Exercises - Calculus

Academic Year 2021-2022

Sheet 12

1. Let $a,\ b \in \mathbb{R}$ with $a < b$ and let $x_0 \in [a,b]$. Let $f : [a,b] \to \mathbb{R}$ be the function such that
$$f(x) = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{if } x \neq x_0 \end{cases}$$
Prove, using only the definition, that f is integrable on $[a,b]$ and that $\displaystyle\int_a^b f(x)dx = 0$

2. Let $f : [a,b] \to \mathbb{R}$ and let $c \in \mathbb{R}$. Prove, using only the definition, that f is Riemann integrable if and only if $f + c$ is Riemann integrable and that, in this case,
$$\int_a^b (f + c) = \int_a^b f + c(b - a).$$

3. Let $f,\ g : [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$. Recall that for any $F : [\inf_{[a,b]} f, \sup_{[a,b]} f] \to \mathbb{R}$ continuous, we have that $F \circ f$ is Riemann integrable. Using this result prove that
   - $|f|$ is Riemann integrable
   - f^2 is Riemann integrable
   - fg is Riemann integrable
     Hint: use the fact that $2fg = (f + g)^2 - f^2 - g^2$
   - $\max\{f, g\}$ and $\min\{f, g\}$ are Riemann integrable
     Hint: use the fact that
     $$\max\{f, g\} = \frac{f + g}{2} + \frac{|f - g|}{2} \quad \text{and} \quad \min\{f, g\} = \frac{f + g}{2} - \frac{|f - g|}{2}$$

4. Let A be any arbitrary set and let $f : A \to \mathbb{R}$. We define
$$f^+ = \max\{f, 0\} \quad \text{and} \quad f^- = -\min\{f, 0\} = \max\{-f, 0\}$$
The function f^+ is called the *positive part* of f, the function f^- is called the *negative part* of f. Prove that
$$f^+ \geq 0, \quad f^- \geq 0, \quad f = f^+ - f^-, \quad |f| = f^+ + f^-.$$

5. Let $f(x) = \sin(x)$, $x \in [-\pi/2, \pi/2]$. Draw the graph of f, of f^+, of f^- and of $|f|$.

6. Let $f : [a,b] \to \mathbb{R}$. Prove that f is Riemann integrable if and only if f^+ and f^- are Riemann integrable. Prove that, in this case,
$$\int_a^b f = \int_a^b f^+ - \int_a^b f^-.$$

7. Let $f : [a,b] \to \mathbb{R}$ be continuous and such that $f \geq 0$. Prove that $\displaystyle\int_a^b f = 0$ if and only if $f(x) = 0$ for any $x \in [a,b]$.

8. Find an example of a function $f : [a,b] \to \mathbb{R}$ such that f is Riemann integrable and there exists no $c \in [a,b]$ such that $f(c)$ is equal to the mean of f on $[a,b]$.

9. Find an example of two functions $f,\ g : [a,b] \to \mathbb{R}$ such that

$$\inf_{[a,b]} f + \inf_{[a,b]} g < \inf_{[a,b]} (f+g)$$

and one where we have

$$\sup_{[a,b]}(f+g) < \sup_{[a,b]} f + \sup_{[a,b]} g$$

10. Let $m,\ n \in \mathbb{N}$ with no common factors and such that $n > 1$ is odd. Prove that

$$\int x^{\frac{m}{n}}\,dx = \frac{x^{\frac{m}{n}+1}}{\frac{m}{n}+1} + c, \quad c \in \mathbb{R},\ x \in \mathbb{R}$$

and that

$$\int x^{-\frac{m}{n}}\,dx = \begin{cases} \frac{x^{-\frac{m}{n}+1}}{-\frac{m}{n}+1} + c_+ & x > 0 \\ \\ \frac{x^{-\frac{m}{n}+1}}{-\frac{m}{n}+1} + c_- & x < 0 \end{cases}, \quad c_\pm \in \mathbb{R},\ x \in \mathbb{R}\backslash\{0\}$$

