Numerical Methods

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Problem 1

Let $f_n \in span\{e_1, e_2, \ldots, e_n\}$ and $f_{n+1} \in span\{e_1, e_2, \ldots, e_n, e_{n+1}\}$. We consider then:

$$\begin{aligned} d^2(f, f_n) &= \langle f - f_n, f - f_n \rangle = \langle f, f \rangle + \langle f_n, f_n \rangle - 2\langle f, f_n \rangle \\ &= \|f\|^2 + \langle \sum_{i=1}^{n} c_i e_i, \sum_{i=1}^{n} c_i e_i \rangle - 2\langle f, \sum_{i=1}^{n} c_i e_i \rangle \\ &= \|f\|^2 + \sum_{i=1}^{n} c_i^2 - 2\sum_{i=1}^{n} c_i \langle f, e_i \rangle \end{aligned}$$

It is shown in the notes that $c_i = \langle f, e_i \rangle$. We have then:

$$d^2(f, f_n) = \|f\|^2 - \sum_{i=1}^{n} \langle f, e_i \rangle \langle f, e_i \rangle$$

Similarly, we have that:

$$d^2(f, f_{n+1}) = \|f\|^2 - \sum_{i=1}^{n+1} \langle f, e_i \rangle \langle f, e_i \rangle$$

Since the inner product is always equal to or greater than 0, we have that

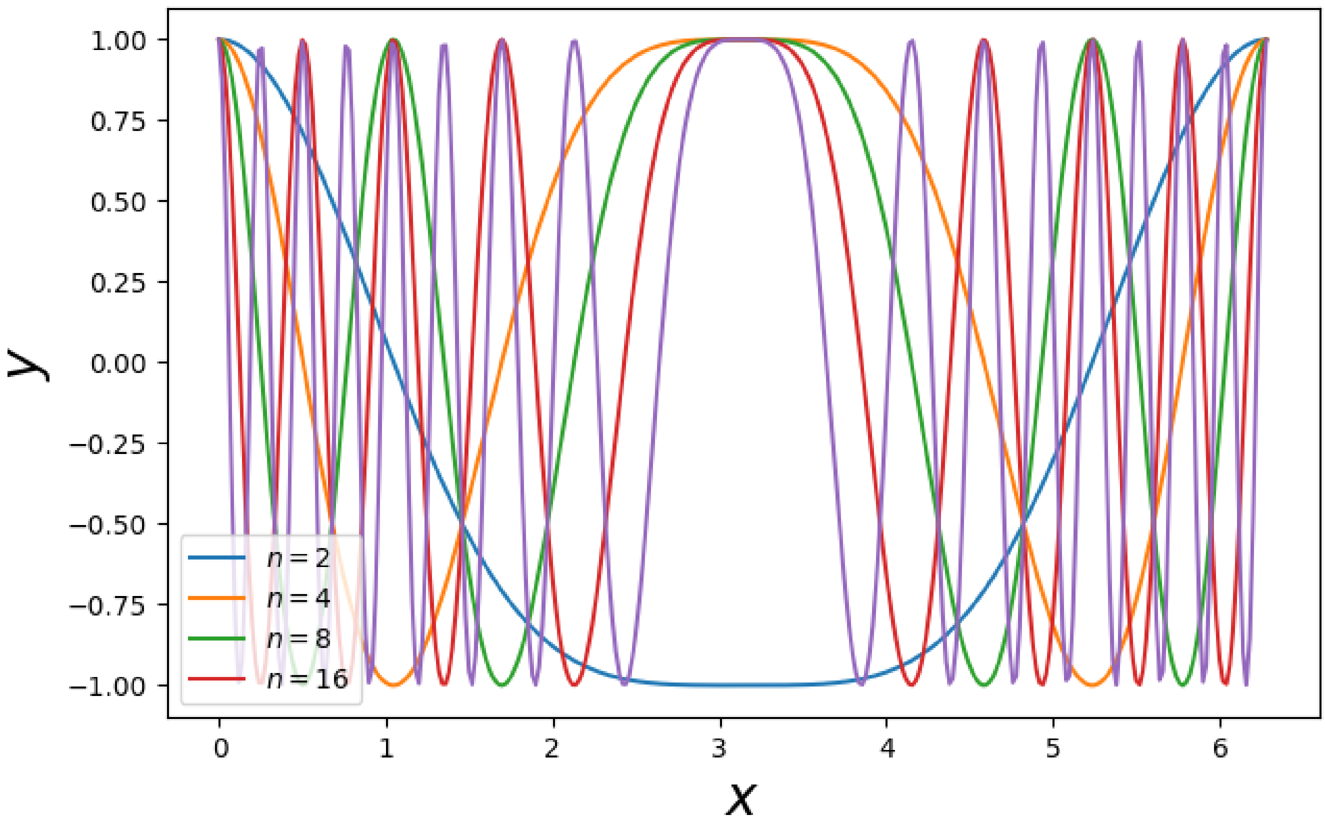
$$\begin{aligned} &\sum_{i=1}^{n+1} \langle f, e_i \rangle \langle f, e_i \rangle \geq \sum_{i=1}^{n} \langle f, e_i \rangle \langle f, e_i \rangle \\ &\implies -\sum_{i=1}^{n+1} \langle f, e_i \rangle \langle f, e_i \rangle \leq -\sum_{i=1}^{n} \langle f, e_i \rangle \langle f, e_i \rangle \\ &\implies \|f\|^2 - \sum_{i=1}^{n+1} \langle f, e_i \rangle \langle f, e_i \rangle \leq \|f\|^2 - \sum_{i=1}^{n} \langle f, e_i \rangle \langle f, e_i \rangle \\ &\implies d^2(f, f_n) \geq d^2(f, f_{n+1}) \\ &\implies d(f, f_n) \geq d(f, f_{n+1}) \end{aligned}$$

