Representing Graphs and Graph Isomorphism

Course Code: 00090 Course Title: Discrete Mathematics

Dept. of Computer Science
Faculty of Science and Technology

Lecturer No:	**16**	**Week No:**	**9**	**Semester:**	**Summer 21-22**
Lecturer:	*Md. Mahmudur Rahman (mahmudur@aiub.edu)*				

Lecture Outline

Representing Graphs and Graph Isomorphism (8.3)

- Graph Representation:
  - Adjacency lists
  - Adjacency matrices
  - Incidence matrices

- Graph Isomorphism

Objectives and Outcomes

- Objectives: To understand three different representation of a Graph, to understand graph invariants and graph isomorphism.

- Outcomes: This students are expected to be able to represent a graph using adjacency list, adjacency matrix, and incidence matrix; be able to determine whether two graphs are isomorphic.

Representing Graphs

- One way to represent a graph without multiple edges is to **list** all the edges of the graph.
- Another way to represent a graph with no multiple edges is to use **adjacency lists**, which specify the vertices that are adjacent to each vertex of the graph.
- **Adjacency Lists**: A table with 1 row per vertex, listing its adjacent vertices.
- **Directed Adjacency Lists**: A table with 1 row per node, listing the *terminal nodes* of each edge *incident from that node*.

Representing a Graph with an Adjacency List

- EXAMPLE 1: Use **adjacency lists** to describe the simple graph given in Figure 1.

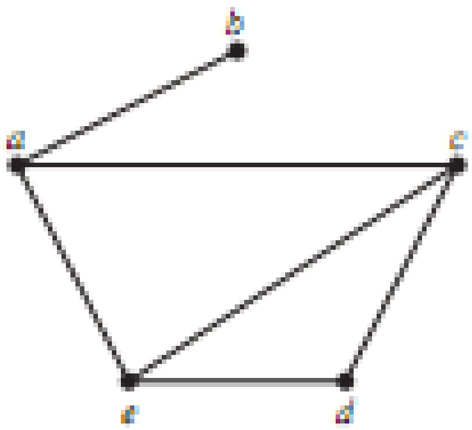


FIGURE 1 A Simple Graph.

TABLE 1 An Adjacency List for a Simple Graph.

Vertex	*Adjacent Vertices*
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

Representing a Graph with an Adjacency List

- EXAMPLE 2: Represent the directed graph shown in Figure 2 by listing all the vertices that are the terminal vertices of edges starting at each vertex of the graph.

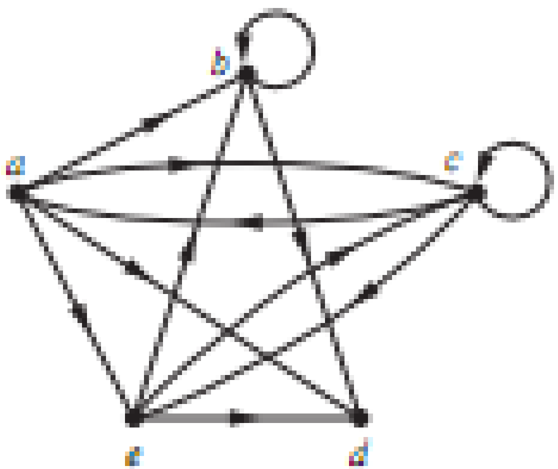


FIGURE 2 A Directed Graph.

TABLE 2 An Adjacency List for a Directed Graph.

Initial Vertex	*Terminal Vertices*
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

Representing Graphs using Matrices

- Two types of **matrices** commonly used to represent graphs –
    1) **Adjacency** matrix
    2) **Incidence** matrix

- Adjacency matrix: A matrix representing a graph using the adjacency of vertices.
- Incidence matrix: A matrix representing a graph using the incidence of edges and vertices.

Adjacency matrices

- Suppose that G = (V, E) is a simple graph where |V|= n. Suppose that the vertices of G are listed arbitrarily as v_1, v_2, …., v_n.
  The **adjacency matrix A** of G, with respect to this listing of the vertices, is the *n x n* zero-one matrix with 1 as its (i,j)th entry when v_i and v_j are adjacent, and 0 as its (i,j)th entry when they are not adjacent.
- In other words, if its adjacency matrix is **A** = [a_{ij}], then

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \text{otherwise} \end{cases}$$

Example 3: Representing a graph using an Adjacency Matrix

EXAMPLE 3 Use an adjacency matrix to represent the graph shown in Figure 3.

