Chapter-2

System of Linear Equations (SLE)

In mathematics, linear systems are the basis and a fundamental part of linear algebra, a subject which is used in most parts of modern mathematics. Computational algorithms for finding the solutions are an important part of numerical linear algebra, and play a prominent role in engineering, physics, chemistry, computer science, and economics. A system of linear equations is a group of two or more linear equations containing the same variables. In a system of equations there is more than one unknown since the equations contain more than one variable. We will explore applications that involve systems of linear equations and look at how to set up a system of equations with given information. Systems of linear equations arise in a wide variety of applications. System of linear equations arises in the problem of polynomial curve fitting, network analysis and analysis of an electric circuit and the linear programming problem etc. System of linear equations also arises when we want to solve mixture problems and distance-rate-time problem. One of the most frequent occasions where linear systems of n equations in n unknowns arise is in least-squares optimization problems. Least squares problems lead to square (i.e. $n \times n$) linear systems of equations. Also systems of linear equations arise in the problem of graph theory and cryptography**.** In cryptanalysis (breaking codes mathematically) we use linear in solving systems of equations related to both a grammar and language in cipher text.

Linear equation:
An equation in two or more variables (unknowns) is linear if it contains no products of unknowns or exponent of each unknown is 1.

Example:

1. $2x + 3y = 8$ (linear)
2. $x_1 + x_2 + ... + x_n = 1$ (linear)
3. $x^2 + 4x = 8$ (non – linear)

Solution:
A solution of linear equation $a_1x_1 + a_2x_2 + ... + a_nx_n = b$ is a sequence of n numbers $\alpha_1, \alpha_2 ..., \alpha_n$ such that the equation is satisfied when we substitute $x_1 = \alpha_1, x_2 = \alpha_2, ..., x_n = \alpha_n$. The set of all such solutions of the linear equation is called a solution set.

$$\left.\begin{aligned} x + y = 2 \\ x + y = 0 \end{aligned}\right\} \text{has no solution.}$$

System of linear equations:

A group of m linear equations of n variables $x_1, x_2, ..., x_n$ are of the form

$$\left.\begin{array}{ccccccc} a_{11}x_1 + & a_{12}x_2 + & . & . & . & +a_{1n}x_n = b_1 \\ a_{21}x_1 + & a_{22}x_2 + & . & . & . & +a_{2n}x_n = b_2 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{m1}x_1 + & a_{m2}x_2 + & . & . & . & +a_{mn}x_n = b_m \end{array}\right\} \text{............(1)}$$

is known as system of linear equation. Here the co-efficient a_{ij}, $i = 1,2,...m$, $j = 1,2,...,n$ of the variable and the free term b_i, $i = 1,2,...,m$ are real numbers.
By a solution (set) of a system means such a set of real numbers that satisfies each equation in a system.

Solution of a system of linear equations:
A sequence of numbers $\alpha_1, \alpha_2 ..., \alpha_n$ is called solution of the system of linear equations given by (1) if $\alpha_1, \alpha_2 ..., \alpha_n$ is a solution of every equation in the system.

Degenerate and non-degenerate linear equation:
A linear equation is said to be **degenerate** if it has the form $0x_1 + 0x_2 + ... + 0x_n = b$. That is, if every coefficient of the variable is equal to zero. The solution of such a generate linear equation is as follows:

(i) If the constant $b \neq 0$, then the above equation has no solution.

(ii) If the constant $b = 0$, then every vector $u = (\alpha_1, \alpha_2, ..., \alpha_n)$ is a solution of the above equation.

The general linear equation $a_1x_1 + a_2x_2 + ... + a_nx_n = b$ is called **non-degenerate** linear equation.

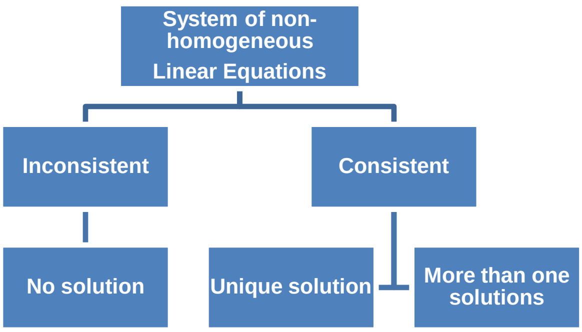
Consistent and inconsistent equations:
A system of linear equations is called consistent if it has at least one set of solution. A system of linear equations is called inconsistent if it has no solution.

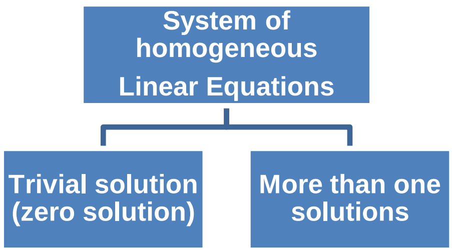
Consistency theorem: The system of linear equations $AX = B$ (m equations and n unknowns) is consistent (i.e. there is at least one solution of the system) if the coefficient matrix A and the augmented matrix $(A|B)$ have the same rank.

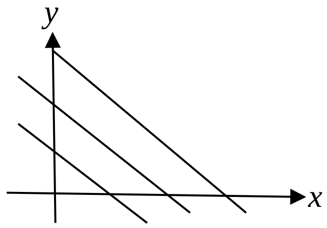
Determinate and Indeterminate:
A consistent system is called determinate if it has a unique solution and indeterminate if it has more than one solution.
An indeterminate system of linear equation always has an infinite number of solutions.

Then 3 cases arise:

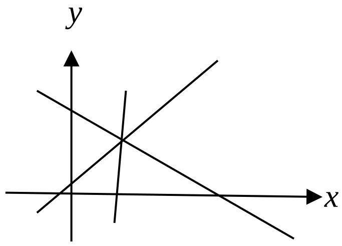
- ► SLE is inconsistent $\Rightarrow$ straight lines do not intersect (i.e., parallel);
- ► SLE has a unique solution $\Rightarrow$ all straight lines pass through a single point;
- ► SLE is redundant $\Rightarrow$ actually one straight line, with which others coincide, exists.



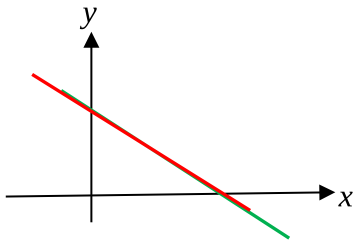




An inconsistent system
(no common point)



A unique system
(only 1 common point)



Infinitely many solution system
(overlapping lines)

Example: Following augmented matrices illustrate the consistency of the linear system.

(i) $[\mathrm{A}|\mathrm{B}]=\left[\begin{array}{ccc|c}1 & 2 & 3 & 2\\ 0 & 1 & 4 & 3\\ 0 & 0 & 1 & 2\end{array}\right]$,
$\operatorname{rank}(A)=\operatorname{rank}[A|B]=3$
So, this system is consistent.

(ii) $[\mathrm{A}|\mathrm{B}]=\left[\begin{array}{ccc|c}1 & 2 & 3 & 2\\ 0 & 1 & 4 & 3\\ 0 & 0 & 0 & 2\end{array}\right]$,
$\operatorname{rank}(A)=2$; $\operatorname{rank}[A|B]=3$
So, this system is inconsistent. There is no solution for this system.

(iii) $[\mathrm{A}|\mathrm{B}]=\left[\begin{array}{ccc|c}1 & 2 & 3 & 2\\ 0 & 1 & 4 & 3\\ 0 & 0 & 0 & 0\end{array}\right]$.
$\operatorname{rank}(A)=2$;
$\operatorname{rank}[A|B]=2$
So, this system is consistent but infinitely many solutions.

Example: Test the consistency of the following system of linear equations with the help of the rank of the matrix

$$\begin{aligned}3x_1+4x_2-x_3+2x_4&=1\\ x_1-2x_2+3x_3+x_4&=2\\ 3x_1+14x_2-11x_3+x_4&=3\end{aligned}\ .$$

Solution: The corresponding augmented matrix is

$$\left(\begin{array}{cccc|c}3 & 4 & -1 & 2 & 1\\ 1 & -2 & 3 & 1 & 2\\ 3 & 14 & -11 & 1 & 3\end{array}\right)\underset{r_1\leftrightarrow r_2}{\sim}\left(\begin{array}{cccc|c}1 & -2 & 3 & 1 & 2\\ 3 & 4 & -1 & 2 & 1\\ 3 & 14 & -11 & 1 & 3\end{array}\right)$$

$$\begin{array}{l}\widetilde{r_2}\to r_2-3r_1\\ \widetilde{r_3}\to r_3-3r_1\end{array}\left(\begin{array}{cccc|c}1 & -2 & 3 & 1 & 2\\ 0 & 10 & -10 & -1 & -4\\ 0 & 20 & -20 & -2 & -3\end{array}\right)$$

$$\underset{r_3\to r_3-2r_2}{\sim}\left(\begin{array}{cccc|c}1 & -2 & 3 & 1 & 2\\ 0 & 10 & -10 & -1 & -4\\ 0 & 0 & 0 & 0 & 5\end{array}\right)\underset{r_2\to\frac{1}{10}r_2}{\sim}\left(\begin{array}{cccc|c}1 & -2 & 3 & 1 & 2\\ 0 & 1 & -1 & -\frac{1}{10} & -\frac{2}{5}\\ 0 & 0 & 0 & 0 & 1\end{array}\right)$$

Now $\operatorname{rank}(A)=2, \operatorname{rank}(A|b)=3$. Since $\operatorname{rank}(A)\neq\operatorname{rank}(A|b)$, the system is inconsistent. It has no solution.

Example: Test the consistency of the following system of linear equations with the help of the rank of the matrix, If consistent solve the system.

$$\begin{aligned} 2x+2y+3z&=2\\ -x+3y+z&=1\,.\\ x-y+z&=3 \end{aligned}$$

Solution: The corresponding augmented matrix is

$$\left(\begin{array}{ccc|c} 2 & 2 & 3 & 2\\ -1 & 3 & 1 & 1\\ 1 & -1 & 1 & 3 \end{array}\right) \begin{array}{l} \widetilde{r_2}\to 2r_2+r_1\\ \widetilde{r_3}\to 2r_3-r_1 \end{array} \left(\begin{array}{ccc|c} 2 & 2 & 3 & 2\\ 0 & 8 & 5 & 4\\ 0 & -4 & -1 & 4 \end{array}\right)$$

$$\widetilde{r_3}\to 2r_3+r_2 \quad \left(\begin{array}{ccc|c} 2 & 2 & 3 & 2\\ 0 & 8 & 5 & 4\\ 0 & 0 & 3 & 12 \end{array}\right)$$

Now $\operatorname{rank}(A)=3, \operatorname{rank}(A|b)=3$. Since $\operatorname{rank}(A)=\operatorname{rank}(A|b)$, the system is consistent.

Echelon matrix can be written to system of linear equations

$$2x+2y+3z=2,\quad 8y+5z=4, 3z=12$$
$$=> z=4,\ \ y=-2,\qquad x=-3\quad \mathit{Solved.}$$

Example: A system of linear equations with exactly one solution

Consider the system

$2x-y=1$

$3x+2y=12$

Solving the first equation for y in terms of x, we obtain the equation

$y=2x-1$

Substituting this expression for y into the second equation yields

$3x+2(2x-1)=12$

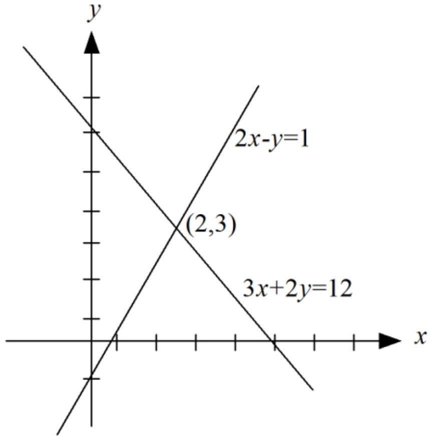
$\Rightarrow 3x+4x-2=12$

$\Rightarrow 7x=14$

$\therefore x=2$

Finally, substituting this value of x into the expression for y gives $y=2(2)-1=3$

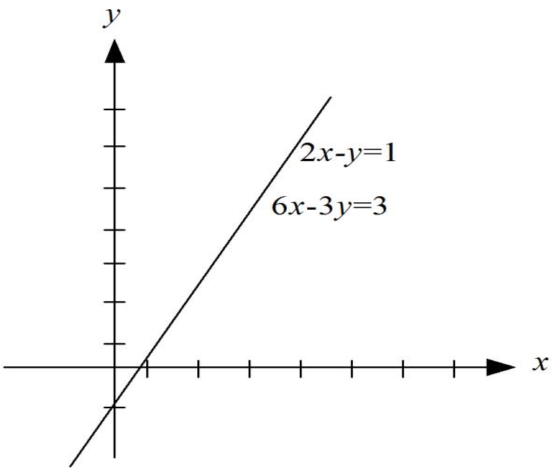
Graphical representation of the system of linear equations for unique solution (One solution):



Therefore, the unique solution of the system is given by $x = 2$ and $y = 3$. Geometrically, the two lines represented by the two equations that make up the system intersect at the point $(2,3)$. So, the solutions are $x = 2$ and $y = 3$.

Example: A system of linear equations which are coincident has infinitely many solutions:
(Graphical representation of the system for infinitely many solutions)

Consider the system $2x - y = 1$; $6x - 3y = 3$.



Solving the first equation for y in terms of x, we obtain the equation
$y = 2x - 1$
Substituting this expression for y into the second equation yields

$$6x - 3(2x - 1) = 3$$
$$\Rightarrow 6x - 6x + 3 = 3$$
$$\therefore 0 = 0$$

Which is a true statement. This result follows from the fact that the second equation is equivalent to the first. Our computations have revealed that the system of two equations is equivalent to the single equation $2x - y = 1$. Thus, any ordered pair of numbers (x, y) satisfying the equations $2x - y = 1$ or $y = 2x - 1$ constitutes a solution to the system.

