WHAT IS A LINEAR EQUATION

ANY EQUATION THAT CAN BE WRITTEN AS

$x_1, x_2, \dots, x_N \in \mathbb{R}$

↳ REAL NUMBERS

THAT SATISFIES

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + \dots + a_N x_N = b$$

WHERE $a_1 \dots a_N \in \mathbb{R}$ $b \in \mathbb{R}$

YES	NO
$3x - \sqrt{2}x_2 - 5x_3 = -1$	$x_1 x_2 + x_3 = 2$
$x_1 + x_2 - \pi x_3 + 72x_4 = 5$	$\sqrt{x_1} + x_2 = 2$
	$\sin(x_1) + x_2^2 = 5$

A LINEAR SYSTEM IS A SET OF LINEAR EQUATION

SEARCH $x_1, x_2, \dots x_N \in \mathbb{R}$ THAT SATISFIES

$$
\begin{array}{l}
1^{ST} \rightarrow \\
\\
\\
M^{TH} \rightarrow
\end{array}
\begin{cases}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1N}x_N = b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2N}x_N = b_2 \\
\vdots \qquad\qquad \vdots \\
a_{M1}x_1 + a_{M2}x_2 + a_{M3}x_3 + \dots a_{MN}x_{MN} = b_M
\end{cases}
$$

N VARIABLES

M EQUATIONS

$a_{11}, a_{12} \dots a_{1N}, a_{21}, a_{22}, \dots a_{2N}, a_{MN} \in \mathbb{R}$ $\quad b_1, b_2, \dots b_M \in \mathbb{R}$

EXAMPLE

$N = 3 \quad M = 2$

$$
\begin{array}{l} 1^{ST} \\ 2^{ND} \end{array}
\begin{cases}
x_1 - \sqrt{2}x_2 + 5x_3 = -1 \\
3x_1 + 0x_2 - 6x_3 = 2
\end{cases}
$$

$N = 5 \quad M = 3$

$$
\begin{cases}
x_1 + 2x_2 - 3x_3 + 4x_4 = -3 \\
x_1 - \sqrt{2}x_4 = 6 \\
x_2 + 3x_5 = 0
\end{cases}
$$

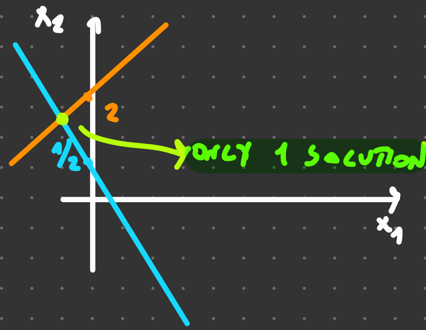
THE SOLUTION OF A LINEAR SYSTEM IS ANY VALUES $\{x_1, x_2, x_N\}$ THAT SATISFIES ALL THE EQUATION IN THE SYSTEM

IS THERE A SOLUTION? IS UNIQUE?

REPRESENT GRAPHICALLY

$N=2 \quad M=2$

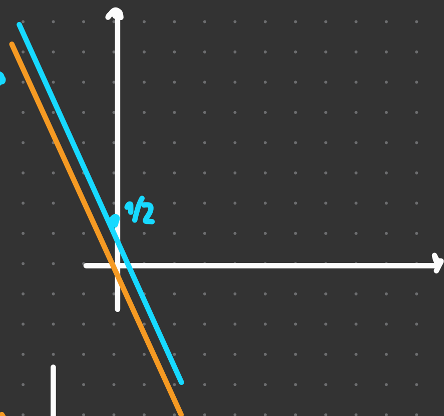
$$\begin{cases} x_1 + 2x_2 = 1 \Rightarrow x_2 = \frac{1}{2} - \frac{1}{2}x_1 \\ x_1 - x_2 = -2 \Rightarrow x_2 = x_1 + 2 \end{cases}$$



$N=2 \qquad M=2$

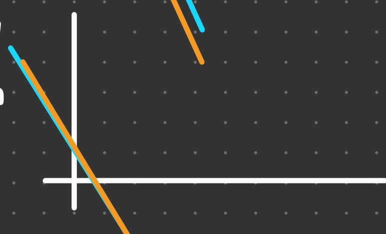
$$\begin{cases} x_1 + 2x_2 = 1 \Rightarrow x_2 = \frac{1}{2} - \frac{1}{2}x_1 \\ 2x_1 + 4x_2 = 0 \Rightarrow x_2 = -\frac{1}{2}x_1 \end{cases}$$

