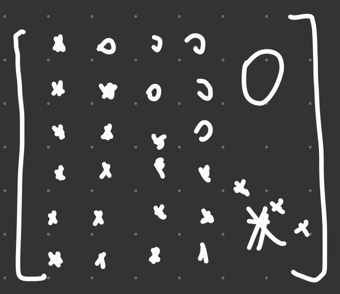
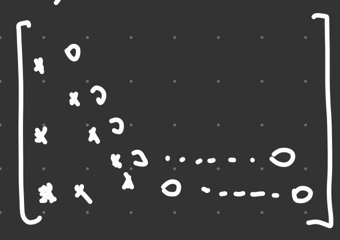
DEFINITION

USUALLY L FOR LOWER

A MATRIX L IS LOWER TRIANGULAR IF ALL ENTRIES IN POSITION (i, j) ARE ZERO FOR $j > i$

ROW COLUMN

(WHENEVER THE POSITION NUMBER ASSOCIATED TO THE COLUM (j) IS BIGGER THAN THE POSITION NUMBER ASSOCIATED TO THE ROW, THEN THE CORRISPONDING ENTRY IS ZERO)

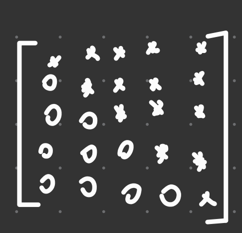
DEFINITION

USUALLY U FOR UPPER

A MATRIX U IS UPPER TRIANGULAR IF ALL ENTRIES IN POSITION (i, j) ARE ZERO FOR $i > j$

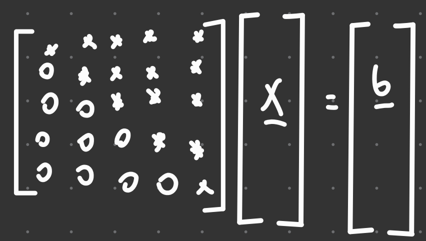
ROW COLUMN

(WHENEVER THE POSITION NUMBER ASSOCIATED TO THE ROW IS BIGGER THAN THE POSITION NUMBER ASSOCIATED TO THE COLUMN, THEN THE CORRISPONDING ENTRY IS ZERO)

OBSERVATION ALL ECHELON FORM MATRIXES ARE UPPER TRIANGULAR

ECHELON FORM



LINEAR SYSTEM $A\underline{x} = \underline{b}$ WITH A UPPER TRIANGULAR OR LOWER TRIANGULAR ARE VERY EASY TO SOLVE BY SUBSTITUTION

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 1/2 & 0 \\ 3 & -2 & 1 \end{bmatrix} = A \qquad \underline{b} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

$$A\underline{x} = \underline{b} \qquad \begin{array}{c} 1^{ST} \\ 2^{ND} \\ 3^{RD} \end{array} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 4 \\ -1 & 1/2 & 0 & 1 \\ 3 & -2 & 1 & 3 \end{array}\right]$$

$$\begin{array}{l} 1^{ST} \\ 2^{ND} \\ 3^{RD} \end{array} \begin{cases} 2x_1 = 4; \quad x_1 = 2 \\ -x_1 + \frac{1}{2}x_2 = 1; \quad x_2 = 6 \\ 3x_1 - 2x_2 + x_3 = 3 \quad x_3 = 9 \end{cases}$$

OBSERVATION

THE PRODUCT OF LOWER TRIANGULAR MATRIXES IS A LOWER TRIANGULAR MATRIX. THE INVERSE OF LOWER TRIANGULAR MATRIX (IF IT EXISTS) IS A LOWER TRIANGULAR MATRIX THE ANALOGOUS HOLDS FOR UPPER MATRIXES

LU FACTORIZATION

• SIMPLIFICATION: ASSUME THAT THE MATRIX A CAN BE TRANSFORMED IN EF WITHOUT THE NEED OR ROW ECHANGE OPERATIONS

$A \in \mathbb{R}^{m \times n}$

A IN ECHELON FORM → U BECAUSE IS IN UPPER TRIANGULAR

$$\underbrace{E_p \dots E_2 E_1}_{\text{ELEMENTARY MATRIX (SAME AS ROW OPERATION)}} A = U \qquad E_1, E_2, \dots E_p \in \mathbb{R}^{m \times n}$$

