$+$ $\times$

$-$ $\div$

EXERCISE PROVE THAT THE ∂S IS ALWAYS A CLOSE SET

BOUNDERY of S

$$\partial S = \overline{S} \cap \overline{S^c}$$

OTHER REMARKS OF DEFINITION OF LIMIT

- $\lim_{A \ni x \to x_0} f(x) = y^0 = (y_1^0, \dots y_n^0) \iff \lim_{A \ni x \to x^0} f_i(x) = y_i^0 \quad \forall i = 1, \dots N$

- LET $C > 0$ BE A CONSTANT. WE CAN REPLACE $< \delta$ WITH $\leq \delta$ AND $< \varepsilon$ WITH $< \varepsilon, \leq \varepsilon, < C\varepsilon, \leq C\varepsilon$

- WE CONSIDER $x \in A$ WITH $x \neq x^0$. SO

x^0 COULD NOT BELONG TO A, SO f COULD NOT BE DEFINED IN x^0. EVEN IF f IS DEFINED IN x^0, THE VALUE OF f IN x^0 IS IRRELEVANT FOR THE LIMIT

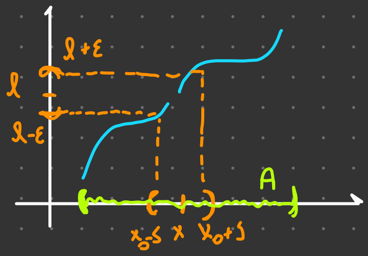
THE CASE $\boxed{M = 1}$ AND $\boxed{M = 1}$

$\boxed{M = 1}$

- $\lim_{A \ni x \to x^0} f(x) = l$ IN $\mathbb{R}$ IF

$\forall \varepsilon > 0 \quad \exists \delta > 0$ SUCH THAT $\forall x \in A$ WITH $0 < d(x, x^0) < \delta$

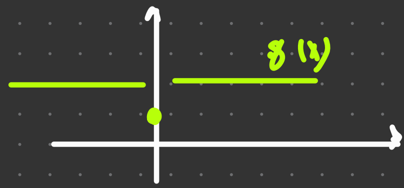
WE HAVE $|f(x) - l| < \varepsilon$ OR $f(x) \in (l - \varepsilon, l + \varepsilon)$



EXAMPLE

$$g(x)\begin{cases} 2 & x\neq 0 \\ 1 & x=0 \end{cases}$$

WHAT HAPPEN TO 0 IS IRRELEVANT



$$\lim_{x\to 0} g(x) = 2$$

- $\lim_{A\ni x\to x^0} f(x) = \pm\infty$ IF

$\forall M \gtrless 0 \;\exists \delta (>0)$ SUCH THAT $\forall x\in A$ WITH

$0<d(x,x^0)<\delta$ WE HAVE $f(x) \gtrless M$

$\boxed{M=1}$ $f: A \subseteq \mathbb{R} \to \mathbb{R}$ ($n=1$ FOR SIMPLICITY)

$x_0 \in \mathbb{R}$ SUCH THAT x_0 IS AN ACCUMULATION POINT FOR A FROM THE RIGHT (LEFT). WE CAN DEFINE, IF IT EXIST

$$\lim_{A\ni x\to x_0^{\pm}} f(x) = l \in [-\infty,+\infty]$$

IF

- $x \to x_0^+$, $l \in \mathbb{R}$

  $\forall \varepsilon > 0$ $\exists \delta > 0$ SUCH THAT $\forall x \in A$ WITH $x_0 < x < x_0 + \delta$

  WE HAVE $l - \varepsilon < f(x) < l + \varepsilon$

- $x \to x_0^-$ $l = -\infty$

  $\forall M < 0$ $\exists \delta > 0$ SUCH THAT $\forall x \in A$ WITH $x_0 - \delta < x < x_0$

  WE HAVE $f(x) < M$

- IF A IS UNBOUND FROM ABOVE BELOW WE CAN DEFINE, IF IT EXIST

  $\lim_{x \to \pm\infty} f(x) = l \in [-\infty, +\infty]$

IF

- $x \to +\infty$, $l \in \mathbb{R}$

  $\forall \varepsilon > 0$ $\exists L > 0$ SUCH THAT $\forall x \in A$ WITH $x > L$ WE HAVE

