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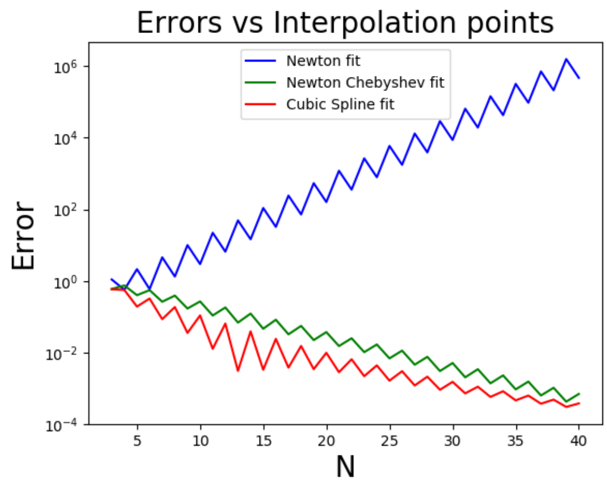
Exercise 4



Let us denote the newton interpolation polynomial as P, the newton interpolation with chebyshev zeros as Q and the natural cubic spline interpolation as S. The first thing we notice by looking at the generated graph is that the S offers the least error, Q stays small, but above S for all values of N and P offers the greatest deviation from the function. With this in mind we will address each interpolation individually.

Newton Interpolating Polynomial P

We know that the general form of a newton interpolating polygon for $N = n + 1$ points:

$$P_n(x) = a_0 + a_1(x - x_1) + a_2(x - x_1)(x - x_2) + \ldots + a_n(x - x_1)(x - x_2)\ldots(x - x_{n-1})(x - x_n)$$

Since we take our interval and pick points that are a constant distance away from their neighbor, the polynomial will be centered at $x = 0$. What this means is that near the boundaries, the polynomial will increase at a much larger rate than near $x = 0$. This means that when the polynomial reaches x_1 or x_n, it will continue increasing at rate based on its order, until it reaches the end of the interval at $x = -5$ or $x = 5$. As we increase the number of points, the order of the polynomial increases, thus the rate at which the polynomial increases after it reaches x_1 or x_n increases. This means the value the polynomial reaches at $x = -5$ or $x = 5$ grows with very increase in the order of the polynomial. Since f has a minimum at $x = -5$ and $x = 5$, a maximum error is achieved at $x = -5$ or at $x = 5$

Intuitively, when more points are added into the interpolation, the polynomial is forced pass through more and more specific points are are closer and closer together. However, near the edge of the boundary the polynomial has a faster changing curvature. In order to fit through all those points the polynomial must "oscillate" really quickly. This is a case of Runge Phenomenon and is the cause of our error.

From the consideration in class, we know that the error of a polynomial of this form is given as follows:

$$f(x) - P_n(x) = \frac{(x - x_1)(x - x_2)\ldots(x - x_N)}{N!} f^{(N)}(c) \qquad (1)$$

As mentioned prior, empirically we have observed that $x = -5$ or 5 yields the largest error for P for $N = 3\ldots40$. T. Let us evaluate the polynomial at -5. From this we obtain:

$$(x - x_1)(x - x_2)\ldots(x - x_N) = (-5 + x_1)(-5 - x_2)\ldots(-5 - x_N)$$

In our case the points $x_1, \ldots x_N$ are in the interval $[-5, 5]$. Since they are equally spaced, we know that for an interpolation of N points we have:

$$x_1 = -5 + \frac{10}{2N} \quad x_2 = -5 + \frac{10}{2N} + \frac{10}{N} = -5 + \frac{5+10}{N}, \quad \dots \quad , x_n = -5 + \frac{5+10(n-1)}{N}$$

So for any j we have:

$$x_j = -5 + \frac{10j-5}{N}$$

This implies then:

$$(x-x_1)(x-x_2)\dots(x-x_N) = \left(\frac{5}{N}\right)\left(\frac{15}{N}\right)\dots\left(\frac{10N-5}{N}\right)$$

Placing this in equation (1) we obtain:

$$f(x) - P_n(x) = \frac{\prod_{j=0}^{N}(5+10j)}{N^N N!} f^{(n)}(c) = \frac{\prod_{j=1}^{n}(5+10j)}{N^n n!} f^{(n)}(c)$$

Checked empirically, the ratio

$$\frac{\prod_{j=0}^{N}(5+10j)}{N^N N!} < 1 \quad \text{for all } N > 12$$

This implies that after some value of N in the range $3 < N < 12$, the majority of the error begins to be supplied by the value $f^{(n)}(c)$.

Newton Interpolating Polynomial with Chebyshev zeros Q

For this polynomial, our $x_1, x_2 \dots x_n$ are the zeros of the Chebyshev polynomial transferred to the interval $[-5, 5]$. For a newton polynomial of degree n this gives the best N points to interpolate through. Unlike the interpolating polynomial P, Q uses x values that are more densely packed at the edges of the polynomial. This means that when the order of the polynomial increases and its curvature near the boundary increases, it is forced to fit through more points. This disallows the polynomial to oscillate near the boundary, and thus reduces error.

Here we get a bit more specific. The error of this interpolation is, again, given by:

$$f(x) - P_n(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_N)}{N!} f^{(N)}(c)$$

However our choice of interpolation points allow us to have the following inequality:

$$|(x-x_1)\dots(x-x_N)| \leq \frac{\left(\frac{b-a}{2}\right)^N}{2^{N-1}} = \frac{5^N}{2^{N-1}} \qquad (2)$$

This means that:

$$f(x) - P_n(x) \leq \frac{5^N}{2^{N-1}N!} f^{(N)}(c)$$

Empirically, I found that for $N \geq 6$, $5^N < 2^{N-1}N!$. Thus as N increases past 6, $\frac{5^N}{2^{N-1}N!}$ begins tending towards zero. The derivative $f^{(n)}(c)$ not big enough to compete with $\frac{5^N}{2^{N-1}N!}$ is not able to keep up, and thus the error keeps decreasing.

The question is then what happens for $N = 3, 4, 5$. For these values of N , $\frac{5^n}{2^{n-1}n!} > 1$. I empirically checked that $f^{(5)}(c) < 101$, $f^{(4)}(c) \leq 24$, $f^{(3)}(c) < 4.7$. This would suggest that the error for these points should be relatively high , however, the bounds we are using are inequalities, and thus our error is able to be much smaller. This implies that the c which satisfies our error inequality is such that $f^{(N)}(c)$ is small. The Chebyshev interpolation is thus able to pick points that would guarantee that $f^{(N)}$ is evaluated at points c for which the result would be small.

Cubic Splicing Interpolation S

The cubic splice S yields the best results by far. This is because with addition of a point, we are not increasing the number of points interpolated by a polynomial in S, but rather bring the interpolated points of each polynomial in S closer together. This means that the more points we have, the smaller of an interval each polynomial in S has to cover. A smaller interval means less space for the polynomial to "oscillate" in as well as more correct points to go through. Thus, by continuously increasing the number of sampled points, we are able to get better and better precision.