In particular, by assigning the value t to x, where t is any real number, we find that $y = 2t - 1$ and so the ordered pair $(t, 2t - 1)$ is a solution of the system. The variable t is called a parameter. For example, setting $t = 0$ gives the point $(0, -1)$ as a solution of the system, and setting $t = 1$ gives the point $(1, 1)$ as another solution. Since t represents any real number, there are infinitely many solutions of the system. Geometrically, the two equations in the system represent the same line, and all solutions of the system are points lying on the line (Figure). Such a system is said be dependent.

Example: A system of linear equations that has no solution:
Consider the system
$2x - y = 1$
$6x - 3y = 12$
Solving the first equation for y in terms of x, we obtain the equation
$y = 2x - 1$

Substituting this expression for y into the second equation yields

$$6x-3(2x-1)=12$$

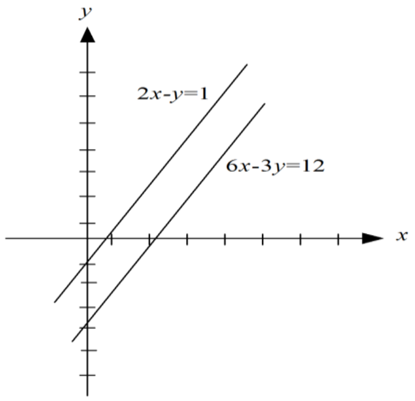
$$\Rightarrow 6x-6x+3=12 \qquad \therefore 0=9$$

which is clearly untrue. Thus, there is no solution to the system of equations.

$y=2x-1$

$y=2x-4$

Graphical representation of the system of linear equations for no solution:



It has been observed that these two lines are parallel to each other.

Homogeneous and nonhomogeneous linear equation:

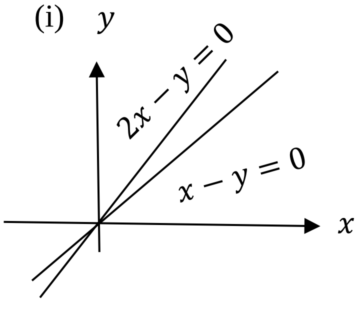
A system of linear equations is called homogeneous if all the constant terms $b_1, b_2 ...., b_n$ of the Non-homogeneous system are zero such as the system has the form:

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \; . \; . \; . \; + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \; . \; . \; . \; + a_{2n}x_n = 0 \\ \quad . \qquad . \qquad . \; . \; . \qquad . \\ \quad . \qquad . \qquad . \; . \; . \qquad . \\ \quad . \qquad . \qquad . \; . \; . \qquad . \\ a_{m1}x_1 + a_{m2}x_2 + \; . \; . \; . \; + a_{mn}x_n = 0 \end{array}$$

Homogeneous system of linear equations has two types of solutions. They are

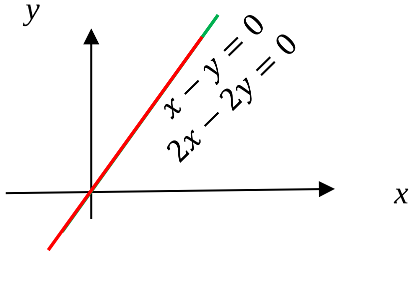
(i) Trivial (zero) solution (all $x_i = 0$)
(ii) More than one solutions

(i)



Trivial (zero solution)

(ii)



More than one solutions

Matrices and system of linear equations:

The system of linear equations (1) can be written in the matrix form.

$$\begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{m1} & a_{n2} & . & . & . & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ . \\ . \\ b_m \end{bmatrix}$$

or simply $A\,X = B$ ……………………………………….(2)

where co-efficient matrix, $A = (a_{i\,j})$, variable matrix, $X = (x_i)$ and constant matrix, $B = (b_i)$

The associated homogeneous system of (1) is $A\,X = 0$.

The system (1) also can be written in augmented matrix form $[A|B]$ or $[A \vdots B]$.

There are three commonly used methods to solve system of linear equations:

1. Using inverse matrix,
2. Using elementary row operations (Gaussian elimination and Gauss-Jordan elimination),
3. Cramer's rule.

Example: Given that

$$\left.\begin{aligned} 2x + 3y - 4z &= 7 \\ x - 2y - 5z &= 3 \end{aligned}\right\} \qquad (1)$$

The augmented matrix of the above system of linear equations is

$$\begin{bmatrix} 2 & 3 & -4 \vdots 7 \\ 1 & -2 & -5 \vdots 3 \end{bmatrix}.$$

Solution of linear equation by applying matrices:

m (no. of linear equations) $= n$(no. of variables) for the system of linear equations:

Consider, m (no. of linear equations) $= n$(no. of variables) for the system of linear equations (2). Let, D be the determinant of the matrix A. we have to evaluate the determinant. If $D = 0$, A is singular. So A^{-1} doesn't exist and hence the system has no solution. If $D \neq 0$, A is nonsingular. So, A^{-1} exists and hence the system has a solution. Now multiplying both sides of (3) by A^{-1}, we have

$A^{-1}AX = A^{-1}B$

$IX = A^{-1}B$

$\therefore X = A^{-1}B$ since $\begin{matrix} A^{-1}A = I \\ I\,X = X \end{matrix}$ and $B = [l_i]$.

That is,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{12} & . & . & . & A_{1n} \\ A_{21} & A_{22} & . & . & . & A_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ A_{n1} & A_{n2} & . & . & . & A_{nn} \end{bmatrix}\begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ \cdot \\ \cdot \\ l_n \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ \cdot \\ \cdot \\ m_n \end{bmatrix} \text{ (say)}$$

Where determinant of the matrix, A is $|A|$. Then, $x_1 = m_1, x_2 = m_2, x_3 = m_3, \ldots\ldots\ldots\ldots x_n = m_n$ (say) is a solution of the given system of *n* linear equations.

It is to be noted that the solution of the system of equation can also be found by reducing the augmented matrix of the given system to reduced echelon form.

m (no. of linear equations) $<$ n (no. of unknowns or variables) of the following system of linear equations:

After reduced the system of linear equations (1) into echelon form,

(i) Number of variable(s) is equal to the number of equation(s) gives the unique solution
(ii) Number of variable(s) is greater than the number of equation(s) gives more than one solution.

Example of the algorithm

Suppose the goal is to find and describe the set of solutions to the following system of linear equations:

$2x + y - z = 8 \qquad (r_1)$

$-3x - y + 2z = -11 \qquad (r_2)$

$-2x + y + 2z = -3 \qquad (r_3)$

The table below is the row reduction process applied simultaneously to the system of equations, and its associated augmented matrix. The row reduction procedure may be summarized as follows: eliminate *x* from all equations below r_1, and then eliminate *y* from all equations below r_2. This will put the system into triangular form. Then, using back-substitution, each unknown can be solved.

System of equations row operations augmented matrix

$$\begin{aligned} 2x+y-z&=8\\ -3x-y+2z&=-11\\ -2x+y+2z&=-3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 2 & 1 & -1 & 8\\ -3 & -1 & 2 & -11\\ -2 & 1 & 2 & -3 \end{array}\right]$$

$$\begin{aligned} 2x+y-z&=8\\ y+z&=2\\ 2y+z&=5 \end{aligned} \qquad \begin{aligned} r_2&\to 2r_2+3r_1\\ r_3&\to r_3+r_1 \end{aligned} \qquad \left[\begin{array}{ccc|c} 2 & 1 & -1 & 8\\ 0 & 1 & 1 & 2\\ 0 & 2 & 1 & 5 \end{array}\right]$$

$$\begin{aligned} 2x+y-z&=8\\ y+z&=2\\ -z&=1 \end{aligned} \qquad r_3\to r_3-2r_2 \qquad \left[\begin{array}{ccc|c} 2 & 1 & -1 & 8\\ 0 & 1 & 1 & 2\\ 0 & 0 & -1 & 1 \end{array}\right]$$

The matrix is now in echelon form

$$\begin{aligned} 2x+y\quad&=7\\ y\quad&=3\\ -z&=1 \end{aligned} \qquad \begin{aligned} r_1&\to r_1-r_3\\ r_2&\to r_2+r_3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 2 & 1 & 0 & 7\\ 0 & 1 & 0 & 3\\ 0 & 0 & -1 & 1 \end{array}\right]$$

$$\begin{aligned} 2x\quad&=4\\ y\quad&=3\\ -z&=1 \end{aligned} \qquad r_1\to r_1-r_2 \qquad \left[\begin{array}{ccc|c} 2 & 0 & 0 & 4\\ 0 & 1 & 0 & 3\\ 0 & 0 & -1 & 1 \end{array}\right]$$

$$\begin{aligned} x\quad&=2\\ y\quad&=3\\ z&=-1 \end{aligned} \qquad \begin{aligned} r_1&\to \frac{1}{2}r_1\\ r_2&\to -r_2 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2\\ 0 & 1 & 0 & 3\\ 0 & 0 & 1 & -1 \end{array}\right]$$

The solution is $z=-1$, $y=3$, and $x = 2$. So, there is unique solution to the original system of equations.

Example: Solve the following system of equations using Gaussian elimination method

$$\begin{aligned} 2x+y+3z&=1 && (r_1)\\ 2x+6y+8z&=3 && (r_2)\\ 6x+8y+18z&=5 && (r_3) \end{aligned}$$

$$\begin{aligned} r_2&\to r_2-r_1\\ r_3&\to r_3-3r_1 \end{aligned} \qquad \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1\\ 0 & 5 & 5 & 2\\ 0 & 5 & 9 & 2 \end{array}\right]$$

$$r_3\to r_3-r_2 \qquad \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1\\ 0 & 5 & 5 & 2\\ 0 & 0 & 4 & 0 \end{array}\right]$$

From this stage, we can get the solution by back solving

$$z=0$$
$$5y+5(0)=2$$
$$\Rightarrow y=\frac{2}{5}$$
and $2x+\frac{2}{5}+3(0)=1 \qquad \Rightarrow x=\frac{3}{10}$

So, the solution is:

$$(x,y,z)=\left(\frac{3}{10},\frac{2}{5},0\right)$$

Example: Solve the following system of equations using elementary row operations

$$\begin{aligned} 3x+y-6z&=-10 && (r_1)\\ 2x+y-5z&=-8 && (r_2) \end{aligned}$$

$6x - 3y + 3z = 0 \qquad (r_3)$

$r_2 \to 3r_2 - 2r_1$
$r_3 \to r_3 - 2r_1$

$$\left[\begin{array}{ccc|c} 3 & 1 & -6 & -10 \\ 0 & 1 & -3 & -4 \\ 0 & 1 & -3 & -4 \end{array}\right]$$

$r_3 \to r_3 - r_2$

$$\left[\begin{array}{ccc|c} 3 & 1 & -6 & -10 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let, $z = a$, where a is a free variable.
We have,
$y = 3a - 4$
$\therefore x + 3z - 4 - 6z = -10$
$\Rightarrow x = a - 2$
So, the general solution of the system is $(x, y, z) = (a - 2, 3a - 4, a)$.
For particular solution, putting $a = 1$ ($putting\ any\ suitable\ value$)

$$(x, y, z) = (-1, -1, 1)$$

Example: Solve the following system of equations using Gaussian elimination method

$x \qquad + z = 1 \qquad (r_1)$
$x + y + z = 2 \qquad (r_2)$
$x - y + z = 1 \qquad (r_3)$

$r_2 \to r_2 - r_1$
$r_3 \to r_3 - r_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$r_3 \to r_3 + r_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The third row $'0 = 1'$ it does not exist. So, the system is inconsistent. That means the system has no solution.

Example: Solve the following system using Gauss-Jordan elimination method.

$$2x - y + z = 1$$
$$x + 4y - 3z = -2$$
$$3x + 2y - z = 0$$

__**Solution**__

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 1 & 4 & -3 & -2 \\ 3 & 2 & -1 & 0 \end{array}\right] \underset{r_1 \leftrightarrow r_2}{\sim} \left[\begin{array}{ccc|c} 1 & 4 & -3 & -2 \\ 2 & -1 & 1 & 1 \\ 3 & 2 & -1 & 0 \end{array}\right] \begin{array}{c} \widetilde{r_2} \to r_2 - 2r_1 \\ \\ \widetilde{r_3} \to r_3 - 3r_1 \end{array} \left[\begin{array}{ccc|c} 1 & 4 & -3 & -2 \\ 0 & -9 & 7 & 5 \\ 0 & -10 & 8 & 6 \end{array}\right]$$

$$\underset{r_2 \to r_2 - r_3}{\sim} \left[\begin{array}{ccc|c} 1 & 4 & -3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & -10 & 8 & 6 \end{array}\right] \underset{r_3 \to r_3 + 10r_2}{\sim} \left[\begin{array}{ccc|c} 1 & 4 & -3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -2 & -4 \end{array}\right]$$

$$\underset{r_3 \to -\frac{1}{2}r_3}{\sim} \left[\begin{array}{ccc|c} 1 & 4 & -3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \underset{r_1 \to r_1 - 4r_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{matrix} \widetilde{r_1} \to r_1 - r_3 \\ \widetilde{r_2} \to r_2 + r_3 \end{matrix} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Hence the solution is $x = 0, y = 1, z = 2.$

Example: Solve the following system of equations using matrix inversion and justify your answer.

$$\left.\begin{matrix} 2x + y = 1 \ \ (r_1) \\ x - 2y = 3 \ \ (r_2) \end{matrix}\right\} \ (1)$$

Solution:

System (1) is written in the matrix form

$$\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$AX = B$

$X = A^{-1}B$

Where, $A = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, and $B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

The determinant of the matrix A is $\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = -6 \neq 0$

So, the matrix A is non-singular and A^{-1} exists.

Now $A^{-1} = \frac{1}{-5}\begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix}$

And $X = A^{-1}B$

$$= \frac{1}{-5}\begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$= \frac{1}{-5}\begin{bmatrix} -5 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore x = 1$ and $y = -1$

Verification:

$$r_1 : L.H.S = 2x + y = 2.1 - 1 = 1 = R.H.S$$
$$r_1 : L.H.S = x - 2y = 1 - 2(-1) = 3 = R.H.S$$

Example: Solve the following system of equations using matrix inversion.