  $l - \varepsilon < f(x) < l + \varepsilon$

- $x \to -\infty$, $l = -\infty$

  $\forall M < 0$ $\exists L < 0$ SUCH THAT $\forall x \in A$ WITH $x < L$

  WE HAVE $f(x) < M$

PROPOSITION

LET $f : A \subseteq \mathbb{R} \to \mathbb{R}$ ($M = 1$ FOR SIMPLICITY)

x_0 ACC. POINT FOR A FROM THE LEFT AND FROM THE RIGHT

LET $l \in [-\infty, +\infty]$ THEN

$$\exists \lim_{A \ni x \to x_0} f(x) = l \iff \exists \lim_{A \ni x \to x_0^+} = l \text{ AND } \exists \lim_{A \ni x \to x_0^-} f(x) = l$$

PROOF ONLY $l \in \mathbb{R}$ "$\Rightarrow$"

HYP $\exists \lim_{x \to x_0} f(x) = l$, THAT IS, $\forall \varepsilon > 0\ \exists \delta > 0$ SUCH THAT

$\forall x \in A$ WITH $0 < d(x, x_0) < \delta$ WE HAVE $f(x) \in (l - \varepsilon, l + \varepsilon)$

THESIS $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = l$, THAT IS,

$\forall \varepsilon > 0\ \exists \delta_1 > 0$ SUCH THAT $\forall x \in A$ WITH $x_0 < x < x_0 + \delta_1$ WE HAVE $f(x) \in (l - \varepsilon, l + \varepsilon)$

AND $\forall \varepsilon > 0\ \exists \delta_2 > 0$ SUCH THAT $\forall x \in A$ WITH

$x_0 - \delta_2 < x < x_0$ WE HAVE $f(x) \in (l - \varepsilon, l + \varepsilon)$

FIX $\varepsilon > 0$ PICK $\delta_1 = \delta_2 = \delta > 0$ THEN $0 < d(x, x_0) < \delta$

IS EQUIVALENT TO SAY $x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta) =$

$(x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$

" $\Leftarrow$ " fix $\varepsilon > 0$ pick $\delta = \min\{\delta_1, \delta_2\}$ then

$\forall x \in A$ with $0 < d(x, x_0) < \delta$ we have either

$x_0 - \delta_1 \leq x_0 - \delta < x < x_0$ or

$x_0 < x < x_0 + \delta \leq x_0 + \delta_1$

hence $f(x) \in (l - \varepsilon, l + \varepsilon)$

THEOREM (BRIDGE THEOREM)

$f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ $\quad x^0$ acc. point for A. Then

$$\lim_{A \ni x \to x^0} f(x) = l \in [-\infty, +\infty]$$

$\Longleftrightarrow$

$\forall \{x^n\}_{n \in \mathbb{N}} \subseteq A \setminus \{x^0\}$ such that $x^n \to x^0$ as $n \to +\infty$

sequence

we have $\lim_n f(x^n) = l$

PROOF on $\langle\!$ " $\Rightarrow$ " with $l \in \mathbb{R}$

we know $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $\forall x \in A$ with

$0 < d(x, x^0) < \delta$ we have $|f(x) - l| < \varepsilon$

LET US FIX $\varepsilon > 0$ AND $\delta > 0$ CORRESPONDING

LET $\{x^n\}_{n\in\mathbb{N}} \subseteq A \setminus \{x^0\}$ SUCH THAT $\lim_n x^n = x^0$

THAT IS, $\lim_n d(x^n, x^0) = 0$ SO $\exists\, n_0 \in \mathbb{N}$ SUCH THAT

$\forall n \geq n_0$ WE HAVE $0 < d(x^n, x^0) < \delta$ HENCE

$|f(x) - l| < \varepsilon$

REMARK SAME RESULT OF BRIDGE THEOREM

- $\lim_{x\to x_0^{\pm}}$ $\quad \forall \{x_n\}_{n\in\mathbb{N}} \subseteq A$ WITH $x_n \to x_0^{\pm}$

- $\lim_{x\to\pm\infty}$ $\quad \forall \{x_n\}_{n\in\mathbb{N}} \subseteq A$ WITH $x_n \to \pm\infty$

EXAMPLES

- LET $c \in \mathbb{R}$ AND $f(x) = c \quad \forall x \in \mathbb{R}$

$$\forall x_0 \in \mathbb{R} \quad \lim_{x\to x_0} f(x) = \lim_{x\to x_0} c = c$$

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} c = c$$

- LET $f(x) = x \quad \forall x \in \mathbb{R}$

$\forall x_0 \in \mathbb{R} \quad \lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0$

