Chapter - 3
Even and Odd function

A function $y = f(x)$ is called even if $f(x) = f(-x)$, for all x in its domain. The graph of an even function is symmetric with respect to the y-axis. For example, any constant function, $\cos x, \sec x, x^{2n}$ $(n = 0, \pm 1, \pm 2, \pm 3 \ldots)$ are even functions.

A function $y = f(x)$ is called odd if $f(x) = -f(-x)$, for all x in its domain. The graph of an odd function is symmetric with respect to the origin. For example, $\sin x, \tan x, \cot x, \csc x, x^{2n+1}$ $(n = 0, \pm 1, \pm 2, \pm 3 \ldots)$ are odd functions.

Most functions, however, are neither even nor odd.

Arithmetic Combinations of Even and Odd Functions

Operations	Even and Even	Odd and Odd	Even and Odd
$+/-$	Even	Odd	Neither
$\times/\div$	Even	Even	Odd

Calculus Properties of Even and Odd Functions

Suppose the function $y = g(x)$ is an even function, continuous on $-L < x < L$, then

$$\int_{-L}^{L} g(x)dx = 2\int_{0}^{L} g(x)dx$$

Suppose the function $y = g(x)$ is an odd function, continuous on $-L < x < L$, then

$$\int_{-L}^{L} g(x)dx = 0.$$

Exercise 3.1

1. Determine the period of the following functions.

(a) $f(x) = \sin 5x$, (b) $f(x) = \tan 7x$, (c) $f(x) = \cos^2 x$, (d) $f(x) = \sin^2 x - \cos^2 x$,

(e) $f(x) = \sin 2x + \cos 3x$.

(a) Ans: $\frac{2\pi}{5}$. (b) Ans: $\pi/7$. (c) Ans: π. (d) Ans: π. (e) Ans: 2π.

2. Determine whether the following functions are even, odd or neither.

(a) $f(x) = \sin x$, (b) $f(x) = x^3 - x$, (c) $f(x) = \tan x$, (d) $f(x) = e^{x^2}$,

(e) $f(x) = \cos x$, (f) $f(x) = \frac{e^{3x}+e^{-3x}}{2}$, (g) $f(x) = \frac{e^{x}-e^{-x}}{2}$, (h) $f(x) = x^3$.

Periodic Function:

Let $T > 0$. A function $f(x)$ is said to be a periodic function if $f(x+nT) = f(x),\ n \in Z$. where T is called period of *f*.

Example 01:
Describe the function shown in figure 1 with period 2 in two different ways:
1. By considering its values on the interval $0 < x < 2$.
2. By considering its values on the interval $-1 < x < 1$.

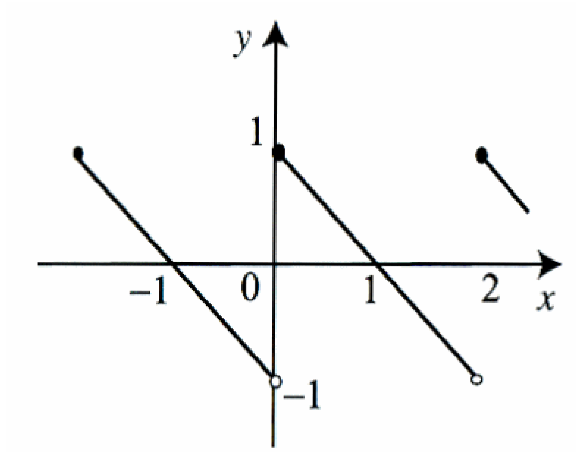


Figure 1: A function of period 2

Solution:
1. On the interval $0 < x < 2$, the function is a portion of the line $y = -x+1$ thus $f(x) = -x+1$ if $0 < x < 2$. The relation $f(x+2) = f(x)$ describes *f(x)* for all other values of *x*.
2. On the interval $-1 < x < 1$, the function consists of two lines. So we have

$$f(x) = \begin{cases} -x-1 & \text{if } -1 \le x < 0 \\ -x+1 & \text{if } 0 \le x < 1 \end{cases}$$

The relation $f(x+2) = f(x)$ describes *f* for all other values of *x*.

Example: $f(x) = \sin x$

$$\therefore\ f(x+2\pi) = \sin(x+2\pi) = \sin x$$

So, $f(x)$ is periodic function and period $T = 2\pi$.

Extension to an odd and even periodic function:

Let $y=f(x)$ is function defined on $0 \le x \le L$.

The odd extensions of $f(x)$ are, $g(x)=\begin{cases} f(x) & \text{if } 0 \le x \le L \\ -f(-x) & \text{if } -L \le x \le 0. \end{cases}$

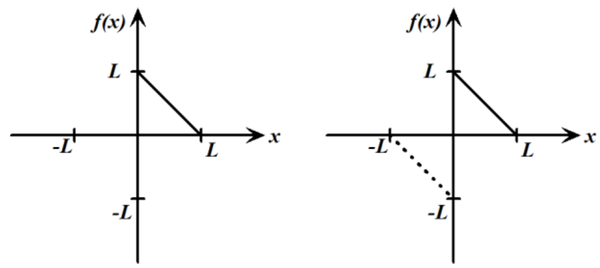


Figure 2: Graph of function $f(x)$ and its odd extension

Similarly, the odd extension of any piecewise function is as follows:

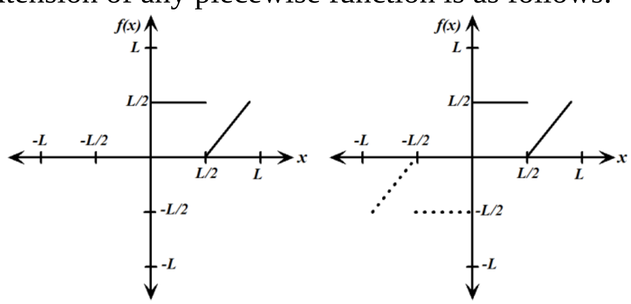


Figure 3: Graph of piecewise function $f(x)$ and its odd extension

The even extensions of $f(x)$ are, $g(x)=\begin{cases} f(x) & \text{if } 0 \le x \le L \\ f(-x) & \text{if } -L \le x \le 0. \end{cases}$

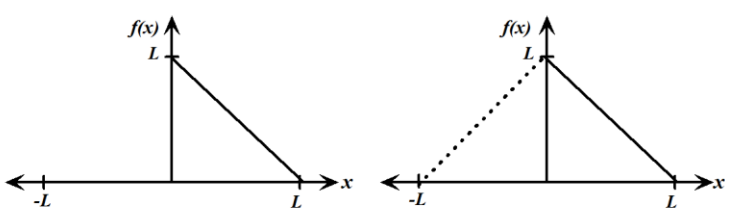


Figure 4: Graph of function $f(x)$ and its even extension

Similarly, the even extension of any piecewise function is as follows:

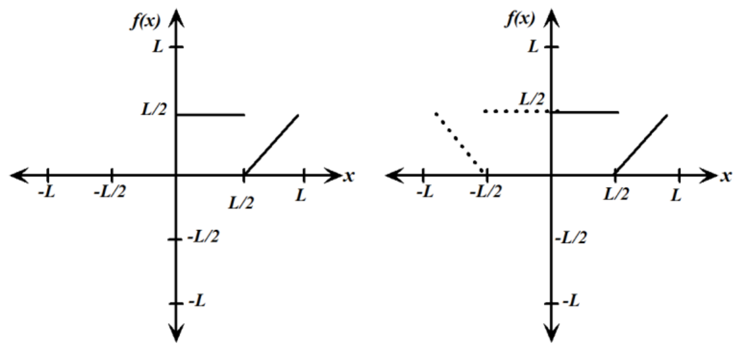


Figure 5: Graph of piecewise function *f(x)* and its even extension

Example 02: Sketch the even extension of the function $f(x) = x^3$ on $0 \leq x \leq$ L.

Solution:

The even extension of the function is,

$$g(x) = \begin{cases} x^3 & \text{if } 0 \leq x \leq L \\ -x^3 & \text{if } -\mathrm{L} \leq \mathrm{x} \leq 0. \end{cases}$$

The sketch of the function and the even extension is,

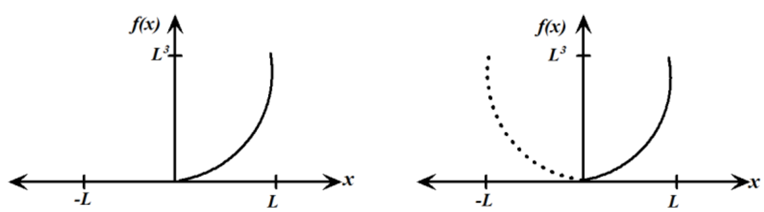


Figure 6: Graph of function *g(x)* and its even extension

Example 03: Sketch the odd extension of the function $f(x) = 1 + x^2$ on $0 \leq x \leq$ L.

Solution:

The odd extension of the function is,

$$g(x) = \begin{cases} 1 + x^2 & \text{if } 0 \leq x \leq L \\ -1 - x^2 & \text{if } -\mathrm{L} \leq x \leq 0. \end{cases}$$

The sketch of the function and the odd extension are,

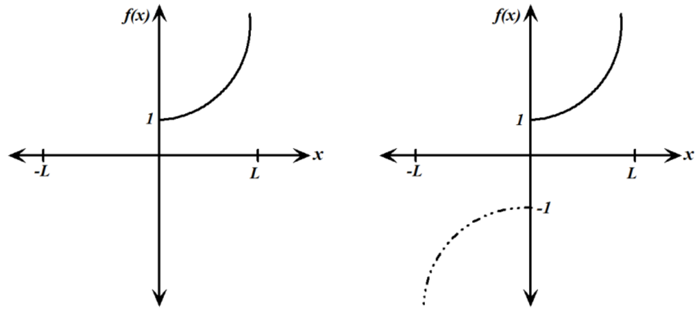


Figure 7: Graph of piecewise function *g(x)* and its odd extension

Some useful formulas:

$$\sin n\pi = 0$$

$$\cos n\pi = (-1)^n, \text{ for all } n \in \mathbb{Z}.$$

If $n = 2k+1$ is an odd number then $k \in \mathbb{Z}$

$$\sin\left(\frac{n\pi}{2}\right) = \sin\left(\frac{(2k+1)\pi}{2}\right) = (-1)^k \text{ and}$$

$$\cos\left(\frac{n\pi}{2}\right) = \cos\left(\frac{(2k+1)\pi}{2}\right) = 0.$$

Useful technique for integration by parts:

$$\int_0^2 x^2 \cos\left(\frac{\pi n x}{2}\right) dx$$

sign	Differentiation	Integration
+	x^2	$\frac{\cos \pi n x}{2}$
-	$2x$	$\frac{2\sin\left(\frac{\pi n x}{2}\right)}{\pi n}$
+	2	$-\frac{4\sin\left(\frac{\pi n x}{2}\right)}{\pi^2 n^2}$
-	0	$-\frac{8\sin\left(\frac{\pi n x}{2}\right)}{\pi^3 n^3}$

$$\therefore \int_0^2 x^2 \cos\left(\frac{\pi n x}{2}\right) dx = \left[\frac{2x^2}{\pi n}\sin\left(\frac{\pi n x}{2}\right) + \frac{8x}{\pi^2 n^2}\cos\left(\frac{\pi n x}{2}\right) - \frac{16}{\pi^3 n^3}\sin\left(\frac{\pi n x}{2}\right)\right]_0^2$$

$$= 0 + \frac{16}{\pi^2 n^2}\cos \pi n - 0 \qquad = \frac{16}{\pi^2 n^2}(-1)^n .$$

Full range Fourier series in Real form

The Fourier series is named in honor of Jean-Baptiste Joseph Fourier (1768–1830), who made important contributions to the study of trigonometric series. Fourier introduced the series for the purpose of solving the heat equation in a metal plate. In mathematics, it decomposes any periodic function or periodic signal into the weighted sum of a (possibly infinite) set of simple oscillating functions, namely sines/cosines and both (or, equivalently, complex exponentials). The fields of electronics, quantum mechanics, and electrodynamics all make heavy use of the Fourier series. Additionally, other methods based on the Fourier series, such as the FFT (Fast Fourier Transform – a form of a Discrete Fourier Transform [DFT]), are particularly useful for the fields of Digital Signal Processing (DSP).

