$+ \quad \times$

$- \quad \div$

COMPUTATIONAL RULES FOR DERIVATIVES

$f, g : (a,b) \to \mathbb{R}$ $\quad -\infty \leq a < b \leq +\infty$ $\quad x_0 \in (a,b)$

SUPOSE f, g ARE DIFFERENTIABLE IN x_0 AND LET $f'(x_0)$ AND $g'(x_0)$ BE THE CORRESPONDING DERIVATES

THEN

1) $\forall c \in \mathbb{R}$ (cf) IS DIFF. IN x_0 AND

$$(cf')(x_0) = c f'(x_0)$$

2) $f+g$ AND $f-g$ ARE DIFF. IN x_0 AND

$$(f+g)'(x_0) = f'(x_0] + g'(x_0) \text{ AND}$$

$$(f-g)'(x_0) = f'(x_0) - g'(x_0)$$

3) (fg) IS DIFF. IN x_0 AND

$$(fg')(x_0) = f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0]$$

4) IF $g(x_0) \neq 0$, THEN $\frac{f}{g}$ IS DIFF. IN x_0 AND

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)\, g(x_0) - f(x_0)\, g'(x_0)}{(g(x_0))^2}$$

REM g DIFF IN $x_0 \Rightarrow g$ CONT IN $x_0 \Rightarrow$

$g \neq 0$ IS A NEIGHBOURHOOD OF x_0, THAT IS $\exists\, \delta_0 > 0$

SUCH THAT $(x_0 - \delta_0, x_0 + \delta_0) \subseteq (a, b)$ AND $g(x) \neq 0$

$\forall x \in (x_0 - \delta_0, x_0 + \delta_0)$

PROOF 1) $h \neq 0$ SUCH THAT $x_0 + h \in (a, b)$

$$\frac{(cf)(x_0+h) - (cf)(x_0)}{h} = \frac{cf(x_0+h) - cf(x_0)}{h} =$$

$$c\left[\frac{f(x_0+h) - f(x_0)}{h}\right] \to cf'(x_0)$$

2) $$\frac{(f+g)(x_0+h) - (f+g)(x_0)}{h} =$$

$$\frac{f(x_0+h) + g(x_0+h) - (f(x_0) + g(x_0))}{h} =$$

$$\frac{f(x_0+h) - f(x_0)}{h} + \frac{g(x_0+h) - g(x_0)}{h} = f'(x_0) + g'(x_0)$$

$f - g = f + (-1)g$ SUM $c = -1$

3) $$\frac{(fg)(x_0+h)-(fg)(x_0)}{h} = \frac{f(x_0+h)g(x_0+h)-f(x_0)g(x_0)}{h}$$

$$\frac{f(x_0+h)g(x_0+h) - f(x_0+h)g(x_0) + f(x_0+h)g(x_0) - f(x_0)g(x_0)}{h}$$

$$= \underbrace{f(x_0+h)}_{\substack{h\to 0 \\ f(x_0)}} \underbrace{\frac{g(x_0+h)-g(x_0)}{h}}_{\substack{h\to 0 \\ g'(x_0)}} + \underbrace{g(x_0)}_{\substack{h\to 0 \\ g(x_0)}} \underbrace{\frac{f(x_0+h)-f(x_0)}{h}}_{\substack{h\to 0 \\ f'(x_0)}}$$

f DIFF IN $x_0 \Rightarrow f$ CONT IN x_0

$$\to f(x_0)g'(x_0) + g(x_0)f'(x_0)$$

4) LET US BEGIN WITH $1/g$ $(g(x_0) \neq 0)$

$$\frac{\left(\frac{1}{g}\right)(x)-\left(\frac{1}{g}\right)(x_0)}{x-x_0} = \frac{\frac{1}{g(x)}-\frac{1}{g(x_0)}}{x-x_0} =$$

g DIFF IN $x_0 \Rightarrow g$ CONT IN x_0

$$\frac{g(x_0)-g(x)}{x-x_0} \cdot \frac{1}{g(x)g(x_0)} = -\underbrace{\frac{g(x)-g(x_0)}{x-x_0}}_{\substack{x\to x_0 \\ g'(x_0)}} \underbrace{\frac{1}{g(x)g(x_0)}}_{\frac{1}{g(x_0)^2}}$$