$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} x = \pm\infty$

- LET $f(x) = |x| \quad \forall x \in \mathbb{R}$

$\forall x_0 \in \mathbb{R} \quad \lim_{x \to x_0} |x| = |x_0|$

FIX $\varepsilon > 0$ PICK $\delta = \varepsilon$, IN FACT, $\big||x| - |x_0|\big| \leq |x - x_0|$

$\lim_{x \to \pm\infty} |x| = +\infty$

- $\lim_{x \to +\infty} \frac{1}{x} = 0$

$\forall \varepsilon > 0$ LET $L = \frac{1}{\varepsilon}$ THEN $\forall x \in \mathbb{R}$ WITH $x > L$ WE HAVE $\left|\frac{1}{x} - 0\right| = \frac{1}{x} < \frac{1}{L} = \varepsilon$

EXERCISE USING DEFINITON PROVE

$\lim_{x \to -\infty} \frac{1}{x} = 0$; $\lim_{x \to \pm\infty} \frac{1}{x^2} = 0$

- $x_0 \neq 0$ THE $\lim_{x \to x_0} \frac{1}{x} = \frac{1}{x_0}$ AND $\lim_{x \to x_0} \frac{1}{x^2} = \frac{1}{x_0^2}$

LET $\{x_n\}_{n \in \mathbb{N}}$ SUCH THAT $x_n \to x_0$ AND $(x_n \neq x_0)$

$(x_n \neq 0$ DEFINETLY$)$

$\lim_{n} \frac{1}{x_n} = \frac{1}{x_0}$ AND $\lim_{n} \frac{1}{x_n^2} = \frac{1}{x_0^2} =$

IMPORTAN REMARK BRIDGE THEOREM $\Rightarrow$

UNIQUENESS OF THE LIMIT FOR FUNCTIONS!

- $\lim_{x \to 0} \frac{x}{|x|}$

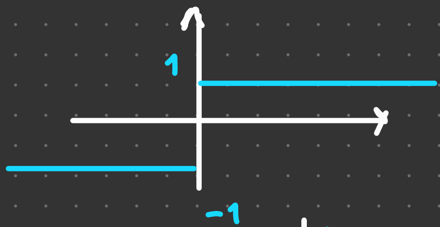
$\frac{x}{|x|}$ SIGN OF x, $x \neq 0$

$$\text{SIGN} = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases}$$

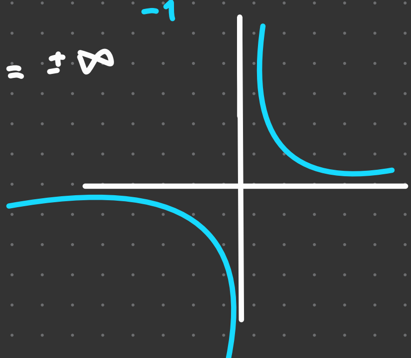
$\lim_{x \to 0^+} \frac{x}{|x|} = \lim_{x \to 0^+} 1 = 1$; $\lim_{x \to 0^-} \frac{x}{|x|} = \lim_{x \to x_0^-} (-1) = -1$

TWO LIMITS ARE DIFFERENT

$\nexists \lim_{x \to 0} \frac{x}{|x|}$



- $\nexists \lim_{x \to 0} \frac{1}{x}$ BECAUSE $\lim_{x \to 0^{\pm}} \frac{1}{x} = \pm\infty$



$x_n \to 0^+ \Rightarrow \frac{1}{x_n} \to +\infty$

$\nexists \lim_{x \to 0} \frac{1}{x^2} = +\infty \quad x_n \to 0,\ x_n \neq 0 \quad \frac{1}{x_n^2} \to +\infty$

CONTINUOUS FUNCTION

DEFINITION $f: A \subseteq \mathbb{R}^n \to \mathbb{R}^N$ FUNCTION

LET $x^0 \in A$ f IS CONTINUOUS IN x^0

IF $\forall \varepsilon > 0$ $\exists \delta > 0$ SUCH THAT $\forall x \in A$ WITH

$0 \leq d(x, x^0) < \delta$ WE HAVE $d(f(x), f(x^0)) < \varepsilon$

REMARK $d(x, x^0) < \delta \iff x \in B_\delta(x^0)$

$\boxed{N = 1}$ $d(x, x^0) < \delta \iff x \in (x_0 - \delta, x_0 + \delta)$

- f IS CONTINUOUS IN x^0 IF AND ONLY IF f_i IS CONTINUOUS IN x^0 $\forall i = 1, \dots M$

