Numerical Methods

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Problem 1

We begin by creating a newton coefficient tree as follows:

$$\begin{array}{lllll}
f[x-h] & & & & \\
 & f[x-h \quad x] & & & \\
f[x] & & f[x-h \quad x \quad x+h] & & \\
 & f[x \quad x+h] & & f[x-h \quad x \quad x+h \quad x+2h] & \\
f[x+h] & & f[x \quad x+h \quad x+2h] & & \\
 & f[x+h \quad x+2h] & & & \\
f[x+2h] & & & &
\end{array}$$

We now evaluate each of the terms:

$$\begin{aligned}
f[x] &= f(x)\\
f[x-h] &= f(x-h)\\
f[x+h] &= f(x+h)\\
f[x+2h] &= f(x+2h)\\
f[x-h \quad x] &= \frac{f(x)-f(x-h)}{h}\\
f[x \quad x+h] &= \frac{f(x+h)-f(x)}{h}\\
f[x+h \quad x+2h] &= \frac{f(x+2h)-f(x+h)}{h}\\
f[x-h \quad x \quad x+h] &= \frac{\frac{f(x+h)-f(x)}{h}-\frac{f(x)-f(x-h)}{h}}{2h} = \frac{f(x+h)-2f(x)+f(x-h)}{2h^2}\\
f[x \quad x+h \quad x+2h] &= \frac{\frac{f(x+2h)-f(x+h)}{h}-\frac{f(x+h)-f(x)}{h}}{2h} = \frac{f(x+2h)-2f(x+h)+f(x)}{2h^2}\\
f[x-h \quad x \quad x+h \quad x+2h] &= \frac{\frac{f(x+2h)-2f(x+h)+f(x)}{2h^2}-\frac{f(x+h)-2f(x)+f(x-h)}{2h^2}}{3h} = \frac{f(x+2h)-3f(x+h)+3f(x)-f(x-h)}{6h^3}
\end{aligned}$$

Now we examine the following the polynomial approximations of a function the derivative of function f, with newton polynomial interpolated at $x-h, x, x+h$.

$$\begin{aligned}
f(y) \approx P_2(y) &= f[x-h] + (y-x+h)f[x-h \quad x] + (y-x)(y-x+h)f[x-h \quad x \quad x+h]\\
\implies f'(y) \approx P_2'(y) &= 0 + f[x-h \quad x] + (2y-2x+h)f[x-h \quad x \quad x+h]\\
&= \frac{f(x)-f(x-h)}{h} + (2y-2x+h)\frac{f(x+h)-2f(x)+f(x-h)}{2h^2}
\end{aligned}$$

We now let $y = x$ to obtain:

$$
\begin{aligned}
f'(x) \approx P_2'(x) &= \frac{f(x)-f(x-h)}{h} + \frac{f(x+h)-2f(x)+f(x-h)}{2h} \\
&= \frac{2f(x)-2f(x-h)+f(x+h)-2f(x)+f(x-h)}{2h} \\
&= \frac{f(x+h)-f(x-h)}{2h}
\end{aligned}
$$

Now we examine the following the polynomial approximations of a function the derivative of function f, with newton polynomial interpolated at $x-h, x, x+h, x+2h$.

$$
\begin{aligned}
f(y) \approx P_3(y) = f[x-h] &+ (y-x+h)f[x-h \;\; x] + (y-x)(y-x+h)f[x-h \;\; x \;\; x+h] \\
&+ (y-x-h)(y-x)(y-x+h)f[x-h \;\; x \;\; x+h \;\; x+2h]
\end{aligned}
$$

$$
\begin{aligned}
\implies f'(y) \approx P_3'(y) &= 0 + f[x-h \;\; x] + (2y-2x+h)f[x-h \;\; x \;\; x+h] \\
&\quad + (3y^2-6xy+3x^2-h^2)f[x-h \;\; x \;\; x+h \;\; x+2h] \\
&= \frac{f(x)-f(x-h)}{h} + (2y-2x+h)\frac{f(x+h)-2f(x)+f(x-h)}{2h^2} \\
&\quad + (3y^2-6xy+3x^2-h^2)\frac{f(x+2h)-3f(x+h)+3f(x)-f(x-h)}{6h^3}
\end{aligned}
$$

Now we let $x=y$ to obtain:

$$
\begin{aligned}
f'(x) \approx P_3'(x) &= \frac{f(x)-f(x-h)}{h} + \frac{f(x+h)-2f(x)+f(x-h)}{2h} - \frac{f(x+2h)-3f(x+h)+3f(x)-f(x-h)}{6h} \\
&= \frac{6f(x)-6f(x-h)+3f(x+h)-6f(x)+3f(x-h)-f(x+2h)+3f(x+h)-3f(x)+f(x-h)}{6h} \\
&= \frac{-2f(x-h)-3f(x)+6f(x+h)-f(x+2h)}{6h}
\end{aligned}
$$

