New Solvable Variants of the Goldfish Many-Body Problem

By Francesco Calogero

A technique to manufacture *solvable* variants of the "goldfish" many-body problem is introduced, and several many-body problems yielded by it are identified and discussed, including cases featuring *multiperiodic* or *isochronous* dynamics.

1. Introduction

NOTATION 1. *Above and hereafter N is (unless otherwise indicated) an arbitrary integer, $N \geq 2$, the (generally complex) numbers $z_n \equiv z_n(t)$ are the dependent variables, t ("time") is the independent variable, superimposed dots denote time-differentiations, and indices such as n, m, ℓ run over the integers from* 1 *to N unless otherwise indicated (see, for instance, in* (1a) *the limitation $\ell \neq n$ on the N values of ℓ). Below we often omit to indicate explicitly the t-dependence of various quantities, when this can be done without causing misunderstandings. We indicate with* $\mathbf{i}$ *the imaginary unit (so that $\mathbf{i}^2 = -1$) and with an underlined letter N-vectors; so that, for instance, the N-vector $\underline{z}$ has the N components z_n. We occasionally use the Kronecker symbol, with its standard definition: $\delta_{mn} = 1$ for $m = n$, $\delta_{mn} = 0$ for $m \neq n$. And let us mention the standard convention according to which an empty sum vanishes and an empty product equals unity, that is, $\sum_{j=J}^{K}(f_j) = 0$, $\prod_{j=J}^{K}(f_j) = 1$ if $K < J$.*

*Address for correspondence: Francesco Calogero, Physics Department, University of Rome "La Sapienza", 00185 Rome, Italy; e-mail: francesco.calogero@roma1.infn.it

This paper is dedicated to Mark Ablowitz, with my best wishes on the occasion of his 70th birthdate.

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The original “goldfish” many-body model [1–4] is characterized by the *translation-invariant* equations of motion

$$\ddot{z}_n = \sum_{\ell=1,\,\ell\neq n}^{N} \left(\frac{2\,\dot{z}_n\,\dot{z}_\ell}{z_n - z_\ell} \right). \tag{1a}$$

Clearly this system of N Ordinary Differential Equations (ODEs) can be interpreted as the *Newtonian* equations of motion of N unit-mass point-particles moving in the complex z-plane and interacting pairwise with forces equal to twice the product of the velocities of the two interacting particles divided by their mutual distance.

Remark 1. The Newtonian equations of motions (1a) are Hamiltonian, being for instance produced (as the reader can easily verify) by the following Hamiltonian:

$$H(\underline{\zeta};\underline{z}) = \sum_{n=1}^{N} \left[\exp(\zeta_n) \prod_{\ell=1,\,\ell\neq n}^{N} (z_n - z_\ell)^{-1} \right], \tag{1b}$$

where of course $\underline{\zeta}$, respectively, $\underline{z}$ denote the N-vectors of components ζ_n, respectively, z_n and ζ_n is the canonical momentum conjugated to the canonical variable z_n.

Remark 2. The solution of the initial-values problem for the goldfish N-body model (1a) is provided [1–4] by the N roots $z_n \equiv z_n(t)$ of the following, rather neat, algebraic equation in the variable z (which is actually polynomial of degree N, as seen after multiplication by $\prod_{m=1}^{N}[z - z_m(0)]$):

$$\sum_{\ell=1,\,\ell\neq n}^{N} \left[\frac{\dot{z}_\ell(0)}{z - z_\ell(0)} \right] = \frac{1}{t}. \tag{1c}$$

Remark 3. A simple generalization of the goldfish equations of motion (1a), featuring the arbitrary parameter ω (and reducing to (1a) for $\omega = 0$), reads as follows:

$$\ddot{z}_n = \mathbf{i}\,\omega\,\dot{z}_n + \sum_{\ell=1,\,\ell\neq n}^{N} \left(\frac{2\,\dot{z}_n\,\dot{z}_\ell}{z_n - z_\ell} \right). \tag{2a}$$

These equations of motion are produced by the Hamiltonian

$$H(\underline{\zeta};\underline{z}) = \sum_{n=1}^{N} \left[-\mathbf{i}\,\omega\,z_n + \exp(\zeta_n) \prod_{\ell=1,\,\ell\neq n}^{N} (z_n - z_\ell)^{-1} \right]. \tag{2b}$$

The solution of the corresponding initial-values problem is provided by the N roots $z_n \equiv z_n(t)$ of the following, again rather neat, algebraic (actually polynomial of degree N) equation in the variable z:

$$\sum_{\ell=1,\,\ell\neq n}^{N}\left[\frac{\dot{z}_\ell\,(0)}{z-z_\ell\,(0)}\right]=\frac{\mathbf{i}\,\omega}{\exp\left(\mathbf{i}\,\omega\,t\right)-1}. \tag{2c}$$

Hence—whenever the parameter ω is real and nonvanishing (as we generally assume hereafter)—this model is *isochronous*: all its solutions are *completely periodic*, with period $T=2\pi/\omega$; or possibly, due to an exchange of the particle positions through the motion, with a period which is a (generally small: see [5]) *integer* multiple of T.

