Euler and Hamilton Paths

Course Code: 00090 Course Title: Discrete Mathematics

Dept. of Computer Science
Faculty of Science and Technology

Lecturer No:	**18**	**Week No:**	**10**	**Semester:**	
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Lecture Outline

8.5 Euler and Hamilton Paths

- Euler Paths & Circuits (Edge based)
- Hamilton Paths & Circuits (Vertex based)

Objectives and Outcomes

- Objectives: To understand terms Euler path, Euler circuit, Hamilton path, Hamilton circuit; to determine whether a graph contains Euler or Hamilton path or circuit.

- Outcomes: The students are expected to be able to explain the terms Euler path, Euler circuit, Hamilton path, Hamilton circuit with examples; be able to determine whether a given graph contains Euler or Hamilton path or circuit.

Euler paths and circuits

- ***Euler path:*** An Euler path in G is a **simple path** containing **every edge** of G.

- ***Euler circuit:*** An Euler circuit in a graph G is **a simple circuit** containing **every edge of G.**

Euler paths and circuits

- **Theorem 1**: A connected multigraph with at least two vertices has an **Euler circuit** iff each of its vertices has even degree.

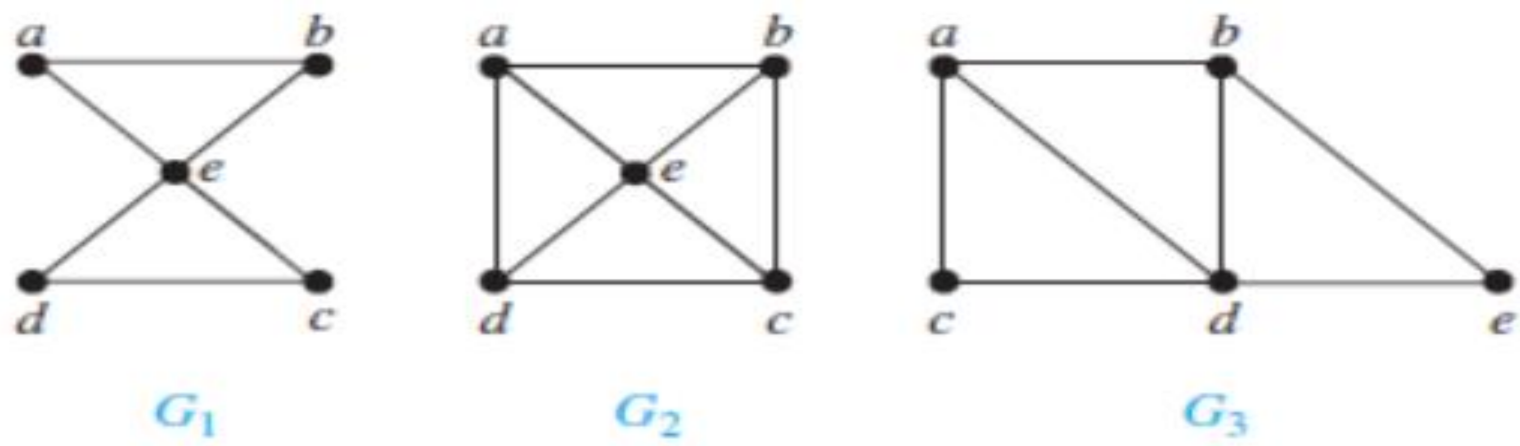
- **Theorem 2**: A connected multigraph has an **Euler path** (but not an Euler circuit) iff it has **exactly** two vertices of **odd degree**.

Euler paths and circuits

- A graph contains **Euler circuit** if **all of its vertices have even degre**e.
- A graph contains **Euler path if exactly two of it's vertices have odd degree**. End points of the Euler path are the two vertices with odd degree.
- A graph does NOT contain any Euler path or Euler circuit if the graph contains more than two vertices of odd degree.
- **Euler circuit and Euler path are mutual exclusive** (i.e., if a graph contains Euler circuit, it does not contain Euler path ; if a graph contains Euler path, it does not contain Euler circuit).
- An Euler path is a path that uses every **edge** of a graph exactly once.
- An Euler circuit is a circuit that uses every **edge** of a graph exactly once.
- An Euler path starts and ends at **different** vertices.
- An Euler circuit starts and ends at the **same** vertex.

