$+$ $\times$

$-$ $\div$

① MOMENT GENERATING FUNCTION (MGF)

> The moment generating function (MGF) of a random variable is a function that encodes the moments of the random variable. It is defined as the expected value of the exponential function of the random variable, evaluated at a given value of the parameter t:
>
> MGF(t) = E[exp(tX)]
>
> where X is the random variable, and t is a real number. The MGF exists for all values of t for which the expected value is finite.
>
> The MGF can be used to compute the moments of a random variable. For example, the kth moment of X can be obtained by taking the kth derivative of the MGF at t = 0:
>
> m_k = MGF^(k)(0)
>
> where m_k is the kth moment of X.

FOR RANDOM VARIABLES:

$X : \Omega \to \mathbb{R} \qquad M_X(z) = E[e^{zX}] \quad z \in \mathbb{R}$

EXAMPLE 1:

DISCRETE

FIX $X \sim \text{Poisson}(\lambda) \quad \Rightarrow P[X = n] = e^{-\lambda}\dfrac{\lambda^n}{n!} \quad n \in \mathbb{N}$

λ FIXED PARAMETER

$X : \Omega \to \mathbb{N}_0 \subseteq \mathbb{R}$

$M_X(z) = E[e^{zX}]$

$z \in \mathbb{R}$ FIXED

REMEMBER THAT

$E[g(X)] = \sum_{n=0}^{+\infty} g(n) P[X=n]$

$g(x) = e^{zx} \qquad M_X(z) = \sum_{n=0}^{+\infty} e^{zn}\dfrac{e^{-\lambda}\lambda^n}{n!} = e^{-\lambda}\sum_{n=0}^{+\infty}\dfrac{(e^z\lambda)^n}{n!} = e^{-\lambda} e^{e^z \lambda}$

↳ REMEMBER THAT

$\sum_{n=0}^{+\infty} \dfrac{x^n}{n!} = e^x = e^{\lambda(e^z - 1)}$

EXAMPLE 2:

CONTINOUS

$X \sim \exp(\lambda)$

DENSITY FUNCTION

$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{IF } x > 0 \\ 0 & \text{OTHERWISE} \end{cases}$

$M_X(z) = E[e^{zX}]$

$z \in \mathbb{R}$ FIXED

$E[g(X)] = \int_{-\infty}^{+\infty} g(x) f_X(x)\,dx$

$g(x) = e^{zx}$

$e^{a+b} = e^a e^b$

$e^{a-b} = e^a e^{-b}$

$M_X(z) = \int_{-\infty}^{+\infty} e^{zx} f_X(x)\,dx = \int_0^{+\infty} e^{zx}\lambda e^{-\lambda x}\,dx = \lambda \int_0^{+\infty} e^{-(\lambda - z)x}\,dx$

- IF $\lambda - z > 0 \iff z < \lambda \Rightarrow M_X(z) = \lambda \dfrac{1}{\lambda - z} = \dfrac{\lambda}{\lambda - z}$

$\int_0^{+\infty} e^{-\alpha x}\,dx = \dfrac{1}{\alpha} \quad (\alpha > 0)$

$\int_0^{+\infty} e^{\alpha x}\,dx = +\infty$

EXAMPLE 3:

CONTINOUS

$X \sim N(m, \sigma^2)$

GAUSSIAN (NORMAL) DISTRIBUTION

$f_X(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\dfrac{(x-m)^2}{2\sigma^2}\right\} \quad x \in \mathbb{R}$

$M_X(z) = E[e^{zX}] = \int_{-\infty}^{+\infty} e^{zx}\dfrac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\dfrac{(x-m)^2}{2\sigma^2}\right\}dx = \dfrac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{+\infty}\exp\left\{\underbrace{-\dfrac{(x-m)^2}{2\sigma^2} + zx}\right\}dx$

COMPLETE THE SQUARE

$a^2 - 2ab + b^2 - b^2$

ANALYZE THIS

$-\left[\dfrac{x^2 - 2mx + m^2 - 2\sigma^2 zx}{2\sigma^2}\right] = -\left[\dfrac{x^2 - 2x(m + \sigma^2 z) + m^2}{2\sigma^2}\right] = -\left[\dfrac{\boxed{x^2 - 2x(m+\sigma^2 z) \pm (m+\sigma^2 z)^2} + m^2}{2\sigma^2}\right]$