Suppose $f(x)$ is a periodic function with a period $T = 2L$ or is defined on the interval $-L < x < L$ (where L could be the length of a violin string or the length of a rod in heat conduction and so on). Then the *Fourier series* representation of $f(x)$ is a trigonometric series (that is, it is an infinite series consists of sine and cosine terms) of the form,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] \qquad \dots \quad (1)$$

where

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx \qquad \dots \quad (2)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad (n = 1, 2, 3, \dots) \qquad \dots \quad (3)$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad (n = 1, 2, 3, \dots) \qquad \dots \quad (4)$$

The coefficients a_0, a_n and b_n are called the Fourier coefficients of $f(x)$.

Note that the cosine functions are even, while the sine functions are odd.

If $f(x)$ is an even function then the integrand in (4) is odd, so $b_n = 0$ for all *n*, leaving a ***Fourier cosine series (and perhaps a constant term) only for f (x)***.

If $f(x)$ is an odd function then the integrand in (2) and (3) are odd, so $a_0 = a_n = 0$ for all *n*, leaving a ***Fourier sine series only for f (x).***

Example 04:

Find a Fourier series for $f(x) = x$, $-2 < x < 2$, $f(x + 4) = f(x)$.

Solution:

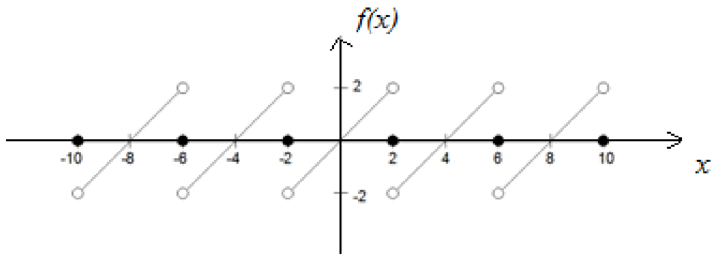


Figure 8: Graph of periodic function *f(x)*

Here, $T = 2L = 4,$ hence $L = 2.$

$$\therefore \ a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2}\int_{-2}^{2} x\,dx = 0 \text{ (odd function)}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_{-2}^{2} x\cos\left(\frac{n\pi x}{2}\right)dx = 0 \ \textbf{(}\text{integrand is odd}\textbf{)}$$

Again,

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_{-2}^{2} x\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_{0}^{2} x\sin\left(\frac{n\pi x}{2}\right)dx$$

sign	D	I
+	x	$\sin\left(\frac{\pi n x}{2}\right)$
-	1	$-\frac{2}{\pi n}\cos\left(\frac{\pi n x}{2}\right)$
+	0	$-\frac{4}{\pi^2 n^2}\sin\left(\frac{\pi n x}{2}\right)$

$$= \left[-\frac{2x}{\pi n}\cos\left(\frac{\pi n x}{2}\right) + \frac{4}{\pi^2 n^2}\sin\left(\frac{\pi n x}{2}\right)\right]_0^2$$

$$= \left[-\frac{4}{\pi n}\cos(n\pi) + \frac{4}{\pi^2 n^2}\sin(n\pi)\right]$$

$$= \frac{4}{\pi n}(-1)^{n+1}$$

Now, we know the Fourier series of $f(x)$ in the interval $-L < x < L$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

Therefore, the Fourier series for *f (x)* is

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{\pi n x}{2}\right).$$

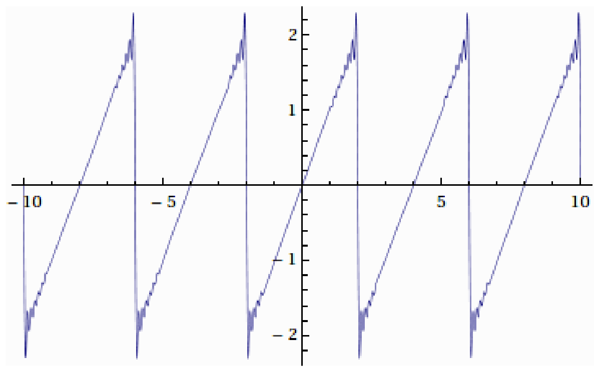
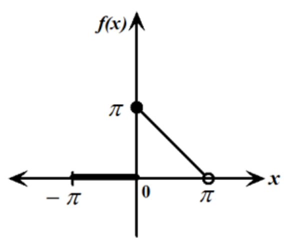


Figure 9: The graph of the partial sum of the first 30 terms of the above Fourier series

Example 05: Compute the first 4 components of the trigonometric Fourier series for the wave form below



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Solution: From the figure we can construct the function as

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \pi - x & 0 \le x < \pi \end{cases}$$

Here, $T = 2L = 2\pi,$ hence $L = \pi.$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\,dx$$

$$= 0 + \frac{1}{\pi}\left[\frac{(\pi - x)^2}{-2}\right]_0^{\pi} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 + \frac{1}{\pi}\int_0^{\pi}(\pi - x)\cos nx\,dx$$
$$= \frac{1}{\pi}\left[\frac{n(\pi - x)\sin nx - \cos nx}{n^2}\right]_0^{\pi} = \frac{1-(-1)^n}{n^2\pi}$$

Sign	D	I
+	$\pi - x$	$\cos nx$
-	-1	$\frac{1}{n}\sin nx$
+	0	$-\frac{1}{n^2}\cos nx$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0 + \frac{1}{\pi}\int_0^{\pi}(\pi - x)\sin nx\,dx$$
$$= \frac{1}{\pi}\left[\frac{n(\pi - x)\cos nx + \sin nx}{-n^2}\right]_0^{\pi} = \frac{1}{n}$$

sign	D	I
+	$\pi - x$	$\sin nx$
-	-1	$-\frac{1}{n}\cos nx$
+	0	$-\frac{1}{n^2}\sin nx$

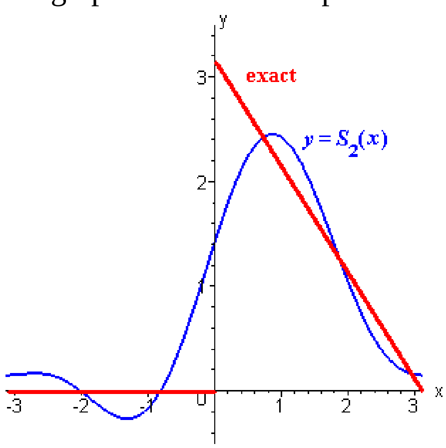
Therefore the Fourier series for *f (x)* is

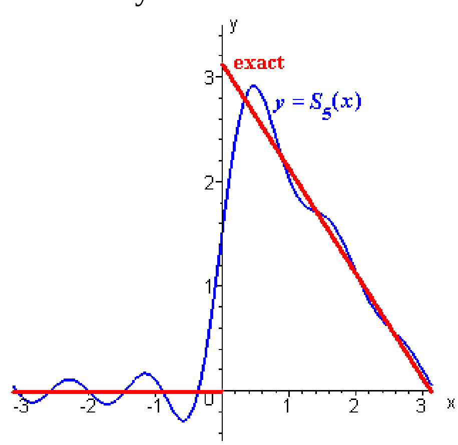
$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\left(\frac{1-(-1)^n}{n^2\pi}\right)\cos nx + \frac{1}{n}\sin nx\right] \qquad (-\pi < x < \pi)$$

Now, the first few partial sums in the Fourier series are

$$S_0 = \frac{\pi}{4}$$
$$S_1 = \frac{\pi}{4} + \frac{2}{\pi}\cos x + \sin x$$
$$S_2 = \frac{\pi}{4} + \frac{2}{\pi}\cos x + \sin x + \frac{1}{2}\sin 2x$$
$$S_3 = \frac{\pi}{4} + \frac{2}{\pi}\cos x + \sin x + \frac{1}{2}\sin 2x + \frac{2}{9\pi}\cos 3x + \frac{1}{3}\sin 3x \text{ and so on.}$$

The graphs of successive partial sums approach *f (x)* more closely.





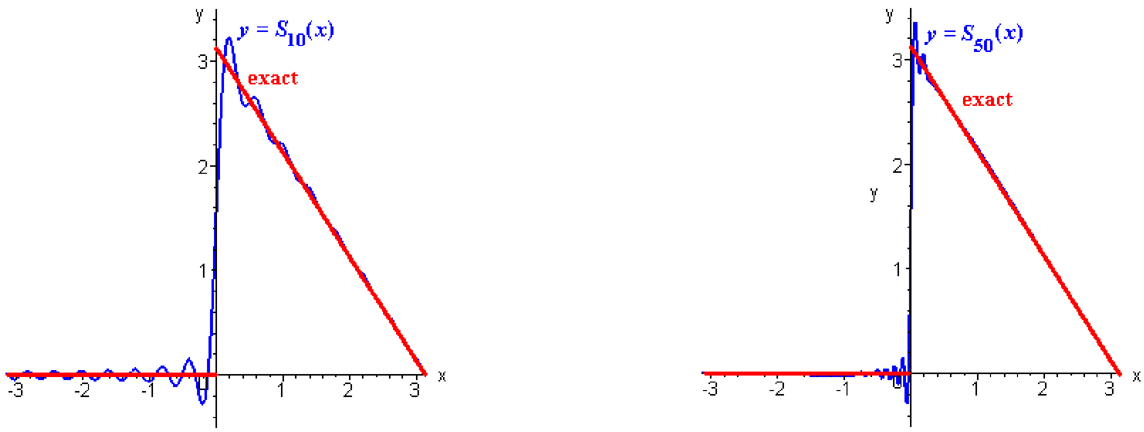


Figure 9: The graph of the partial sum of the first 50 terms of the above Fourier series

Example 06: Find the Fourier series expansion for the standard square wave,

$$f(x) = \begin{cases} -1 & (-1 < x < 0) \\ +1 & (0 \le x < +1) \end{cases}$$

Solution:

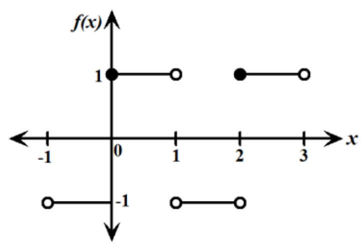


Figure 10: The graph of function *f(x)*

Here, $T = 2L = 2$, hence $L = 1$.

$$\therefore \quad a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0}(-1)\,dx + \int_{0}^{1} 1\,dx = [-x]_{-1}^{0} + [x]_{0}^{1} = -1 + 1 = 0$$

$$\therefore \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)\,dx = \int_{-1}^{0} -\cos(n\pi x)\,dx + \int_{0}^{1}\cos(n\pi x)\,dx$$

$$= -\frac{1}{n\pi}[\sin n\pi x]_{-1}^{0} + \frac{1}{n\pi}[\sin n\pi x]_{0}^{1} = 0$$

$$\therefore \; b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{1}\int_{-1}^{1} f(x)\sin(n\pi x)\,dx$$

$$= \int_{-1}^{0} -\sin(n\pi x)\,dx + \int_{0}^{1}\sin(n\pi x)\,dx$$

$$= \frac{1}{n\pi}\left[\cos(n\pi x)\right]_{-1}^{0} - \frac{1}{n\pi}\left[\cos(n\pi x)\right]_{0}^{1}$$

$$= \frac{2\left[1-(-1)^n\right]}{n\pi}$$

Therefore the Fourier series of *f(x)* is

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\sin n\pi x\right] = \frac{2}{\pi}\sum_{n=odd}^{\infty}\left[\frac{2}{n}\sin(n\pi x)\right]$$

The graphs of the third and ninth partial sums (containing two and five non-zero terms respectively) are displayed here, together with the exact form for $f(x)$, with a **periodic extension** beyond the interval (–1, +1) that is appropriate for the square wave.

