1 & 2\\ 2 & -1\end{vmatrix}=-5$

$A_{21}=(-1)^{2+1}\begin{vmatrix}1 & 1\\ -1 & 3\end{vmatrix}=-4$, $A_{22}=(-1)^{2+2}\begin{vmatrix}1 & 1\\ 2 & 3\end{vmatrix}=1$, $A_{23}=(-1)^{2+3}\begin{vmatrix}1 & 1\\ 2 & -1\end{vmatrix}=3$

$A_{31}=(-1)^{3+1}\begin{vmatrix}1 & 1\\ 2 & -3\end{vmatrix}=-5$, $A_{32}=(-1)^{3+2}\begin{vmatrix}1 & 1\\ 1 & -3\end{vmatrix}=4$, $A_{33}=(-1)^{3+3}\begin{vmatrix}1 & 1\\ 1 & 2\end{vmatrix}=1$

$$\text{adj}(A)=\begin{bmatrix}3 & -9 & -5\\ -4 & 1 & 3\\ -5 & 4 & 1\end{bmatrix}^T=\begin{bmatrix}3 & -4 & -5\\ -9 & 1 & 4\\ -5 & 3 & 1\end{bmatrix}.$$

Inverse of a matrix using co-factor:

The inverse of a non-singular matrix A is given by $A^{-1}=\frac{1}{|A|}\text{adj}A$ $(|A|\neq 0)$

Example: Compute the inverse of the matrix $A=\begin{pmatrix}1 & 0 & -4\\ -2 & 2 & 5\\ 3 & -1 & 2\end{pmatrix}$ and also verify your answer.

We know, $A^{-1}=\frac{\text{adj}(A)}{|A|}$ here $\text{inv}(A)=A^{-1}$

$$|A|=\begin{vmatrix}1 & 0 & -4\\ -2 & 2 & 5\\ 3 & -1 & 2\end{vmatrix}=1(4+5)-0(-4-15)-4(2-6)=25$$

$$\text{adj}A=\begin{pmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{pmatrix}^T=\begin{pmatrix}A_{11} & A_{21} & A_{31}\\ A_{12} & A_{22} & A_{32}\\ A_{13} & A_{23} & A_{33}\end{pmatrix}$$

$A_{11}=(-1)^{1+1}\begin{vmatrix}2 & 5\\ -1 & 2\end{vmatrix}=9$, $A_{12}=(-1)^{1+2}\begin{vmatrix}2 & 5\\ 3 & 2\end{vmatrix}=19$, $A_{13}=(-1)^{1+3}\begin{vmatrix}-2 & 2\\ 3 & -1\end{vmatrix}=-4$

$A_{21}=(-1)^{2+1}\begin{vmatrix}0 & -4\\ -1 & 2\end{vmatrix}=4$, $A_{22}=(-1)^{2+2}\begin{vmatrix}1 & -4\\ 3 & 2\end{vmatrix}=14$, $A_{23}=(-1)^{2+3}\begin{vmatrix}1 & 0\\ 3 & -1\end{vmatrix}=1$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 0 & -4 \\ 2 & 5 \end{vmatrix} = 8$, $A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & -4 \\ -2 & 5 \end{vmatrix} = 3$, $A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 0 \\ -2 & 2 \end{vmatrix} = 2$

$\therefore \operatorname{adj} A = \begin{pmatrix} 9 & 4 & 8 \\ 19 & 14 & 3 \\ -4 & 1 & 2 \end{pmatrix}$

$\therefore A^{-1} = \dfrac{\operatorname{adj} A}{|A|}$

$= \dfrac{1}{25}\begin{pmatrix} 9 & 4 & 8 \\ 19 & 14 & 3 \\ -4 & 1 & 2 \end{pmatrix}$

Verification: $AA^{-1} = I = A^{-1}A$

$$A^{-1}A = \frac{1}{25}\begin{bmatrix} 9 & 4 & 8 \\ 19 & 14 & 3 \\ -4 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -4 \\ -2 & 2 & 5 \\ 3 & -1 & 2 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 25 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Example: Whether following matrices are invertible or not?

$(a)\begin{pmatrix} -2 & 9 \\ 0 & 1 \\ 5 & -3 \end{pmatrix}$ and $(b)\begin{pmatrix} 2 & -2 & 4 \\ 3 & 2 & 1 \\ 0 & -1 & 1 \end{pmatrix}$.

