Exercises - Calculus

Academic Year 2021-2022

Sheet 11

1. Determine how many times it is differentiable the function
$$f(x) = \begin{cases} x^3 & \text{se } x \geq 0, \\ (-x)^3 & \text{se } x < 0. \end{cases}$$

2. Write the Taylor polynomial of order n of the function f at the point x_0 where
   (a) $f(x) = 2 + x + 3x^2 - x^3$, $x_0 = 0$ and $x_0 = 1$ and $n = 2$. Do the two polynomials coincide?
   (b) $f(x) = \sin(3x)$, $x_0 = \pi/6$ and $n = 5$
   (c) $f(x) = \arctan(1/x)$, $x_0 = 1$ and $n = 3$
   (d) $f(x) = \log(x^2) + x^2 + x$, $x_0 = 1$ and $n = 3$

3. Write the Taylor formula of order n with Peano remainder of the function f at the point x_0 where
   (a) $f(x) = \log(1 + 3x)$, $x_0 = 0$ and $x_0 = 1$ and $n = 4$
   (b) $f(x) = \cos(x^2)$, $x_0 = 0$ and $x_0 = \sqrt{\pi}$ and $n = 3$
   (c) $f(x) = \log(1 + \sin(x))$, $x_0 = 0$ and $n = 3$
   (d) $f(x) = e^{-x^2}$, $x_0 = 0$ and $x_0 = -1$ and $n = 3$

4. Write the Taylor formula of order n with Lagrange remainder of the function f at the point x_0 where
   (a) $f(x) = e^{x-1}$, $x_0 = 2$ and $n = 3$
   (b) $f(x) = x^3 + \tan(x)$, $x_0 = \pi$ and $n = 2$
   (c) $f(x) = e^x - 1 - \sin(x)$, $x_0 = 1$ and $n = 2$
   (d) $f(x) = (e^{3(x+1)} - 1)\sin(x + 1)$, $x_0 = -1$ and $n = 2$
   (e) $f(x) = \dfrac{\cos(x^2)}{2} + 3x^2$, $x_0 = \sqrt{\pi/3}$ and $n = 2$.

5. Compite, if it exists, the following limit
   (a) $\displaystyle\lim_{x\to 0} \frac{\sin(x) + \log(1 - x)}{x^2}$
   (b) $\displaystyle\lim_{x\to 0} \frac{x\cos x - \sin x}{x^2}$
   (c) $\displaystyle\lim_{x\to 0} \frac{1 - e^{-\sin x}}{1 + x - \cos x}$

(d) $\lim_{x\to 0} \dfrac{e^{\sin x} - 1 - x}{\log(\cos x)}$

(e) $\lim_{x\to 0} \dfrac{\log(1-x^4)}{e^{x^2} - 1 - x^2}$

(f) $\lim_{x\to 0} \dfrac{\log(1-x^3)}{e^{x^2} - 1 - x^2}$

(g) $\lim_{x\to 0} \dfrac{\log(x^2 - \sin^2 x + 1)}{e^{x^2} - 1 - x^2}$

(h) $\lim_{x\to 0} \dfrac{x\left(2e^{-x} - 2 + 2x - x^2\right)}{(\cos(x) - 1)^2}$

(i) $\lim_{x\to 0} \dfrac{\log(1+x^2) - x^2}{2x^3 \sin(x)}$

(j) $\lim_{x\to 0} \dfrac{e^x - 1 - \sin(x) - x^2/2}{\log(1+x^3)}$

(k) $\lim_{x\to 0} \dfrac{e^{x^2} - 1 - x^2 + 3x^4}{\cos(2x^2) - 1}$

(l) $\lim_{x\to 0^+} \dfrac{2\cos(\sqrt{x}) - 2 + x}{(\arctan(3x))^2}$

6. Let, for the parameter $b \in \mathbb{R}$,

$$f(x) = \begin{cases} \dfrac{\sin(4x)}{e^{2x} - 1} & \text{se } x < 0 \\ 2b & \text{se } x = 0 \\ b^2 - 2b\dfrac{\log(1+x) - x}{x^2} & \text{se } x > 0 \end{cases}$$

Determine, if they exist, the values of $b \in \mathbb{R}$ such that the function f is continuous on the whole $\mathbb{R}$.

7. Let, for the parameter $a \in \mathbb{R}$,

$$f(x) = \begin{cases} \dfrac{\log(1-2x)}{\arctan(3x)} & \text{se } x < 0 \\ a & \text{se } x = 0 \\ -\dfrac{9}{4}a^3 + 24a\dfrac{2\cos(\sqrt{x}) - 2 + x}{x^2} & \text{se } x > 0 \end{cases}$$

Determine, if they exist, the values of $a \in \mathbb{R}$ such that the function f is continuous on the whole $\mathbb{R}$.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable 4 times in 0. Assume that $f'(0) = f''(0) = 0$. Prove that

- if $f'''(0) \neq 0$, then 0 is not a local extremum point for f

- if $f'''(0) = 0$ and $f^{(4)}(0) \neq 0$, then 0 is a strict local extremum point for f. Establish whether 0 is a strict local minimum or maximum point depending on the sign of $f^{(4)}(0)$

9. Establish whether the following series are converging

   (a) $\displaystyle\sum_{n=1}^{\infty} \left(\log(1+\sqrt{n}) - \log(\sqrt{n}) + \frac{1-2\sqrt{n}}{2n} \right)$

   (b) $\displaystyle\sum_{n=1}^{\infty} \left(\sin\left(\frac{1}{\sqrt{n}}\right) - \frac{1}{\sqrt{n}} \right)$

   (c) $\displaystyle\sum_{k=1}^{\infty} (\arctan(1/k) - 1/k)$

2A) $f(x) = 2 + x + 3x^2 - x^3$ $\quad x_0 = 0 \quad N = 2$

$x_0 = 1 \quad N=2$

$$f(0) + \frac{f'(0)}{1!}x = 2 + \frac{1+6x-3x^2}{1}\cancel{(x-1)}$$

$x_0 = 0;\ 3$

$x_0 = 1;\ 6$

2B) $f(x) = \sin(3x)$ $\quad x_0 = \frac{\pi}{6} \quad N=5$

$$f\left(\frac{\pi}{6}\right) + \frac{f'\left(\frac{\pi}{6}\right)\left(x-\frac{\pi}{6}\right)}{1!} + \frac{f''\left(\frac{\pi}{6}\right)}{2!}\left(x-\frac{\pi}{6}\right)^2 + \frac{f'''\left(\frac{\pi}{6}\right)\left(x-\frac{\pi}{6}\right)^3}{3!} + \frac{f^{IV}\left(x-\frac{\pi}{6}\right)^4}{4!}$$

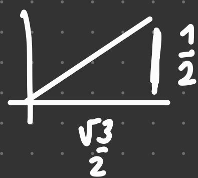
$f(x) = \sin(3x)$

$f'(x) = 3\cos(3x)$

$f''(x) = -9\sin(3x)$

$f'''(x) = -27\cos(3x)$

$f^{IV}(x) = 81\sin(3x)$



$$\frac{\overset{27}{\cancel{81}}}{2} \cdot \cancel{4\cdot 3\cdot 2}$$

$$\frac{1}{2} + \frac{3\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right) - \frac{9}{4}\left(x-\frac{\pi}{6}\right)^2 + \frac{\frac{-\overset{9}{\cancel{27}}\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)^3}{\cancel{3!}} + \frac{\frac{\overset{27}{\cancel{81}}}{2\cdot 8}\left(x-\frac{\pi}{6}\right)^4}{\cancel{4!}}$$