$2x - y + 3z = 53$
$4x - z = -53$
$3x + 3y + 2z = 106$

Solution: We write down the given system as

$$\begin{pmatrix} 2 & -1 & 3 \\ 4 & 0 & -1 \\ 3 & 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 53 \\ -53 \\ 106 \end{pmatrix}$$

$\Rightarrow \quad A\,X = B \qquad$ [say]

$\Rightarrow \quad X = A^{-1}\,B \qquad$ [since A^{-1}exists]

$$\Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{53}\begin{pmatrix} 3 & 11 & 1 \\ -11 & -5 & 14 \\ 12 & -9 & 4 \end{pmatrix}\begin{pmatrix} 53 \\ -53 \\ 106 \end{pmatrix} = \begin{pmatrix} -6 \\ 22 \\ 29 \end{pmatrix} \qquad \text{[after finding } A^{-1}\text{]}$$

$\Rightarrow \quad x =-6, y = 22, z = 29$ is the required solution.

Example: Using matrix inversion solve the system of linear equations

$x_1 + 2x_2 + 3x_3 = 5$
$2x_1 + 5x_2 + 3x_3 = 3$.
$x_1 + 8x_3 = 17$

Solution:

The system of equations can be written in the matrix form as $AX = B$, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$. The solution can be written as $X = A^{-1}B$. Let us find A^{-1} using elementary row operations.

$$[A \vdots I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \underset{\substack{r_2 \to r_2 - 2r_1 \\ r_3 \to r_3 - r_1}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \underset{r_3 \to r_3 + 2r_2}{\sim}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \underset{r_1 \to r_1 - 2r_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \underset{r_1 \to r_1 + 9r_3}{\sim}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \underset{\substack{r_2 \to r_2 - 3r_3 \\ r_3 \to -r_3}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Now, $X = A^{-1}B = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ $\quad x_1 = 1, x_2 = -1$ and $x_3 = 2$.

Cramer's Rule:

Let a system of linear equations is given $AX = B$ and $|A| = D$. This system is

(a) inconsistent if $B \neq 0$ but $D = 0$;
(b) consistent and redundant if $B = 0$ and $D = 0$;
(c) consistent and unique if $D \neq 0$; and in this case the solution is given by

$$x_i = \frac{D_i}{D} \; (i = 1, 2, \ldots\ldots, n),$$

D_i can be obtained by replacing i^th column by right hand side.

Explicit formulas for small systems

Consider the linear system $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$

which in matrix format is $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Assume $a_1b_2 - b_1a_2$ nonzero. Then, with help of determinants x and y can be found with Cramer's rule as

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{c_1b_2 - b_1c_2}{a_1b_2 - b_1a_2} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{a_1c_2 - c_1a_2}{a_1b_2 - b_1a_2}.$$

The rules for 3×3 matrices are similar. Given

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

which in matrix format is

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Then the values of x, y and z can be found as follows:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \text{ and } z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Similar idea can be extended for $n \times n$ systems

Example:

Let us demonstrate Cramer's rule with the following system:

$x + 2y + 3z = 1$

$-x \qquad + 2z = 2$
$\quad -2y + z \ = -2$

Step 1:

The coefficient matrix of this system is $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 0 & -2 & 1 \end{bmatrix}$

Note that the matrix is square (it has 3 rows and 3 columns), and so we may proceed with the next step of Cramer's rule.

Step 2:

Now find the determinant of the coefficient matrix A; use the matrix manipulator in the tools box if you would like help in this computation. You should get $|A| = 12$. This is not zero, so Cramer's rule may be applied here.

Step 3:

$A_x = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ -2 & -2 & 1 \end{bmatrix}$ and its determinant is $|A_x| = -20$. Therefore $x = \frac{|A_x|}{|A|} = -\frac{20}{12} = -\frac{5}{4}$

Step 4:

Using the same method, the values for the remaining 2 variables, x and y, are computed below:

$A_y = \begin{bmatrix} 1 & 1 & 3 \\ -1 & 2 & 2 \\ 0 & -2 & 1 \end{bmatrix}$ and its determinant is $|A_y| = 13$. Therefore $y = \frac{|A_y|}{|A|} = -\frac{13}{12}$

$A_z = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 0 & -2 & -2 \end{bmatrix}$ and its determinant is $|A_z| = 2$. Therefore $z = \frac{|A_z|}{|A|} = \frac{2}{12} = \frac{1}{6}$

Example: Verify whether the following system of linear equations is consistent or not. If consistent then solve the system using Cramer’s rule. Also check your answer.

$2x - 3y - 5z = 40$
$17x + 14y - 22\,z = 22$
$15x + 17y - 17z = -18$.

Solution:

$D = \begin{vmatrix} 2 & -3 & -5 \\ 17 & 14 & -22 \\ 15 & 17 & -17 \end{vmatrix} = 0$ and $\neq \mathbf{0}$. Therefore, this SLE is either inconsistent or more than one solutions.

Example:

$2x - 5y + 6z = -27$
$10x - 11y - 9z = 0$
$-3x \qquad + 2z = 16$

Solution:

$D = \begin{vmatrix} 2 & -5 & 6 \\ 10 & -11 & -9 \\ -3 & 0 & 2 \end{vmatrix} = -277$ and $B \neq 0$ and, therefore, this SLE has a unique solution.

Now $D_1 = \begin{vmatrix} -27 & -5 & 6 \\ 0 & -11 & -9 \\ 16 & 0 & 2 \end{vmatrix}$; $D_2 = \begin{vmatrix} 2 & -27 & 6 \\ 10 & 0 & -9 \\ -3 & 16 & 2 \end{vmatrix}$; $D_3 = \begin{vmatrix} 2 & -5 & -27 \\ 10 & -11 & 0 \\ -3 & 0 & 16 \end{vmatrix}$.

$= 2370$; $= 1059$; $= 1339.$

Thus $x = -\dfrac{2370}{277}$; $y = -\dfrac{1059}{277}$; $z = -\dfrac{1339}{277}$.

Example:
Solve the following system of linear equations using Cramer's rule
$2x + y + z = 3$
$x - y - z = 0$
$x + 2y + z = 0$
We have the left-hand side of the system with the variables (the "coefficient matrix") and the right-hand side with the answer values. Let D be the determinant of the coefficient matrix of the above system, and let D_x be the determinant formed by replacing the x-column values with the answer-column values: Evaluating each determinant, we get:

$$D = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3$$

$$D_x = \begin{vmatrix} 3 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 2 & 1 \end{vmatrix} = 3, \quad D_y = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{vmatrix} = -6, \quad D_z = \begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{vmatrix} = 9$$

So, according to Cramer's rule:

$$x = \frac{D_x}{D} = \frac{3}{3} = 1, \quad y = \frac{D_y}{D} = \frac{-6}{3} = -2 \text{ and } z = \frac{D_z}{D} = \frac{9}{3} = 3$$

Example:
Solve the following system of linear equations using Crammer's rule
$2x + 3y + z = 10$
$x - y + z = 4$
$4x - y - 5z = -8$.

Solution: Each unknown will be the quotient of the determinant obtained by substituting the answers in the right sides of the equations for the coefficients of the unknown divided by the determinant formed by taking the coefficients on the left sides of the equations.

$$x = \frac{\begin{vmatrix} 10 & 3 & 1 \\ 4 & -1 & 1 \\ -8 & -1 & -5 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & 1 \\ 4 & -1 & -5 \end{vmatrix}} = \frac{84}{42} = 2, \; y = \frac{\begin{vmatrix} 2 & 10 & 1 \\ 1 & 4 & 1 \\ 4 & -8 & -5 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & 1 \\ 4 & -1 & -5 \end{vmatrix}} = \frac{42}{42} = 1, \; z = \frac{\begin{vmatrix} 2 & 3 & 10 \\ 1 & -1 & 4 \\ 4 & -1 & -8 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & 1 \\ 4 & -1 & -5 \end{vmatrix}} = \frac{126}{42} = 3$$

Example:
Determine the value(s) of λ and μ such that the following system of linear equations has (i) no solution, (ii) more than one solution, and (iii) a unique solution.

$$\begin{cases} x+y+z=6 \\ x+2y+3z=10 \\ x+2y+\lambda z=\mu \end{cases}$$

Solution: The corresponding augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right) \underset{\substack{r_2 \to r_2 - r_1 \\ r_3 \to r_3 - r_2}}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & 6 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 1 & \mu - 10 \end{array}\right)$$

The above system is in echelon form. Now we consider the following three cases:

(i) If $\lambda = 3$ and $\mu \neq 10$ then third equation of (1) is of the form $0 = a$, where $a = \mu - 10 \neq 0$ which is not true. So, the system is inconsistent. Thus, the system has no solution for $\lambda = 3$ and $\mu \neq 10$.

(ii) If $\lambda = 3$ and $\mu = 10$ then third equation of (1) is vanishes and the system will be in echelon form having two equations in three variables. So, it has $3 - 2 = 1$ free variables which is z. Hence the given system has more than one solution for $\lambda = 3$ and $\mu = 10$.

(iii) For a unique solution, the coefficient of z in the 3rd equation must be non-zero i.e., $\lambda \neq 3$ and μ may have any value. Therefore, the given system has unique solution for $\lambda \neq 3$ and arbitrary values of μ.

Example:

Determine the value(s) of λ and μ such that the following system of linear equations has (i) no solution (ii) more than one solution and (iii) a unique solution.

$$\begin{cases} x+y-z=1 \\ 2x+3y+\lambda z=3. \\ x+\lambda y+3z=2 \end{cases}$$

Solution: The given system of liner equations is $\begin{cases} x+y-z=1 \\ 2x+3y+\lambda z=3 \\ x+\lambda y+3z=2 \end{cases}$

Reduce the system to echelon form by means of elementary row operations,

$$\begin{cases} x+y-\quad z=1 \\ y+\quad (\lambda+2)z=1 \\ (\lambda-1)y+4z=1 \end{cases} \quad \begin{bmatrix} L_2' = L_2 - 2L_1 \\ L_3' = L_3 - L_1 \end{bmatrix}$$

$$\begin{cases} x+y-\quad z=1 \\ y+\quad (\lambda+2)z=1 \\ \{4-(\lambda-1)(\lambda+2)\}z=2-\lambda \end{cases} \quad \left[L_3' = L_3 - (\lambda-1)L_2\right]$$

$$\begin{cases} x+y-\quad z=1 \\ y+\quad (\lambda+2)z=1 \\ \left(6-\lambda-\lambda^2\right)z=2-\lambda \end{cases}$$

$$\begin{cases} x+y-\quad z=1 \\ \quad y+\quad (\lambda+2)z=1 \qquad \cdots(1) \\ (\lambda+3)(2-\lambda)z=2-\lambda \end{cases}$$

The above system is in echelon form. Now we consider the following three cases:

(i) From third equation of (1), we see that if $\lambda+3=0$ or $\lambda=-3$ then the equation becomes $0=5$, which is contradiction. Therefore, the system is inconsistent if $\lambda=-3$. Thus, the system has no solution for $\lambda=-3$.

(ii) We know, if the number of variables is greater than the number of equations, then the system has more than one solution. From third equation of (1), we see that if $\lambda=2$ then it becomes $0=0$. In this case the system has three variables with two equations. So, the given system has more than one solution for $\lambda=2$.

(iii) We know, if the number of variables and the number of equations be equal, then the system has unique solution. The system (1) has a unique solution
$(\lambda+3)(2-\lambda)\neq 0 \Rightarrow \lambda\neq -3, \lambda\neq 2.$

Example:
Determine the value(s) of λ and μ such that the following system of linear equations has (i) no solution, (ii) more than one solution, and (iii) a unique solution.

$$\begin{cases} 2x+3y+z=5 \\ 3x-y+\lambda z=2 \\ x+7y-6z=\mu \end{cases}$$

Solution: The given system of liner equations is

$$\begin{cases} 2x+3y+z=5 \\ 3x-y+\lambda z=2 \\ x+7y-6z=\mu \end{cases}$$

Reduce the system to echelon form by means of elementary operations,

$$\begin{cases} 2x+3y+\quad z=5 \\ -11y+\quad (2\lambda-3)z=-11 \\ 11y-13z=2\mu-5 \end{cases} \quad \begin{bmatrix} L_2'=2L_2-3L_1 \\ L_3'=2L_3-L_1 \end{bmatrix}$$

$$\begin{cases} 2x+3y+\quad z=5 \\ -11y+\quad (2\lambda-3)z=-11 \\ 2(\lambda-8)z=2(\mu-8) \end{cases} \quad \begin{bmatrix} L_3'=L_3+L_2 \end{bmatrix}\cdots(1)$$

The above system is in echelon form. Now we consider the following three cases:

(i) For a unique solution, the coefficient of z in the 3rd equation of (1) must be non-zero i.e., $\lambda\neq 8$ and μ may have many values. Therefore, the given system has unique solution for $\lambda\neq 8$ and arbitrary values of μ.

(ii) If $\lambda=8$ and $\mu=8$ then third equation of (1) is vanishes and the system will be in echelon form having two equations in three variables. So, it has $3-2=1$ free variables which is z. Hence the given system has more than one solution for $\lambda=8$ and $\mu=8$..

(iii) If $\lambda = 8$ and $\mu \neq 8$ then third equation of (1) is of the form $0 = a$, where $a = \mu - 8 \neq 0$ which is not true. So, the system is inconsistent. Thus, the system has no solution for $\lambda = 8$ and $\mu \neq 8$.