FOR LIMITS AND CONTINUITY WE CAN RESTRICT OURSELVES

TO $N=1$ BY WORKING COMPONENT BY COMPONENT f_i

THEOREM: $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^N;\ x^0 \in A$

- IF x^0 IS AN ISOLATED POINT OF A, THAN f IS CONTINUOUS IN x^0

- IF x^0 IS AN ACCUMULATION POINT OF A, THEN f IS CONTINUOUS IN $x^0 \iff \exists \lim_{x \to x^0} f(x) = f(x^0)$

IN BOTH CASES

f IS CONTINUOUS IN $x^0 \iff \forall \{x^n\}_{n \in \mathbb{N}} \subseteq A$ WITH $x^n \to x^0$ AS $n \to +\infty$ WE HAVE $f(x^n) \to f(x^0)$

PROOF $f : A \subseteq \mathbb{R} \to \mathbb{R}$ $x_0 \in A$ ACC. POINT

FROM LEFT TO RIGHT

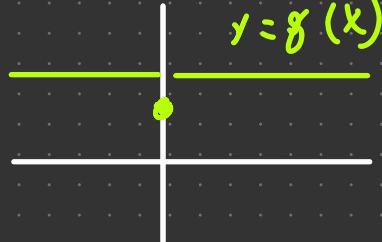
f CONTINUOUS AT x_0

$\iff$

$\exists \lim_{x \to x_0^-} f(x) = f(x_0)$ AND $\exists \lim_{x \to x_0^+}$ + COMPLETARE

EXAMPLE

$$g(x) = \begin{cases} 2 & x \neq 0 \\ 1 & x = 0 \end{cases}$$



g IS NOT CONTINUOUS IN 0

$\exists \lim_{x \to x_0} g(x) = 2$; $\exists \lim_{x \to 0^+} g(x) = 2 \Rightarrow \exists \lim_{x \to 0} g(x) = 2$

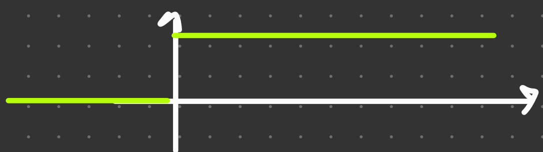
BUT $2 \neq g(x) = 1$

IF $\exists \lim_{x \to x_0} f(x) = l \in \mathbb{R}$ BUT $l \neq f(x_0)$

WE SAY THE DISCONTINUITY IS REMOVABLE

???

2) $f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$ HAVISVE FUNCTION



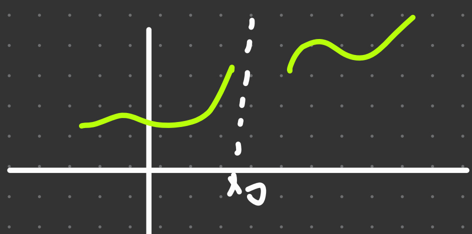
$\exists \lim_{x \to 0^+} f(x) = 1$

$\exists \lim_{x \to 0^-} f(x) = 0$

BUT $1 \neq 0$ SO $\nexists \lim_{x \to 0} f(x)$

- IF $\exists \lim_{x \to x_0^-} f(x) = l_- \in \mathbb{R}$, $\exists \lim_{x \to x_0^+} f(x) = l_+ \in \mathbb{R}$

BUT $l_- \neq l_+$ WE SAY THAT WE HAVE A JUMP DISCONTINUITY



DEFINITION $f: A \subseteq \mathbb{R}^N \to \mathbb{R}^N$ IS CONTINUOUS ON A IF IT IS CONTINUOUS IN x^0 $\forall x^0 \in A$

EXAMPLE

- $c \in \mathbb{R}$ $f(x) = c$ IS CONTINUOUS ON $\mathbb{R}$
- $f(x) = x$ IS CONTINUOUS ON $\mathbb{R}$
- $f(x) = |x|$ IS CONTINUOUS ON $\mathbb{R}$
- $f(x) = \frac{1}{x}$ IS CONTINUOUS ON ITS DOMAIN $\mathbb{R} \setminus \{0\}$
- IF $x_0 = 0$, $f(x) = \frac{1}{x}$ IS **NOT** DEFINED IN $x_0 = 0$

WE CAN TALK ABOUT LIMIT $x \to 0$ BUT <u>NOT</u> CONTINUITY

EXTENSION BY CONTINUITY $f: A \subseteq \mathbb{R}^N \to \mathbb{R}$

ASSUME $x^0 \notin A$ AND IT IS AN ACCUMULATION POINT FOR A. ASSUME $\exists \lim_{A \ni x \to x^0} f(x) = \underline{\underline{l \in \mathbb{R}}}$

WE CAN DEFINE $\tilde{f}: A \cup \{x^0\} \to \mathbb{R}$ SUCH THAT

$$\tilde{f}(x) = \begin{cases} f(x) & x \in A \\ l & x = x^0 \end{cases}$$

$\tilde{f}$ IS CALLED BY CONTINUITY OF f IN x^0, NOTE THAT $\tilde{f}$ IS CONTINUOUS IN x^0!