$= -\left[\dfrac{x^2 - 2x(m+\sigma^2 z) + (m+\sigma^2 z)^2 - (\cancel{m^2} + \sigma^4 z^2 + 2m\sigma^2 z) + \cancel{m^2}}{2\sigma^2}\right] = \underbrace{-\left[\dfrac{[x - (m+\sigma^2 z)]^2}{2\sigma^2}\right]}_{\text{DEPENDS ON } X} + \underbrace{\dfrac{\sigma^{\cancel{4}} z^2 + 2m\cancel{\sigma^2} z}{2\cancel{\sigma^2}}}_{\text{DO NOT DEPENDS ON } X}$

$M_X(z) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \underbrace{\exp\left\{\dfrac{\sigma^2 z^2}{2} + mz\right\}}_{\text{TAKE OUT, IS A CONSTANT}} \int_{-\infty}^{+\infty} \underbrace{\exp\left\{-\dfrac{(x - (m+\sigma^2 z))^2}{2\sigma^2}\right\}}_{\tilde{f}(x)} dx$

$\tilde{f}(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\dfrac{(x - (m+\sigma^2 z))^2}{2\sigma^2}\right\} \sim N(m + \sigma^2 z, \sigma^2)$

$M(z) = \exp\left\{\dfrac{\sigma^2 z^2}{2} + mz\right\} \quad \int_{-\infty}^{+\infty} \tilde{f}(x)\,dx = 1$

$X \sim N(m, \sigma^2)$

$M_X(z) = \exp\left\{\dfrac{\sigma^2 z^2}{2} + mz\right\}$

MGF FOR RANDOM VECTOR

$(X,Y): \Omega \to \mathbb{R}$

SCALAR PRODUCT

$M_{(X,Y)}(z,w) = E[e^{zX+wY}] = E[e^{(z,w)\cdot(X,Y)}]$

↳ REMEMBER

$A=(a_1,a_2)$ $b=(b_1,b_2) \in \mathbb{R}$

$A\cdot B = a_1 b_1 + a_2 b_2$

$\underbrace{(z,w) \in \mathbb{R}^2}_{\text{VECTOR OF NUMBERS}}$ $(X,Y) \in \mathbb{R}^2$ RANDOM VECTOR

$(z,w)\cdot(X,Y) = zX + wY$

PLAN A

$f(x,y)$ = JOINT DENSITY

$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy$

x FIXED FIRST MARGINAL

$f_Y(y) = \int_{-\infty}^{+\infty} f_{(X,Y)}(x,y)\,dx$

y FIXED SECOND MARGINAL

PLAN B

$f_{(x,y)} \xrightarrow{MGF} M_{(X,Y)}(z,w) = E[e^{zX+wY}] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{zx+wy} f_{(x,y)}(x,y)\,dx\,dy$

WE HAVE $M_{(X,Y)}(z,w)$

- IF WE PUT $w=0$ INTO $E[e^{zX+wY}] \to E[e^{zX}] = M_X(z)$

↳ MOMENT GENERATING FUNCTION OF X

- IF WE PUT $z=0$ INTO $E[e^{zX+wY}] \to E[e^{wY}] = M_Y(w)$

1) EVALUATE $M_{X,Y}$
2) EVALUATE M_X, M_Y
3) DEFINE f_X, f_Y FROM M_X, M_Y

SUM OF INDEPENDENT R.V.