11. For the following functions f compute the indefinite integral $\displaystyle\int f$ on their domains of definition

    (a) $f(x) = 3x^2 - 2x + 1$
    (b) $f(x) = 4x^3 + \cos(x) + \sqrt[3]{x}$
    (c) $f(x) = e^{3x} - \sin(x+1)$
    (d) $f(x) = \dfrac{1}{1+x}$
    (e) $f(x) = \dfrac{x}{1-x}$
    (f) $f(x) = \dfrac{x^2-2x}{x+1}$
    Hint: use the division between two polynomials

12. Compute the following definite integrals

    (a) $\displaystyle\int_0^1 (x^2 - 3x + 5)dx$
    (b) $\displaystyle\int_{-1}^2 5x^{3/5}dx$
    (c) $\displaystyle\int_1^2 (3e^{x-1} + 2x)dx$

(d) $\int_0^{\pi/3} (\cos(3x) + \sin(3x))dx$

(e) $\int_0^{1/2} \frac{2}{\sqrt{1-x^2}} dx$

(f) $\int_1^2 \frac{1}{x} dx$ and $\int_{-e^2}^{-e} \frac{1}{x} dx$

13. Compute the area of the following planar regions E

(a) $E = \{(x, y) \in \mathbb{R}^2 : \ x \in [0, 1], \ 0 \leq y \leq \sqrt{x}\}$

(b) $E = \left\{(x, y) \in \mathbb{R}^2 : \ x \in [-\pi/4, \pi/4], \ -x^2 \leq y \leq \frac{1}{\cos^2(x)}\right\}$

(c) $E = \{(x, y) \in \mathbb{R}^2 : \ x \in [-1, 0], \ x^3 \leq y \leq e^{3x}\}$

(d) $E = \{(x, y) \in \mathbb{R}^2 : \ x \in [-1, 1], \ -1 + x^2 \leq y \leq 1 - x^2\}$

(e) $E = \{(x, y) \in \mathbb{R}^2 : \ x \in [-\pi/2, \pi/2], \ -1 \leq y \leq \cos(x)\}$

(f) $E = \left\{(x, y) \in \mathbb{R}^2 : \ x \in [-1, 1], \ 0 \leq y \leq \frac{x^2 + 5}{x^2 + 1}\right\}$

14. Let $E = \left\{(x, y) \in \mathbb{R}^2 : \ y \in [0, 1], \ \frac{y^2}{2} \leq x \leq e^y - 1\right\}$. Draw the set E and compute its area.

15. Prove that there exists a constant $c \in \mathbb{R}$ such that

$$\arcsin(x) = -\arccos(x) + c, \quad x \in [-1, 1].$$

Then compute the value of c.

1. Let $a, b \in \mathbb{R}$ with $a < b$ and let $x_0 \in [a, b]$. Let $f : [a, b] \to \mathbb{R}$ be the function such that

$$f(x) = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{if } x \neq x_0 \end{cases}$$

Prove, using only the definition, that f is integrable on $[a, b]$ and that $\int_a^b f(x)dx = 0$

$$f(x) = \begin{cases} 1 & \text{IF } x = x_0 \\ 0 & \text{IF } x \neq x_0 \end{cases} \qquad f(x) \in \mathbb{R}([a,b])$$

$$\text{IFF } \sup_P s(P) = \inf_P S(P)$$

$$s(P) = \sum_{j=1}^{m} m_j \Delta x_j$$

$$S(P) = \sum_{j=1}^{m} M_j \Delta x_j \qquad S(P) = I(f) = \int_a^b f(x)dx$$

$$l_N = \frac{b-a}{N}$$

$$s(P) = \sum_{j=1}^{m} \underset{\substack{\| \\ 0\ \forall j}}{m_j} \overset{\substack{l_N \\ \|}}{\Delta x_j} = 0$$

