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EIGENVALUE $\lambda \in \mathbb{R}$ EIGENVECTORS $\underline{x} \in \mathbb{R}^n$ $\underline{x} \neq \underline{0}$ $A \in \mathbb{R}^{n \times n}$

$$A\underline{x} = \lambda \underline{x}$$

$\underline{x}$ IS AN EIGENVECTOR OF A THEN $\alpha \underline{x}$ IS AN EIGENVECTOR OF A (ASSOCIATED TO THE SAME EIGENVALUE λ) FOR ANY $\alpha \in \mathbb{R}, \alpha \neq 0$

$\lambda \in \mathbb{R}$ $A\underline{x} = \lambda \underline{x}$ $(\underline{x} \neq 0)$ EQUIVALENT TO $(A - \lambda I)\underline{x} = \underline{0}$ $(\underline{x} \neq \underline{0})$

$\lambda \in \mathbb{R}$, EIGENVALUE FIND ALL $\underline{x}$ EIGENVECTORS OF A ASSOCIATED TO λ

EXCLUDED

$$\left\{ \begin{array}{l} \underline{x} \in \mathbb{R}^n : A\underline{x} = \lambda \underline{x} \\ \underline{x} \neq \underline{0} \end{array} \right\} = \left\{ \begin{array}{l} \underline{x} \in \mathbb{R}^n : (A - \lambda I)\underline{x} = \underline{0} \\ \underline{x} \neq \underline{0} \end{array} \right\} = \mathrm{Nul}(A - \lambda I) / \{\underline{0}\}$$

DEFINITION $A \in \mathbb{R}^{n \times n}$, $\lambda \in \mathbb{R}$ EIGENVALUE OF A. WE CALL THE EIGENSPACE OF A ASSOCIATED TO λ $\mathrm{Nul}(A - \lambda I)$ WHICH IS A SUBSPACE OF $\mathbb{R}^n$. IT CORRESPONDS TO ALL EIGENVECTORS OF A ASSOCIATED TO λ PLUS THE $\underline{0}$ VECTOR

EXAMPLE

$$A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$$

1) IS $\lambda = 8$ AN EIGENVALUE?

2) IF YES, SHOW THE EIGENSPACE ASSOCIATED TO λ (FIND A BASIS FOR SUCH EIGENSPACE) WHAT IS DIMENSION?

3) REPEAT THE EXERCISE FOR $\lambda = 3$

1) $\underline{\lambda = 8}$

$$A - \lambda I = A - 8I = \begin{bmatrix} -4 & 2 & 3 \\ -1 & -7 & -3 \\ 2 & 4 & 1 \end{bmatrix} \cdot (-1) \quad \begin{bmatrix} 1 & 7 & 3 \\ -4 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix} \begin{matrix} -4 \\ 2 \end{matrix} \quad \begin{bmatrix} 1 & 7 & 3 \\ 0 & 30 & 15 \\ 0 & -10 & -5 \end{bmatrix} \begin{matrix} 1/15 \\ \cdot(-\frac{1}{5}) \end{matrix} \quad \begin{bmatrix} 1 & 7 & 3 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 7 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \underline{x}$$

$8I$ MEANS TAKE A AND SUBTRACT 8 IN THE DIAGONAL

FREE VARIABLE

$\Downarrow$

YES $\lambda = 8$ IS AN EIGENVALUE

2) $\mathrm{Nul}(A - 8I)$

$2x_2 + x_3 = 0$; $x_2 = -\frac{1}{2} x_3$

$x_1 + 7x_2 + 3x_3 = 0$; $x_1 = \frac{1}{2} x_3$

$$\begin{cases} x_1 = \frac{1}{2} x_3 \\ x_2 = -\frac{1}{2} x_3 \\ x_3 = x_3 \in \mathbb{R} \end{cases} \qquad \underline{x} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} x_3 \in \mathbb{R}$$

BASIS OF $(A - 8I)$ IS $\left\{ \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} \right\}$ THE DIMENSION IS 1

3) $\lambda = 3$

$$A - \lambda I = A - 3I = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{bmatrix} \begin{matrix} -1 \\ 2 \end{matrix}$$

ECHELON FORM

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

FREE VARIABLES

YES $\lambda = 3$ IS AN EIGENVALUE

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 0 \; ; \; x_1 = -2x_2 - 3x_3 \\ 0 = 0 \\ 0 = 0 \end{cases} \qquad \begin{cases} x_1 = -2x_2 - 3x_3 \\ x_2 = x_2 \in \mathbb{R} \\ x_3 = x_3 \in \mathbb{R} \end{cases}$$

$$\underline{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} x_3 \quad x_2, x_3 \in \mathbb{R}$$