Euler paths and circuits

- **EXAMPLE 1 (p.572):** Which of the undirected graphs in Figure below have an Euler circuit? Of those that do not, which have an Euler path?



Euler paths and circuits

- **Solution**:
- The graph G1 has an Euler circuit, for example, a, e, c, d, e, b, a.
- Neither of the graphs G2 or G3 has an Euler circuit.
- However, G3 has an Euler path, namely, a, c, d, e, b, d, a, b. G2 does not have an Euler path.

Determining Euler Circuit/Path: Example

- **EXAMPLE 2 (p.572):** Which of the directed graphs in Figure 4 have an Euler circuit? Of those that do not, which have an Euler path?

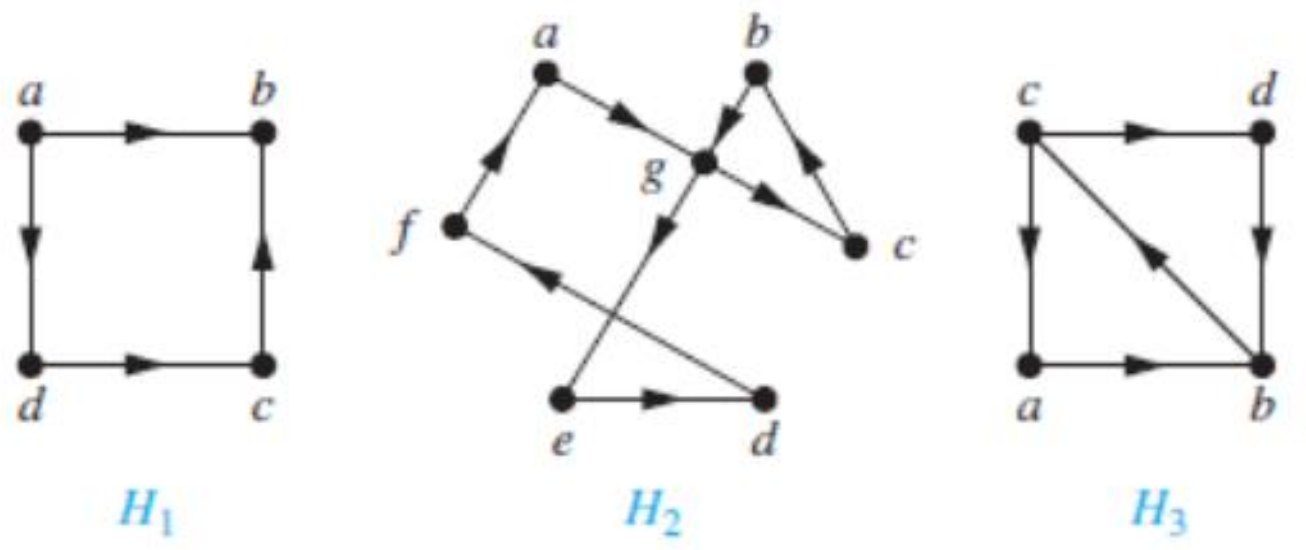


Figure 4

Determining Euler Circuit/Path: Example

- Solution:
- The graph H_2 has an Euler **circuit**, for example, a, g, c, b, g, e, d, f, a.
- Neither H_1 nor H_3 has an Euler **circuit**
- H_3 has an Euler **path**, namely, c, a, b, c, d, b, but H_1 does not have Euler **path**

Determining Euler Circuit/Path: Example

- **Example 4(p.575)**:Which graphs shown in Figure 7 have an Euler path?

