Assignment 2 Report

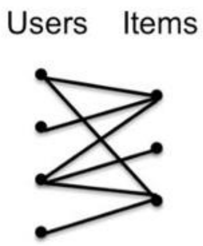
Name(s):

ID(s):

In each the following parts, you should include relevant mathematical derivations, matrix computations, and theoretical analysis. Based on my provided prompts, you are expected to analyze the results and summarize your insights with proper mathematical justification.

Consider a user-item bipartite graph where each edge in the graph between user U to item I, indicates that user U likes item I. We also represent the ratings matrix for this set of users and items as R, where each row in R corresponds to a user and each column corresponds to an item. If user i likes item j, then $R_{i,j} = 1$, otherwise $R_{i,j} = 0$. Also assume we have m users and n items, so matrix R is $m \times n$.

Let's define a matrix P, $m \times n$, as a diagonal matrix whose i-th diagonal element is the degree of user node i, $i.e.$ the number of items that user i likes. Similarly, a matrix Q, $n \times n.$, is a diagonal matrix whose i-th diagonal element is the degree of item node i or the number of users that liked item i. See figure below for an example.



$$R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Figure 1: User-Item bipartite graph.

Part 1: Cosine Similarity Matrix Derivation

Let's define the item similarity matrix, S_I, $n \times n$, such that the element in row i and column j is the cosine similarity of item i and item j which correspond to column i and column j of the matrix R. Show that $S_I = Q^{-1/2} R^T R Q^{-1/2}$, where $Q^{-1/2}$ is defined by $Q_{rc}^{-1/2} = 1/\sqrt{Q_{rc}}$ for all nonzero entries of the matrix, and 0 at all other positions.

Repeat the same question for user similarity matrix, S_U where the element in row i and column j is the cosine similarity of user i and user j which correspond to row i and row j of the matrix R. That is, your expression for S_U should also be in terms of some combination of R, P, and Q.

Your Answer Should Include:

- Step-by-step mathematical derivation for S_I
- Step-by-step mathematical derivation for S_U
- Explanation of the role of normalization matrices $P^{-1/2}$and $Q^{-1/2}$
- Verification that your expressions produce valid cosine similarity values

Remember: Cosine similarity of two vectors $u$and $v$is defined as:

$$\text{cos-sim}(u, v) = \frac{u \cdot v}{\| u \| \| v \|}$$

Part 2: Collaborative Filtering Matrix Derivation

1.The recommendation method using user-user collaborative filtering for user u, can be described as follows: for all items s, compute $r_{u,s} = \sum_{x \in \text{users}} \text{cos-sim}(x, u) \times R_{xs}$ and recommend the k items for which $r_{u,s}$ is the largest.

2.Similarly, the recommendation method using item-item collaborative filtering for user u can be described as follows: for all items s, compute $r_{u,s} = \sum_{x \in \text{items}} R_{ux} \times \text{cos-sim}(x, s)$ and recommend the k items for which $r_{u,s}$ is the largest.

Let's define the recommendation matrix, Γ, $m \times n$, such that $\Gamma(i, j) = r_{i,j}$. Find Γ for both item-item and user-user collaborative filtering approaches, in terms of R, $P$and Q.

Your Answer Should Include:

- Mathematical derivation of Γ for user-user collaborative filtering
- Mathematical derivation of Γ for item-item collaborative filtering
- Matrix-level expressions (not individual element definitions)
- Verification using the hint: For the item-item case, $\Gamma = RQ^{-1/2}R^T RQ^{-1/2}$
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Part 3: Implementation and Analysis

1.Describe your implementation approach for computing the similarity matrices and recommendation matrices. What numerical considerations did you account for (e.g., handling zero divisions, sparse matrices)?

2.What insights did you gain from the mathematical derivations? How do the theoretical expressions translate to practical computational considerations?

Part 4: Extra Credit (optional)

If you explored alternative similarity metrics, different normalization approaches, or extended the theoretical analysis beyond the basic requirements, you may describe them here. The mathematical rigor of your extensions and the depth of your analysis will be considered as extra credit.