# Representing Relations
# The Pigeonhole Principle

Course Code: 00090 Course Title: Discrete Mathematics


**Dept. of Computer Science**
**Faculty of Science and Technology**


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| **Lecturer No:** | **14** | **Week No:** | **8** | **Semester:** | **Summer 21-22** |
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# Lecture Outline

## 7.3 Representing Relations

- Representing Relations using Matrices (zero-one matrices)
- Representing Relations using Directed graph (Digraph)

## 5.2 The Pigeonhole Principle

# Objectives and Outcomes

- Objectives: To understand how to represent a relation using a zero-one matrix and directed graph (digraph), to understand the Pigeonhole principle and it's applications.

- Outcomes: The students are expected to be able represent a relation using a zero-one matrix and digraph; be able to determine whether a relation is reflexive, symmetric, antisymmetric, and/or transitive by analyzing a zero-one matrix or digraph that represents the relation; be able to explain the  Pigeonhole principle and and its applications.

# Representing Relations

- There are many ways to represent a relation between finite sets. One way is to list its **ordered pairs**, another way is to use a **table** (we have covered those already)
- In this section we will discuss two alternative methods of representing relations –
  - Representing Relations using **Matrices** (**zero-one matrices**)
  - Representing Relations using **Directed graph** (**Digraph**)
- All Relations we study in this section are **Binary Relations**.

# Representing Relations Using Matrices

- A relation between finite sets can be represented using a zero-one matrix.
- Suppose ***R*** is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$
  - The elements of the two sets can be listed in any particular arbitrary order. When $A = B$, we use the same ordering.

- The relation ***R*** is represented by the matrix $M_R = [m_{ij}]$, where

  The matrix representing ***R* has a 1 as its (*i*,*j*) entry when $a_i$ is related to $b_j$** and a **0 if $a_i$ is not related to $b_j$.**

## Representing Relations Using Matrices: Example 1

- **Example 1:** Suppose that $A = \{1,2,3\}$ and $B = \{1,2\}$.

  Let ***R*** be the relation from $A$ to $B$ containing $(a, b)$ if $a \in A$, $b \in B$, and $\boldsymbol{a > b}$. What is the **matrix representing *R*** (assuming the ordering of elements is the same as the increasing numerical order)?

- **Solution**: Because $R = \{(2,1), (3,1),(3,2)\}$, the matrix is

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

- **Note:** The matrix of a relation ***R*** from $A$ to $B$ is dependent on the **orderings of the *A* and *B***

## Representing Relations Using Matrices: Example 2

**Example 2:** Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$.

**Which ordered pairs** are in the relation ***R*** represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

**Solution:** Because ***R*** consists of those ordered pairs $(a_i, b_j)$ with $m_{ij} = 1$, it follows that:

$\boldsymbol{R} = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), \{(a_3, b_3), (a_3, b_5)\}$

# Matrices of Relations on Sets

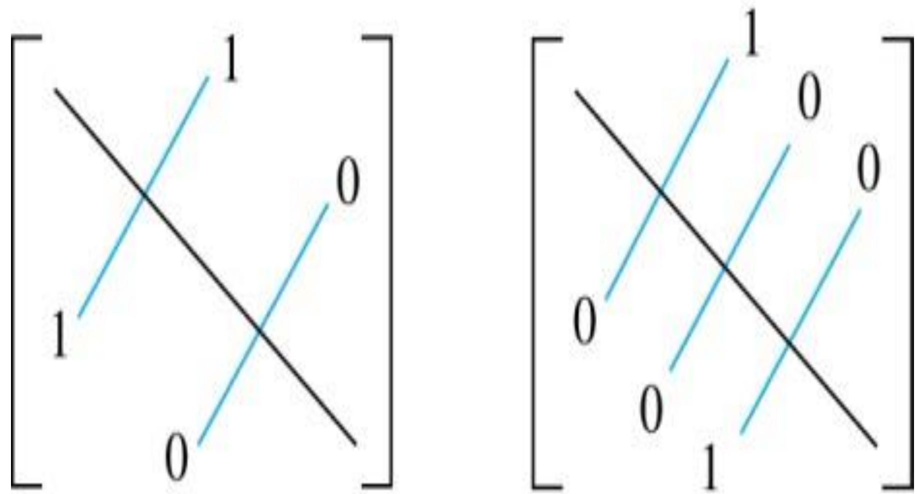
- If ***R*** is a ***reflexive*** relation, all the elements on the **main diagonal of $M_R$ are equal to 1.**
- The matrix of an ***antisymmetric*** relation has the property that

