Graphs

Course Code: 00090 Course Title: Discrete Mathematics

Dept. of Computer Science
Faculty of Science and Technology

Lecturer No:	**15**	**Week No:**	**9**	**Semester:**	**Summer 21-22**
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Lecture Outline

Graphs and Graph Models (8.1)

- Multigraph
- Pseudograph
- Simple directed graph
- Directed Multigraph
- Mixed graph

Graph Terminology and Special Types of Graphs (8.2)

- Basic terminology
- Adjacent vertices
- Degree of a vertex
- In-degree of a vertex
- Out-degree of a vertex
- Isolated vertex
- Pendant vertex
- The Handshaking Theorem
- Some Special Simple Graphs
- Bipartite Graphs

Objectives and Outcomes

- Objectives: To understand basic terminologies of graph with examples, Handshaking theorem for undirected and directed graphs, some special types of graphs, bipartite graph and complete bipartite graph.
- Outcomes: The students are expected to be able explain graph terminologies, be able to find out degree of vertices and prove Handshaking theorem, be able to draw Complete graph, Cycle, Wheel, n-cube, be able to determine whether a graph is bipartite using graph coloring.

Directed Graph

- Definition 2: A directed graph(or ***digraph***) (*V,E*) consists of a nonempty set of vertices *V* and a set of directed edges *E*.
- Each directed edge is associated with an ordered pair of vertices.
- The directed edge associated with the ordered pair (*u,v*) is said to *start* at ***u*** and *end* at ***v***.

Graph Terminology : Different Types of Graphs

- **Simple Graph**: An ***undirected*** graph with no multiple edges or loops is called a simple graph.
- **Multigraph**: An ***undirected*** graph that may contain multiple edges connecting the same vertices but no loops.
- **Pseudograph**: An ***undirected*** graph that may contain multiple edges and loops is called a pseudograph.

Graph Terminology : Different Types of Graphs

- **Simple Directed graph**: When a ***directed*** graph has no loops and has no multiple directed edges, it is called a simple directed graph.
- **Directed multigraph**: A graph with ***directed** edges* that may contain multiple directed edges is called a directed multigraph.
- **Mixed Graph**: A graph with *both **directed** and **undirected** edges* is called a mixed graph. A mixed graph may contain loop(s).
- **Loop**: An edge that connect a vertex to itself is called a loop.

Graph Terminology : Different Types of Graphs

Table 1: Graph Terminology

Type	Edges	Multiple Edges Allowed?	Loops Allowed?
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed Multigraph	Directed	Yes	Yes
Mixed graph	Directed and Undirected	Yes	Yes

Graph Terminology and Special Types of Graphs (8.2)

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  - In-degree of a vertex
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Basic Terminology

- **Definition**1: Two vertices *u* and *v* in an undirected graph *G* are called ***adjacent*** (or ***neighbors***) in *G* if *u* and *v* are endpoints of an edge of *G.*
- If *e* is associated with {*u, v*}, the edge *e* is called ***incident*** with the vertices *u* and *v*.
- The edge ***e*** is also said to ***connect*** ***u*** and ***v.***
- The vertices ***u*** and ***v*** are called ***endpoints*** of an edge associated with {***u, v***}.

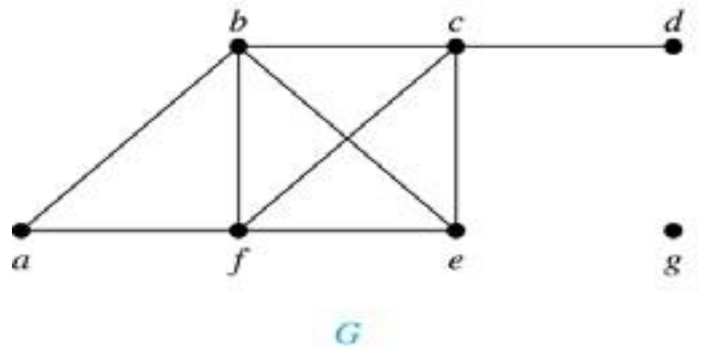
Basic Terminology

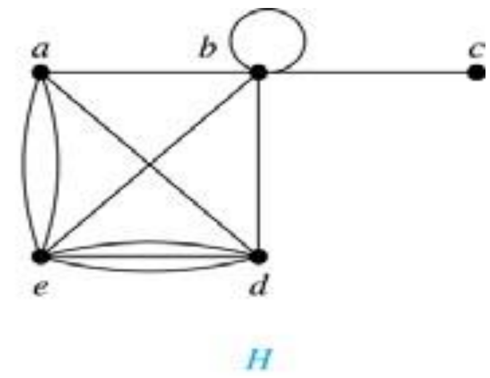
- **Definition 3:** The ***degree of a vertex in an undirected graph*** is the ***number of edges incident with it***, **except** that a ***loop*** at a vertex contributes ***twice*** to the degree of that vertex.
  - The degree of the vertex *v* is denoted by deg(*v*)

- **Isolated vertex**: A vertex of degree zero is called isolated.
- **Pendant vertex**: A vertex is pendant if and only if it has degree one.