Problem 3

From Problem 2 we have that:

$$
\begin{aligned}
P'(y) = \frac{f(x)-f(x-h)}{h} &+ (2y-2x+h)\frac{f(x+h)-2f(x)+f(x-h)}{2h^2} \\
+(3y^2-6xy+3x^2-h^2)&\frac{f(x+2h)-3f(x+h)+3f(x)-f(x-h)}{6h^3} \\
\implies P^{(3)}(y) &= \frac{f(x+2h)-3f(x+h)+3f(x)-f(x-h)}{h^3}
\end{aligned}
$$

We now evaluate the Taylor expansion of all of the terms in the numerator of $P^{(3)}(y)$:

$$
\begin{array}{llllllll}
-1f(x-h) & = & -f(x) & +hf'(x) & -\dfrac{h^2f''(x)}{2} & +\dfrac{h^3f^{(3)}(x)}{6} & -\dfrac{h^4f^{(4)}(c)}{24} \\
3f(x) & = & +3f(x) & +0 & -0 & +0 & -0 \\
-3f(x+h) & = & -3f(x) & -3hf'(x) & -\dfrac{3h^2f''(x)}{2} & -\dfrac{3h^3f^{(3)}(x)}{6} & -\dfrac{3h^4f^{(4)}(c)}{24} \\
1f(x+2h) & = & -f(x) & +2hf'(x) & +\dfrac{4h^2f''(x)}{2} & +\dfrac{8h^3f^{(3)}(x)}{6} & +\dfrac{16h^4f^{(4)}(c)}{24}
\end{array}
$$

where $x-h<c<x+2h$. We now combine the terms together to obtain:

$$\frac{f(x+2h)-3f(x+h)+3f(x)-f(x-h)}{h^3}=\frac{h^3f^{(3)}(x)}{h^3}+\frac{12h^4f^{(4)}(c)}{24h^3}$$

We therefore have the third derivative of f defined as:

$$f^{(3)}(x)=\frac{f(x+2h)-3f(x+h)+3f(x)-f(x-h)}{h^3}-h\frac{f^{(4)}(c)}{2}$$

We can see that this expression has an error term scaling with $O(h)$.

We now observe what happens when we change the evaluating value of our points by changing $x=z-\frac{h}{2}$. We have then:

$$P^{(3)}(y)=\frac{f(z+\frac{3h}{2})-3f(z+\frac{h}{2})+3f(z-\frac{h}{2})-f(z-\frac{3h}{2})}{h^3}$$

We now apply the same process to obtain the following:

$$\begin{aligned}
-1f(z-\frac{3h}{2}) &= -f(z) &&+\frac{3hf'(z)}{2} &&-\frac{9h^2f''(z)}{8} &&+\frac{27h^3f^{(3)}(z)}{48} &&-\frac{81h^4f^{(4)}(x)}{216}+\frac{243h^5f^{(5)}(c)}{3840}\\
3f(z-\frac{h}{2}) &= +3f(z) &&-\frac{3hf'(z)}{2} &&+\frac{3h^2f''(z)}{8} &&-\frac{3h^3f^{(3)}(z)}{48} &&+\frac{3h^4f^{(4)}(x)}{216}-\frac{3h^5f^{(5)}(c)}{3840}\\
-3f(z+\frac{h}{2}) &= -3f(z) &&-\frac{3hf'(z)}{2} &&-\frac{3h^2f''(z)}{8} &&-\frac{3h^3f^{(3)}(z)}{48} &&-\frac{3h^4f^{(4)}(x)}{216}-\frac{3h^5f^{(5)}(c)}{3840}\\
1f(z+\frac{3h}{2}) &= +f(z) &&+\frac{3hf'(z)}{2} &&+\frac{9h^2f''(z)}{8} &&+\frac{27h^3f^{(3)}(z)}{48} &&+\frac{81h^4f^{(4)}(x)}{216}+\frac{243h^5f^{(5)}(c)}{3840}
\end{aligned}$$

where $z-\frac{3h}{2}<c<z+\frac{3h}{2}$. We now combine these terms to obtain:

$$\frac{f(z+\frac{3h}{2})-3f(z+\frac{h}{2})+3f(z-\frac{h}{2})-f(z-\frac{3h}{2})}{h^3}=\frac{h^3f^{(3)}(z)}{h^3}+\frac{h^5f^{(5)}(z)}{8h^3}$$

We therefore have that the third derivative of f at z is given by:

$$f^{(3)}(z)=\frac{f(z+\frac{3h}{2})-3f(z+\frac{h}{2})+3f(z-\frac{h}{2})-f(z-\frac{3h}{2})}{h^3}-\frac{h^2f^{(5)}(z)}{8}$$

We can see that this expression has an error term scaling with $O(h^2)$.

Thus, by selecting equally spaced points, we are able to get the error to scale with an additional order with respect to h.

