Connectivity

Course Code: 00090 Course Title: Discrete Mathematics

Dept. of Computer Science
Faculty of Science and Technology

Lecturer No:	**17**	**Week No:**	**10**	**Semester:**	
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Lecture Outline

Connectivity 8.4

- Paths
- Circuits
- Simple path and Simple circuits
- Connectedness in undirected Graph
- Connectedness in directed Graph
- Graph Isomorphism : checking simple circuit of certain length

Objectives and Outcomes

- Objectives: To understand the terms path, circuit, simple path, simple circuit; to understand connectedness in undirected and directed graphs, to determine whether two graphs are isomorphic.

- Outcomes: The students are expected to be able to explain path, circuit, simple path, simple circuit; be able to determine whether an undirected graph is connected or whether a directed graph is weakly or strongly connected; be able to determine whether two graphs are isomorphic.

Paths

- A **path** is a sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph.

Paths

- **Definition1**: Let *n* be a nonnegative integer and G an undirected graph. A **path** of length *n* from *u* to *v* in G is a sequence of *n* edges $e_1, e_2, \ldots, e_n$ of G such that

  e_1 is associated with $\{x_0, x_1\}$, e_2 is associated with $\{x_1, x_2\}$, and so on, with e_n associated with $\{x_{n-1}, x_n\}$, where $x_0=u$ and $x_n=v$.
- The path is a ***circuit*** if it begins and ends at the same vertex, that is, if u = v, and has length greater than zero.
- The path or circuit is said to *pass through the vertices* $x_1, x_2, \ldots, x_{n-1}$ or *traverses the edges* $e_1, e_2, \ldots, e_n$.
- A **path/circuit** is ***simple*** if it does not contain the same edge more than once.

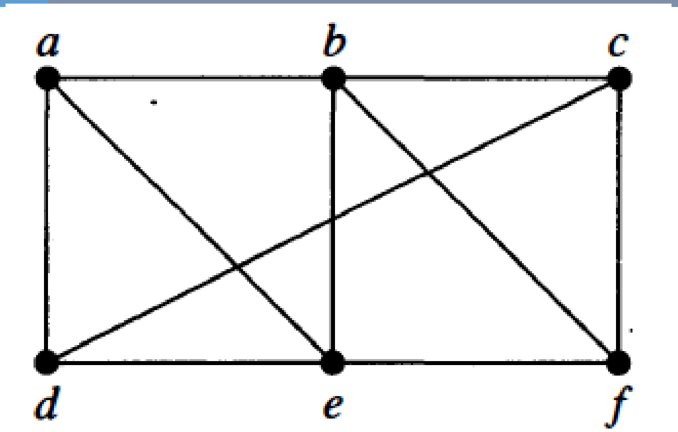
Paths in Directed Graph

- Same as in undirected graphs, but the path must go in the direction of the arrows.

Connectedness in Undirected Graphs

- An ***undirected graph*** is called ***connected*** if there is a **path** between every pair of distinct vertices of the graph

Example 2 (p.561)



- In the simple graph shown in Figure, a , d, c, f, e is a **simple path** of length 4, because {a , d}, {d, c } , {c, f}, and { f, e} are all edges.
- However, d, e, c, a is not a path, because {e, c} is not an edge.
- Note that b, c, f, e, b is a circuit of length 4 because {b, c}, {c, f}, {f, e}, and {e, b} are edges, and this path begins and ends at b.
- The path a, b, e, d, a, b, which is of length 5, is **not simple** because it contains the edge {a, b} twice.

Example 6 (p.563)