Example 1

- <u>Example 1</u> : What are the degrees of the vertices in the graphs G and H?





<u>Solution</u>:

G: deg(a) = 2, deg(b) = deg(c)= deg(f)=4, deg(d)=1, deg(e) = 3, and deg(g)= 0

H: deg(a) = 4, deg(b) = deg(e) = 6, deg(c) = 1, deg(d) = 5.

The Handshaking Theorem

- **Theorem 1** (The Handshaking Theorem):

  Let G = (V,E) be an ***undirected graph*** with ***e*** edges. Then

$$\mathbf{2}\boldsymbol{e} = \sum_{\in} \mathbf{deg}(\boldsymbol{v})$$

Note: This applies even if multiple edges and loops are present

Example 2

- Example 2: How many edges are there in a graph with 10 vertices each of degree six?

- **Solution**: Because the sum of the degrees of the vertices is $6.10 = 60$, it follows that $2e=60$.

  Therefore, $e = 30$

Theorem 2

- **Theorem 2:** *An undirected graph has an even number of vertices of odd degree.*

- Example: If a graph has 5 vertices, can each vertex have degree 3?
- **Solution**: This is not possible by the Handshaking theorem, because the sum of the degrees of the vertices 3.5 = 15 is odd.

Initial vertex & Terminal Vertex

- **Definition 3**: When (u, v) is an edge of the graph G with directed edges, *u is said to be* ***adjacent to v*** and *v is said to be* ***adjacent from u***.
- The vertex ***u*** is called the ***initial vertex*** of (u, v) and

  v is called the ***terminal/end vertex*** of (u, v).

- ***Note***: The initial vertex and terminal vertex of a loop are the same.

In-degree & Out-degree of a vertex

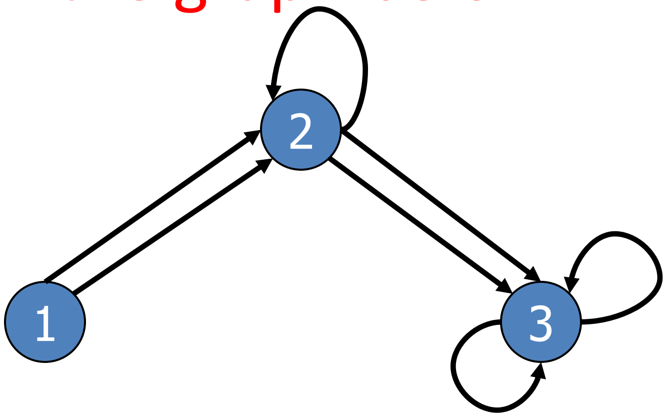
- **Definition 4**: In a graph with directed edges the ***in-degree of a vertex v***, denoted by **$deg^{-}(v)$**, is the number of edges with v as their terminal vertex.

  The ***out-degree of v***, denoted by **$deg^{+}(v)$**, is the number of edges with v as their initial vertex.

- ***Note***: A **loop** at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.

Example: In-degree & Out-degree of vertices of a graph

- Question: What are in-degrees and out-degrees of all the vertices in the graph below?



Solution:

$deg^{-}(1) = 0$

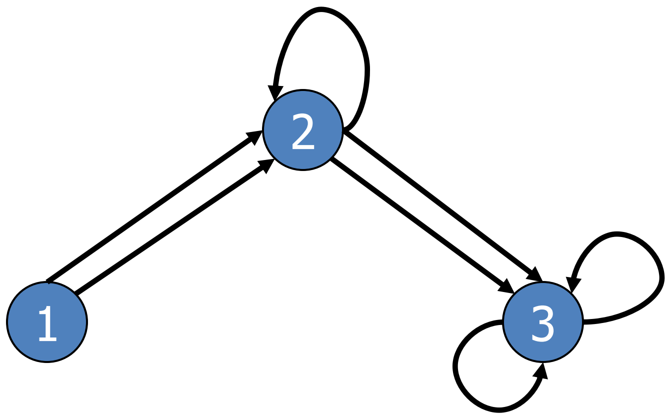
$deg^{-}(2) = 3$

$deg^{-}(3) = 4$

$deg^{+}(1) = 2$

$deg^{+}(2) = 3$

$deg^{+}(3) = 2$



- Practice Yourself: Example 4

Theorem 3

- **Theorem 3:** Let G = (V, E) be a graph with *directed edges*. Then

$$\sum_{\in} \deg^{-}(v) = \sum_{\in} \deg^{+}(v) = |E|$$

➔ Handshaking Theorem for directed graph

Some Special Simple Graphs: Complete Graph (K_n)

- The **complete graph on *n* vertices**, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.
- The graph , for n = 1, 2, 3, 4, 5, 6 are displayed in the following figure:

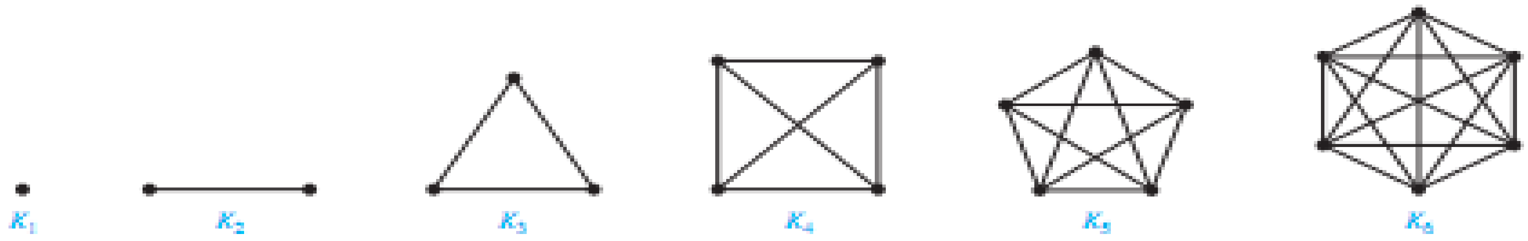


FIGURE 3 The Graphs K_n for $1 \leq n \leq 6$.

Some Special Simple Graphs: Cycles (C_n)

- The **Cycle C_n**, n>=3, consists of n vertices v_1, v_2, ...v_n and edges {v_1 , v_2 }, {v_2, v_3}, ... {v_{n-1}, v_n }, and {v_n, v_1 }
- The cycles for C_3 , C_4 , C_5 , and C_6 are displayed in the following figure:

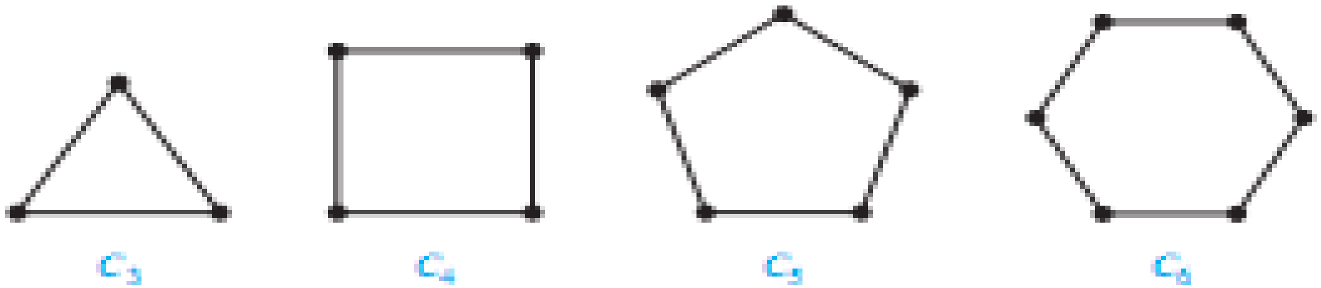
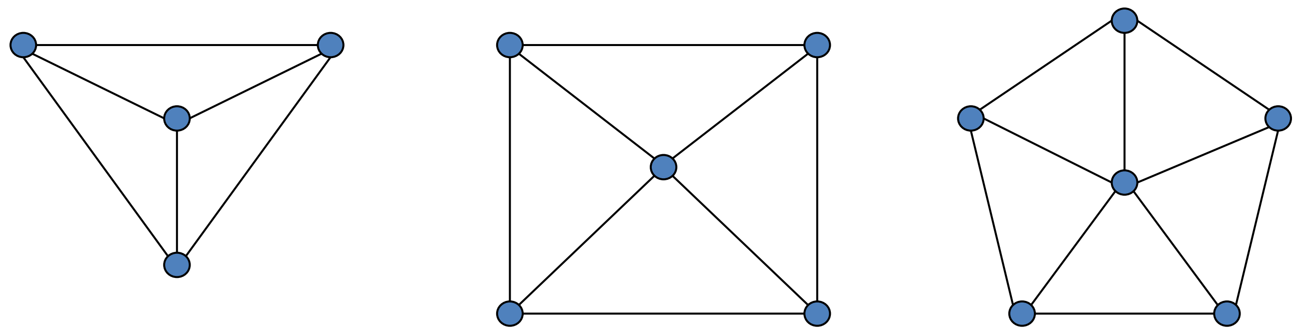