$$\to -\frac{g'(x_0)}{(g(x_0))^2}$$

$$\left(\frac{f}{g}\right) = f \cdot \frac{1}{g}$$

$$\left(\frac{f}{g}\right)'(x_0) = f'(x_0)\frac{1}{g(x_0)} + f(x_0)\cdot\left(\frac{1}{g}\right)'(x_0) =$$

$$= \frac{f'(x_0)}{g(x_0)} + f(x_0)\left(-\frac{g'(x_0)}{((g(x_0))^2}\right) = \frac{f'(x_0)\,g(x_0) - f(x_0)\,g'(x_0)}{(g(x_0))^2}$$

EXERCISE $f(x) = x^2 = x \cdot x \quad \forall x \in \mathbb{R}$

$f'(x) \quad 1 \cdot x + x \cdot 1 = 2x \quad \forall x \in \mathbb{R}$

$N \geq 1 \quad N \in \mathbb{N} \quad f(x) = x^N = x \cdot x^{N-1} \quad \forall x \in \mathbb{R}$

$$f'(x_0) = \frac{d\,x^N}{dx}\; 1\; x^{N-1} + x \cdot \frac{d\,x^{N-1}}{dx}$$

BY INDUCTION PROVE THAT

$$\frac{d\,x^N}{dx} = N\,x^{N-1} \quad \forall N \in \mathbb{N} \quad (x^0 = 1)$$

- $\forall n \in \mathbb{N} \cup \{0\}$ $P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

POLYNOMIAL OF DEGREE AT MOST N

P_n IS DIFF. ON $\mathbb{R}$ AND $\forall x \in \mathbb{R}$

$P_n'(x) = a_n n x^{n-1} + a_{n-1}(n-1)x^{n-1} + \dots a_2 2x \, a_1$

POLYNOMIAL OF DEGREE AT MOST N-1

$Q_m(x) = b_m x^m + \dots + b_1 x + b_0$ $m \in \mathbb{N}$ $b_m \neq 0$

POLYNOMIAL OF DEGREE m

$$\frac{P_n(x)}{Q_m(x)}$$ x SUCH THAT $Q_m(x) \neq 0$

$\forall$ x IN THE DOMAIN, SO SUCH THAT $Q_m(x) \neq 0$ WE HAVE THAT $\frac{P_n}{Q_n}$ IS DIFF. AND

$$\left(\frac{P_n}{Q_n}\right)'(x) = \frac{P_n'(x) Q_m(x) - Q_m'(x) P_n(x)}{(Q_m(x))^2}$$

- $n \in \mathbb{N}$ $\quad f(x) = \frac{1}{x^n} = x^{-n}$ $\quad x \neq 0$

$\forall x \in \mathbb{R} \setminus \{0\}$

$$\frac{d\,x^{-n}}{dx} = \frac{-n x^{n-1}}{(x^n)^2} = -n x^{-n-1} \quad \forall x \in \mathbb{R} \setminus \{0\} \quad \forall n \in \mathbb{N}$$

- $\forall a \in \mathbb{Z}$ $\forall x \in \mathbb{R}$ WITH $x \neq 0$ IF $a < 0$ WE HAVE