$Z = X+Y$ UNDER ASSUMPTION THAT X,Y ARE INDEPENDENT

PROBLEM: FIND f_Z

SOLUTION: $f_Z(t) = \int_{-\infty}^{+\infty} f_{(X,Y)}(x, t-x)\,dx$

$M_{X+Y}(\underset{\in\mathbb{R}}{z}) = E[e^{z(X+Y)}] = E[e^{zX}\cdot e^{zY}] = E[e^{zX}]\cdot E[e^{zY}] = M_X(z)\cdot M_Y(z)$

X,Y INDEP $\Rightarrow$ $\varphi(X)$ INDEP. OF $\psi(Y)$

$U = e^{zX}$, $V = e^{zY}$ ARE INDEPENDENT

PROPOSITION: X, Y INDEPENDENT $\Rightarrow M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$

MULTIVARIATE (BIVARIATE) GAUSSIAN DISTRIBUTION

$(X,Y) \sim N(\underline{m}, \underline{\underline{C}})$

- $\underline{m} \in \mathbb{R}^2$ MEAN (VECTOR)
- $\underline{\underline{C}} = 2 \times 2$ COVARIANCE MATRIX $\rightarrow$ SYMMETRIC POSITIVE DEFINED

$$\underline{\underline{C}} = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_y\sigma_x & \sigma_y^2 \end{bmatrix}$$

$\rho \in (-1,1) =$ CORRELATION COEFFICENT

$$\rho = \frac{COV(X,Y)}{\sqrt{VAR(X)VAR(Y)}}$$

$\sigma_x^2 = VAR(X)$

$\sigma_y^2 = VAR(Y)$

$$f_{(X,Y)}(x,y) = \frac{1}{2\pi} \frac{1}{\sqrt{\det \underline{\underline{C}}}} \exp\left\{ -\frac{1}{2} \, {}^{t}(\underline{t}-\underline{m}) \, \underline{\underline{C}}^{-1} (\underline{t}-\underline{m}) \right\}$$

t TRANSPOSE

$\underline{\underline{C}}^{-1}$: INVERSE OF $\underline{\underline{C}}$

$\underline{t} = (x,y)$

QUADRATIC FORMS

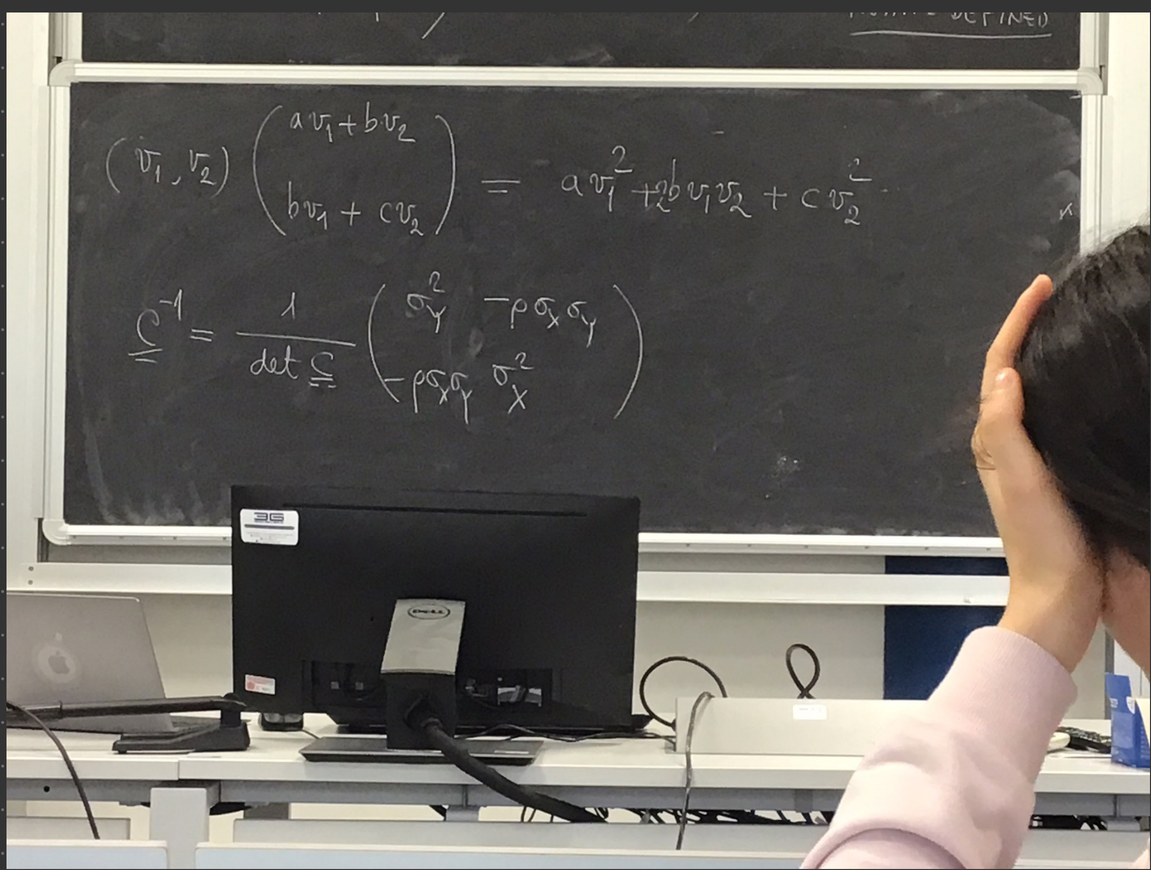
$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ $\quad \underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} =$ COLUMN VECTOR