Solution: We order the vertices as a, b, c, d. The matrix representing this graph is

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

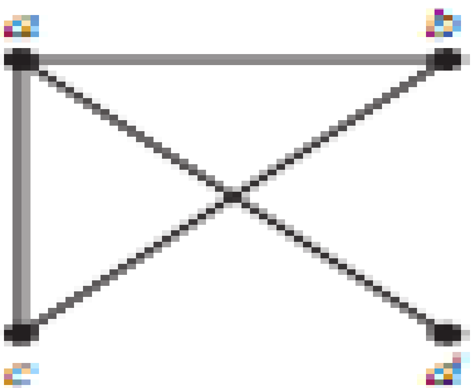


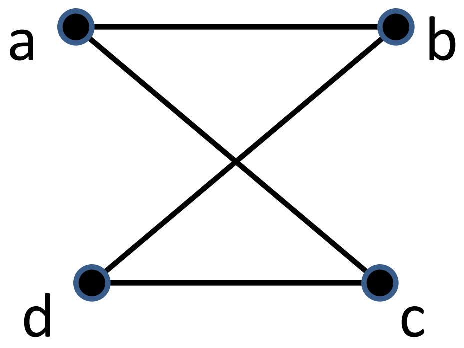
FIGURE 3

Given an Adjacency matrix, draw the graph

- **<u>Example 4</u>** : Draw a graph with the adjacency matrix below.

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

<u>Solution</u>:

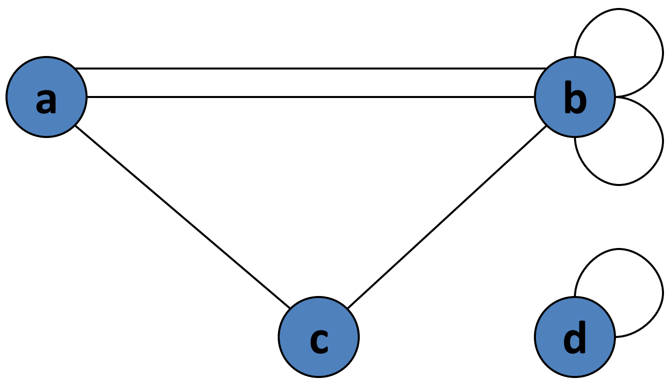


Adjacency matrices

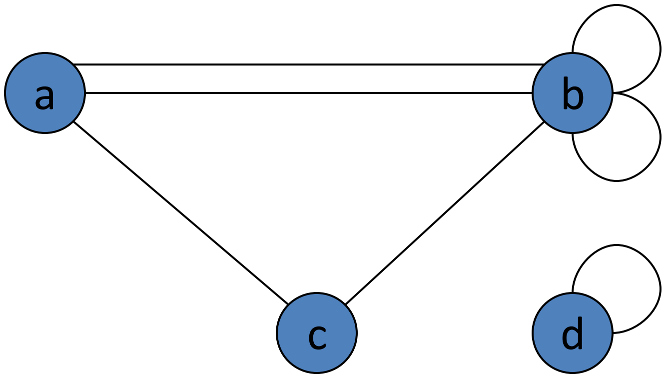
- Adjacency matrix can also be used to represent **undirected graphs** with **loops** and with **multiple edges**.
  - A loop at the vertex a_i is represented by 1 at the (i,j)th position of the adjacency matrix.
  - When multiple edges are present, the adjacency matrix is no longer zero-one matrix, because the (i,j)th entry of this matrix equals the number of edges that are associated to $\{a_i, a_j\}$
  - **All undirected graphs**, including multigraphs and pseudographs, **have symmetric adjacency matrices**

Adjacency matrices

- ***Example*** : What's the adjacency matrix of the following graph?



Adjacency matrices



Solution:

$$\begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Practice at Home

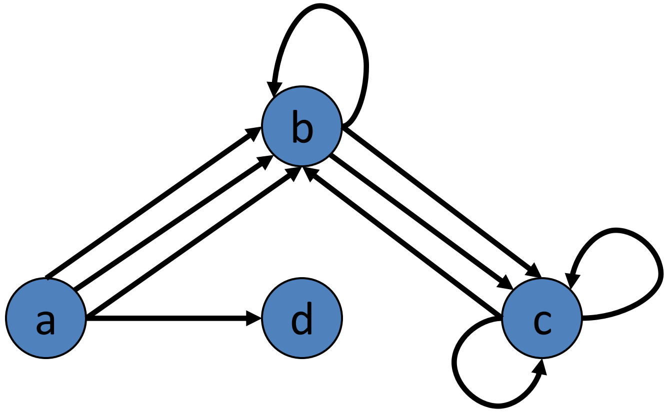
- **Example 5** : Use an adjacency matrix to represent the pseudograph shown in **Figure 5**.