ALL THE $E_1, \dots, E_p$ ARE LOWER TRIANGULAR, THERFORE $(E_p \dots E_2 E_1)$ IS LOWER TRIANGULAR

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}$$

$$A = \underbrace{(E_p \dots E_2 E_1)^{-1} (E_p \dots E_2 E_1)}_{\text{IDENTITY MATRIX}} A = (E_p \dots E_2 E_1)^{-1} U \Rightarrow A = LU$$

IDENTITY MATRIX $E_1 \cdot E_1^{-1} = I_N$

UPPER TRIANGULAR MATRIX

LOWER TRIANGULAR MATRIX

• $A \in \mathbb{R}^{m \times n}$ (THE SIMPLIFICATION ABOVE HOLDS)

$A = LU$ $\quad L$ LOWER TRIANGULAR

U UPPER TRIANGULAR (EF OF A)

HOW IS L CALCULATED?

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix} \begin{matrix} 1/2 \\ 3/2 \end{matrix} = \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 10 & -1 \end{bmatrix} -5 = \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix} = U$$

ECHELON FORM of A

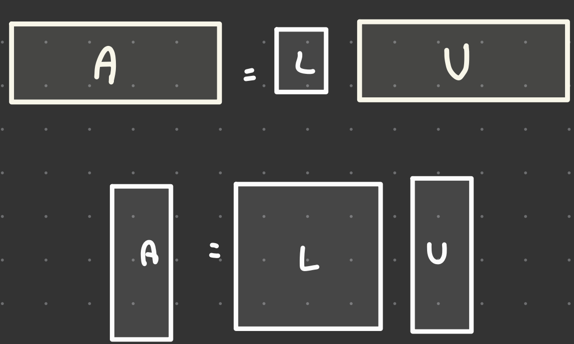
IF YOU HAVE MISSING INFORMATION (NOT ALL STEPS NECESSARY TO DO EF) THEN PUT A ZERO IN THE CORRISPONDING PLACE

$$L = \begin{bmatrix} 1 & 0 & 0 \\ ? & 1 & 0 \\ ? & ? & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -5 & 1 \end{bmatrix} = L$$

$$\boxed{A = LU}$$

• ONE STARTS WITH $L \in \mathbb{R}^{m \times m}$ → SQUARE MATRIX, LOWER TRIANGULAR, WITH ALL 1 ON THE DIAGONAL. THE REMAINING ENTRIES ARE FILLED ONE BY ONE DURING THE ROW REDUCTION ALGHORITHM. WHENEVER A MULTIPLIER $S \in \mathbb{R}$ IS USED TO MAKE THE ENTRY (i, j) OF A EQUAL TO ZERO (BY ROW OPERATION) WE FILL THE CORRESPONDING ENTRY (i, j) OF L WITH THE NUMBER S. ALL THE REMAINING ENTRIES OF L ARE SET TO ZERO.

$$\begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots \; 1 \end{bmatrix}$$



$$A = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix} \begin{matrix} 3 \\ -1/2 \end{matrix} = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & -6 & 10 & -1 \end{bmatrix} \begin{matrix} -2 \end{matrix} = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1/2 & -2 & 1 \end{bmatrix}$$

$A = LU \qquad A\underline{x} = \underline{b} \qquad A \in \mathbb{R}^{m \times n} \qquad \underline{x} \in \mathbb{R}^n \qquad \underline{b} \in \mathbb{R}^m$

L IS A SQUARE MATRIX WITH 1 ON DIAGONALS

$$L\underbrace{U\underline{x}}_{\text{VECTOR}} = \underline{b} \quad \Rightarrow \quad \begin{cases} L\underline{y} = \overset{\text{KNOWN}}{\underline{b}} & \leftarrow \text{FIRST COMPUTE } \underline{y} \in \mathbb{R}^m \\ \underset{\text{KNOWN}}{U}\underline{x} = \underline{y} & \leftarrow \text{THEN USE } \underline{y} \text{ TO COMPUTE } \underline{x} \end{cases}$$

$U\underline{x} = \underline{y} \quad \underline{y} \in \mathbb{R}^m$

$$A = \begin{bmatrix} 2 & -1 & 2 \\ -6 & 0 & -2 \\ 8 & -1 & 5 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

