Nonlinear differential algorithm to compute all the zeros of a generic polynomial

Titas Geryba

The Background

Def. A polynomial is a mathematical expression involving a sum of powers in one or more variables multiplied by coefficients.

$$P_N(z) = c_N z^N + c_{N-1} z^{N-1} + c_{N-2} z^{N-2} + \ldots + c_1 z + c_0$$

Polynomial

We can factor a polynomial and rewrite it in the form:

$$P_N(z) = c_N (z - x_1)(z - x_2) \ldots (z - x_{N-1})(z - x_N)$$

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$$P_N(z) = c_N (z - x_1)(z - x_2) \ldots (z - x_{N-1})(z - x_N)$$

$$P_N(z) = 0$$

The Background

Polynomials of degree 2

$$ax^2 + bx + c = 0.$$

have the following general solution Polynomial

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The Background

Polynomials of degree 3

$$ax^3 + bx^2 + cx + d = 0.$$

have the following general solution Polynomial

$$\begin{aligned} x \quad = \quad & \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} \\ + \quad & \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} \quad - \quad \frac{b}{3a}. \end{aligned}$$

The Background

Polynomials of degree 4

$$ax^4 + bx^3 + cx^2 + dx + e = 0.$$

have the following general solution: Polynomial

$$\cdots$$

The Background

Abel–Ruffini theorem

there is no algebraic solution to the general polynomial equations of degree five or higher with arbitrary coefficients.

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of degree five or higher with arbitrary coefficients.

Numerics!

The Algorithm

We will consider Monic polynomials as follows:

$$P_N\left(z;\vec{c},\underline{x}\right) = z^N + \sum_{m=1}^{N}\left(c_m\, z^{N-m}\right) = \prod_{n=1}^{N}\left(z - x_n\right)$$

The Algorithm

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We then introduce that coefficients and zeros can change over a variable t :

$$p_N\left(z;\vec{\gamma}\left(t\right),\underline{y}\left(t\right)\right) = z^N + \sum_{m=1}^{N}\left[\gamma_m\left(t\right)\, z^{N-m}\right] = \prod_{n=1}^{N}\left[z - y_n\left(t\right)\right]$$

The Algorithm

$$\sum_{m=1}^{N}\left[\gamma_m(t)\ z^{N-m}\right]=\prod_{n=1}^{N}\left[z-y_n(t)\right]$$

Abel–Ruffini theorem

$$\dot{y}_n(t)=-g(t)\ \left\{\prod_{\ell=1,\ \ell\neq n}^{N}\left[y_n(t)-y_\ell(t)\right]^{-1}\right\}\ \sum_{m=1}^{N}\left\{\left[c_m-\gamma_m(0)\right]\ \left[y_n(t)\right]^{N-m}\right\}$$

$$g(t)=\frac{\dot{f}(t)}{f(T)-f(0)}\ \ \text{implying}\int_0^T dt\ g(t)=1\,.$$

The Algorithm

$$\dot{y}_n(t) = -g(t) \left\{ \prod_{\ell=1,\ \ell\neq n}^{N} [y_n(t) - y_\ell(t)]^{-1} \right\} \sum_{m=1}^{N} \left\{ [c_m - \gamma_m(0)] \, [y_n(t)]^{N-m} \right\}$$

$$g(t) = \frac{\dot{f}(t)}{f(T) - f(0)} \quad \text{implying} \int_0^T dt\, g(t) = 1 \,.$$

$$F(t) = \begin{cases} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \vdots \\ \dot{y}_n(t) \end{cases}$$

The Algorithm

$$\dot{y}_n(t) = -g(t) \left\{ \prod_{\ell=1,\ \ell\neq n}^{N} \left[y_n(t) - y_\ell(t) \right]^{-1} \right\} \sum_{m=1}^{N} \left\{ \left[c_m - \gamma_m(0) \right] \left[y_n(t) \right]^{N-m} \right\}$$

The Algorithm

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$y_n(t)$ are our initial values for the time zeros.

The Algorithm

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Abel–Ruffini theorem

$y_n(t)$ are our initial values for the time zeros.

$$\gamma_m(0) = (-1)^m \sum_{n_1 > n_2 > \cdots > n_m = 1}^{N} \left[y_{n_1}(0)\ y_{n_2}(0) \cdots y_{n_m}(0) \right]$$

The Algorithm

$$\dot{y}_n(t) = -g(t) \left\{ \prod_{\ell=1,\ \ell\neq n}^{N} [y_n(t) - y_\ell(t)]^{-1} \right\} \sum_{m=1}^{N} \left\{ [c_m - \gamma_m(0)]\ [y_n(t)]^{N-m} \right\}$$

$$g(t) = \frac{\dot{f}(t)}{f(T) - f(0)} \quad \text{implying} \int_0^T dt\, g(t) = 1\,.$$

$$F(t) = \begin{cases} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \vdots \\ \dot{y}_n(t) \end{cases}$$

The Algorithm

$$g(t) = \frac{\dot{f}(t)}{f(T) - f(0)} \quad \text{implying} \int_0^T dt\, g(t) = 1$$

$$f(t) = t \Rightarrow g(t) = \frac{1}{f(1) - f(0)} = 1$$

$$T = 1 \Rightarrow \int_0^T dt\, g(t) = 1$$

The Algorithm

1. Generate the initial guesses

2. Calculate value

$$\gamma_m(0) = (-1)^m \sum_{n_1 > n_2 > \cdots > n_m = 1}^{N} \left[y_{n_1}(0)\ y_{n_2}(0) \cdots y_{n_m}(0) \right]$$

3. Integrate the system of equations numerically

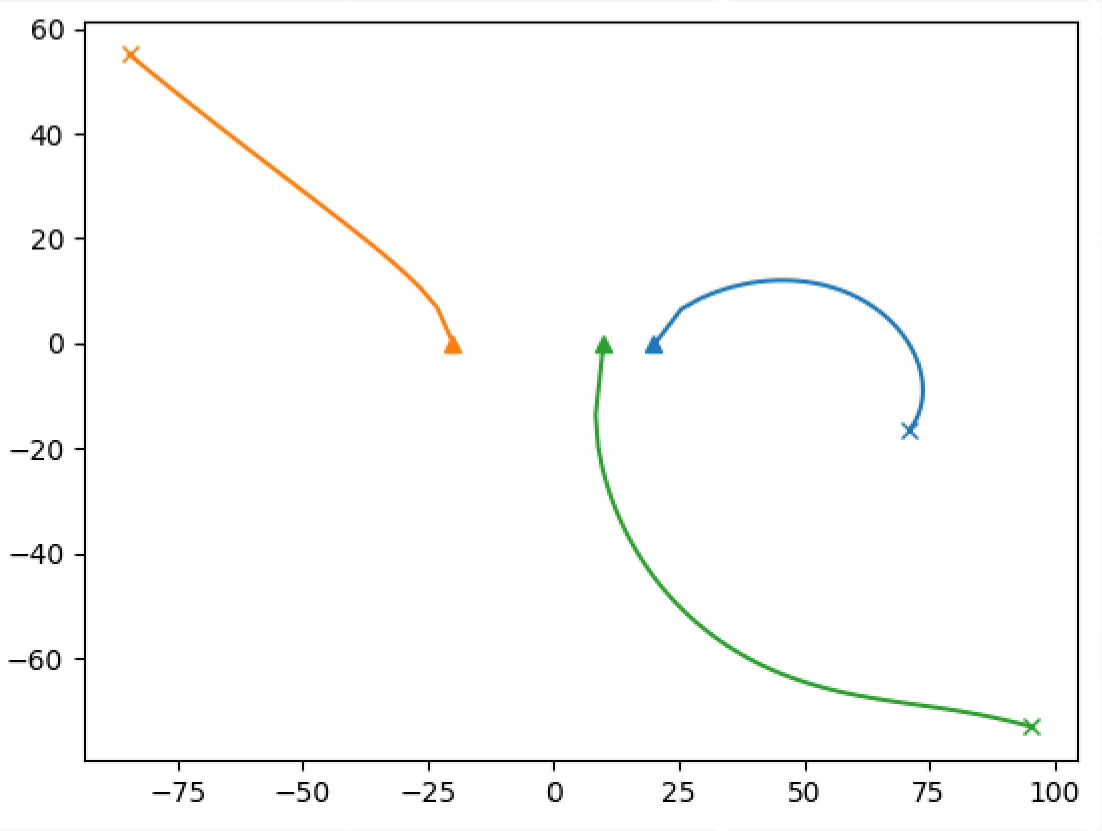
$$\dot{y}_n(t) = -g(t) \left\{ \prod_{\ell=1,\ \ell \neq n}^{N} \left[y_n(t) - y_\ell(t) \right]^{-1} \right\} \sum_{m=1}^{N} \left\{ \left[c_m - \gamma_m(0) \right] \left[y_n(t) \right]^{N-m} \right\}$$

$$F(t) = \begin{cases} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \vdots \\ \dot{y}_n(t) \end{cases}$$

$g(t) = 1, \qquad T = 1$

Example 1

$$x^3 - 15x^2 - 150^x + 1000$$
$$= (x-20)(x+20)(x-10)$$



Iteration 1

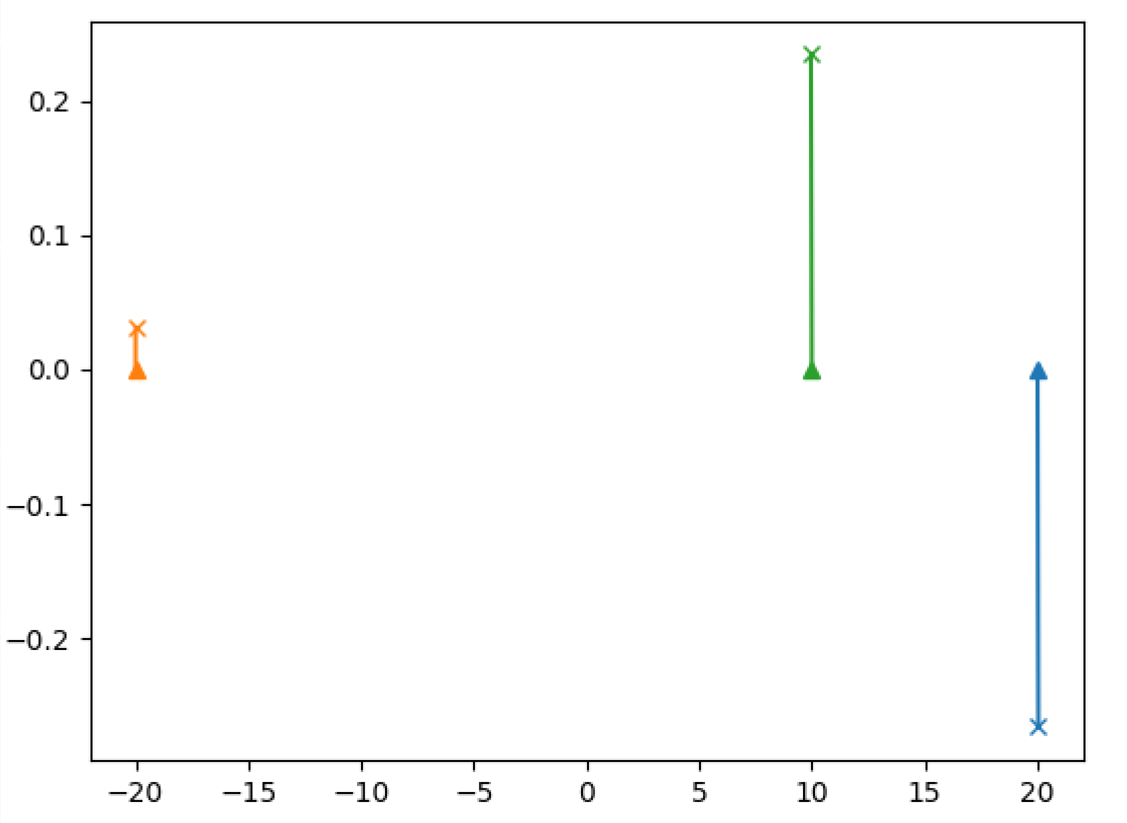
$$-100 \leq Re(y_i(0)) \leq 100$$
$$-100 \leq Im(y_i(0)) \leq 100$$

Initial Value	After Iteration
71.1406749978-16.2823174277i	20.0370960114-0.266170648667j
-84.7349639578+55.126702689j	-19.9976154713+0.0311076576057j
95.5650319505-73.1030055883j	9.9605194599+0.235062991061j

Example 1

$$x^3 - 15x^2 - 150^x + 1000$$
$$= (x-20)(x+20)(x-10)$$

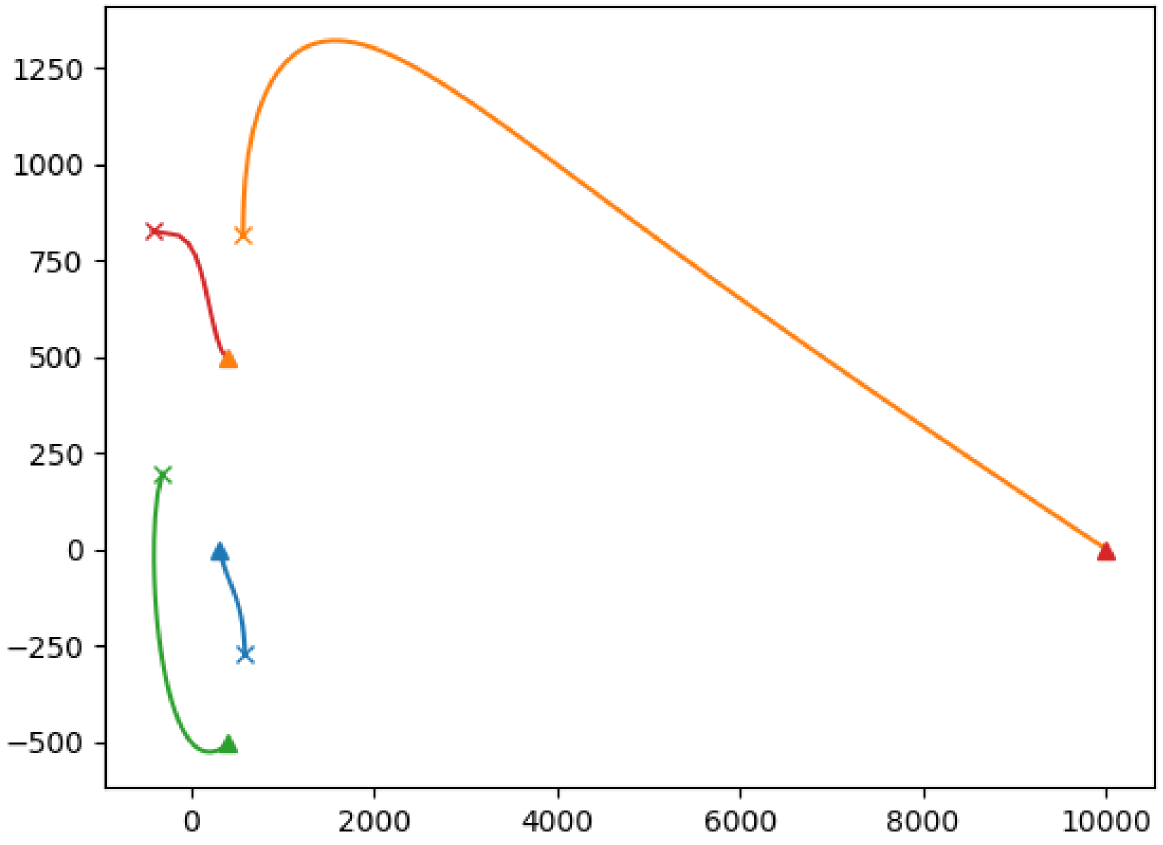
Iteration 2



Initial Value	After Iteration
20.0370960114-0.266170648667j	20+1.75207071074e-16j
-19.9976154713+0.0311076576057j	-20-2.21719344273e-17j
9.9605194599+0.235062991061j	10+1.95156391047e-17j