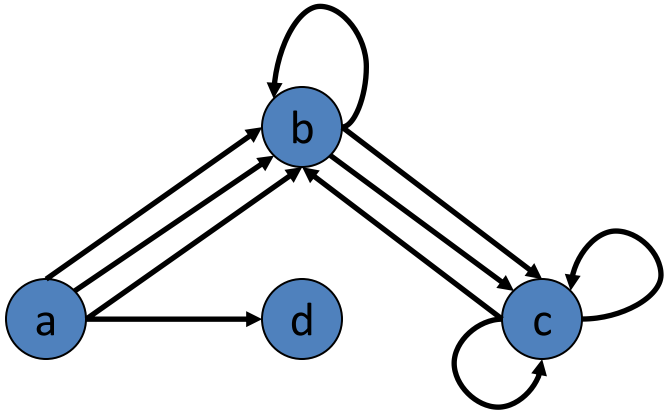
Adjacency matrices for Directed Graphs

- Adjacency matrices can also be used to represent **directed multigraphs**.
  - However, such matrices are not zero-one matrices when there are multiple edges in the same direction connecting two vertices
- In the adjacency matrix for a directed graph, a_{ij} equals the number of edges that are associated to (v_i, v_j).

Adjacency matrices for directed graphs

Example : What is the adjacency matrix?





Solution:

$$\begin{pmatrix} 0 & 3 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Incidence matrices

- **Incidence matrix**: A matrix representing a graph using the **incidence** of **edges** and **vertices**.
- Let G = (V, E) be an undirected graph. Suppose that $v_1, v_2, \ldots,$ v_n are the vertices and $e_1, e_2, \ldots, e_m$ are the edges of G. Then the incidence matrix with respect to this ordering of V and E is the **n x m matrix**, **M = [m_{ij}],** where

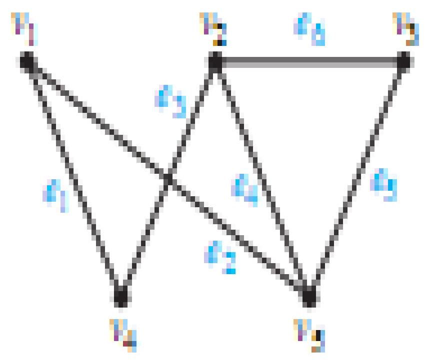
$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i, \\ 0 & \text{otherwise} \end{cases}$$

Example 6: Representing a graph with an Incidence Matrix

$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

EXAMPLE 6 Represent the graph shown in Figure 6 with an incidence matrix.

Solution: The incidence matrix is



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

Example 7 : Representing a graph with an Incidence

EXAMPLE 7 Represent the pseudograph shown in Figure 7 using an incidence matrix.

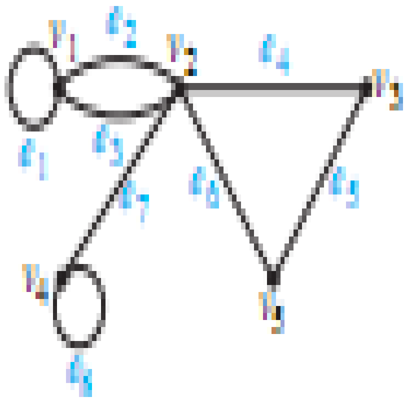


FIGURE 7
A Pseudograph.

Solution: The incidence matrix for this graph is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array}
\begin{array}{c}
\begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}
\end{array}.$$

Isomorphism of Graphs

- **Definition 1**: The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic **if there is a one-to-one and onto function *f* from V_1 to V_2** with the property that a and b are adjacent in G_1 iff f(a) and f(b) are adjacent in G_1, for all a and b in V_1.

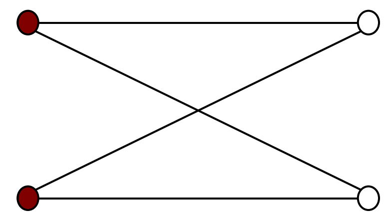
  Such a function ***f*** is called an **isomorphism**.