- CALCULATE $A = LU$
- SOLVE $A\underline{x} = \underline{b}$ USING LU FACTORIZATION

$$A = \begin{bmatrix} 2 & -1 & 2 \\ -6 & 0 & -2 \\ 8 & -1 & 5 \end{bmatrix} \begin{matrix} -3 \\ 4 \end{matrix} = \begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 3 & -3 \end{bmatrix} \begin{matrix} -1 \end{matrix} = \begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix}$$

$$\begin{cases} L\underline{y} = \underline{b} \\ U\underline{x} = \underline{y} \end{cases}$$

$$\overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & -1 & 1 \end{bmatrix}} \overset{\underline{y}}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}} = \overset{\underline{b}}{\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}} \qquad \begin{cases} y_1 = 1 \quad \leftarrow \text{START FROM THE FIRST ROW} \\ -3y_1 + y_2 = 0; \quad y_2 = 3 \\ 4y_1 - y_2 + y_3 = 4; \quad y_3 = 3 \end{cases}$$

ALWAYS UNIQUE

$$\underline{y} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

$$\overset{U}{\begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{bmatrix}} \overset{\underline{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

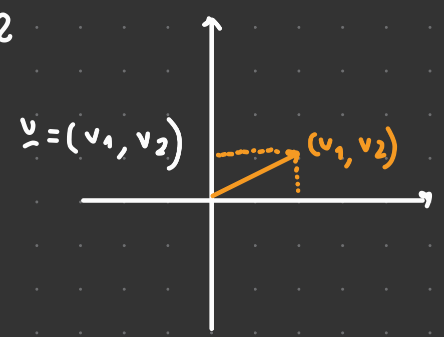
$$\begin{cases} x_3 = 3 \quad \leftarrow \text{START FROM THE LAST ROW} \\ -3x_2 + 4x_3 = 3; \quad x_2 = 3 \\ 2x_1 - x_2 + 2x_3 = 1; \quad x_1 = -1 \end{cases} \qquad \underline{x} = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}$$

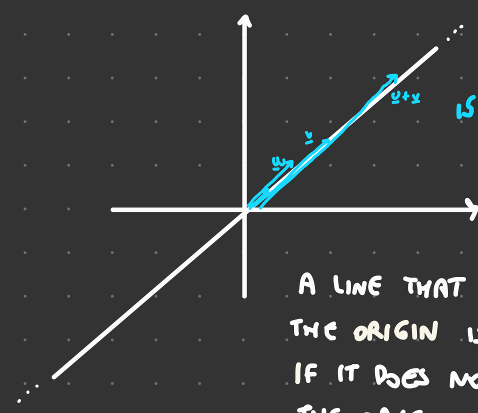
SUBSPACE OF $\mathbb{R}^N$

H A SUBSET OF $\mathbb{R}^N$ IS CALLED A LINEAR SUBSPACE IF:

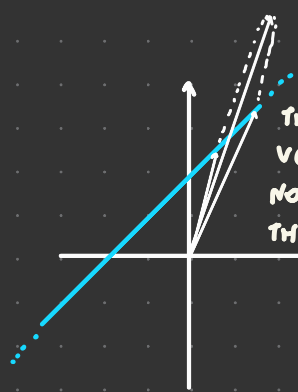
1. IF $\underline{u}, \underline{v} \in H$ THEN ALSO $\underline{u} + \underline{v} \in H$
2. IF $\underline{u} \in H$ THEN ALSO $c\underline{u} \in H$ FOR ANY REAL NUMBER $c \in \mathbb{R}$

$\mathbb{R}^2$





A LINE THAT PASSES TROUGH THE ORIGIN IS A SUBSPACE, IF IT DOES NOT GO TROUGH THE ORIGIN IT IS NOT A SUBSPACE



THE SUM OF 2 VECTORS DOES NOT BELONGS TO THE SUBSPACE

$A \in \mathbb{R}^{m \times n}$

$$A = [\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n]$$

"CONE"

$\mathrm{Col}(A) = \mathrm{SPAN}\{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n\} \subseteq \mathbb{R}^m$ IT IS A LINEAR SUBSPACE OF $\mathbb{R}^m$

$c_1\underline{a}_1 + c_2\underline{a}_2 + \dots + c_n\underline{a}_n \Rightarrow$ IS A LINEAR SUBSPACE

