Matrices

Mathematical Induction

Course Code: CSC 1204 Course Title: Discrete Mathematics

Dept. of Computer Science
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Lecture Outline

3.8 Matrices

- Definition
- Notation
- Arithmetic Operations of Matrices
- Boolean Operations of Matrices

4.1 Mathematical Induction

- Proof by Mathematical Induction

Objectives and Outcomes

- **Objectives**: To understand matrix and matrix notation, to perform arithmetic and Boolean operations of matrices, to prove by mathematical induction.
- **Outcomes**: Students are expected to be able explain matrix and matrix notations; be able to perform arithmetic and Boolean operations of matrices; be able to prove a formula or inequality using mathematical induction.

Matrices

- **Definition 1**: A matrix is a rectangular array of numbers. A matrix with m rows and n columns is called an $m \times n$ matrix.
  - The plural of **matrix** is **matrices**
- A matrix with the same number of rows as columns is called a ***square*** (or ***square matrix***)
- Two matrices are ***equal*** if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.
- **Example 1:** The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$ is a 3×2 matrix.

Notation

- Let *m* and *n* be positive integers and let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & & . \\ . & . & & . \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

- The *i* th row of **A** is the $1 \times n$ matrix $[a_{i1}, a_{i2},\ldots,a_{in}]$. The *j* th column of **A** is the $m \times 1$ *matrix:* $$\begin{bmatrix} a_{1j} \\ a_{2j} \\ . \\ . \\ a_{mj} \end{bmatrix}$$
- The (*i, j*)th *element* or *entry* of **A** is the element a_{ij}.
- We can use **A** = $[a_{ij}]$ to denote the matrix with its (*i,j*)th element equal to a_{ij}.

Matrix Arithmetic : Addition and Subtraction

- The **sum of two matrices** of the **same size** is obtained by adding elements in the corresponding positions.
  - The **subtraction of two matrices** of the **same size** is obtained by subtracting elements in the corresponding positions
- Only matrices with the same dimensions can be added and subtracted. The resulting matrix has the same dimension of the two matrices being added or subtracted.
- ***Note***: Matrices of different sizes cannot be added/subtracted

Addition of two Matrices

- **Definition 3**: Let **A** = [a_{ij}] and **B** = [b_{ij}] be $m \times n$ matrices. The sum of **A** and **B**, denoted by **A + B**, is the $m \times n$ matrix that has $a_{ij} + b_{ij}$ as its (i,j)th element.

  In other words , **A + B** = [$a_{ij} + b_{ij}$]

Example

$$1.\ \begin{bmatrix} -2 & 0 & 4 \\ 3 & -10 & 12 \\ 3 & -2 & -2 \end{bmatrix} + \begin{bmatrix} -4 & 6 & 0 \\ -15 & 2 & -4 \\ 6 & 7 & 1 \end{bmatrix}$$

Add the corresponding elements in each matrix

$$\begin{bmatrix} -2-4 & 0+6 & 4+0 \\ 3-15 & -10+2 & 12-4 \\ 3+6 & -2+7 & -2+1 \end{bmatrix}$$

$$\begin{bmatrix} -6 & 6 & 4 \\ -12 & -8 & 8 \\ 9 & 5 & -1 \end{bmatrix}$$

Example 2 (p. 258)

$$A=\begin{bmatrix}1 & 0 & -1\\ 2 & 2 & -3\\ 3 & 4 & 0\end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix}3 & 4 & -1\\ 1 & -3 & 0\\ -1 & 1 & 2\end{bmatrix}$$

$$A+B=\begin{bmatrix}4 & 4 & -2\\ 3 & -1 & -3\\ 2 & 5 & 2\end{bmatrix}$$

Product of two Matrices

- **Definition 4**: Let **A** be $m \times k$ matrix and **B** be $k \times n$ matrix. The product of **A** and **B**, denoted by **AB**, is the $m \times n$ matrix with its (i,j)th element equal to the sum of the products of the corresponding entries from the ith row of **A** and the jth column of **B**.
- In other words, If **AB** = $[c_{ij}]$, then $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ik}b_{kj}$.

