YOUTUBE PROOF

LET $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ BE A SEQUENCE OF REAL NUMBERS

THE ONLY SEQUENCES THAT HAVE A LIMIT ARE THE CONVERGENT ONE

IF $\{a_n\}_{n\in\mathbb{N}}$ IS CONVERGENT, IT CAN NOT CONVERGE TO TWO DIFFERENT LIMITS

- SUPPOSE THAT $\{a_n\}_{n\in\mathbb{N}}$ CONVERGES TO $a \in \mathbb{R}$ AND $\{a_n\}_{n\in\mathbb{N}}$ CONVERGES ALSO TO $b \in \mathbb{R}$, THAT IS

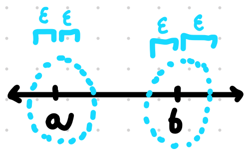
$$a_n \to a \qquad a_n \to b$$

PROOF BY CONTRADICTION

(ASSUME THAT $a \neq b$ TO PROVE A CONTRADICTION, SO $a = b$)

SFC SUPPOSE FOR CONTRADICTION

- $a \neq b$ (SO THAT THE LIMIT IS NOT UNIQUE)



ALSO $b > a$

LET $\varepsilon > 0$ SUCH THAT $\varepsilon = \frac{b-a}{2}$

THE TERMS OF OUR SEQUENCE WILL ALL EVENTUALLY BE LESS THAN $\frac{b-a}{2}$ AWAY FROM THE LIMIT a AND $\frac{b-a}{2}$ AWAY FROM THE LIMIT b

- THEN $\exists N_1 \in \mathbb{N}$ SUCH THAT

$$\forall n > N_1, \ |a_n - a| < \varepsilon \Rightarrow |a_n - a| < \frac{b-a}{2}$$

- THEN $\exists N_2 \in \mathbb{N}$ SUCH THAT

$$\forall n > N_2, \ |a_n - b| < \varepsilon \Rightarrow |a_n - b| < \frac{b-a}{2}$$

TO GUARANTEE BOTH THE RESTRICTION $\forall n > N_1$ AND $\forall n > N_2$

WE HAVE TO BE OVER THE MAXIMUM BETWEEN N_1 AND N_2

LET $N = \max\{N_1, N_2\}$. THEN $\forall n > N$

$$|a_n - a| < \frac{b-a}{2} \qquad\qquad |a_n - b| < \frac{b-a}{2}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$-\frac{b-a}{2} < a_n - a < \frac{b-a}{2} \qquad\qquad -\frac{b-a}{2} < a_n - b < \frac{b-a}{2}$$

$$\frac{a-b}{2} < a_n - a < \frac{b-a}{2} \qquad\qquad \frac{a-b}{2} < a_n - b < \frac{b-a}{2}$$

$$\frac{3a-b}{2} < a_n < \frac{b+a}{2} \qquad\qquad \frac{a+b}{2} < a_n < \frac{3b-a}{2}$$

SINCE $a_n < \frac{b+a}{2}$ AND $\frac{a+b}{2} < a_n$ WE HAVE A CONTRADICTION

SO a MUST BE EQUAL TO b AND THE LIMIT OF A CONVERGENT SEQUENCE IS UNIQUE

IF A SEQUENCE IS CONVERGENT IT CONVERGES TO EXACTLY ONE LIMIT

UNIQUENESS OF THE LIMIT

LET $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ BE A SEQUENCE OF REAL NUMBERS.

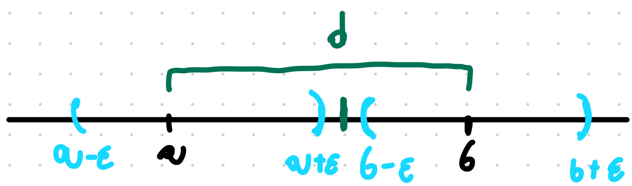
IF $\{a_n\}_{n\in\mathbb{N}}$ IS CONVERGENT, IT CAN NOT CONVERGES TO TWO DIFFERENT LIMITS, THAT IS.

IF $\exists\, a,b \in \mathbb{R}$ SUCH THAT $\lim_n a_n = a$ AND $\lim_n a_n = b$
THEN $a = b$

PROOF

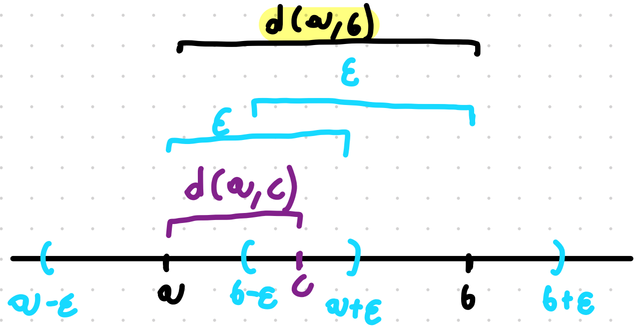
LET $a \neq b$ THEN $d = d(a,b) = |b-a| > 0$

TAKE $0 < \varepsilon \leq d/2$, THEN $\underbrace{(a-\varepsilon, a+\varepsilon)}_{= B_\varepsilon(a)} \cap \underbrace{(b-\varepsilon, b+\varepsilon)}_{= B_\varepsilon(b)} = \emptyset$



IF FACT, BY CONTRADICTION, ASSUME $B_\varepsilon(a) \cap B_\varepsilon(b) \neq \emptyset$

SO $\exists\, x \in B_\varepsilon(a) \cap B_\varepsilon(b)$, THAT IS $d(x,a) < \varepsilon$ AND $d(x,b) < \varepsilon$



BUT $0 < d = d(a,b) \le d(a,x) + d(x,b) < \varepsilon + \varepsilon = \underbrace{2\varepsilon \le d}_{\text{FOR HYPOTESIS } 2\varepsilon \le d} = d(a,b) \neq \emptyset$

BY CONTRADICTION ASSUME $a \neq b$. PICK $0 < \varepsilon \le d/2$

- $\lim_N a_N = a \Rightarrow \exists N_0 \in \mathbb{N}$ S.T. $\forall N \ge N_0$ WE HAVE $a_N \in B_\varepsilon(a)$
- $\lim_N a_N = b \Rightarrow \exists M_0 \in \mathbb{N}$ S.T. $\forall N \ge M_0$ WE HAVE $a_N \in B_\varepsilon(b)$

$\Rightarrow \forall N \ge K_0 = \max(N_0, M_0)$ WE HAVE $a_N \in B_\varepsilon(a)$ AND $a_N \in B_\varepsilon(b)$, THAT IS $a_N \in B_\varepsilon(a) \cap B_\varepsilon(b)$

CONTRADICTION SYNCE

$B_\varepsilon(a) \cap B_\varepsilon(b) = \emptyset$