Example 2

$$(z-300)(z+400+500i)(z+400-500i)(z-9999)$$



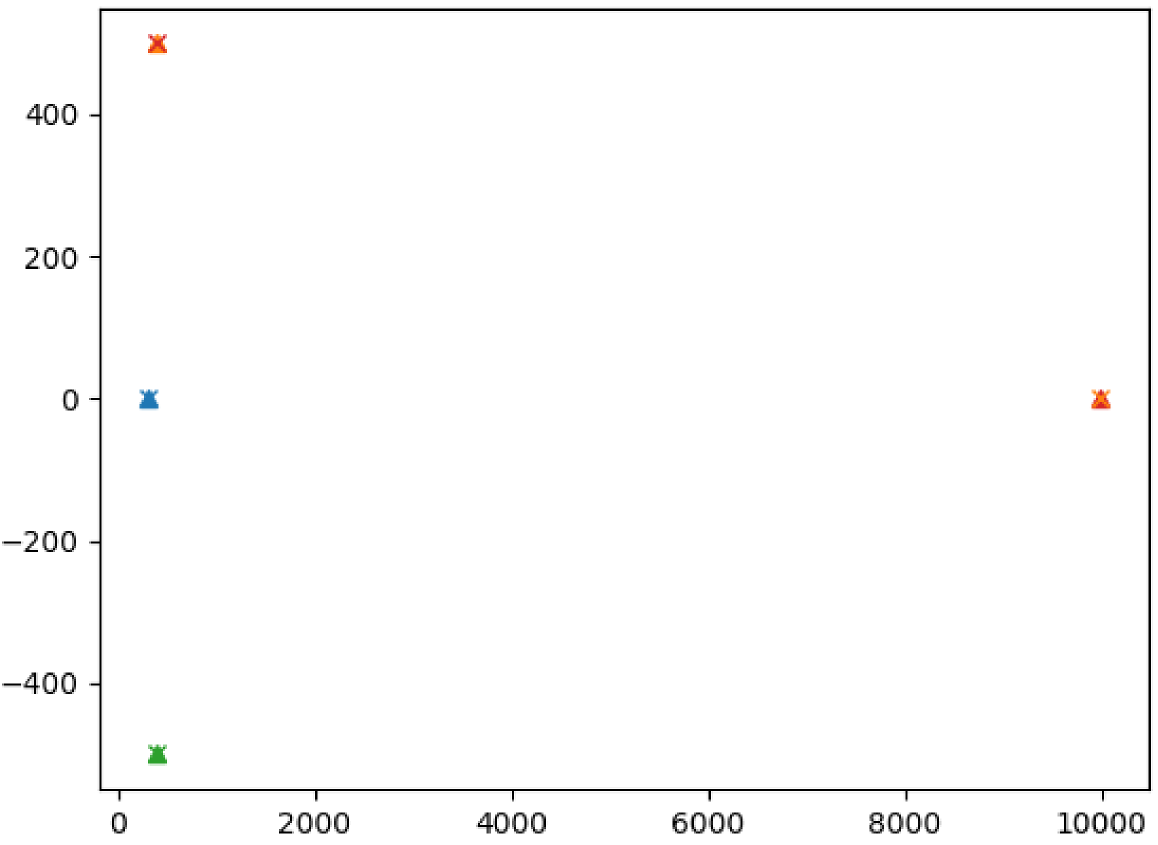
Iteration 1

$$-10000 \leq Re(y_i(0)) \leq 10000$$
$$-10000 \leq Im(y_i(0)) \leq 10000$$

Initial Value	After Iteration
585.655713746-269.955507343i	300.001015787+0.000710725589028i
564.148717961+816.963408841j	9998.99930425+0.000165669875692i
-314.12545548+197.401514585i	400.000338707-500.000649204i
-426.088792188+825.795241733i	399.999341253+499.999772808i

Example 2

$$(z-300)(z+400+500i)(z+400-500i)(z-9999)$$



Iteration 2

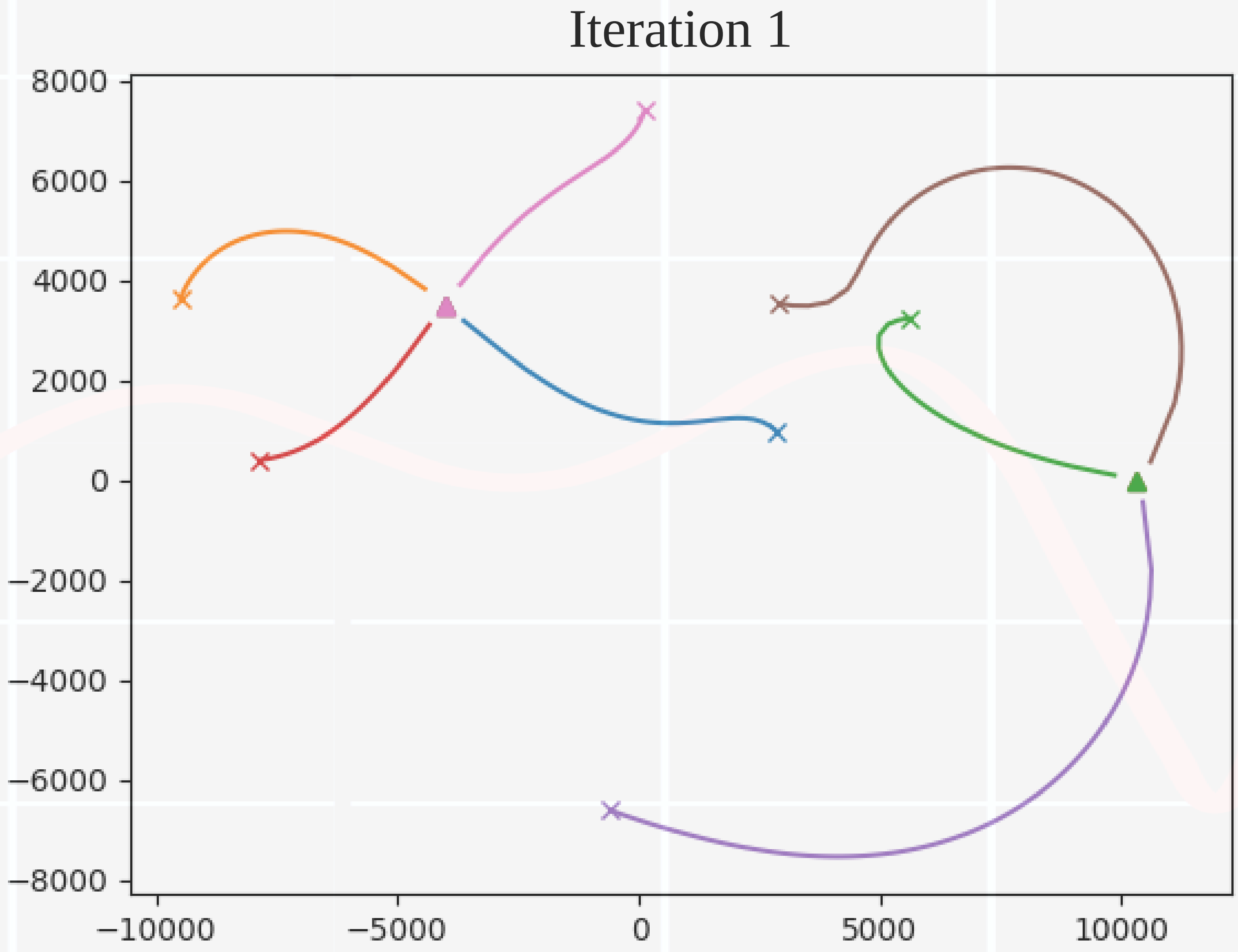
Initial Value	After Iteration
300.001015787+0.000710725589028i	300.00000000000011-8.6230971161845894e-14j
9998.99930425+0.000165669875692i	9998.9999999999854+7.464394626691454e-15i
400.000338707-500.000649204i	400-499.99999999999983i
399.999341253+499.999772808i	400+499.99999999999994i

Example 3

$$(z - 10300.33)^3 (z + 4021.33 - 3500.45i)^4$$

$-10000 \leq Re(y_i(0)) \leq 10000$
$-10000 \leq Im(y_i(0)) \leq 10000$

Initial Values
2870.49552818+961.06154354i
-9495.1097591+3621.26902329i
5609.31336913+3230.88764027i
-7853.83812259+394.252835218i
-576.092464347-6617.06027871i
2911.0754522+3528.7642718i
121.515534542+7383.42701134i



Example 3

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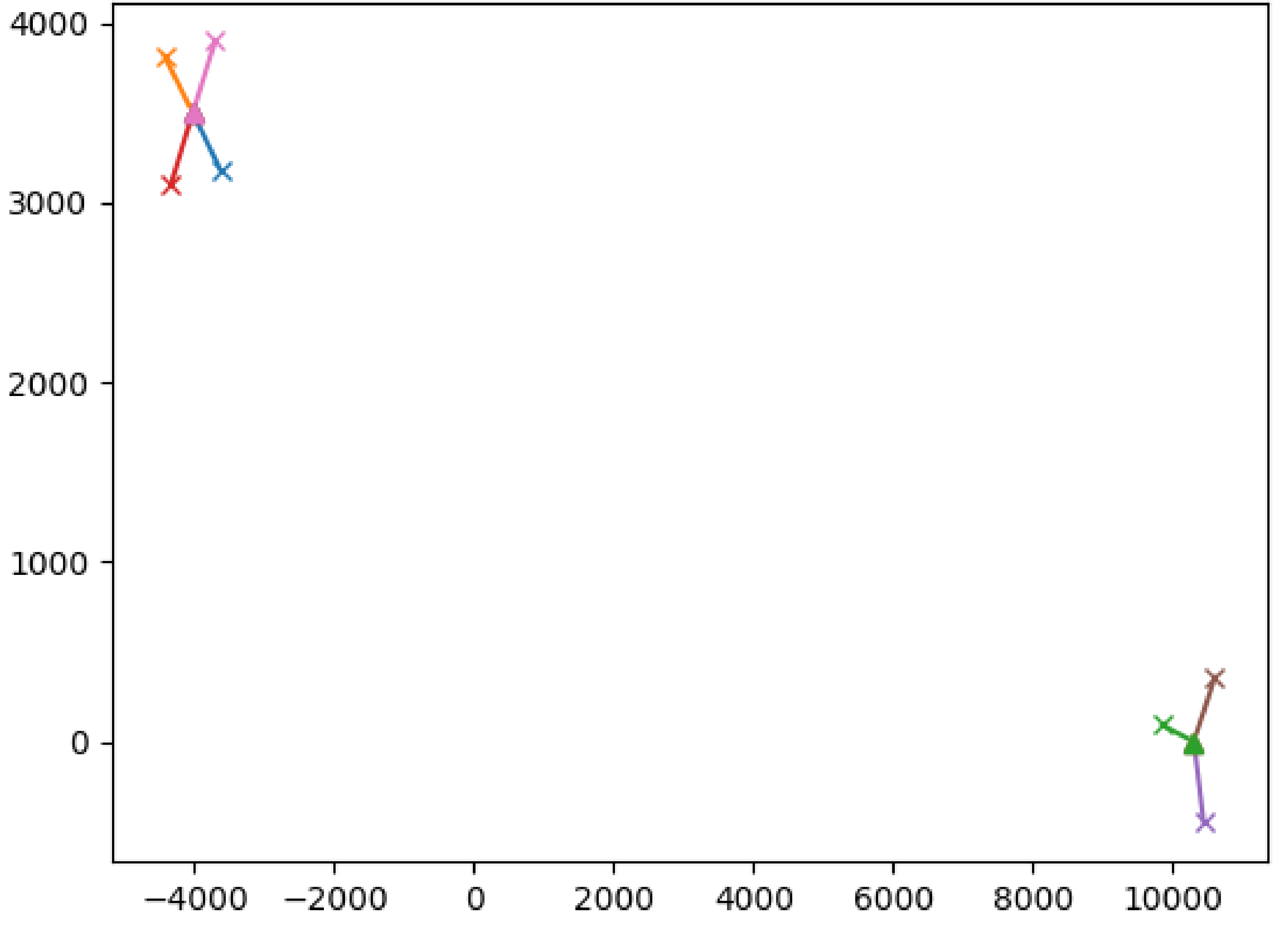
Iteration 2