  if $m_{ij} \neq 1$ with $i = j$, then $m_{ji} = 0$. In other words, either $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$
  - For antisymmetry, there can never be two 1's symmetrically placed about the main diagonal
- ***R*** is a ***symmetric relation***, if the **matrix is symmetric.**
  - if and only if $m_{ij} = m_{ji}$ (for all pairs of integers $i$ and $j$ with $i = 1,2,\ldots..,n$ and $j = 1,2,\ldots..n$)
  - if and only if $\mathbf{M_R} = (\mathbf{M_R})^t$
  - ***Note*:** $(\mathbf{M_R})^t$ is the transpose of $\mathbf{M_R}$ which is obtained by interchanging rows and columns of $\mathbf{M_R}$

# Zero-One Matrices for Different Types of Relations

$$\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

**Reflexive**




(a) Symmetric (b) Antisymmetric

# Example of a Relation on a Set

**Example 3**: Suppose that the relation ***R*** on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is ***R*** reflexive, symmetric, and/or antisymmetric?

**Solution**: Because all the diagonal elements are equal to 1, ***R* is reflexive**.

Because $M_R$ is symmetric, ***R* is symmetric.**

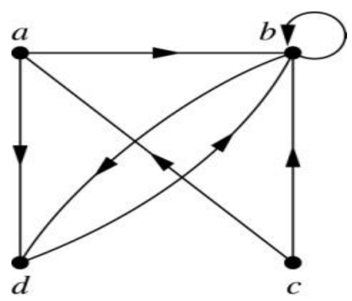
***R*** is **not antisymmetric** because both $m_{1,2}$ and $m_{2,1}$ are 1.

# Representing Relations Using Digraphs

**Definition:** A *directed graph*, or *digraph*, consists of a set $V$ of *vertices* (or *nodes*) together with a set $E$ of ordered pairs of elements of $V$ called *edges* (or *arcs*). The vertex $a$ is called the *initial vertex* of the edge $(a, b)$, and the vertex $b$ is called the *terminal vertex* of this edge.

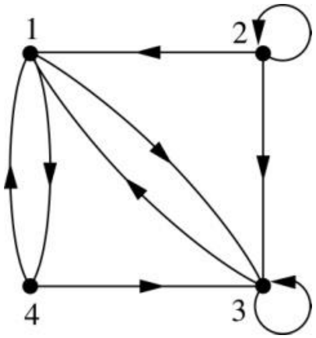
- An edge of the form $(a, a)$ is called a *loop*.

**Example 7:** A **drawing** of the directed graph with vertices $a$, $b$, $c$, and $d$, and edges $(a, b)$, $(a, d)$, $(b, b)$, $(b, d)$, $(c, a)$, $(c, b)$, and $(d, b)$ is shown below.



# Representing Relations Using Digraphs: Example

Example: What are the **ordered pairs** in the relation represented by this directed graph?

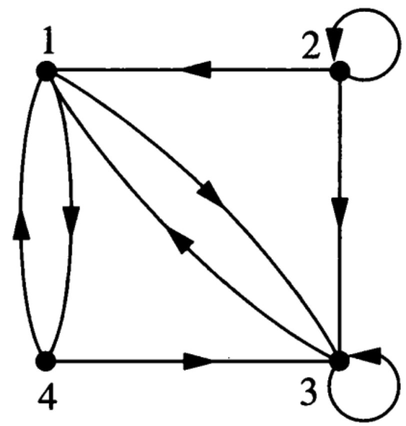



**Solution**: The **ordered pairs** in the relation are

(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3)

## Representing Relations Using Digraphs: Example

- Example 9 : What are the **ordered pairs** in the relation *R* represented by this graph?




- **Solution**:

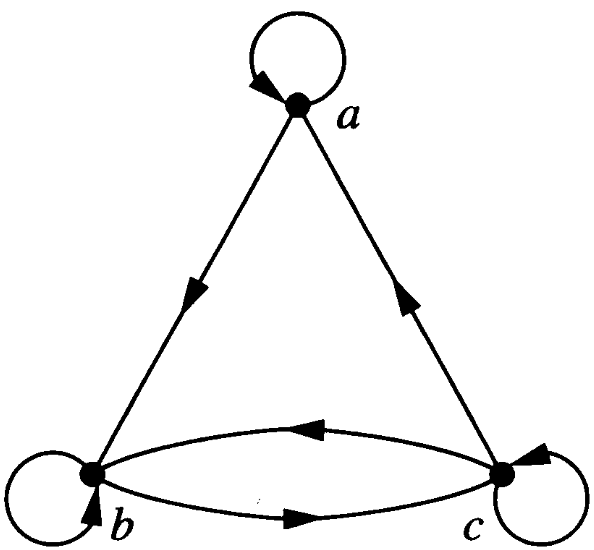
  R = {(1,3), (1,4), (2,1), (2,2), (2,3), (3,1), (3,3), (4,1), (4,3)}

