Question 1

Claim: $(\exists m \in N)(\exists n \in N)(3m + 5n = 12)$

Proof: By direct proof.

First, we test the equation using different values for m and n.

Note: m & n cannot be 0.

A. Let $m = 2$ and $n = 1$. Then $3(2) + 5(1) = 11$

B. Let $m = 1$ and $n = 2$. Then $3(1) + 5(2) = 13$

Check the difference (B – A) to determine the number that can't be reached using natural numbers: $13 – 11 = 2$

Check the divisibility of 3 and 5 by 2:

- $3/2 = 1.5$
- $5/2 = 2.5$

Both the above answers are not natural numbers.

Test above values to determine whether m & n should be 1.5 or 2.5:

Let $m = 1.5$ and $n = 1.5$. Then $3(1.5) + 5(1.5) = 12$

Let $m = 2.5$ and $n = 2.5$. Then $3(2.5) + 5(2.5) = 20$

Conclusion: Therefore, natural numbers are insufficient to solve the claim, the claim is false.

QED

Question 2

Claim: The sum of any five consecutive integers is divisible by 5.

Proof: By induction.

Step 1: Check A(1) by doing some examples.

Let A = n1 + n2 + n3 + n4 + n5.

Let a = (1,2,3,4,5)

- A(a) = 1+2+3+4+5 = 15
- 15/5 = 5

Let b = (10,11,12,13,14)

- A(b) = 10+11+12+13+14 = 60
- 60/5 = 12

Both a and b are divisible by 5.

Step 2 (Induction Step): Assume A(1) holds, therefore $(\forall n \in Z)[A(n) \Rightarrow A(n+1)]$.

Let (a+1) = (2,3,4,5,6)

- A(a) = 2+3+4+5 = 20
- 20/5 = 4

Let (b+1) = (11,12,13,14,15)

- A(b) = 11+12+13+14+15 = 65
- 65/5 = 13

Conclusion: Hence, by the principal of mathematical induction, the identity holds for all n. Thus, the claim is true.

QED

Question 3

Claim: For any integer n, the number $n^2 + n + 1$ is odd. That is $(\forall n \in Z)[n^2 + n + 1 = a \mid a/2 = p/q + r]$.

Proof: by Induction.

Step 1: Check A(1) by doing some examples.

Le n = 3.

Then: (3)2 + 3 + 1 = 13.

Then: 13/2 = [12/2 + 0.5] – Written in the form p/q + r.

Le n = 5.

Then: (4)2 + 4 + 1 = 21.

Then: 21/2 = [20/2 + 0.5] – Written in the form p/q + r.

Step 2 (Induction Step): Assume A(1) holds, therefore $(\forall n \in Z)[A(n) \Rightarrow A(n+1)]$.

Then: $(n+1)^2 + (n+1) +1$

Then: $(n+1)(n+1) + (n+1) +1$

Then: $n^2 + 2n + 1 + n +1 + 1$

Then: $n^2 + 3n + 3$

Using the quadratic formula: $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$\text{when } ax^2+bx+c=0$

n = (-3 ± √-3)/2

This indicates that n cannot be divided into pairs.

Conclusion: Hence, by mathematical induction, the identity holds for all n. Thus, the claim is true.

QED

Question 4

Claim: Every odd natural number is of one of the forms 4n + 1 or 4n + 3, where n is an integer.

Proof: by Induction.

Step 1: Check A(1) by doing some examples.

Let n = 1		Let n = 2
- Then: 4(1) + 1 = 5	&	Then: 4(2) + 1 = 9
- 5 is odd.		9 is odd.
- Then: 4(1) + 3 = 7	&	Then: 4(2) + 3 = 11
- 7 is odd		11 is odd

Using 1 and 3 we can get four consecutive odd numbers (5,7,9,11).

Step 2 (Induction Step): Assume A(1) holds, therefore (∀n ϵ Z)[A(n) => A(n+1)].

Then: 4(n+1) + 1

Then: 4n + 4 + 1

Then: 4n + 5

Then (divide by 2 to determine if 5 is divisible evenly):

- (4n/2) + (5/2)
- 2n + 2.5, thus, 5 is not divisible by two, therefore A(n+1) + 1 remains odd.

Then: 4(n+1) + 3 => 4n + 7

Then (divide by 2 to determine if 7 is divisible evenly):

- Then (4n/2) + (7/2)
- 2n + 3.5: Thus, 7 is not divisible by two, therefore A(n+1) + 3 remains odd.

Conclusion: Hence, by mathematical induction, all odd numbers can be obtained by adding 1 or 3 to 4n.

QED

Question 5

Claim: Prove that one and only one out of n, $n + 2$ and $n + 4$ is divisible by 3, where n is any positive integer.

Proof: by Cases.

Let $r = 0, 1, 2$,

Let q be the quotient.

Therefore, any number is in the form of $3q$, $3q+1$ or $3q+2$.

Case 1: Let $n =3q$

$n = 3q = 3(q)$ is divisible by 3,

$n + 2 = 3q + 2$ is not divisible by 3.

$n + 4 = 3q + 4 = 3(q + 1) + 1$ is not divisible by 3.

Case 2: if $n =3q + 1$

$\Rightarrow n = 3q + 1$ is not divisible by 3.

$\Rightarrow n + 2 = 3q + 1 + 2 = 3q + 3 = 3(q + 1)$ is divisible by 3.