USUALLY $\tilde{f}$ IS CALLED f

SAME FOR REMOVABLE DISCONTINUITY

$$x^0 \in A \quad \lim_{x \to x^0} f(x) = l \in \mathbb{R} \text{ with } l \neq f(x^0)$$

DEFINE

$$\tilde{f}(x) = \begin{cases} f(x) & x \in A \setminus \{x^0\} \\ l & x = x^0 \end{cases}$$

WHICH IS CONTINUOUS AT x^0

PROPERTIES OF LIMITS AND CONTINUOUS FUNCTION

PERMANENCE OF SIGN THEOREM (LIMITS)

$$f : A \subseteq \mathbb{R}^n \to \mathbb{R}$$

x^0 ACC POINT FOR A. SUPPOSE $\exists \lim_{x \to x^0} f(x) = l \in [-\infty, +\infty]$

THEN

a) IF $l \underset{<}{>} 0$ (INCLUDING $\underline{+}\infty$), THEN $\exists\, \delta > 0$ SUCH THAT

$\forall\, x \in A$ WITH $0 < d(x, x^0) < \delta$ $\quad (\forall x \in A \cap B_\delta(x^0) \setminus \{x^0\})$

WE HAVE $f(x) \underset{<}{>} 0$

a1) if $f(x) \underset{\le}{\ge} 0 \quad \forall x \in A \setminus \{x^0\}$

OR $\forall x \in A \cap (B_{r_0}(x^0) \setminus \{x^0\})$ for some $r_0 > 0$

then $\lim_{x \to x^0} f(x) = l \underset{\le}{\ge} 0$

PROOF a1) $l > 0$ $l \in \mathbb{R}$

fix $\varepsilon > 0$, such that $l - \varepsilon > 0$ then $\exists \delta > 0$

such that $\forall x \in A$ with $0 < d(x, x^0) < \delta$ we

have $0 < l - \varepsilon < f(x) < l + \varepsilon$

PERMANENCE OF THE SIGN THEOREM (CONTINUOUS FUNCTIONS)

$f: A \subseteq \mathbb{R}^N \to \mathbb{R}$ $x^0 \in A$ f continuous in x^0

if $f(x^0) \underset{<}{>} 0$ then $\exists \delta > 0$ such that

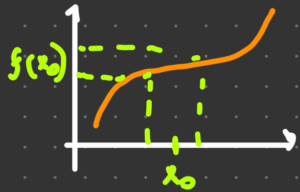
$\forall x \in A$ with $d(x, x^0) < \delta$ (that is, $\forall x \in A \cap B_\delta(x^0)$)

we have $f(x) \underset{<}{>} 0$

PROOF $f(x^0) > 0$ let $\varepsilon > 0$ such that $f(x_0) - \varepsilon > 0$

then $\exists \delta > 0$ such that $\forall x \in A \cap B_\delta(x^0)$ we

have $0 < f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon$



IMPORTANT THEOREM $f: \mathbb{R}^N \to \mathbb{R}^N$

f is continuous on $\mathbb{R}^N$ $\Leftrightarrow$

$\forall A_1 \subseteq \mathbb{R}^N$ OPEN we have $f^{-1}(A_1)$ is open in $\mathbb{R}^N$

$\forall C_1 \subseteq \mathbb{R}^N$ CLOSED we have $f^{-1}(C_1)$ is closed in $\mathbb{R}^N$

IDEA: $M=1$ A_1 open "$\Rightarrow$"

if $x \in f^{-1}(A_1)$ then $f(x) = y \in A_1$

$\Rightarrow \exists \varepsilon > 0$ such that $(y-\varepsilon, y+\varepsilon) \subseteq A_1$