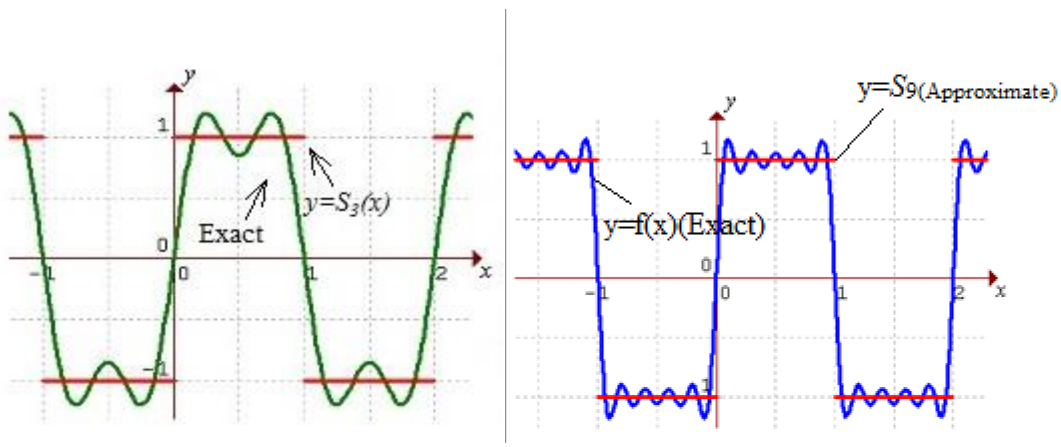


Figure 11: The graph of the partial sum of the first 9 terms of the above Fourier series

Example 07: Find the Fourier series for the function $f(x)$ defined by

$$f(x) = \begin{cases} 1-x^2 & (-1 \le x < 0) \\ 0 & (0 \le x < 1) \end{cases}$$

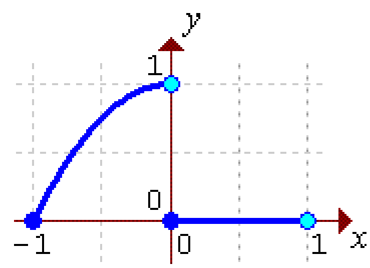


Figure 12: Graph of *f(x)*

Solution:

Here, $T = 2L = 2$, hence $L = 1$.

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0}(1-x^2)\,dx + \int_{0}^{1} 0\,dx$$

$$= \left[x - \frac{x^3}{3}\right]_{-1}^{0} + 0 = (0-0) - \left(-1-\left(-\tfrac{1}{3}\right)\right) = +\frac{2}{3}$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos\left(\frac{n\pi x}{1}\right)dx$$

$$= \int_{-1}^{0}(1-x^2)\cos(n\pi x)\,dx + 0$$

Sign	D	I
+	$1-x^2$	$\cos(n\pi x)$
-	$-2x$	$\frac{1}{n\pi}\sin(n\pi x)$
+	-2	$-\frac{1}{n^2\pi^2}\cos(n\pi x)$
-	0	$-\frac{1}{n^3\pi^3}\sin(n\pi x)$

$$= \left[\left(\frac{(1-x^2)}{n\pi} + \frac{2}{(n\pi)^3}\right)\sin(n\pi x) - \frac{2x}{(n\pi)^2}\cos(n\pi x)\right]_{-1}^{0}$$

$$= (0-0) - \left(0 - \frac{-2}{(n\pi)^2}\cos(n\pi)\right) = \frac{-2}{(n\pi)^2}(-1)^n$$

$$b_n = \int_{-1}^{1} f(x)\sin\left(\frac{n\pi x}{1}\right)dx$$

$$= \int_{-1}^{0}(1-x^2)\sin(n\pi x)\,dx + 0$$

Sign	D	I
+	$1-x^2$	$\sin(n\pi x)$
-	$-2x$	$-\frac{1}{n\pi}\cos(n\pi x)$
+	-2	$-\frac{1}{n^2\pi^2}\sin(n\pi x)$
-	0	$\frac{1}{n^3\pi^3}\cos(n\pi x)$

$$= \left[-\left(\frac{(1-x^2)}{n\pi} + \frac{2}{(n\pi)^3}\right)\cos(n\pi x) - \frac{2x}{(n\pi)^2}\sin(n\pi x)\right]_{-1}^{0}$$

$$= -\left(\frac{1}{n\pi} + \frac{2}{(n\pi)^3} - 0\right) - \left(-\left(0 + \frac{2}{(n\pi)^3}\right)\cos(n\pi) - 0\right)$$

$$= \frac{2}{(n\pi)^3}\left((-1)^n - 1\right) - \frac{1}{n\pi}$$

The Fourier series of $f(x)$ on $[-L, +L]$ in general is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

Therefore

$$f(x) = \frac{1}{3} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^{n+1}}{(n\pi)^2}\cos(n\pi x) + \left(2.\frac{(-1)^n - 1}{(n\pi)^2} - \frac{1}{n\pi}\right)\sin(n\pi x)\right]$$

The first few terms of this series are

$$\begin{aligned} f(x) &= \frac{1}{3} + \frac{2}{\pi^2}\cos(\pi x) - \left(\frac{4}{\pi^3} + \frac{1}{\pi}\right)\sin(\pi x) \\ &\quad - \frac{1}{2\pi^2}\cos(2\pi x) - \frac{1}{2\pi}\sin(2\pi x) \\ &\quad + \frac{2}{9\pi^2}\cos(3\pi x) - \left(\frac{4}{27\pi^3} + \frac{1}{3\pi}\right)\sin(3\pi x) + \ldots \end{aligned}$$

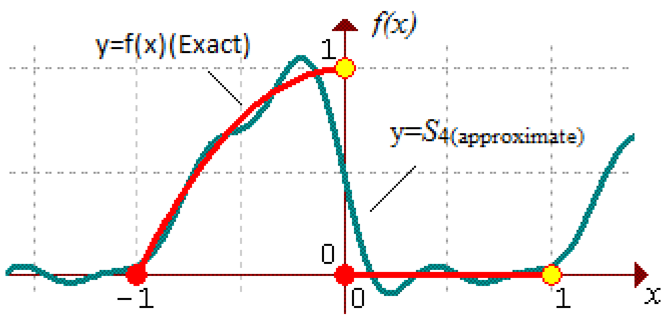


Figure 13: The graph of the partial sum of the first 4 terms of the above Fourier series

Example 08: Find the Fourier series for the function $f(x)$ defined on the interval $[-1, 1]$ by

$$f(x) = 1 - |x|$$

Solution:

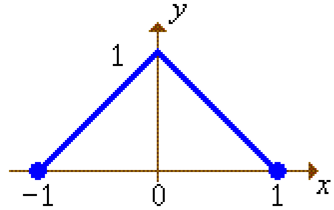


Figure 14: The Graph of *f(x)*

Here, $T = 2L = 2$, hence $L = 1$.

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = 2\int_0^1 f(x)\,dx \quad (\because \text{ symmetry})$$

$$= 2\int_0^1 (1-x)\,dx = 2\left[\frac{(1-x)^2}{-2}\right]_0^1 = -0+1 = 1$$

$$a_n = 2\int_0^1 (1-x)\cos(n\pi x)\,dx$$

$$= 2\left[\frac{(1-x)}{n\pi}\sin(n\pi x) - \frac{1}{(n\pi)^2}\cos(n\pi x)\right]_0^1$$

$$= 2\left(\left(0 - \frac{(-1)^n}{(n\pi)^2}\right) - \left(0 - \frac{1}{(n\pi)^2}\right)\right)$$

$$= \frac{2\left(1-(-1)^n\right)}{(n\pi)^2} = \begin{cases} 0 & (n \text{ even}) \\ \dfrac{4}{(n\pi)^2} & (n \text{ odd}) \end{cases}$$

Sign	D	I
$+$	$\longrightarrow 1-x$	$\cos(n\pi x)$
-	$\longrightarrow -1$	$\frac{1}{n\pi}\sin(n\pi x)$
$+$	0	$-\frac{1}{n^2\pi^2}\cos(n\pi x)$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin\left(\frac{n\pi x}{1}\right)dx = 0 \ (\because \text{integrand is odd})$$

Therefore the Fourier series is

$$f(x) = \frac{1}{2} + \frac{2}{\pi^2}\sum_{n=odd}^{\infty}\left[\frac{2}{n^2}\cos(n\pi x)\right]$$

The first few terms of this series are

$$f(x) = \frac{1}{2} + \frac{4}{\pi^2}\left(\cos(\pi x) + \frac{\cos(3\pi x)}{9} + \frac{\cos(5\pi x)}{25} + \ldots\right)$$

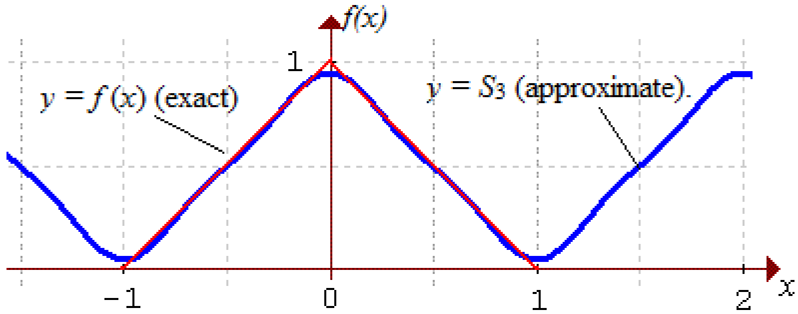


Figure 15: The graph of *f(x)* and the third partial sum S_3

Full range Fourier series in Complex form

Consider, $f(x)$ be defined in the interval, $[c-L, c+L]$. Fourier series of the function, $f(x)$ in real form:

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\left(\frac{e^{i\frac{n\pi x}{L}}+e^{-i\frac{n\pi x}{L}}}{2}\right)+b_n\left(\frac{e^{i\frac{n\pi x}{L}}-e^{-i\frac{n\pi x}{L}}}{2i}\right)\right] \qquad [\text{Since } \cos(nx)=\frac{e^{inx}+e^{-inx}}{2}\ \&\sin(nx)=\frac{e^{inx}-e^{-inx}}{2i}]$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(\frac{a_n-ib_n}{2}\right)e^{i\frac{n\pi x}{L}}+\sum_{n=1}^{\infty}\left(\frac{a_n+ib_n}{2}\right)e^{-i\frac{n\pi x}{L}}$$

$$=C_0+\sum_{n=1}^{\infty}C_{-n}e^{i\frac{n\pi x}{L}}+\sum_{n=1}^{\infty}C_ne^{-i\frac{n\pi x}{L}}$$

$$=\sum_{n=1}^{\infty}C_{-n}e^{i\frac{n\pi x}{L}}+C_0+\sum_{n=1}^{\infty}C_ne^{-i\frac{n\pi x}{L}}$$

$$=...+C_{-3}e^{i\frac{3\pi x}{L}}+C_{-2}e^{i\frac{2\pi x}{L}}+C_{-1}e^{i\frac{\pi x}{L}}+C_0e^{i\frac{0\pi x}{L}}+C_1e^{-i\frac{\pi x}{L}}+C_2e^{-i\frac{2\pi x}{L}}+C_3e^{-i\frac{3\pi x}{L}}+....$$

$$=\sum_{n=-\infty}^{\infty}C_ne^{-i\frac{n\pi x}{L}}$$

$$where\, C_0=\frac{a_0}{2}, C_{-n}=\frac{a_n-ib_n}{2}, C_n=\frac{a_n+ib_n}{2}$$

If $f(x)$ be the periodic function of period 2L in the interval, $[c-L, c+L]$, Fourier series of the function, $f(x)$ in complex form can be written in the following form:

$$f(x)=\sum_{n=-\infty}^{\infty}C_ne^{-i\frac{n\pi x}{L}}.$$

Here $C_n=\frac{1}{2L}\int_{c-L}^{c+L}f(x)e^{i\frac{n\pi x}{L}}dx; n=0,\pm1,\pm2,\pm3,.....$

The coefficient, C_n is called complex Fourier coefficient. The complex form of Fourier series is algebraically simpler and more symmetric. Therefore, it is often used in physics and other sciences.

Example 09: Find the complex form of Fourier series of function, $f(x)=x^2$ in the region, $-\pi\le x\le\pi$.