  - When two simple graphs are isomorphic, there is a one-to-one correspondence between vertices of the two graphs that preserves the adjacency relationship
  - Two graphs are isomorphic iff they are identical except for their node names
  - Isomorphism of simple graphs is an equivalence relation

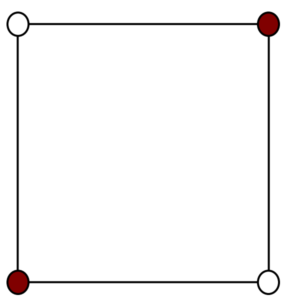
Isomorphism of Graphs

- Intuitively, two graphs are isomorphic if can bend, stretch and reposition vertices of the first graph, until the second graph is formed. Etymologically, *isomorphic* means "same shape".

e.g.: Can twist or relabel:

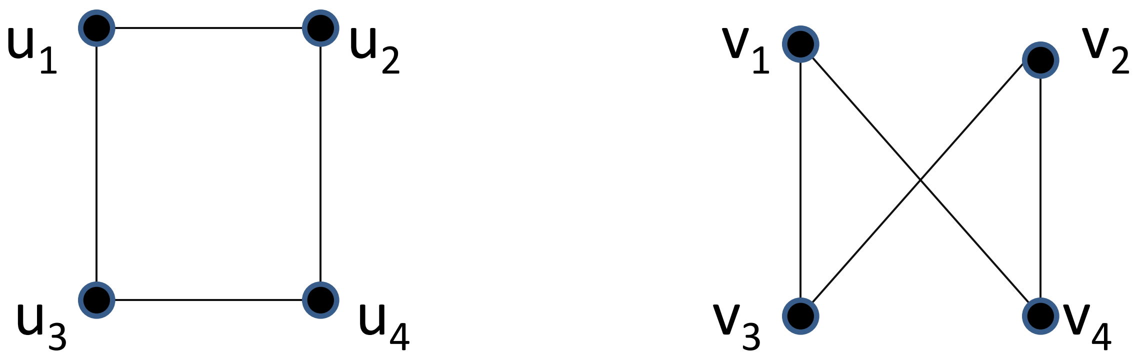


to obtain:



Example 8

- Show that the graphs G=(V,E) and H=(W,F) are isomorphic.



G H

Solution of Example 8

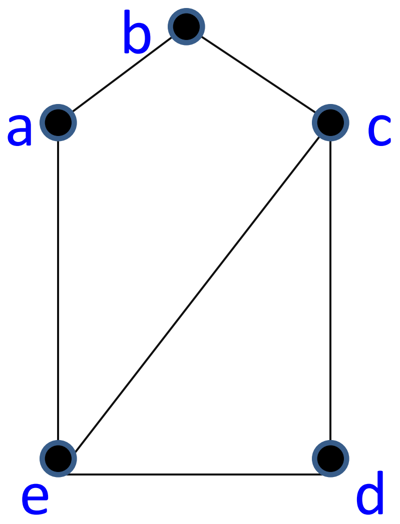
- The function f with $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, *and* $f(u_4)= v_2$ is a one-to-one correspondence between V and W. To see that this correspondence preserves adjacency, note that adjacent vertices in G are u_1 and u_2, u_1 and u_3, u_2 and u_4, and each of the pairs $f(u_1)= v_1$ *and* $f(u_2) = v_4$, $f(u_1)= v_1$ *and* $f(u_3)= v_3$, $f(u_2)= v_4$ *and* $f(u_4)= v_2$, *and* $f(u_3)= v_3$ *and* $f(u_4)= v_2$ *are adjacent in H.*
- So, the graphs G and H are isomorphic.

Isomorphism of Graphs

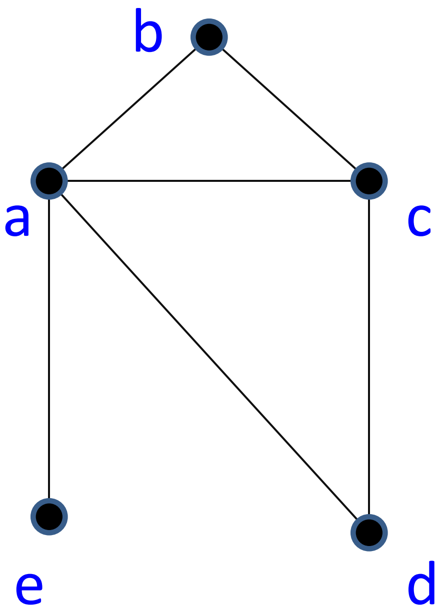
- It is often difficult to determine whether two simple graphs are isomorphic.
- Sometimes it is not hard to show that two graphs are **not isomorphic**.
- A property preserved by isomorphism of graph is called a ***graph invariant***.
- **Isomorphic simple graphs must have** –
  - The same # of vertices
  - The same # of edges
  - The degrees of the vertices must be the same and degrees of adjacent vertices must be same
  - Same length of simple circuit(s), if any

Example 9

- Show that the following two graphs G and H are not isomorphic.