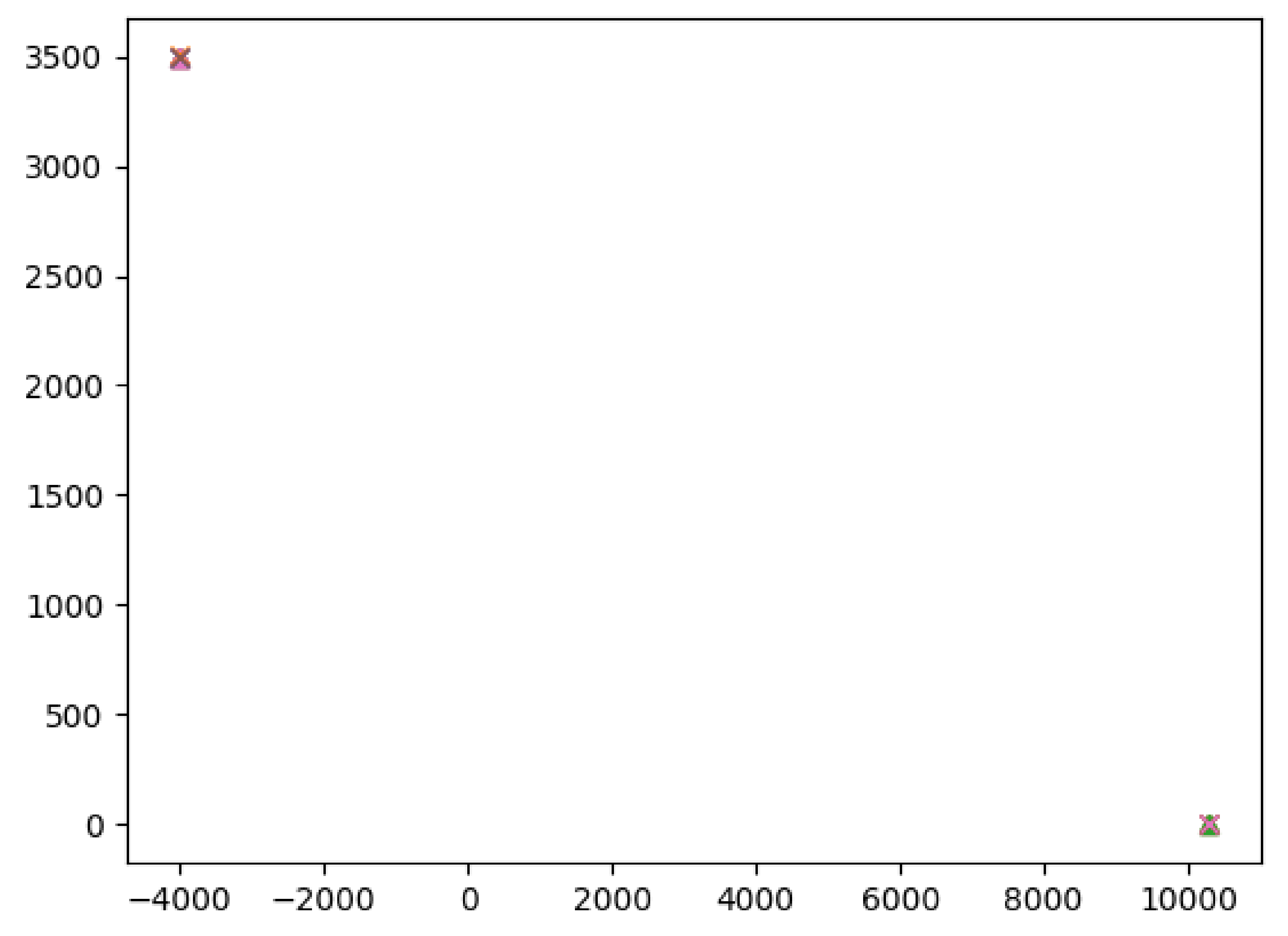


Example 3

$$(z - 10300.33)^3(z + 4021.33 - 3500.45i)^4$$

Values after 10 iterations
-4020.25991852+3500.53122138i
-4022.40025185+3500.36861989i
10300.2346611-0.000941070591542i
-4021.24861409+3499.37991271i
10300.3784842-0.082096481233i
10300.3768546+0.0830375518274i
4021.41121554+3501.52024603i

Iteration 10



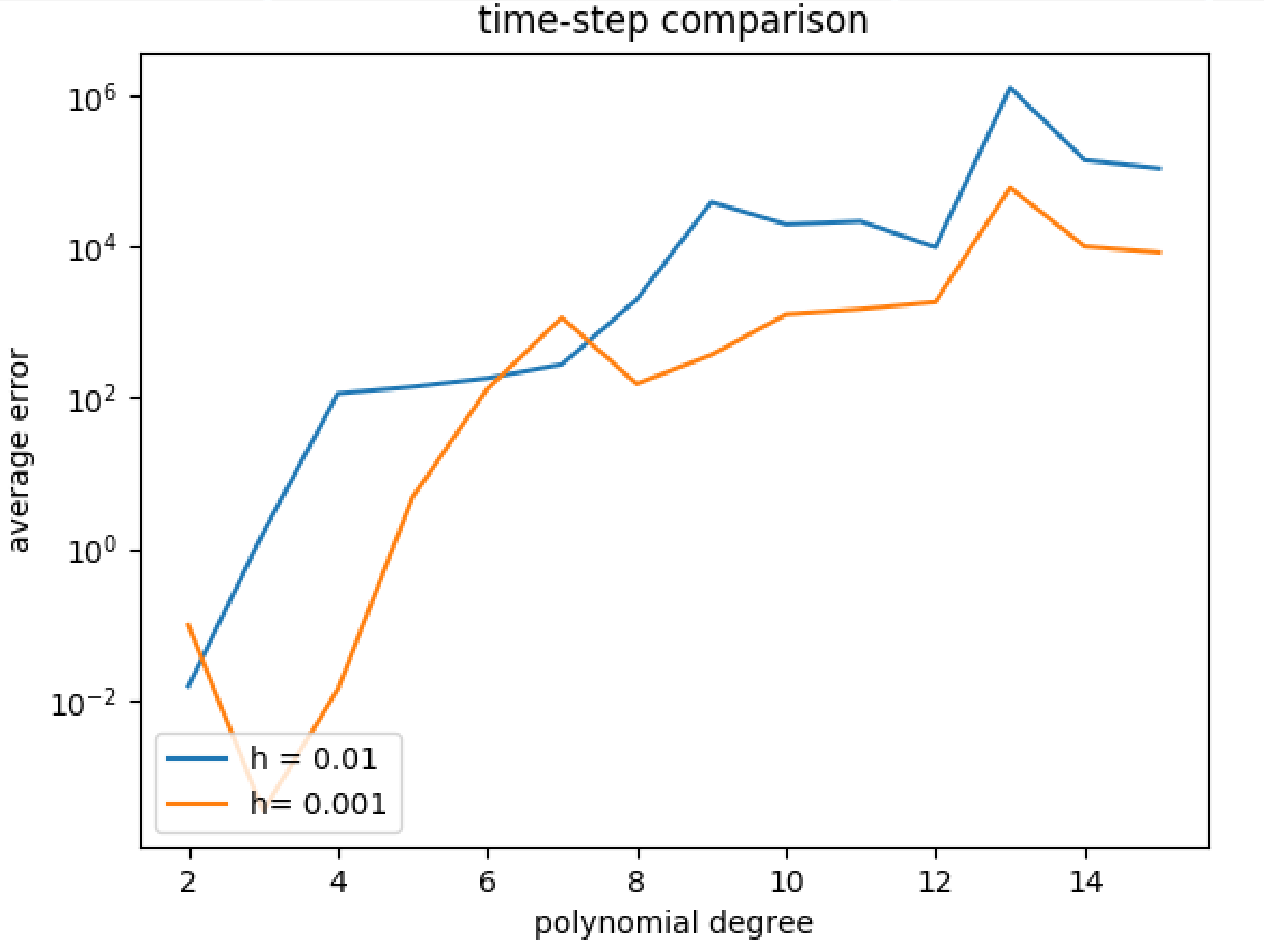
Factors we can alter

1. Time step
2. Iteration number
3. Initial zero guess

Time Step

1.

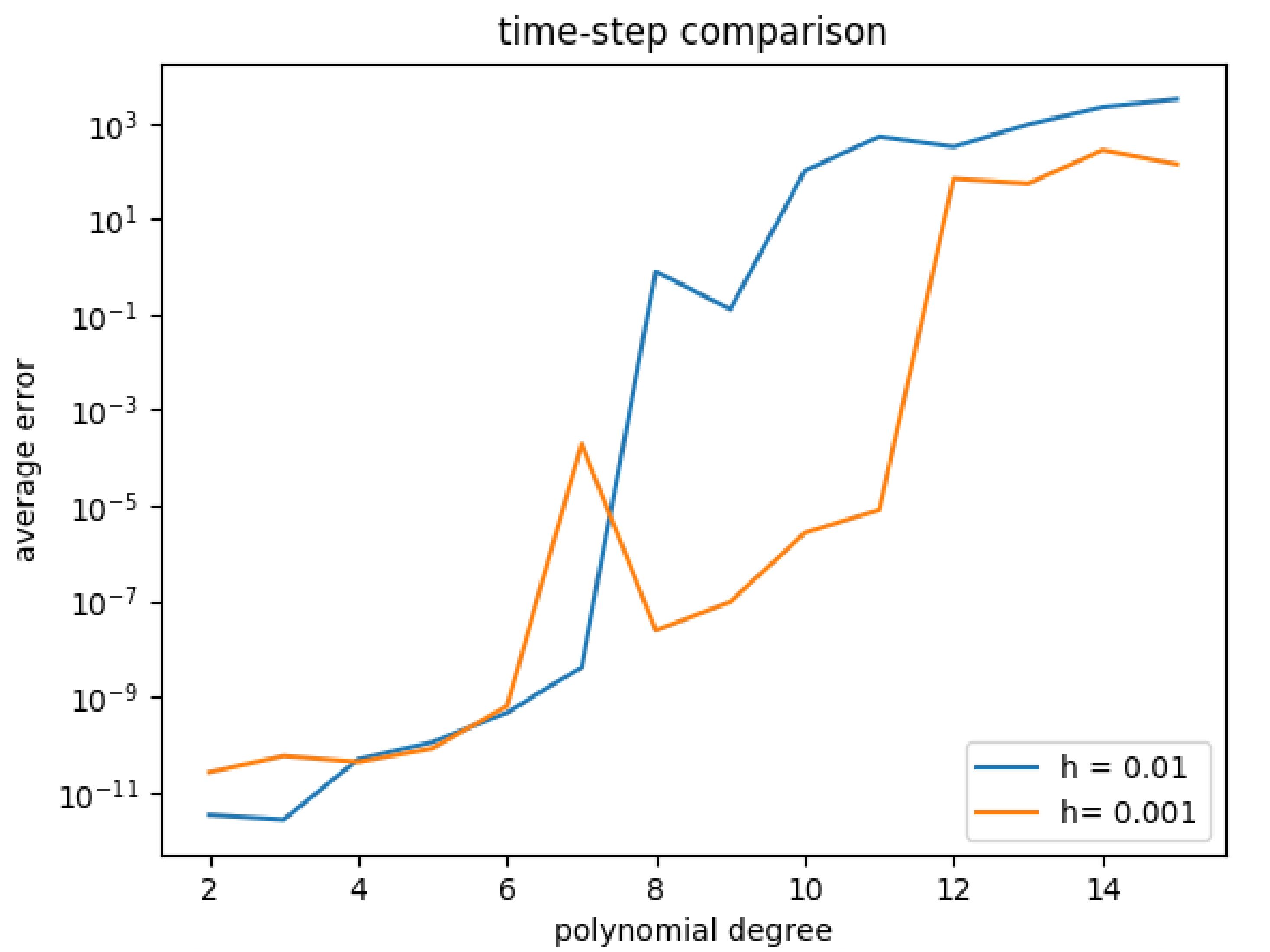
Time step



$$-1000 \leq Re(y_i(0)) \leq 1000$$
$$-1000 \leq Im(y_i(0)) \leq 1000$$
$$-1000 \leq Re(x_i(0)) \leq 1000$$
$$-1000 \leq Im(x_i(0)) \leq 1000$$

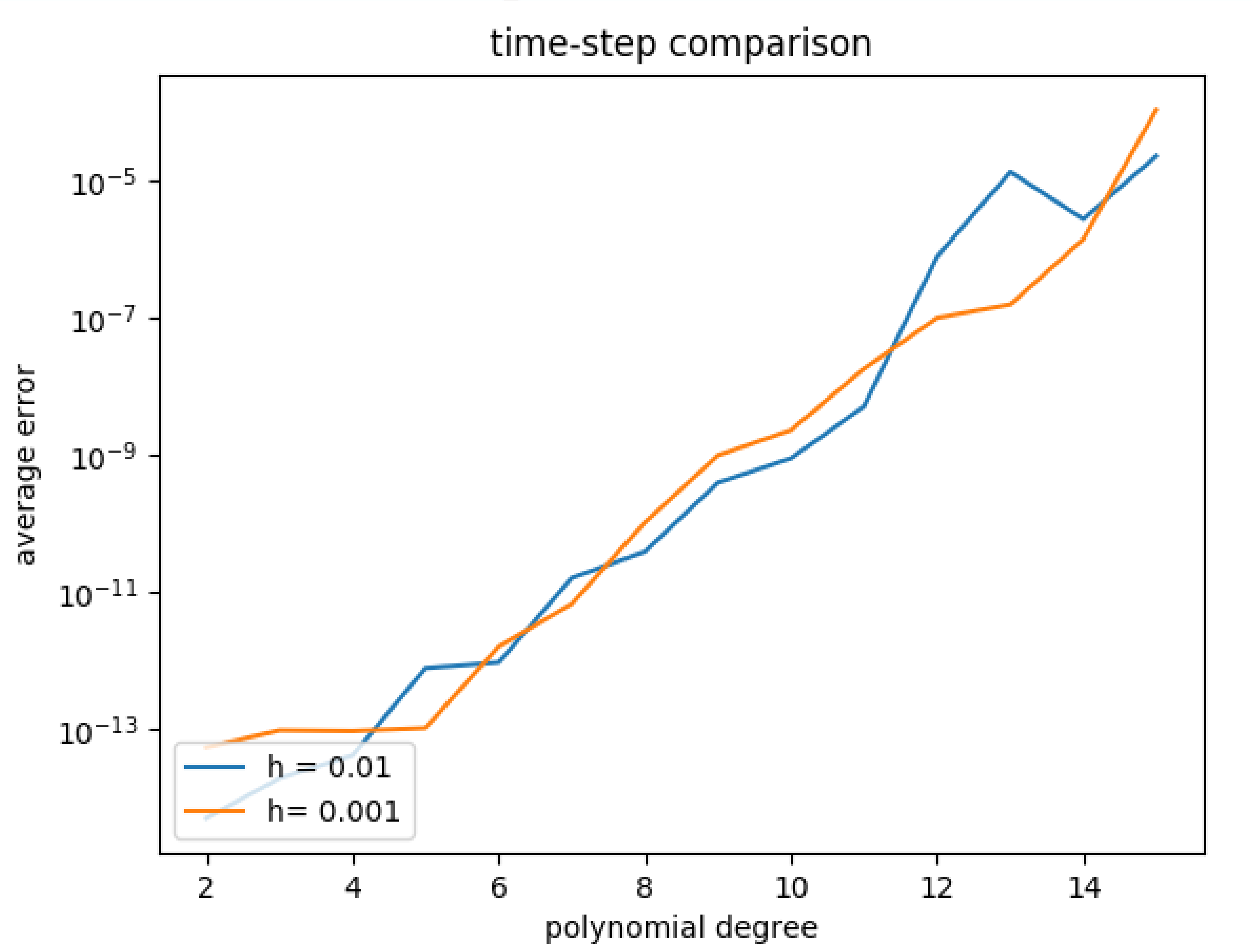
1 iteration

Time step



$-1000 \leq Re(y_i(0)) \leq 1000$
$-1000 \leq Im(y_i(0)) \leq 1000$
$-1000 \leq Re(x_i(0)) \leq 1000$
$-1000 \leq Im(x_i(0)) \leq 1000$
3 iterations

