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Problem 2

Our ODE is given by the following:

$$\begin{cases} y'' &= \sin(y') \\ y(0) &= 1 \\ y(1) &= -1 \end{cases}$$

We can rewrite this ODE into a system of ODEs by introducing a new variable $v(t)$ as follows:

$$\begin{cases} v' &= \sin(y') \\ y' &= v \\ y(0) &= 1 \\ y(1) &= -1 \end{cases}$$

We now know from finite differences that:

$$f''(x) = \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} + O(h^2) \tag{1}$$

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2) \tag{2}$$

We apply (1) to $v'(t) = y''(t)$ to obtain:

$$y''(t) = \frac{y(t+h) + y(t-h) - 2y(t)}{h^2} + O(h^2) \tag{3}$$

We then apply (2) to $v(t) = y'(t)$ to obtain:

$$y'(t) = \frac{y(x+h) - y(x-h)}{2h} + O(h^2)$$

From here, we notice that $y''(t) = v'(t) = \sin(v) = \sin(y')$. We can therefore write:

$$y''(t) = \sin(y'(t)) = \sin\left(\frac{y(x+h) - y(x-h)}{2h} + O(h^2)\right) \tag{4}$$

We now equate (3) and (4) to obtain the following:

$$y''(t) = \frac{y(t+h) + y(t-h) - 2y(t)}{h^2} + O_1(h^2) = \sin\left(\frac{y(x+h) - y(x-h)}{2h} + O_2(h^2)\right)$$

Rearranging we obtain:

$$\frac{y(t+h) + y(t-h) - 2y(t)}{h^2} - \sin\left(\frac{y(t+h) - y(t-h)}{2h} + O_2(h^2)\right) = -O_1(h^2)$$

We now examine the term $O_1(h^2)$ on the right hand side of the above equation. This term came from the finite difference expression for y''. This means that $O_1(h^2) = \frac{h^2}{12}y^{(4)}(c)$ for some c between 0 and 1. This means that we have:

$$\left|\frac{y(t+h) + y(t-h) - 2y(t)}{h^2} - \sin\left(\frac{y(t+h) - y(t-h)}{2h} + O_2(h^2)\right)\right| \leq |\frac{h^2}{12}y^{(4)}(c)|$$

We now examine the following:

$$\begin{aligned} &y''(t) = \sin(y'(t)) \\ &\implies y'''(t) = \cos(y'(t))y''(t) = \cos(y'(t))\sin(y'(t)) = \frac{1}{2}\sin(2y'(t)) \\ &\implies y^{(4)}(t) = \frac{1}{2}(\cos(2y'(t))2y''(t)) = \cos(2y'(t))\sin(y'(t)) \end{aligned}$$

From here we can observe that for any value of t, we have that:

$$|y^{(4)}(t)| = |\cos(2y'(t))\sin(y'(t))| \leq 1$$

Thus we have that $|y^{(4)}(c)| \leq 1$, implying that:

$$\left|\frac{y(t+h)+y(t-h)-2y(t)}{h^2} - \sin\left(\frac{y(t+h)-y(t-h)}{2h} + O_2(h^2)\right)\right| \leq \frac{h^2}{12}$$

Let $\frac{y(t+h)-y(t-h)}{2h} = X$. We now rewrite the above expression as follows:

$$\left|\frac{y(t+h)+y(t-h)-2y(t)}{h^2} - \sin(X)\right| \leq \left|\frac{h^2}{12} + \sin(X + O_2(h^2)) - \sin(X)\right| \tag{5}$$

Now we need to study the function

$$g(x) = |\sin(x+h) - \sin(x)|$$

where $h < 1$. Empirically it was observed that $g(x) < h$. We can thus rewrite (5):

$$\left|\frac{y(t+h)+y(t-h)-2y(t)}{h^2} - \sin\left(\frac{y(t+h)-y(t-h)}{2h}\right)\right| \leq \left|\frac{h^2}{12} + h^2\right| = \left|\frac{13h^2}{12}\right|$$

We can rewrite this inequality in discrete form. Let $w_i = f(ih)$. We have then:

$$\left|\frac{w_{i+1}+w_{i-1}-2w_i}{h^2} - \sin\left(\frac{w_{i+1}-w_{i-1}}{2h}\right)\right| \leq \left|\frac{13h^2}{12}\right| \tag{6}$$

or less precisely,

$$\left|\frac{w_{i+1}+w_{i-1}-2w_i}{h^2} - \sin\left(\frac{w_{i+1}-w_{i-1}}{2h}\right)\right| = O(h^2) \tag{7}$$

We can now check whether the y coordinates we generate by the overshooting are correct by first confirming that (7) is satisfied. Once this is confirmed, we can look at whether (6) is satisfied. This is done in the python file by graphing the results. It is seen that this both (6) and (7) are satisfied in the code.