Relations and Their Properties

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Dept. of Computer Science
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Lecture Outline

7.1 Relations and Their Properties

- Relations and Functions
- Properties of Relations
  - Reflexive Relations
  - Symmetric Relations
  - Antisymmetric Relations
  - Transitive Relations
- Combining Relations
- Composite of Relations

Objectives and Outcomes

- Objectives: To understand the Relations and the difference between function and relation, to analyze a relation to determine whether it contains certain property, how to combine two relations, how to find the composite of two relations.
- Outcomes: The students are expected to be able to explain relation and how it is differs from function; be able to determine whether a relation is reflexive, whether it symmetric, whether it is antisymmetric and/or whether it is antisymmetric; be able to combine two relations; be able to find out the composite relations of two relations.

Introduction

- The most direct way to express a relationship between elements of two sets is to use ordered pairs made up of two related elements. For this reason, **sets of ordered pairs are called *binary relations*.**
- In this section, we introduce the basic terminology used to describe ***binary relations***.
- We can use relations to solve problems involving communications networks, project scheduling, and identifying elements in sets with common properties.

Binary Relations

Definition: Let A and B be sets. A ***binary relation*** *from A to B* is a subset of $A \times B$.

- *In other words, a* ***binary relation*** from ***A*** to ***B*** is a set ***R*** of ordered pairs where the first element of each ordered pair comes from ***A*** and the second element comes from ***B***.
- We use the notation $a \, R \, b$ to denote that $(a, b) \in R$

  When (a, b) belongs to R, a is said to be **related to** b *by* R.

Note: $a \not R \, b$ means a is **not** related to b by R, i.e., $(a, b) \notin R$

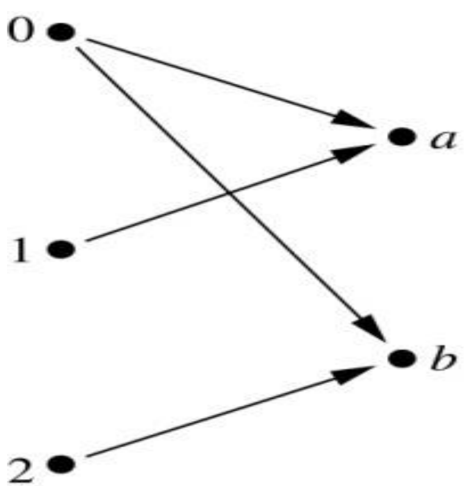
Example 3

- Let *A* = {0,1,2} and *B* = {*a*, *b*} .

Then {(0, *a*), (0, *b*), (1,*a*) , (2, *b*)} is a **relation** from *A* to *B*.

This means, for instance, 0 ***R*** a , but that $1 \not{\boldsymbol{R}}\ b$

- Relations can be represented **graphically** or using a **table**:



R	*a*	*b*
0	×	×
1	×	
2		×

Note: If a relation is given as a table, the **domain** consists of the members of the first column and the **range** consists of the members of the second column.

Functions as Relations

- Recall that A function ***f*** from a set ***A*** to a set ***B*** assigns exactly one element of ***B*** to each element of ***A***. The graph of *f* is the set of ordered pairs *(a, b)* such that ***b = f(a).***
- Because the graph of ***f*** is a **subset** of ***A X B***, it is a **relation** from ***A*** to ***B.***

  Moreover, the graph of a function has the property that every element of *A* is the first element of exactly one ordered pair of the graph.
- Conversely, if *R* is a relation from *A to B* such that every element in *A* is the first element of exactly one ordered pair of *R,* then a function can be defined with *R* as its graph. This can be done by assigning to an element *a* of ***A*** the unique element $b \in$ ***B*** such that ***(a, b)∈R.***

Functions VS Relations

- A ***relation*** can be used to express a ***one-to-many relationship*** between the elements of the sets ***A*** and ***B****,* where an element of ***A*** may be related to more than one element of ***B****.*
- A ***function*** represents a relation where exactly one element of ***B*** is related to each element of ***A.***
- **Relations** are more general than **functions**. A function is a relation where **exactly one element** of *B* is related to each element of *A.*

Relations on a Set

- Relations from a set A to itself are of special interest.

Definition 2: A *relation on a set A* is a relation from *A to A.*

In other words, a relation on a set *A is a* subset of $A \times A$.

Example: Suppose that $A = \{a, b, c\}$.

Then $R = \{(a, a),(a, b), (a, c)\}$ is a relation on A.

Relations on a Set(cont.)

- **Example 4:** Let A be the set $\{1, 2, 3, 4\}$.

  Which ordered pairs are in the **relation**

  R $= \{(a, b) \mid a \text{ divides } b\}$?

- **Solution:** Because (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b, we see that

  $R = \{(1,1), (1, 2), (1,3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$

Relations on a Set (cont.)

- **Example 6:** How many relations are there on a set with *n* elements?