$$S(P) = \sum_{j=1}^{m} M_j \Delta x_j =$$

$$I_n \ni x_0 \quad \Pi_{n=1} \qquad \Rightarrow I(f) = 0$$

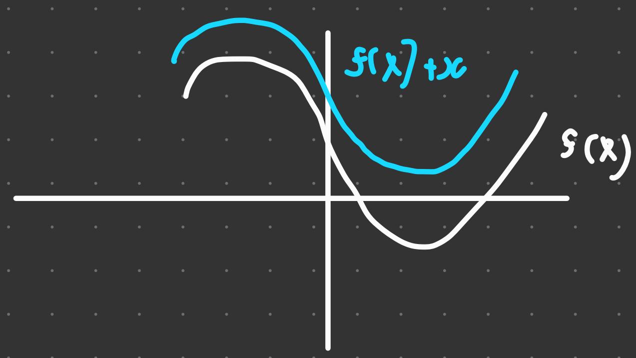
2. Let $f : [a, b] \to \mathbb{R}$ and let $c \in \mathbb{R}$. Prove, using only the definition, that f is Riemann integrable if and only if $f + c$ is Riemann integrable and that, in this case,

$$\int_a^b (f + c) = \int_a^b f + c(b - a).$$

$f : [a, b] \to \mathbb{R} \qquad c \in \mathbb{R}$

$f + c$ IS REIMANN INTEGRABLE $\iff$ f IS R. INT

$$\int_a^b f + c = \int_a^b f + c(b-a)$$

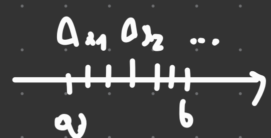


$\inf f = m \qquad \inf (f+c) = m + c$

$\sup f = M \qquad \sup (f+c) = M + c$

f IS R. INT. $\iff \sup\limits_P \{ s(P) = \sum_j m_j \Delta x_j \} = \inf\limits_P \{ S(P) \}$

$f + c$ IS R. INT $\iff \sup\limits_P \{ s(P) = \sum_j \underbrace{(m_j + c)\Delta x_j}_{\sum_j m_j \Delta x_j + \underbrace{\sum_j c \Delta x_j}_{c = \sum_j \Delta x_j}} \} = \inf\limits_P \{ S(P) \}$



$\iff \sup\limits_P \{ s(P) = \sum_j m_j \Delta x_j + c(b-a) \}$

$\Updownarrow$

$\inf\limits_P \{ s(P) = \sum_j m_j \Delta x_j \} + c(b-a)$

3. Let $f,\, g : [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$. Recall that for any $F : [\inf_{[a,b]} f, \sup_{[a,b]} f] \to \mathbb{R}$ continuous, we have that $F \circ f$ is Riemann integrable. Using this result prove that

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(c) $f(x) = e^{3x} - \sin(x+1)$

(d) $f(x) = \dfrac{1}{1+x}$

(e) $f(x) = \dfrac{x}{1-x}$

(f) $f(x) = \dfrac{x^2 - 2x}{x+1}$

Hint: use the division between two polynomials

a) $\int 3x^2 - 2x + 1 = \int 3x^2 - \int 2x + \int 1 = 3\int x^2 - 2\int x + \int 1 = \frac{3x^3}{3} - \frac{2x^2}{2} + x + c = x^3 - x^2 + x + c \quad c \in \mathbb{R}$

b) $\int 4x^3 + \cos(x) + \sqrt[3]{x} = 4\int x^3 + \int \cos(x) + \int x^{1/3} = x^4 + \sin(x) + \frac{3x^{4/3}}{4} + c = x^4 + \sin(x) + \frac{3x^{4/3}}{4} + c \quad c \in \mathbb{R}$

c) $\int \left(e^{3x} - \sin(x+1)\right) dx = \int e^{3x}\, dx - \int \sin(x+1)\, dx = \frac{e^{3x}}{3} + \cos(x+1) + c \quad c \in \mathbb{R}$

d) $\int \frac{1}{1+x}\, dx = \ln|1+x| + \begin{cases} c^+ & \text{if } x > 0 \\ c_- & \text{if } x < 0 \end{cases}$

e) $\int \frac{x}{1-x}\, dx = -\int \frac{x}{x-1}\, dx = -\int \frac{x-1+1}{x-1}\, dx = -\int \frac{x-1}{x-1}\, dx - \int \frac{1}{x-1}\, dx = -x - \ln|x-1| + c \quad c \in \mathbb{R}$

f) $\int \frac{x^2 - 2x}{x+1}\, dx = \int \frac{x^2}{x+1}\, dx - 2\int \frac{x}{x+1}\, dx = \int \frac{x^2}{x+1}\, dx - 2\int \frac{x+1-1}{x+1}\, dx = \int \frac{x^2}{x+1}\, dx \underbrace{- 2\int 1\, dx}_{-2x} + \underbrace{2\int \frac{1}{x+1}\, dx}_{2\ln|x+1|} + c$

$$\begin{array}{l|l} x^2 - 2x + 0 & x+1 \\ \hline x^2 + x & x - 3 \\ \hline / \ -3x & \\ \ \ -3x - 3 & \\ \hline / \ \ 3 & \end{array}$$