$\Rightarrow \exists \delta > 0$ such that $B_\delta(x) \subseteq f^{-1}(A_1)$

$\Rightarrow f^{-1}(A_1)$ open

COMPARISON THEOREM

$f: A \subseteq \mathbb{R}^N \to \mathbb{R}$, $g: A \subseteq \mathbb{R}^N \to \mathbb{R}$

x^0 acc. point for A

Suppose $f(x) \le g(x)$ $\forall x \in A \setminus \{x^0\}$ then

a) if $\exists \lim_{x \to x^0} f(x) = +\infty$, then $\exists \lim_{x \to x_0} g(x) = +\infty$

b) IF $\exists \lim_{x \to x^0} g(x) = -\infty$, THEN $\exists \lim_{x \to x^0} f(x) = -\infty$

c) IF $\exists \lim_{x \to x^0} f(x) = a \in [-\infty, +\infty]$ AND $\exists \lim_{x \to x^0} g(x) b \in [-\infty, +\infty]$

THEN $a \leq b$

SANDWICH THEOREM $f, g, h : A \subseteq \mathbb{R}^N \to \mathbb{R}$

x^0 ACC. POINT FOR A $\quad f(x) \leq h(x) \leq g(x) \quad \forall x \in A \setminus \{x^0\}$

IF FOR SOME $l \in [-\infty, +\infty]$ WE HAVE

$\lim_{x \to x^0} f(x) = l$ AND $\lim_{x \to x^0} g(x) = l$, THEN

$$\exists \lim_{x \to x^0} h(x) = l$$

PROOF $l \in \mathbb{R}$ ($l = \pm\infty$, FOLLOW BY COMPARISON a), b) ABOVE

TWO WAYS

1) WE KNOW

$\forall \varepsilon > 0 \; \exists \delta_1 > 0$ SUCH THAT $\forall x \in A$ WITH $0 < d(x, x^0) < \delta_1$ WE HAVE $l - \varepsilon < f(x) < l + \varepsilon$

$\forall \varepsilon > 0 \; \exists \delta_2 > 0$ SUCH THAT $\forall x \in A$ WITH $0 < d(x, x^0) < \delta_2$ WE HAVE $l - \varepsilon < g(x) < l + \varepsilon$

FIX $\varepsilon > 0$ CHOOSE $\delta = \min\{\delta_1, \delta_2\} > 0$ THEN

$\forall x \in A$ WITH $0 < d(x, x^0) < \delta$ WE HAVE

$$l - \varepsilon < f(x) \leq h(x) \leq g(x) < l + \varepsilon$$

$$\Rightarrow \lim_{x \to x^0} h(x) = l$$

2) $\forall \{x^n\}_{n \in \mathbb{N}} \subseteq A \setminus \{x^0\}$ SUCH THAT $x^n \to x^0$

WE KNOW $\lim_n f(x^n) = l$ AND $\lim_n g(x^n) = l$

BUT $f(x^n) \leq h(x^n) \leq g(x^n) \Rightarrow \lim_n h(x^n) = l$

$$\Rightarrow \lim_{x \to x^0} h(x) = l$$

THEOREM $f: A \subseteq \mathbb{R}^N \to \mathbb{R}$ $\quad g: A \subseteq \mathbb{R}^N \to \mathbb{R}$

x^0 ACC. POINT FOR A. SUPPOSE THAT

$\exists \lim_{x \to x^0} f(x) = a \in \mathbb{R}$ AND $\exists \lim_{x \to x^0} g(x) = b \in \mathbb{R}$

THEN

1) $\forall c \in \mathbb{R}$, $\lim_{x \to x_0} (cf) = ca$

2) $\lim_{x \to x^0} (f+g)(x) = a+b$; $\lim_{x \to x^0} (f-g)(x) = a-b$

3) $\lim_{x \to x^0} (fg)(x) = ab$

4) IF $b \neq 0$ $\lim_{x \to x_0} \left(\frac{f}{g}\right)(x) = \frac{a}{b}$

NOTE: $\exists r_0 > 0$ SUCH THAT $\forall x \in A \cap (B_r(x^0) \setminus \{x^0\})$
$g(x) \neq 0$

THEOREM: $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ $g : A \subseteq \mathbb{R}^n \to \mathbb{R}$

$x^0 \in A$ ASSUME f, g CONTINUOUS IN x^0. THEN

1) $\forall c \in \mathbb{R}$ cf IS CONT IN x^0

2) $f+g$ AND $f-g$ ARE CONT IN x^0

3) $f \cdot g$ IS CONT IN x^0

4) IF $g(x^0) \neq 0$ THEN $\frac{f}{g}$ CONT IN x^0

NOTE $\frac{f}{g}$ WELL-DEFINED IN A NEIGHBOURHOOD OF x^0

$\exists r_0 > 0$ SUCH THAT $\forall x \in A \cap B_{r_0}(x^0)$ WE HAVE $g(x) \neq 0$

PROOF x^0 IS ISOLATED OK

- x^0 ACCUM POINT

3) fg CONT IN $x^0 \iff \lim_{x \to x^0} (fg)(x) = f(x^0)\, g(x^0)$
$$\underset{(fg)(x^0)}{\parallel}$$

BUT $\lim_{x \to x^0} f(x) = f(x^0)$ AND $\lim_{x \to x^0} g(x) = g(x^0)$

$\overset{?}{\Rightarrow} \lim_{x \to x^0} f(x)g(x) = f(x^0)g(x^0)$??