FIGURE 4 The Cycles C_3, C_4, C_5, and C_6.

Some Special Simple Graphs: Wheels (W_n)

- We obtain the **Wheel (W_n)** when we add an additional vertex to the **Cycle C_n** , for n>=3, and connect this new vertex to each of the n vertices in C_n , by new edges.
  - The ***wheel W_n*** is just a cycle graph with an extra vertex in the middle
- The Wheels ***W_3 , W_4 , W_5*** *are displayed in the figure below:*



Some Special Simple Graphs: n-Cubes(Q_n)

- The ***n*-dimensional hypercube**, **or *n*-cube**, denoted by Q_n, is the graph that has vertices representing the 2^n bit strings of length n.
  - Two vertices are adjacent iff the bit strings that they represent differ in exactly one bit position.
  - The graphs Q_1, Q_2 , and Q_3 are displayed in the following figure:

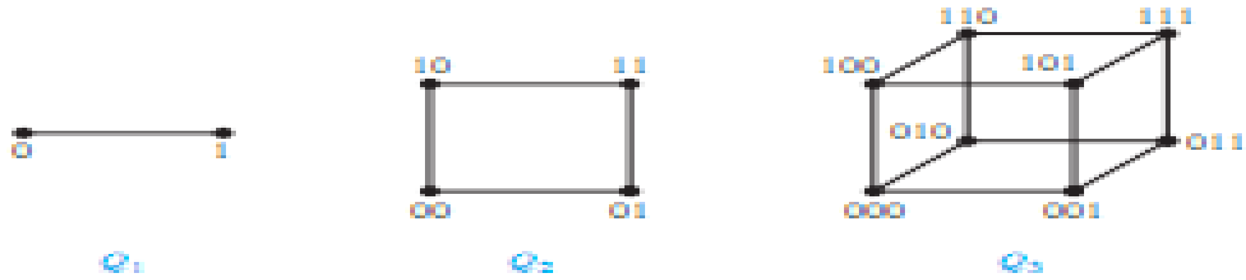


FIGURE 6 The n-cube Q_n, $n = 1, 2, 3$.

Bipartite graphs

- **Definition 5**: A simple graph G is called ***bipartite*** if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2

- **Note**: If each vertex of V_1 is connected to each vertex of V_2, then it is called **complete bipartite graph** and it is denoted by $K_{m,n}$ where m is the number of vertices in V_1 and n is the number of vertices in V_2