$\int \frac{x^2 - 2x}{x+1}\, dx =$

$= \int x - 3 + \frac{3}{x+1}\, dx$

$\frac{x^2}{2} - 3x + 3\ln|x+1| + c$

$\int \left(x + \frac{1}{x+1} - 1\right) dx = \frac{x^2}{2} + \ln|x+1| - x + c$

$\frac{x^2}{2} + \ln|x+1| - x - 2x + 2\ln|x+1| + c = \frac{x^2}{2} - 3x + 3\ln|x+1| + c$

12. Compute the following definite integrals

(a) $\int_0^1 (x^2 - 3x + 5)dx$

(b) $\int_{-1}^2 5x^{3/5}dx$

(c) $\int_1^2 (3e^{x-1} + 2x)dx$

a) $$\int_0^1 (x^2-3x+5)dx = \left[\frac{x^3}{3} - \frac{3}{2}x^2 + 5x\right]_0^1 = \frac{1}{3} - \frac{3}{2} + 5 = \frac{2-9+30}{6} = \frac{23}{6}$$

b) $$\int_{-1}^2 5x^{3/5}\,dx = \left[\frac{5\,x^{8/5}}{8/5}\right]_{-1}^2 = \left[\frac{25\,x^{8/5}}{8}\right]_{-1}^2 = \frac{25}{2^3}\,2^{8/5} - \frac{25}{8}\overset{+1}{(-1)^{8/5}} =$$

c) $$\int_1^2 (3e^{x-1} + 2x)dx = \left[3e^{x-1} + x^2\right]_1^2 = 3e + 4 - 3 - 1 = 3e$$

d) $$\int_0^{\pi/3} (\cos(3x) + \sin(3x))dx = \left[\frac{\sin(3x)}{3} - \frac{\cos(3x)}{3}\right]_0^{\pi/3} = \frac{\sin(\pi)}{3} - \frac{\cos(\pi)}{3} - \frac{\sin(0)}{3} + \frac{\cos(0)}{3} = 0 + \frac{1}{3} - 0 + \frac{1}{3} = \frac{2}{3}$$

e) $$\int_0^{1/2} \frac{2}{\sqrt{1-x^2}}dx = 2\int_0^{1/2} \overbrace{\frac{1}{\sqrt{1-x^2}}dx}^{!!!} = \left[2\arcsin(x)\right]_0^{1/2} = 2\arcsin(1/2) - \underbrace{2\arcsin(0)}_{0} = \pi/3$$

f1) $$\int_1^2 \frac{1}{x}dx = \left[\ln|x|\right]_1^2 = \ln(2) - \ln(1) = \ln(2)$$

f2) $$\int_{-e^2}^{-e} \frac{1}{x}dx = \left[\ln|x|\right]_{-e^2}^{-e} = \ln|e| - \ln|e^2| = \ln|e| - 2\ln|e| = 1 - 2 = -1$$

13. Compute the area of the following planar regions E

(a) $E = \{(x, y) \in \mathbb{R}^2 : \ x \in [0, 1], \ 0 \le y \le \sqrt{x}\}$

(b) $E = \left\{(x, y) \in \mathbb{R}^2 : \ x \in [-\pi/4, \pi/4], \ -x^2 \le y \le \dfrac{1}{\cos^2(x)}\right\}$

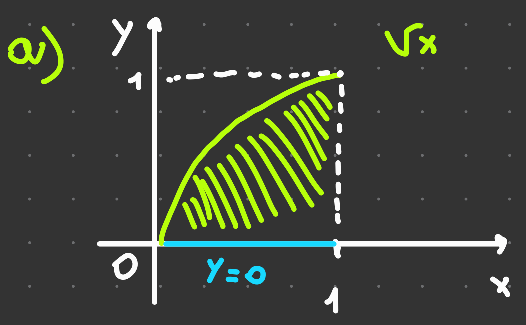
(c) $E = \{(x, y) \in \mathbb{R}^2 : \ x \in [-1, 0], \ x^3 \le y \le e^{3x}\}$