# Determining which Properties a Relation has from its Digraph

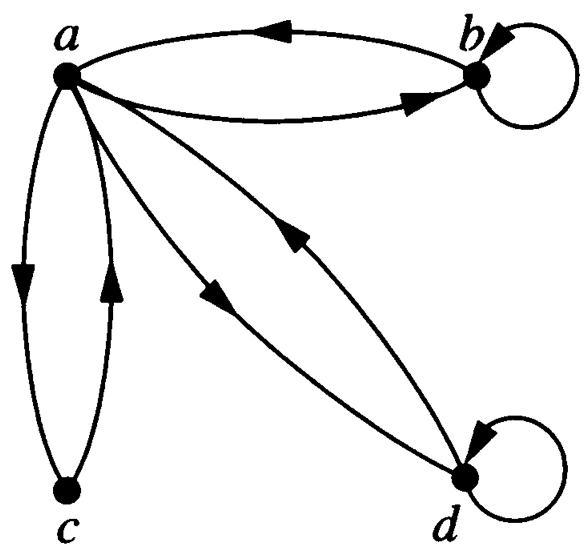
- ***Reflexivity***: A **loop** must be present **at ALL vertices** in the graph.
- ***Symmetry***: If $(a, b)$ is an edge, then so is $(b, a)$.
- ***Antisymmetry:*** Between any two vertices there is at most one directed edge.
- ***Transitivity***: If $(a, b)$ and $(b, c)$ are edges, then so is $(a, c)$.

# Example of a Relation on a Set

**Example 10:** Determine whether the relations for the directed graphs shown below are **reflexive, symmetric, antisymmetric, and/or trainsitive.** **[Solution ➔ next slide]**




(a) Directed graph of *R*




(b) Directed graph of *S*

# Solution of Example 10

- **Solution of (a):**
- **Reflexive**; because there are loops at every vertex
- **Not symmetric**; because there is an edge from *a* to *b*, but not one from *b* to *a*
- **Not antisymmetric**; because there is an edge from *b* to *c* and an edge from *c* to *b*
- **Not transitive**; because there is an edge from *a* to *b* and an edge from *b* to *c*, but no edge from *a* to *c*.

# Solution of Example 10

- **Solution of (b):**
- **Not Reflexive**; because loops are not present at every vertex.
- **Symmetric**; because every edge between distinct vertices is accompanied by an edge in the opposite direction.
- **Not antisymmetric**; because there is an edge from *a* to *b* and an edge from *b* to *a.*
- **Not transitive**; because (*c, a* ) and (*a, b* ) belongs to *S,* *but* (*c, b*) does not belong to *S*

# Class Work

- Draw the **digraph** and the **matrix** of the relation $R = \{(1, 2), (1, 3), (2, 2), (2, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 4)\}$ on the set $A = \{1, 2, 3, 4\}$.