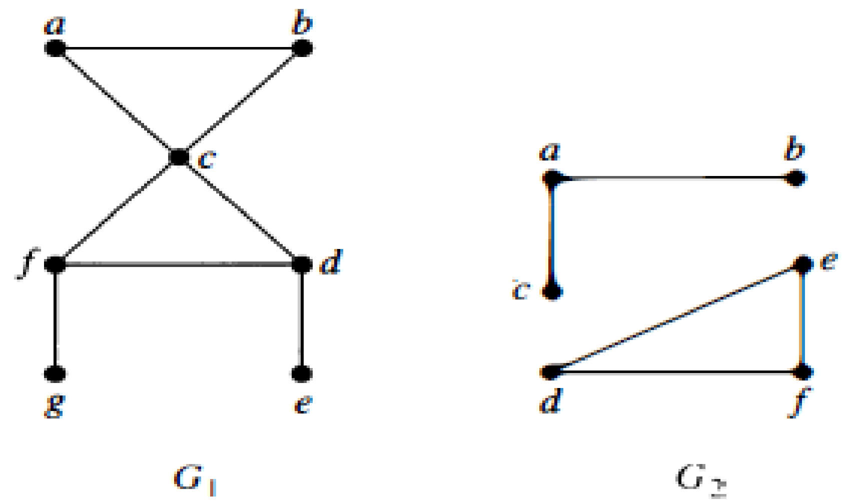


Figure 3: The Graphs G_1 and G_2

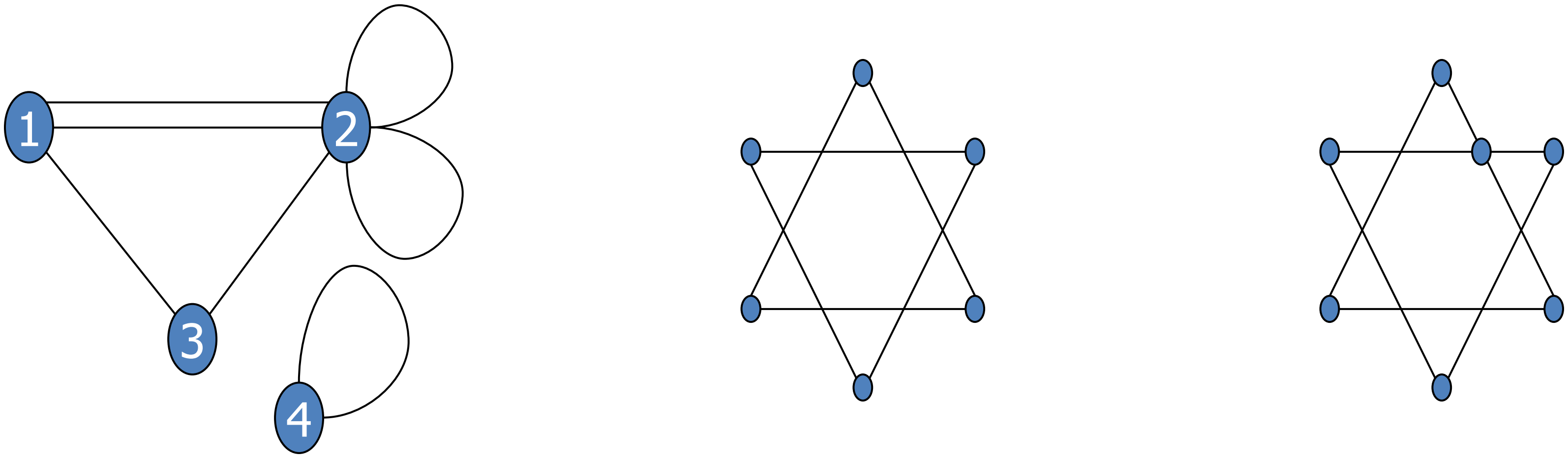
- The graph G_1 in Figure 3 is **connected**, because for every pair of distinct vertices there is a path between them.
- However, the graph G_2 in Figure 3 is **not connected**. For instance, there is no path in G2 between vertices *a* and *d*.

Connectivity

- Connectivity is a basic concept of graph theory. It defines whether a graph is connected or disconnected. Without connectivity, it is not possible to traverse a graph from one vertex to another vertex.
- A graph is said to be connected graph if there is a path between every pair of vertex. From every vertex to any other vertex there must be some path to traverse. This is called the connectivity of a graph.
- A graph is said to be disconnected, if there exists multiple disconnected vertices and edges.
- Graph connectivity theories are essential in network applications, routing transportation networks, network tolerance etc.

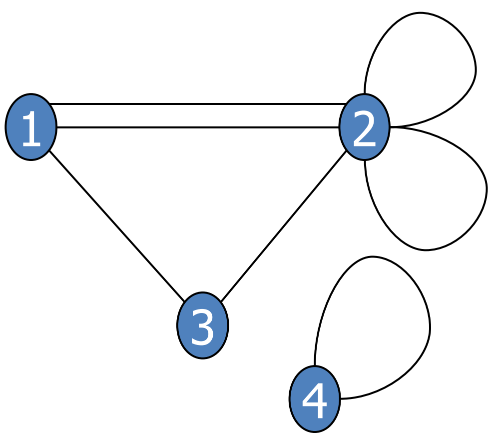
Connectivity

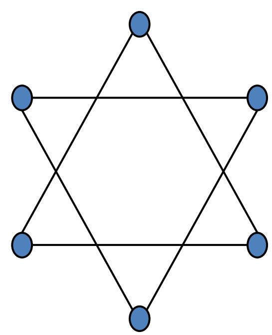
- <u>Question</u>: Which of the following graphs are connected?