G



H

Solution of Example 9

- **Solution :** Both G and H have five vertices and six edges. However, H has a vertex of degree one, namely e, whereas G has no vertices of degree one.

  It follows that G and H are not isomorphic.

- There are other answers too!

  What are the other answers?

Isomorphism of Graphs

- **Note**: The number of vertices, the number of edges, and the number of vertices of each degree are all invariants under isomorphism. If any of these quantities differ in two simple graphs, these graphs cannot be isomorphic.
- However, when these invariants are the same, it does not necessarily mean that the two graphs are isomorphic. There are NO useful sets of invariants currently known that can be used to determine whether simple graphs are isomorphic.

Graph Isomorphism : Negative Examples

Q1: Why are the following graphs not isomorphic?

Graph Isomorphism : Negative Examples

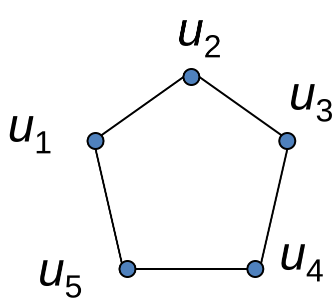
A1: 1^{st} graph has more vertices than 2^{nd}. So the graphs are not isomorphic. What are other answers?

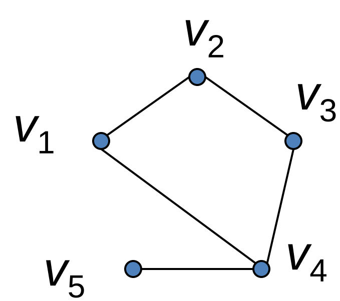
Q2: Why are the following graphs not isomorphic?

Graph Isomorphism : Negative Examples

A2: 1^{st} graph has more edges than 2^{nd}. What are other answers?

Q3: Why are the following graphs not isomorphic?

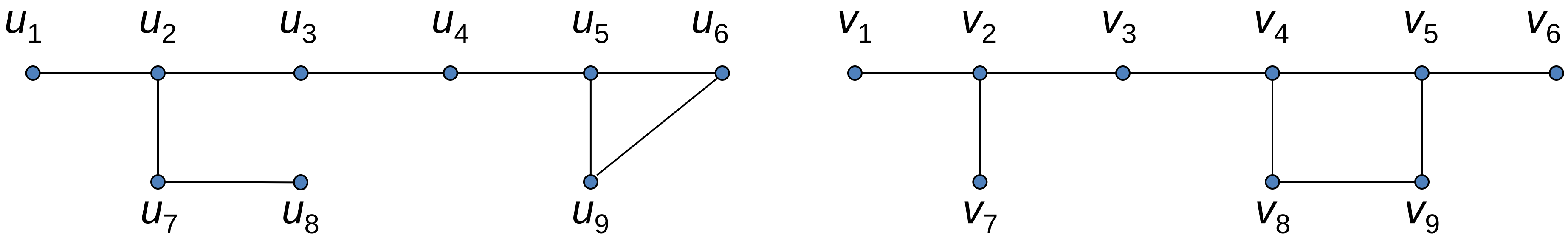




Graph Isomorphism : Negative Examples

A3: 2nd graph has vertex of degree 1, 1st graph doesn't. So the graphs are not isomorphic. What are other answers?

Q4: Why are the following graphs non-isomorphic?



Graph Isomorphism : Negative Examples

- A4: 1st graph has 2 vertices of degree 1, whereas the 2nd graph has 3 vertices of degree 1.

  So the graphs are not isomorphic.

  What are other answers?

Practice @ Home

- Example 10
- Example 11
- **Relevant Exercises from your text book**

Books

- **Rosen, K. H., & Krithivasan, K. (2012). Discrete mathematics and its applications: with combinatorics and graph theory. Tata McGraw-Hill Education. (7th Edition)**
- Liu, C. L. (1986). Elements of discrete mathematics. Tata McGraw-Hill Education.

References

1. Discrete Mathematics, *Richard Johnsonbaugh*, Pearson education, Inc.
2. Discrete Mathematical Structures, *Bernard Kolman*, *Robert C. Busby*, *Sharon Ross,* Prentice-Hall, Inc.
3. *SCHAUM'S outlines Discrete Mathematics(2nd edition)*, by *Seymour Lipschutz*, *Marc Lipson*

- An online tutorial https://www.geeksforgeeks.org/graph-and-its-representations/
- Another online tutorial https://www.gatevidyalay.com/graph-isomorphism/
- Personal blog http://www.rogerfhouse.com/math-articles/graph-isomorphism.pdf