Remark 4. By setting $z_n(t)=x_n(t)+\mathbf{i}\,y_n(t)$ with $x_n(t)$ and $y_n(t)$ real and by then introducing the N real two-vectors $\vec{r}_n=(x_n,\ y_n)\equiv\vec{r}_n(t)$ (or equivalently, the three-vectors $\vec{r}_n=(x_n,\ y_n,\ 0)$) the equations of motion (2a) read

$$\ddot{\vec{r}}_n=\omega\,\hat{o}\wedge\dot{\vec{r}}_n$$
$$+\,2\sum_{\ell=1,\,\ell\neq n}^{N}\left[\frac{\dot{\vec{r}}_n\left(\dot{\vec{r}}_\ell\cdot\vec{r}_{n\ell}\right)+\dot{\vec{r}}_\ell\left(\dot{\vec{r}}_n\cdot\vec{r}_{n\ell}\right)-\vec{r}_{n\ell}\left(\dot{\vec{r}}_n\cdot\dot{\vec{r}}_\ell\right)}{r_{nl}^2}\right],$$

$$\vec{r}_{n\ell}\equiv\vec{r}_n-\vec{r}_\ell\,,\quad r_{nl}^2=\vec{r}_{n\ell}\cdot\vec{r}_{n\ell}, \tag{2d}$$

where $\hat{o}$ is the unit three-vector orthogonal to the xy-plane, $\hat{o}=(0,\ 0,\ 1)$, and the symbols "$\cdot$", respectively, "$\wedge$" indicate the standard scalar, respectively, (three-dimensional) vector product, so that $r_{n\ell}^2=(x_n-x_\ell)^2+(y_n-y_\ell)^2$ and $\hat{o}\wedge\vec{r}_n=(-y_n,\ x_n,\ 0)$. Note the *rotation-invariant*—and, of course, *translation-invariant*—character of these *real* Newtonian equations of motion in the xy -plane.

Several *solvable* generalizations of the goldfish model, characterized by Newtonian equations of motion featuring additional forces besides those appearing in the right-hand side of (1a), are known: see, for instance, [2–4].

Remark 5. Above and hereafter we call a many-body model *solvable* if its initial-values problem can be solved—possibly after performing *quadratures*—by *algebraic* operations, such as finding the N zeros of a known polynomial of degree N (of course such an algebraic equation can be *explicitly* solved only for $N\leq 4$).

In this paper, a simple technique is introduced, which allows to identify and investigate additional *solvable* models "of goldfish type"; although not characterized by quite as neat, and quite as neatly solvable, equations of motion as those reported above. The models treated in this paper are presumably *new*, in spite of the simplicity of the technique to identify them (see below); and are perhaps *interesting* (as all solvable models tend to be), in spite of their inferior elegance (for instance, they generally feature many-body forces). To allow the reader to form immediately a personal opinion we report, already in this introductory section, four examples of such *solvable* models (see Subsection 1.1). More general classes of such models, and the technique to identify and solve them, are reported in Section 2. Section 3 indicates possible further developments, yielding still more general and presumably less elegant *solvable* models. Some technical developments are confined to two Appendices, to avoid interruptions in the flow of the presentation.

1.1. A representative selection of solvable *many-body problems of goldfish type*

In this subsection, we report with minimal commentary four examples of *solvable* many-body problems of goldfish type.

EXAMPLE 1. *The solution of the initial-values problem for the three-body system characterized by the following Newtonian equations of motion featuring two-body velocity-dependent forces of goldfish type and in addition three-body velocity-independent forces,*

$$\ddot{z}_n = \frac{2\,\dot{z}_n\,\dot{z}_{n+1}}{z_n - z_{n+1}} + \frac{2\,\dot{z}_n\,\dot{z}_{n+2}}{z_n - z_{n+2}} + \left[(z_n - z_{n+1})\,(z_n - z_{n+2})\right]^{-1} \cdot$$
$$\cdot \left[\omega_1^2\,(z_1 + z_2 + z_3)\,z_n^2 - \omega_2^2\,(z_1\,z_2 + z_2\,z_3 + z_3\,z_1)\,z_n + \omega_3^2\,z_1\,z_2\,z_3\right],$$
$$n = 1, 2, 3 \mathrm{mod}(3), \tag{3a}$$

is provided (see Example 5 with $N = 3$) by the three zeros $z_n(t)$ of the following cubic polynomial in the variable z:

$$\psi\,(z;t) = z^3 + \sum_{m=1}^{3} \left\{\left[c_m\,(0)\,\cos(\omega_m\,t) + \dot{c}_m\,(0)\,\omega_m^{-1}\,\sin(\omega_m\,t)\right] z^{3-m}\right\}, \tag{3b}$$

where

$$c_1\,(0) = -\sum_{n=1}^{3} [z_n\,(0)]\,,\quad c_2\,(0) = \frac{1}{2} \sum_{n,m=1,\,n\neq m}^{3} [z_n\,(0)\,z_m\,(0)]\,,$$

$$c_3\,(0) = -\frac{1}{6} \sum_{n,m,\ell=1;\,n\neq m\neq \ell}^{3} [z_n\,(0)\,z_m\,(0)\,z_\ell\,(0)]\,;$$

$$\dot{c}_1\,(0) = -\sum_{n=1}^{3} [\dot{z}_n\,(0)]\,,\quad \dot{c}_2\,(0) = \sum_{n,m=1,\,n\neq m}^{3} [\dot{z}_n\,(0)\,z_m\,(0)]\,,$$

$$\dot{c}_3\,(0) = -\frac{1}{2} \sum_{n,m,\ell=1;\,n\neq m\neq \ell}^{3} [\dot{z}_n\,(0)\,z_m\,(0)\,z_\ell\,(0)]\,. \tag{3c}$$

Hence, all solutions of this three-body model are multiply periodic if the 3*, a priori arbitrary, parameters* ω_1^2, ω_2^2, ω_3^2 *are all positive; if moreover* $\omega_n = j_n\,\omega$, $n = 1,\ 2,\ 3$, *with* ω *an arbitrary nonvanishing real number and* $j_1,\ j_2,\ j_3$ *arbitrary nonvanishing integers, it is isochronous, all its solutions being then completely periodic with period* $T = 2\pi/\omega$ *(or possibly—due to the exchange of the zeros* $z_n(t)$ *through the motion—the period might be a, generally small, integer multiple of* T*: see [5]).*

EXAMPLE 2.