Problem 4

Part 1

The reason that the error of the midpoint approximation described in this problem grows in $O(h^2)$ is because of symmetry and breaking the integration interval (a,b) into sub-intervals. First we examine symmetry. Let $w=a+(b-a)/2$. By the Taylor series centered at w we have:

$$f(x) = f(w) + (x-w)f'(w) + \frac{1}{2}(x-w)^2 f''(c_x)$$

where $c_x \in (a,b)$ and depends on the variable x. We now evaluate the integral of both sides:

$$\int_a^b f(x)dx = \int_a^b f(w)dx + (x-w)\int_a^b (x-w)dx + \frac{1}{2}\int_a^b f''(c_x)(x-w)^2 dx$$

Now, by the intermediate value theorem, $f''(c_x) = f(c)$ for some $c \in (a,b)$. Also we denote $h = b-a$. We have then:

$$\begin{aligned}
&= (b-a)f(w) + (x-w)\int_a^b (x-w)dx + \frac{1}{2}f''(c_x)\int_a^b (x-w)^2 dx \\
&= hf(w) + 0 + \frac{f''(c_x)}{6}\Big[(x-w)^3\Big]_a^b \\
&= hf(w) + \frac{hf''(c_x)}{6} \cdot \big(3w^2 - 3\,(b+a)\,w + b^2 + ab + a^2\big)
\end{aligned}$$

This is where our symmetry comes in handy. We substitute $w = a + (b-a)/2$ to obtain:

$$\begin{aligned}
&= hf(w) + \frac{hf''(c_x)}{6} \cdot \big(3(a+(b-a)/2)^2 - 3\,(b+a)\,(a+(b-a)/2) + b^2 + ab + a^2\big) \\
&= hf(w) + \frac{hf''(c_x)}{6} \cdot \frac{a^2 - 2ab + b^2}{4} \\
&= hf(w) + \frac{hf''(c_x)}{24} \cdot (b-a)^2 \\
&= hf(w) + \frac{hf''(c_x)}{24} \cdot h^2
\end{aligned}$$

Thus we conclude that:

$$\int_a^b f(x)dx = hf(w) + \frac{h^3}{24}f''(c)$$

This implies that the error of the midpoint formula grows with $O(h^3)$, where $h = (b-a)$.

Now that we know the midpoint approximation over an interval (a,b), we split (a,b) into N sub-intervals $(a+(k-1)h, a+(k)h)$ where $0 < k \le N$ and where $h = (b-a)/N$. We can then write:

$$\int_a^b f(x)dx = \sum_{i=1}^{N}\int_{a+(i-1)h}^{a+ih} f(x)dx = \sum_{i=1}^{N}\left(hf(w_i) + h^3\frac{f''(c_i)}{24}\right)$$

where $a+(i-1)h < c_i < a+ih$ and $w_i = a + (i - \frac{1}{2})h$. We now separate the sum as follows:

$$\sum_{i=1}^{N} hf(w_i) + h^3 f''(c_i) = h\sum_{i=1}^{N} f(w_i) + h^3\sum_{i=1}^{N}\frac{f''(c_i)}{24}$$

By the generalized intermediate theorem, for some c such that $a < c < b$ we have:

$$\sum_{i=1}^{N} f''(c_i) = f''(c)$$

We can therefore write:

$$\sum_{i=1}^{N} f(w_i) + h^3 \sum_{i=1}^{N} f''(c_i) = \sum_{i=1}^{N} f(w_i) + h^3 N \frac{f''(c)}{24}$$

We now remember that $N = \frac{b-a}{h}$. We have thus:

$$\int_a^b f(x)dx = \sum_{i=1}^{N} f(w_i) + h^3 \frac{b-a}{h} f''(c) = \sum_{i=1}^{N} f(w_i) + h^2(b-a)\frac{f''(c)}{24}$$

Thus we conclude that by using the midpoint method on small partitions on our interval (a, b), we get an error that is $O(h^2)$, where $h = (b - a)/N$. We will show that the error of the described trapezoid method is also $O(h^2)$ with $h = (b - a)/N$ in the answer of part 2.

Part 2

The error of the trapezoid integral approximation over interval a, b is given by:

$$\text{Error} = \int_a^b E(x)dx = -\frac{(b-a)^3}{12} f''(c)$$

We are dividing our interval (a, b) into N pieces, so let $h = \frac{(b-a)}{N}$. We have then:

$$\int_a^b E(x)dx = \sum_{i=1}^{N} \int_{a+(i-1)h}^{a+ih} E(x)dx = -\sum_{i=1}^{N} \frac{(h)^3}{12} f''(c_i)$$

Where $a + (i - 1)h < c_i < a + ih$. We examine the expression further:

$$-\sum_{i=1}^{N} \frac{(h)^3}{12} f''(c_i) = -N\frac{(h)^3}{12} \sum_{i=1}^{N} f''(c_i)$$

We replace now $N = (b - a)/h$:

$$= \frac{(a-b)h^2}{12} \sum_{i=1}^{N} f''(c_i)$$

Then, by the generalized intermediate theorem, for some c such that $a < c < b$ we have:

$$\sum_{i=1}^{N} f''(c_i) = f''(c)$$

We have thus:

$$\text{Error}(h) = \int_a^b E(x)dx = \frac{(a-b)h^2}{12} f''(c)$$

Proving that our approximation indeed has $O(h^2)$ error, where $h = (b - a)/N$.