

### Existence and uniqueness theorem:

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—that is, if and only if an echelon form of the augmented matrix has *no* row of the form

$$[0 \ \cdots \ 0 \ b] \quad \text{with } b \text{ nonzero}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

### Equivalent ways to write a linear system:

If  $A$  is an  $m \times n$  matrix, with columns  $\mathbf{a}_1, \dots, \mathbf{a}_n$ , and if  $\mathbf{b}$  is in  $\mathbb{R}^m$ , the matrix equation

$$A\mathbf{x} = \mathbf{b} \tag{4}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \tag{5}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n \ \mathbf{b}] \tag{6}$$

### Span of the columns of $A$ :

Let  $A$  be an  $m \times n$  matrix. Then the following statements are logically equivalent. That is, for a particular  $A$ , either they are all true statements or they are all false.

- For each  $\mathbf{b}$  in  $\mathbb{R}^m$ , the equation  $A\mathbf{x} = \mathbf{b}$  has a solution.
- Each  $\mathbf{b}$  in  $\mathbb{R}^m$  is a linear combination of the columns of  $A$ .
- The columns of  $A$  span  $\mathbb{R}^m$ .
- $A$  has a pivot position in every row.

### Decomposition of the solution of a linear system:

Suppose the equation  $A\mathbf{x} = \mathbf{b}$  is consistent for some given  $\mathbf{b}$ , and let  $\mathbf{p}$  be a solution. Then the solution set of  $A\mathbf{x} = \mathbf{b}$  is the set of all vectors of the form  $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$ , where  $\mathbf{v}_h$  is any solution of the homogeneous equation  $A\mathbf{x} = \mathbf{0}$ .

### Characterization of linearly independent set:

An indexed set  $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  of two or more vectors is linearly dependent if and only if at least one of the vectors in  $S$  is a linear combination of the others. In fact, if  $S$  is linearly dependent and  $\mathbf{v}_1 \neq \mathbf{0}$ , then some  $\mathbf{v}_j$  (with  $j > 1$ ) is a linear combination of the preceding vectors,  $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$ .

### Cases that are surely linearly dependent:

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  in  $\mathbb{R}^n$  is linearly dependent if  $p > n$ .

If a set  $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  in  $\mathbb{R}^n$  contains the zero vector, then the set is linearly dependent.

### Invertibility implies uniqueness:

If  $A$  is an invertible  $n \times n$  matrix, then for each  $\mathbf{b}$  in  $\mathbb{R}^n$ , the equation  $A\mathbf{x} = \mathbf{b}$  has the unique solution  $\mathbf{x} = A^{-1}\mathbf{b}$ .

### Properties of the inverse operator:

- a. If  $A$  is an invertible matrix, then  $A^{-1}$  is invertible and

$$(A^{-1})^{-1} = A$$

- b. If  $A$  and  $B$  are  $n \times n$  invertible matrices, then so is  $AB$ , and the inverse of  $AB$  is the product of the inverses of  $A$  and  $B$  in the reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}$$

- c. If  $A$  is an invertible matrix, then so is  $A^T$ , and the inverse of  $A^T$  is the transpose of  $A^{-1}$ . That is,

$$(A^T)^{-1} = (A^{-1})^T$$

### Row equivalence for invertible matrixes:

An  $n \times n$  matrix  $A$  is invertible if and only if  $A$  is row equivalent to  $I_n$ , and in this case, any sequence of elementary row operations that reduces  $A$  to  $I_n$  also transforms  $I_n$  into  $A^{-1}$ .

### The rank theorem:

The dimensions of the column space and the row space of an  $m \times n$  matrix  $A$  are equal. This common dimension, the rank of  $A$ , also equals the number of pivot positions in  $A$  and satisfies the equation

$$\text{rank } A + \dim \text{Nul } A = n$$

### The invertible matrix theorem:

Let  $A$  be a square  $n \times n$  matrix. Then the following statements are equivalent. That is, for a given  $A$ , the statements are either all true or all false.