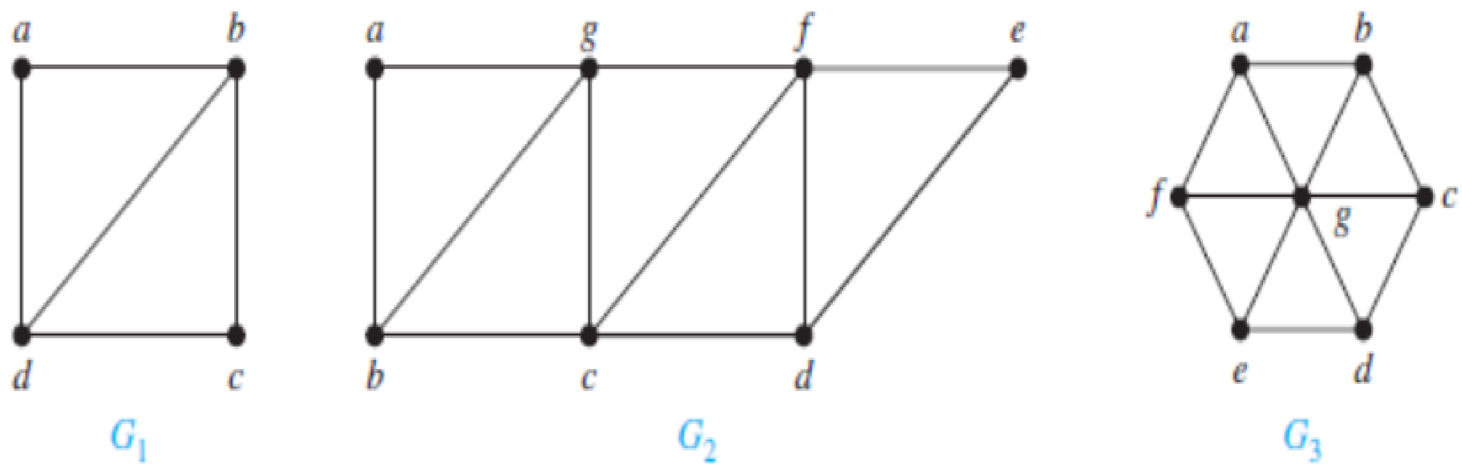


FIGURE 7 **Three Undirected Graphs.**

Determining Euler Circuit/Path: Example

- **Solution**:
- **G_1** contains exactly two vertices of odd degree, namely, b and d. Hence, it has an Euler path that must have b and d as its endpoints.
  - One such Euler path is d, a, b, c, d, b.
- Similarly,**G_2** has exactly two vertices of odd degree, namely, b and d. So it has an Euler path that must have b and d as endpoints.
  - One such Euler path is b, a, g, f, e, d, c, g, b, c, f, d.
- G_3 has no Euler path, because it has six vertices of odd degree.

Hamilton Paths and Circuits

- **Hamilton path**: A **simple path** in a graph that passes through every vertex exactly once is called a *Hamilton path*.
- **Hamilton circuit**: A **simple circuit** in a graph G that passes through every vertex exactly once is called a *Hamilton circuit*.
- **Note**: A Hamilton path/circuit does not necessarily pass through all the edges of the graph

Hamilton Paths and Circuits

<u>EXAMPLE 5(p.577)</u>: Which of the simple graphs in Figure 10 have a Hamilton circuit or, if not, a Hamilton path?

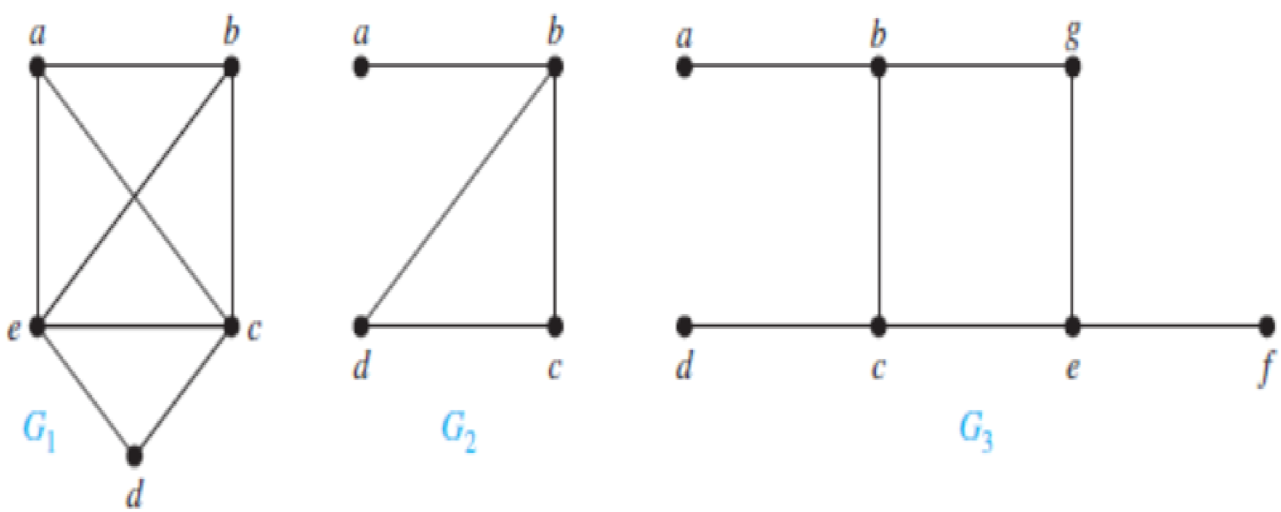


Figure 10 : Three Simple Graphs

Hamilton Paths and Circuits

Solution:

- G_1 has a Hamilton circuit: a, b, c, d, e, a.
- There is no Hamilton circuit in G_2 (this can be seen by noting that any circuit containing every vertex must contain the edge {a, b} twice),

  but G_2 does have a Hamilton path, namely, a, b, c, d.
- G_3 has neither a Hamilton circuit nor a Hamilton path, because any path containing all vertices must contain one of the edges {a, b},{e, f }, and {c, d} more than once.

Necessary & Sufficient Criteria for Hamilton Circuits

- There are **NO** known simple **necessary** and **sufficient** criteria for the existence of Hamilton circuits. However, many theorems are known that give **sufficient conditions** for the existence of Hamilton circuits.
- Certain properties can be used to show that a graph has no Hamilton circuit. For instance, a graph with a vertex of degree one cannot have a Hamilton circuit, because in a Hamilton circuit, each vertex is incident with two edges in the circuit. Moreover, if a vertex in the graph has degree two, then both edges that are incident with this vertex must be part of any Hamilton circuit.
- When a Hamilton circuit is being constructed and this circuit has passed through a vertex, then all remaining edges incident with this vertex, other than the two used in the circuit, can be removed from consideration.
- A Hamilton circuit cannot contain a smaller circuit within it.

Hamilton Paths and Circuits

<u>EXAMPLE 6</u> (p.577): Show that neither graph displayed in Figure 11 has a Hamilton circuit.

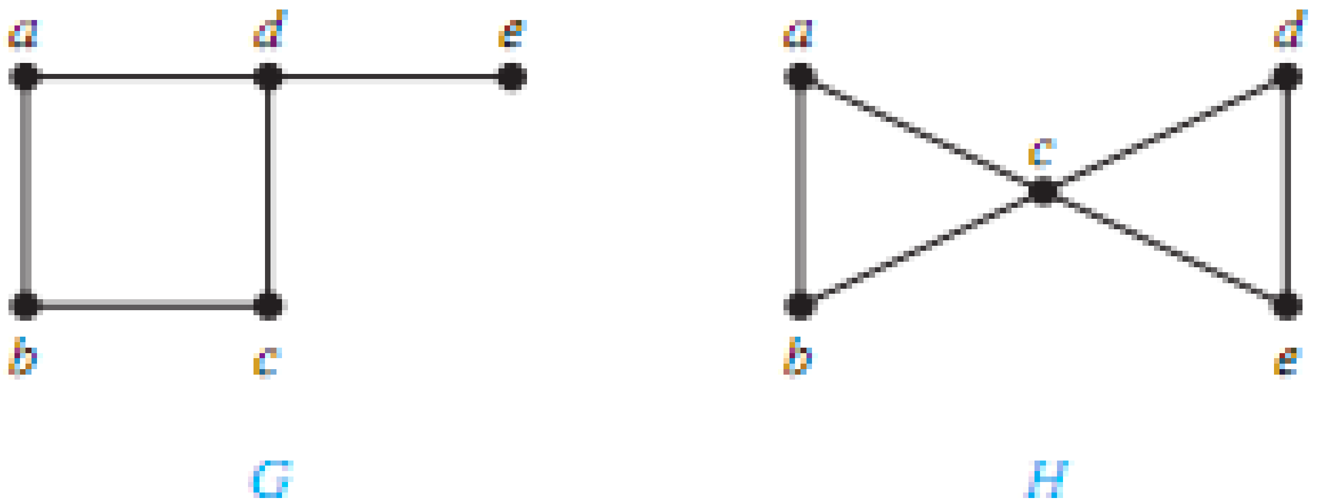


Figure 11

- ***Solution:***
- *There is no Hamilton circuit in G because G has a vertex of degree one, namely, e.*
- Now consider *H. Because the degrees of the vertices a, b, d, and e are all two, every edge* incident with these vertices must be part of any Hamilton circuit. It is now easy to see that no Hamilton circuit can exist in *H*, *for any Hamilton circuit would have to contain four edges* incident with *c, which is impossible.*

- Extra Question: Is there any Hamilton paths in the graphs of Example 6?
- **Answer**: Yes, both graph contains Hamilton paths.
- In graph G ➔ *e, d, a, b, c*
- In graph H ➔ *a, b, c, d, e*

- **Question**: What are other answers (Hamilton paths)?