BASIS FOR EIGENSPACE OF A ASSOCIATED TO THE EIGENVALUE 3 IS $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

THE DIMENSION = 2

$A \in \mathbb{R}^{n \times n}$

DOES THE MATRIX HAVE THE EIGENVALUE $\lambda = 0$?

$$(A - \lambda I)\underline{x} = \underline{0} \qquad \underline{x} \in \mathbb{R}^n, \; \underline{x} \neq \underline{0}$$

YES IFF $A\underline{x} = \underline{0}$ HAS A NON TRIVIAL SOLUTION

YES IFF A IS NOT INVERTIBLE

OBSERVATION A MATRIX $A \in \mathbb{R}^{n \times n}$ IS INVERTIBLE IF AND ONLY IF IT DOES NOT HAVE EIGENVALUE $\lambda = 0$

- GIVEN A MATRIX $A \in \mathbb{R}^{n \times n}$ HOW CAN I FIND THE EIGENVALUES?

$\lambda \in \mathbb{R}$ SUCH THAT $(A - \lambda I)\underline{x} = \underline{0}$ HAS NON-TRIVIAL SOLUTIONS $(\underline{x} \neq \underline{0})$

THIS CONDITION IS TRUE IF AND ONLY IF $(A - \lambda I)$ IS NOT INVERTIBLE

WHICH IN TURN IS EQUIVALENT TO $\det(A - \lambda I) = 0$

THEOREM THE EIGENVALUES OF A MATRIX $A \in \mathbb{R}^{n \times n}$ (SQUARE) CORRESPOND TO THE ZEROS (ROOTS) OF THE EQUATION $\det(A - \lambda I) = 0$

THIS IS CALLED THE CHARACTERISTIC EQUATION

EXAMPLES

$$\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$$

FIND ALL THE EIGENVALUES OF THE TWO MATRIXES

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0 \; ; \; \det\begin{pmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{pmatrix} \quad (2-\lambda)(2-\lambda) - 7 \cdot 7 \; ; \; (2-\lambda)^2 = 49 \; ; \; 2 - \lambda = \pm 7 \; ; \; \lambda = -5, \; \lambda = 9$$

THE EIGENVALUES OF $\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$ ARE 9 AND -5

$$\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \qquad \det\begin{pmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{pmatrix} = 0 \qquad (2-\lambda)(4-\lambda) - (1)(-1) = \lambda^2 + 8 - 4\lambda - 2\lambda + 1 = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 \longrightarrow \lambda = 3$$

THE EIGENVALUE OF $\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$ IS 3

$$\begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \end{bmatrix} \qquad \det\begin{bmatrix} (x-\lambda) & x & x & x & x \\ x & (x-\lambda) & x & x & x \\ x & x & (x-\lambda) & x & x \\ x & x & x & (x-\lambda) & x \\ x & x & x & x & (x-\lambda) \end{bmatrix} = 0$$

ALWAYS GET A POLYNOMIAL

ITS DEGREE IS ALWAYS n (DIMENSION OF THE MATRIX)

THEOREM $A \in \mathbb{R}^{n \times n}$ THE EIGENVALUES OF A CORRESPOND TO THE ROOTS OF THE CHARACTERISTIC EQUATION $\det(A - \lambda I) = 0$

THE LEFT HAND SIDE IS A POLYNOMIAL OF DEGREE n IN λ.

THERFORE A MATRIX A HAS AT MOST n EIGENVALUES (COUNTED WITH THE MULTIPLICITY)

DEFINITION THE ROOT MULTIPLICITY OF AN EIGENVALUE λ IS CALLED ITS ALGEBRAIC MULTIPLICITY

OBSERVATION IF $A \in \mathbb{R}^{n \times n}$ IS UPPER OR LOWER TRIANGULAR ITS EIGENVALUES ARE THE VALUES ON THE DIAGONAL

$$\begin{bmatrix} 3 & 1 & 7 & 1 \\ 0 & 8 & 0 & -4 \\ 0 & 0 & -2 & 5 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad \det\begin{bmatrix} 3-\lambda & 1 & 7 & 1 \\ 0 & (8-\lambda) & 0 & -4 \\ 0 & 0 & (-2-\lambda) & 5 \\ 0 & 0 & 0 & (3-\lambda) \end{bmatrix} = (3-\lambda)(8-\lambda)(-2-\lambda)(3-\lambda) = 0$$

$\lambda = 3 \quad \lambda = 8 \quad \lambda = -2 \quad \lambda = 3$

$3, 8, -2$

EX

$$A = \begin{bmatrix} -7 & 2 & 4 \\ -6 & 0 & 6 \\ -4 & 2 & 1 \end{bmatrix}$$

① FIND ALL THE EIGENVALUES OF A

② FOR EACH EIGENVALUE FIND A BASIS FOR THE ASSOCIATED EIGENSPACE (DIMENSION?)