Solution:

The Fourier series of function in complex form in the interval [c-L,c+L] is given by

$$f(x)=\sum_{n=-\infty}^{\infty}C_ne^{-\frac{in\pi x}{L}}.$$

$$Here, L = \pi.$$

$$\therefore f(x) = x^2 = \sum_{n=-\infty}^{\infty} C_n e^{-inx} \qquad (1)$$

$$Now, C_n = \frac{1}{2L}\int_{c-L}^{c+L} f(x)e^{\frac{in\pi x}{L}}dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{\frac{in\pi x}{\pi}}dx \qquad \left[c - L = -\pi \ \&\ c + L = \pi, we\, have\ c = 0. So\, L = \pi\ \right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{inx}dx \qquad \left[\because\ f(x) = x^2\right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\left[\cos(nx) + i\sin(nx)\right]dx \qquad \left[\because e^{inx} = \cos(nx) + i\sin(nx)\right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,dx + \frac{i}{2\pi}\int_{-\pi}^{\pi} x^2\sin(nx)\,dx$$

$$= \frac{2}{2\pi}\int_{0}^{\pi} x^2\cos(nx)\,dx + \frac{i}{2\pi}(0) \qquad \left[\because\ x^2\sin(nx)\, is\, an\, odd\ function, therefore \int_{-\pi}^{\pi} x^2\sin(nx)\,dx = 0\ \right]$$

$$= \frac{1}{\pi}\left[\frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) - \frac{2}{n^3}\sin(nx)\right]_0^{\pi} \qquad \left[use\ formula: \int uvdx = u\int vdx - \int(\frac{d}{dx}u)\int vdx\right]$$

$$= \frac{1}{\pi}\left[\frac{2\pi}{n^2}(-1)^n\right] \qquad \left[\because \sin(n\pi) = 0\, \&\, cos(n\pi) = (-1)^n\right]$$

$$C_n = \frac{2}{n^2}(-1)^n \qquad ;\ n \neq 0$$

$$If\ n = 0,\ then\, C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx$$

$$= \frac{2}{2\pi}\int_{0}^{\pi} x^2 dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{\pi^2}{3}$$

Eq. (1) becomes,

$$f(x) = \frac{\pi^2}{3} + \sum_{n=-\infty}^{n=\infty} \frac{2(-1)^n}{n^2}e^{-inx}; n \neq 0$$

This is the required complex form of Fourier series.

Exercise 3.2

1. Sketch the graph and find the Fourier coefficients and then Fourier series of the function of $f(x) = x^2$ in the interval $-\pi \le x \le \pi$.

Ans: $a_0 = \dfrac{2\pi^2}{3}$, $a_n = \dfrac{4(-1)^n}{n^2}$, $b_n = 0.$

2. Sketch the graph and obtain the Fourier series of the function
$f(x) = \begin{cases} 0 & \text{when } -\pi < x < 0 \\ 1 & \text{when } 0 < x < \pi \end{cases}.$

Ans: $a_0 = 1$, $a_n = 0$, $b_n = -\dfrac{1}{\pi n}[(-1)^n - 1] = \begin{cases} 0 & \text{when } n = \text{even} \\ \dfrac{2}{\pi n} & \text{when } n = \text{odd} \end{cases}.$

3. Sketch the graph and obtain the Fourier series of the function
$f(x) = \begin{cases} -x & \text{when } -\pi < x < 0 \\ x & \text{when } 0 < x < \pi \end{cases}.$

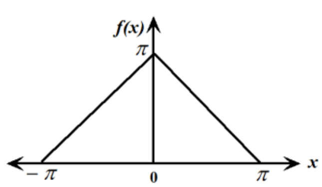
Ans: $a_0 = \pi$, $b_n = 0$, $a_n = \dfrac{2}{\pi n^2}[(-1)^n - 1] = \begin{cases} 0 & \text{when } n = \text{even} \\ \dfrac{4}{\pi n^2} & \text{when } n = \text{odd} \end{cases}.$

4. Sketch the graph and obtain the Fourier series of the function $f(x) = |x|$ in the interval $-\pi \le x \le \pi$.

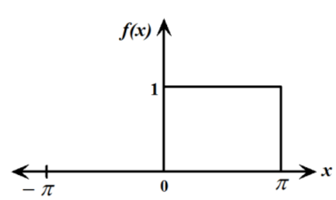
Ans: $a_0 = \pi$, $a_n = \begin{cases} \dfrac{-4}{\pi n^2} & \text{when } n = 1, 3, 5, \dots \\ 0 & \text{when } n = 2, 4, 6, \dots \end{cases}$, $b_n = 0$.

5. W rite down the functions corresponding to the following figures. Also compute the first few components of the trigonometric Fourier series.

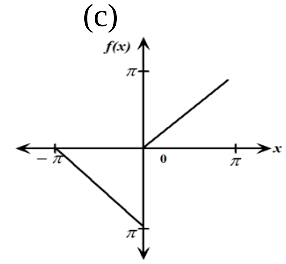
(a)



(b)



(c)



6. Find the complex form of Fourier series of the following function:
   (i) $f(x) = 2x$ in the region, $-\pi \le x \le \pi$.
   (ii) $f(x) = x^2$ in the region, $\pi \le x \le \pi$.
   (iii) $f(x) = x^2 - x$ in the region, -$\pi \le x \le \pi$.

Half-Range Fourier series

If *f(x)* and *f ′(x)* are piecewise continuous functions defined on the interval $0 \le x \le L$, then *f(x)* can be extended into an even periodic function, F, of period $2L$, such that $f(x) = F(x)$ on the interval $[0, L]$, and whose Fourier series is, therefore, a cosine series.

Similarly, *f(x)* can be extended into an odd periodic function of period $2L$, such that $f(x) = F(x)$ on the interval $(0, L)$, and whose Fourier series is, therefore, a sine series.
The process that such extensions are obtained is often called cosine /sine series *half-range expansions*.
A Fourier series for $f(x)$, valid on $[0, L]$, may be constructed by extension of the domain to $[-L, L]$.
An odd extension of *f(x)* of the period *2L* leads to a **Fourier sine series**:

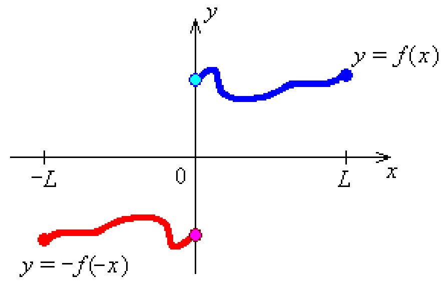


Figure 01: Odd extension of *f(x)*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$a_0 = 0, \quad a_n = 0 \quad n = 1, 2, 3, \ldots$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad (n = 1, 2, 3, \ldots)$$

An even extension of *f(x)* of period *2L* leads to a **Fourier cosine series**:

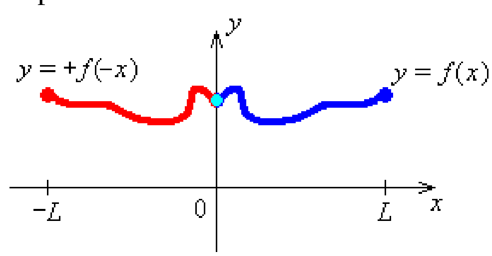


Figure 02: Even extension of *f(x)*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where, $a_0 = \frac{2}{L}\int_0^L f(x)dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, 3, \ldots$ and $b_n = 0$

Example 01: Find the Half range Fourier sine and cosine series for

$$f(x)=\begin{cases} 2x, & 0<x<\frac{1}{2} \\ 2-2x, & \frac{1}{2}<x<1 \end{cases}$$

Solution:

Half range Fourier sine series (Odd periodic extension)

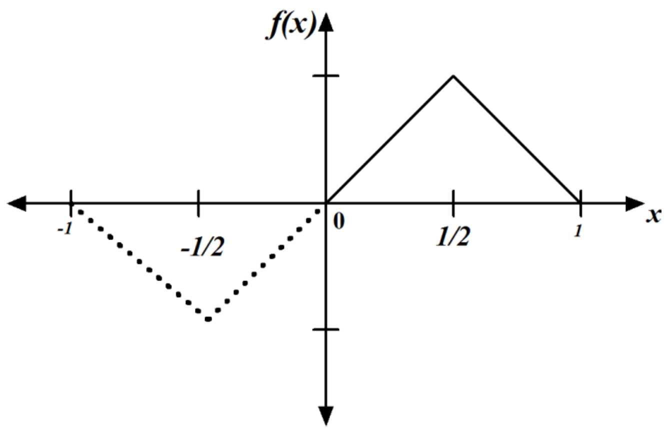


Figure 03: Odd extension of *f(x)*

Here, $T=2L=2$, hence $L=1$.
An odd extension of $f(x)$ is required to the interval [–1, 1].
$a_n = 0$ for all n.

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$=\frac{2}{1}\left[\int_0^{\frac{1}{2}} 2x\sin(n\pi x)dx+\int_{\frac{1}{2}}^{1}(2-2x)\sin(n\pi x)dx\right]$$

Sign	D	I
+	$\longrightarrow 2x$	$\sin(n\pi x)$
-	$\longrightarrow 2$	$-\frac{1}{n\pi}\cos(n\pi x)$
+	0	$-\frac{1}{n^2\pi^2}\sin(n\pi x)$

$$=\left[\frac{-4x}{n\pi}\cos(n\pi x)+\frac{4}{n^2\pi^2}\sin(n\pi x)\right]_0^{\frac{1}{2}}+\left[-\frac{(4-4x)}{n\pi}\cos(n\pi x)-\frac{4}{n^2\pi^2}\sin(n\pi x)\right]_{\frac{1}{2}}^{1}$$

$$=\frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

Sign	D	I
+	$2-2x$	$\sin(n\pi x)$
-	-2	$\frac{-1}{n\pi}\cos(n\pi x)$
+	0	$-\frac{1}{n^2\pi^2}\sin(n\pi x)$

Therefore the half range Fourier sine series for $f(x)$ on [0, 1] (which is also the Fourier series for $f(x) = x$ on [–1, 1]) is

$$f(x)=\sum_{n=odd}^{\infty}\frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin(n\pi x)$$

<u>Half range Fourier cosine series (Even periodic extension)</u>

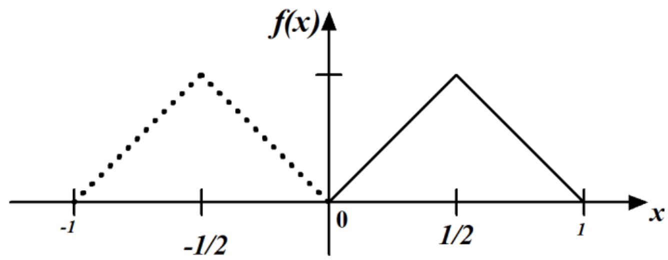


Figure 04: Even extension of *f(x)*

Here, $T=2L=2$, hence $L=1$.
An even extension of $f(x)$ is required to the interval [–1, 1].
$b_n = 0$ for all n.

$$a_0=\frac{2}{L}\int_0^L f(x)dx,\quad a_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx,\quad n=1,2,3,\ldots$$

Sign	D	I
+	$2x$	$\cos(n\pi x)$
-	2	$\frac{1}{n\pi}\sin(n\pi x)$
+	0	$-\frac{1}{n^2\pi^2}\cos(n\pi x)$

$$a_0 = \frac{2}{L}\int_0^L f(x)dx = \frac{2}{1}\left[\int_0^{\frac{1}{2}} 2x dx + \int_{\frac{1}{2}}^{1} (2-2x)dx\right] = \frac{1}{2} \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{1}\left[\int_0^{\frac{1}{2}} 2x\cos(n\pi x)dx + \int_{\frac{1}{2}}^{1} (2-2x)\cos(n\pi x)dx\right]$$

Sign	D	I
+	$2-2x$	$\cos(n\pi x)$
-	-2	$\frac{1}{n\pi}\sin(n\pi x)$
+	0	$-\frac{1}{n^2\pi^2}\cos(n\pi x)$