$\{x^n\}_{n \in \mathbb{N}} \in A \setminus \{x^0\}$ such that $x^n \to x^0$

WE KNOW $\lim_n f(x^n) = \underset{\uparrow \mathbb{R}}{f(x^0)}$ AND $\lim_n g(x^n) = \underset{\uparrow \mathbb{R}}{g(x^0)}$

$\Rightarrow \lim_n f(x^n)g(x^n) = f(x^0)g(x^0)$

REMARK

LIMIT AND CONTINUITY ARE LOCAL PROPERTIES, THAT IS, DEPEND ON THE BEHAVIOUR OF f IN A NEIGHBOURHOOD OF x^0. SO IN $A \cap B_r(x^0)$ FOR SOME $r_0 > 0$ (FOR THE LIMIT, EXCEPT x^0 ITSELF)

IN THE ASSUMPTION IT IS ENOUGH TO REQUIRE THEM

$\forall x \in A \cap (B_{r_0}(x^0) \underbrace{\setminus \{x^0\}}_{\text{LIMITS}})$ FOR SOME $r_0 > 0$

$\boxed{A \subseteq \mathbb{R}}$, $x_0 \in \mathbb{R}$

$x \to x_0^+ \quad A \cap (x_0, x_0 + r_0)$ FOR SOME $r_0 > 0$

$x \to x_0^- \quad A \cap (x_0 - r_0, x_0)$ FOR SOME $r_0 > 0$

$x \to +\infty \quad A \cap (a_0, +\infty)$ FOR SOME $a_0 > 0$

$x \to -\infty \quad A \cap (-\infty, b_0)$ FOR SOME $b_0 < 0$

DEFINITION

$f: A \subseteq \mathbb{R}^n \to \mathbb{R}^N$ WE SAY THAT f IS **BOUNDED** IF $f(A)$ IS BOUNDED, THAT IS $\exists K > 0$ SUCH THAT

$$\| f(x) \| \leq K \quad \forall x \in A$$

THEOREM

$f: A \subseteq \mathbb{R}^N \to \mathbb{R}, \quad g: A \subseteq \mathbb{R}^n \to \mathbb{R}^M$

x^0 ACC. POINT FOR A

f BOUNDED; g INFINITESIMAL AS $x \to x^0$, THAT IS,

$$\lim_{x \to x^0} g(x) = 0 \in \mathbb{R}^M$$

$$\lim_{x \to x^0} f(x)\, g(x) = 0 \in \mathbb{R}^M$$

PROOF

$$\lim_{x \to x_0} g(x) = 0 \in \mathbb{R}^N \iff \lim_{x \to x_0} \| g(x) \| = 0$$

THEN

$$0 \leq \| f(x)\, g(x) \| = \underbrace{| f(x) |}_{\leq K} \, \| g(x) \| \leq K \| g(x) \| \to 0$$

EXERCISE: $f: A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ BOUNDED

$g: A \subseteq \mathbb{R}^n \to \mathbb{R}$ INFINITESIMAL AS $x \to x^0$

PROVE THAT gf IS INFINITESIMAL AS $x \to x^0$

NOTE $f: \mathbb{N} \to \mathbb{R}$ $f(n) = a_n$

$$\lim_{\mathbb{N} \ni x \to +\infty} f(x) = \lim_n a_n$$

LIMIT OF MONOTONE FUNCTION

$I = (a, b)$ OPEN INTERVAL $-\infty \leq a < b \leq +\infty$

$f: (a, b) \to \mathbb{R}$ MONOTONE

THEN $\exists \lim\limits_{x \to a^+} f(x)$ AND $\exists \lim\limits_{x \to b^-} f(x)$

(IF $a = -\infty$ $\lim\limits_{x \to a^+} = \lim\limits_{x \to -\infty}$; IF $b = +\infty$ $\lim\limits_{x \to b^-} = \lim\limits_{x \to +\infty}$

- f INCREASING

$$\lim_{x \to a^+} f(x) = \inf_{x \in (a,b)} f(x) \qquad \lim_{x \to b^-} f(x) = \sup_{x \in (a,b)} f(x)$$