Solution:

(a). Definition of inverse is for square matrix only. It is a rectangular matrix. So, it has no inverse.

(b). It is a square matrix. It may be singular or non-singular. To become sure, we calculate the determinant of the matrix. We name the matrix 'A'.

$\therefore \quad |A| = \begin{vmatrix} 2 & -2 & 4 \\ 3 & 2 & 1 \\ 0 & -1 & 1 \end{vmatrix} = 0.$

So, the matrix 'A' is singular & hence the inverse does not exist in this case.

Example: Find inverse of $B = \begin{pmatrix} 2 & 0 & -3 \\ 0 & 9 & 2 \\ -5 & 2 & 4 \end{pmatrix}$ and also verify the answer.

Now $|B| = \begin{vmatrix} 2 & 0 & -3 \\ 0 & 9 & 2 \\ -5 & 2 & 4 \end{vmatrix} = -71 \neq 0$. $\therefore$ B is non-singular, and B^{-1} exists.

We know: The cofactor matrix, $\operatorname{cof}(B) = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}$ & $\operatorname{adj}(B) = [\operatorname{cof}(B)]^T$.

$B_{11} = (-1)^{1+1}\begin{vmatrix} 9 & 2 \\ 2 & 4 \end{vmatrix} = 1\,(36-4) = 32$ $\quad B_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 2 \\ -5 & 4 \end{vmatrix} = -1\,(0+10) = -10$ $\quad B_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 9 \\ -5 & 2 \end{vmatrix} = 1\,(0+45) = 45$

$B_{21} = (-1)^{2+1}\begin{vmatrix} 0 & -3 \\ 2 & 4 \end{vmatrix} = -1\,(0+6) = -6$ $\quad B_{22} = (-1)^{2+2}\begin{vmatrix} 2 & -3 \\ -5 & 4 \end{vmatrix} = 1\,(8-15) = -7$ $\quad B_{23} = (-1)^{2+3}\begin{vmatrix} 2 & 0 \\ -5 & 2 \end{vmatrix} = -1\,(4-0) = -4$

$B_{31} = (-1)^{3+1}\begin{vmatrix} 0 & -3 \\ 9 & 2 \end{vmatrix} = 1\,(0+27) = 27$ $\quad B_{32} = (-1)^{3+2}\begin{vmatrix} 2 & -3 \\ 0 & 2 \end{vmatrix} = -1\,(4+0) = -4$ $\quad B_{33} = (-1)^{3+3}\begin{vmatrix} 2 & 0 \\ 0 & 9 \end{vmatrix} = 1\,(18-0) = 18$

Thus $\operatorname{cof}(B) = \begin{pmatrix} 32 & -10 & 45 \\ -6 & -7 & -4 \\ 27 & -4 & 18 \end{pmatrix}$ & $\operatorname{adj}(B) = \begin{pmatrix} 32 & -6 & 27 \\ -10 & -7 & -4 \\ 45 & -4 & 18 \end{pmatrix}$.

$$\therefore B^{-1} = \frac{1}{-71}\begin{pmatrix} 32 & -6 & 27 \\ -10 & -7 & -4 \\ 45 & -4 & 18 \end{pmatrix} = \begin{pmatrix} -32/71 & 6/71 & -27/71 \\ 10/71 & 7/71 & 4/71 \\ -45/71 & 4/71 & -18/71 \end{pmatrix}.$$

Verification: $BB^{-1} = I = B^{-1}B$

$$B^{-1}B = \frac{1}{-71}\begin{bmatrix} 32 & -6 & 27 \\ -10 & -7 & -4 \\ 45 & -4 & 18 \end{bmatrix}\begin{bmatrix} 2 & 0 & -3 \\ 0 & 9 & 2 \\ -5 & 2 & 4 \end{bmatrix} = \frac{1}{-71}\begin{bmatrix} -71 & 0 & 0 \\ 0 & -71 & 0 \\ 0 & 0 & -71 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Row Echelon Form (REF)

A matrix is in row echelon form (ref) when it satisfies the following conditions.

1. Nonzero rows (rows with at least one nonzero element) are above any rows of all zeros.
2. The leading coefficient (the first nonzero number from the left, also called the pivot) of a nonzero row is always strictly to the right of the leading coefficient of the row above it.