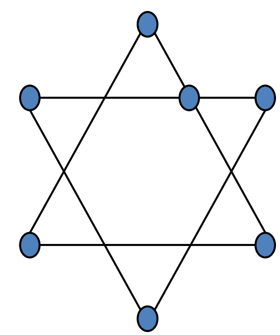


Connectivity

Answer: First and second are disconnected. Last one is connected.





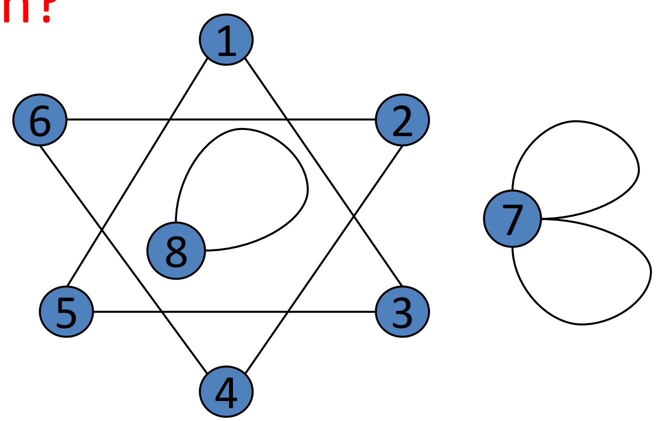


Connected Components

- A connected component of a graph G is a connected subgraph of G that is not proper subgraph of another connected subgraph of G.
- A connected component of a graph G is a maximal connected subgraph of G.
  - A graph G that is not connected has two or more connected components that are disjoint and have G as their union

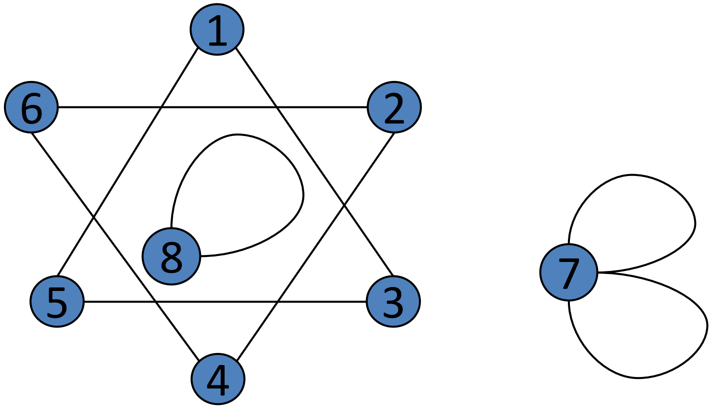
Connected Components

- **Definition**: A ***connected component*** in a graph *G* is a set of vertices such that all vertices in the set are connected to each other and every possible connected vertex is included.
- **Question** : What are the connected components of the following graph?



Connected Components

- **Answer**: The connected components are {1,3,5}, {2,4,6}, {7} and {8} as one can see visually by pulling components apart:



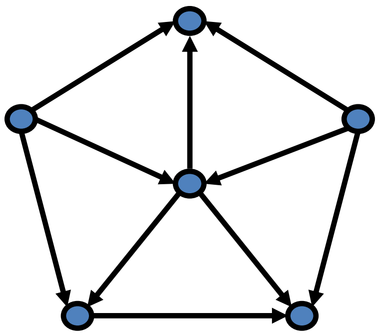
Connectedness in Directed Graphs

- A **directed graph** is ***strongly connected*** if there is a path from *a* to *b* and from *b* to *a* whenever *a* and *b* are vertices in the graph.
  - For a directed graph to be strongly connected there must be a sequence of directed edges from any vertex in the graph to any other vertex
- A **directed graph** is ***weakly connected*** if there is a path between every two vertices in the underlying undirected graph.
- **Note**: Any strongly connected directed graph is also weakly connected, but the opposite is **not true**.

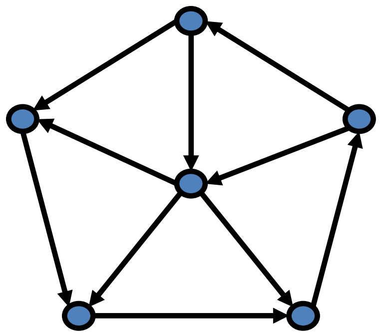
EXAMPLE: Are the directed graphs G and H shown in Figure strongly connected? Are they weakly connected?