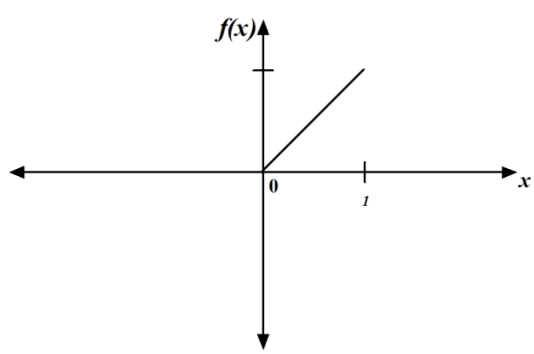
$$= \left[\frac{-4x}{n\pi}\sin(n\pi x) + \frac{4}{n^2\pi^2}\cos(n\pi x)\right]_0^{\frac{1}{2}} + \left[\frac{(4-4x)}{n\pi}\sin(n\pi x) - \frac{4}{n^2\pi^2}\cos(n\pi x)\right]_{\frac{1}{2}}^{1}$$

$$= \frac{4}{n^2\pi^2}\left[2\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi) - 1\right]$$

Therefore the half range Fourier cosine series for $f(x)$ on [0, 1] (which is also the Fourier series for $f(x) = x$ on [–1, 1]) is

$$f(x) = \frac{1}{2} + \sum_{n=even}^{\infty} \frac{4}{n^2\pi^2}\left[2\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi) - 1\right]\cos(n\pi x)$$

Example 02: Find the first few terms of half range Fourier sine and cosine series for the wave form below



Solution:

From the figure we can construct the function as

$$f(x) = x, \quad 0 \le x \le 1$$

<u>Half range Fourier sine series (Odd periodic extension)</u>

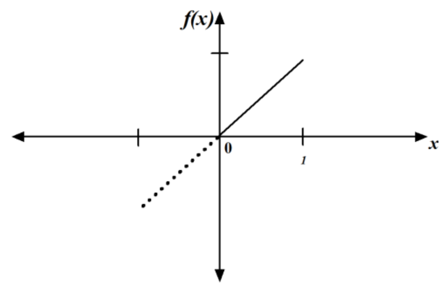


Figure 05: Odd extension of *f(x)*

Here, $T = 2L = 2$, hence $L = 1$.

An odd extension of $f(x)$ is required to the interval [–1, 1].
$a_n = 0$ for all n.

$$b_n = \frac{2}{1}\int_0^1 x \sin\left(\frac{n\pi x}{1}\right) dx, \quad (n = 1, 2, 3, \dots)$$

$$\Rightarrow b_n = 2\left[-\frac{x}{n\pi}\cos\left(\frac{n\pi x}{1}\right) + \frac{1}{(n\pi)^2}\sin\left(\frac{n\pi x}{1}\right)\right]_0^1$$

$$= \frac{2}{n\pi} \times (-1)^{n+1}$$

Sign	D	I
+	x	$\sin(n\pi x)$
-	1	$-\frac{1}{n\pi}\cos(n\pi x)$
+	0	$-\frac{1}{n^2\pi^2}\sin(n\pi x)$

Therefore the Fourier sine series for $f(x) = x$ on [0, 1] (which is also the Fourier series for $f(x) = x$ on [–1, 1]) is

$$f(x) = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(n\pi x)}{n\pi}$$

Or

$$f(x) = \frac{2}{\pi}\left[\sin(\pi x) - \frac{\sin(2\pi x)}{2} + \frac{\sin(3\pi x)}{2} - \frac{\sin(4\pi x)}{2} + \cdots\right]$$

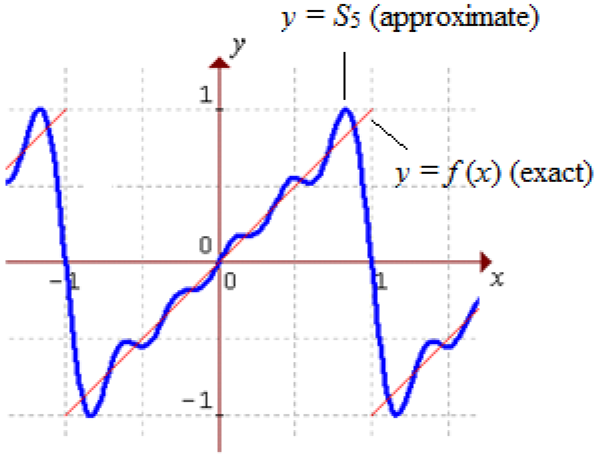


Figure 06: The graph of *y=f(x)* and the partial sum of the first 5 terms of the above Fourier series

Half range Fourier cosine series (Even periodic extension)

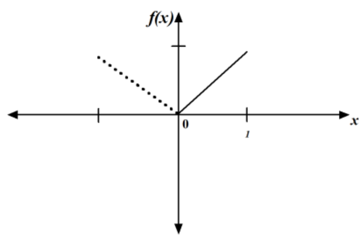


Figure 07: Even extension of *f(x)*

Here, $T = 2L = 2$, hence $L = 1$.

The even extension of $f(x)$ is required to the interval [–1, 1].

$b_n = 0$ for all n.

Evaluating the Fourier cosine coefficients,

$$a_0 = \frac{2}{1}\int_0^1 x\,dx = \left[x^2\right]_0^1 = 1$$

Sign	*D*	*I*
+	x	$\cos(n\pi x)$
-	1	$\frac{1}{n\pi}\sin(n\pi x)$

and $a_n = \frac{2}{1}\int_0^1 x\cos\left(\frac{n\pi x}{1}\right)dx, \quad (n = 1, 2, 3, \ldots)$

$+$	0	$-\frac{1}{n^2\pi^2}\cos(n\pi x)$

$$\Rightarrow \quad a_n = 2\left[\frac{x}{n\pi}\sin(n\pi x) + \frac{1}{(n\pi)^2}\cos(n\pi x)\right]_0^1 = \frac{2\left((-1)^n - 1\right)}{(n\pi)^2}$$

Evaluating the first few terms,

$$a_0 = 1, \quad a_1 = \frac{-4}{\pi^2}, \quad a_2 = 0, \quad a_3 = \frac{-4}{9\pi^2}, \quad a_4 = 0, \quad a_5 = \frac{-4}{25\pi^2}, \quad a_6 = 0, \ldots$$

or
$$a_n = \begin{cases} 1 & (n = 0) \\ \frac{-4}{(n\pi)^2} & (n = 1, 3, 5, \ldots) \\ 0 & (n = 2, 4, 6, \ldots) \end{cases}$$

Therefore the Fourier cosine series for $f(x) = x$ on [0, 1] (which is also the Fourier series for $f(x)$ on [–1, 1]) is

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos\left((2k-1)\pi x\right)}{(2k-1)^2}$$

or

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\left(\cos\pi x + \frac{\cos 3\pi x}{9} + \frac{\cos 5\pi x}{25} + \frac{\cos 7\pi x}{49} + \ldots\right)$$

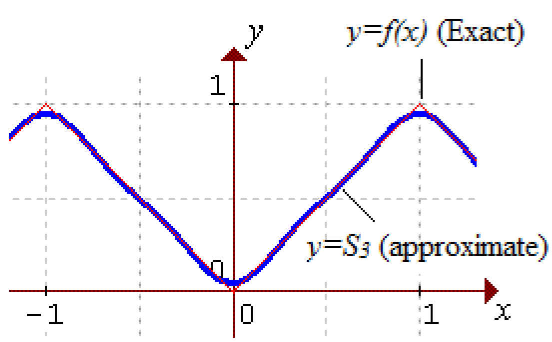


Figure 08: The graph of $y=f(x)$ and the partial sum of the first 3 terms of the above Fourier series.

Example 03: Find the half range Fourier cosine series for the function $f(x)$ defined on the interval [0, 2] by

$$f(x) = 4 - x^2$$

Solution:

The first few terms of this series are

$$f(x) = \frac{8}{3} + \frac{16}{\pi^2}\left[\cos\left(\frac{\pi x}{2}\right) - \frac{1}{4}\cos(\pi x) + \frac{1}{9}\cos\left(\frac{3\pi x}{2}\right) + \cdots\right]$$

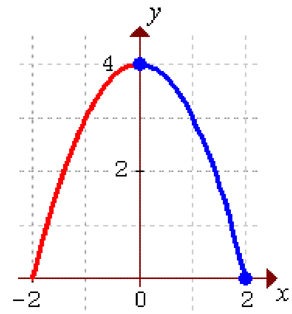


Figure 09: Even Extension of *f(x)*

Here, $T = 2L = 4$, hence $L = 2$.
An even extension of $f(x)$ is required
$b_n = 0$ for all n.

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^2 (4 - x^2)\,dx$$

$$= 2\left[4x - \frac{x^3}{3}\right]_0^2 = 8 - \frac{8}{3} = \frac{16}{3}$$

$$a_n = \int_0^2 (4 - x^2)\cos\left(\frac{n\pi x}{2}\right)dx =$$

$$\left[\left((4 - x^2)\left(\frac{2}{n\pi}\right) + 2\left(\frac{2}{n\pi}\right)^3\right)\sin\left(\frac{n\pi x}{2}\right) - 2x\left(\frac{2}{n\pi}\right)^2\cos\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$= \left(\left(0 - \frac{16(-1)^n}{(n\pi)^2}\right) - (0 - 0)\right) = \frac{16(-1)^{n+1}}{(n\pi)^2}$$

Sign	D	I
+	$4 - x^2$	$\cos\left(\frac{n\pi x}{2}\right)$
-	$-2x$	$\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$
+	-2	$-\frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)$
-	0	$-\frac{8}{n^3\pi^3}\sin\left(\frac{n\pi x}{2}\right)$

Therefore the Fourier series is

$$f(x) = \frac{8}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n^2}\cos\left(\frac{n\pi x}{2}\right)\right)$$

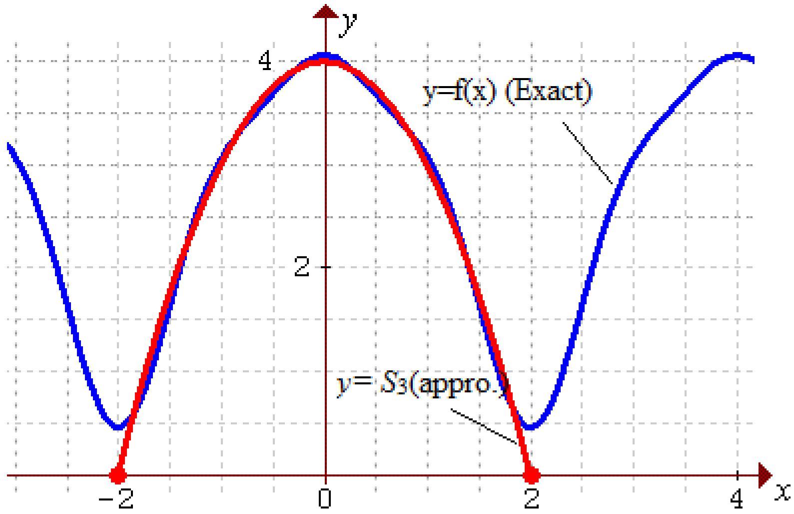


Figure 10: The graph of $y=f(x)$ and the partial sum of the first 3 terms of the above Fourier series

Example 04: Find the half range Fourier sine series for the function $f(x)$ defined on the interval [0, 2] by

$$f(x) = 2x - x^2$$

Solution:

Here, $T = 2L = 4$, hence $L = 2$. An odd extension of $f(x)$ is required.