- a.  $A$  is an invertible matrix.
- b.  $A$  is row equivalent to the  $n \times n$  identity matrix.
- c.  $A$  has  $n$  pivot positions.
- d. The equation  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution.
- e. The columns of  $A$  form a linearly independent set.
- f. The linear transformation  $\mathbf{x} \mapsto A\mathbf{x}$  is one-to-one.
- g. The equation  $A\mathbf{x} = \mathbf{b}$  has at least one solution for each  $\mathbf{b}$  in  $\mathbb{R}^n$ .
- h. The columns of  $A$  span  $\mathbb{R}^n$ .
- i. The linear transformation  $\mathbf{x} \mapsto A\mathbf{x}$  maps  $\mathbb{R}^n$  onto  $\mathbb{R}^n$ .
- j. There is an  $n \times n$  matrix  $C$  such that  $CA = I$ .
- k. There is an  $n \times n$  matrix  $D$  such that  $AD = I$ .
- l.  $A^T$  is an invertible matrix.

- m. The columns of  $A$  form a basis of  $\mathbb{R}^n$ .
- n.  $\text{Col } A = \mathbb{R}^n$
- o.  $\dim \text{Col } A = n$
- p.  $\text{rank } A = n$
- q.  $\text{Nul } A = \{\mathbf{0}\}$
- r.  $\dim \text{Nul } A = 0$

### The basis theorem:

Let  $V$  be a  $p$ -dimensional vector space,  $p \geq 1$ . Any linearly independent set of exactly  $p$  elements in  $V$  is automatically a basis for  $V$ . Any set of exactly  $p$  elements that spans  $V$  is automatically a basis for  $V$ .

### Stability result for linear systems: NALS: stability of a linear system????

### Properties of determinants:

Let  $A$  and  $B$  be  $n \times n$  matrices.

- a.  $A$  is invertible if and only if  $\det A \neq 0$ .
- b.  $\det AB = (\det A)(\det B)$ .
- c.  $\det A^T = \det A$ .
- d. If  $A$  is triangular, then  $\det A$  is the product of the entries on the main diagonal of  $A$ .
- e. A row replacement operation on  $A$  does not change the determinant. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same scalar factor.

### Similar matrices theorem:

If  $n \times n$  matrices  $A$  and  $B$  are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

### The diagonalization theorem:

An  $n \times n$  matrix  $A$  is diagonalizable if and only if  $A$  has  $n$  linearly independent eigenvectors.

In fact,  $A = PDP^{-1}$ , with  $D$  a diagonal matrix, if and only if the columns of  $P$  are  $n$  linearly independent eigenvectors of  $A$ . In this case, the diagonal entries of  $D$  are eigenvalues of  $A$  that correspond, respectively, to the eigenvectors in  $P$ .

### The QR factorization:

If  $A$  is an  $m \times n$  matrix with linearly independent columns, then  $A$  can be factored as  $A = QR$ , where  $Q$  is an  $m \times n$  matrix whose columns form an orthonormal basis for  $\text{Col } A$  and  $R$  is an  $n \times n$  upper triangular invertible matrix with positive entries on its diagonal.

### Equivalent statements for least square problems:

Let  $A$  be an  $m \times n$  matrix. The following statements are logically equivalent:

- The equation  $A\mathbf{x} = \mathbf{b}$  has a unique least-squares solution for each  $\mathbf{b}$  in  $\mathbb{R}^m$ .
- The columns of  $A$  are linearly independent.
- The matrix  $A^T A$  is invertible.

When these statements are true, the least-squares solution  $\hat{\mathbf{x}}$  is given by

$$\hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b} \quad (4)$$

### QR factorization for least square problems:

Given an  $m \times n$  matrix  $A$  with linearly independent columns, let  $A = QR$  be a QR factorization of  $A$  as in Theorem 12. Then, for each  $\mathbf{b}$  in  $\mathbb{R}^m$ , the equation  $A\mathbf{x} = \mathbf{b}$  has a unique least-squares solution, given by

$$\hat{\mathbf{x}} = R^{-1} Q^T \mathbf{b} \quad (6)$$