Weakly Connected

Strongly Connected



G



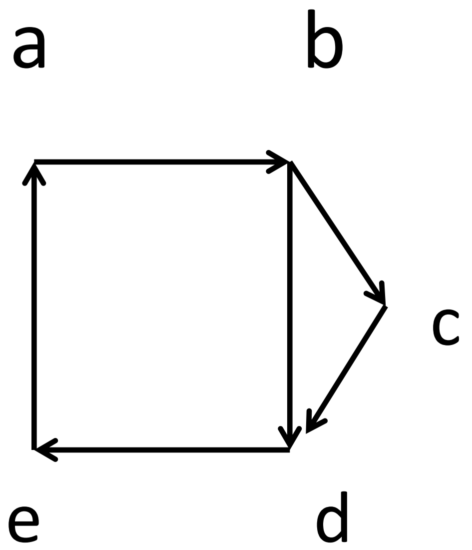
H

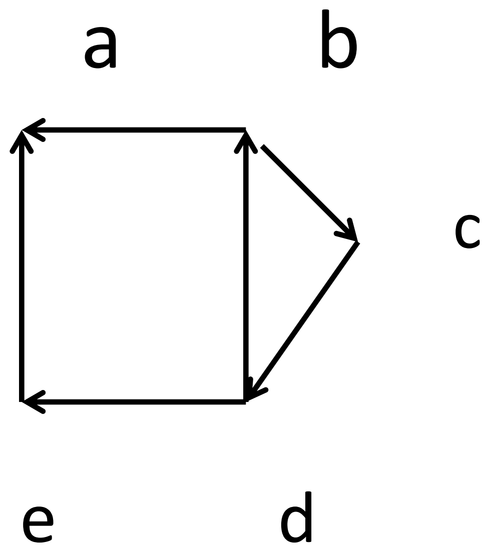
H is strongly connected because there is a path between any two vertices in this directed graph. Hence, H is also weakly connected.

The graph G is not strongly connected. There is no directed path from a vertex to other vertex in this graph.

Example

- Graph **G** is **strongly connected** (hence G is also weakly connected). Graph **H** is not strongly connected, however H is weakly connected.





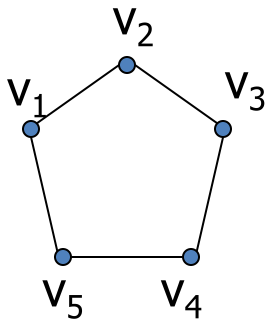
Graph **G** Graph **H**

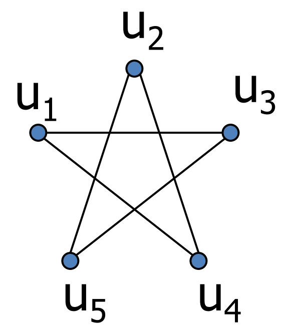
Paths and Isomorphism

- There are several ways that **paths** & **circuits** can help determine whether two graphs are isomorphic.
  - For example, the existence of a simple circuit of a particular length is a useful **invariant** that can be used to show that two graphs are not isomorphic
- A useful **isomorphic invariant** for simple graphs is the **existence of a simple circuit of length k**, where k is a positive integer greater than 2.

Graph Isomorphism – Example

Determine whether the graphs G and H are isomorphic.





G **H**

Solution

At first, you check all the graph invariants between the two graphs you know so far. If there is any mismatch, then the graphs are not isomorphic.

However, if all the graph invariants are same, then you go for mapping (i.e., mapping the vertices of one graph to another).

Here, all the graph invariants are same (you check it!).

Now you go for mapping…

$f(v_1) = u_1$, $f(v_2) = u_3$, $f(v_3) = u_5$, $f(v_4) = u_2$, $f(v_5) = u_4$

The **function *f*** is **bijective** (i.e., ***one-to-one*** as well as ***onto***).

So, the graphs G and H are isomorphic.

Example 14(p.565): Determine whether the graphs G and H shown in Figure 8 are isomorphic.