$\Downarrow$

IT IS STILL A LINEAR SUBSPACE IF YOU MULTIPLY ALL BY c

$\mathrm{NUL}(A) = \{$ SET OF ALL SOLUTIONS $\underline{x}$ OF THE HOMOG. EQ. $A\underline{x} = \underline{0}\} \subseteq \mathbb{R}^N$ THIS IS A LINEAR SUBSPACE OF $\mathbb{R}^N$

$\underline{x}, \underline{y} \in \mathrm{NUL}(A)$

$A(\underline{x}+\underline{y}) = A\underline{x} + A\underline{y} = \underline{0} + \underline{0} = \underline{0}$ $\qquad \underline{x} \in \mathrm{NUL}(A) \quad c\underline{x}$

$$A(c\underline{x}) = cA\underline{x} = c\underline{0} = \underline{0}$$

$A\underline{x} = \underline{b}$ $\quad \underline{x}$ ESISTS IF AND ONLY IF $\underline{b} \in \mathrm{SPAN}\{\underline{a}_1, \underline{a}_2, \dots \underline{a}_n\} = \mathrm{COL}(A)$

$A\underline{x} = \underline{b}$ IF I HAVE A SOLUTION $\underline{x}$, HOW MANY OF THEM?

IF $A\underline{x} = \underline{0}$ HAS ONLY THE TRIVIAL SOLUTION $\underline{x} = \underline{0}$ THEN THE SOLUTION $A\underline{x} = \underline{b}$ IS UNIQUE

IF $A\underline{x} = \underline{0}$ HAS INFINITE SOLUTIONS THEN ALSO $A\underline{x} = \underline{b}$ IT WILL HAVE INFINITE SOLUTIONS

$$A^{4\times 3} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -2 & 4 & 2 \\ 1 & -2 & -1 \end{bmatrix}$$

1) EXPRESS "EXPLICITY" ALL THE VECTORS IN $\mathrm{COL}(A)$
2) SAY IF $\underline{b} = [1 \ 1 \ 1 \ 1]$ IS IN $\mathrm{COL}(A)$ YES
3) SAY IF $\underline{x} = [1 \ 1 \ -1]$ IS IN $\mathrm{NUL}(A)$
4) EXPRESS ALL VECTORS IN $\mathrm{NUL}(A)$ "EXPLICITY"

$\mathrm{COL}(A) \subseteq \mathbb{R}^4$

$\mathrm{NUL}(A) \subseteq \mathbb{R}^3$

$\boxed{1}$ $\mathrm{COL}(A) = c_1 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 4 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 2 \\ 2 \\ -1 \end{bmatrix}$ $\quad c_1, c_2, c_3 \in \mathbb{R}$

$\boxed{2}$ $\underline{b} \in \mathrm{SPAN}\{\underline{a}_1, \underline{a}_2, \underline{a}_3\}$ $\quad A\underline{x} = \underline{b}$

$\underline{x} \in \mathbb{R}^3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 2 & 2 & 1 \\ -2 & 4 & 2 & 1 \\ 1 & -2 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 2 & 2 & 1 \\ 0 & 6 & 6 & 3 \\ 0 & -3 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3/2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & \boxed{3/2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

NO SOLUTION $\underline{x}$ THERFORE $\underline{b}$ IS NOT IN $Col(A)$

3

$$A\underline{x} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -2 & 4 & 2 \\ 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

THERFORE $\underline{x} \in Nul(A)$

4 SOLVE $A\underline{x} = \underline{0}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ -2 & 4 & 2 & 0 \\ 1 & -2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 6 & 6 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right] \sim \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \end{array}$$

INFINITE SOLUTION

x_3 FREE VARIABLE

$$\begin{cases} 2x_2 + 2x_3 = 0; & x_2 = -x_3 \\ x_1 + x_2 + 2x_3 = 0; & x_1 = -x_3 \end{cases} \qquad \begin{cases} x_1 = x_1 \in \mathbb{R} \\ x_2 = -x_3 \\ x_3 = -x_3 \end{cases} \qquad \underline{x} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} x_3 \quad x_3 \in \mathbb{R}$$

THE SPACE $Nul(A)$ IS ANY VECTOR IN THE FORM $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} x_3 \quad x_3 \in \mathbb{R}$