THEY ARE // NO SOLUTION



$$\begin{cases} x_1 + 2x_2 = 1 \Rightarrow x_2 = 1/2 - 1/2\, x_1 \\ 2x_1 + 4x_2 = 2 \Rightarrow x_2 = \frac{1}{2} - \frac{1}{2}x_1 \end{cases}$$

INFINITE SOLUTIONS



- ONE SOLUTION $\quad x_1$ —
- NO SOLUTION $\quad$ VOID $\emptyset$
- INFINITE SOLUTIONS

DEF THE SOLUTION SET OF A LINEAR SYSTEM IS THE SET OF ALL $\{x_1, x_2, \dots x_N\}$ THAT SATISFIES ALL THE EQUATIONS IN THE SYSTEM

HOW TO SOLVE LINEAR SYSTEM

$$\begin{cases} x_1 - 3x_3 = 8 \\ x_2 + 5x_3 = -2 \\ 2x_1 + 2x_2 + 9x_3 = 7 \end{cases}$$

3 OPERATIONS THAT WILL $\boxed{\text{NOT}}$ CHANGE THE SOLUTION OF THE LINEAR SET

1. EXCHANGE TWO OPERATIONS

SAME AS
$$\begin{cases} x_2 + 5x_3 = -2 \\ x_1 - 3x_3 = 8 \\ 2x_1 + 2x_2 + 9x_3 = 7 \end{cases}$$

2. TAKE ANY NUMBER $\neq 0$ MULTIPLY ANY THE EQUATIONS BY THAT NUMBER $\alpha \in \mathbb{R}$ $\alpha \neq 0$

SAME AS
$$\begin{cases} 3x_1 - 9x_3 = 24 \quad (\times 3) \\ x_2 + 5x_3 = -2 \\ 2x_1 + 2x_2 + 9x_3 = 7 \end{cases}$$

3. SUBTRACT ANY EQUATION MULTIPLIED BY $\alpha \in \mathbb{R}$ (EVEN 0) TO ANY OTHER EQUATION

SAME AS
$$\begin{cases} x_1 - 3x_3 = 8 \\ x_1 + x_2 + 5x_3 = -2 \quad (\times 2) \\ (2 - 2.0)x_1 + (2 - 2.1)x_2 + (9 - 5.2)x_3 = \end{cases}$$

$(7-(2(-3))$

TAKE

$$\begin{cases} x_1 - 3x_3 = 8 \\ x_2 + 5x_3 = -2 \\ 2x_1 + 2x_2 + 9x_3 = 7 \end{cases}$$

SUBTRACT THE 1^{ST} EQ MUTIPLY BY 2 TO THE 3RD EQU

!!! WHEN DOING THE SUBTRACTION OF THE FIRST LINE A WITH THE SECOND LINE B

- KEEP B THE SAME
- MULTIPLY B BY A NUMBER k
- DO THE DIFFERENCE $A - Bk$ ADD IT INSTEAD OF A

$$\begin{cases} x_1 - 3x_3 = 8 \\ x_2 + 5x_3 = -2 \\ \boxed{0x_1} + 2x_2 + 15x_3 = -9 \end{cases}$$

KILLED THIS VARIABLE

SUBTRACT THE 2^{ND} EQ. MUTIPLY BY 2 TO THE 3^{RD} EQUATION

$$\begin{cases} x_1 - 3x_2 = 8 \\ x_2 + 5x_3 = -2 \\ \boxed{0x_2} + 5x_3 = -5 \end{cases} \Rightarrow \begin{cases} x_1 = 8 + 3(-1) = 5 \\ x_2 = 3 \\ x_3 = -1 \end{cases}$$

KILLED

A **MATRIX** $C \in \mathbb{R}^{m \times n}$ IS A COLLECTION OF REAL NUMBERS ORGANISED INTO A RECTANGULAR ARRAY WITH M ROWS AND N COLUMS

ROWS M ☰ COLUMS N |||

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1N} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2N} \\ c_{31} & c_{32} & c_{33} & \cdots & c_{3N} \\ \vdots & & & & \vdots \\ c_{M1} & c_{M2} & c_{M3} & \cdots & c_{MN} \end{bmatrix}$$