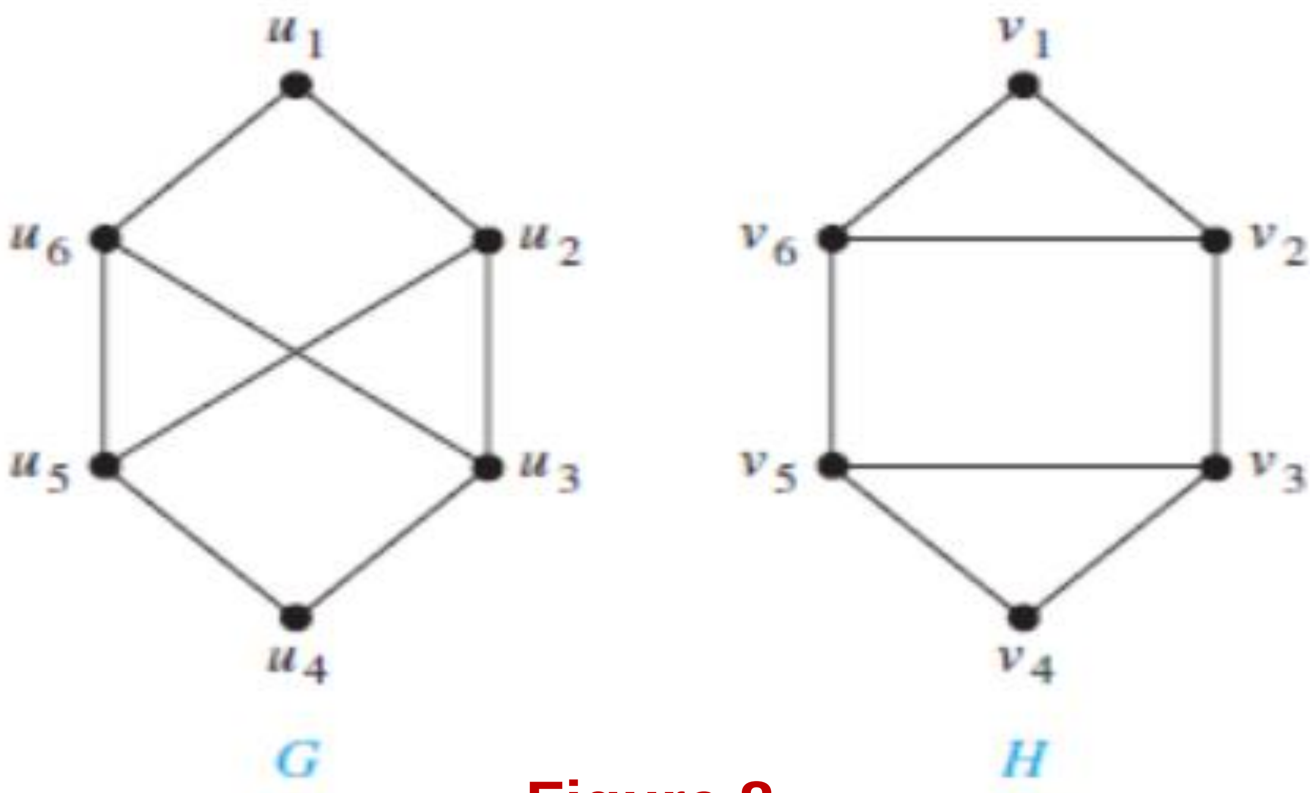


Figure 8

Solution

- Both G and H have six vertices and eight edges.
- Each has four vertices of degree three, and two vertices of degree two.
- So, the three invariants –number of vertices, number of edges, and degrees of vertices —all agree for the two graphs. [**Note**: you must check other graph invariants too]
- However, H has a simple circuit of length three, namely, v_1, v_2, v_6, v_1 , whereas G has no simple circuit of length three, as can be determined by inspection (all simple circuits in G have length at least four).
- Because the existence of a simple circuit of length three is an isomorphic invariant, **G** and **H** are **not isomorphic**.

Example 15 (p.566) : Determine whether the graphs G and H shown in Figure 9 are isomorphic.