- **Solution**: A relation on a set ***A*** is a subset of ***A X A.*** Because ***A X A*** has n^2 elements when ***A*** has *n* elements and a set with *m* elements has 2^m subsets, there are **2^{n^2} subsets of *A X A.***

  Thus, **there are 2^{n^2} relations on a set with *n* elements.**

- For example, there are $2^{3^2} = 2^9 = 512$ relations on the set {*a*, *b*, *c*}

Relations on a Set (cont.)

Example 5: Consider these relations on the set of integers:

$R_1 = \{(a,b) \mid a \leq b\}$,

$R_2 = \{(a,b) \mid a > b\}$,

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a,b) \mid a = b\}$,

$R_5 = \{(a,b) \mid a = b + 1\}$,

$R_6 = \{(a,b) \mid a + b \leq 3\}$.

Which of these relations contain each of the pairs (1,1), (1, 2), (2, 1), (1, −1), and (2, 2)?

Solution: Checking the conditions that define each relation,

we see that the pair (1,1) is in R_1, R_3, R_4 , and R_6;

(1,2) is in R_1 and R_6;

(2,1) is in R_2, R_5, and R_6;

(1, −1) is in R_2, R_3, and R_6 ; (2,2) is in R_1, R_3, and R_4.

Properties of Relations

- There are several properties that are used to classify relations on a set. We will introduce the most important of these here.
  - ***Reflexive***
  - ***Symmetric***
  - ***antisymmetric***
  - ***Transitive***

Reflexive Relation

- ***Definition:*** A relation ***R*** on a set ***A*** is called reflexive

  if $(a, a) \in R$ for **every element** $a \in A$.
- Using quantifiers, a relation on the set A is reflexive

  if $\forall a\ ((a, a) \in R)$, where universe of discourse is the set of **ALL elements in *A*.**
- In a reflexive relation, **every element** is related to itself.

  i.e. ***a R a* for all *a* ∈ *A***

- ***Example:*** The relation $\boldsymbol{R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}}$ on the set $\{1, 2, 3\}$, is reflexive.

Determining whether a Relation is Reflexive

Example7 : Consider the following relations on {1, 2, 3, 4}:

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$

$R_2 = \{(1,1), (1,2), (2,1)\}$

$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$

$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$

$R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$,

$R_6 = \{(3,4)\}$.

Which of these relations are reflexive?

Solution: The **relations R_3 and R_5 are reflexive** because they both contain **ALL pairs of the form (a, a)**, namely (1,1) (2,2), (3,3) and (4,4).

Reflexive Relation: Another Example

- **Example 8 (modified)**: The following relations on the set of integers are ***reflexive***:

  $R_1 = \{(a, b) \mid a \leq b\}$,

  $R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$,

  $R_4 = \{(a, b) \mid a = b\}$.

  The following relations are NOT reflexive:

  $R_2 = \{(a, b) \mid a > b\}$ (note that $3 \not> 3$),

  $R_5 = \{(a, b) \mid a = b + 1\}$ (note that $3 \neq 3 + 1$),

  $R_6 = \{(a, b) \mid a + b \leq 3\}$ (note that $4 + 4 \not\leq 3$).

Reflexive Relation: More Examples

- **Example 9:** Is the "***divides***" relation on the set of **positive integers** reflexive?
  - Solution: Yes. Because $a \mid a$ whenever a is a positive integer, the "divides" relation is reflexive.

- **Question**: Is the "***divides***" relation on the set of **integers** reflexive?
  - Solution: No. Because $0 \nmid 0$ (0 does not divide 0)

Symmetric Relation

- **Definition**: A relation ***R*** on a set ***A*** is called ***symmetric*** **if *(b, a)* $\in$ *R* whenever *(a, b)* $\in$ *R***, for all *a, b* $\in$ *A*.

- **Example:** The relation ***R = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,3), (1, 4)}*** on *the set {1, 2, 3, 4}* is symmetric.

Antisymmetric Relation

Definition: A relation R on a set A such that for all $a, b \in A$ if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.

- In other words, R is **antisymmetric** if whenever a = b, then $a \not R b$ or $b \not R a$.
- It follows that R is **not antisymmetric** if we have a and b in A, $a \neq b$, and both $a\ R\ b$ or $b\ R\ a$.
- **Note:** The terms ***symmetric*** and ***antisymmetric*** are NOT opposite, because a relation can have both of these properties or may lack both of them.

  { (1,1), (2,2) } ➔ the relation is both symmetric & antisymmetric

  { (0,1), (1,2), (2,1)} ➔ the relation is neither symmetric nor antisymmetric

Symmetric & Antisymmetric Relation: Example

Example 10 : Consider the following relations on {1, 2, 3, 4}:

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$

$R_2 = \{(1,1), (1,2), (2,1)\}$

$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$

$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$

$R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$,

$R_6 = \{(3,4)\}$.

Which of the relations are ***symmetric*** *and which are* ***antisymmetric***?