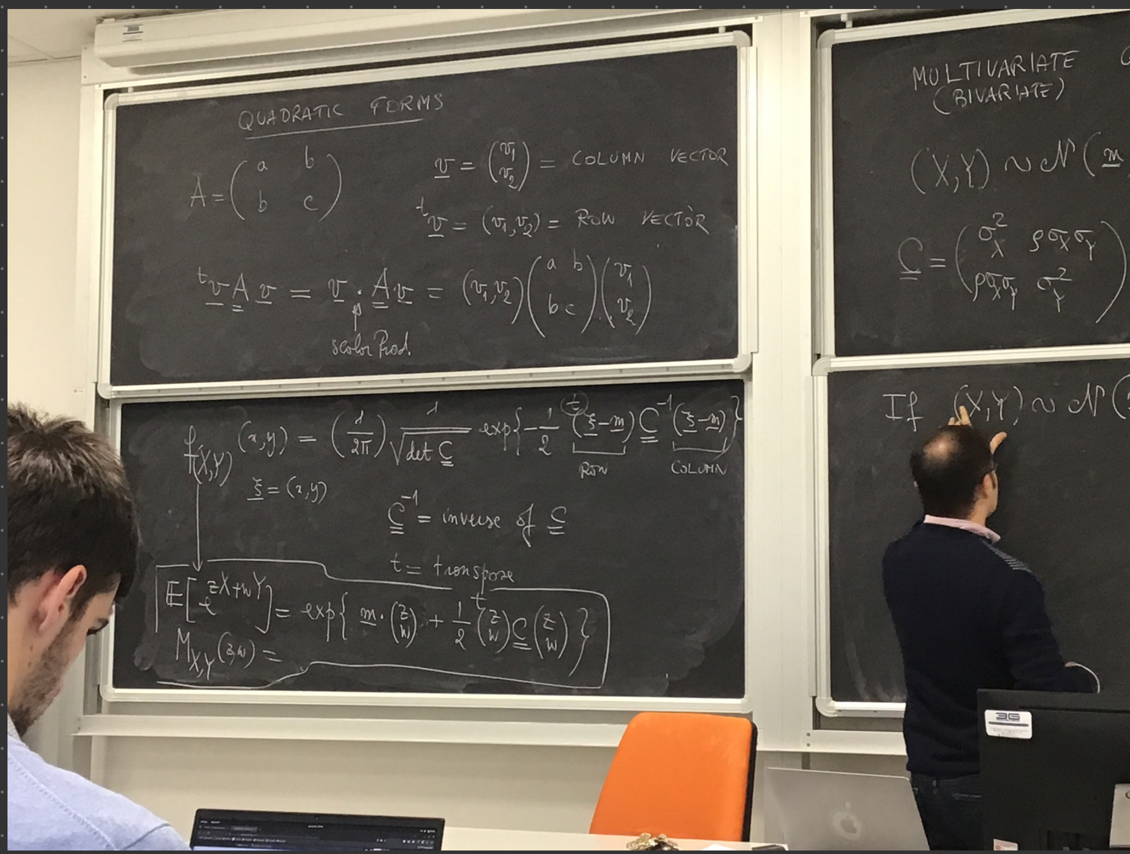
${}^{t}\underline{v} = (v_1, v_2) =$ ROW VECTOR