$a_n = 0$ for all n.

$$b_n = \frac{2}{2}\int_0^2 \left(2x - x^2\right)\sin\left(\frac{n\pi x}{2}\right)dx =$$

$$\left[\left(-\left(\frac{2}{n\pi}\right)\left(2x-x^2\right) - 2\left(\frac{2}{n\pi}\right)^3\right)\cos\left(\frac{n\pi x}{2}\right) + \left(\frac{2}{n\pi}\right)^2\left(2-2x\right)\sin\left(\frac{n\pi x}{2}\right)\right]_0^2$$

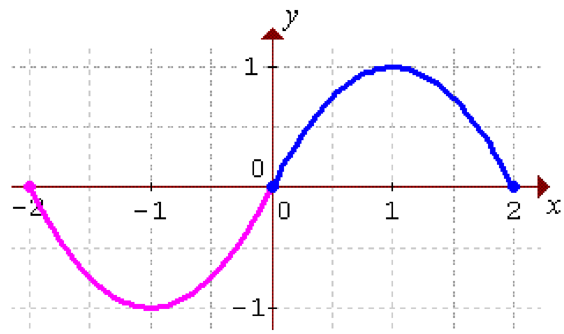


Figure 11: Odd extension of $f(x)$

$$= \left(\left(0 - 2\left(\frac{2}{n\pi}\right)^3\right)\cos(n\pi) + 0\right) - \left(\left(0 - 2\left(\frac{2}{n\pi}\right)^3\right) + 0\right)$$

$$= 2\left(\frac{2}{n\pi}\right)^3 \left(1 - (-1)^n\right) = \begin{cases} 0 & (n \text{ even}) \\ \dfrac{32}{(n\pi)^3} & (n \text{ odd}) \end{cases}$$

Sign	D	I
+	$2x - x^2$	$\sin\left(\frac{n\pi x}{2}\right)$
-	$2 - 2x$	$\frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)$
+	-2	$-\frac{4}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)$
-	0	$\frac{8}{n^3\pi^3}\cos\left(\frac{n\pi x}{2}\right)$

Therefore the Fourier series is

$$f(x) = \frac{32}{\pi^3}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}\sin\left(\frac{(2k-1)\pi x}{2}\right)$$

The first few terms of this series are

$$f(x) = \frac{32}{\pi^3}\left(\sin\left(\frac{\pi x}{2}\right) + \frac{1}{27}\sin\left(\frac{3\pi x}{2}\right) + \frac{1}{125}\sin\left(\frac{5\pi x}{2}\right) + \ldots\right)$$

The partial sum of just the first *three* non-zero terms yields an excellent approximation everywhere. The graph of $y=f(x)$ and the third partial sum $y=S_3$ illustrates:

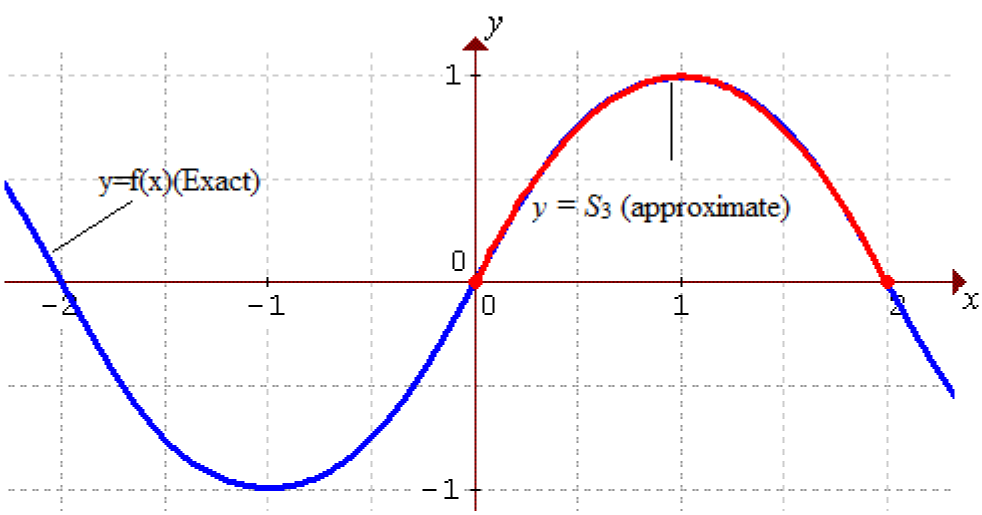


Figure 12: The graph of *f(x)* and the partial sum of the first 3 terms of the above Fourier series

Exercise: 3.3

1. Sketch the graph and express $f(x) = x$ as a half range Fourier sine and cosine series in the interval $0 < x < 2$.

   $$\text{Ans}: f_s(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(\frac{n\pi}{2})x, \quad f_c(x) = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\{(-1)^n - 1\}}{n^2}\cos(\tfrac{n\pi}{2})x$$

2. Sketch the graph and express $f(x) = x^2$ as a half range Fourier sine and cosine series in the interval $0 < x < \pi$.

$$\text{Ans: } f_s(x) = \sum_{n=1}^{\infty}\frac{2}{\pi}\left[\frac{\pi^2}{n}(-1)^{n+1} + \frac{2}{n^3}\{(-1)^n - 1\}\right]\sin nx$$

$$\text{and } f_c(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^{n+1}\cos nx.$$

3. Sketch and find the half range Fourier sine and cosine series of

   $$f(x) = \begin{cases} x, & 0 < x < 4 \\ 8 - x, & 4 < x < 8. \end{cases}$$

   Ans: $f_s(x) = \frac{32}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{8}$, and $f_c(x) = \frac{16}{\pi^2}\sum_{n=1}^{\infty}[\frac{2\cos\frac{n\pi}{2} - \cos n\pi - 1}{n^2}]\cos\frac{n\pi x}{8}$.

4. Sketch and find the half range Fourier sine and cosine series of

   $$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 2, & 1 < x < 2 \end{cases}.$$

   Ans: $f_s(x) = \sum_{n=1}^{\infty}\left(\frac{2}{n\pi} - \frac{4}{n\pi}(-1)^n\right)\sin\frac{n\pi x}{2}$, $\quad f_c(x) = \frac{3}{2} - \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\frac{n\pi}{2}\cos\frac{n\pi x}{2}$.

5. Sketch and find the half range Fourier sine and cosine series of $f(x) = \pi - x$ in the interval $0 < x < \pi$.

   Ans: $f_s(x) = 2\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right]$,

   and $f_c(x) = \frac{\pi}{2} + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$.

Fourier Integral

Fourier integral is a formula for the decomposition of a non-periodic function into harmonic components whose frequencies range over a continuous set of values.

Let $f(x)$ is a periodic function with a period $T = 2L$ or is defined on the interval $-L < x < L$. Then the *Fourier series* representation of $f(x)$ is a trigonometric series (that is, it is an infinite series consists of sine and cosine terms) of the form,

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

$$= a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(w_n x) + b_n \sin(w_n x)\right), \quad w_n = \frac{n\pi}{L}$$

$$= \sum_{n=0}^{\infty}\left(a_n \cos(w_n x) + b_n \sin(w_n x)\right)$$

Note that, $\Delta w = w_{n+1} - w_n = \dfrac{(n+1)\pi}{L} - \dfrac{n\pi}{L} = \dfrac{\pi}{L}$

$$f(x) = \frac{1}{\pi}\sum_{n=0}^{\infty}\left[\cos(w_n x)\Delta w \int_{-L}^{L} f(v)\cos(w_n v)dv + \sin(w_n x)\Delta w \int_{-L}^{L} f(v)\sin(w_n v)dv\right]$$

As, $L \to \infty, \Delta w \to 0, \Rightarrow \sum(\ )\Delta w \to \int(\ )dw.$

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\left[\cos(wx)\int_{-\infty}^{\infty} f(v)\cos(wv)dv + \sin(wx)\int_{-\infty}^{\infty} f(v)\sin(wv)dv\right]dw$$

$$f(x) = \int_{0}^{\infty}\left[\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(wv)dv\cos(wx) + \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(wv)dv\sin(wx)\right]dw$$

$$\therefore\ f(x) = \int_{0}^{\infty}\left[A(w)\cos(wx) + B(w)\sin(wx)\right]dw.$$

Hence, the Fourier integral of non-periodic function *f(x)* but piecewise continuous in any infinite interval as follows:

$$f(x) = \int_{0}^{\infty}\left[A(w)\cos(wx) + B(w)\sin(wx)\right]dw$$

Where, $A(w) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(v)\cos(wv)dv$ and $B(w) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(v)\sin(wv)dv$

Fourier cosine integral:

$$f(x) = \int_0^\infty [A(w)\cos(wx) + B(w)\sin(wx)]dw$$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(wv)dv \text{ and } B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(wv)dv$$

If the function *f(x)* is even then and writing $v = x$,

$$A(w) = \frac{2}{\pi}\int_0^\infty f(x)\cos(wx)dx \text{ (Even) and } B(w) = 0 \text{ (odd)}$$

Hence, the **Fourier cosine integral** of *f(x)* is

$$f(x) = \int_0^\infty [A(w)\cos(wx)]dw$$

Fourier sine integral:

$$f(x) = \int_0^\infty [A(w)\cos(wx) + B(w)\sin(wx)]dw$$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(wv)dv \text{ and } B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(wv)dv$$

If the function *f(x)* is odd and writing $v = x$ then,

$$A(w) = 0 \text{ (odd) and } B(w) = \frac{2}{\pi}\int_0^\infty f(x)\sin(wx)dx \text{ (Even)}$$

Hence, the **Fourier sine integral** of *f(x)* is

$$f(x) = \int_0^\infty [B(w)\sin(wx)]dw$$

Note that:

$$\int e^{ax}\sin bx dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2 + b^2}$$

$$\int e^{ax}\cos bx dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2 + b^2}$$

Example: 01

Find the Fourier integral of $f(x) = e^{-2x}$ where $x > 0$ and $f(x) = -f(-x)$ and hence prove that $\int_0^\infty \frac{w\sin(wx)}{w^2+4}dw = \frac{\pi}{2}e^{-2x}$

Solution:

Here $f(x)=e^{-2x}$

$$A(w)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(wx)dx=0 \because f(x)=-f(-x)$$

$$B(w)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(wx)dx=\frac{2}{\pi}\int_{0}^{\infty} e^{-2x}\sin(wx)dx \ =\frac{2}{\pi}\left[\frac{e^{-2x}\left(-2\sin(wx)-w\cos(wx)\right)}{w^2+4}\right]_0^{\infty}$$

$$=\frac{2}{\pi}\left(\frac{w}{w^2+4}\right)$$

Hence the Fourier integral of *f(x)* is,

$$f(x)=\int_0^{\infty}\left[A(w)\cos(wx)+B(w)\sin(wx)\right]dw$$

$$\therefore e^{-2x} =\int_0^{\infty}\left[\frac{2}{\pi}\left(\frac{w}{w^2+4}\right)\sin(wx)\right]dw$$

Now,

$$\int_0^{\infty}\frac{w\sin(wx)}{w^2+4}dw=\frac{\pi}{2}e^{-2x} \text{ (Proved)}$$

Example: 02 Find the Fourier integral of the function

$$f(x)=\begin{cases}0 & ,\text{when } x<-3\\ 1 & ,\text{when } -3<x<3\\ 0 & ,\text{when } x>3\end{cases}.$$

Solution:

$$A(w)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(wx)dx=\frac{1}{\pi}\int_{-\infty}^{-3}0.\cos(wx)dx+\frac{1}{\pi}\int_{-3}^{3}1.\cos(wx)dx+\frac{1}{\pi}\int_{3}^{\infty}0.\cos(wx)dx$$

$$=\frac{2}{\pi}\int_0^3 1.\cos(wx)dx=\frac{2}{\pi}\left[\frac{\sin(wx)}{w}\right]_0^3=\frac{2\sin(3w)}{\pi w}$$

$$B(w)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(wx)dx=\frac{1}{\pi}\int_{-\infty}^{-3}0.\sin(wx)dx+\frac{1}{\pi}\int_{-3}^{3}1.\sin(wx)dx+\frac{1}{\pi}\int_{3}^{\infty}0.\sin(wx)dx$$

$$\therefore B(w)=0$$

Hence the Fourier integral of *f(x)* is,

$$f(x)=\int_0^{\infty}\left[A(w)\cos(wx)+B(w)\sin(wx)\right]dw$$

$$\therefore f(x)=\int_0^{\infty}\left[\left(\frac{2\sin 3w}{\pi w}\right)\cos(wx)\right]dw$$