$$\ddot{z}_n = \sum_{\ell=1,\,\ell\neq n}^{N}\left(\frac{2\,\dot{z}_n\,\dot{z}_\ell}{z_n - z_\ell}\right)$$
$$+\left[\prod_{\ell=1,\,\ell\neq n}^{N}(z_n - z_\ell)\right]^{-1} c_N\,\left[1 + k_N^2 - 2\,(k_N\;c_N\,/\,\mu_N)^2\right], \qquad (4a)$$

with

$$c_N = (-)^N \prod_{n=1}^{N}(z_n)\,. \qquad (4b)$$

Here, k_N *is an arbitrary real parameter in the range* $0 < k_N < 1$, *and* μ_N *is an arbitrary nonvanishing parameter. The solvable character of this model—calling into play the Jacobian elliptic function* $\mathrm{sn}(\tau;\kappa)$*—is demonstrated in Section 2, because this model corresponds to Example 7 with* $\mu_m = \delta_{mN}\,\mu_N$.

EXAMPLE 3.

$$\ddot{z}_n = (2\,\lambda + 1)\;\mathbf{i}\,\omega\,\dot{z}_n - \lambda\;(\lambda + 1)\;\omega^2\,z_n$$
$$+\sum_{\ell=1,\,\ell\neq n}^{N}\left[\frac{2\;(\dot{z}_n - \mathbf{i}\,\lambda\,\omega\,z_n)\;(\dot{z}_\ell - \mathbf{i}\,\lambda\,\omega\,z_\ell)}{z_n - z_\ell}\right]$$
$$-\left[\prod_{\ell=1,\,\ell\neq n}^{N}(z_n - z_\ell)\right]^{-1}\alpha\;(c_r)^p\;(z_n)^{N-r}\,, \qquad (5a)$$

with

$$\lambda = \frac{2}{(p-1)\,r}, \qquad (5b)$$

$$c_r = \frac{(-)^r}{r!}\sum_{n_1,n_2,\ldots,n_r=1;\;n_1\neq n_2\neq\ldots\neq n_r}^{N}\left[\prod_{\ell=1}^{r}(z_{n_\ell})\right]. \qquad (5c)$$

Here, α is an arbitrary (possibly complex) constant, r is an arbitrary integer in the range $1 \leq r \leq N$, p is an arbitrary integer ($p \neq 1$), and ω is an arbitrary real and nonvanishing parameter.
The solutions $z_n(t)$ of this model are the N zeros of a time-dependent polynomial $\psi(z,t)$, of degree N in z, which is periodic in t (as explained in Section 2: see in particular Example 8). Hence this model is isochronous, all its solutions being completely periodic with a fixed period independent of the initial data.

EXAMPLE 4.

$$\ddot{z}_n = (2\,\lambda + 1)\,\mathbf{i}\,\omega\,\dot{z}_n - \lambda\,(\lambda + 1)\,\omega^2\,z_n + \sum_{\ell=1,\,\ell\neq n}^{N} \left[\frac{2\,(\dot{z}_n - \mathbf{i}\,\lambda\,\omega\,z_n)\,(\dot{z}_\ell - \mathbf{i}\,\lambda\,\omega\,z_\ell)}{z_n - z_\ell}\right] - \left[\prod_{\ell=1,\,\ell\neq n}^{N} (z_n - z_\ell)\right]^{-1} \beta\,[\tilde{c}_s]^q\,z_n^{N-s}; \tag{6a}$$

$$\lambda = \frac{2 - q}{(q-1)\,s}\,, \tag{6b}$$

$$\tilde{c}_s = \frac{(-)^s}{(s-1)!} \sum_{n_1,n_2,\ldots,n_s=1;\,n_1\neq n_2\neq\ldots\neq n_s}^{N} \left[\left(\dot{z}_{n_1} - \mathbf{i}\,\lambda\,\omega\,z_{n_1}\right)\prod_{\ell=2}^{s}(z_{n_\ell})\right]. \tag{6c}$$

Here β is an arbitrary (possibly complex) constant, ω is an arbitrary (real, nonvanishing) parameter, s is an arbitrary integer in the range from 1 to N ($1 \leq s \leq N$) and q is an arbitrary integer ($q \neq 1$, $q \neq 2$). Also this model is isochronous, see Example 9.

2. Results

As in previous treatments [2–4], our main tool is the relationship among the N coefficients $c_m(t)$ and the N zeros $z_n(t)$ of a time-dependent monic polynomial $\psi(z;t)$ of degree N in the variable z:

$$\psi\,(z;t) = z^N + \sum_{m=1}^{N}\left[c_m\,(t)\,z^{N-m}\right], \tag{7a}$$

$$\psi\,(z;t) = \prod_{n=1}^{N}\left[z - z_n\,(t)\right]. \tag{7b}$$

The N coefficients c_m are of course given by the following *explicit* expressions in terms of the N zeros z_n:

$$c_1 = -\sum_{n=1}^{N} (z_n)\,, \quad c_2 = \frac{1}{2} \sum_{n,m=1,\, n\neq m}^{N} (z_n\, z_m)\ , \ldots ,$$

$$c_m = \frac{(-)^m}{m!} \sum_{n_1,n_2,\ldots,n_m=1;\, n_1\neq n_2\neq\ldots\neq n_m}^{N} \left[\prod_{\ell=1}^{m} \left(z_{n_\ell}\right) \right], \ldots ,$$

$$c_N = (-)^N \prod_{n=1}^{N} (z_n)\,; \tag{8a}$$

while the N zeros z_n are of course only defined up to permutations and can be *explicitly* expressed in terms of the coefficients c_m only for $N \leq 4$.