Reduced Row Echelon Form (RREF) or Canonical form

A matrix is in reduced row echelon form (also called row canonical form) if it satisfies the additional condition

3. The first non-zero element in each row, called the leading entry, is 1, that is, every leading co-efficient is 1 and above and below all entries are zero in this column. Identity matrix is called standard canonical matrix.

The following matrix is in row echelon form, but not in reduced row echelon form

$$\begin{bmatrix} 1 & 9 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

However, the matrix below is not in row echelon form, as the leading coefficient of row 3 (that is 6) is not strictly to the right of the leading coefficient of row 2 (that is 4) and the main diagonal is not made up of only 1s

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 4 & 7 & 2 \\ 0 & 6 & 0 & 0 \end{bmatrix}.$$

The matrix in reduced row-echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example: Find row echelon forms (REF) of the following matrix, and then convert it to reduced row echelon form (RREF) or canonical form by using elementary row operations.

$$\begin{pmatrix} 2 & -3 & 1 & 2 \\ 4 & -6 & 2 & 4 \\ -4 & 6 & -1 & 0 \end{pmatrix}.$$

Solution:

$$\begin{pmatrix} 2 & -3 & 1 & 2 \\ 4 & -6 & 2 & 4 \\ -4 & 6 & -1 & 0 \end{pmatrix} \begin{matrix} \widetilde{r_2} \longrightarrow r_2 - 2r_1 \\ \widetilde{r_3} \longrightarrow r_3 + 2r_1 \end{matrix} \begin{pmatrix} 2 & -3 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{pmatrix} \underset{r_2}{\sim} \longrightarrow r_2 \leftrightarrow r_3 \begin{pmatrix} 2 & -3 & 1 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \leftarrow \text{REF.}$$

$$\begin{pmatrix} 2 & -3 & 1 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underset{r_1}{\sim} \longrightarrow r_1 - r_2 \begin{pmatrix} 2 & -3 & 0 & -2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underset{r_1}{\sim} \longrightarrow \frac{r_1}{2} \begin{pmatrix} 1 & -3/2 & 0 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \leftarrow \text{RREF.}$$

Example: Find row echelon forms (REF) of the following matrix, and then convert it to reduced row echelon form (RREF) using elementary row operations.

$$\begin{pmatrix} -2 & 1 & 1 \\ 0 & 8 & -4 \\ 3 & 2 & -5 \end{pmatrix}.$$

Solution:

$$\begin{pmatrix} -2 & 1 & 1 \\ 0 & 8 & -4 \\ 3 & 2 & -5 \end{pmatrix} \underset{r_3}{\sim} \longrightarrow 3r_1 + 2r_3 \begin{pmatrix} -2 & 1 & 1 \\ 0 & 8 & -4 \\ 0 & 7 & -7 \end{pmatrix} \underset{r_3}{\sim} \longrightarrow 8r_3 - 7r_2 \begin{pmatrix} -2 & 1 & 1 \\ 0 & 8 & -4 \\ 0 & 0 & -28 \end{pmatrix} \leftarrow \text{REF.}$$

$$\begin{pmatrix} -2 & 1 & 1 \\ 0 & 8 & -4 \\ 0 & 0 & -28 \end{pmatrix} \underset{r_3}{\sim} \longrightarrow r_3/-28 \begin{pmatrix} -2 & 1 & 1 \\ 0 & 8 & -4 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} \widetilde{r_1} \longrightarrow r_1 - r_3 \\ \widetilde{r_2} \longrightarrow r_2 + 4r_3 \end{matrix} \begin{pmatrix} -2 & 1 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underset{r_2}{\sim} \longrightarrow r_2/8 \begin{pmatrix} -2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underset{r_2}{\sim} \longrightarrow r_1 - r_2 \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underset{r_1}{\sim} \longrightarrow r_1/-2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \leftarrow \text{RREF.}$$

Example: Find echelon forms of the following matrix $\begin{bmatrix} 4 & -1 \\ 2 & 0 \\ 5 & 2 \end{bmatrix}$.

Solution: $\begin{bmatrix} 4 & -1 \\ 2 & 0 \\ 5 & 2 \end{bmatrix} \begin{matrix} r_2 \longrightarrow 2r_2 - r_1 \\ \sim \\ r_3 \longrightarrow 4r_3 - 5r_1 \end{matrix} \begin{bmatrix} 4 & -1 \\ 0 & 1 \\ 0 & 13 \end{bmatrix} \quad r_3 \overset{\sim}{\longrightarrow} r_3 - 13r_2 \begin{bmatrix} 4 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \leftarrow$ REF.

$$r_1 \overset{\sim}{\longrightarrow} r_1 + r_2 \begin{bmatrix} 4 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} r_1 \overset{\sim}{\longrightarrow} \frac{r_1}{4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \leftarrow \text{RREF.}$$

Inverse of a matrix using elementary row operations (also called the Gauss-Jordan method).