(d) $E = \{(x, y) \in \mathbb{R}^2 : \ x \in [-1, 1], \ -1 + x^2 \le y \le 1 - x^2\}$

(e) $E = \{(x, y) \in \mathbb{R}^2 : \ x \in [-\pi/2, \pi/2], \ -1 \le y \le \cos(x)\}$

(f) $E = \left\{(x, y) \in \mathbb{R}^2 : \ x \in [-1, 1], \ 0 \le y \le \dfrac{x^2 + 5}{x^2 + 1}\right\}$

a)



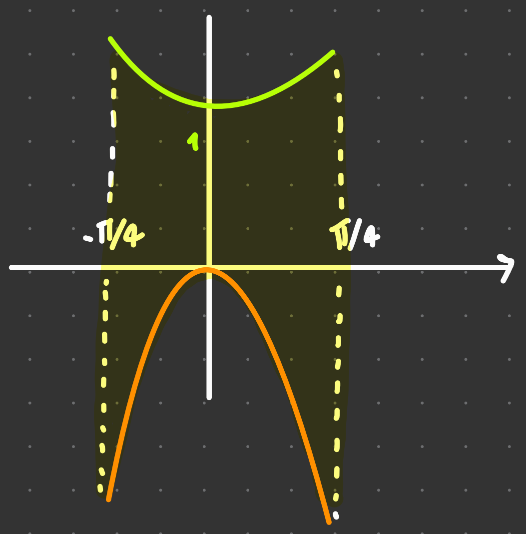
STEPS

1. THE INTERVALS IN WHICH x AND y VARY
2. DRAW THE SET E
3. COMPUTE THE AREA

AREA $\int_{x=0}^{x=1} y_{HIGH} - y_{LOW}$

$$\int_0^1 \sqrt{x} - 0 \, dx = \frac{2x^{3/2}}{3}\Big|_0^1 = \boxed{\frac{2}{3}}$$

b) $x \in [-\pi/4, \pi/4]$ $\quad -x^2 \le y \le \dfrac{1}{\cos^2(x)}$

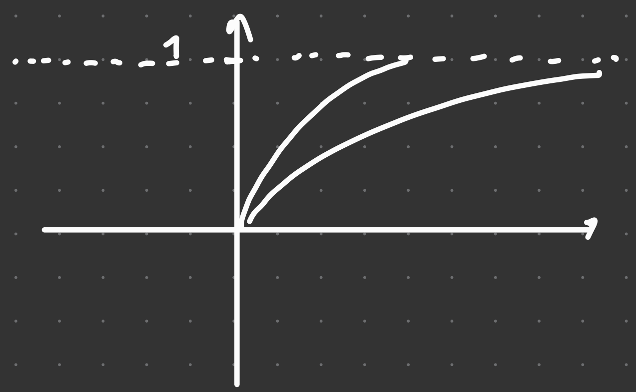


$\tan(\pi/4) = 1 \quad \tan(-\pi/4) = 1$

$$\int_{-\pi/4}^{\pi/4} \left(\frac{1}{\cos^2(x)} - (-x^2)\right) dx = \left[\tan(x) + \frac{x^3}{3}\right]_{-\pi/4}^{\pi/4} = \boxed{2 + \frac{2}{3}\left(\frac{\pi}{4}\right)^3}$$

14. Let $E = \left\{(x,y) \in \mathbb{R}^2 : \ y \in [0,1], \ \frac{y^2}{2} \le x \le e^y - 1\right\}$. Draw the set E and compute its area.

$y \in [0,1]$ $\qquad \frac{y^2}{2} \le x \le e^y - 1$



$\hookrightarrow y \le \sqrt{2x}$

$y \ge \ln(1+x)$

$$\int_{y=0}^{y=1} \left(\left(e^y - 1\right) - \frac{y^2}{2}\right) dy = e - \frac{13}{6}$$

UTTO CULO