$$\frac{d\,x^{a}}{dx} = a x^{a-1}$$

$a > 0$ $\quad a = n \in \mathbb{N}$; $a < 0$ $\quad a = -n,\ n \in \mathbb{N}$

$a = 0$ $\quad x^0 = 1 \Rightarrow$ ITS DERIVATIVE IS $0 = 0\, x^{-1}$

$a = 1$ $\quad x^1 = x \Rightarrow$ ITS DERIVATIVE $1 = 1 \cdot x^0$

DIFFERENTIABILITY OF COMPOSITION OF FUNCTION

$f : (a, b) \to \mathbb{R}$ $\quad -\infty \leq a < b \leq +\infty$

$g : (c, d) \to \mathbb{R}$ $\quad -\infty \leq c < d \leq +\infty$

SUPPOSE $f((a,b]) \subseteq (c,d)$ SO THAT IT IS

WELL DEFINED THE FUNCTION

$$h = g \circ f : (a,b] \to \mathbb{R}$$

LET $x_0 \in (a,b]$ AND $y_0 = f(x_0) \in (c,d)$ AND ASSUME

f IS DIFF IN x_0 WITH DERIVATIVE $f'(x_0)$, AND

g IS DIFF IN y_0 WITH DERIVATIVE $g'(y_0)$

THEN $(g \circ f)$ IS DIFF IN x_0 AND

$$\boxed{(g \circ f)'(x_0) = g'(y_0) \cdot f'(x_0) = g'(f(x_0))\, f'(x_0)}$$ CHAIN RULE

IDEA OF THE PROOF

HAP $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$ $\lim_{y \to y_0} \frac{g(y)-g(y_0)}{y-y_0} = g'(y_0)$

WE WANT TO COMPUTE

$$\lim_{x \to x_0} \frac{(g \circ f) - (g \circ f)(x_0)}{x - x_0} =$$

$$\lim_{x \to x_0} \frac{g(f(x_0)) - g(f(x_0))}{x - x_0}$$

$$\frac{g(f(x)) - g(y_0)}{x - x_0} \text{``}=\text{''} \frac{g(f(x)) - g(y_0)}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0} \text{''}$$

$y = f(x)$

$$\lim_{x \to x_0} \frac{g(f(x_0)) - g(y_0)}{f(x) - y_0} \text{``}=\text{''} \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0}$$

CAREFUL WHEN $f(x) = f(x_0)$!!

EXAMPLE

$$\lim_{x \to 0} \frac{\sin(x) - \sin(0)}{x - 0} = \lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

$$\frac{\sin(x+h) - \sin(x)}{h} = \frac{\sin(x)\cos(h) + \sin(x)\cos(x) - \sin(x)}{h}$$

$$= \sin(x) \underbrace{\left[\frac{\cos(h) - 1}{h} \right]}_{\downarrow\, h \to 0 \atop 0} + \cos(x) \underbrace{\frac{\sin(h)}{h}}_{h \to 0 \atop 1} \to \cos(x)$$

$x \in \mathbb{R}$

SIN IS DIFF. IN $\mathbb{R}$ AND

$$\boxed{\frac{d}{dx}\sin(x) = \cos(x) \quad \forall x \in \mathbb{R}}$$

• COS IS DIFF IN $\mathbb{R}$ AND

$$\boxed{\frac{d}{dx}\cos(x) = -\sin(x) \quad \forall x \in \mathbb{R}}$$

$$\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$$

$$f(x) = \frac{\pi}{2} - x \qquad g(y) = \sin(y)$$

$$f'(x) = -1 \qquad g'(y) = \cos(y)$$

$$\frac{d\cos(x)}{dx} = g'(f(x))\, f'(x) = \cos\left(\pi/2 - x\right)(-1) = -\sin(x)$$

• $$\lim_{x \to 0} \frac{e^x - e^0}{x - 0} = \lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

$$\frac{e^{x+h} - e^x}{h} = e^x \frac{e^h - 1}{h} \xrightarrow{h \to 0} e^x \, 1 = e^x$$

EXP IS DIFF. IN $\mathbb{R}$ AND

$$\frac{d e^x}{dx} = e^x \quad \forall x \in \mathbb{R}$$

EXERCISE $a > 0$ $a \neq 1$ COMPUTE $\frac{d\, a^x}{dx}$

$a^x = e^{(\log a) x}$

- TAN IS DIFF. AT ANY $x \in \mathbb{R}$ SUCH THAT $\cos(x) \neq 0$

AND $\frac{d \tan(x)}{dx} = \frac{1}{\cos^2(x)} = 1 + \tan^2(x)$ $\forall x \in \mathbb{R}$ WITH $\cos(x) \neq 0$