- **Solution**:

  The relations $\boldsymbol{R_2}$ *and* $\boldsymbol{R_3}$ *are* **symmetric**.

  The relations $\boldsymbol{R_4}$, $\boldsymbol{R_5}$, and $\boldsymbol{R_6}$ are **antisymmetric**.

Question: What about $\boldsymbol{R_1}$? Neither symmetric nor antisymmetric

Transitive Relation

- **Definition:** A relation R on a set A is called ***transitive*** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

- ***Example*** *:* The relation $\boldsymbol{R = \{(1,1), (1,2),(1, 3), (1, 4), (2,2), (2,3), (2, 4), (3, 3), (3,4), (4, 4)\}}$ on *the set* $\{1, 2, 3, 4\}$ is ***transitive.***

Transitive Relation: Example 13

Consider the following relations on {1, 2, 3, 4}:

R_1 = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)}

R_2 = {(1,1), (1,2), (2,1)}

R_3 = {(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)}

R_4 = {(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)}

R_5 = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)},

R_6 = {(3,4)}.

Which of the relations are ***transitive***?

- ***Solution***: **R_4 , R_5 & R_6 : transitive** $\Leftarrow$ verify that if (a, b) and (b, c) belong to this relation then (a, c) belongs also to the relation.
  R_4 transitive since (3,2) and (2,1), (4,2) and (2,1), (4,3) and (3,1), and (4,3) and (3,2) are the only such sets of pairs, and (3,1) , (4,1) and (4,2) belong to R_4.
  Same reasoning for R_5 and R_6.
- **R_1 : not transitive** $\Leftarrow$ (3,4) and (4,1) belong to R_1, but (3,1) does not.
- **R_2 : not transitive** $\Leftarrow$ (2,1) and (1,2) belong to R_2, but (2,2) does not.
- **R_3 : not transitive** $\Leftarrow$ (4,1) and (1,2) belong to R_3, but (4,2) does not.

Transitive Relation: Another Example

- Is the relation $R = \{ (a, a), (b, c), (c, b), (d, d)\}$ on the set $X = \{a, b, c, d\}$ is ***transitive***?

- **Solution**:

  No.

  Because (b, c) and (c, b) are in R, but (b, b) is not in R

Combining Relations

- Because relations from ***A*** to ***B*** are subsets of ***A X B***, two relations from ***A*** to ***B*** can be combined in any way two sets can be combined.

- Given two relations R_1 and R_2, we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.

Combining Relations : Example

- **Example**: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations $R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$ can be combined using basic set operations to form new relations:

$$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$$

$$R_1 \cap R_2 = \{(1,1)\}$$

$$R_1 - R_2 = \{(2,2),(3,3)\}$$

$$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$$

Composite of Relations

- Let ***R*** be a relation from *A* to *B* and ***S*** a relation from *B* to a set *C.* The composite of *R* and *S* is the relation consisting of ordered pairs *(a, c),* where $a \in A$, $c \in C$, and for which there exists an element b $\in B$ such that $(a,b) \in R$ and $(b, c) \in S$.
- We denote the **composite of *R* and *S*** by $S \circ R$

- ***Note:*** Computing the composite of two relations requires that we find elements that are the
  - **second elements** of ordered pairs in the **first relation**, and
  - **first element** of ordered pairs in the **second relation**

Composite of Relations : Example

- ***Example 20 :*** What is the composite of the relations ***R*** and ***S,*** where ***R*** is the relation from {1, 2, 3} to {1, 2, 3, 4} with

  R *= {(1,1), (1,4), (2,3), (3,1), (3,4)}* and ***S*** is the relation from *{1, 2, 3, 4} to {0, 1, 2}* with ***S*** *= {(1,0), (2,0), (3,1), (3,2), (4,1)}?*

- **Solution:** $S \circ R$ is constructed using all ordered pairs in R and ordered pairs in S, where the second element of the ordered pair in R agrees with the first element of the ordered pair in S. For example, the ordered pairs (2,3) in R and (3,1) in S produce the ordered pair (2,1) in $S \circ R$. Computing all the ordered pairs in the composite, we find

  $S \circ R$ = {(1,0), (1,1),(2,1), (2,2),(3,0), (3,1) }

Exercise 30

- Let R be the relation $\{(1,2), (1,3),(2,3), (2,4),(3,1)\}$, and let S be the relation $\{(2,1), (3,1),(3,2), (4,2)\}$.

  Find $\boldsymbol{S \circ R}$

- **Solution: Try out yourself!**

- **Answer:** $\boldsymbol{S \circ R =} \{(1,1), (1,2),(2,1), (2,2)\}$

Books

- Rosen, K. H., & Krithivasan, K. (2012). Discrete mathematics and its applications: with combinatorics and graph theory. Tata McGraw-Hill Education. (7th Edition)
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- Online tutorial
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- University of Pittsburgh
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