Example:
Find the values of k such that the following system of linear equations has non-zero solution.
$x + ky + 3z = 0$
$4x + 3y + kz = 0.$
$2x + y + 2z = 0$

Solution:
The augmented matrix $C = [A\!:\!B]$
$$\sim \begin{bmatrix} 1 & k & 3 & : & 0 \\ 4 & 3 & k & : & 0 \\ 2 & 1 & 2 & : & 0 \end{bmatrix}$$
On interchanging first row and third row, we have
$$\sim \begin{bmatrix} 2 & 1 & 2 & : & 0 \\ 4 & 3 & k & : & 0 \\ 1 & k & 3 & : & 0 \end{bmatrix}$$
Reducing the system to row echelon form by the elementary row operations …
$$\sim \begin{bmatrix} 2 & 1 & 2 & : & 0 \\ 0 & 1 & k-4 & : & 0 \\ 0 & k-\frac{1}{2} & 2 & : & 0 \end{bmatrix} \quad \begin{bmatrix} R_2' = R_2 - 2R_1 \\ R_3' = R_3 - \frac{1}{2}R_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 2 & : & 0 \\ 0 & 1 & k-4 & : & 0 \\ 0 & 0 & 2-\left(k-\frac{1}{2}\right)(k-4) & : & 0 \end{bmatrix} \quad \left[R_3' = R_3 - (k-\tfrac{1}{2})R_2\right]$$
So,
$$2 - \left(k - \frac{1}{2}\right)(k-4) = 0$$
$$\Rightarrow -k^2 + \frac{9}{2}k = 0$$

$$\Rightarrow k(-k + \frac{9}{2}) = 0$$
$$\therefore \ k = 0, \frac{9}{2}$$

Example: A medicine company wishes to produce three types of medicine : type P, Q and R. To manufacture a type P medicine requires 2 minutes each on machine I and II and 3 minutes on machine III. A type of Q medicine requires 2 minutes on machine I, 3 minutes on machine II and 4 minutes of machine III. A type R medicine requires 3 minutes on machine I, 4 minutes on machine II and 3 minutes on machine III. There are 3.5 hours available on machine 1, 4.5 hours available on machine II and 5 hours available on machine III. How many medicine of each type should company make in order to use all the available time?

Solution:
Here, 3.5 hours $= 210$ minutes, 4.5 hours $=\ 270$ minutes and 5 hours $=\ 300$ minutes.

Let x, y and z be the number of medicines of types P, Q and R respectively. Then we have the following system of linear equations: $\begin{cases} 2x + 2y + 3z = 210 \\ 2x + 3y + 4z = 270 \\ 3x + 4y + 3z = 300 \end{cases}$

The augmented matrix of the above system is

$$\begin{bmatrix} 2 & 2 & 3 & : & 210 \\ 2 & 3 & 4 & : & 270 \\ 3 & 4 & 3 & : & 300 \end{bmatrix}$$

Reducing the system to echelon form by the elementary row operations

$$\sim \begin{bmatrix} 2 & 2 & 3 & : & 210 \\ 0 & 1 & 1 & : & 60 \\ 0 & 2 & -3 & : & -30 \end{bmatrix} \begin{bmatrix} r_2' = r_2 - r_1 \\ r_3' = 2r_3 - 3r_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 2 & 3 & : & 210 \\ 0 & 1 & 1 & : & 60 \\ 0 & 0 & -5 & : & -150 \end{bmatrix} [r_3' = r_3 - 2r_1]$$

Hence the solution of the above system is $x = 30, y = 30, z = 30$

Thus, the number of each type of medicine is 30.

Example: Determine the polynomial $p(x) = a_0 + a_1x + a_2x^2$ whose graph passes through the points $(1,4)$, $(2,0)$ and $(3,12)$.

Solution:
Given polynomial $p(x) = a_0 + a_1x + a_2x^2$
Substituting $x = 1, 2$ and 3 into $p(x)$ and the corresponding y – values produces the system of linear equations in the variables a_0, a_1 and a_2 shown below:

$$\begin{cases} p(1) = a_0 + a_1(1) + a_2(1)^2 = a_0 + a_1 + a_2 = 4 \\ p(2) = a_0 + a_1(2) + a_2(2)^2 = a_0 + 2a_1 + 4a_2 = 0 \\ p(3) = a_0 + a_1(3) + a_2(3)^2 = a_0 + 3a_1 + 9a_2 = 12 \end{cases}$$

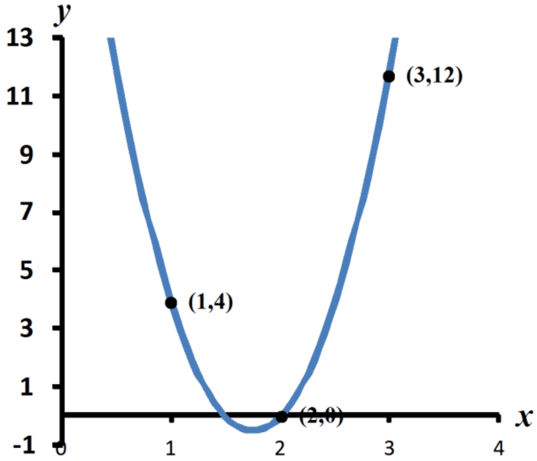
Reducing this system to echelon form by the elementary operations,

$$\begin{cases} a_0 + a_1 + a_2 = 4 \\ a_1 + 3a_2 = -4 \\ a_1 + 5a_2 = 12 \end{cases} \quad \begin{bmatrix} L_2{}' = L_2 - L_1 \\ L3' = L_3 - L_2 \end{bmatrix}$$

$$\begin{cases} a_0 + a_1 + a_2 = 4 \\ a_1 + 3a_2 = -4 \\ 2a_2 = 16 \end{cases} \quad \left[L_3{}' = L_3 - L_2\right]$$

By back substitution method from 3rd equation, we have $a_2 = 8$
From the 2nd equation, we get $a_1 + 24 = -4 \therefore a_1 = -28$
and from 1st equation, we get $a_0 - 28 + 8 = 4 \therefore a_0 = 24$
Hence, the solution of this system is $a_0 = 24$, $a_1 = -28$ and $a_2 = 8$.
So, the polynomial function is $p(x) = 24 - 28x + 8x^2$

The graph p is shown in the following figure:



Example: Find the polynomial that fits the points $(-2, 3), (-1, 5), (0, 1), (1, 4)$ and $(2, 10)$.

Solution: We have provided five points, so we chose a fourth-degree polynomial function

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \cdots\cdots\cdots\cdots\cdots\cdots (1)$$

Substitution the given points into $p(x)$ products the system of linear equations listed below:

$$\begin{aligned} a_0 - 2a_1 + 4a_2 - 8a_3 + 16a_4 &= 3 \\ a_0 - a_1 + a_2 - a_3 + a_4 &= 5 \\ a_0 &= 1 \\ a_0 + a_1 + a_2 + a_3 + a_4 &= 4 \\ a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 &= 10 \end{aligned}$$

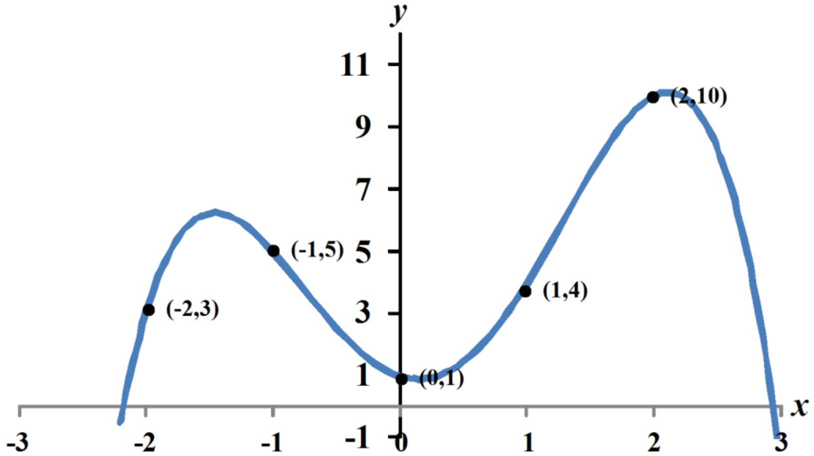
The solution of the above system is

$$a_0 = 1, \quad a_1 = -\frac{5}{24}, \quad a_2 = \frac{101}{24}, a_3 = \frac{18}{24}, a_4 = -\frac{17}{24}$$

Which means the polynomial function is

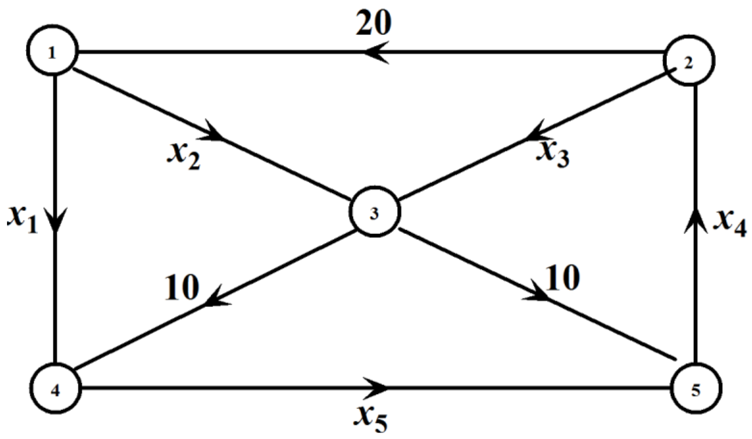
$$p(x) = 1 - \frac{5}{24}x + \frac{101}{24}x^2 + \frac{18}{24}x^3 - \frac{17}{24}x^4 = \frac{1}{24}(24 - 5x + 101x^2 + 18x^3 - 17x^4)$$

The graph of $p(x)$ is shown in the following figure:



Example:

Set up a system of linear equations to represent the network shown in the following figure and solve the system.



Solution:

Each of the network's five junctions gives rise to a linear equation, as shown below:

$x_1 + x_2 = 20$ junction 1

$x_3 + 20 = x_4$ junction 2

$x_2 + x_3 = 10 + 10$ junction 3

$x_1 + 10 = x_5$ junction 4

$x_5 + 10 = x_4$ junction 5

The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & : & 20 \\ 0 & 0 & 1 & -1 & 0 & : & -20 \\ 0 & 1 & 1 & 0 & 0 & : & 20 \\ 1 & 0 & 0 & 0 & -1 & : & -10 \\ 0 & 0 & 0 & -1 & 1 & : & -10 \end{bmatrix}$$

Reduce the system to echelon form by the elementary row operations

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & : & 20 \\ 0 & 1 & 1 & 0 & 0 & : & 20 \\ 0 & 0 & 1 & -1 & 0 & : & -20 \\ 0 & 0 & 0 & -1 & 1 & : & -10 \\ 1 & 0 & 0 & 0 & -1 & : & -10 \end{bmatrix} \quad \begin{bmatrix} R_2 \leftrightarrow R_3 \\ R_4 \leftrightarrow R_5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & : & 20 \\ 0 & 1 & 1 & 0 & 0 & : & 20 \\ 0 & 0 & 1 & -1 & 0 & : & -20 \\ 0 & 0 & 0 & -1 & 1 & : & -10 \\ 0 & -1 & 0 & 0 & -1 & : & -30 \end{bmatrix} \quad \left[R_5{}' = R_5 - R_1\right]$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & : & 20 \\ 0 & 1 & 1 & 0 & 0 & : & 20 \\ 0 & 0 & 1 & -1 & 0 & : & -20 \\ 0 & 0 & 0 & -1 & 1 & : & -10 \\ 0 & 0 & 1 & 0 & -1 & : & -10 \end{bmatrix} \quad \left[R_5{}' = R_5 + R_2\right]$$

$$\left[\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & : & 20 \\ 0 & 1 & 1 & 0 & 0 & : & 20 \\ 0 & 0 & 1 & -1 & 0 & : & -20 \\ 0 & 0 & 0 & 1 & -1 & : & 10 \\ 0 & 0 & 0 & 1 & -1 & : & 10 \end{array}\right] \quad \begin{bmatrix} R_4{}' = (-1)R_4 \\ R_5{}' = R_5 - R_3 \end{bmatrix}$$

$$\left[\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & : & 20 \\ 0 & 1 & 1 & 0 & 0 & : & 20 \\ 0 & 0 & 1 & 0 & -1 & : & -10 \\ 0 & 0 & 0 & 1 & -1 & : & 10 \\ 0 & 0 & 0 & 0 & 0 & : & 0 \end{array}\right] \quad \begin{bmatrix} R_3{}' = R_3 + R_4 \\ R_5{}' = R_5 - R_4 \end{bmatrix}$$

$$\left[\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & : & 20 \\ 0 & 1 & 0 & 0 & 1 & : & 30 \\ 0 & 0 & 1 & 0 & -1 & : & -10 \\ 0 & 0 & 0 & 1 & -1 & : & 10 \\ 0 & 0 & 0 & 0 & 0 & : & 0 \end{array}\right] \quad \left[R_2{}' = R_2 - R_3\right]$$

$$\left[\begin{array}{ccccccc} 1 & 0 & 0 & 0 & -1 & : & -10 \\ 0 & 1 & 0 & 0 & 1 & : & 30 \\ 0 & 0 & 1 & 0 & -1 & : & -10 \\ 0 & 0 & 0 & 1 & -1 & : & 10 \\ 0 & 0 & 0 & 0 & 0 & : & 0 \end{array}\right] \quad \left[R_1{}' = R_1 - R_2\right]$$

The corresponding system of equations are as follows:

$$\begin{cases} x_1 \qquad\qquad\qquad - x_5 = -10 \\ \qquad x_2 \qquad\qquad\quad x_5 = 30 \\ \qquad\qquad x_3 \qquad - x_5 = -10 \\ \qquad\qquad\qquad x_4 - x_5 = 10 \end{cases}$$

The above system is in echelon form having 4 equations in 5 unknowns. So, it has $(5-4)=1$ free variable, which is x_5.

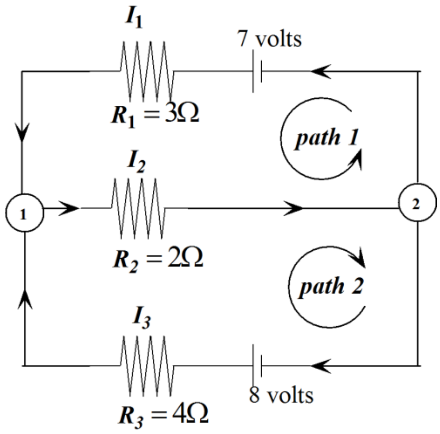
Let $x_5 = t,$ then by back substitution method, we have

$x_4 = t+10,\ x_3 = t-10,\ x_2 = 30-t,\ x_1 = t-10,$ where t is a real number.