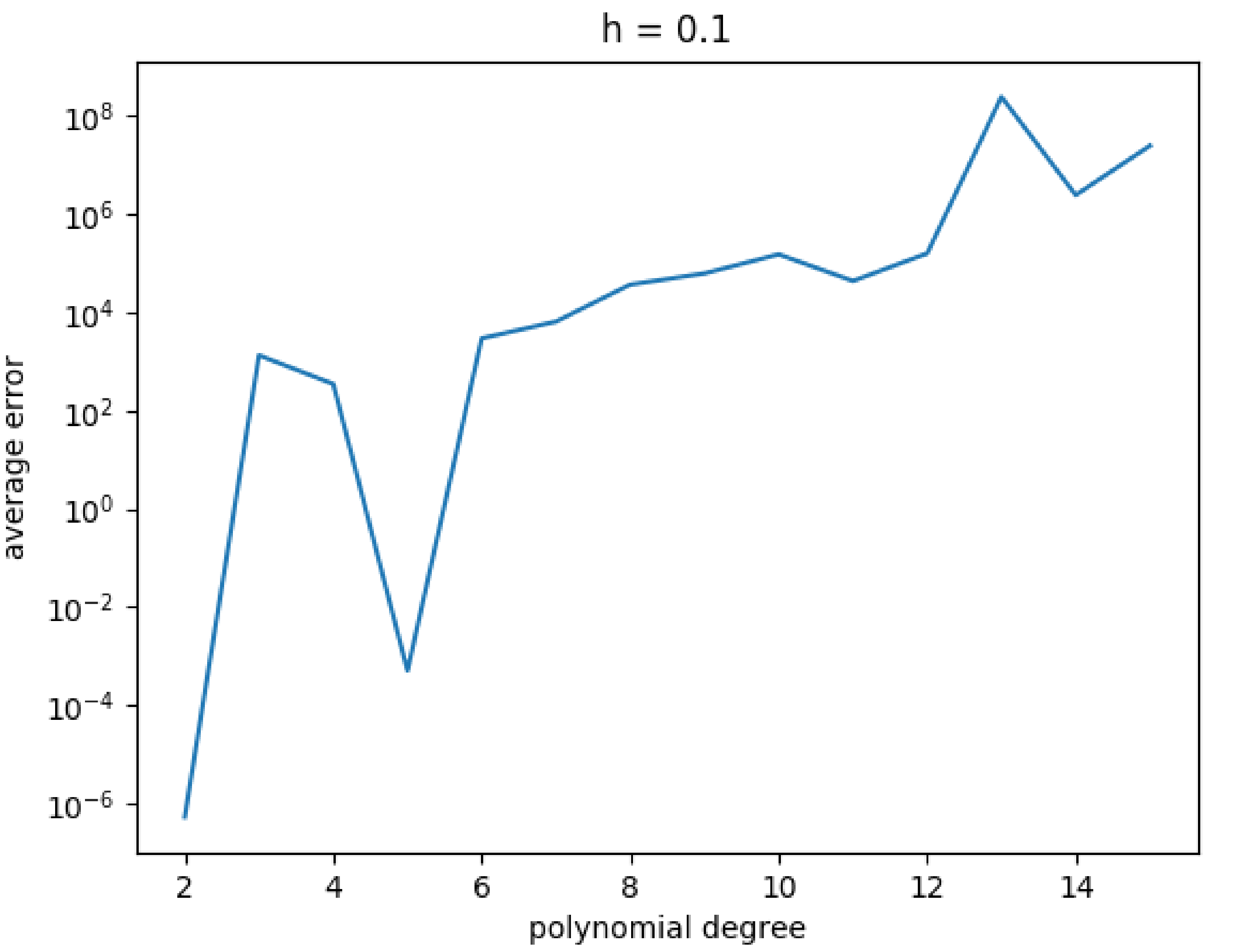
Time step



$$-1000 \le Re(y_i(0)) \le 1000$$
$$-1000 \le Im(y_i(0)) \le 1000$$
$$-1000 \le Re(x_i(0)) \le 1000$$
$$-1000 \le Im(x_i(0)) \le 1000$$

10 iterations

Time step

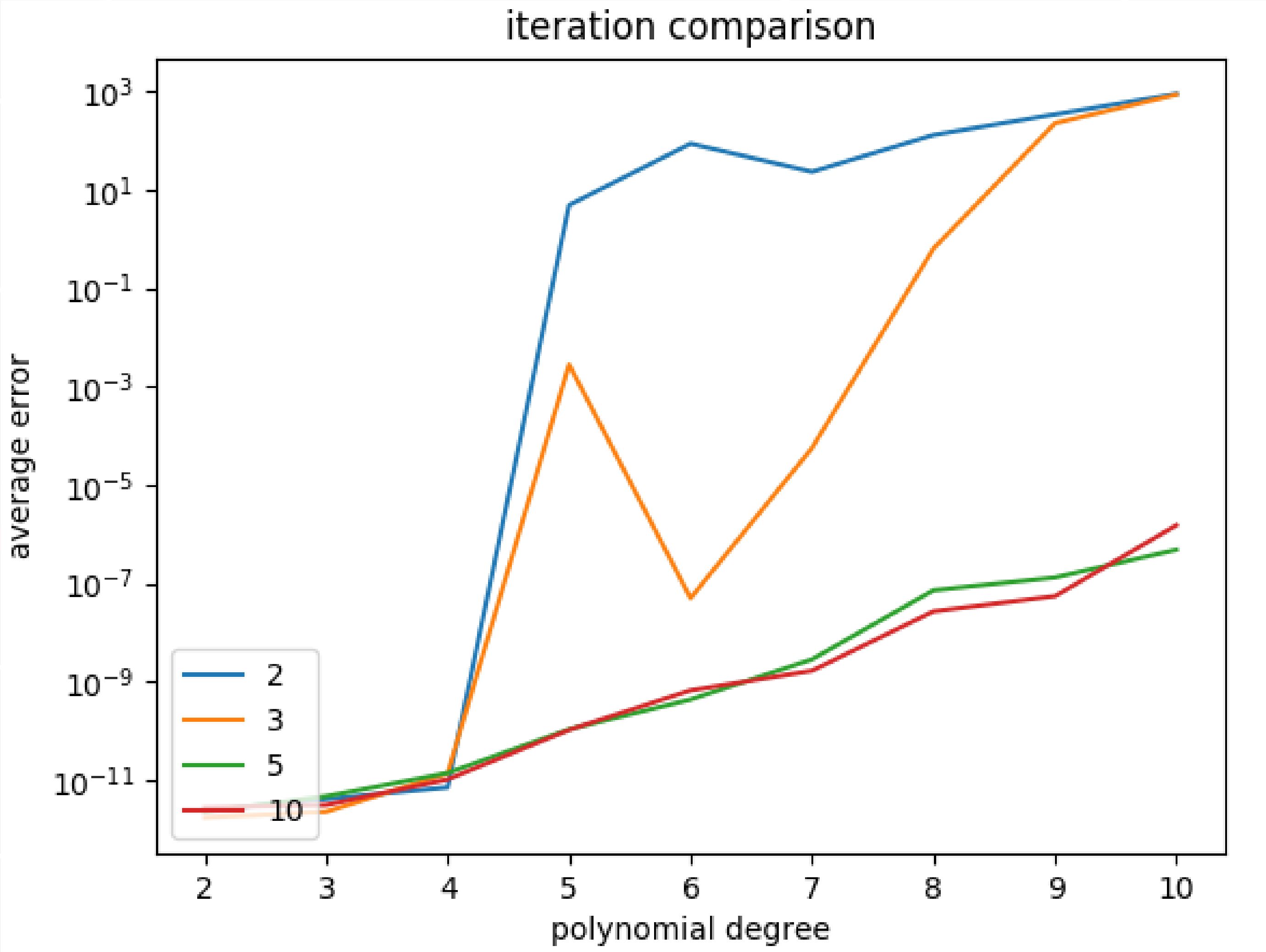


$-1000 \leq Re(y_i(0)) \leq 1000$
$-1000 \leq Im(y_i(0)) \leq 1000$
$-1000 \leq Re(x_i(0)) \leq 1000$
$-1000 \leq Im(x_i(0)) \leq 1000$
10 iterations

Iteration Number

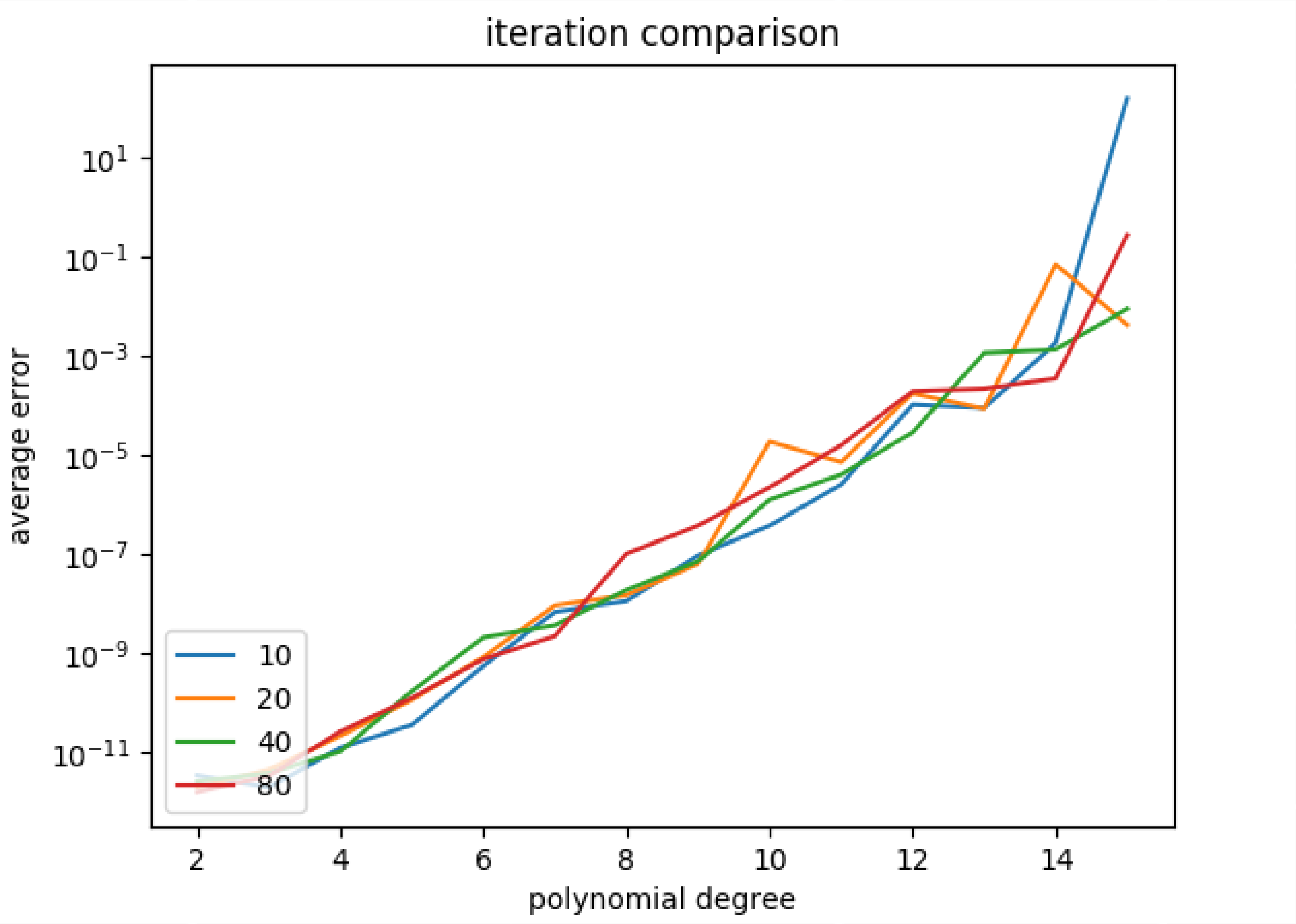
1.

Iteration Number



$$-1000 \leq Re(y_i(0)) \leq 1000$$
$$-1000 \leq Im(y_i(0)) \leq 1000$$
$$-1000 \leq Re(x_i(0)) \leq 1000$$
$$-1000 \leq Im(x_i(0)) \leq 1000$$
$$h = 0.01$$

Iteration Number



$$-1000 \leq Re(y_i(0)) \leq 1000$$
$$-1000 \leq Im(y_i(0)) \leq 1000$$
$$-1000 \leq Re(x_i(0)) \leq 1000$$
$$-1000 \leq Im(x_i(0)) \leq 1000$$
$$h = 0.01$$

Why?