- f DECREASING

$$\lim_{x \to a^+} f(x) = \sup_{x \in (a,b)} f(x); \quad \lim_{x \to b^-} f(x) = \inf_{x \in (a,b)} f(x)$$

PROOF

f INCREASING, $b \in \mathbb{R}$ $\sup\limits_{(a,b)} f = l \in \mathbb{R}$

• $f(x) \leq l \quad \forall x \in (a,b)$

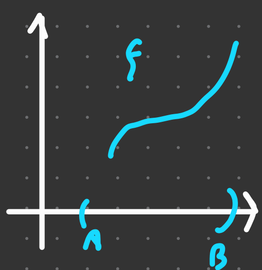
$\forall \varepsilon > 0 \quad \exists x_\varepsilon \in (a,b)$ SUCH THAT $f(x_\varepsilon) > l - \varepsilon$

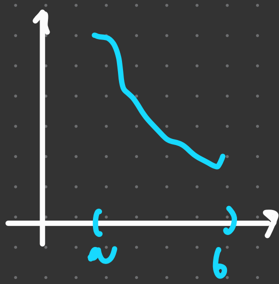
$\Rightarrow \forall\, x_\varepsilon \leq x < b$ WE HAVE

MONOTONICITY

$$l - \varepsilon < f(x_\varepsilon) \leq f(x) \leq l < l + \varepsilon$$

TAKE $\delta > 0$ SUCH THAT $(b - \delta, b) \subseteq (x_\varepsilon, b)$





$\lim\limits_{x \to x^0} f$

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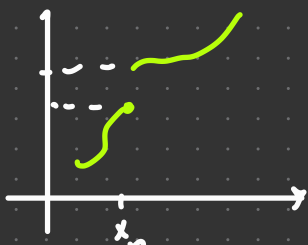

EXAMPLE

LET $P_N(x) = a_N x^N + \dots + a_1 x + a_0$

POLYNOMIAL OF DEGREE $N \geq 0$ $(a_N \neq 0)$

$q_M(x) = b_m x^M + \dots + b_1 x + b_0$

POLYNOMIAL OF DEGREE $M \geq 1$ $(b_m \neq 0)$

QUESTION

$x_0 \in \mathbb{R}$; WHAT IS $\lim_{x \to x_0} P_N(x) = P_N(x^0)$

SINCE P_N IS CONTINUOUS ON $\mathbb{R}^1$

LET $C = \{ x \in \mathbb{R} : q_M(x) = 0 \}$ FINITE SET

$\frac{P_N(x)}{q_M(x)}$ IS DEFINED FOR $x \in \mathbb{R} \setminus C$ AND IS CONTINUOUS ON $\mathbb{R} \setminus C$

SO $\forall x_0 \notin C$ WE HAVE $\lim_{x \to x_0} \frac{P_N(x)}{q_M(x)} = \frac{P_N(x_0)}{q_M(x_0)}$

WHAT ABOUT LIMITS FOR $x \to \pm\infty$?

- $$\lim_{x \to \pm\infty} (x^{4} - 2x^2 + 5) = +\infty$$

EVEN

$$x^4 - 2x^2 + 5 = x^4\underbrace{\left(1 - \frac{2}{x^2} + \frac{5}{x^4}\right)}_{1} \to \text{"}+\infty \cdot 1\text{"} = +\infty$$

$\frac{2}{x^2} \to 0$, $\frac{5}{x^4} \to 0$

- $$\lim_{x \to \pm\infty} (-x^{3} + 2x + 1) = \mp\infty$$

ODD

$$x^3 - 2x + 1 = x^3 \underbrace{\left(-1 + \frac{2}{x^2} + \frac{1}{x^3}\right)}_{-1} = \text{"}\pm\infty(-1)\text{"} = \mp\infty$$

$\frac{2}{x^2} \to 0$, $\frac{1}{x^3} \to 0$

- $$\lim_{x \to \pm\infty} \frac{-2x^4 + 5x - 3}{x^5 + 2x}$$

$$= \lim_{x \to \pm\infty} \frac{x^4}{x^5} \frac{\left(-2 + \frac{5}{x^3} - 3/x^4\right)}{\left(1 + \frac{2}{x^4}\right)} = \lim_{x \to \pm\infty} \frac{1}{x} \frac{\left(-2 + 5/x^3 - 3/x^4\right)}{\left(1 + \frac{2}{x^4}\right)}$$

$$= 0 \cdot (-2) = 0$$