So, this system has an infinite number of solutions.

Example:

Determine the currents I_1, I_2 and I_3 for the electrical network shown in the following figure.



Solution:

Applying Kirchhoff's current law to each junction produces

$I_1 + I_3 = I_2$; Junction 1 or junction 2

and applying Kirchhoff's second law to two paths produces

$R_1I_1 + R_2I_2 = 3I_1 + 2I_2 = 7$ path1

$R_2I_2 + R_3I_3 = 2I_2 + 4I_3 = 8$ path 2

So, we have the following system of three linear equations in the I_1, I_2 and I_3

$$\begin{cases} I_1 & -I_2 & +I_3 & = & 0 \\ 3I_1 & +2I_2 & & = & 7 \\ & 2I_2 & +4I_3 & = & 8 \end{cases}$$

The augmented matrix of the above system is

$$\begin{bmatrix} 1 & -1 & 1 & : & 0 \\ 3 & 2 & 0 & : & 7 \\ 0 & 2 & 4 & : & 8 \end{bmatrix}$$

Reducing the system to echelon form by the elementary row operations

$$\begin{bmatrix} 1 & -1 & 1 & : & 0 \\ 0 & 5 & -3 & : & 7 \\ 0 & 2 & 4 & : & 8 \end{bmatrix} \qquad \left[R_2^{/} = R_2 - 3R_1\right]$$

$$\begin{bmatrix} 1 & -1 & 1 & : & 0 \\ 0 & 5 & -3 & : & 7 \\ 0 & 0 & 26 & : & 26 \end{bmatrix} \qquad \left[R_3^{/} = 5R_3 - 2R_2\right]$$

$$\begin{bmatrix} 1 & -1 & 1 & : & 0 \\ 0 & 5 & -3 & : & 7 \\ 0 & 0 & 1 & : & 1 \end{bmatrix} \qquad \left[R_3^{/} = \frac{1}{26}R_3\right]$$

$$\begin{bmatrix} 1 & -1 & 0 & : & -1 \\ 0 & 5 & 0 & : & 10 \\ 0 & 0 & 1 & : & 1 \end{bmatrix} \qquad \begin{bmatrix} R_1^{/} = R_1 - R_3 \\ R_2^{/} = R_2 + 3R_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & : & -1 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 1 \end{bmatrix} \qquad \left[R_2^{/} = \frac{1}{5}R_2\right]$$

$$\begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 1 \end{bmatrix} \qquad \left[R_1^{/} = R_1 + R_2\right]$$

So, the solution of the above system is $I_1 = 1A$, $I_2 = 2A$ and $I_3 = 1A$.

Example:
Determine the currents I_1, I_2, I_3, I_4, I_5 and I_6 for the electrical network shown in the following figure.

Solution:
Applying Kirchhoff's current law to each junction produces

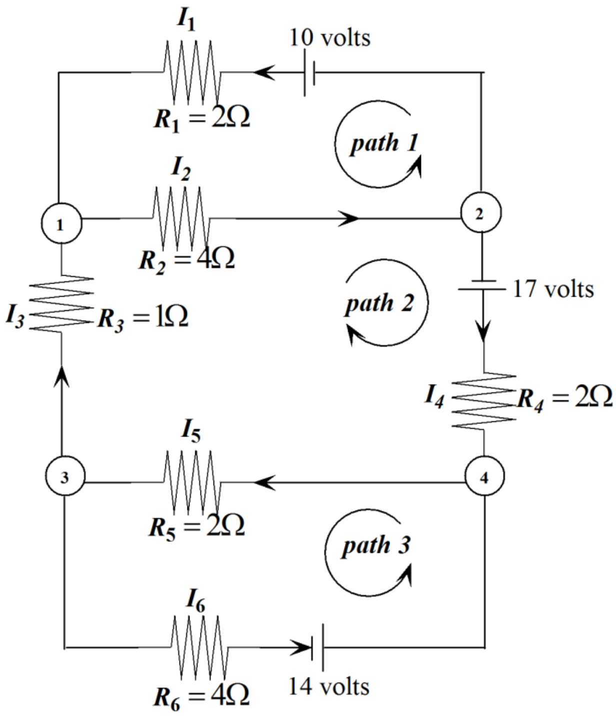
$I_1 + I_3 = I_2$ junction 1

$I_1 + I_4 = I_2$ junction 2

$I_3 + I_6 = I_5$ junction 3

$I_4 + I_6 = I_5$ junction 4

and applying Kirchhoff's second law to the three paths produces



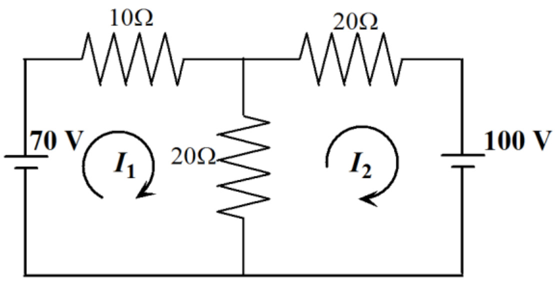
$$\begin{cases} 2I_1 + 4I_2 = 10 & \text{path 1} \\ 4I_2 + I_3 + 2I_4 + 2I_5 = 17 & \text{path 2} \\ 2I_5 + 4I_6 = 14 & \text{path 3} \end{cases}$$

Now we have the following system of seven linear equations in the variables I_1, I_2, I_3, I_4, I_5 and I_6

$$\begin{cases} I_1 & -I_2 & +I_3 & & & & = & 0 \\ I_1 & -I_2 & & +I_4 & & & = & 0 \\ & & I_3 & & -I & +I_6 & = & 0 \\ & & & I_4 & -I_5 & +I_6 & = & 0 \\ 2I_1 & +4I_2 & & & & & = & 10 \\ & 4I_2 & +I_3 & +2I_4 & +2I_5 & & = & 17 \\ & & & & 2I_5 & +4I_6 & = & 14 \end{cases}$$

Using Gauss-Jordan elimination method, we have solution of the above system is
$I_1 = 1, I_2 = 2, I_3 = 1, I_4 = 1, I_5 = 3$ and $I_6 = 2$
which means $I_1 = 1$amp, $I_2 = 2$ amp and $I_3 = 1$amp. $I_4 = 1$amp, $I_5 = 3$amp and $I_6 = 2$ amp.

Example:
Determine the loop currents I_1 and I_2 of the following circuit using mesh analysis.



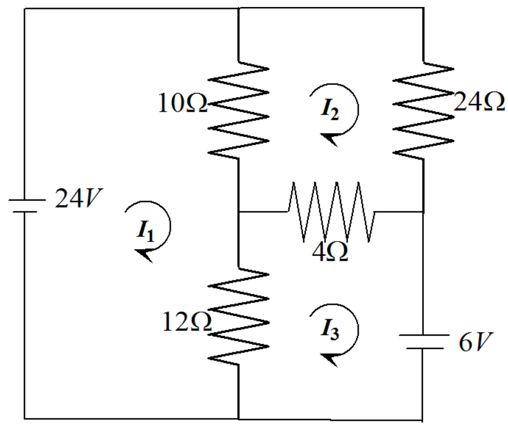
Solution:
For loop 1: $10I_1 + 20I_1 - 20I_2 = 70$
$\Rightarrow 30I_1 - 20I_2 = 70$
$\Rightarrow 3I_1 - 2I_2 = 7$

For loop 2: $20I_2 + 20I_2 - 20I_1 = -100$
$\Rightarrow -20I_1 + 40I_2 = -100$
$\Rightarrow I_1 - 2I_2 = 5$

Thus, the system of linear equations is
$3I_1 - 2I_2 = 7$
$I_1 - 2I_2 = 5$
Therefore, $I_1 = 1\ A$ and $I_2 = -2A$

Example:
Determine the loop currents I_1, I_2 and I_3 of the following circuit using mesh analysis.

Solution:



For loop 1: $10I_1 + 12I_1 - 10I_2 - 12I_3 = 24$

$\Rightarrow 22I_1 - 10I_2 - 12I_3 = 24$

$\Rightarrow 11I_1 - 5I_2 - 6I_3 = 12$

For loop 2: $24I_2 + 10I_2 + 4I_2 - 10I_1 - 4I_3 = 0$

$\Rightarrow -10I_1 + 38I_2 - 4I_3 = 0$

$\Rightarrow 5I_1 - 19I_2 + 2I_3 = 0$

For loop 3: $12I_3 - 12I_1 - 4I_2 = -6$

$\Rightarrow 12I_1 + 4I_2 - 12I_3 = 6$

$\Rightarrow 6I_1 + 2I_2 - 6I_3 = 3$

Thus, the system of linear equation is

$11I_1 - 5I_2 - 6I_3 = 12$

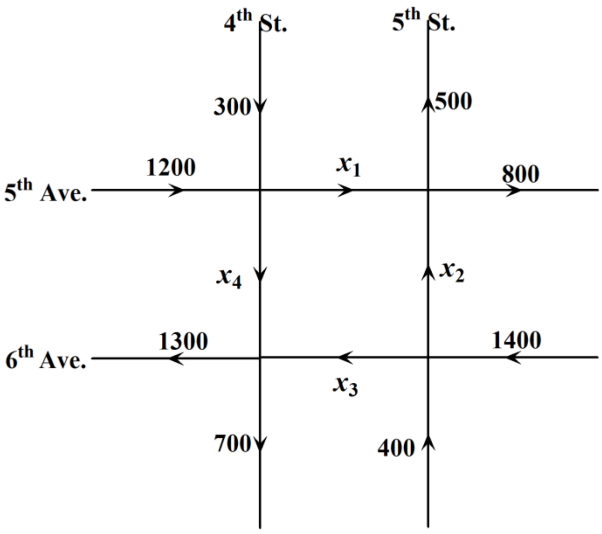
$5I_1 - 19I_2 + 2I_3 = 0$

$6I_1 + 2I_2 - 6I_3 = 3$

Hence solving the system, $I_1 = 3.7\ A,\ I_2 = 1.35\ A, I_3 = 3.66\ A.$

Example:

The following figure shows the flow of downtown traffic in a current city during the rush hours on a typical weekday. The arrows indicate the direction of traffic flow on each-way road, and the average number of vehicles per hour entering and leaving each intersection appears beside each road. 5th Avenue and 6th Avenue can each handle up to 2000 vehicles per hour without causing congestion, whereas the maximum capacity of both 4th street and 5th street is 1000 vehicles per hour. The flow of traffic is controlled by traffic lights installed at each of the four intersections.



(a) Write a general expression involving the rates of flow x_1, x_2, x_3, x_4 and suggest two possible flow patterns that will ensure no traffic congestion.

(b) Suppose the part of 4th street between 5th Avenue and 6th Avenue is to be resurfaced and that traffic flow between the two junctions must therefore be reduced to at most 300 vehicles per hour. Find two possible flow patterns that will result in a smooth flow of traffic.

Solution:

(a) To avoid congestion, all traffic entering an intersection must also leave that intersection. Applying this condition to each of the four intersections in a clockwise direction beginning with the 5th Avenue and 4th Street intersection, we obtain the following equations:

$$\begin{aligned} 1500 &= x_1 + x_4 \\ 1300 &= x_1 + x_2 \\ 1800 &= x_2 + x_3 \\ 2000 &= x_3 + x_4 \end{aligned}$$

This system of four linear equations in the four variables x_1, x_2, x_3, x_4 may be written in the more standard form

$$\begin{array}{cccccc} x_1 & & & + x_4 & = & 1500 \\ x_1 & + x_2 & & & = & 1300 \\ & x_2 & + x_3 & & = & 1800 \\ & & x_3 & + x_4 & = & 2000 \end{array}$$

Using Gauss-Jordan elimination method, we obtain

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1500 \\ 1 & 1 & 0 & 0 & 1300 \\ 0 & 1 & 1 & 0 & 1800 \\ 0 & 0 & 1 & 1 & 2000 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1500 \\ 0 & 1 & 0 & -1 & -200 \\ 0 & 1 & 1 & 0 & 1800 \\ 0 & 0 & 1 & 1 & 2000 \end{array}\right] \qquad [R_2{}' = R_2 - R_1]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1500 \\ 0 & 1 & 0 & -1 & -200 \\ 0 & 0 & 1 & 1 & 2000 \\ 0 & 0 & 1 & 1 & 2000 \end{array}\right] \qquad [R_3{}' = R_3 - R_2]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1500 \\ 0 & 1 & 0 & -1 & -200 \\ 0 & 0 & 1 & 1 & 2000 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad [R_4{}' = R_4 - R_3]$$

The last augmented matrix is in row-reduced form and is equivalent to a system of three linear equations in the four variables x_1, x_2, x_3, x_4. This we may express three of the variables-say, x_1, x_2, x_3 in terms of x_4. Setting $x_4 = t$ (t a parameter), we may write the infinitely many solutions of the system as

$$x_1 = 1500 - t$$
$$x_2 = -200 + t$$
$$x_3 = 2000 - t$$
$$x_4 = t$$

Observe that for a meaningful solution we must have $200 \le t \le 1000$ since x_1, x_2, x_3 and x_4 must all be nonnegative and the maximum capacity of a street is 1000.
For example, picking $t = 300$ gives the flow pattern

$$x_1 = 1200,\ x_2 = 100,\ x_3 = 1700,\ x_4 = 300$$

Selecting $t = 500$ gives the flow pattern

$$x_1 = 1000,\ x_2 = 300,\ x_3 = 1500,\ x_4 = 500$$

(b) In this case, x_4 must not exceed 300. Again, using results of part(a), we find, upon setting $x_4 = t = 300,$ the flow pattern $x_1 = 1200, x_2 = 100, x_3 = 1700, x_4 = 300$ obtained earlier.