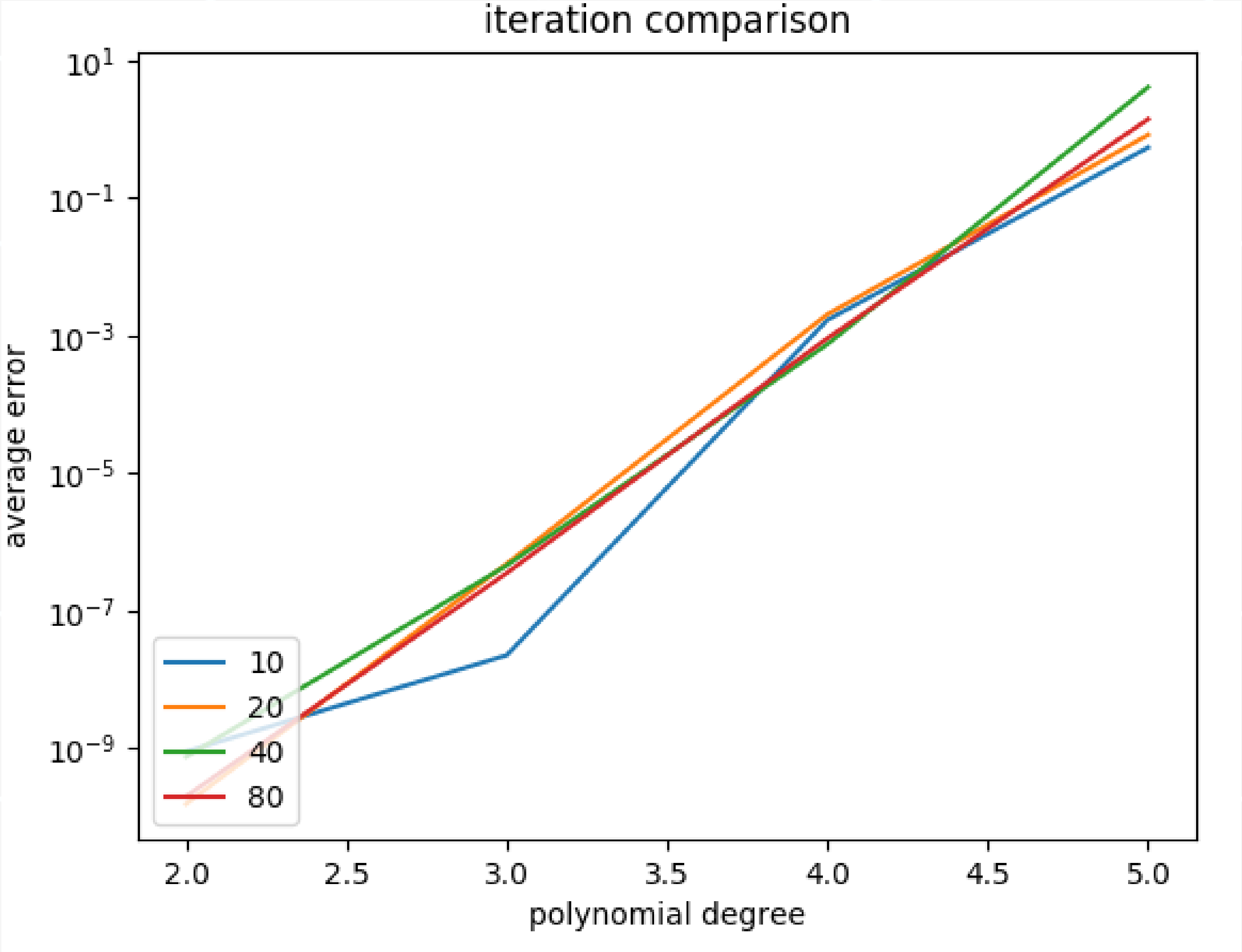
Polynomial Expansion

+

Machine Restrictions

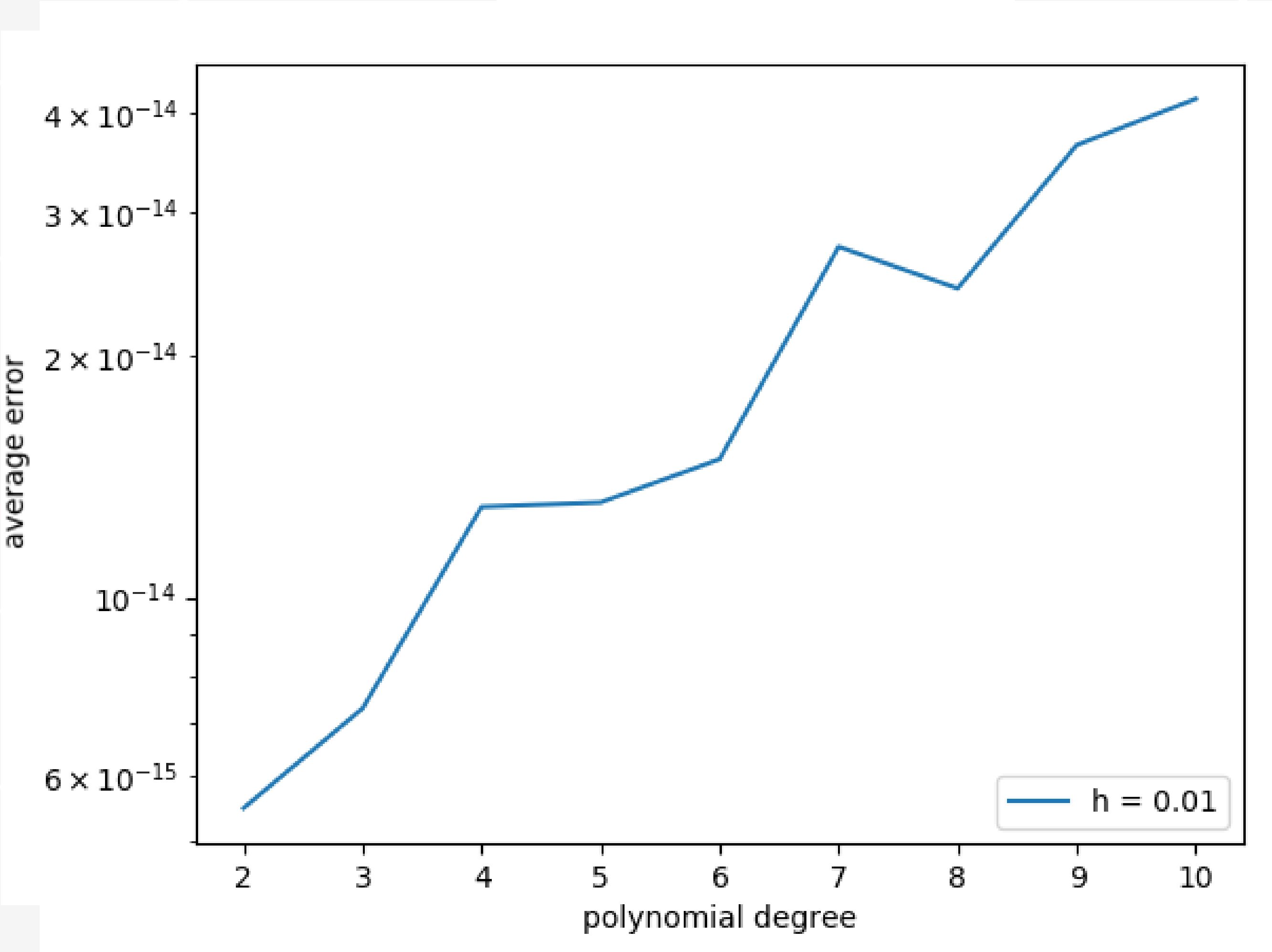
How about evaluating the zero?

How about evaluating the zero?



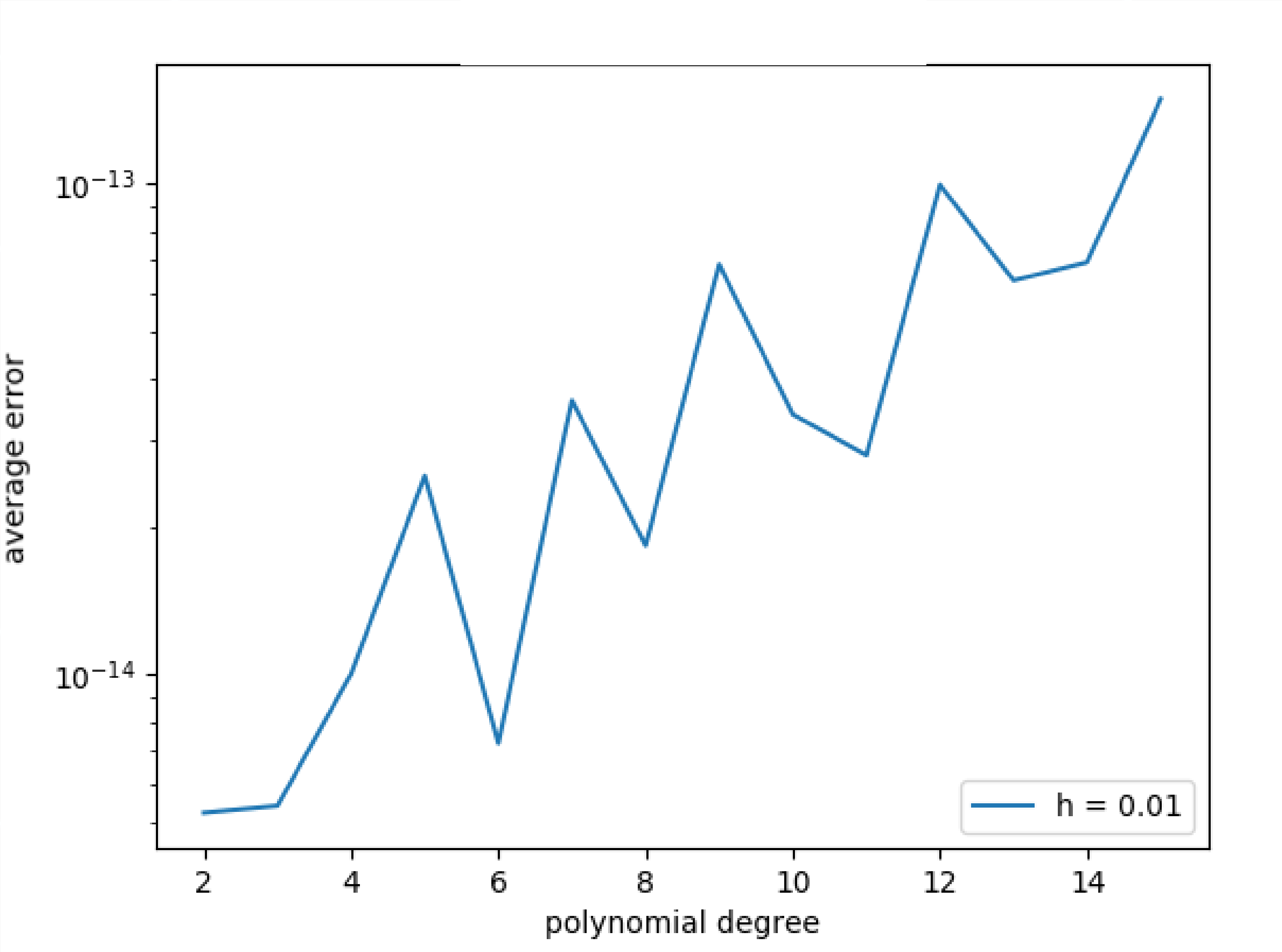
$$-1000 \leq Re(y_i(0)) \leq 1000$$
$$-1000 \leq Im(y_i(0)) \leq 1000$$
$$-1000 \leq Re(x_i(0)) \leq 1000$$
$$-1000 \leq Im(x_i(0)) \leq 1000$$
$$h = 0.01$$

How about evaluating the zero?



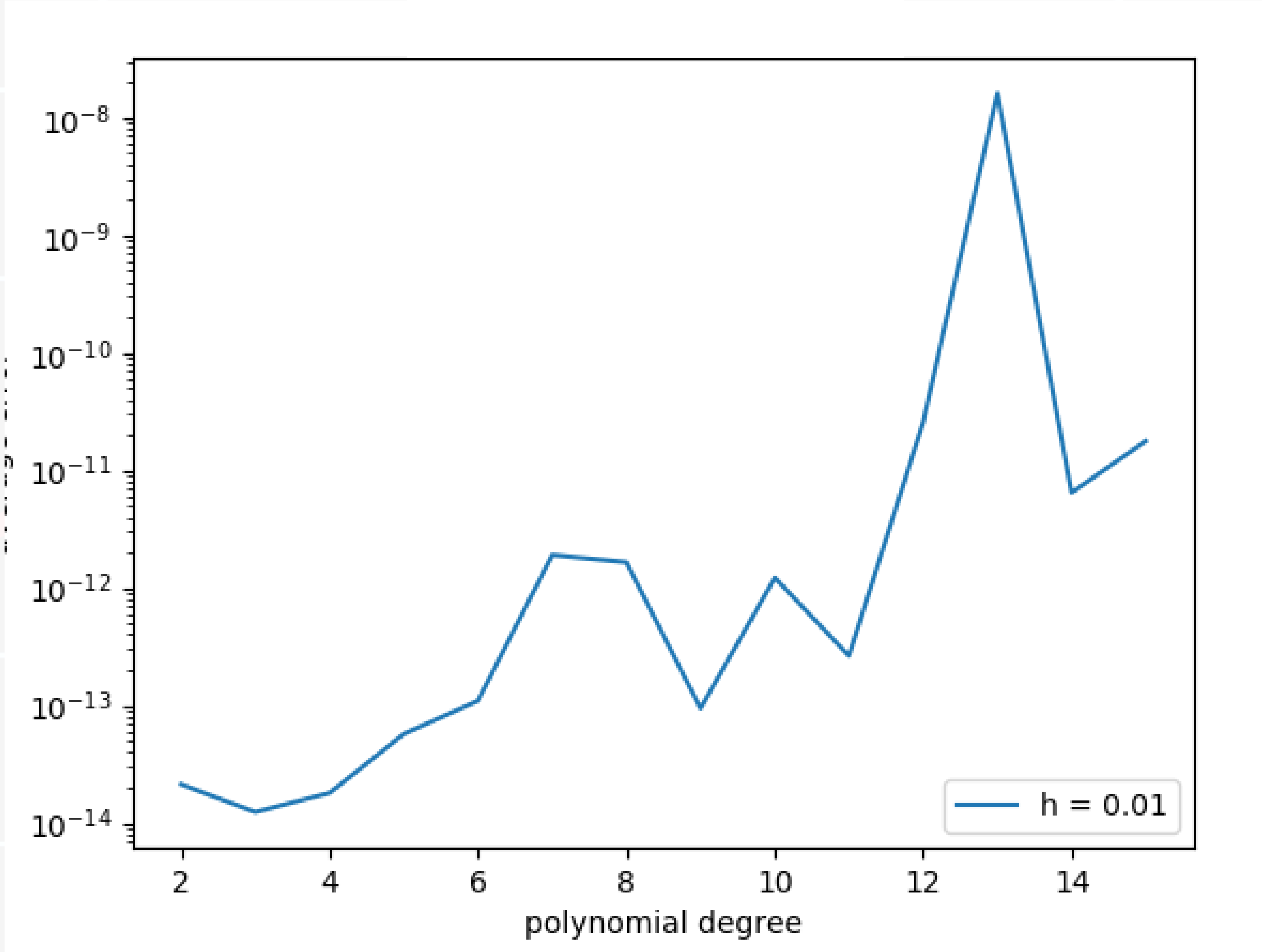
$$-100 \leq Re(y_i(0)) \leq 100$$
$$-100 \leq Im(y_i(0)) \leq 100$$
$$-1.10 \leq Re(x_i(0)) \leq 1.10$$
$$-1.10 \leq Im(x_i(0)) \leq 1.10$$
$$iterations = 10$$

How about evaluating the zero?



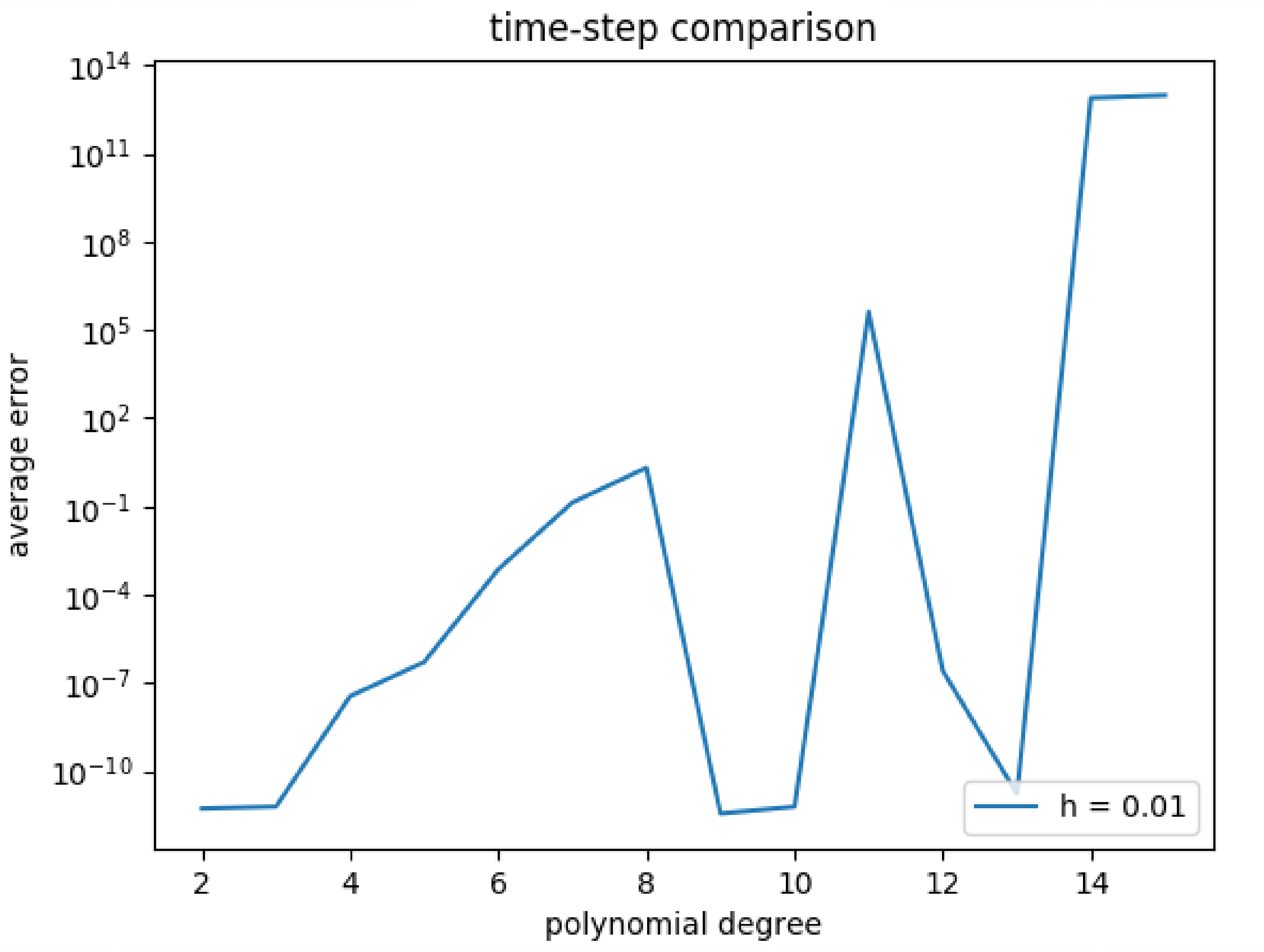
$$-10 \leq Re(y_i(0)) \leq 10$$
$$-10 \leq Im(y_i(0)) \leq 10$$
$$-2 \leq Re(x_i(0)) \leq 2$$
$$-2 \leq Im(x_i(0)) \leq 2$$
$$iterations = 10$$