Let us also note for future reference that these formulas imply the following expressions of the time derivatives $\dot{c}_m$ of the coefficients c_m:

$$\dot{c}_1 = -\sum_{n=1}^{N} (\dot{z}_n)\ , \quad \dot{c}_2 = \sum_{n,m=1,\, n\neq m}^{N} (\dot{z}_n\, z_m)\,, \ldots ,$$

$$\dot{c}_m = \frac{(-)^m}{(m-1)!} \sum_{n_1,n_2,\ldots,n_m=1;\, n_1\neq n_2\neq\ldots\neq n_m}^{N} \left[\dot{z}_{n_1} \prod_{\ell=2}^{m} \left(z_{n_\ell}\right) \right], \ldots ,$$

$$\dot{c}_N = (-)^N \sum_{n=1}^{N} \left[\dot{z}_n \prod_{\ell=1,\, \ell\neq n}^{N} (z_\ell) \right]. \tag{8b}$$

In previous treatments [2–4], one generally focussed on a *linear* and *autonomous* Partial Differential Equation (PDE) satisfied by the polynomial $\psi(z;t)$, and on the relations it entails for the time evolution of the N coefficients $c_m(t)$ and the N zeros $z_n(t)$. Here we focus instead *directly* on the following relations—proven in Appendix A—among the time-derivatives of the zeros $z_n(t)$ and of the coefficients $c_m(t)$:

$$\dot{z}_n = -\left[\prod_{\ell=1,\, \ell\neq n}^{N} (z_n - z_\ell) \right]^{-1} \sum_{m=1}^{N} \left[\dot{c}_m\, z_n^{N-m} \right], \tag{9a}$$

$$\ddot{z}_n = \sum_{\ell=1,\, \ell\neq n}^{N} \left(\frac{2\, \dot{z}_n\, \dot{z}_\ell}{z_n - z_\ell} \right) - \left[\prod_{\ell=1,\, \ell\neq n}^{N} (z_n - z_\ell) \right]^{-1} \sum_{m=1}^{N} \left(\ddot{c}_m\, z_n^{N-m} \right). \tag{9b}$$

It is now plain how to obtain *solvable* many-body problems of goldfish type from the second of these formulas. Indeed, assume that the coefficients $c_m(t)$ satisfy the following *solvable* set of ODEs:

$$\ddot{c}_m = F_m(c_m) + G_m(\dot{c}_m) . \tag{10}$$

Remark 6. Hereafter we assume that the functions $F_m(c_m)$ and $G_m(\dot{c}_m)$ are appropriately assigned so that these N second-order ODEs, (10), can *all* be *solved by algebraic operations* and, if need be, *quadratures* (see Remark 5). Note that, merely for simplicity, here we restricted attention to *decoupled* ODEs characterizing the time evolution of the N coefficients $c_m(t)$; and in the right-hand side of (10) we assumed a *separate* dependence on c_m and $\dot{c}_m$.

Indeed the insertion of (10) in (9b) yields the system of N ODEs

$$\ddot{z}_n = \sum_{\ell=1,\, \ell\neq n}^{N} \left(\frac{2\, \dot{z}_n\, \dot{z}_\ell}{z_n - z_\ell} \right) - \left[\prod_{\ell=1,\, \ell\neq n}^{N} (z_n - z_\ell) \right]^{-1} \sum_{m=1}^{N} \left\{ \left[F_m(c_m) + G_m(\dot{c}_m) \right] z_n^{N-m} \right\} ; \tag{11}$$

and by inserting in these equations the expressions of the coefficients c_m in terms of the N zeros z_n, see (8a)—and likewise the expressions of their time derivatives $\dot{c}_m$ in terms of the N zeros z_n and their time derivatives $\dot{z}_n$, see (8b)—one clearly obtains the Newtonian equations of motion of a many-body model of goldfish type (generally with many-body velocity-dependent forces). Which is *solvable* because the assumed *solvability* of the N ODEs (10) implies that the time-dependent quantities $c_m(t)$ can be considered known, and the N time-dependent quantities $z_n(t)$ are then the N zeros of the *known* time-dependent polynomial (of degree N in the variable z) $\psi(z; t)$, see (7a).

2.1. Examples

In this subsection, we report various representative examples of *solvable* many-body problems of goldfish type.

EXAMPLE 5. *Let, in* (10)*,*

$$F_m(c_m) = -\omega_m^2\, c_m , \quad G_m(\dot{c}_m) = 0, \tag{12a}$$

with the positive quantities ω_m^2 *arbitrarily assigned. Then, the Newtonian equations of motion* (11) *read*

$$\ddot{z}_n = \sum_{\ell=1,\,\ell\neq n}^{N} \left(\frac{2\,\dot{z}_n\,\dot{z}_\ell}{z_n - z_\ell}\right) + \left[\prod_{\ell=1,\,\ell\neq n}^{N} (z_n - z_\ell)\right]^{-1} \sum_{m=1}^{N} \left[\omega_m^2\, c_m\, z_n^{N-m}\right], \tag{12b}$$

of course with the N coefficients c_m *expressed via the N coordinates* z_n *by* (8a).

The solution of this Newtonian N-body model is then provided by the N zeros $z_n(t)$ *of the following polynomial of degree N in z:*

$$\psi\,(z;t) = z^N + \sum_{m=1}^{N} \left\{\left[c_m\,(0)\,\cos\left(\omega_m\,t\right) + \dot{c}_m\,(0)\;\omega_m^{-1}\;\sin\left(\omega_m\,t\right)\right]\,z^{N-m}\right\}, \tag{12c}$$

where the quantities $c_m(0)$ *are expressed in terms of the initial values* $z_n(0)$ *of the coordinates* $z_n(t)$ *by* (8a) *and the quantities* $\dot{c}_m(0)$ *are expressed in terms of the initial values* $z_n(0)$ *of the coordinates* $z_n(t)$ *and the initial values* $\dot{z}_n(0)$ *of the velocities* $\dot{z}_n(t)$ *by* (8b)*. Note that this implies that all solutions of the N -body problem characterized by these Newtonian equations of motion,* (12b)*, are multiply periodic; the model is actually isochronous if* $\omega_m^2 = j_m^2\,\omega^2$ *with* ω^2 *an arbitrary positive parameter and the N numbers* j_m *arbitrary nonvanishing integers.*