(c) Picking $t = 250$ gives the flow pattern $x_1 = 1250, x_2 = 50, x_3 = 1750, x_4 = 250.$

Linear Programming Problem:

The linear programming is the modern method of mathematics to solve the system of linear inequalities. The solution makes the objective linear function a minimum (or maximum) and which satisfies the constraints and non-negative conditions.
General linear programming problems:
Let be Z a linear function by $Z = \sum_{i=1}^{n} c_i x_i \dots \dots \dots \dots \dots \dots \dots (i)$
where c_i is set of n constants.
Let a_{ij} be mn constants and b_i be a set of m constants such that

$$a_{11}x_1 + a_{12}x_2 + \dots \dots \dots \dots \dots \dots + a_{1n}x_n (\leq, =, \geq)\ b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots \dots \dots \dots \dots \dots + a_{2n}x_n (\leq, =, \geq)\ b_2$$
$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$
$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots \dots \dots \dots \dots \dots + a_{mn}x_n (\leq, =, \geq)\ b_m$$

and finally let $x_1 \geq 0, x_2 \geq 0, \dots \dots \dots \dots, x_n \geq 0 \dots \dots \dots \dots \dots . . (3)$

(i) The problem of solving the values of $x_1, x_2, \dots, x_n$ which make Z a minimum (or maximum) and which satisfies equations (2) and (3) is called general linear programming.
(ii) $Z = c_1x_1 + c_2x_2 + \dots \dots \dots + c_nx_n$ is called objective function.
(iii) System of linear inequalities eq$^n(2)$ is called constrains and in eq$^n(3)$ $x_i \geq 0$ is called non negative restriction.
(iv) Solutions: Values of unknowns $x_1, x_2, \dots, x_n$ which the constraints eq$^n(2)$ of a general linear programming problem are called general solutions.
(v) Feasible solution: Any solution if GLPP which satisfies the non-negative restrictions of the problem is called feasible solution of GLPP.
(vi) Optimum solution: Any feasible solution which optimizes (minimizes, maximizes) the objective function is called optimum solution.

Linear programming problem can be solved by (i) Graphical method, (ii) Simplex method (Pivoting method).

Example: A company manufactures and sells two models of lamps, L1 and L2. To manufacture each lamp, the manual work involved in model L1 is 20 minutes and for L2, 30 minutes. The mechanical (machine) work involved for L1 is 20 minutes and for L2, 10 minutes. The manual work available per month is 100 hours and the machine is limited to only 80 hours per month. Knowing that the profit per unit is 15 and 10 for L1 and L2, respectively, determine the quantities of each lamp that should be manufactured to obtain the maximum benefit.

Solution:

Let
x = number of lamps L1
y = number of lamps L2

Objective function $f(x, y) = 15\,x + 10\,y$

Convert the time from minutes to hours.

20 min = 1/3 h 30 min = 1/2 h 10 min = 1/6 h

	L1	L2	Time
Manual	1/3	1/2	100
Machine	1/3	1/6	80

Writing the constraints as a system of inequalities we get

$$\frac{1}{3}x + \frac{1}{2}y \leq 100$$

$$\frac{1}{3}x + \frac{1}{6}y \leq 80$$

As the numbers of lamps are natural numbers, we have $x \geq 0$ & $y \geq 0$

Represent the constraints graphically.

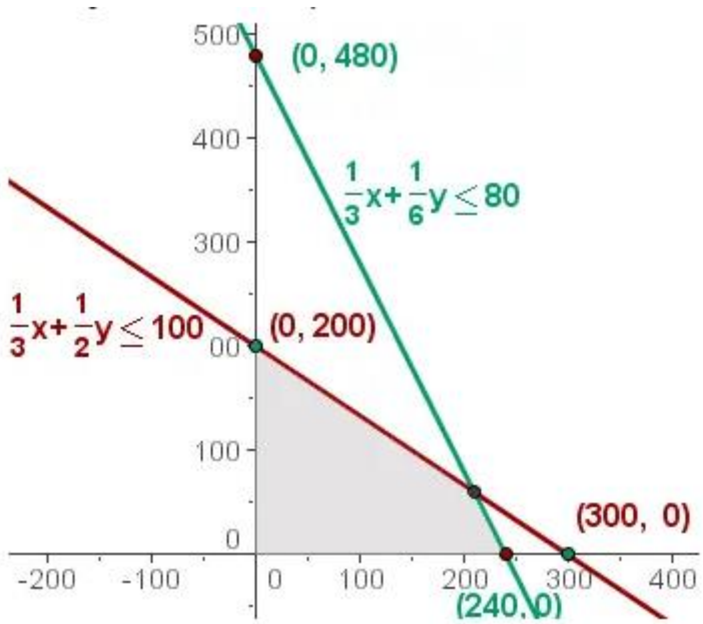
As $x \geq 0$ & $y \geq 0$, work in the first quadrant.

Solve the inequation graphically: $\frac{1}{3}x + \frac{1}{2}y \leq 100$; and take a point on the plane, for example (0,0).

$$\frac{1}{3} \cdot 0 + \frac{1}{2} \cdot 0 \leq 100$$

$$\frac{1}{3} \cdot 0 + \frac{1}{6} \cdot 0 \leq 80$$

The area of intersection of the solutions of the inequalities would be the solution to the system of inequalities, which is the set of feasible solutions.



The optimal solution, if unique, is a vertex. These are the solutions to systems:

$$\frac{1}{3}x + \frac{1}{2}y = 100; \; x = 0 \; (0, 200)$$

$$\frac{1}{3}x + \frac{1}{6}y = 80; \; y = 0(240, 0)$$

$$\frac{1}{3}x + \frac{1}{2}y = 100; \frac{1}{3}x + \frac{1}{6}y = 80(210, 60)$$

To determine which of them has the maximum or minimum values.

In the objective function, place each of the vertices that were determined in the previous step.

$$f(x, y) = 15x + 10y$$

$$f(0, 200) = 15 \cdot 0 + 10 \cdot 200 = 2000$$

$$f(240{,}0) = 15240 + 100 = 3{,}600$$

$$f(210, 60) = 15 \cdot 210 + 10 \cdot 60 = 3750$$

So, (210,60) is our required answer.

Example: A calculator company manufactures two types of calculator: a handheld calculator and a scientific calculator. Statistical data projects that there is an expected demand of at least 100 scientific and 80 handheld calculators each day. Since the company has certain limitations on the production capacity, the company can only manufacture 200 scientific and 170 handheld calculators per day. The company has received a contract to deliver a minimum of 200 calculators per day. If there is a loss of 2 taka on each scientific calculator that you sold and a profit of 5 taka on each handheld calculator, then how many calculators of each type the company should manufacture daily to maximize the net profit?

Solution: To solve this problem, let's first formulate it properly by following the steps.

Step 1: Identify the number of decision variables.

In this problem, since we have to calculate how many calculators of each type should be manufactured daily to maximize the net profit, the number of scientific and handheld calculators each are our decision variables.

Consider,

number of scientific calculators manufactured = x

number of handheld calculators manufactured = y

Step 2: Identify the constraints on the decision variables.

The lower bound, as mentioned in the problem (there is an expected demand of at least 100 scientific and 80 handheld calculators each day) are as follows.

Hence, $x \geq 100$ and $y \geq 80$.

The upper bound owing to the limitations mentioned the problem statement (the company can only manufacture 200 scientific and 170 handheld calculators per day) are as follows:

Hence, $x \leq 200$ and $y \leq 170$.

In the problem statement, we can also see that there is a joint constraint on the values of x and y due to the minimum order on a shipping consignment that can be written as:

$$x + y \geq 200$$

Step 3: Write the objective function in the form of a linear equation.

In this problem, it is clearly stated that we have to optimize the net profit. As stated in the problem(If there is a loss of 2 taka on each scientific calculator that you sold and a profit of 5 taka on each handheld calculator), the net profit function can be written as:

Profit (P) = $\boldsymbol{-2x + 5y}$

Step 4: Explicitly state the non-negativity restriction.

Since the calculator company cannot manufacture a negative number of calculators.

Hence, $\boldsymbol{x \geq 0}$ and $\boldsymbol{y \geq 0}$

Since we have formulated the problem, let's convert the problem into a mathematical form to solve it further.

Maximization of P = $\boldsymbol{-2x + 5y}$

subject to:

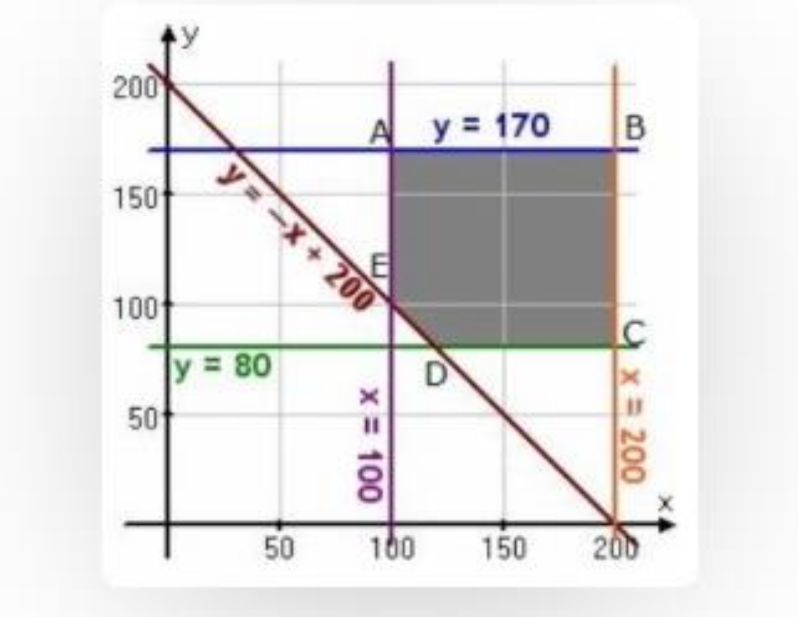
$$100 \leq x \leq 200$$
$$80 \leq y \leq 170$$
$$x + y \geq 200$$

Step 5: Plot the constraints on the graph.

Let's plot all the constraints defined in step 2 on a graph in a similar manner as we plot an equation.

Step 6: Highlight the feasible region on the graph.

After plotting the coordinates on the graph, shade the area that is outside the constraint limits (which is not possible). The highlighted feasible area will look like this:



Step 7: Find the coordinates of the optimum point.

To find the coordinates of the optimum point, we will solve the simultaneous pair of linear equations.

Corner Points	**Equation, P = $-2x + 5y$**
A (100, 170)	**P = 650**
B (200, 170)	**P = 450**
C (200, 80)	**P = 0**
D (120, 80)	**P = 160**
E (100, 100)	**P = 300**

Step 8: Find the optimum point.

The above table shows that the maximum value of P is **650** that is obtained at

(x, y) = A (100, 170).

Cryptographically Problem:

The process to write (encoded) and read (decoded) any secret messages by using matrices is known as Cryptography.

Specific Aims: We will

- **be able to encode a message using matrix multiplication;**
- **decode a coded message using the matrix inverse and matrix multiplication.**

Algorithm to Encode a Message:

- Assign the numbers 1-26 to the letters (capital/small) in the alphabet given below and assign the number 0 to a blank to provide for space between words.

Blank	A	B	C	D	E	F	G	H	I	J	K	L	M	N
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

O	P	Q	R	S	T	U	V	W	X	Y	Z
15	16	17	18	19	20	21	22	23	24	25	26

- Write the provided message corresponds to the sequence of numbers.
- Matrix, "A" (say) can be used as an encoding matrix, if elements are positive integer of the considered matrix and inverse matrix exists.
- Divide the numbers in the sequence into groups of the order of matrix, "A" (or size of the matrix) and use these groups as the columns of a matrix, "B" (say). Proceed down the columns not across the rows.
- Write the provided message corresponds to the sequence. Then, multiply this matrix, "B" on the left by matrix, "A".
- Coded message will be written by picking the elements in each column from left of the matrix "AB".

Algorithm to Decode a Message:

- Find the inverse of encoding matrix, "A", if exists.
- Divide the numbers in the sequence into groups of the order of matrix, "A" (or size of the matrix) and use these groups as the columns of a matrix, "B" (say). Proceed down the columns not across the rows.
- Multiply this matrix, "B" on the left by inverse matrix, "A^{-1}".
- Writing the numbers in the columns of this matrix "$A^{-1}B$" in sequence and using the letters to correspondence numbers given below.

O	P	Q	R	S	T	U	V	W	X	Y	Z	Blank
15	16	17	18	19	20	21	22	23	24	25	26	**0**

A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	2	3	4	5	6	7	8	9	10	11	12	13	14

- These letters give decoded message.

Example: Encode the message **SECRET CODE** by using matrix $A = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$.