①

$\det(A - \lambda I) = 0$

$$\det \begin{bmatrix} -7-\lambda & 2 & 4 \\ -6 & 0-\lambda & 6 \\ -4 & 2 & 1-\lambda \end{bmatrix} = (-7-\lambda)\begin{bmatrix} 0-\lambda & 6 \\ 2 & 1-\lambda \end{bmatrix} - 2\begin{pmatrix} -6 & 6 \\ -4 & 1-\lambda \end{pmatrix} + 4\begin{pmatrix} -6 & 0-\lambda \\ -4 & 2 \end{pmatrix} = (-7-\lambda)(\lambda^2-\lambda-12) - 2(6\lambda-6+24) + 4(-12-4\lambda) =$$

$$= -7\lambda^2 + 7\lambda + \cancel{84} - \lambda^3 + \lambda^2 + \cancel{12\lambda} - \cancel{12\lambda} - \cancel{36} - 16\lambda - \cancel{48} = -\lambda^3 - 6\lambda^2 - 9\lambda = -\lambda(\lambda^2 + 6\lambda + 9)$$

$\lambda = 0$ $\quad 36 - 4(9) = 0$

$\frac{-6}{2} = -3$

$\lambda = 3$

THE EIGENVALUES OF $\begin{bmatrix} -7 & 2 & 4 \\ -6 & 0 & 6 \\ -4 & 2 & 1 \end{bmatrix}$ ARE $\lambda = 0$ AND $\lambda = -3$

$\lambda = 0$

EIGENSPACE

$$\begin{bmatrix} -7 & 2 & 4 \\ -6 & 0 & 6 \\ -4 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 1 \\ -7 & 2 & 4 \\ -4 & 2 & 1 \end{bmatrix} \xrightarrow{7,\ 4} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & -3 \\ 0 & 2 & -3 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

FREE VARIABLE

$-x_1 + x_3 = 0;\ x_1 = x_3$

$2x_2 - 3x_3;\ x_2 = \frac{3}{2}x_3$

$$\begin{cases} x_1 = x_3 \\ x_2 = 3/2\,x_3 \\ x_3 = x_3 \in \mathbb{R} \end{cases} \quad \underline{x} = \begin{bmatrix} 1 \\ 3/2 \\ 1 \end{bmatrix} x_3$$

$\lambda = 0$ BASIS $\left\{ \begin{bmatrix} 1 \\ 3/2 \\ 1 \end{bmatrix} \right\}$ DIMENSION = 1

$\lambda = -3$

$$\begin{bmatrix} -4 & 2 & 4 \\ -6 & 3 & 6 \\ -4 & 2 & 4 \end{bmatrix} \xrightarrow{3/2,\ 1} \begin{bmatrix} -4 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

FREE VARIABLES

$-4x_1 + 2x_2 + 4x_3 = 0;\ x_1 = \frac{1}{2}x_2 + x_3$

$$\begin{cases} x_1 = \frac{1}{2}x_2 + x_3 \\ x_2 = x_2 \in \mathbb{R} \\ x_3 = x_3 \in \mathbb{R} \end{cases}$$

$$\underline{x} = \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3$$

$\lambda = -3$ BASIS $\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ DIMENSION = 2

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$$

① FIND ALL THE EIGENVALUES OF A

② FOR EACH EIGENVALUE FIND A BASIS FOR THE ASSOCIATED EIGENSPACE (DIMENSION?)

① CALCULATION ALREADY DONE, ONE $\lambda = 3$ EIGENVALUE (ALGEBRIC MULTIPLICITY 2)

$$\det \begin{bmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{bmatrix} = (3-\lambda)^2$$

② $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \xrightarrow{1} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad$ 2 - 3 $\quad$ 4 - 3

FREE VARIABLE

$-x_1 + x_2 = 0;\ x_1 = x_2$

$$\begin{cases} x_1 = x_2 \\ x_2 = x_2 \in \mathbb{R} \end{cases} \quad \underline{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2$$

BASIS $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ DIM = 1

DIMENSION

OBSERVATION THE DIMENSION OF THE EIGENSPACE ASSOCIATED TO AN EIGENVALUE λ IS ALWAYS EQUIVALENT OR SMALLER THAN THE ALGEBRIC MULTIPLICITY OF λ