EXAMPLE 6. *Let, in* (10)*,*

$$F_m\,(c_m) = (\mathbf{i}\,\omega_m - \gamma_m)\;c_m\;,\quad G_m\,(\dot{c}_m) = \mathbf{i}\,\omega_m\,\gamma_m\,\dot{c}_m, \tag{13a}$$

with the $2N$ *parameters* ω_m *and* γ_m *a priori arbitrarily assigned (but see below). Then the Newtonian equations of motion* (11) *read*

$$\ddot{z}_n = \sum_{\ell=1,\,\ell\neq n}^{N} \left(\frac{2\,\dot{z}_n\,\dot{z}_\ell}{z_n - z_\ell}\right) - \left[\prod_{\ell=1,\,\ell\neq n}^{N} (z_n - z_\ell)\right]^{-1} \cdot \sum_{m=1}^{N} \left\{\left[(\mathbf{i}\,\omega_m - \gamma_m)\;c_m + \mathbf{i}\,\omega_m\,\gamma_m\,\dot{c}_m\right]\,z_n^{N-m}\right\}, \tag{13b}$$

of course with the N coefficients c_m *expressed via the N coordinates* z_n *by* (8a)*, and their N time-derivatives* $\dot{c}_m$ *expressed via the N coordinates* z_n *and their N time derivatives* $\dot{z}_n$ *via* (8b).

The solution of this Newtonian N-body model is then provided by the N zeros $z_n(t)$ of the polynomial of degree N in z (7a) *with*

$$c_m(t) = c_m(0)\exp(-\gamma_m t) + \left[\frac{\gamma_m c_m(0) + \dot{c}_m(0)}{\gamma_m + \mathbf{i}\,\omega_m}\right]\cdot$$
$$\cdot\left[\cos(\omega_m t) + \mathbf{i}\,\sin(\omega_m t) - \exp(-\gamma_m t)\right], \qquad (13c)$$

where the quantities $c_m(0)$ are expressed in terms of the initial values $z_n(0)$ of the coordinates $z_n(t)$ by (8a) *and the quantities $\dot{c}_m(0)$ are expressed in terms of the initial values $z_n(0)$ of the coordinates $z_n(t)$ and the initial values $\dot{z}_n(0)$ of the velocities $\dot{z}_n(t)$ by* (8b).

Note that this implies that all solutions of the N-body problem characterized by these Newtonian equations of motion, (13b)*, are multiply periodic, respectively, asymptotically multiply periodic, if the $2N$ parameters ω_m and γ_m are all real and satisfy the requirements $\omega_m \neq 0$ and $\gamma_m = 0$, respectively, $\gamma_m > 0$; while the model is isochronous, respectively, asymptotically isochronous, if the N parameters ω_m satisfy the Diophantine restriction $\omega_m = j_m\,\omega$ with the numbers j_m all nonvanishing integers, ω an arbitrary real constant, and $\gamma_m = 0$, respectively, $\gamma_m > 0$ (for a precise definition of asymptotic isochrony, and other examples of models featuring this property, see [6]).*

EXAMPLE 7. *Let, in* (10)*,*

$$F_m(c_m) = -\delta(\mu_m)\,c_m\left[1 + k_m^2 - 2\,(k_m\,c_m\,/\,\mu_m)^2\right], \quad G_m(\dot{c}_m) = 0, \qquad (14a)$$

where $\delta(\mu)$ vanishes if μ vanishes and is unity otherwise ($\delta(\mu) = 0$ if $\mu = 0$, $\delta(\mu) = 1$ if $\mu \neq 0$), the N parameters k_m are positive and less than unity ($0 < k_m < 1$), and the N parameters μ_m are arbitrarily assigned (if some of them are assigned a vanishing value, then the corresponding functions F_m should also be identically set to zero; this being guaranteed by the factor $\delta(\mu_m)$, see (14a)*). Then the solution of the initial-values problem for the ODEs* (10) *read simply $c_m(t) = c_m(0) + \dot{c}_m(0)\,t$ whenever the corresponding μ_m vanishes, $\mu_m = 0$; while when μ_m does not vanish, $\mu_m \neq 0$, as explained in Appendix B,*

$$c_m(t) = a_m\,b_m\,\mathrm{sn}(\mathrm{b_m}\,[\mathrm{t} - \mathrm{t_{0m}}]\,,\,\mathrm{a_m\,k_m})\,, \quad \mathrm{b_m} = \left(\frac{1 + \mathrm{k_m^2}}{1 + \mathrm{a_m^2\,k_m^2}}\right)^{1/2}, \qquad (14b)$$

where the function $\mathrm{sn}(\tau;\kappa)$ is the well-known Jacobian elliptic function (see, for instance, [7]) while the quantities a_m and t_{0m} can be evaluated in terms of the initial data $c_m(0)$ and $\dot{c}_m(0)$. With this assignment, the equations of motion

(11) *read*

$$\ddot{z}_n = \sum_{\ell=1,\, \ell\neq n}^{N} \left(\frac{2\,\dot{z}_n\,\dot{z}_\ell}{z_n - z_\ell}\right) - \left[\prod_{\ell=1,\, \ell\neq n}^{N} (z_n - z_\ell)\right]^{-1} \cdot$$

$$\cdot \sum_{m=1}^{N} \left\{\delta\left(\mu_m\right)\left[-\left(1+k_m^2\right)\, c_m + 2\, k_m^2\, \mu_m^{-2}\, c_m^3\right]\, z_n^{N-m}\right\}. \qquad (14c)$$