Hamilton Paths and Circuits

- **Theorem 3 (DIRAC'S THEOREM):** If G is a simple graph with n vertices with $n>=3$ such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.
- **Theorem 4 (ORE'S THEOREM)**: if G is a simple graph with n vertices with $n>=3$ such that

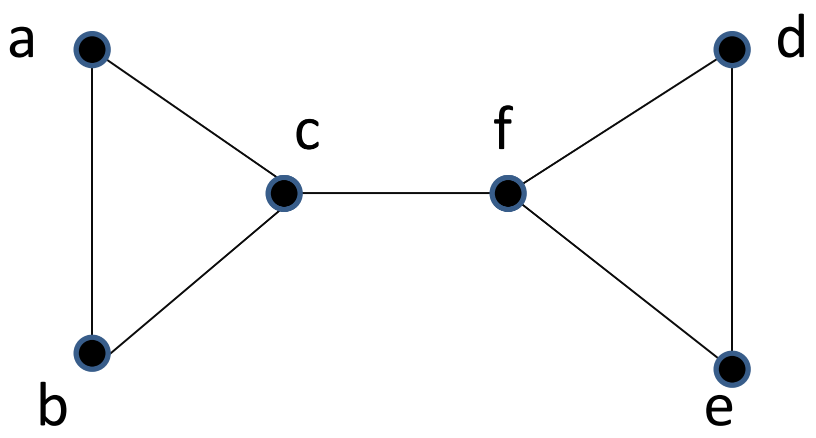
  $deg(u) + deg(v) >= n$ for every pair of non-adjacent vertices u and v in G, then G has a Hamilton circuit.

Hamilton Paths and Circuits

- Both **Ore's Theorem** and **Dirac's Theorem** provide **sufficient conditions** for a connected simple graph to have a Hamilton circuit. However, these theorems do not provide **necessary conditions** for the existence of a Hamilton circuit.
  - For example, the graph C_5 has a Hamilton circuit but does not satisfy the hypotheses of either Ore's Theorem or Dirac's Theorem

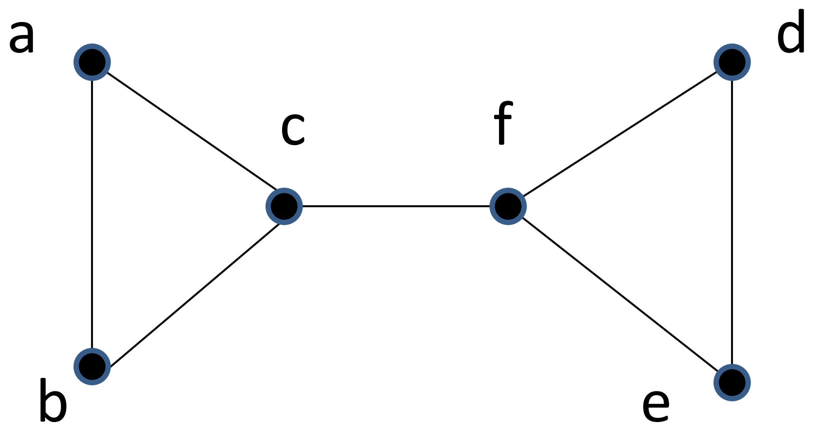
Exercise 37 (p.583)

- Does the following graph have a **Hamilton path**? If so, find such a path. If it does not, give an argument to show why no such path exists.



Exercise 37 (p.645)

- **Solution:**
- This graph has the Hamilton path *a, b, c, f, d, e*.
  This simple path hits each vertex once.



Question: Does the above graph have Hamilton Circuit? Explain.