- **Try it out yourself!**

# Practice @ Home

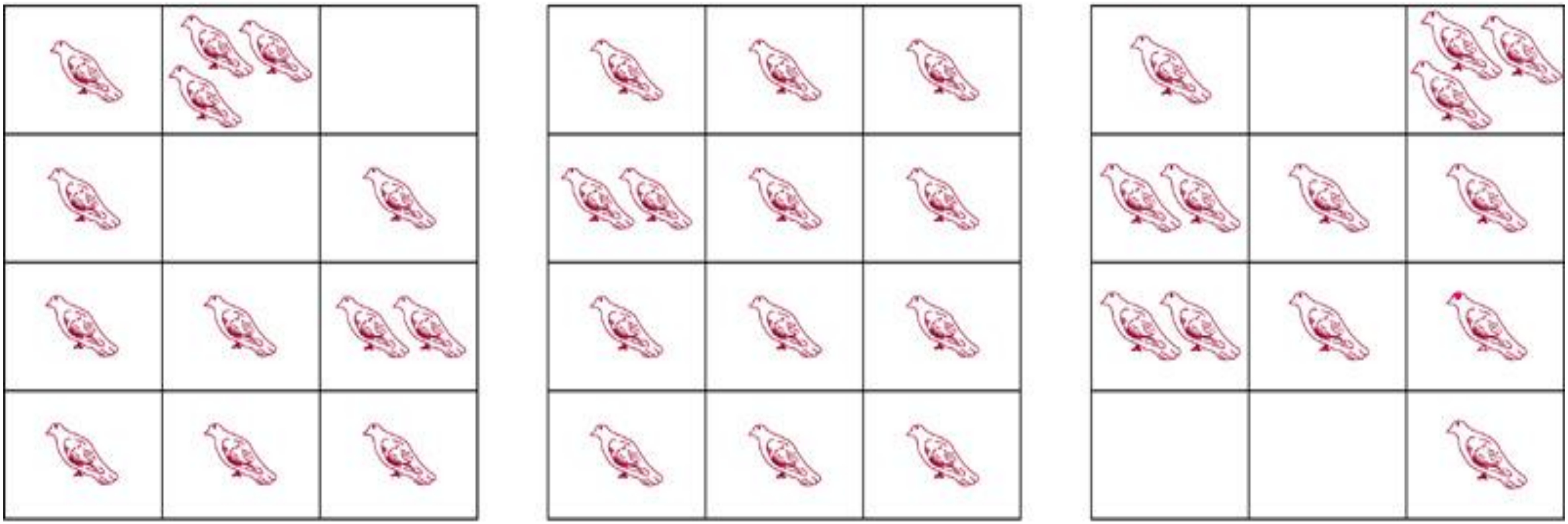
- Relevant Odd-Numbered Exercises from your text book

# 5.2 The Pigeonhole Principle

- If there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it.
- **Theorem 1**: (**The Pigeonhole Principle**) If k is a positive integer and k+1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.
  - Proof (by contraposition)
  - Also called the **Dirichlet drawer principle**

# FIGURE 1 : There Are More Pigeons Than  Pigeonholes





(a) (b) (c)

# The Pigeonhole Principle

- The Pigeonhole Principle can be used to prove a useful corollary about functions.
- **Corollary 1**: A function from a set with $k + 1$ or more elements to a set with $k$ elements is **not one-to-one**.

- **Example 1 (p. 348):** Among any group of 367 people, there must be **at least two** with the **same birthday,** because there are only 366 possible birthdays.

# The Generalized Pigeonhole Principle

- **Theorem 2 (The Generalized Pigeonhole Principle):** If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

- **Example 5 (p.349):** Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

# Example 6

- What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?
- **Solution**: The minimum number of students needed to ensure that at least six students receive the same grade is the smallest integer $N$ such that $\lceil N/5 \rceil = 6$.

  The smallest such integer is $N = 5.5 + 1 = 26$.

***Note:*** $\lceil N/5 \rceil \geq 6$, or, $N \geq 5.5 + 1$, or, $N \geq 26$

$N_{\text{smallest}} = 26$

# Class Work: Exercise 31

- Exercise 31: There are 38 different time periods during which classes at a university can be scheduled. If there are 677 different classes, how many different rooms will be needed?
- Solution: Try it out!

# ***Solution*** of **Exercise 31**

**Solution:** Thinks that the **38 time periods** are the **pigeonholes**, and the 677 classes are the pigeons.

By the generalized pigeonhole principle there is some time period in which at least $\lceil 677/38 \rceil = 18$ classes are meeting.

Since each class must meet in a different room, we need 18 rooms.

# Practice @ Home

- Relevant Odd-Numbered Exercises from your text book

# Books

- Rosen, K. H., & Krithivasan, K. (2012). Discrete mathematics and its applications: with combinatorics and graph theory. Tata McGraw-Hill Education. (7th Edition)

# References


1. Discrete Mathematics, *Richard Johnsonbaugh*, Pearson education, Inc.
2. Discrete Mathematical Structures, *Bernard Kolman*, *Robert C. Busby*, *Sharon Ross,* Prentice-Hall, Inc.
3. *SCHAUM'S outlines Discrete Mathematics(2nd edition)*, by *Seymour Lipschutz*, *Marc Lipson*

- Online tutorial

https://www.tutorialspoint.com/discrete_mathematics/discrete_mathematics_relations.htm

- University of Pittsburgh

https://people.cs.pitt.edu/~milos/courses/cs441/lectures/Class21b.pdf

- Online tutorial

https://www.geeksforgeeks.org/discrete-mathematics-the-pigeonhole-principle/

- National Tsing Hua University

http://www.cs.nthu.edu.tw/~wkhon/math/lecture/lecture06.pdf