- ***Note***: A product of two matrices is defined only when the number of **columns in the first matrix** ***equals*** the **number of rows of the second matrix.**

Example

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}$$

Note: Here, we always multiply the row of first matrix by the column of the second matrix.

e.g. to find the element $a_{3,2}$, multiply the third row of First matrix by the second column of Second matrix

$$a_{3,2} = \begin{matrix} 3 & 1 & 0 \end{matrix} * \begin{matrix} 4 \\ 1 \\ 0 \end{matrix} = 3*4 + 1*1 + 0*0 = 13$$

Transpose of a Matrix

- **Definition**: Let **A** = [a_{ij}] be an *m* x *n* matrix. The *transpose* of **A**, denoted by $\mathbf{A}^t$, is the *n x m* matrix obtained by interchanging the rows and columns of **A**.

- If $\mathbf{A}^t = [\, b_{ij} \,]$, then $b_{ij} = a_{ji}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

Example 7

The transpose of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

Zero-One Matrices

- **Definition**: A matrix with entries that are either 0 or 1 is called *zero-one matrix.*
- Algorithms operating on discrete structures represented by zero-one matrices are based on **Boolean arithmetic** defined by the following **Boolean operations**:

$$b_1 \wedge b_2 = \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$b_1 \vee b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

Join and **Meet** of Zero-One Matrices

- **Definition:** Let **A** = $[a_{ij}]$ and **B** = $[b_{ij}]$ be an $m \times n$ zero-one matrices.

  The ***join*** of **A** and **B** is the zero-one matrix with (i,j)th entry $a_{ij} \vee b_{ij}$. The *join* of **A** and **B** is denoted by **A** ∨ **B**.

  The **meet** of of **A** and **B** is the zero-one matrix with (i,j)th entry $a_{ij} \wedge b_{ij}$. The *meet* of **A** and **B** is denoted by **A** ∧ **B**.

- **Note:** These operations(join and meet) are only possible when **A** and **B** have the same size, just as in the case of matrix addition.

Example 9 (p. 262)

- Find the join and meet of the zero-one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

- **Solution**: The join of **A** and **B** is

$$\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The meet of **A** and **B** is

$$\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Boolean Product of Zero-One Matrices

Definition: Let $\mathbf{A} = [a_{ij}]$ be an $m \times k$ zero-one matrix and $\mathbf{B} = [b_{ij}]$ be a $k \times n$ zero-one matrix. The *Boolean product* of **A** and **B**, denoted by $\mathbf{A} \odot \mathbf{B}$, is the $m \times n$ zero-one matrix with (i,j)th entry c_{ij} where $c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \ldots \vee (a_{ik} \wedge b_{kj})$

Example 9 (p.262): Find the Boolean product of **A** and **B**, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Solution of Example 9

The Boolean product **A** $\odot$ **B** is given by

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

4.1 Mathematical Induction

- Suppose that we have an infinite ladder, and we want to know whether we can reach every step on this ladder.
- We know two things –
  1) We can reach the first rung of the ladder
  2) If we can reach a particular rung of the ladder, then we can reach the next rung
- Can we conclude that we can reach every rung?

4.1 Mathematical Induction

- By (1), we know that we can reach the first rung of the ladder. Moreover, because we can reach the first rung, by (2), we can also reach the second rung; it is the next rung after the first rung.
- Applying (2) again, because we can reach the second rung, we can also reach the third rung.
- Continuing in this way, we can show that we can reach the fourth rung, the fifth rung, and so on.
  - For example, after 100 uses of (2), we know that we can reach the 101st rung