N COLUMS

M ROWS

$$\begin{bmatrix} 4 & 1 & \sqrt{7} & 3 \\ 2 & 1 & -6 & \pi \\ 0 & 3 & 0 & -11 \end{bmatrix}$$

3 ROWS $M=3$

4 COLUM) $N=4$

$\mathbb{R}^{3\times 4}$

$$\begin{bmatrix} \sqrt{2} & 1 \\ 2 & 1 \\ 0 & 6 \\ 0 & 3 \end{bmatrix}$$

$M=4$ $N=2$

$\mathbb{R}^{4\times 2}$

A MARIX IS A MORE COMPACT WAY TO EXPRESS A LINEAR SYSTEM

$$\begin{cases} x_1 - 3x_3 = 8 \\ x_2 + 5x_3 = -2 \\ 2x_1 + 2x_2 + 9x_3 = 7 \end{cases} \Rightarrow \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 2 & 2 & 9 & 7 \end{bmatrix}$$

GIVEN ANY LINEAR SYSTEM IS ASSOCIATED WITH THE AUGMENTED MATRIX

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots a_{1N}x_N = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots a_{2N}x_N = b_N \\ \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ a_{M1}x_1 + a_{M2}x_2 + \dots + a_{MN}x_N = b_M \end{cases}$$

AUGMENTED MATRIX: (CLEARLY THE OPPOSITE HOLDS)

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} & b_1 \\ a_{21} & a_{22} & \dots & a_{2N} & b_2 \\ \vdots & & & & \vdots \\ a_{M1} & a_{M2} & & a_{MN} & b_M \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 2 & 2 & 9 & 7 \end{array}\right] \qquad \begin{cases} x_1 - 3x_3 = 8 \\ x_2 + 5x_3 = -2 \\ 2x_1 + 2x_2 + 9x_3 = 7 \end{cases}$$

ELEMENTARY ROW OPERATIONS

1. EXCHANGE TWO ROWS
2. MULTIPLY ANY ROW BY $\alpha \in \mathbb{R}$ $\alpha \neq 0$
3. SUBTRACT ANY ROW MULTIPLIED BY $\alpha \in \mathbb{R}$ TO ANY OTHER ROW

IF YOU APPLY ELEMENTARY ROW OPERATIONS TO THE AUGMENTED MATRIX ASSOCIATED TO A LINEAR SYSTEM THE NEW AUGMENTED MATRIX IS ASSOCIATED TO A LINEAR SYSTEM W/ THE SAME SOLUTION SET AS THE ORIGINAL ONE

$$\begin{cases} x_2 - 2x_3 = 8 \\ -x_1 + 3x_2 - x_3 = -1 \\ 2x_1 - 8x_2 + 6x_3 = 1 \end{cases} \qquad \begin{bmatrix} 0 & 1 & -2 & 8 \\ -1 & 3 & -1 & -1 \\ 2 & -8 & 6 & 1 \end{bmatrix}$$

CHANGE ROWS 1 AND 3

$$\begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} \begin{bmatrix} 2 & -8 & 6 & 1 \\ -1 & 3 & -1 & -1 \\ 0 & 1 & -2 & 8 \end{bmatrix} \begin{array}{c} (-1/2) \\ \\ \beta + (\frac{1}{2}\alpha) \end{array} \Rightarrow \begin{bmatrix} 2 & -8 & 6 & 1 \\ 0 & -1 & 2 & -1/2 \\ 0 & 1 & -2 & 8 \end{bmatrix} (-1)$$

$$\begin{bmatrix} 2 & -8 & 6 & 1 \\ 0 & -1 & 2 & -1/2 \\ 0 & 0 & 0 & 15/2 \end{bmatrix} \qquad 8 - \frac{1}{2} = \frac{15}{2}$$

THIS EQUATION IS <u>IMPOSSIBLE</u>

→ NO SOLUTION

TWO MATRIX ARE CALLED ROW-EQUIVALENT IF YOU CAN TRANSFORM ONE MATRIX IN THE OTHER ONE BY ELEMENTARY ROW OPERATIONS

OBSERVATION ELEMENTARY ROW OPERATIONS ARE REVERSIBLE

$\boxed{\text{PROPOSITION}}$ IF TWO AUGMENENTED MATRIXES ARE ROW-EQUIVALENT THEN THE SOLUTION OF THE ASSOCIATED LINEAR SYSTEM ARE THE SAME