Examples of Complete Bipartite graphs

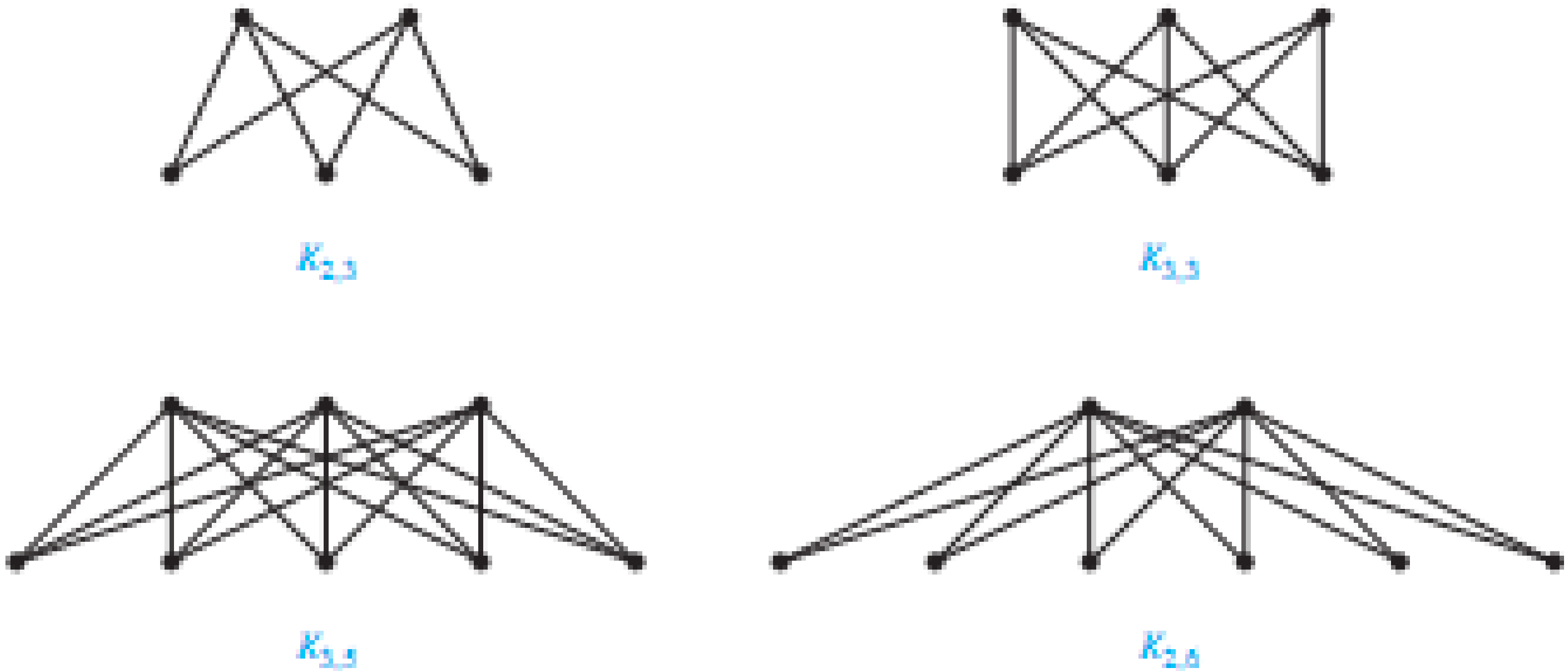


FIGURE 9 Some Complete Bipartite Graphs.

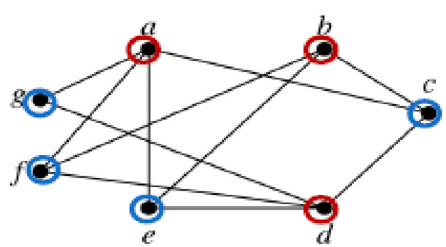
Example 11: Are the graphs G and H are Bipartite?

Bipartite Graphs

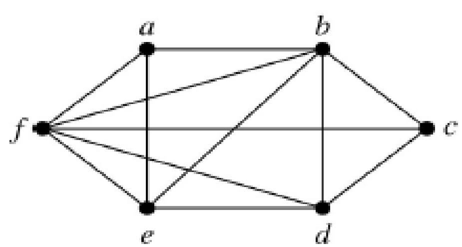
Definition:

An equivalent definition of a bipartite graph is one where it is possible to color the vertices either red or blue so that no two adjacent vertices are the same color.

G is bipartite



G



H

H is not bipartite: if we color *a* red, then its neighbors *f* and *b* must be blue. But f and b are adjacent.

Practice @ Home

- **Relevant Odd-Numbered exercises from text book**

Books

- **Rosen, K. H., & Krithivasan, K. (2012). Discrete mathematics and its applications: with combinatorics and graph theory. Tata McGraw-Hill Education. (7th Edition)**
- Liu, C. L. (1986). Elements of discrete mathematics. Tata McGraw-Hill Education.

References

1. Discrete Mathematics, *Richard Johnsonbaugh*, Pearson education, Inc.
2. Discrete Mathematical Structures, *Bernard Kolman*, *Robert C. Busby*, *Sharon Ross,* Prentice-Hall, Inc.
3. *SCHAUM'S outlines Discrete Mathematics(2nd edition)*, by *Seymour Lipschutz*, *Marc Lipson*

- Deo, N. (2017). Graph theory with applications to engineering and computer science. Courier Dover Publications.
- Radford University Lecture: https://www.radford.edu/~nokie/classes/360/graphs-terms.html