These are the Newtonian equations of motion of an N-body model of goldfish type once the quantities c_m are expressed in terms of the particle coordinates z_n via the formulas (8a)*. And—if no one of the parameters μ_m vanishes—clearly the periodicity of the Jacobian elliptic function* $\mathrm{sn}(\tau;\,\kappa)$ *as function of τ (for* $0<\kappa<1$*, see, for instance, [7]) guarantees that there is a set of initial data yielding multiply periodic solutions: clearly a sufficient condition is that the corresponding values of all the constants a_m be real and satisfy the inequalities* $a_m^2 < 1$ *(the evaluation of these constants a_m in terms of the initial data is discussed in Appendix B).*

EXAMPLE 8. *Let, in* (10)*,*

$$F_m\left(c_m\right) = \delta_{rm}\,\alpha\,\left(c_r\right)^p, \quad G_m\left(\dot{c}_m\right) = 0, \qquad (15a)$$

with r an arbitrary integer (of course in the range from 1 *to N), p an arbitrary integer* ($p \neq 1$)*, and α an arbitrary (possibly complex) parameter. Then the N ODEs* (11) *read*

$$\ddot{z}_n = \sum_{\ell=1,\, \ell\neq n}^{N} \left(\frac{2\,\dot{z}_n\,\dot{z}_\ell}{z_n - z_\ell}\right) - \left[\prod_{\ell=1,\, \ell\neq n}^{N} (z_n - z_\ell)\right]^{-1} \alpha\,\left(c_r\right)^p\,\left(z_n\right)^{N-r}, \qquad (15b)$$

of course with c_r expressed via the N coordinates z_n by (8a)*.*

Clearly, the solution of this Newtonian N-body system of goldfish type is provided by the N zeros $z_n(t)$ of the polynomial of degree N in z (7a) *with $c_r(t)$ solution of the ODE $\ddot{c}_r = \alpha\,(c_r)^p$, which is clearly solvable by quadratures, and, for $m \neq r$, $c_m(t) = c_m(0) + \dot{c}_m(0)\,t$.*

A, perhaps more interesting, isochronous case obtains from this via a well-known trick, see, for instance, [4]—where it is described in sufficient detail to justify our reporting its implications here and below without providing a detailed derivation; it is, by the way, the same trick allowing the generalization from the model (1a) *to the model* (2a)*. The Newtonian equations of motion of this new model read as follows:*

$$\ddot{z}_n = \mathbf{i}\,(2\,\lambda + 1)\,\omega\,\dot{z}_n - \lambda\,(\lambda + 1)\,\omega^2\,z_n$$

$$+\sum_{\ell=1,\,\ell\neq n}^{N}\left[\frac{2\,(\dot{z}_n-\mathbf{i}\,\lambda\,\omega\,z_n)\,(\dot{z}_\ell-\mathbf{i}\,\lambda\,\omega\,z_\ell)}{z_n-z_\ell}\right]$$

$$-\left[\prod_{\ell=1,\,\ell\neq n}^{N}(z_n-z_\ell)\right]^{-1}\alpha\,(c_r)^p\,(z_n)^{N-r}\,, \qquad (16a)$$

with

$$\lambda=\frac{2}{(p-1)\,r}, \qquad (16b)$$

and ω *an arbitrary real parameter* ($\omega\neq 0$; *for* $\omega=0$ *this generalized isochronous model (8) coincides with the, nonisochronous, original model* (15b)*). Here, of course the coefficient* c_r *is expressed via the* N *coordinates* z_n *as follows (see* (8a)*):*

$$c_r=\frac{(-)^r}{r!}\sum_{n_1,n_2,\ldots,n_r=1;\,n_1\neq n_2\neq\ldots\neq n_r}^{N}\left[\prod_{\ell=1}^{r}\left(z_{n_\ell}\right)\right]. \qquad (16c)$$

These equations of motion obtain from (15b) *by first rewriting them via the purely notational change of (dependent and independent) variables* $z_n(t)\Rightarrow\zeta_n(\tau)$ *(and correspondingly* $c_r(t)\Rightarrow\gamma_r(\tau)$*) so that* (15b) *read*

$$\zeta_n''=\sum_{\ell=1,\,\ell\neq n}^{N}\left(\frac{2\,\zeta_n'\,\zeta_\ell'}{\zeta_n-\zeta_\ell}\right)-\left[\prod_{\ell=1,\,\ell\neq n}^{N}(\zeta_n-\zeta_\ell)\right]^{-1}\alpha\,(\gamma_r)^p\,(\zeta_n)^{N-r}\,, \qquad (17)$$

where (above and hereafter) appended primes denote of course differentiations with respect to τ. *Next, we introduce new* z_n *coordinates via the position*

$$z_n\,(t)=\exp(\mathbf{i}\,\lambda\,\omega\,t)\;\zeta_n\,(\tau)\;,\quad \tau=\frac{\exp(\mathbf{i}\,\omega\,t)-1}{\mathbf{i}\,\omega}\;,\quad \lambda=\frac{2}{(p-1)\,r} \qquad (18a)$$

implying

$$\dot{z}_n\,(t)-\mathbf{i}\,\lambda\,\omega\,z_n\,(t)=\exp\left[\mathbf{i}\,(\lambda+1)\,\omega\,t\right]\,\zeta_n'\,(\tau)\,, \qquad (18b)$$

$$\ddot{z}_n\,(t)-\mathbf{i}\,(2\,\lambda+1)\,\omega\,\dot{z}_n\,(t)+\lambda\,(\lambda+1)\,\omega^2\,z_n\,(t)=\exp\left[\mathbf{i}\,(\lambda+2)\,\omega\,t\right]\,\zeta_n''\,(\tau)\,. \qquad (18c)$$

It is then a matter of trivial algebra to verify that (17) *yield* (16a). *Note that the autonomous (i.e., time-independent) character of these equations of motion,* (16a)*, is a consequence of the definition* (16b) *(or, equivalently,* (18a)*) of* λ. *And the isochronous character of the model* (16a) *is of course implied by* (18a) *(see, if need be, [4] and [5]).*