[A | I]

$\downarrow$

Elementary Row Operations

$\downarrow$

[I | A^{-1}]

Example: Find the inverse of $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$.

We start with the matrix A, and write it down with an identity matrix I next to it

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

(This is called the "augmented matrix")
Now reduce the matrix A (the Matrix on the left) into an identity matrix. The goal is to make Matrix A having **1**s on the diagonal and **0**s elsewhere (an identity matrix) and the right hand side comes along for the ride, with every operation being done on it as well.
But we can only do these **"elementary row operations"**:

- **swap** rows
- **multiply** or divide each element in a row by a constant
- replace a row by **adding** or subtracting a multiple of another row to it

And we must do it to the **whole row**, like this:

First, write down the entries of the matrix A, but write them in a double-wide matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & & & \\ 1 & 4 & 3 & & & \\ 1 & 3 & 4 & & & \end{array}\right]$$

In the other half of the double-wide, write the identity matrix:

Augmented matrix, $A|\,I = \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$

Now do matrix row operations to convert the left-hand side of the double-wide into the identity.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \widetilde{\mathrm{r}_2} \to \mathrm{r}_2 - \mathrm{r}_1 \\ \\ \widetilde{\mathrm{r}_3} \to \mathrm{r}_3 - \mathrm{r}_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\widetilde{\mathrm{r}_1} \to \mathrm{r}_1 - 3\mathrm{r}_3 \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 4 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\widetilde{\mathrm{r}_1} \to \mathrm{r}_1 - 3\mathrm{r}_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] = I|\,A^{-1}$$

Now the left-hand side of the double-wide contains the identity, the right-hand side contains the inverse. That is, the inverse matrix is the following:

$$A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Example: Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}.$

Solution:

Step 1: Adjoin the identity matrix to the right side of A:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

Step 2: Apply row operations to this matrix until the left side is reduced to I. The computations are:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \widetilde{r_2} \to r_2 - 2r_1 \\ \\ \widetilde{r_3} \to r_3 - r_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

$$\widetilde{r_3} \to r_3 + 2r_2 \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

$$\begin{array}{r} \widetilde{r_1} \to r_1 + 3r_3 \\ \\ \widetilde{r_2} \to r_2 - 3r_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

$$\begin{array}{r} \widetilde{r_1} \to r_1 - 2r_2 \\ \\ \widetilde{r_3} \to (-1)r_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right].$$

Step 3: Conclusion: the inverse matrix is: $A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}.$

Example: Find A^{-1} using elementary row operations where

$$A = \begin{bmatrix} 1 & -2 & -4 \\ -2 & 0 & -4 \\ 1 & -1 & -1 \end{bmatrix}.$$

Solution: $$[\boldsymbol{A}|\boldsymbol{I}] = \left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ -2 & 0 & -4 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \widetilde{r_2} \to r_2 + 2r_1 \\ \\ \widetilde{r_3} \to r_3 - r_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & -4 & -12 & 2 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 & 1 \end{array}\right]$$

$$\tilde{r_3} \to 4r_3 + r_2 \left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & -4 & -12 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 4 \end{array}\right]$$

$\Rightarrow$ A can't be converted to $\boldsymbol{I}$ by elementary row operation.

$\Rightarrow$ A^{-1}does not exist.

<u>Exercise 1.3</u>

1. Find the inverse of the following matrices (if possible) using elementary row operations (ERO) and co-factor method, also justify your answers.

$a.\ A = \begin{bmatrix} 1 & -3 \\ 6 & -9 \end{bmatrix}$, $b.\ A = \begin{bmatrix} 4 & -2 \\ 7 & 6 \end{bmatrix}$, c. $A = \begin{bmatrix} -6 & 7 \\ 4 & -8 \end{bmatrix}$, $d.\ A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$,

$e.\ B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $f.\ A = \begin{bmatrix} -2 & 3 & 1 \\ 4 & 5 & 8 \\ 9 & -1 & 1 \end{bmatrix}$, $g.\ A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$, $h.\ A = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}$,

$i.\ A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$, $j.\ A = \begin{bmatrix} 1 & 5 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

2. If $A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$, find $AB,\ BA,\ A^{-1}B,\ B^{-1}A$ and $(AB)^{-1}$.

3. If $A = \begin{bmatrix} -1 & -8 & -3 \\ 2 & 0 & 5 \\ 3 & 6 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$, find $A^{-1}B$.