$$\left(\frac{\sin}{\cos}\right)'(x) = \frac{\sin'(x)\cos(x) - \cos'(x)\sin(x)}{\cos^2(x)} =$$

$$= \frac{\cos(x)\cos(x) - (-\sin(x))\sin(x)}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

$$= \frac{1}{\cos^2(x)} = 1 + \tan^2(x)$$

EXAMPLE $f(x) = \sin(3x^2)$ $x \in \mathbb{R}$

$x \to y = 3x^2 \to \sin(y)$

$f'(x) = \sin'(3x^2) \cdot \frac{d\, 3x^2}{dx} = \cos(3x^2) \cdot 6x$

• $f(x) = x^3 e^{2\cos(x)}$ $x \in \mathbb{R}$

$$3x^2 e^{2\cos(x)} + x^3 e^{2\cos(x)}(-2\sin(x))$$

$$\frac{d}{dx} e^{2\cos(x^2)} \Rightarrow e^{2\cos(x^2)} \frac{d}{dx}(2\cos(x^2)) =$$

$$e^{2\cos(x^2)} \cdot (-2\sin(x^2))(2x)$$

DERIVATIVE OF THE INVERSE FUNCTION

REMARK $f: (a, b) \to \mathbb{R}$ $-\infty \leq a < b \leq +\infty$

f CONTINOUS AND INVERTIBLE (THAT IS, INJECTIVE)

THEN $f(a,b)$ IS AN OPEN INTERVAL

RECALL THAT f IS STRICTLY MONOTONE!

THEOREM

$f: (a,b) \to \mathbb{R}$ $-\infty \leq a < b \leq +\infty$

f CONTINOUS AND INVERTIBLE

LET $f^{-1}: (c, d) \to (a, b)$ WITH

$-\infty \leq c < d \leq +\infty$ SUCH THAT $f((a,b)) = (c,d)$

LET $x_0 \in (a,b)$ AND ASSUME f IS DIFFERENTIABLE IN x_0 WITH $f'(x_0) \neq 0$ THEN

f^{-1} IS DIFFERENTIABLE IN $y_0 = f(x_0)$ AND

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

IDEA OF THE PROOF $x \neq x_0 \Rightarrow f(x) \neq f(x_0)$ INJECTIVITY

$$\lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)} \quad (f'(x_0) \neq 0)$$

$$\| $$

$$\frac{f^{-1}(f(x)) - f^{-1}(y_0)}{f(x) - y_0} \underset{y = f(x)}{=} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0}$$

HOW TO RECALL THIS FORMULA

SUPPOSE TO KNOW f^{-1} IS DIFF. IN y_0

THEN $(f^{-1} \circ f)(x) = x$

$$1 = \frac{dx}{dx} = (f^{-1})'(f(x)) \cdot f'(x)$$

$$\Rightarrow (f^{-1})'(f(x)) = \frac{1}{f'(x)} \quad \text{IF } f'(x) \neq 0$$

EXAMPLE $\forall n \in \mathbb{N}$ $\forall x > 0$ $\frac{d x^n}{dx} = n x^{n-1} \neq 0$

so $\frac{d \sqrt[n]{x}}{dx} = \frac{d x^{1/n}}{dx} = \frac{1}{n} x^{1/n - 1}$ $\forall x > 0$ $\forall n \geq 2$

$y \to \sqrt{y}$ $y > 0$ $x = \sqrt{y}$ $y = x^2$ $x > 0$

$$\frac{d \sqrt{y}}{dy} = \frac{1}{\left.\frac{d x^2}{dx}\right|_{x=\sqrt{y}}} = \frac{1}{2\sqrt{y}} = \frac{1}{2} y^{1/2 - 1}$$

- $$\frac{d x^{5/2}}{dx} = \frac{d (\sqrt{x})^5}{dx} = 5 \sqrt{x}^4 \cdot \frac{1}{2\sqrt{x}} = \frac{5}{2} x^{5/2 - 1} \qquad x > 0$$

$$\left.\frac{d x^{5/2}}{dx}\right|_{x=0^+} = \lim_{x \to 0^+} \frac{x^{5/2} - 0^{5/2}}{x - 0} = \lim_{x \to 0^+} x^{5/2 - 1} = 0$$