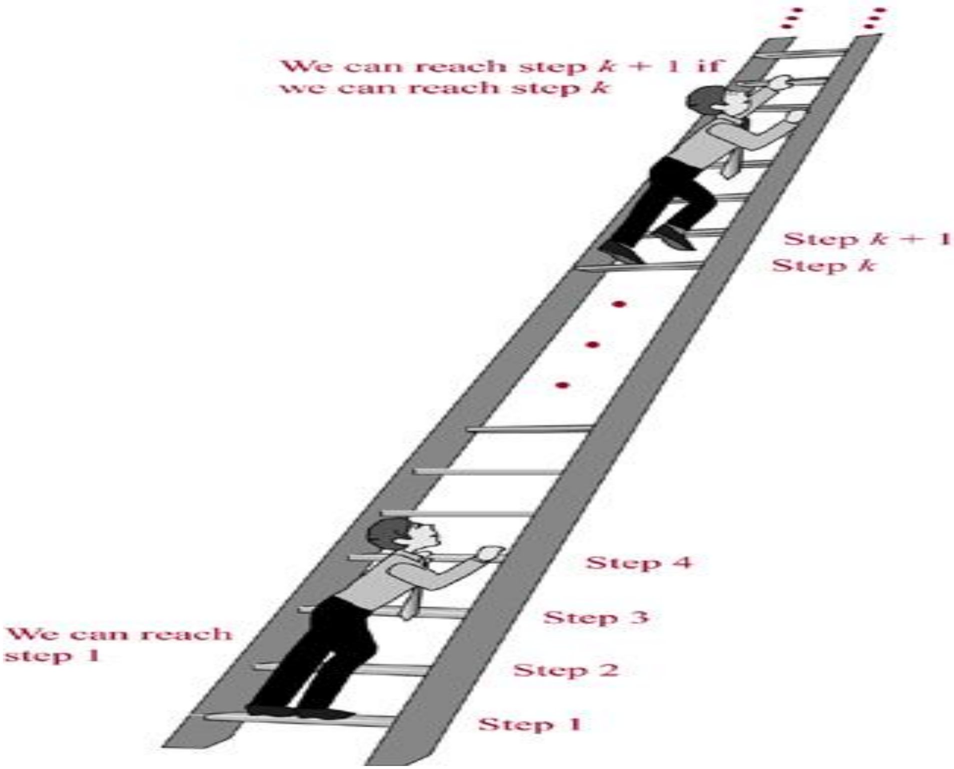
4.1 Mathematical Induction

- Can we conclude that we are able to reach every rung of this infinite ladder?
- **Answer: Yes.**

  We can verify using an important proof technique called **mathematical induction**

  We can show that $P(n)$ is true for every positive integer n, where $P(n)$ is the statement that we can reach the nth rung of the ladder.

FIGURE 1 : Climbing an Infinite Ladder



Ways to Remember How Mathematical Induction Works

E.g.

- Climbing an infinite ladder
- People telling secrets
- Infinite row of dominoes

Mathematical Induction

- Mathematical induction is an extremely important proof technique that can be used to prove assertions of this type.
- Mathematical induction is used extensively to prove results about a large variety of discrete objects.
  - For example, it is used to prove results about the complexity of algorithms, the correctness of certain types of computer programs, theorems about graphs and trees, as well as a wide range of identities and inequalities

Mathematical Induction

- Mathematical induction can be used only to prove results obtained in some other way. It is ***not*** a tool for discovering formulae or theorems.
    - The principle of mathematical induction is a useful tool for proving that a certain predicate is true for all natural numbers
    - It cannot be used to discover theorems, but only to prove them

Mathematical Induction

- **Principle of Mathematical Induction:** To prove that P(*n*) is true for all positive integers *n*, where P(*n*) is a propositional function, we complete two steps:
  1) **BASIS STEP**: We verify that P(1) is true.
  2) **INDUCTIVE STEP** : We show that the conditional statement $P(k) \rightarrow P(k+1)$ is true for all positive integers k
     - **Inductive hypothesis**: P(k) is true

- $[\, P(1) \wedge \forall k(P(k) \rightarrow P(k+1))] \rightarrow \forall n\, P(n)$

Example 1

- **Show that if n is a positive integer, then 1 + 2 + ……… + n = n(n+1)/2**

- **Solution**: Let *P(n)* be the proposition that the sum of the first n positive integers is n(n+1)/2.