Euler vs. Hamilton Paths & Circuits

- On the surface, there is a one-word difference between **Euler paths/circuits** and **Hamilton paths/circuits**: The former covers ***all edges***; the latter covers ***all vertices***.

- **Euler** path/circuit == > main concern **Edge**
  - ALL the EDGES must be visited exactly ONCE

- **Hamilton** path/circuit ==> main concern **Vertex**
  - ALL the VERTICES must be visited exactly ONCE

Euler vs. Hamilton Paths & Circuits

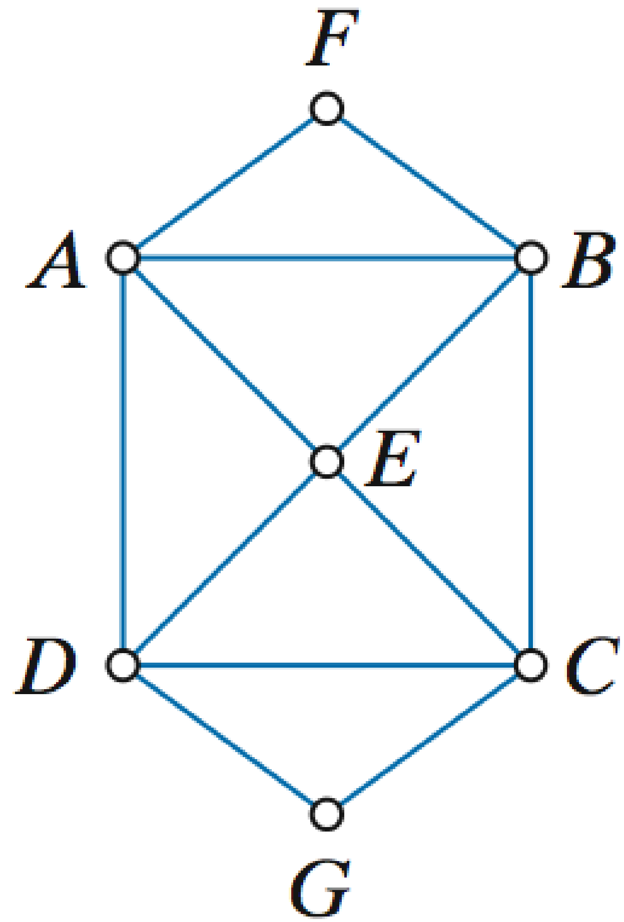
- **Note:** **I*f a graph has a Hamilton circuit, then it automatically has a Hamilton path*** –the Hamilton circuit can always be truncated into a Hamilton path by dropping the last vertex of the circuit. (For example, the Hamilton circuit *A, F, B, C, G, D, E, A* can be truncated into the Hamilton path *A, F, B, C, G, D, E*.)

  ==> See next slide

- **Contrast** this with the **mutually exclusive relationship between Euler circuits and Euler paths**: If a graph has an Euler circuit it cannot have an Euler path and vice versa.

Example

The figure shows a graph that (1) has Euler circuits (the vertices are all even-degree) and (2) has Hamilton circuits.



One such Hamilton circuit is

A, *F*, *B*, *C*, *G*, *D*, *E*, *A* – there are plenty more.

Can you identify an Euler circuit in this graph?

Practice @ Home

- **Relevant Odd-Numbered Exercises**
- 19, 21, 31, 35, 37, 39, 43, 47

Books

- **Rosen, K. H., & Krithivasan, K. (2012). Discrete mathematics and its applications: with combinatorics and graph theory. Tata McGraw-Hill Education. (7th Edition)**
- Liu, C. L. (1986). Elements of discrete mathematics. Tata McGraw-Hill Education.

References

1. Discrete Mathematics, *Richard Johnsonbaugh*, Pearson education, Inc.
2. Discrete Mathematical Structures, *Bernard Kolman*, *Robert C. Busby*, *Sharon Ross,* Prentice-Hall, Inc.
3. *SCHAUM'S outlines Discrete Mathematics(2nd edition)*, by *Seymour Lipschutz*, *Marc Lipson*

- Online learning platform

https://courses.lumenlearning.com/math4liberalarts/chapter/introduction-euler-paths/

- University of Hawaii materials

http://courses.ics.hawaii.edu/ReviewICS241/morea/graphs/Graphs5-QA.pdf

- Online learning

https://www.geeksforgeeks.org/mathematics-euler-hamiltonian-paths/