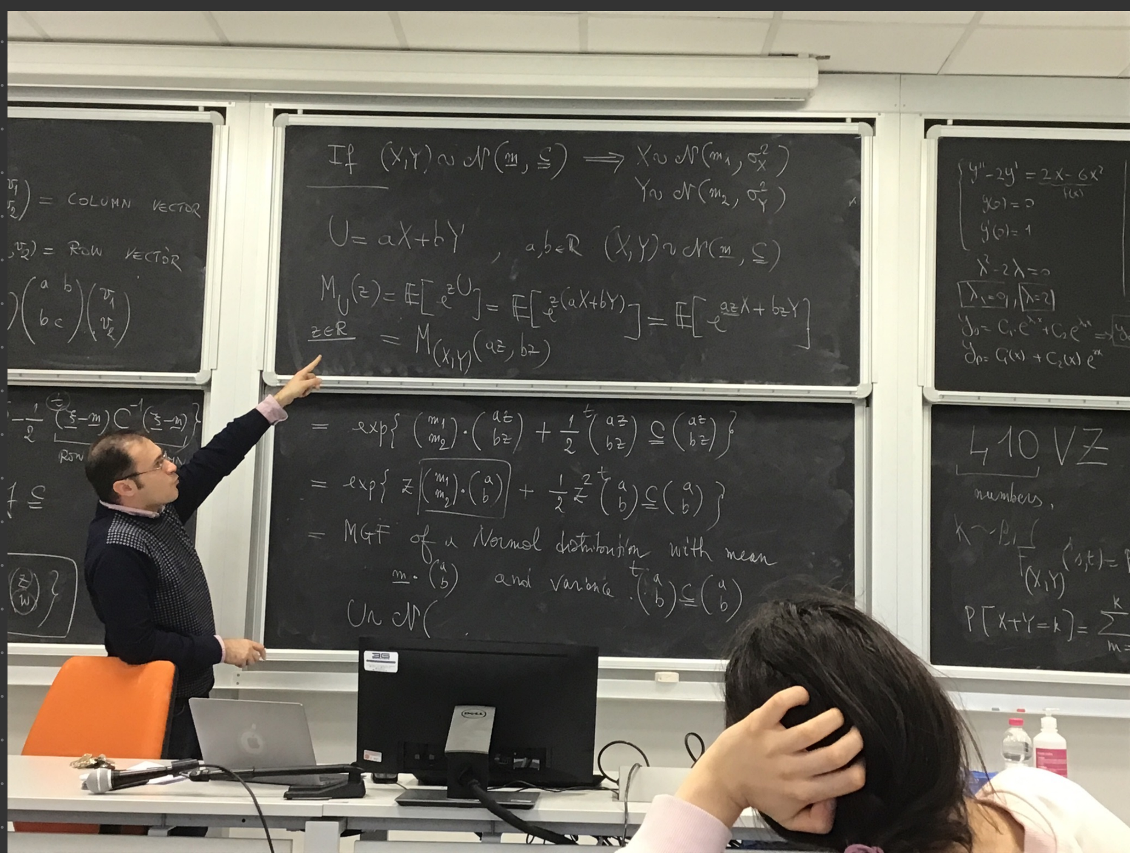
$${}^{t}\underline{v} \underline{\underline{A}} \underline{v} = \underline{v} \cdot \underline{\underline{A}} \underline{v} = (v_1, v_2) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