Solution:

Step 1: The provided message "**SECRET CODE** " corresponds to the sequence

S	**E**	**C**	**R**	**E**	**T**	**space**	**C**	**O**	**D**	**E**
19	**5**	**3**	**18**	**5**	**20**	**0**	**3**	**15**	**4**	**5**

Step 2: Divide these numbers in the sequence into groups of 2 (based on the size of given matrix) and use these groups as the columns (proceed down the columns) of a matrix, B of two rows. Thus,

$$B = \begin{bmatrix} 19 & 3 & 5 & 0 & 15 & 5 \\ 5 & 18 & 20 & 3 & 4 & 0 \end{bmatrix}$$

Step 3: Now,

$$AB = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 19 & 3 & 5 & 0 & 15 & 5 \\ 5 & 18 & 20 & 3 & 4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4(19)+3(5) & 4(3)+3(18) & 4(5)+3(20) & 4(0)+3(3) & 4(15)+3(4) & 4(5)+3(0) \\ 1(19)+1(5) & 1(3)+1(18) & 1(5)+1(20) & 1(0)+1(3) & 1(15)+1(4) & 1(5)+1(0) \end{bmatrix}$$

$$AB = \begin{bmatrix} 91 & 66 & 80 & 9 & 72 & 20 \\ 24 & 21 & 25 & 3 & 19 & 5 \end{bmatrix}$$

Step 4: Therefore, the coded message is **91 24 66 21 80 25 9 3 72 19 20 5.**

Example: The encoded message is 7 **6 28 20 23 5**. Decode this message by using matrix, $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Solution:

Step 1: When elements of the encoding matrix, A are positive and find inverse matrix of A.

$$A^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

Step 2: The encoded message is 7 **6 28 20 23 5.**

Since the encoding matrix, A is 2×2, make a matrix "C" having two rows by picking two numbers from the left of encoded message as columns of matrix "C". We have,

$$C = \begin{bmatrix} 7 & 28 & 23 \\ 6 & 20 & 5 \end{bmatrix}$$

Step 3: Now,

$$A^{-1}C = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 7 & 28 & 23 \\ 6 & 20 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 20 & 5 \\ 1 & 8 & 18 \end{bmatrix}$$

Step 4: Writing the numbers in the columns of this matrix in sequence and using the letters to correspondence numbers. Thus,

6 1 20 8 5 18

F A T H E R

Exercise 2

1. Test whether the systems of linear equations are consistent. If consistent find the solutions of the system. Also check your answer by direct substitution.

a.
$$\begin{aligned} x_1 - 3x_2 &= 2 \\ -3x_1 + 9x_2 &= 3 \end{aligned}$$
Ans: Inconsistent

b.
$$\begin{aligned} 2x_1 - 3x_2 &= -2 \\ 2x_1 + x_2 &= 1 \\ 3x_1 + 2x_2 &= 1 \end{aligned}$$
Ans: Inconsistent

c.
$$\begin{aligned} -2x_2 + 3x_3 &= 1 \\ 3x_1 + 6x_2 - 3x_3 &= -2 \\ 6x_1 + 6x_2 + 3x_3 &= 5 \end{aligned}$$
Ans: Inconsistent

d.
$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 5 \\ -x_1 + 3x_2 + 5x_3 &= 3 \\ -4x_1 + 2x_2 - 6x_3 &= 3 \end{aligned}$$
Ans: Inconsistent

e.
$$\begin{aligned} -3x + 2y - 3z &= -8 \\ 2x - y + z &= 4 \\ x + 2y - 4z &= -2 \end{aligned}$$
Ans. $(2,\ 2,\ 2)$

f.
$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$
Ans: $(3, 1,\ 2)$

g.
$$\begin{aligned} x + y + 2z &= 1 \\ y + z &= 1 \\ -2x + 3y + z &= 3 \end{aligned}$$
Ans. $(-1, 0, 1)$

h.
$$\begin{aligned} x + 2y + 4z &= -13 \\ 3x - y + z &= 5 \\ x + 3y &= 3 \end{aligned}$$
Ans. $(3,\ 0,\ -4)$

i.
$$\begin{aligned} x_1 - x_2 + 2x_3 &= 5 \\ 2x_1 + x_2 - x_3 &= 2 \\ 2x_1 - x_2 - x_3 &= 4 \end{aligned}$$
Ans: $(2, -1, 1)$

j.
$$\begin{aligned} 2x - y - 5z &= 4 \\ x + y + z &= -3 \\ -x - 4y + z &= 4 \end{aligned}$$
Ans: $(-1, -1, -1)$

k.
$$\begin{aligned} 5x + 3y &= 19 \\ 2x - 7y &= -17 \end{aligned}$$
Ans: $(2, 3)$

l.
$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 2 \\ -x_1 + x_2 - 2x_3 &= -1 \\ x_1 - 4x_2 - 2x_3 &= 8 \end{aligned}$$
Ans: $(2, -1, -1)$

m.
$$\begin{aligned} 2x - 3y + 4z &= 8 \\ 3x + 4y - 5z &= -4 \\ 5x - 7y + 6z &= 9 \end{aligned}$$
Ans: $(1, 2, 3)$

n.
$$\begin{aligned} 3x + y + 2z &= 14 \\ 2y + 5z &= 22 \\ 2x + 5y - z &= -22 \end{aligned}$$
Ans: $(2, -4, 6)$

o.
$$\begin{aligned} x + y + z &= 0 \\ x - y + z &= 0 \\ x + y - z &= 0 \end{aligned}$$
Ans: $(0,\ 0, 0)$

p.
$$\begin{aligned} -x + y - z &= 0 \\ 3x - y + 2z &= -2 \\ 2x + 4y + 3z &= 2 \end{aligned}$$
Ans: $(-2, 0, 2)$

q.
$$\begin{aligned} x + 2y + 3z &= 1 \\ x + 3y + 6z &= 3 \\ 2x + 6y + 13z &= 5 \end{aligned}$$
Ans: $(-6, 5, -1)$

r.
$$\begin{aligned} x + y + z &= -1 \\ x - y + z &= -5 \\ 2x + y - z &= 5 \end{aligned}$$
Ans: $(0, 2, -3)$

s.
$$\begin{aligned} x + 2y + 3z &= 5 \\ 2x + 5y + 3z &= 3 \\ x + 8z &= 17 \end{aligned}$$
Ans: $(1, -1, 2)$

t.
$$\begin{aligned} 3x + 2y - z &= -15 \\ 5x + 3y + 2z &= 0 \\ 3x + y + 3z &= 11 \\ -6x - 4y + 2z &= 30 \end{aligned}$$
Ans: $(-4, 2, 7)$

u.
$$\begin{aligned} -x + 2y + 2z &= -2 \\ 3x + 2y - z &= 9 \\ x + 4y + z &= 5 \end{aligned}$$
Ans: $(2, 1, -1)$

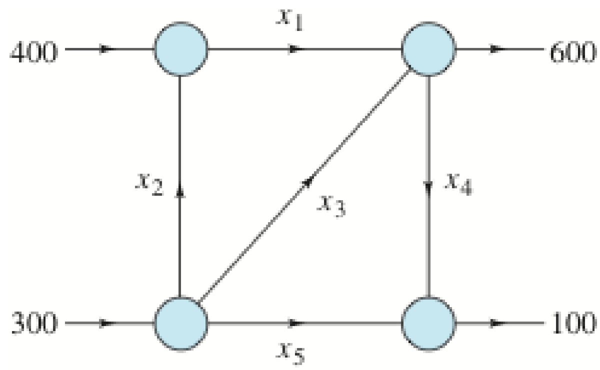
2. Determine the value(s) of λ such that the following system of linear equations has (i) no solution, (ii) more than one solution, and (iii) a unique solution.
$$\begin{cases} \lambda x + y + z = 1 \\ x + \lambda y + z = 1 \\ x + y + \lambda z = 1 \end{cases}.$$
Ans: (i) $\lambda = -2$, (ii) $\lambda = 1$, (iii) $\lambda \neq 1, \lambda \neq -2$.

3. Determine the value(s) of λ such that the following system of linear equations has (i) no solution, (ii) more than one solution, and (iii) a unique solution.
$$\begin{cases} x - 3z = -3 \\ 2x + \lambda y - z = -2 \\ x + 2y + \lambda z = 1 \end{cases}.$$
Ans: (i) $\lambda = -5$, (ii) $\lambda = 2$, (iii) $\lambda \neq 2, \lambda \neq -5$.

4. Determine the value(s) of λ and μ such that the following system of linear equations has (i) no solution, (ii) more than one solutions and (iii) a unique solution.
$$\begin{aligned} x + y + z &= 2 \\ x + 3y + \lambda z &= 6. \\ x + 2y + 3z &= \mu \end{aligned}$$
Ans: (i) $\lambda = 5,\ \mu \neq 4$, (ii) $\lambda = 5,\ \mu = 4$, (iii) $\lambda \neq 5$.

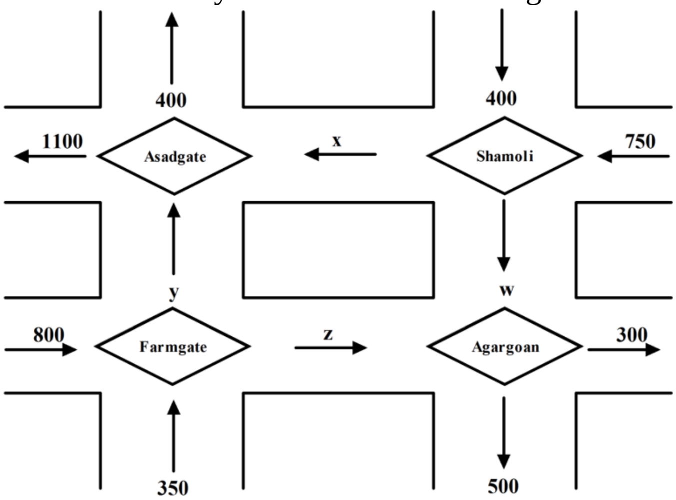
5. Determine the value(s) of λ such that the following system of linear equations has (i) no solution, (ii) more than one solution, and (iii) a unique solution.
$$\begin{cases} x + y + \lambda z = 1 \\ x + \lambda y + z = \lambda \\ \lambda x + y + z = \lambda^2 \end{cases}.$$
Ans: (i) $\lambda = -2$, (ii) $\lambda = 1$, (iii) $\lambda \neq 1, \lambda \neq -2$.

6. The network in the figure below shows the traffic flow (in vehicles per hour) over the several one-way streets. Determine the general flow pattern for the network.



Ans: Here x_3 and x_5 are free variables.

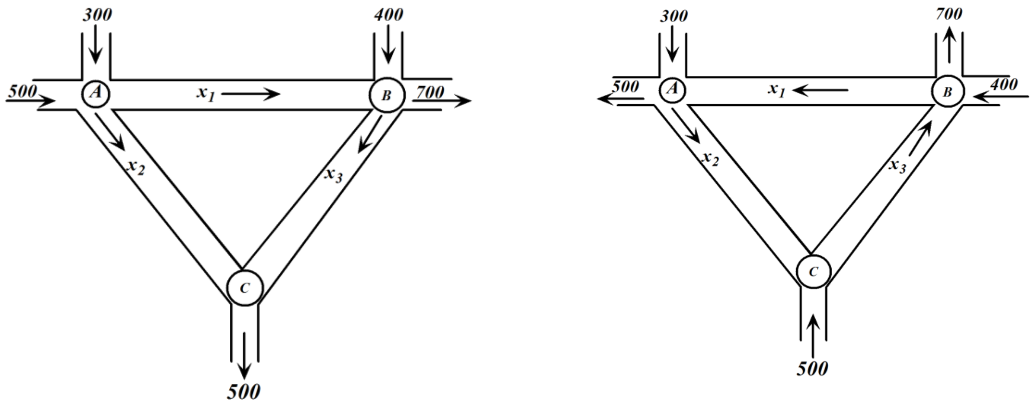
7. The network in the figure below shows the traffic flow (in vehicles per hour) over the several one-way streets. Determine the general flow pattern for the network.



8. The network in the figure below shows the traffic flow (in vehicles per hour) over the several one-way streets. Determine the general flow pattern for the network.

a.

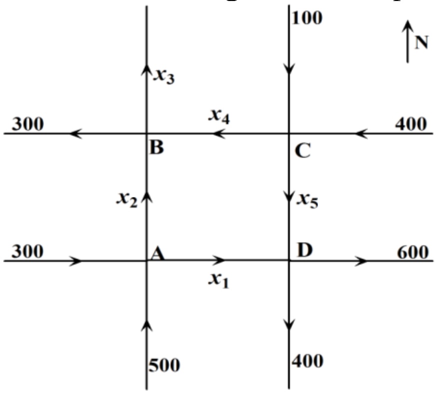
b.



Ans: (a) Here x_3 is the free variable. The system has an infinite number of solutions. But to remove negativity x_3 must be between 0 to 500.

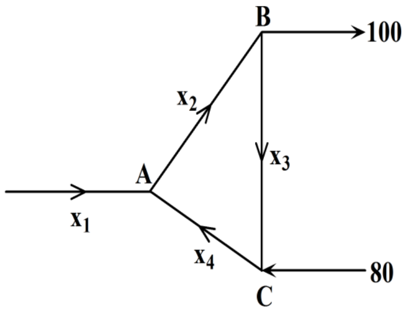
(b) Here x_3 is the free variable. The system has an infinite number of solutions. But to remove negativity x_3 must be greater than 500.

9. The network in the figure below shows the traffic flow (in vehicles per hour) over the several one-way streets. Determine the general flow pattern for the network.



Ans: Here x_5 is the free variable. The system has an infinite number of solutions. But to remove negativity x_5 must start from 200.

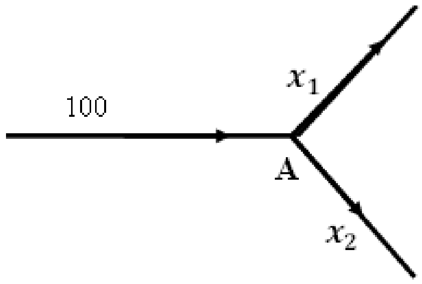
10. Find the general flow pattern of the network system in the figure. Assuming that the flows are all nonnegative, what is the smallest possible value for x_4.



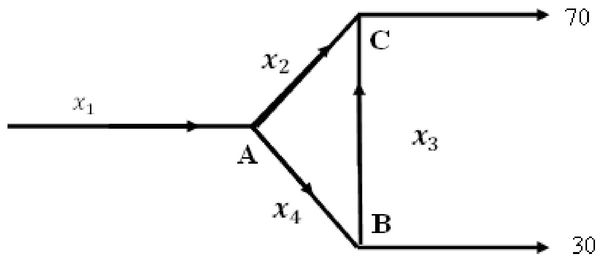
11. A company produces three products x, y, z every day. Their total production on a certain day is 45 tons. It is found that the production of z exceeds the production of x by 8 tons while the total production of 'x' and 'z' is twice the production of 'y'. Determine the production level of each product.

   Ans: $x = 1;\ y = 15;\ z = 19.$

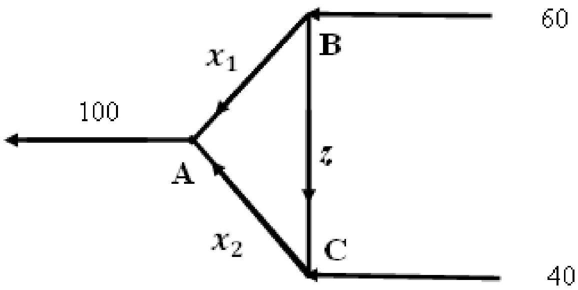
12. Construct the system of linear equations from the following diagrams, reduced the system to echelon form and finally find the general flow pattern, where x_i is the number of cars.