Example: 03 Find the Fourier integral of the function

$$f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Solution: The Fourier integral of *f(x)* is,

$$f(x) = \int_0^\infty [A(w)\cos(wx) + B(w)\sin(wx)]dw$$

where,

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(wx)dx = \frac{1}{\pi}\int_{-\infty}^{0} 0.\cos(wx)dx + \frac{1}{\pi}\int_0^\infty e^{-x}\cos(wx)dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}\left(-\cos(wx) + w\sin(wx)\right)}{w^2+1}\right]_0^\infty = \frac{1}{\pi(1+w^2)}$$

and

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(wx)dx = \frac{1}{\pi}\int_{-\infty}^{0} 0.\sin(wx)dx + \frac{1}{\pi}\int_0^\infty e^{-x}.\sin(wx)dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}\left(-\sin(wx) - w\cos(wx)\right)}{w^2+1}\right]_0^\infty = \frac{w}{\pi(1+w^2)}$$

Hence the Fourier integral of *f(x)* is,

$$f(x) = \int_0^\infty [A(w)\cos(wx) + B(w)\sin(wx)]dw$$

$$\therefore\ f(x) = \int_0^\infty \left[\frac{1}{\pi(1+w^2)}\cos(wx) + \frac{w}{\pi(1+w^2)}\sin(wx)\right]dw$$

Example: 04

Find the Fourier **sine integral** of the function

$$f(x) = \begin{cases} x, & \text{when } 0 < x < 2 \\ 0, & \text{when } x > 2 \end{cases}.$$

Solution:

We know, for Fourier sine integral $A(w) = 0$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(wx)dx = \frac{2}{\pi}\int_0^\infty f(x)\sin(wx)dx = \frac{2}{\pi}\int_0^2 x.\sin(wx)dx + \frac{2}{\pi}\int_2^\infty 0.\sin(wx)dx$$

$$= \frac{2}{\pi}\left[-x\frac{\cos(wx)}{w} + \frac{\sin(wx)}{w^2}\right]_0^2 = \frac{2}{\pi w^2}\left(\sin 2w - 2w\cos 2w\right)$$

Hence the Fourier sine integral of *f(x)* is,

$$f(x) = \int_0^\infty [B(w)\sin(wx)]dw = \int_0^\infty \left[\frac{2}{\pi w^2}\left(\sin 2w - 2w\cos 2w\right)\sin(wx)\right]dw$$

Exercise: 3.4

1. Find the Fourier integral of the function

$$f(x) = \begin{cases} 0 & \text{when } x < -1 \\ 1 - x & \text{when } -1 \le x \le 1 \\ 0 & \text{when } x \ge 1 \end{cases}.$$

Ans: $f(x) = \frac{1}{\pi}\int_0^\infty \left[\frac{2}{\pi w}\sin w\cos wx + \frac{2}{\pi}\left(\frac{1}{w}\cos w - \frac{1}{w^2}\sin w\right)\sin wx\right] dw.$

2. Find the Fourier integral of the function

$$f(x) = \begin{cases} 0 & \text{when } x < 0 \\ \frac{1}{2} & \text{when } x = 0 \\ e^{-x} & \text{when } x > 0 \end{cases}.$$

Ans: $f(x) = \frac{1}{\pi}\int_0^\infty \frac{\cos ux + u\sin ux}{1+u^2} du.$

3. Find the Fourier integral of $f(x) = e^{-kx}$ when $x > 0$ and $f(-x) = -f(x)$ for $k > 0$ and hence prove that $\int_0^\infty \frac{w\sin(wx)}{w^2+k^2}dw = \frac{\pi}{2}e^{-kx}, k > 0$.

4. Find the Fourier integral of $f(x) = e^{-kx}$ when $x > 0$ and $f(-x) = f(x)$ for $k > 0$ and hence prove that $\int_0^\infty \frac{\cos(wx)}{w^2+k^2}dw = \frac{\pi}{2k}e^{-kx}, k > 0$.

Fourier Transform

The Fourier Transform is a generalization of the Fourier Series. Strictly speaking it only applies to continuous and aperiodic functions. The Fourier Transform converts a set of time domain data vectors into a set of frequency domain vectors. The Fourier transform is called the frequency domain representation of the original signal. The term Fourier transforms refers to both the frequency domain representation and the mathematical operation that associates the frequency domain representation to a function of time.

Finite Fourier sine transforms:

From half range Fourier sine series of $f(x)$ in the interval $0 < x < L$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{L}\right) \quad \ldots \quad \ldots (1)$$

Where $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{\pi n x}{L}\right)dx = \frac{2}{L}F_s\left(f(x)\right) = \frac{2}{L}F_s(n)$ and

$F_s\left(f(x)\right) = \int_0^L f(x)\sin\left(\frac{\pi n x}{L}\right)dx$ is called finite Fourier sine transform.

From equation (1) ,we get

$f(x) = \frac{2}{L}\sum_{n=1}^{\infty} F_s\left(f(x)\right)\sin\left(\frac{\pi n x}{L}\right)$ is called inverse finite Fourier sine transform of $F_s(f(x))$.

Finite Fourier cosine transforms:

From half range Fourier cosine series of $f(x)$ in the interval $0 < x < L$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n x}{L}\right) \quad \ldots \quad \ldots (2)$$

Where $a_0 = \frac{2}{L}\int_0^L f(x)\,dx = \frac{2}{L}F_c(0)$ and $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{\pi n x}{L}\right)dx = \frac{2}{L}F_c(n)$ and

$F_c(n) = \int_0^L f(x)\cos\left(\frac{\pi n x}{L}\right)dx$ is called finite Fourier cosine transform.

From equation (2), we get

$f(x) = \frac{1}{L}F_c(0) + \frac{2}{L}\sum_{n=1}^{\infty} F_c(n)\cos\left(\frac{\pi n x}{L}\right)$ is called inverse finite Fourier cosine transform of $F_c(n)$.

Infinite Fourier sine transforms:

For an **odd** function $f(x)$, the Fourier integral is the Fourier sine integral

$$f(x)=\int_0^\infty [B(w)\sin(wx)]dw\ldots\ldots(3) \text{ where } B(w)=\frac{2}{\pi}\int_0^\infty f(v)\sin(wv)dv\ldots\ldots(4)$$

We now set $B(w)=\sqrt{\frac{2}{\pi}}\hat{F}_s(w)$. Then from (4) writing $v=x$ we have

$$\hat{F}_s(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(wx)dx$$

This is called **infinite Fourier sine transform** of $f(x)$. Similarly, from (3) we have,

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{F}_s(w)\sin(wx)dw$$

This is called the **inverse infinite Fourier sine transform** of $\hat{F}_s(w)$.

Infinite Fourier cosine transforms:

For an **even** function $f(x)$, the Fourier integral is the Fourier cosine integral

$$f(x)=\int_0^\infty [A(w)\cos(wx)]dw\ldots\ldots(5) \text{ where } A(w)=\frac{2}{\pi}\int_0^\infty f(v)\cos(wv)dv\ldots\ldots(6)$$

We now set $A(w)=\sqrt{\frac{2}{\pi}}\hat{F}_C(w)$. Then from (4) writing $v=x$ we have

$$\hat{F}_C(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(wx)dx$$

This is called **infinite Fourier cosine transform** of $f(x)$. Similarly, from (5) we have,

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{F}_C(w)\cos(wx)dw$$

This is called the **inverse infinite Fourier cosine transform** of $\hat{F}_C(w)$.

Example: 01

Find the Fourier sine transform of $f(x)=\begin{cases} x & ,\text{when } 0<x<\dfrac{\pi}{2} \\ \pi - x & ,\text{when } \dfrac{\pi}{2}<x<\pi \end{cases}$.

Solution: Here $L=\pi$

We know the finite Fourier sine transforms of $f(x)$ is

$$F_S(f(x))=\int_0^L f(x)\sin\left(\frac{\pi n x}{L}\right)dx \quad =\int_0^{\pi} f(x)\sin nx\,dx$$

$$=\int_0^{\frac{\pi}{2}} x\sin nx\,dx+\int_{\frac{\pi}{2}}^{\pi}(\pi-x)\sin nx\,dx$$

$$=\left[-x\frac{\cos nx}{n}+\frac{\sin nx}{n^2}\right]_0^{\frac{\pi}{2}}+\left[-(\pi-x)\frac{\cos nx}{n}-\frac{\sin nx}{n^2}\right]_{\frac{\pi}{2}}^{\pi}$$

$$=-\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right)+\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)+\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right)+\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)$$

$$=\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)+\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)$$

$$=\frac{2}{n^2}\sin\left(\frac{n\pi}{2}\right)$$

$$\therefore F_s(f(x))=\frac{2}{n^2}\sin\left(\frac{n\pi}{2}\right) \quad \text{(Ans.)}$$

Example: 02 Find the Fourier cosine transform of $f(x)=\begin{cases} 1 & \text{when } 0<x<\dfrac{\pi}{2} \\ -1 & \text{when } \dfrac{\pi}{2}<x<\pi \end{cases}$.

Solution: Here $L=\pi$

We know the finite Fourier cosine transforms of $f(x)$ is

$$F_C(n)=\int_0^L f(x)\cos\left(\frac{\pi n x}{L}\right)dx$$

$$=\int_0^{\pi} f(x)\cos nx\,dx = \int_0^{\frac{\pi}{2}} 1.\cos nx\,dx+\int_{\frac{\pi}{2}}^{\pi} -1.\cos nx\,dx$$

$$=\left[\frac{\sin nx}{n}\right]_0^{\frac{\pi}{2}}-\left[\frac{\sin nx}{n}\right]_{\frac{\pi}{2}}^{\pi} = \frac{1}{n}\sin\left(\frac{n\pi}{2}\right)+\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)$$

$$= \frac{2}{n}\sin\left(\frac{n\pi}{2}\right)$$

$$F_c(n) = \frac{2}{n}\sin\left(\frac{n\pi}{2}\right).$$

Example 03: Find the Fourier sine and cosine transform of $f(x) = \begin{cases} x & \text{when } 0 < x < 1 \\ 2 & \text{when } 1 < x < 2 \\ 0 & \text{when } x > 2 \end{cases}$

Solution: We know the **infinite Fourier sine transform** is,

$$\hat{F}_s(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(nx)dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^1 x\sin(nx)dx + \int_1^2 2\sin(nx)dx + \int_2^\infty 0.\sin(nx)dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\left(-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right)_0^1 - \frac{2}{n}\left(\cos(nx)\right)_1^2\right]$$

$$\therefore \hat{F}_s(n) = \sqrt{\frac{2}{\pi}}\left[-\frac{1}{n}\cos n + \frac{1}{n^2}\sin n - \frac{2}{n}\left(\cos(2n) - \cos n\right)\right]$$

We know the **infinite Fourier cosine transform** is,

$$\hat{F}_C(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(nx)dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^1 x\cos(nx)dx + \int_1^2 2\cos(nx)dx + \int_2^\infty 0.\cos(nx)dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\left(\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right)_0^1 + \frac{2}{n}\left(\sin(nx)\right)_1^2\right]$$

$$\therefore \hat{F}_C(n) = \sqrt{\frac{2}{\pi}}\left[\frac{1}{n}\sin n + \frac{1}{n^2}\cos n - \frac{1}{n^2} + \frac{2}{n}\left(\sin(2n) - \sin n\right)\right]$$

Exercise: 3.5

Sketch the graph and then find the (a) finite Fourier sine transform, and (b) finite Fourier cosine transform of the following functions:

1. $f(x) = 2x$ where $0 < x < 4$.

Ans: $f_s(n) = \frac{32}{n\pi}(-1)^{n+1}$, $f_c(n) = \frac{32}{n^2\pi^2}[(-1)^n - 1]$.

2. $f(x) = x^2, \quad 0 < x < b$.

Ans: $f_s(n) = \begin{cases} \frac{2b^3}{n^3\pi^3}(\cos n\pi - 1) - \frac{b^3}{n\pi}\cos n\pi\,, & n = 1,2,3,.. \\ \frac{b^3}{3}, & n = 0 \end{cases}$, $f_c(n) = \frac{2b^3}{n^2\pi^2}(\cos n\pi - 1)$.

3. Sketch the graph and then find the (a) infinite Fourier sine transform, and (b) infinite Fourier cosine transform of $f(x) = \begin{cases} 1 & \text{when } 0 \le x < 1 \\ 0 & \text{when } x \ge 1 \end{cases}$

Ans: $f_s(n) = \sqrt{\frac{2}{\pi}}\frac{1}{n}(1 - \cos n)$, $f_c(n) = \sqrt{\frac{2}{\pi}}\frac{\sin n}{n}$.