$\Rightarrow n + 4 = 3q + 1 + 4 = 3q + 5 = 3(q + 1) + 2$ is not divisible by 3.

Case 3: if $n = 3q + 2$

$n =3q + 2$ is not divisible by 3.

$n + 2 = 3q + 2 + 2 = 3q + 4 = 3(q + 1) + 1$ is not divisible by 3.

$n + 4 = 3q + 2 + 4 = 3q + 6 = 3(q + 2)$ is divisible by 3.

Conclusion: Hence, by proof by cases, the identity holds for one and only one.

QED

Question 6

Claim: The only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.

Proof: by Cases and Contradiction.

Let n, n + 2, n + 4 be three natural numbers that come one after the other.

If it happens that 3 does not divide n.

- Then: n = 3q + 1 or n = 3q + 2, for some q.
- Case 1 (3q + 1): n + 2 = 3q + 3, thus 3 divides n.
- Case 2 (3q + 2): n + 4 = 3q + 6, thus 3 divides n.

The fact that 3 divides two of the three numbers, means they cannot all be prime numbers.

Conclusion: Therefore, by proof by cases and contradiction, 3,5,7 are the only prime numbers evenly distributed.

QED

Question 7

Claim: For any natural number n, $2 + 2^2 + 2^3 + . . . + 2^n = 2^{n+1} – 2$.

Proof: by Induction.

Step 1:

Let n = 4, then $2 + 2^2 + 2^3 + 2^4 = 2^{4+1} – 2 = 30$.

Let n = 5, then $2 + 2^2 + 2^3 + 2^4 + 2^5 = 2^{5+1} – 2 = 62$.

Step 2 (Induction Step): Assume A(1) holds, therefore $(\forall n \in Z)[A(n) \Rightarrow A(n+1)]$.

We add 2^{n+1} to both sides for A(n+1).

Then: $2 + 2^2 + 2^3 + . . . + 2^n + 2^{n+1} = 2^{n+1} – 2 + 2^{n+1}$.

Then: LHS = $2*2^{n+1} – 2$.

Then: LHS = 2^{n+1} -2.

Conclusion: Hence, by the mathematical principal of induction, the identity holds for any natural number n.

QED

Question 8

Claim:

Prove (from the definition of a limit of a sequence) that if the sequence $\{a_n\}_{n=1}^{\infty}$ tends to limit L as $n \to \infty$, then for any fixed number $M > 0$, the sequence $\{Ma_n\}_{n=1}^{\infty}$ tends to the limit ML.

Proof: by arbitrary number.

Let Ɛ be an arbitrary number > 0.

We know that for a sequence [$\{a_n\}_{n=1}^{\infty}$] tends to the limit [a] as n -> ∞.

Formally: $(\forall \varepsilon>0)(\exists n \in N)(\forall m \geq n)[|a_m - a|<\varepsilon]$.

Then: If we transform the sequence [$\{a_n\}_{n=1}^{\infty}$] and the limit [a] by any fixed number M > 0, the difference | M*am – M*a| is still < Ɛ.

Formally: $(\forall \varepsilon>0)(\exists n \in N)(\forall m \geq n)[|Ma_m - Ma|<\varepsilon]$.

Therefore: [$\{M^*a_n\}_{n=1}^{\infty}$] tends to the limit [M*a] as n -> ∞.

Conclusion: Hence, for an arbitrary number Ɛ > 0, the sequence [$\{Ma_n\}_{n=1}^{\infty}$] tends to the limit ML as n -> ∞.

QED

Question 9

Claim:

Given an infinite collection $A_n, n = 1, 2, \ldots$ of intervals of the real line, their *intersection* is defined to be

$$\bigcap_{n=1}^{\infty} A_n = \{x \mid (\forall n)(x \in A_n)\}$$

Give an example of a family of intervals $A_n, n = 1, 2, \ldots$, such that $A_{n+1} \subset A_n$ for all n and $\bigcap_{n=1}^{\infty} A_n = \emptyset$. Prove that your example has the stated property.

Proof:

Let $A_n = (0, 1/m)$

If m = 1, then $A_1 = (0, 1)$

Then: $[\bigcap_{n=1}^{\infty} A_n] \subset A_1$.

Therefore, any element of the above intersect should be an element within (0, 1).

However, if $x \in (0, 1)$ then there exists a number m such that $1/m < x$.

Then: $x \notin A_n$, thus $x \notin [\bigcap_{n=1}^{\infty} A_n]$

Conclusion: Hence, $[\bigcap_{n=1}^{\infty} A_n] = \{ \}$.

QED

Question 10

Claim:

Give an example of a family of intervals $A_n, n = 1, 2, \ldots$, such that $A_{n+1} \subset A_n$ for all n and $\bigcap_{n=1}^{\infty} A_n$ consists of a single real number. Prove that your example has the stated property.

Proof:

Let $A_n = (-1/m, 0]$

If $m = 1$, then $A_1 = (-1, 0]$

Then: $[\bigcap_{n=1}^{\infty} A_n] \subset A_1$.

Therefore, any element of the above intersect should be an element within (-1, 0].

However, if $x \in (-1, 0]$ then there exists a number m such that $x < -1/m < x$.

Then: $x \in A_n$, thus $x \in [\bigcap_{n=1}^{\infty} A_n]$

Conclusion: Hence, $[\bigcap_{n=1}^{\infty} A_n] = \{0\}$.

QED