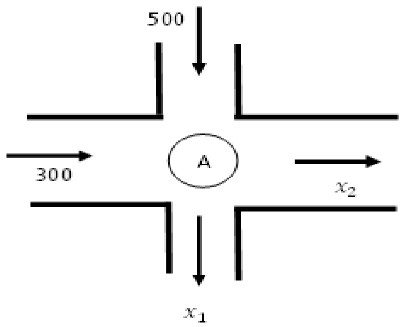
(i)



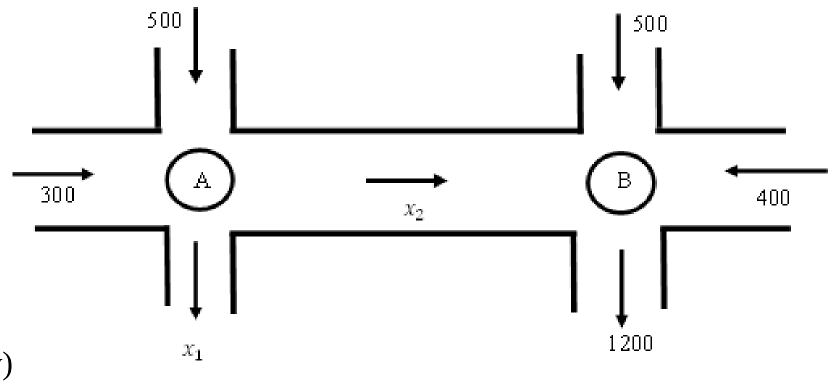
(ii)



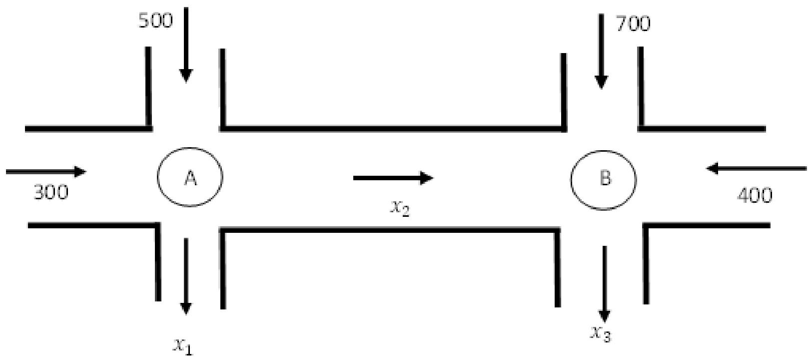
(iii)



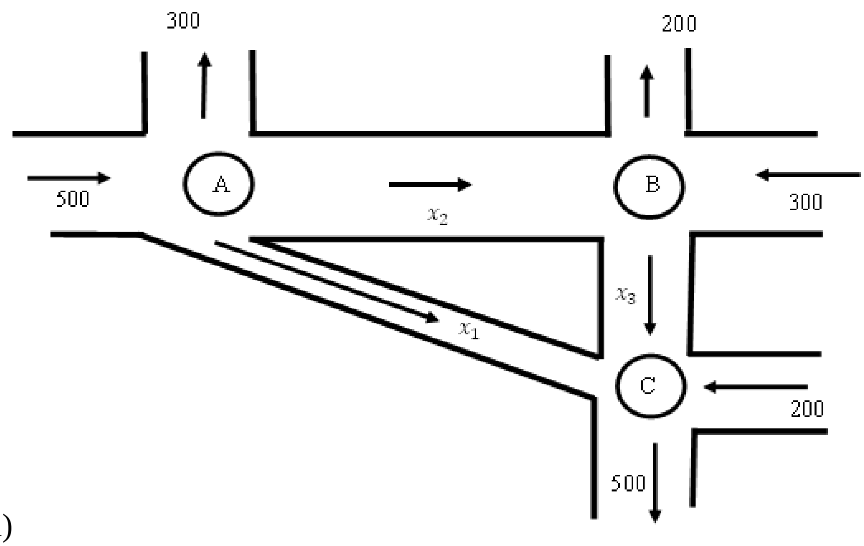
(iv)



(v)

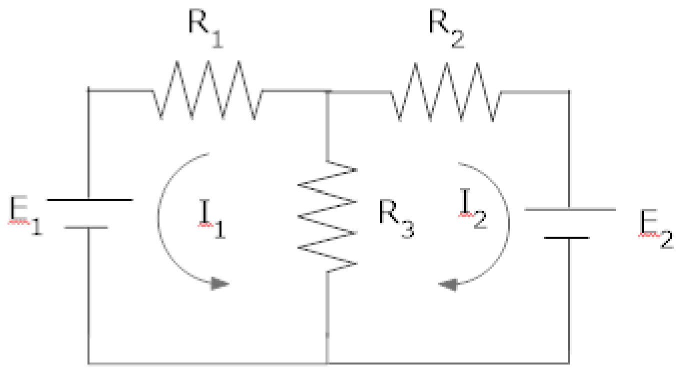


(vi)



(vii)

13. Determine the loop currents I_1 and I_2 of the following circuit using mesh analysis.
Where $E_1 = 28$, $E_2 = 7$ V, $R_1 = 4\ \Omega$, $R_2 = 1\Omega$, $R_3 = 2\Omega$.



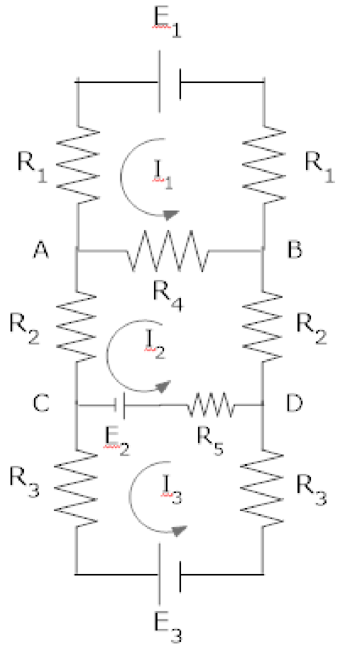
Answer: $I_1 = -5\ A$, $I_2 = 1\ A$.

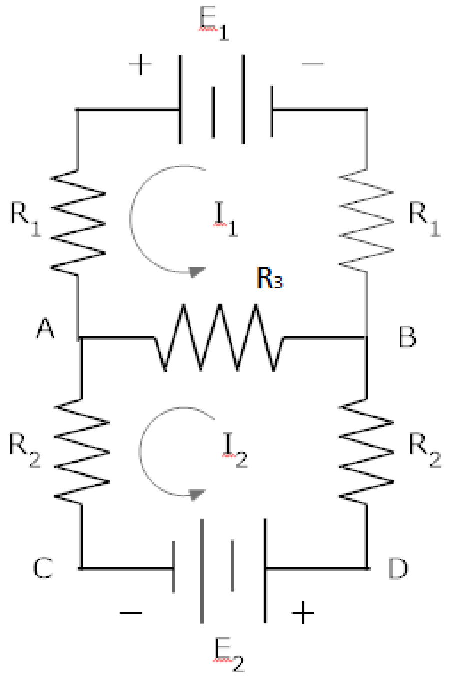
14. Determine the loop currents using mesh analysis.

(a) $E_1 = 15$ V, $E_2 = 6$ V, $E_3 = 10$ V, $R_1 = 1\ \Omega$, $R_2 = 2\ \Omega$, $R_3 = 2\Omega$, $R_4 = 2\Omega$, $R_5 = 1\Omega$.

(b) $E_1 = 15$ V, $E_2 = 6$ V, $R_1 = 1\Omega$, $R_2 = 2\Omega$, $R_3 = 2\Omega$.

a) b)





Ans: (a) $I_1 = 4.638\ A, I_2 = 1.776\ A,\ I_3 = -2.844\ A,$ (b) $I_1 = 5.1\ A, I_2 = 2.7A.$

15. Find the minimum and maximum values of the given objective function, subject to the indicated constraints.
Objective function:
$$z = 3x + 2y$$
Constraints:
$$x \geq 0$$
$$y \geq 0$$
$$x + 3y \leq 15$$
$$4x + y \leq 16$$
16. Find the minimum and maximum values of the given objective function, subject to the indicated constraints.
Objective function:
$$z = 4x + 3y$$
Constraints:
$$x \geq 0$$
$$y \geq 0$$
$$2x + 3y \geq 6$$
$$3x - 2y \leq 9$$
$$x + 5y \leq 20$$

17. The company for production of electronic chips produces 2 typesof graphics cards (C1, C2), that are produced from 2 types of machines (M1, M2). The M1 machine produces a C1 graphics card for 1 min, C2 for 2 min. The M2 machine produces a C1 graphics card for 3 min, C2 for 2 min. TheC1 graphics card is sold for

\$20, C2 for \$35. Determine the number of graphics cards (x for C1, y for C2) which maximizes the revenue generated from the production of graphics cards for an hour.

(Hints: $x + 2y \leq 60, 3x + 2y \leq 60, x \geq 0, y \geq 0$ and the optimization equation is: z = $20x + 35y$)

Answer: The maximum value is: $z(0,30) = 1050$.

18. You are a civil engineer designing a bridge. The walkway needs to be made of wooden planks. You are able to use either Sitka spruce planks (which weigh 3 pounds each), basswood planks (which weigh 4 pounds each), or a combination of both. The total weight of the planks must be between 600 and 900 pounds in order to meet safety code.If Sitka spruce planks cost \$3.25 each and basswood planks cost \$3.75 each, how many of each plank should you use to minimize cost while stillmeeting building code?

    (Hints: $3x + 4y \geq 600$, $3x + 4y \leq 900, x \geq 0, y \geq$ and the optimization equation is: z = $3.25x + 3.75y$)

    Answer: The minimum value is: $z(0,150)$ =562.50.

19. A storage solutions company manufactures large and small file folder cabinets. Large cabinets require 50 pounds of metal to fabricate and small cabinets require 30 pounds, but the company has only 450 pounds of metal on hand. If the company can sell each large cabinet for \$70 and each small cabinet for \$58, how many of each cabinet should it manufacture in order to maximize income?

    (Hints: $50x + 30y \leq 450, x \geq 0, y \geq 0$ and the optimization equation is: z = $70x + 58y$)

    Answer: The maximum value is: $z(0,15) = 870$.

20. A transport company has two types of trucks, Type A and Type B. Type A has a refrigerated capacity of 20 m^3 and a non-refrigerated capacity of 40 m^3 while Type B has the same overall volume with equal sections for refrigerated and non-refrigerated stock. A grocer needs to hire trucks for the transport of 3,000 m^3 of refrigerated stock and 4,000 m^3 of non-refrigerated stock. The cost per kilometer of a Type A is 30 and 40 for Type B. How many trucks of each type should the grocer rent to achieve the minimum total cost?

21. A company specializes in the production of three products chain, ring, bangles. The three products require metal and labour where supplies are limited. Construct mathematical model using data given in the following table:

	Unit of Metal required	Unit of Labour required	Price (Tk. In lac)
Chain	3	2	1
Ring	2	1	1
Bangles	1	2	3
Available Resources	3	2	

22. A manufacturer of leather belts makes three types of belts A, B, and C which are processed on three machines M_1, M_2 and M_3. Belt A requires 2 hours on machine M_1 and 3 hours on machine M_3. Belt B requires 3 hours on machine M_1, 2 hours on machine M_2 and 2 hours on machine M_3.Belt C requires 5 hours on machine M_2 and 4 hours on machine M_3. There are 8 hours of time per day available on machine M_1, 10 hours of time per day available on machine M_2 and 15 hours of time per day available on machine M_3. The profit gained from belt A is Tk. 300 per unit, belt B is Tk. 500 per unit and belt C is Tk. 400 per unit. Formulate the linear programming problem to maximize the profit.

23. A dietician mixes together three kinds of food say P, Q and R in such a way that the mixture contains at least 6 units of vitamin A, 7 units of vitamin B, 12 units of vitamin C and 9 units of vitamin D. The cost of 1 kg food P, 1 kg food Q and 1 kg food R is Tk. 500, Tk. 800 and Tk. 300 respectively. Formulate the linear programming problem to minimize the cost to buy the mentioned foods. The vitamin contents of 1 kg food P, 1 kg food Q and 1 kg food R are given in the following table:

	Vitamin A	Vitamin B	Vitamin C	Vitamin D
P	1	1	1	2
Q	2	1	3	1
R	1	2	2	3

24. Encode the message **DO PRACTICE** by using matrix, $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

25. Encode the message **HONESTY IS THE BEST POLICY** by using matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ given above.

26. Encrypt the message **GOOD STUDENTS** by the provided matrix, $B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$.

27. The encoded message is **28 13 28 20 23 5**. Decode this message by using matrix, $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Supplementary:

MATLab command for finding unique solution (if exists) of a system of equation:

$$\begin{aligned} 3x + 2y - z &= 20 \\ 2x + 3y - 3z &= 7 \\ x - y + 6z &= 41 \end{aligned}$$

Ans: (5,6,7)

For	Input Command	Output
Coefficient matrix: $A = \begin{pmatrix} 3 & 2 & -1 \\ 2 & 3 & -3 \\ 1 & -1 & 6 \end{pmatrix}$	>> A = [3 2 -1;2 3 -3;1 -1 6]	A = 3 2 -1 2 3 -3 1 -1 6
Right hand side matrix: $B = \begin{pmatrix} 20 \\ 7 \\ 41 \end{pmatrix}$	>> B= [20;7;41]	B = 20 7 41
checking whether there exists a unique solution or not!	>> if det(A)~=0 disp ('There exists a unique solution for the given system.') else disp ('There is no unique solution for the given system.') end	There exists a unique solution for the given system.
Solution set, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ where, $X = A^{-1}B$	>> X=inv(A)*B	X = 5.0000 6.0000 7.0000

References:

- Linear Programming by Thomas S. Ferguson
- Linear Programming by George B. Dantzig, Mukund N. Thapa
- Operations Research by Ravindran, Phillips & Solberg
- Operations Research by H. Taha