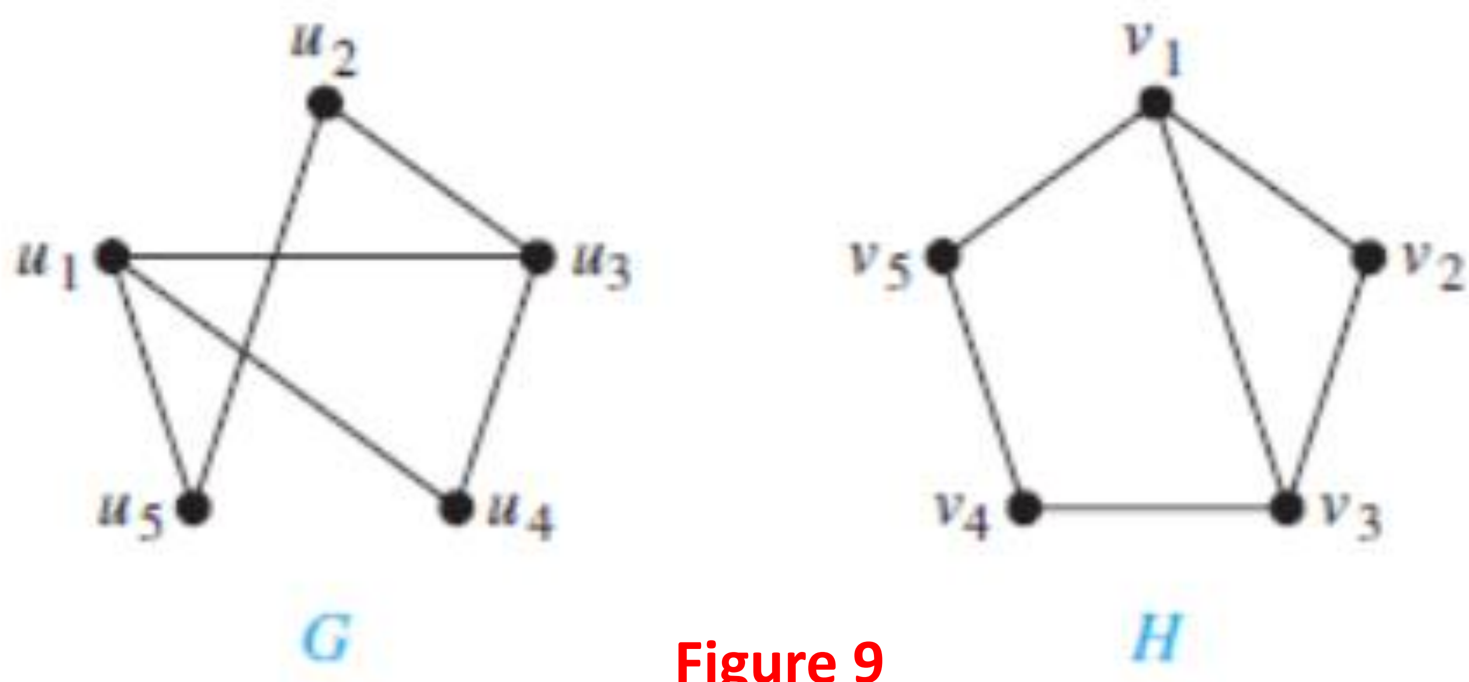


Figure 9

Solution

- Both G and H have five vertices and six edges.
- Each has two vertices of degree three, and three vertices of degree two.
- Both have a simple circuit of length three, a simple circuit of length four, and a simple circuit of length five.
- Because all these isomorphic invariants agree here, so

  G and **H** **may be isomorphic**.
- …..continued to next slide

Finding of possible isomorphism:

- ………. we will follow paths that go through all vertices so that the corresponding vertices in the two graphs have the same degree.

- For example, the paths u_1, u_4, u_3, u_2, u_5 *in G and* v_3, v_2, v_1, v_5, v_4 in H both go through every vertex in the graph; start at a vertex of degree three; go through vertices of degrees two, three, and two, respectively; and end at a vertex of degree two.
- By following these paths through the graphs, we can define the mapping *f* with
  - $f(u_1) = v_3$
  - $f(u_2) = v_5$
  - $f(u_3) = v_1$
  - $f(u_4) = v_2$
  - $f(u_5) = v_4$
- ***It shows that f is an isomorphism, so G and H are isomorphic.***

Books

- **Rosen, K. H., & Krithivasan, K. (2012). Discrete mathematics and its applications: with combinatorics and graph theory. Tata McGraw-Hill Education. (7th Edition)**
- Liu, C. L. (1986). Elements of discrete mathematics. Tata McGraw-Hill Education.

References

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2. Discrete Mathematical Structures, *Bernard Kolman*, *Robert C. Busby*, *Sharon Ross,* Prentice-Hall, Inc.
3. *SCHAUM'S outlines Discrete Mathematics(2nd edition)*, by *Seymour Lipschutz*, *Marc Lipson*

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- Duke University https://www2.cs.duke.edu/courses/spring11/cps102/notes/lec23.pdf