  We must do two things to prove that *P(n)* is true for n = 1, 2, 3, ….. Namely, we must show that P(1) is true and that the conditional statement P(*k*) implies P(k+1) is true for *k* = 1, 2, 3,

  BASIS STEP: P(1) is true, because 1 = 1(1+1)/2

  INDUCTIVE STEP: For the inductive hypothesis, we assume that P(k) holds for an arbitrary positive integer k. That is we assume that

  $1 + 2 + \ldots + k = k(k+1)/2$ (INDUCTIVE HYPOTHESIS)

Example 1

- Under this assumption, it must be shown that *P(k+1)* is true, namely, that
  $1 + 2 + \ldots\ldots + k + (k+1) = (k+1)[(k+1)+1]/2 =$ **$(k+1)(k+2)/2$** is also true.

  Adding **(k+1)** to both sides of the equation in P(k), we obtain

  $$1 + 2 + \ldots\ldots + k + (k+1) = k(k+1)/2 + (k+1)$$
  $$= [k(k+1) + 2(k+1)] / 2$$
  $$= \mathbf{(k+1)(k+2)/2}$$

This last equation shows that P(*k*+1) is true under the assumption that P(*k*) is true. This completes the inductive step.

We have completed the basis step and inductive step, so by mathematical induction we know that P(n) is true for all positive integers n. That is, we have proven that $1 + 2 + \ldots. + n = n(n+1)/2$ for all positive integers n.

Example 3

- Use mathematical induction to show that $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$ for all **nonnegative integers *n***.
- **Solution**: Let P(n) be the proposition that $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$ for the integer n.

  BASIS STEP: P(0) is true because $2^0 = 1 = 2^1 - 1$

  INDUCTIVE STEP: For the inductive hypothesis, we assume that P(k) is true.

  That is, we assume that $1 + 2 + 2^2 + \ldots + 2^k = 2^{k+1} - 1$

  To carry out the inductive step using this assumption, we must show that when we assume that *P(k)* is true, then *P(k+1)* is also true. That is we must show that

  $1 + 2 + 2^2 + \ldots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 1 = \mathbf{2^{k+2} - 1}$, assuming the inductive hypothesis *P(k)*.

Example 3 (Cont.)

- Under the assumption of *P(k)*, we see that

$1 + 2 + 2^2 + \ldots\ldots + 2^k + 2^{k+1} = (1 + 2 + 2^2 + \ldots\ldots + 2^k) + 2^{k+1}$

$= (2^{k+1} - 1) + 2^{k+1}$

$= 2.\ 2^{k+1} - 1$

$= \mathbf{2^{k+2} - 1}$

Because we have completed the basis step and the inductive step, by mathematical induction we know that *P(n)* is true for all nonnegative integers *n*.

That is, $1 + 2 + 2^2 + \ldots\ldots + 2^n = 2^{n+1} - 1$ for all nonnegative integers n.

Example 5

- **Example 5**: Use mathematical induction to prove the inequality $n < 2^n$ for all **positive integers** $\boldsymbol{n}$.
- **Solution**: Let $P(n)$ be the proposition that $n < 2^n$.

  BASIS STEP: $P(1)$ is true, because $1 < 2^1 = 2$

  INDUCTIVE STEP: Assume $P(k)$ is true for all positive integer k, that is, $k < 2^k$ (Inductive Hypothesis)

  We need to show that if $k < 2^k$, then $k+1 < 2^{k+1}$

  Now, $k + 1 < 2^k + 1$ [adding 1 to both sides of $k < 2^k$]

  $\leq 2^k + 2^k = 2.\ 2^k = 2^{k+1}$ [note: $1 \leq 2^k$]

  This shows that $P(k+1)$ is true, i.e., $k+1 < 2^{k+1}$

  Because we have completed both the basis step and the inductive step, by the principle of mathematical induction we have shown that $n < 2^n$ is true for all positive integers n.

