LINEAR SYSTEM $\longleftrightarrow$ AUGMENTED MATRIX $\xrightarrow{\text{ROW REDUCTION ALGORITM}}$ AVG MATRIX IN ECHELON FORM $\longrightarrow$ FIND THE SOLUTION SET

$$\begin{bmatrix} \boxed{*} & * & * & * & * \\ 0 & \boxed{*} & * & * & * \\ 0 & 0 & \boxed{*} & * & * \\ 0 & 0 & 0 & \boxed{*} & * \end{bmatrix}$$

ANY MATRIX CAN BE TRANSFORMED IN ECHELON FORM (EF) BY THE ROW REDUCTION ALGORITHM

↳ SEQUENCE OF ELEMENTARY ROW EQUATION

IMPORTANT ANY MATRIX IS ROW EQUIVALENT TO A MATRIX IN EF

SUCH EF MATRIX IS NOT UNIQUE

THERE MIGTH BE MANY EF MATRIX THAT ARE ROW EQUIVALENT TO THE STARTING MATRIX

$$\begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & 3 & 10 \\ 2 & 4 & 8 \end{bmatrix} \xleftrightarrow{R.E} \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{3} & 8 \\ 0 & 0 & \boxed{2} \end{bmatrix} \xleftrightarrow{R.E} *2 \begin{bmatrix} \boxed{2} & 4 & 6 \\ 0 & \boxed{3} & 8 \\ 0 & 0 & \boxed{2} \end{bmatrix}$$

$\updownarrow$ 1°-3° R.E

$$\begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{3} & 10 \\ 0 & 0 & \boxed{2} \end{bmatrix}$$

ALL ARE ROW EQUIVALENT (RE)

THE PATTER OF THE LEADING ENTRY IS ALWAYS

BUT THE POSITION OF THE LEADING ENTRY OF ALL ROWS OF ALL SUCH EF MATRIXES ARE THE SAME

(AND TAKE THE NAME OF PIVOT POSITION)

A PIVOT POSITION IS THE POSITION OF THE LEADING ENTRY OF ALL ROWS OF AN EF MARIX

$N = M = 3$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$\begin{array}{cccc} \multicolumn{4}{c}{\left[\begin{matrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{matrix}\right]} \\ \downarrow & \downarrow & \downarrow & \\ x_1 & x_2 & x_3 & \end{array}$$

$\circledast$ = DIFFERENT FROM 0

$$\left[\begin{array}{ccc|c} \circledast & * & * & * \\ 0 & \circledast & * & * \\ 0 & 0 & \circledast & * \end{array}\right] \qquad \begin{matrix} \downarrow \\ x_3 \end{matrix}$$

$$\left[\begin{matrix} \circledast & * & * & * \\ 0 & \circledast & * & * \\ 0 & 0 & 0 & \circledast \end{matrix}\right]$$

$\downarrow$

NO SOLUTION
LAST EQ. IS IMPOSSIBLE

$$\begin{array}{ccccc} \multicolumn{5}{c}{\left[\begin{array}{cccc|c} \circledast & * & * & * & * \\ 0 & \circledast & * & * & * \\ 0 & 0 & \circledast & * & * \end{array}\right]} \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ x_1 & x_2 & x_3 & x_4 & \end{array}$$

REWRITE x_3 AS A FUNCTION OF x_4

$\downarrow$

INFINITE SOLUTIONS

$$\left[\begin{matrix} \circledast & * & * & * & * \\ 0 & \circledast & * & * & * \\ 0 & 0 & \circledast & * & * \\ 0 & 0 & 0 & \circledast & * \end{matrix}\right]$$

THEOREM (EXISTENCE AND UNIQUENESS THEOREM, VERBOSE VERSION)

AUGMENTED MATRIX ASSOCIATED TO A LINEAR SYSTEM IN EF

1) IF THERE IS AT LEAST ROW IN WHICH ALL ENTRIES ARE 0 APART THE LAST ONE WHICH IS DIFFERENT FROM 0, THEN THE SOLUTION SET IS VOID NO SOLUTION

$\downarrow$

IN THIS CASE THE LINEAR SYSTEM IS CALLED INCONSISTENT

2) OTHERWISE THE SYSTEM IS CONSISTENT (THERE MAY BE ONE OR INFINITE SOLUTIONS)