Rank of a matrix

After reduced the matrix to echelon form, the maximum number of independent (non-zero) rows (or columns) in the matrix is called rank of the given matrix.

Procedure: Reduce the given matrix A to row echelon form using elementary row operations (transformations). The number of nonzero rows of the echelon matrix is the rank of the given matrix.

Example: Reduce the matrix to echelon form and find the rank of the matrix

$$\mathrm{A} = \begin{bmatrix} 2 & 4 & 1 & 3 \\ -1 & -2 & 1 & 0 \\ 0 & 0 & 2 & 2 \\ 3 & 6 & 2 & 5 \end{bmatrix}.$$

Solution: To determine the row-rank of A we proceed as follows.

$$\mathrm{A} = \begin{bmatrix} 2 & 4 & 1 & 3 \\ -1 & -2 & 1 & 0 \\ 0 & 0 & 2 & 2 \\ 3 & 6 & 2 & 5 \end{bmatrix} \underset{r_3 \to \frac{1}{2} r_3}{\overset{r_1 \to r_1 + r_2}{\sim}} \begin{bmatrix} 1 & 2 & 2 & 3 \\ -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 3 & 6 & 2 & 5 \end{bmatrix} \underset{r_4 \to r_4 - 3r_1}{\overset{r_2 \to r_2 + r_1}{\sim}} \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -4 & -4 \end{bmatrix}$$

$$\underset{r_4 \to \frac{r_4}{-4}}{\overset{r_2 \to \frac{r_2}{3}}{\sim}} \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \underset{r_4 \to r_4 - r_2}{\overset{r_3 \to r_3 - r_2}{\sim}} \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So, the rank of the matrix A is 2.

Example: Reduce the matrix to echelon form and find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

Solution: To determine the row-rank of A, we proceed as follows.

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} \underset{r_3 \to r_3 - r_1}{\overset{r_2 \to r_2 - 2r_1}{\sim}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \underset{r_3 \to r_3 - r_2}{\sim} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

Here, there are 3 non-zero rows in row echelon form of A. So, the rank of the matrix A is 3 .

Example: Find the rank of the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution: The matrix A is already in reduced row echelon form. By counting the number of non-zero rows, we say its rank is 3.

Example: Find the rank of the matrix $A = \begin{bmatrix} -2 & 3 \\ 0 & 4 \\ -2 & 7 \end{bmatrix}$.

Solution: Here we have

$$\begin{bmatrix} -2 & 3 \\ 0 & 4 \\ -2 & 7 \end{bmatrix} \underset{r_3 \to r_3 - r_1}{\sim} \begin{bmatrix} -2 & 3 \\ 0 & 4 \\ 0 & 4 \end{bmatrix} \underset{r_3 \to r_3 - r_2}{\sim} \begin{bmatrix} -2 & 3 \\ 0 & 4 \\ 0 & 0 \end{bmatrix}$$

The echelon form of 'A' contains 2 pivots. Hence $\text{rank}(A) = 2$.

Exercise 1.4

1. Reduce the following matrices to echelon form and find the rank of the following matrices:

$$M = \begin{pmatrix} 6 & 2 & 0 & 4 \\ -2 & -1 & 3 & 4 \\ -1 & -1 & 6 & 10 \end{pmatrix},\ B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & 2 \\ 5 & 9 & 12 & 14 \end{pmatrix} \text{ and } P = \begin{pmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{pmatrix}.$$

   Ans: 2 Ans: 2 Ans: 2

2. Find the rank of the following matrices:

   i) $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 1 \\ 0 & 3 & 0 \end{bmatrix}$, ii) $A = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 4 & 2 \\ -1 & 2 & 6 \end{bmatrix}$. iii) $A = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 3 & 4 \\ -1 & 2 & 6 \end{bmatrix}$.

   Ans: 2 Ans: 2 Ans: 3

References:

- Linear Algebra and its applications by David C Lay
- Linear Algebra and its applications by Lay and Lay