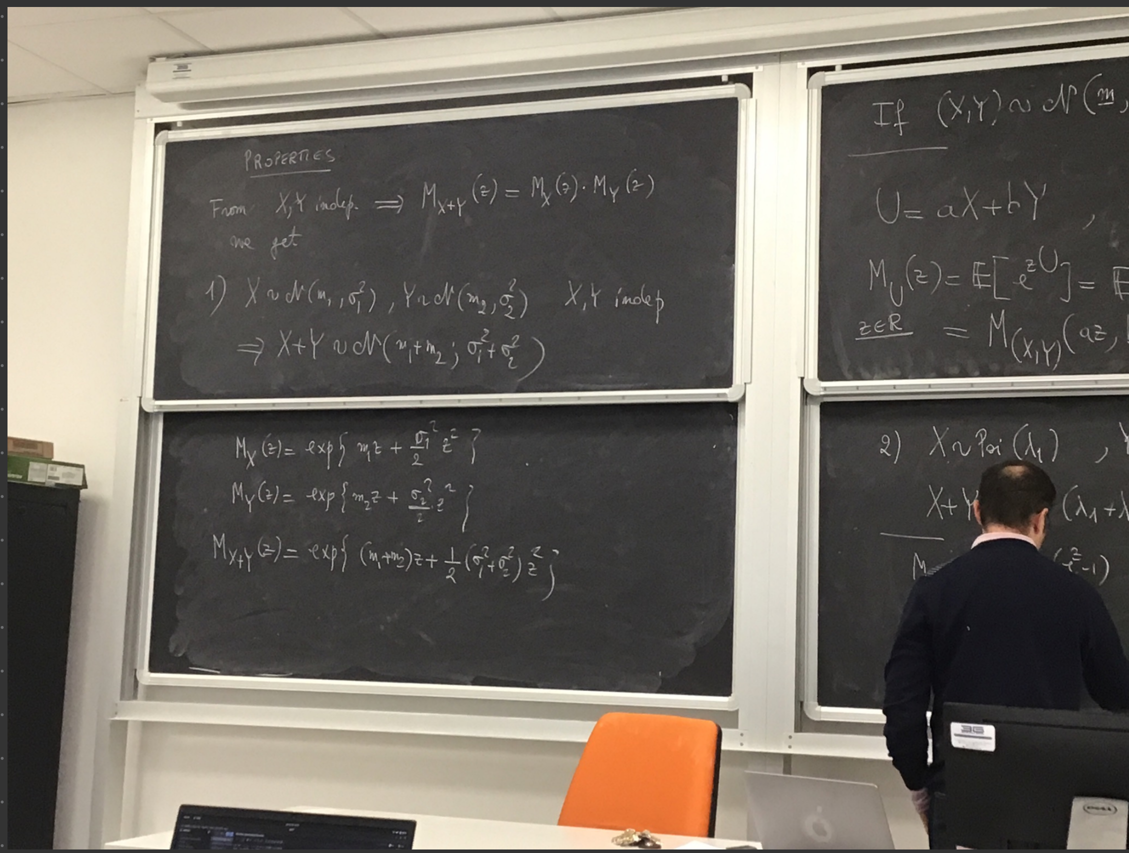
$\cdot$ SCALAR PRODUCT

$$(v_1, v_2) \begin{pmatrix} av_1 + bv_2 \\ bv_1 + cv_2 \end{pmatrix} = av_1^2 + 2bv_1v_2 + cv_2^2$$




QUADRATIC FORMS
COLUMN VECTOR
ROW VECTOR
scalar prod.
MULTIVARIATE
(BIVARIATE)
If
ROW
COLUMN
inverse of
t = transpose


If (X,Y) ~ N(m, C) ⟹ X ~ N(m₁, σ²ₓ)
Y ~ N(m₂, σ²ᵧ)
U = aX+bY , a,b ∈ R
COLUMN VECTOR
ROW VECTOR
MGF of a Normal distribution with mean
and variance
410 VZ
numbers


PROPERTIES
From X,Y indep. ⟹ $M_{X+Y}(z) = M_X(z) \cdot M_Y(z)$
we get
1) $X \sim N(m_1, \sigma_1^2)$, $Y \sim N(m_2, \sigma_2^2)$ X,Y indep
⟹ $X+Y \sim N(m_1+m_2; \sigma_1^2+\sigma_2^2)$
$M_X(z) = \exp\{m_1 z + \frac{\sigma_1^2}{2} z^2\}$
$M_Y(z) = \exp\{m_2 z + \frac{\sigma_2^2}{2} z^2\}$
$M_{X+Y}(z) = \exp\{(m_1+m_2)z + \frac{1}{2}(\sigma_1^2+\sigma_2^2) z^2\}$
$U = aX + bY$
$z \in \mathbb{R}$