Problem 2

To examine this problem we remind ourselves the definition of $|C_k|$ in terms of real coefficients:

$$|C_k| = \left| \frac{A_k - iB_k}{2} \right| \qquad \text{where } A_k = \frac{1}{\pi}\int_0^{2\pi} f(x)cos(kx) \text{ and } B_k = \frac{1}{\pi}\int_0^{2\pi} f(x)sin(kx)$$

The two important quantities that we want to observe are $f(x)sin(kx)$ and $f(x)cos(kx)$, as they determine the value of the integrals. Let us observe the graphs of $f(x) = \cos(n\sin(\frac{x}{2})\pi)$ for $n = 2, 4, 8, 16$:



An important feature to notice is that the frequency of oscillations in the graph increases as n increases. This means that the value of $f(x)$ changes at a faster rate when n increases. Now take two consecutive peaks with oppositely signed ordinates. Denote these peaks p_1 and p_2 with absicass x_1 and x_2 respectively. All peaks have an ordinate of 1 or -1. A higher n implies these peaks are closer together. Let n be small, implying small frequency of oscillations and distant consecutive peaks. If we take either $\sin(kx)$ or $\cos(kx)$ with a low k, it will imply that the $f(x)\sin(kx)$ and $f(x)\cos(kx)$ will have unknown ordinates at x_1 and x_2. However, we can increment k to increase the frequency of $\sin(kx)$ and $\cos(kx)$. We want to increase k to increase this frequency up to a point where:

$$\cos(kp_1) \approx -\cos(kp_2) \qquad \sin(kp_1) \approx -\sin(kp_2) \qquad (1)$$

This will allow the integral of one oscillation of the product of $f(x)$ and $sin(kx)$ and $f(x)$ and $cos(kx)$ to be close to 0.

When n is small, it only takes a small k value to make sure condition (1) is fulfilled. However, when n increases, a larger k is required to counteract the large frequency of $f(x)$. This is why the value of k_{cut} increases as n increases.

Problem 3

At the highest resolution $h = \frac{2\pi}{512}$. An example of a finite difference formula that gives an error of order 2 is given by:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{6}f^{'''}(c) \qquad \text{where } 0 < c < 2\pi$$

We see that the error term contains $f^{'''}(c)$. For c we pick the term such that $f^{'''}(c)$ is the maximum of the function over this interval. Numerically, I found that $f^{'''}(c) < 245$. Thus we now evaluate the error term:

$$\text{Error} = O(h^2)f^{'''}(c) \approx 245h^2 = \frac{245 \cdot 4\pi^2}{\cdot 512^2} \approx 0,03689656186320$$

This implies that we should expect the result to be accurate to 1 decimal space.

We now consider a fourth order difference formula. We know that the error term of such a formula would be in the form:

$$O(h^4)f^{(5)}(c) \qquad \text{where } 0 < c < 2\pi$$

Numerically, we find that $f^{(5)}(c) < 10200$. We have thus:

$$\text{Error} = O(h^4)f^{(5)}(c) \approx \frac{16\pi^4}{512^4} \cdot 10200 \approx 2,31334177904488719024623602960 44e-4.$$

This implies that we can expect the result to be accurate to 3 decimal places.

Problem 4

It is mentioned in the notes that the discrete Fourier transformation has good approximations for functions that are 2π periodic and smooth. We will show that the function given in the problem is not smooth when considered as a periodic function.

First we show that the function is periodic on $[0, 2\pi]$:

$$f(0) = \frac{1}{1 + (0 - \pi)^2} = \frac{1}{1 + \pi^2} = \frac{1}{1 + (\pi)^2} = f(2\pi)$$

However we will now consider the derivative:

$$f'(x) = -\frac{2\,(x - \pi)}{\left((x - \pi)^2 + 1\right)^2}$$

We now evaluate the function at the endpoints:

$$f'(0) = \frac{2\pi}{(x^2 + 1)^2} \qquad f'(2\pi) = -\frac{2\pi}{(x^2 + 1)^2}$$

We see then that $f'(0) \neq f'(2\pi)$ implying that the function is not smooth, its first derivative at $x = 2k\pi$ for $k \in Z$ being undefined.

In the problem we are trying to find the derivative of the function at n given points which are evenly spaced in the interval $[0, 2\pi)$. Since the derivative at 0 is undefined, it is no surprise that for all $n > 3$ the biggest error between the actual derivative and the Fourier approximation of the derivative is the same and occurs at $x = 0$ (for $n = 3$ there is a derivative approximation that is worse than the false value at $x = 0$).