WE CAN SHOW $m, n \in \mathbb{N}$ WITH NO COMMON FACTORS

(a) $\frac{d}{dx} x^{\pm m/n} = \pm \frac{m}{n} x^{\pm m/n - 1}$ $\forall x > 0$

(b) $\left.\frac{d}{dx} x^{m/n}\right|_{x=0^+} = 0$ if $\frac{m}{n} > 1$

IF N IS ODD (a) HOLDS $\forall x \neq 0$

(b) $\left. \frac{d}{dx} x^{m/n} \right|_{x=0} = 0$ IF $\frac{m}{n} > 1$

$\frac{d\, e^x}{dx} = e^x$ $\log (0,+\infty) \to \mathbb{R}$ IS DIFF. IN $(0,+\infty)$

$\frac{d}{dx} \log(x) = \frac{1}{x}$ $\forall x > 0$

$\frac{d}{dy} \log(y) = \frac{1}{e^{\log(y)}} = \frac{1}{y}$ $\forall y > 0$

- $b \neq 0$ $b \in \mathbb{R}$

x^b $\boxed{x > 0}$

$\frac{d\, x^b}{dx} = \frac{b}{x} e^{(\log x) b} = b\, x^{b-1}$ $\boxed{x > 0}$

IF $b > 1$ THE $\left. \frac{d\, x^b}{dx} \right|_{x = 0^+} = 0$

$\frac{d}{dx} \cos(x) = -\sin(x) \Rightarrow \text{ARCOS}: [-1, 1] \to [0, \pi]$

IS DIFF $\forall x \in (-1, 1)$ AND

$$\frac{d}{dx} \text{ARCOS}(x) = -\frac{1}{\sqrt{1-x^2}} \quad \forall x \in (-1, 1)$$

IN FACT $y_0 \in (-1, 1)$

$$\frac{d}{dx} \text{ARCOS}(y_0) = \frac{1}{-\sin(\text{ARCOS}(y_0))} =$$

$x_0 = \text{ARCOS}(y_0) \in (0, \pi)$ SO $\sin(x_0) > 0$

AND $\sin(x_0) = \sqrt{1 - \cos^2(x_0)}$ BUT

$\cos(x_0) = y_0$

$$= \frac{-1}{\sqrt{1 - \cos(\text{ARCCOS}(y_0))^2}} = -\frac{1}{\sqrt{1 - y_0^2}}$$

- $\frac{d}{dx} \sin(x) = \cos(x) \Rightarrow \text{ARCSIN}\ [-1, 1] \to [-\pi/2, \pi/2]$

IS DIFF. $\forall x \in (-1, 1)$ AND

$$\boxed{\frac{d\, \arcsin(x)}{dx} = \frac{1}{\sqrt{1-x^2}} \quad \forall x \in (-1,1)}$$

$y_0 \in (-1,1)$

$$\frac{d\, \arcsin(y_0)}{dy} = \frac{1}{\cos(\arcsin(y_0))}$$

$y_0 \in (-1,1) \Rightarrow x_0 = \arcsin(y_0) \in (-\pi/2, \pi/2)$ so

$\cos(x_0) > 0 \Rightarrow \cos(x_0) = \sqrt{1-\sin^2(x_0)}$ AND

$\sin(x_0) = \sin(\arcsin(y_0)) = y_0$

- $$\frac{d \tan}{dx} = 1 + \tan^2(x) > 0 \quad \forall x \in (-\pi/2, \pi/2)$$

$\Rightarrow \arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ IS DIFF.

$\forall x \in \mathbb{R}$ AND $\frac{d}{dy} \arctan(x) = \frac{1}{1+x^2} \quad \forall x \in \mathbb{R}$

$y_0 \in \mathbb{R}$ $$\frac{d\, \arctan(y_0)}{dy} = \frac{1}{1+\tan^2(\arctan(y_0))} =$$

$$= \frac{1}{1+y_0^2}$$