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix} \begin{matrix} (4) \\ (6) \end{matrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{bmatrix} \left(5/3\right) \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

WANT TO HAVE 0 HERE NOW

$-15-(-9)(5/3)$

$-10-(-6)\,5/3$

THIS EQUATION IS USELESS

$$\left[\begin{array}{cccc|c} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{array}\right]$$

WANT TO HAVE THESE AS 0

A MATRIX IS IN **ECHELON FORM (EF)** IF IT SATISFIES THESE CONDITIONS

- ALL ROWS FULL OF ZEROS ARE AT THE BOTTOM
- THE **LEADING ENTRY** (LEFT MOST NON-ZERO ENTRY) IS ON THE RIGTH OF THE LEADING ENTRY OF THE ROWS ABOVE

$$\begin{bmatrix} \boxed{*} & * & * & * & * & * & \cdots & * \\ 0 & \boxed{*} & * & * & * & * & \cdots & * \\ 0 & 0 & 0 & \boxed{*} & * & * & \cdots & * \\ 0 & 0 & 0 & 0 & \boxed{*} & * & * & * \\ 0 & & 0 & & 0 & & \cdots & 0 \end{bmatrix}$$

$*$ ⇒ GENERAL NUMBER

0 ⇒ ZERO

$\boxed{*}$ ⇒ DIFFERENT THAN ZERO

$$\begin{bmatrix} 0 & \boxed{*} & * & * & * & * \\ 0 & 0 & \boxed{*} & * & * & * \\ \boxed{*} & 0 & 0 & 0 & * & * \end{bmatrix}$$ NOT ECHELON FORM

$$\begin{bmatrix} \boxed{*} & 0 & 0 & 0 & * & * \\ 0 & \boxed{*} & * & * & * & * \\ 0 & 0 & \boxed{*} & * & * & * \end{bmatrix}$$ ECHELON FORM

OBSERVATION → EACH LEADING ENTRY OF A MATRIX IN EF HAS ALL ZEROS BELOW

(ALL ENTRIES ON THE SAME COLUMN THAT ARE BELOW ARE ZERO)

$$\left[\begin{array}{cccc|c} 0 & -1 & -4 & 3 & 3/4 \\ -2 & 2 & -2 & 0 & 1 \\ -1 & 3 & 3 & 2 & 0 \\ -3 & -2 & -2 & 0 & 27 \end{array}\right] \qquad \left[\begin{array}{cccc|c} -1 & 3 & 3 & 2 & 0 \\ -2 & 2 & -2 & 0 & 1 \\ 0 & -1 & -4 & 3 & 3/4 \\ -3 & -2 & -2 & 0 & 27 \end{array}\right] \quad 2 \quad 3$$

WANT 0

$$\left[\begin{array}{ccccc} -1 & 3 & 3 & 2 & 0 \\ 0 & \boxed{-4} & -8 & -4 & 1 \\ 0 & -1 & -4 & 3 & 3/4 \\ 0 & -11 & -11 & -6 & 27 \end{array}\right] \quad (1/4) \quad (11/4)$$

- SUBTRACT 2^{ND} EQ $\times$ $1/4$ TO THE 3^{RD}
- SUBTRACT 2^{ND} EQ $\times$ $11/4$ TO THE 4^{TH}

$$\left[\begin{array}{cccc|c} -1 & 3 & 3 & 2 & 0 \\ 0 & -4 & -8 & -4 & 1 \\ 0 & 0 & \boxed{-2} & 4 & 1/2 \\ 0 & 0 & 11 & 5 & 27-11/4 \end{array}\right] \quad \cdot -11/2$$

$\hookrightarrow -6 + 11$

$$\left[\begin{array}{cccc|c} -1 & 3 & 3 & 2 & 0 \\ 0 & -4 & -8 & -4 & 1 \\ 0 & 0 & -2 & 4 & 1/2 \\ 0 & 0 & 0 & 27 & 27 \end{array}\right]$$

$$27 - 11/4 - \left(1/2 \cdot \left(-11/2\right)\right) = 27 - 11/4 + 11/4$$