(Modified)Exercise 3

- ***Modified* Exercise 3:** Use mathematical induction to show that $1^2 + 2^2 + \ldots.. + n^2 = n(n+1)(2n+1) / 6$ for **all positive integers *n*.**

- **Solution:** Let P(n) be the proposition $1^2 + 2^2 + \ldots.. + n^2 = n(n+1)(2n+1)/6$

BASIS STEP: P(1) is true, because $1^2 = 1.2.3/6$

INDUCTIVE STEP: We assume that $1^2 + 2^2 + \ldots.. + k^2 = k(k+1)(2k+1)/6$ (Inductive Hypothesis)

We want to show that $1^2 + 2^2 + \ldots.. + k^2 + (k + 1)^2 =$ $\mathbf{(k+1)(k+2)(2k+3)/6}$

Now, $1^2 + 2^2 + \ldots.. + k^2 + (k + 1)^2 = k(k+1)(2k+1)/6 + (k + 1)^2$

$= (k+1)/6\ [k(2K+1) + 6(K+1)]$

$= (k+1)/6\ [2k^2 + 7k + 6]$

$= (k+1)/6\ [2k^2 + 4k + 3k + 6]$

$= (k+1)/6\ [2k^2 + 4k + 3k + 6]$

$=$ $\mathbf{(k+1)(k+2)(2k+3)/6}$ So, inductive step is tru.

Therefore, $1^2 + 2^2 + \ldots.. + n^2 = n(n+1)(2n+1)/6$ for all positive integers n.

Exercise 5

Exercise 5: Prove that $1^2 + 3^2 + 5^2 + \ldots + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$ whenever *n* is a **nonnegative integer**.

Solution: Let P(n) be the proposition $1^2 + 3^2 + 5^2 + \ldots + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$

BASIS STEP: P(0) is true, because $1^2 = 1.1.3/3$

INDUCTIVE STEP: We assume that $1^2 + 3^2 + 5^2 + \ldots + (2k+1)^2 = (k+1)(2k+1)(2k+3)/3$ (Inductive Hypothesis)

We want to show that $1^2 + 3^2 + 5^2 + \ldots + (2k+1)^2 + (2k+3)^2 = (k+2)(2k+3)(2k+5)/3$

Now, $(1^2 + 3^2 + 5^2 + \ldots + (2k+1)^2) + (2k+3)^2 = (k+1)(2k+1)(2k+3)/3 + (2k+3)^2$

$= (2k+3)/3\ [\ (k+1)(2k+1) + 3(2k+3)\ ]$

$= (2k+3)/3\ [\ (2k^2 + 3k + 1) + (6k + 9)]$

$= (2k+3)/3\ [\ 2k^2 + 9k + 10\ ]$

$= (2k+3)/3\ [2k^2 + 4k + 5k + 10\ ] = (2k+3)/3\ [2k(k+2) + 5(k+2)]$

$= (k+2)(2k+3)(2k+5)/3$

Therefore inductive step is true

So, $1^2 + 3^2 + 5^2 + \ldots + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$, whenever *n* is a nonnegative integer

Practice @ Home

- **Relevant odd-numbered Exercises** from your text book

Books

1. *Discrete Mathematics and its applications with combinatorics and graph theory (7th edition)* by Kenneth H. Rosen [Indian Adaptation by KAMALA KRITHIVASAN], published by McGraw-Hill

References

1. Discrete Mathematics, *Richard Johnsonbaugh*, Pearson education, Inc.
2. Discrete Mathematical Structures, *Bernard Kolman*, *Robert C. Busby*, *Sharon Ross,* Prentice-Hall, Inc.
3. *SCHAUM'S outlines Discrete Mathematics(2nd edition)*, by *Seymour Lipschutz*, *Marc Lipson*