EXAMPLE 9. *Let, in* (10)*,*

$$F_m(c_m) = 0, \quad G_m(\dot{c}_m) = \delta_{sm}\, \beta\, (\dot{c}_s)^q, \tag{19a}$$

with s an arbitrary integer (of course in the range from 1 *to N), q an arbitrary integer (*$q \neq 1$*), and β an arbitrary (possibly complex) parameter. Then the N ODEs* (11) *read*

$$\ddot{z}_n = \sum_{\ell=1,\, \ell\neq n}^{N} \left(\frac{2\, \dot{z}_n\, \dot{z}_\ell}{z_n - z_\ell} \right) - \left[\prod_{\ell=1,\, \ell\neq n}^{N} (z_n - z_\ell) \right]^{-1} \beta\, (\dot{c}_s)^q\, z_n^{N-s}, \tag{19b}$$

of course with $\dot{c}_s$ *expressed via the N coordinates* z_n *and their N time-derivatives* $\dot{z}_n$ *via* (8b)*.*

Clearly the solution of this Newtonian N-body system of goldfish type is provided by the N zeros $z_n(t)$ *of the polynomial of degree N in z* (7a) *with* $c_m(t) = c_m(0) + \dot{c}_m(0)\, t$ *if* $m \neq s$ *and* $c_s(t)$ *solution of the ODE* $\ddot{c}_s = \beta\, (\dot{c}_s)^q$*, the initial-values problem of which is explicitly solvable:*

$$c_s(t) = c_s(0) + \left(\frac{1-q}{2-q} \right) C^{-1} \left[(1 + C\, t)^{(2-q)/(1-q)} - 1 \right] \dot{c}_s(0),$$
$$C = \beta\, [\dot{c}_s(0)]^{q-1}. \tag{19c}$$

An isochronous variant of this model can be obtained via the same trick used in Example 4. Given the similarity of the technique, we do not feel the need to detail here the derivation of this model, the equations of motion of which have been already reported above, see (4). We only indicate that the model and its solution obtain via the following change of variables, analogous to (18a) *except for a different definition of the parameter λ (here and below we assume of course* $q \neq 1$, $q \neq 2$*):*

$$z_n(t) = \exp(\mathbf{i}\, \lambda\, \omega\, t)\, \zeta_n(\tau), \quad \tau = \frac{\exp(\mathbf{i}\, \omega\, t) - 1}{\mathbf{i}\, \omega}, \quad \lambda = \frac{2-q}{(q-1)\, s}. \tag{20}$$

This change of dependent and independent variables must of course be applied to the model (19b) *and the solution* (19c) *after the purely notational change of variables* $z_n(t) \Rightarrow \zeta_n(\tau)$ *(and correspondingly* $c_s(t) \Rightarrow \gamma_s(\tau)$, $\dot{c}_s(t) \Rightarrow \gamma_s'(\tau)$*) has been performed. The alert reader will have no difficulty to figure out the corresponding modifications of the initial data. Note that in this case the solution is reduced to finding the N zeros of an explicitly known time-dependent polynomial of degree N in z, the time evolution of which is clearly periodic with period* $T = 2\pi s\, |(q-1)/\omega|$ *(or smaller, if* $|2-q| = js$ *with j an integer larger than unity,* $j > 1$*).*

3. Outlook

Whoever read with any attention the previous sections of this paper must have realized that all the examples presented above have been selected with the minimal goal to illustrate the kind of *solvable* many-body problems characterized by Newtonian equations of motion of goldfish type that can be manufactured via the technique introduced in this paper: a *direct* approach which is perhaps too simple to justify the claim that it is *quite novel*. Purpose and scope of this last section is to indicate tersely seven points at which the above developments could be performed in more general contexts than those employed in the preceding sections, thereby yielding more general instances of *solvable* dynamical systems.

(i) While the main idea of this approach has been to infer from *solvable* time evolutions of the N coefficients $c_m(t)$ of a polynomial $\psi(z, t)$ of degree N in z the corresponding time evolutions of the N zeros $z_n(t)$ of $\psi(z, t)$ (see (7)), and then interpreting the N quantities $z_n(t)$ as the coordinates of N particles moving in the complex z-plane, it might of course be possible to consider other nonlinear relations from N coefficients $c_m(t)$ and N coordinates $z_n(t)$: for instance by considering N -dimensional functional spaces spanned by other basic functions than polynomials of the variable z.

(ii) Models based on systems of *first-order* ODEs might also have been considered to begin with (see (9a)) rather than systems of *second-order* ODEs (see (9b) and (11)); possibly passing subsequently to *second-order* ODEs of Newtonian type for the *real* quantities $x_n(t)$ or $y_n(t)$ such that $z_n(t) = x_n(t) + \mathbf{i}\, y_n(t)$. In this connection, let us also mention—with thanks to a Referee—that the well-known *first-order* equations of motion characterizing the dynamics of point vortices are somewhat analogous to—although actually different from—those that are easily obtainable from (9a), and feature the *same equilibria* , which are therefore related to *zeros* of polynomials, see, for instance, [8, 9] and references therein.

(iii) Consideration could be focused on equations describing *real* motions in the xy-plane rather than *complex* motions in the complex z-plane (with $z = x + \mathbf{i}\, y$; see *Remark 4*); including the identification of cases in which such *real* motions performed by the real two-vectors $\vec{r}_n = (x_n, y_n) \equiv \vec{r}_n(t)$ are *rotation-invariant* in the xy-plane, corresponding to system of ODEs satisfied by the *complex* variables $z_n(t) \equiv x_n(t) + \mathbf{i}\, y_n(t)$ which are themselves *invariant* under the phase transformation $z_n(t) \Rightarrow z_n(t)$ $\exp(\mathbf{i}\,\theta)$ with θ an arbitrary *time-independent real* parameter.