How about evaluating the zero?



$$-10 \leq Re(y_i(0)) \leq 10$$
$$-10 \leq Im(y_i(0)) \leq 10$$
$$-5 \leq Re(x_i(0)) \leq 5$$
$$-5 \leq Im(x_i(0)) \leq 5$$
$$iterations = 10$$

How about evaluating the zero?



$$-100 \leq Re(y_i(0)) \leq 100$$
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$$-100 \leq Re(x_i(0)) \leq 100$$
$$-100 \leq Im(x_i(0)) \leq 100$$
$$iterations = 10$$

Initial Zero Guess

1.

Initial Zero Guess

Observation

$$(z - 10300.33)^3(z + 4021.33 - 3500.45i)^4$$

Initial Guess $y_j(0) = a_j + b_j\, i \; where \; a, b \in [-1, 1]$

Initial Zero Guess

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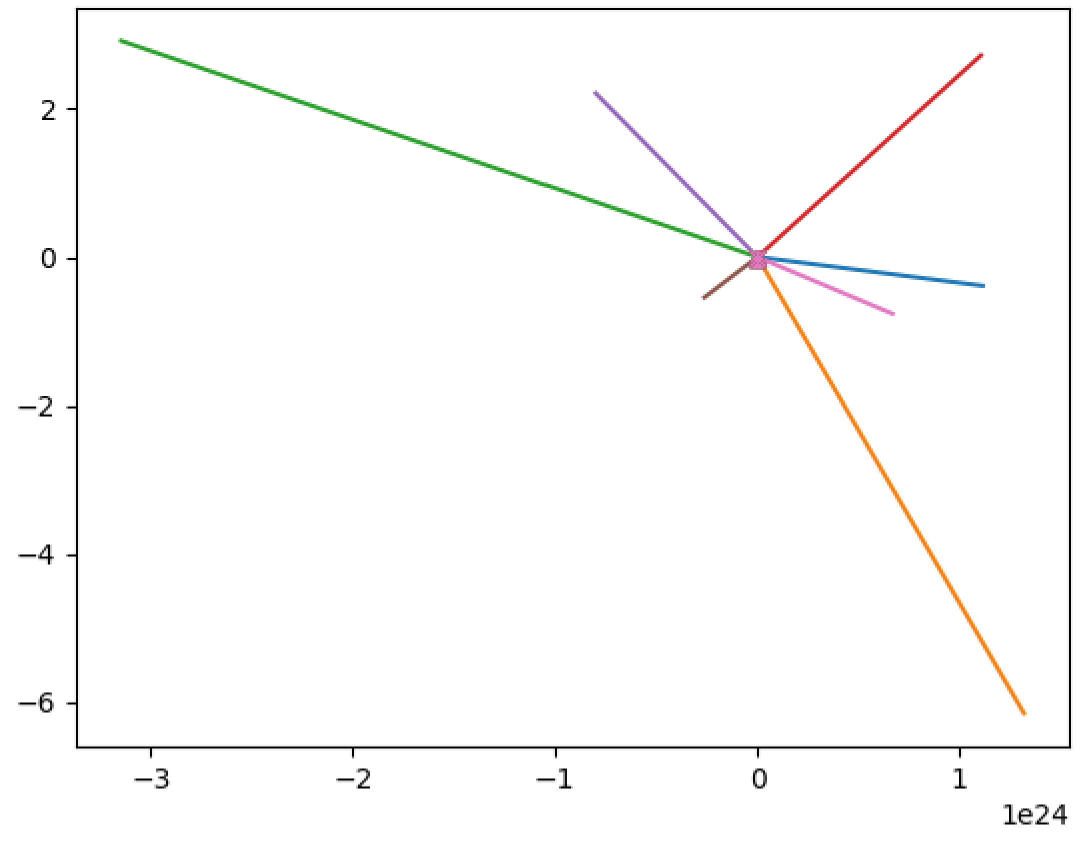
Iteration 1



Initial Zero Guess

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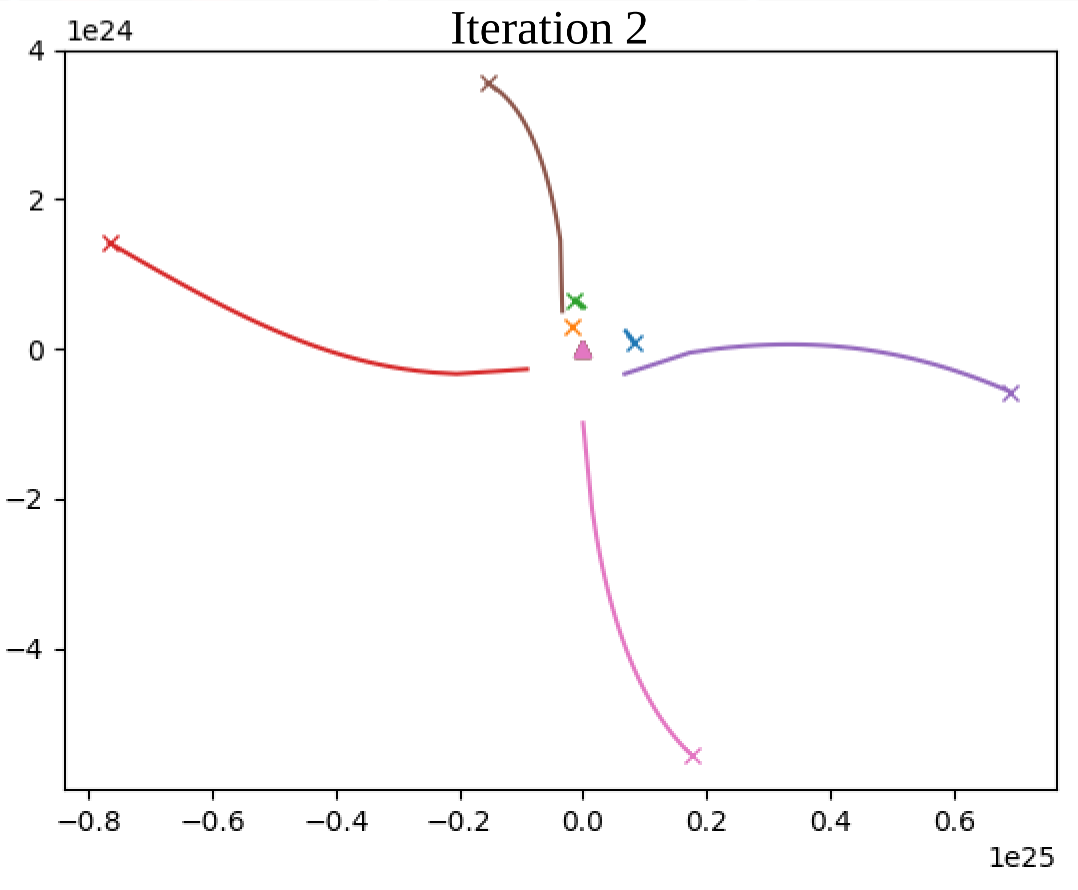
$$(z - 10300.33)^3 (z + 4021.33 - 3500.45i)^4$$



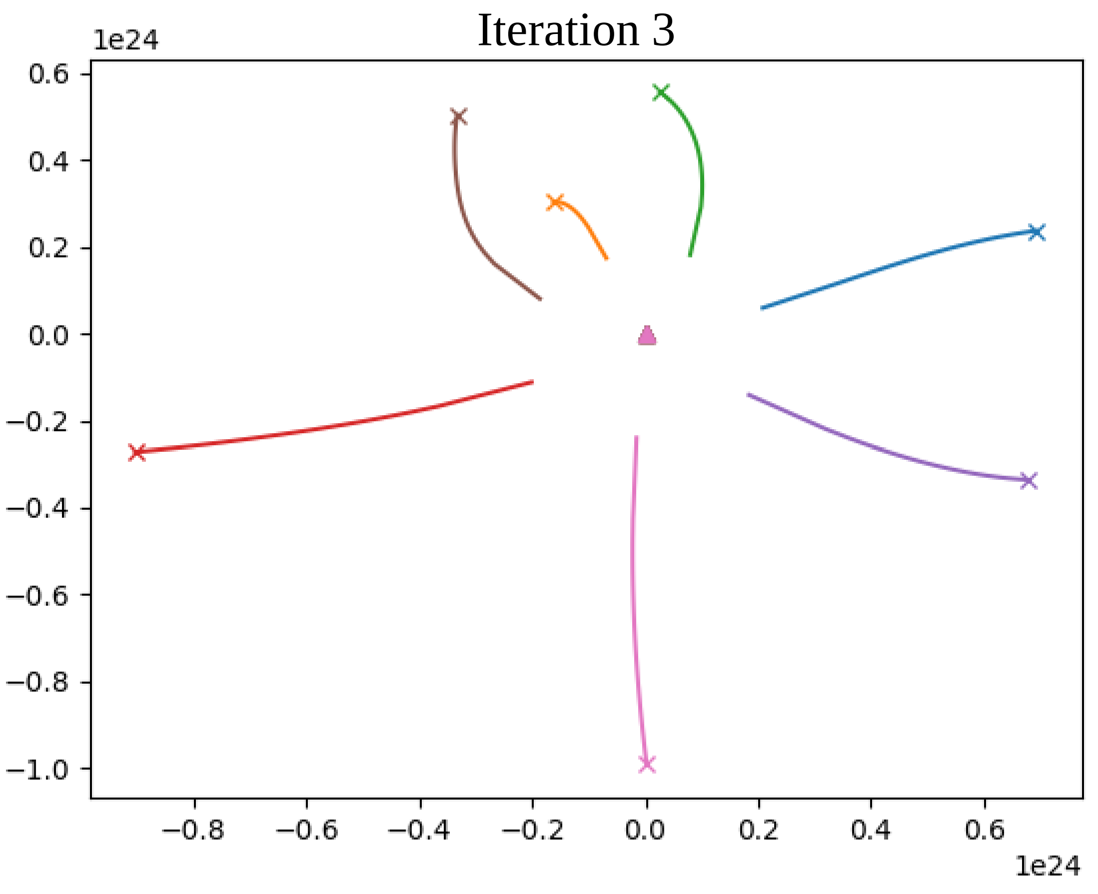
Initial Guess $y_j(0) = a_j + b_j i$ *where* $a, b \in [-1, 1]$