4. Sketch the graph and then find the (a) infinite Fourier sine transform, and (b) infinite Fourier cosine transform of $f(x) = \begin{cases} x & \text{when } 0 < x < 1 \\ 2 - x & \text{when } 1 < x < 2 \\ 0 & \text{when } x > 2 \end{cases}$.

Ans: $f_s(n) = \sqrt{\frac{2}{\pi}}\frac{1}{n^2}(2\sin n - \sin 2n)$, $f_c(n) = \frac{1}{n^2}\sqrt{\frac{2}{\pi}}\,(2\cos n - \cos 2n - 1)$.

5. Sketch the graph and then find the (a) infinite Fourier sine transform, and (b) infinite Fourier cosine transform of $f(x) = e^{-x}, \quad x \ge 0$.

Ans: $f_s(n) = \sqrt{\frac{2}{\pi}}\frac{n}{n^2+1}$, $f_c(n) = \sqrt{\frac{2}{\pi}}\frac{1}{n^2+1}$.

Application of Fourier transform

Solutions of partial differential equations (Boundary Value Problem) by Fourier transform:

Finite Fourier transforms of partial derivatives

sine transformation cosine transformation

Selection of finite sine and cosine transform:

We shall decide the choice of finite sine or cosine transform by the form of boundary conditions, such that

(a) The conditions $U(0,t)$ and $U(l,t)$, that is finite sine transformation.

(b) The conditions $U(0,t)$ and $U_x(l,t)$ or $\dfrac{\partial U(l,t)}{\partial x}$, that is finite cosine transformation.

where U are the functions of x and t.

Example: 15

Use the Fourier transformation to solve the following boundary value problem

$\dfrac{\partial U}{\partial t}=\dfrac{\partial^2 U}{\partial x^2}; U(0,t)=U(\pi,t)=0; t>0$ and $U(x,0)=2x$ where $0<x<\pi$.

Solution: Given that

$$\frac{\partial U}{\partial t}=\frac{\partial^2 U}{\partial x^2}$$

Taking both sides finite sine transformation

$$\int_0^\pi \frac{\partial U}{\partial t}\sin nx dx=\int_0^\pi \frac{\partial^2 U}{\partial x^2}\sin nx\,dx \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Let

$$u=u(n,t)=\int_0^\pi U(x,t)\sin nx\,dx$$

$$\frac{\partial u}{\partial t}=\int_0^\pi \frac{\partial U}{\partial t}\sin nx dx=\int_0^\pi \frac{\partial^2 U}{\partial x^2}\sin nx\,dx \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

[from equation (ii)]

$$= \frac{\partial U}{\partial x} \sin nx \Big|_0^{\pi} - n \int_0^{\pi} \frac{\partial U}{\partial x} \cos nx \, dx$$

$$= 0 - nU(x,t) \cos nx \Big|_0^{\pi} - n^2 \int_0^{\pi} \frac{\partial U}{\partial x} \sin nx \, dx = 0 - 0 - n^2 u$$

$$= -n^2 u$$

$$\therefore \frac{\partial u}{\partial t} = -n^2 u \quad \Rightarrow \frac{\partial u}{u} = -n^2 \partial t \quad \Rightarrow \ln u = -n^2 t + \ln A$$

[Integrating]

$$u(n,t) = Ae^{-n^2 t} \text{.....................}(iii)$$

$$u(n,0) = A \quad \text{[when } t = 0\text{]}$$

$$\Rightarrow \int_0^{\pi} U(x,0) \sin nx dx = A$$

$$\Rightarrow A = \int_0^{\pi} 2x \sin nx dx = \frac{-2x \cos nx}{n} \Big|_0^{\pi} + \frac{2}{n} \int_0^{\pi} \cos nx dx$$

$$\therefore A = -\frac{2\pi}{n} \cos n\pi$$

Now, from equation (iii), we get

$$u(n,t) = -\frac{2\pi}{n} \cos n\pi e^{-n^2 t} \text{.....................(iii)}$$

So, sine transformation is

$$U(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \left(-\frac{2\pi}{n} \cos n\pi e^{-n^2 t}\right) \sin nx$$

.

Exercise: 3.6

Use the finite Fourier transform to solve the following boundary value problems:

(1) $\frac{\partial U}{\partial t} = 3 \frac{\partial^2 U}{\partial x^2}$ with $U(0,t) = U(2,t) = 0$ and $U(x,0) = x,$ where $t > 0$ and $0 < x < 2$.

Ans: $U(x,t) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin \frac{n\pi x}{2} e^{-\frac{3}{4} n^2 \pi^2 t}$.

(2) $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$ with $U(0,t) = U(\pi,t) = 0$ and $U(x,0) = 2x,$ where $t > 0$ and $0 < x < \pi$.

Ans: $U(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{-2\pi}{n} \cos n\pi \, e^{-n^2 t}\right) \sin nx$.

(3) $\frac{\partial U}{\partial t}=\frac{\partial^2 U}{\partial x^2}$ with $U(0,t)=U(6,t)=0$ and

$U(x,0)=\begin{cases}1, & 0<x<3\\ 0, & 3<x<6\end{cases}$ where $t>0$ and $0<x<6$.

Ans: $U(x,t)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(1-\cos\frac{n\pi}{2}\right)e^{-\frac{n^2\pi^2 t}{36}}\sin\frac{n\pi x}{6}$.

Discrete Fourier Transform:

Discrete Fourier Transforms are helpful in digital signal processing for making convolution and many other signal manipulations.

A physical process can be described either in the time domain, by some quantity $h(t)$, or else in the frequency domain, that is $H(f)$, with $-\infty<f<\infty$. For many purposes, one can relate $h(t)$ and $H(f)$ by the equation of Fourier transform and inverse Fourier transform given below:

$$H(f)=\int_{-\infty}^{\infty}h(t)\,e^{i2\pi ft}dt \qquad (1)$$

$$h(t)=\int_{-\infty}^{\infty}H(f)\,e^{-i2\pi ft}df \qquad (2)$$

If t is considered as time (seconds) then f will be frequency (cycles/seconds) in the above equations. On the other hand, if h is considered as a function of x (in meters) then H will be a function of inverse wavelength (cycles/meter).
In most real type problems, $h(t)$ is sampled (i.e. it's value is recorded) at evenly spaced intervals, say Δ, in time. Then, the sequence of N consecutive sampled values is,

$$h_k=h(t_k),\qquad t_k=k\Delta,\qquad k=0,1,2,3,\dots,N-1.$$

For simplicity, we will hereafter consider that N is even. The reciprocal of the time interval, Δ, is called the sampling rate. For any sampling interval Δ, there is a special frequency, called Nyquist critical frequency (f_c), is given by

$$f_c=\frac{1}{2\Delta}.$$

Sampling theorem: If a continuous function $h(t)$ is sampled at an interval Δ, then $H(f)=0$ for all $|f|\geq f_c$.
With N numbers of sampled values, we will evidently be able to produce no more than N numbers of output. So instead of trying to estimate the Fourier transform $H(f)$ at all values of f in the range $-f_c$ to f_c, let us seek estimates only at the discrete values,

$$f_n=\frac{n}{N\Delta}\,,\qquad n=-\frac{N}{2},\dots,-3,-2,-1,0,1,2,3,\dots\frac{N}{2}. \qquad (3)$$

The extreme values of n given above correspond exactly to the lower and upper limits of the Nyquist critical frequency range. It is noticed that there are $N+1$, not N, values of f_n. However, it is turned out that f_n is equal both at $n=-\frac{N}{2}$ and $\frac{N}{2}$. This reduces the count to N.
The Fourier transform in equation (1), can be discretely written as

$$H(f_n) \approx \sum_{k=0}^{N-1} h(t_k)\, e^{i2\pi f_n t_k}\Delta = \Delta \sum_{k=0}^{N-1} h(t_k)\, e^{i2\pi \frac{n}{N\Delta}k\Delta} = \Delta \sum_{k=0}^{N-1} h(t_k)\, e^{i2\pi k\frac{n}{N}}$$

The discrete Fourier transform(DFT) of the N points h_k, is then defined as the discrete sum,

$$H_n \equiv \sum_{k=0}^{N-1} h(t_k)\, e^{i2\pi k\frac{n}{N}}.$$

From the above definition, the DFT does not depend on any dimensional parameter, such as the time scale Δ. The relationship between continuous and discrete Fourier transform can be written as, $H(f_n) \approx \Delta H_n$ where f_n is given by equation (2). However, one can easily verify that, $H_{-n} = H_{N-n}\,, n = 1, 2, 3, \dots, \frac{N}{2}$. With this conversion in mind, one can understand that n in H_n vary from 0 to $N-1$ (one complete period). Following this convention, we can write,

$$f_n = \begin{cases} 0 & n = 0 \\ 0 < f_n < f_c & 1 \le n \le N/2 - 1 \\ -f_c < f_n < 0 & N/2 + 1 \le n \le N - 1 \\ f_c \; or - f_c & n = N/2 \end{cases}$$

The formula for the inverse discrete Fourier transform (IDFT), which recovers the set h_k's exactly from the H_n's, is:

$$h_k \equiv \frac{1}{N} \sum_{n=0}^{N-1} H_n\, e^{-i2\pi k\frac{n}{N}}.$$

Notice that the only differences between DFT and IDFT are changing the sign in the exponential and dividing the answer by N.

Example -1: Find N-point DFT for $h_k = \delta[k] + 0.9\,\delta[k-6]$ where the unit impulse function, $\delta[k]$, is defined as $\delta[k] = \begin{cases} 1 & for\ k = 0 \\ 0 & for\ k \neq 0. \end{cases}$ Assume that $t_k = k\Delta = k$ for $\Delta = 1$.

Solution: The discrete Fourier transform is defined as

$$\begin{aligned} H_n &\equiv \sum_{k=0}^{N-1} h_k\, e^{i2\pi k\frac{n}{N}} \\ &= \sum_{k=0}^{N-1} \{\delta[k] + 0.9\,\delta[k-6]\}\, e^{i2\pi k\frac{n}{N}} \\ &= \delta[0]e^{0} + 0.9\,\delta[6-6]e^{i2\pi 6\frac{n}{N}} \\ &= 1 + 0.9e^{i12\pi\frac{n}{N}} \\ &= 1 + 0.9\left[\cos\left(12\pi\frac{n}{N}\right) + i\,\sin\left(12\pi\frac{n}{N}\right)\right] \qquad [\because e^{i\theta} = \cos\theta + i\sin\theta] \end{aligned}$$

where $n = 0, 1, 2, \dots, N-1$. (…Ans)

<u>Note:</u> If $N = 8$ then 8-point DFT is given by $H_n = 1 + 0.9\left[\cos\left(\frac{3\pi}{2}n\right) + i\,\sin\left(\frac{3\pi}{2}n\right)\right]$

n	0	1	2	3	4	5	6	7
H_n	1.9	$1 - 0.9i$	0.1	$1 + 0.9i$	1.9	$1 - 0.9i$	0.1	$1 + 0.9i$
$\lvert H_n \rvert$	1.9	1.3454	0.1	1.3454	1.9	1.3454	0.1	1.3454

Exercise 3.7

Find N-point DFT for the following signal h_k:

(i) $h_k = 7\delta[k]$

(ii) $h_k = 5\delta[k-7]$

(iii) $h_k = 6\delta[k] + 10\ \delta[k-5]$

(iv) $h_k = \delta[k-2] + 5\ \delta[k-6]$

Ans: (i) 7, (ii) $5\left[\cos\frac{14\pi n}{N} + i\sin\frac{14\pi n}{N}\right], n = 0,1,2, \dots \dots, N-1,$

(iii) $6 + 10\ \left[\cos\frac{10\pi n}{N} + i\sin\frac{10\pi n}{N}\right], n = 0,1,2, \dots \dots, N-1,$

(iv) $\left[\cos\frac{4\pi n}{N} + i\sin\frac{4\pi n}{N}\right] + 5\left[\cos\frac{12\pi n}{N} + i\sin\frac{12\pi n}{N}\right], n = 0,1,2, \dots \dots, N-1.$

Reference:

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