(iv) Other special functions could be used instead of the particular Jacobian elliptic function used in Example 7.

(v) The restriction to decoupled equations could be by-passed (see (10) and *Remark 6*), considering instead *any* set of N *solvable* ODEs satisfied by the N coefficients $c_m(t)$. Opening thereby the way to the following observation.

(vi) Because the basic idea of the approach employed in this paper is to transform *solvable* evolution equations satisfied by N quantities (which we called $c_m(t)$) into *solvable* evolution equation satisfied by N quantities (which we called $z_n(t)$) nonlinearly related to each other in a *solvable* manner, this technique might also be *iterated* an arbitrary number of times—that is, applied sequentially, identifying the variables $c_m(t)$ for the next cycle with the variables $z_n(t)$ produced by the previous cycle. This approach—which could be implemented with different specific techniques at every cycle of the iteration—will yield many more *solvable* systems of N ODEs, which are however likely to be less and less neat hence less and less interesting (except possibly in purely mathematical contexts).

(vii) Finally, in the case of Newtonian equations of motions, the question is always open and interesting of their possibly allowing a Hamiltonian formulation, and in such cases of their treatment in a *quantal* (rather than *classical*) context.

Appendix A: Relations among the Time Derivatives of the Zeros and the Coefficients of a Time-Dependent Polynomial

The starting point to prove the relation (9a) are the two relations

$$\psi_t\,(z;t) = \sum_{m=1}^{N} \left[\dot{c}_m\, z^{N-m}\right], \tag{A.1a}$$

$$\psi_t\,(z;t) = -\sum_{m=1}^{N} \left[\dot{z}_m \prod_{\ell=1,\,\ell\neq m}^{N} (z - z_\ell)\right], \tag{A.1b}$$

which clearly obtain by time-differentiation of (7a), respectively, of (7b). They imply the relation

$$\sum_{m=1}^{N} \left[\dot{z}_m \prod_{\ell=1,\,\ell\neq m}^{N} (z - z_\ell)\right] = -\sum_{m=1}^{N} \left[\dot{c}_m\, z^{N-m}\right], \tag{A.1c}$$

and it is plain that, for $z = z_n$, this formula yields (9a).
Likewise, an additional time-differentiation of (A.1a) yields

$$\psi_{tt}\,(z;t) = \sum_{m=1}^{N} \left(\ddot{c}_m\, z^{N-m}\right), \tag{A.2a}$$

while an additional time-differentiation of (A.1b) yields

$$\psi_{tt}\,(z;t) = \sum_{m=1}^{N} \left\{\ddot{z}_m \left[\prod_{\ell=1,\,\ell\neq m}^{N} (z - z_\ell)\right]\right\}$$

$$- \sum_{\ell_1,\ell_2=1,\,\ell_1\neq\ell_2}^{N} \left\{ \dot{z}_{\ell_1}\,\dot{z}_{\ell_2} \left[\prod_{\ell'=1,\,\ell'\neq\ell_1,\ell_2}^{N} (z - z_{\ell'}) \right] \right\}$$

$$= - \sum_{m=1}^{N} \left(\ddot{c}_m\, z^{N-m} \right), \qquad \text{(A.2b)}$$

where the second equality is implied by (A.2a). It is then again plain that, for $z = z_n$, one gets (9b).

Appendix B: Solution of the Initial-Value Problem for the Ordinary Differential Equation (ODE) $\ddot{c}(t) = -(1 + k^2)\, c(t) + 2\, k^2\, [c(t)]^3$

It is well known that a solution of the ODE

$$\ddot{c} = -(1 + k^2)\, c + 2\, k^2\, c^3 \qquad \text{(B.1a)}$$

is provided by the Jacobian elliptic function sn(t; k) (see, for instance, [7]); but this is *not* the *general* solution of this ODE, which we could not find in the literature, but can be easily obtained, reading

$$c\,(t) = a\, b\, \mathrm{sn}\,(b\, [t - t_0]\,,\, a\, k)\,, \quad b^2 = \frac{1 + k^2}{1 + a^2\, k^2}, \qquad \text{(B.1b)}$$

with a and t_0 two *arbitrary* (*a priori* complex) parameters. Clearly this becomes the solution of the initial-values problem of the ODE (B.1a) provided these two parameters a and t_0 are fixed by the two conditions

$$c\,(0) = a\, b\, \mathrm{sn}\,(-b\, t_0,\, a\, k)\,, \quad b^2 = \frac{1 + k^2}{1 + a^2\, k^2}, \qquad \text{(B.2a)}$$

$$\dot{c}\,(0) = \left[\frac{a\, \left(1 + k^2\right)}{1 + a^2\, k^2} \right] \mathrm{sn}'\,(-b\, t_0,\, a\, k)\,, \qquad \text{(B.2b)}$$

where (see, for instance, [7])

$$[\mathrm{sn}'\,(\tau,\, \kappa)]^2 = \left\{1 - [\mathrm{sn}\,(\tau,\, \kappa)]^2\right\} \left\{1 - k^2\, [\mathrm{sn}\,(\tau,\, \kappa)]^2\right\} \qquad \text{(B.2c)}$$

so that

$$[\dot{c}\,(0)]^2 = \left\{a^2\, b^2 - [c\,(0)]^2\right\} \left\{b^2 - k^2\, [c\,(0)]^2\right\}. \qquad \text{(B.2d)}$$

From these formulas, (B.2a) and (B.2d), it is possible to obtain explicit expressions of a^2 and t_0 in terms of $c(0)$, $\dot{c}(0)$, and k, but they are not very illuminating hence we forsake their display.

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Physics Department, University of Rome "La Sapienza";
Istituto Nazionale di Fisica Nucleare, Sezione di Roma

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