Initial Zero Guess

Observation 1

$$(z - 10300.33)^3 (z + 4021.33 - 3500.45i)^4$$

Initial Guess $y_j(0) = a_j + b_j i \text{ where } a, b \in [-1, 1]$



Initial Zero Guess

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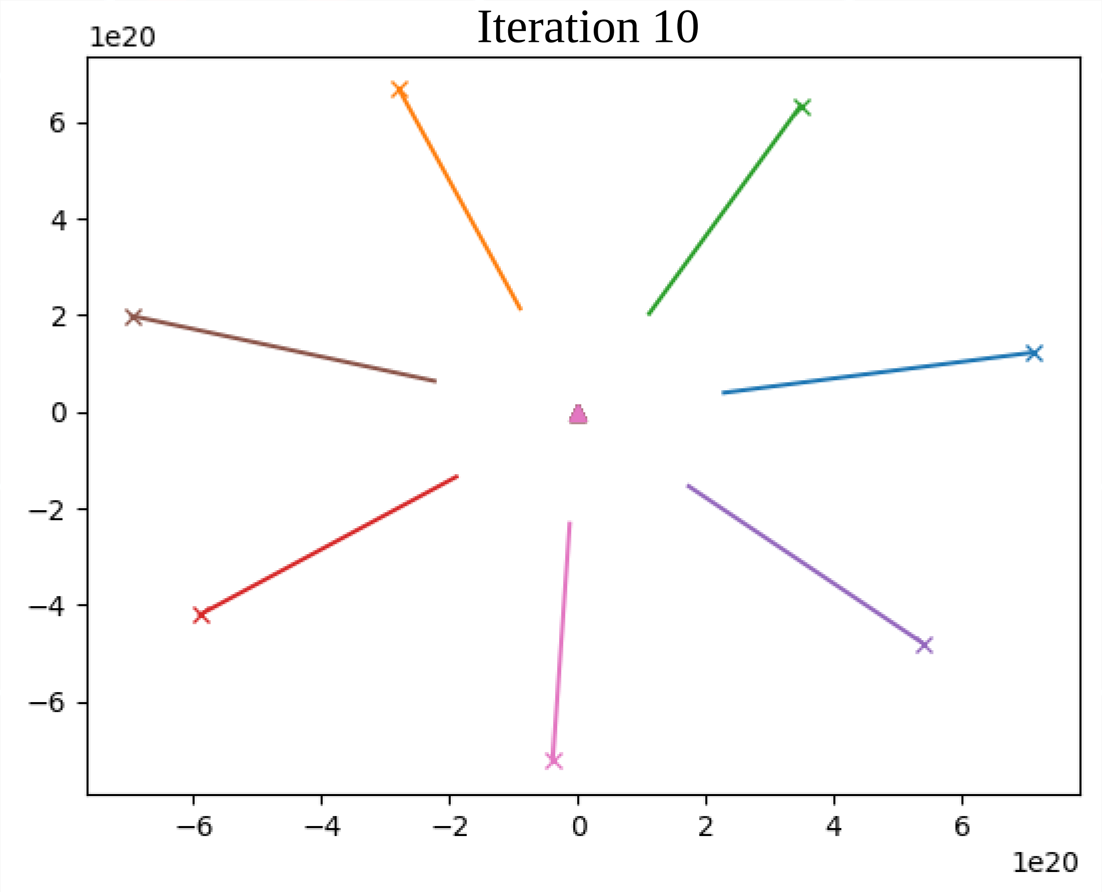
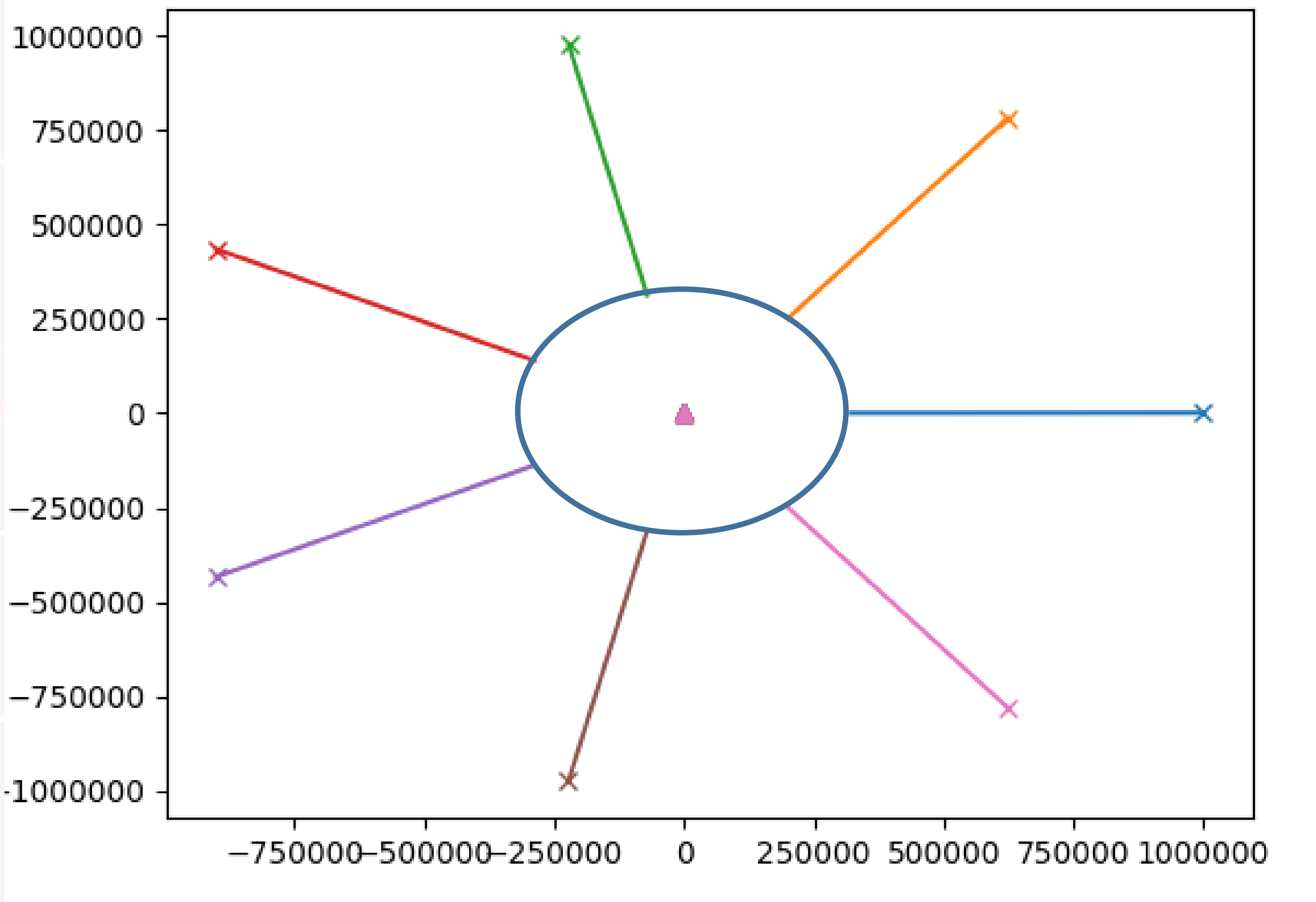
Initial Guess $y_j(0) = a_j + b_j i \text{ } where \text{ } a, b \in [-1, 1]$



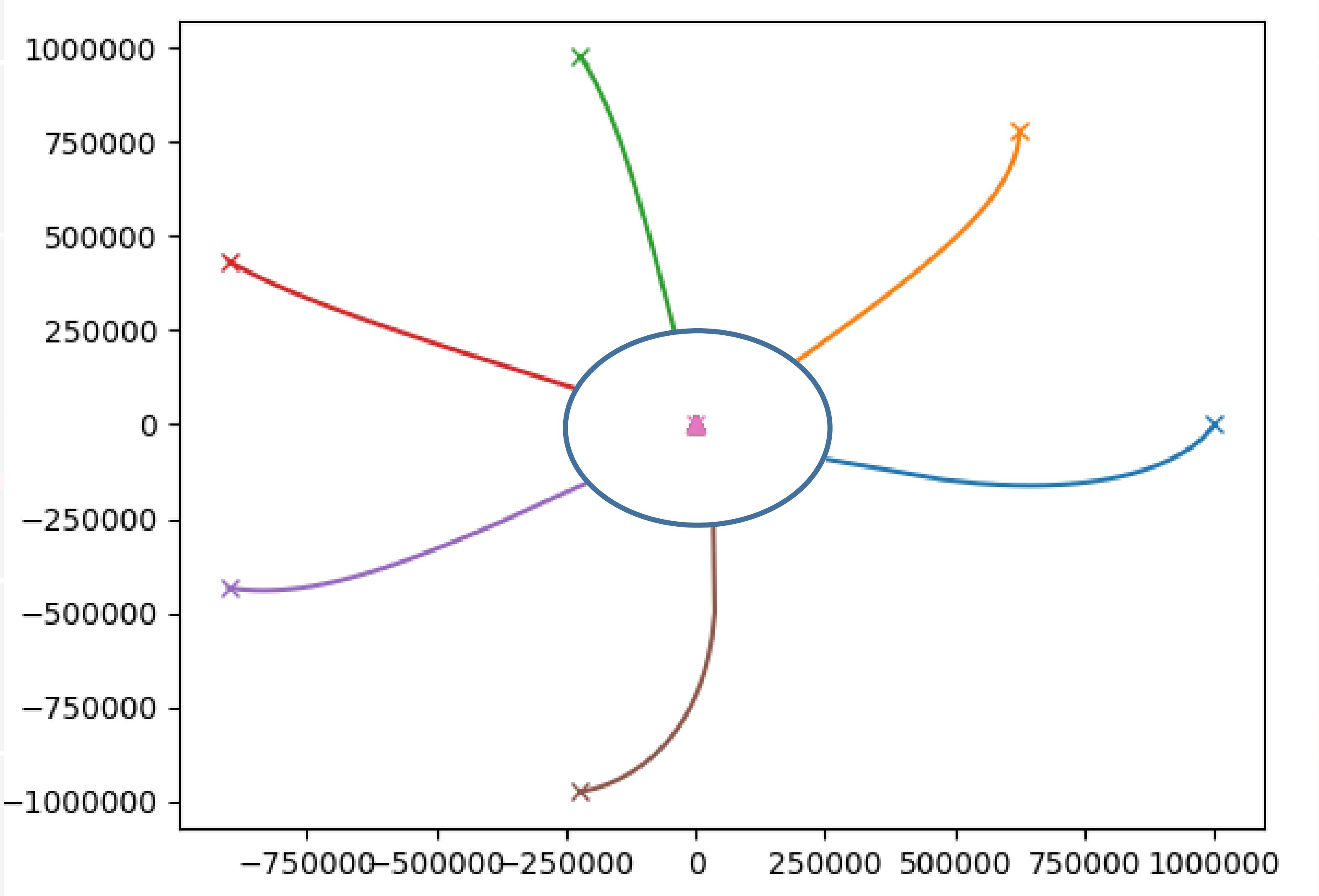
Initial Zero Guess

Observation 2



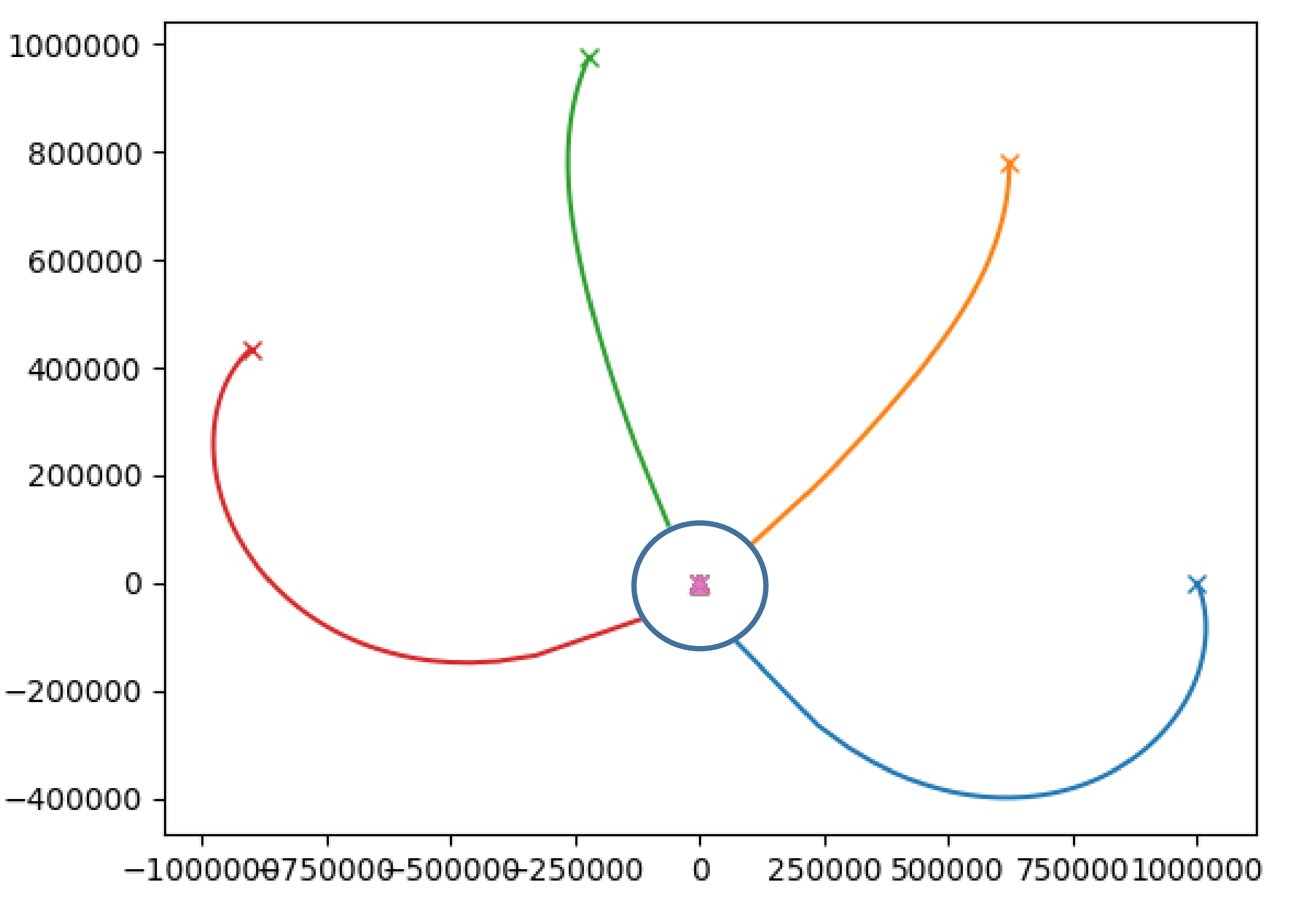
Initial Zero Guess

Observation 2



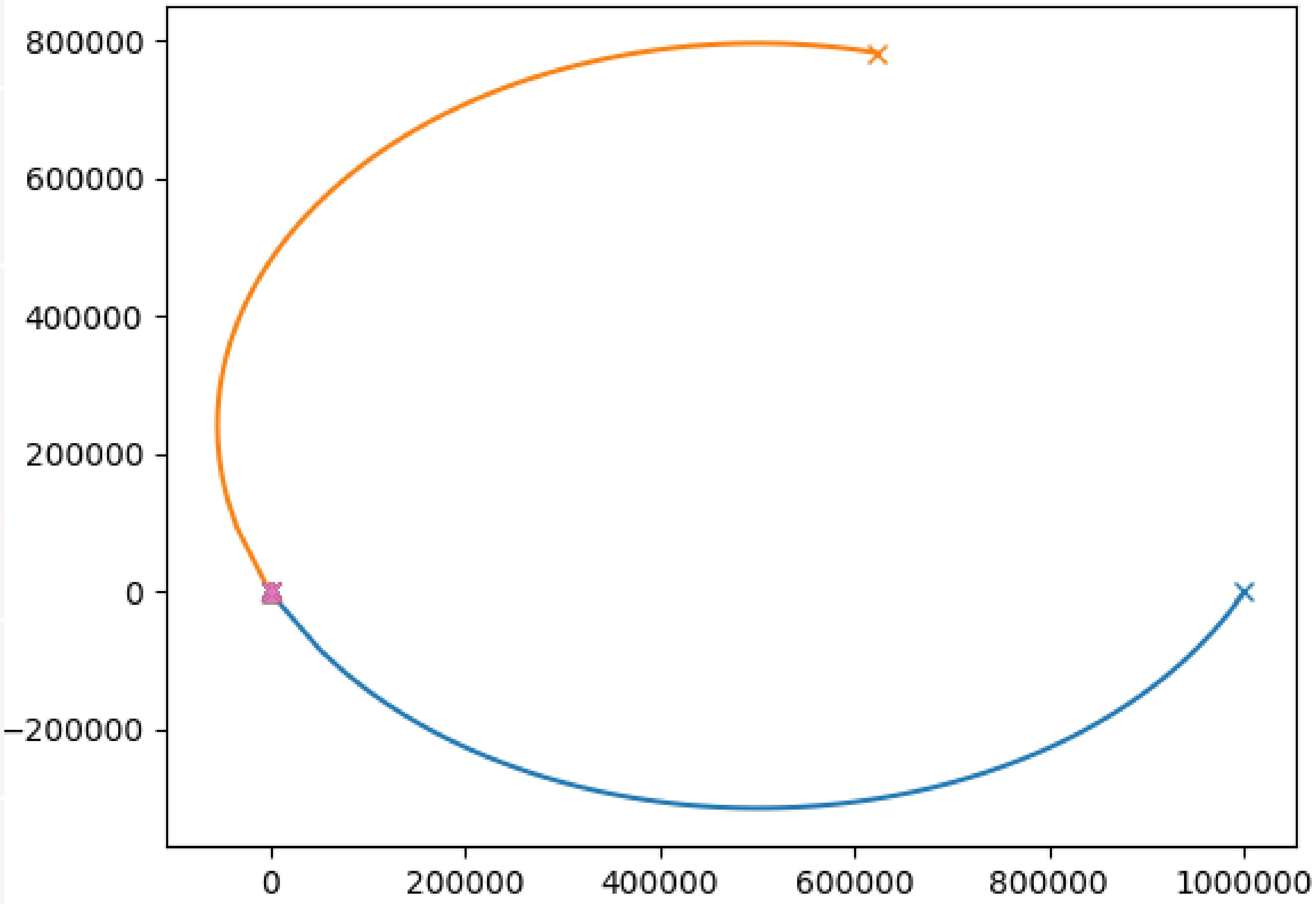
Initial Zero Guess

Observation 2



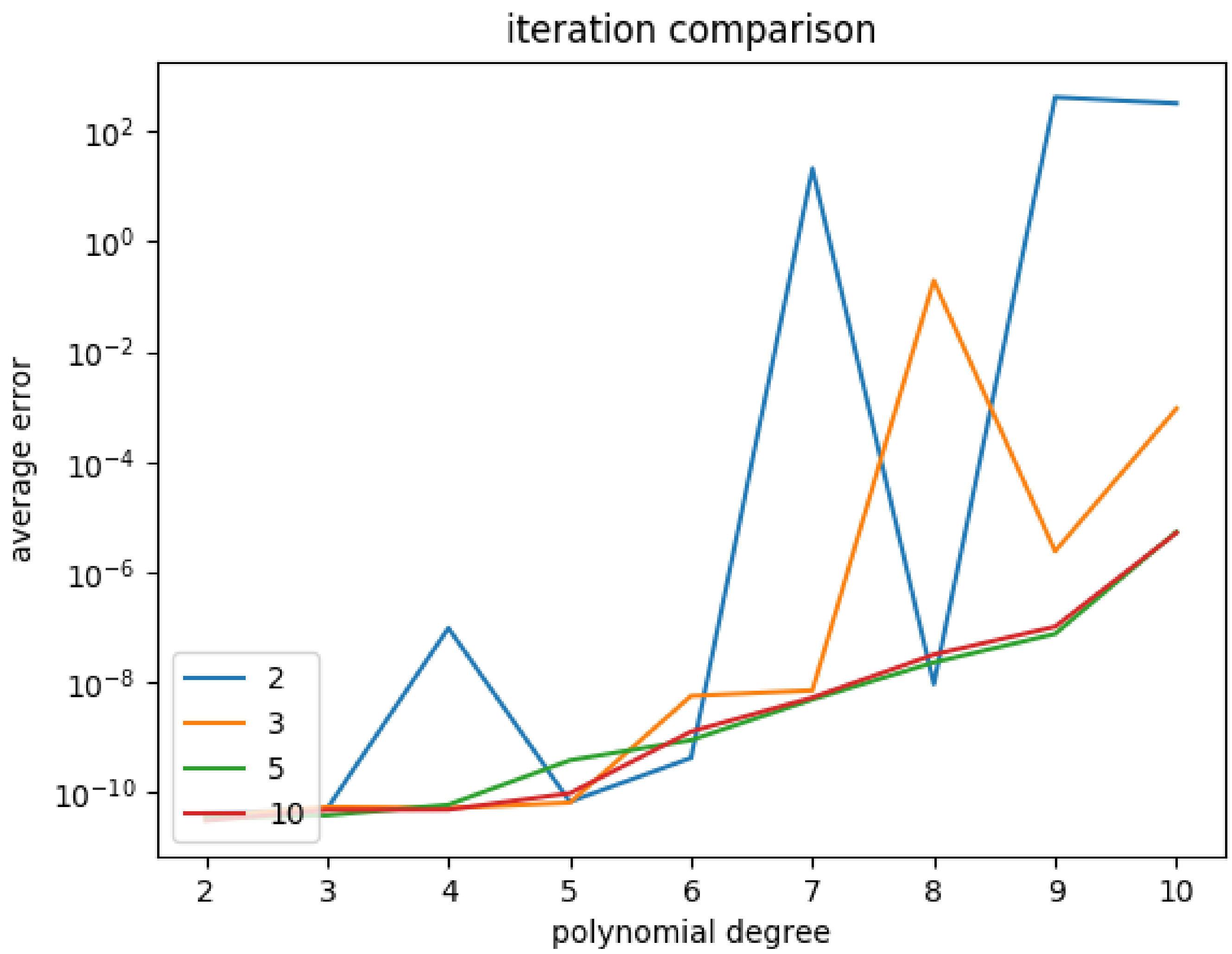
Initial Zero Guess

Observation 2

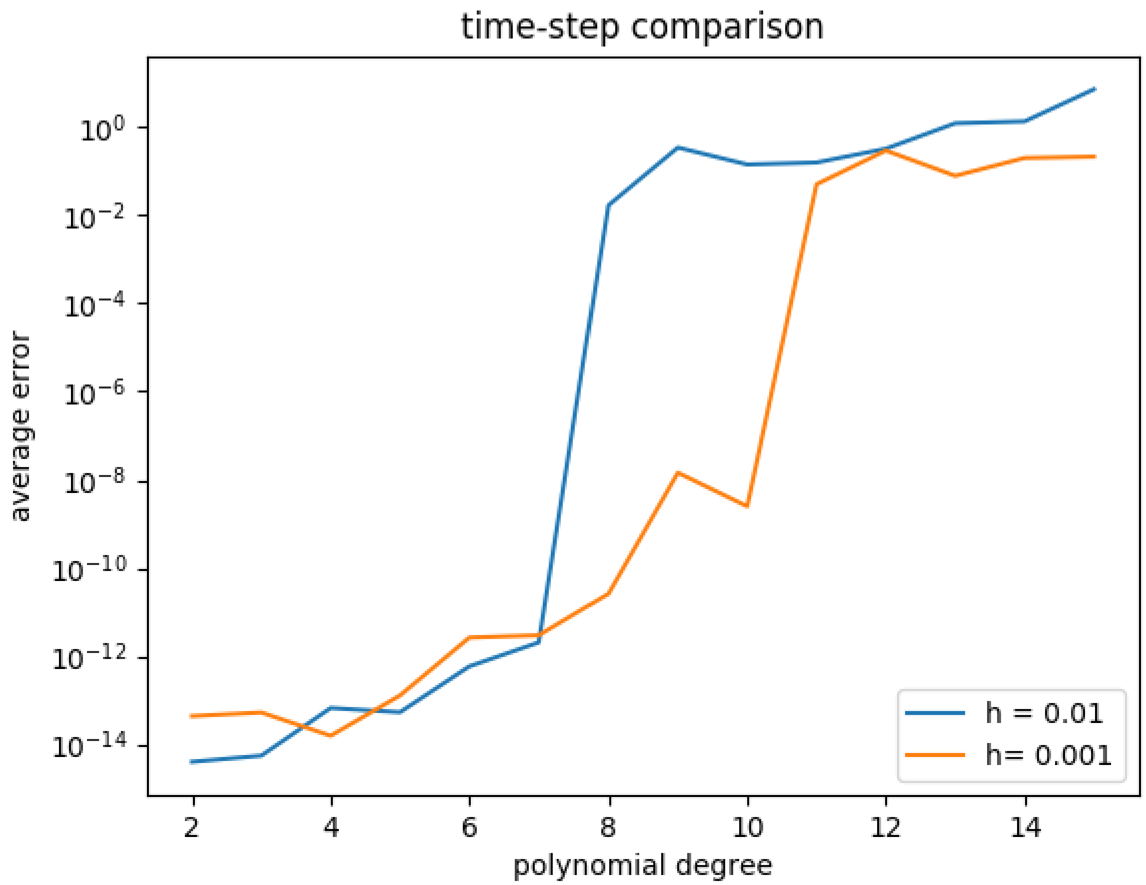


Some Concluding thoughts…

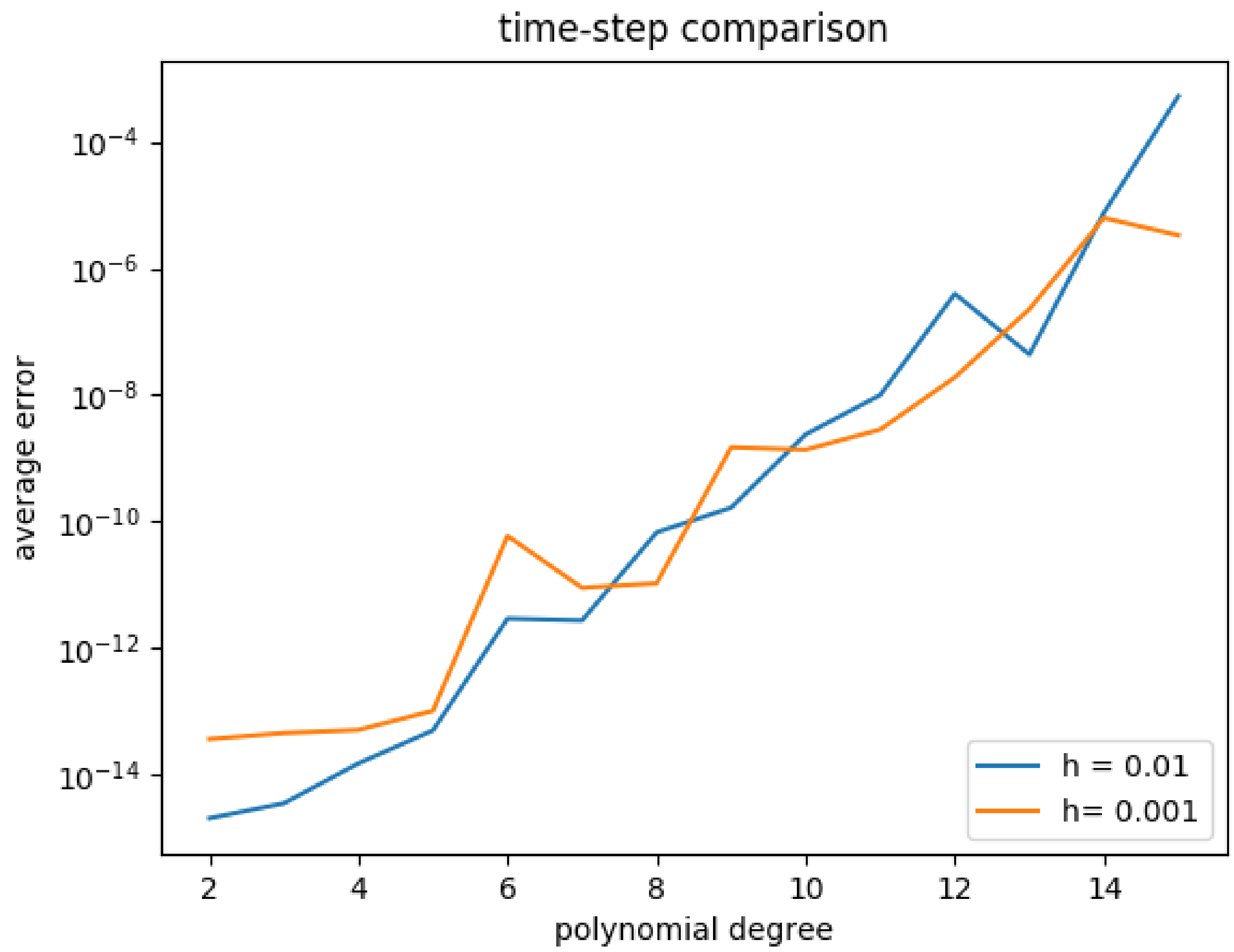
Extra slides in case needed



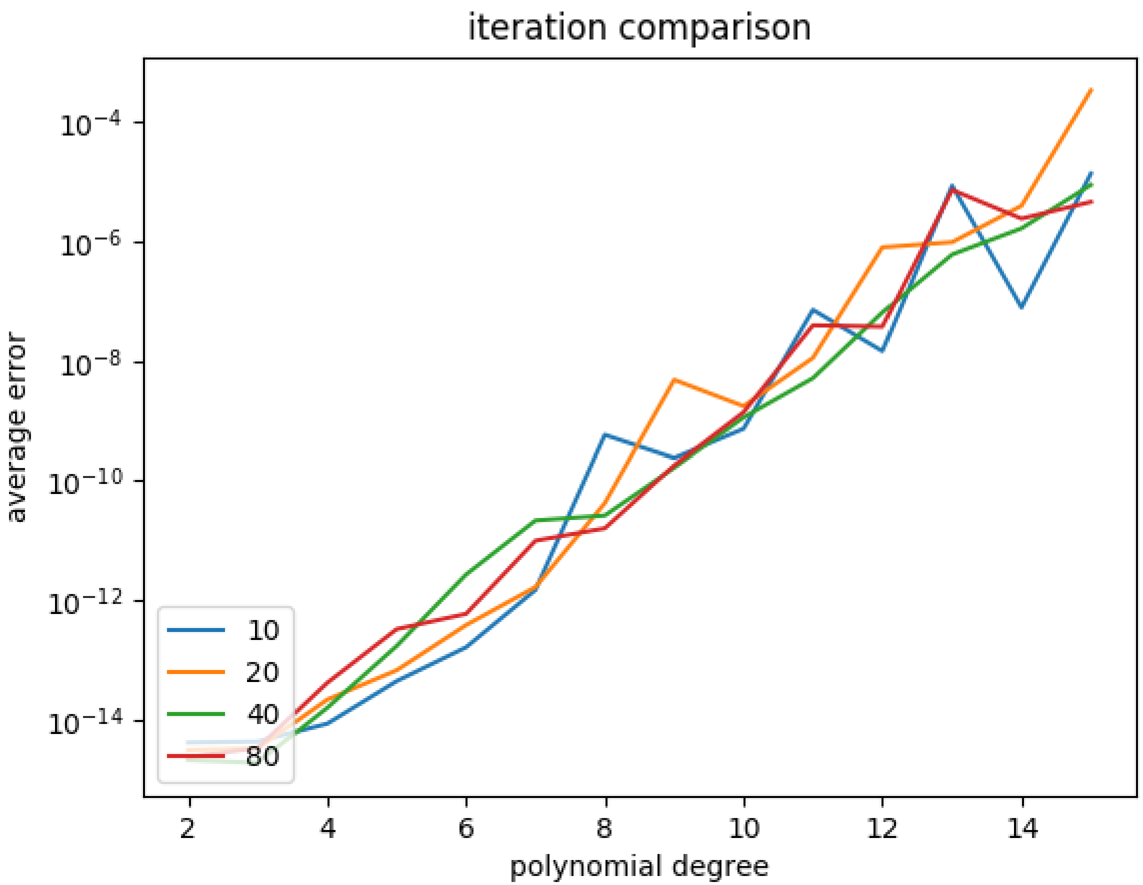
$$-1000 \leq y_i(0) \leq 1000$$
$$-1000 \leq x_i \leq 1000$$
$$h = 0.001$$



$$-10 \leq y_i(0) \leq 10$$
$$-1.000001 \leq x_i \leq 1.000001$$
$$10\ iterations$$



$$-10 \leq y_i(0) \leq 10$$
$$-1.000000001 \leq x_i \leq 1.000000001$$
$$10 \ iterations$$



$$-10 \le y_i(0) \le 10$$
$$-1.000001 \le x_i \le 1.000001$$
$$h = 0.01$$