- IF EACH COLUM (EXCLUDING THE LAST ONE) HAS A PIVOT POSITION, THEN THE SOLUTION IS UNIQUE AND YOU CAN FIND BY "BACK PROPAGATION"
- OTHERWISE THE SOLUTION ARE INFINITE AND CAN BE EXPRESSED AS A SET PARAMETRIZED BY THE VARIABLES ASSOCIATED TO THE COLUM WITHOUT A PIVOT POSITION (FREE VARIABLES)

> IF YOU HAVE MORE COLUMS THAN ROWS YOU ARE SURE THAT YOU WILL HAVE NO SOLUTION OR INFINITE SOLUTIONS (CAN'T HAVE ONLY ONE SOLUTION)

$$\left[\begin{array}{ccc|c} \boxed{1} & -1 & 2/3 & 2/3 \\ -2 & 2 & -2 & 0 \\ 2 & 1 & 1 & 1 \\ 3 & 1 & 1 & -1 \end{array}\right] \begin{array}{c} \\ -2 \\ 2 \\ 3 \end{array} = \begin{bmatrix} \boxed{1} & -1 & 2/3 & 2/3 \\ 0 & 0 & -2/3 & 4/3 \\ 0 & 3 & -1/3 & -1/3 \\ 0 & 4 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2/3 & 2/3 \\ 0 & 3 & -1/3 & -1/3 \\ 0 & 0 & -2/3 & 4/3 \\ 0 & 4 & -1 & -3 \end{bmatrix} =$$

• TRANSFORM TO EF

$$= \begin{bmatrix} \boxed{1} & -1 & 2/3 & 2/3 \\ 0 & \boxed{9} & -1 & -1 \\ 0 & 0 & -2 & 4 \\ 0 & 4 & -1 & -3 \end{bmatrix} \begin{array}{c} \\ \cdot 3 \\ \cdot 3 \\ 4/9 \end{array} = \begin{bmatrix} 1 & -1 & 2/3 & 2/3 \\ 0 & 9 & -1 & -1 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & -5/9 & -\frac{23}{9} \end{bmatrix} \cdot 9 = \begin{bmatrix} \boxed{1} & -1 & 2/3 & 2/3 \\ 0 & \boxed{9} & -1 & -1 \\ 0 & 0 & \boxed{-2} & 4 \\ 0 & 0 & -5 & -23 \end{bmatrix} \begin{array}{c} \\ \\ \\ -5/2 \end{array} =$$

$-1 + \frac{4}{9}$ $\quad -3 + 4/9$

$$= \begin{bmatrix} \boxed{1} & -1 & 2/3 & 2/3 \\ 0 & \boxed{9} & -1 & -1 \\ 0 & 0 & \boxed{-2} & 4 \\ 0 & 0 & 0 & -33 \end{bmatrix}$$

NO SOLUTION

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & -2 & 1 \\ 2 & -2 & -2 & -2 \\ -3 & 0 & 1 & -2 \\ 1 & 1 & 1 & 3 \end{array}\right] \begin{array}{c} \\ 2 \\ -3 \\ 1 \end{array} = \left[\begin{array}{ccc|c} \boxed{1} & 2 & -2 & 1 \\ 0 & -6 & 2 & -4 \\ 0 & 6 & -5 & 1 \\ 0 & -1 & 3 & 2 \end{array}\right] = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & -1 & 3 & 2 \\ 0 & 6 & -5 & 1 \\ 0 & -6 & 2 & -4 \end{bmatrix} \begin{array}{c} \\ \\ -6 \\ 6 \end{array}$$

$$= \begin{bmatrix} \boxed{1} & 2 & -2 & 1 \\ 0 & \boxed{-1} & 3 & 2 \\ 0 & 0 & 13 & 13 \\ 0 & 0 & -16 & -16 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \begin{array}{c} \\ \\ \\ -1 \end{array} = \left[\begin{array}{ccc|c} \boxed{1} & 2 & -2 & 1 \\ 0 & \boxed{-1} & 3 & 2 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

IS IN EF

← THIS EQ. WAS USELESS

UNIQUE SOLUTION

(BECAUSE EACH COLUMN HAS A PIVOT POSITION)

3RD ROW: $x_3 = 1$

2ND ROW: $-x_2 + 3x_3 = 2; \quad x_2 = 1$

1ST ROW: $x_1 + 2x_2 - 2x_3 = 1; \quad x_1 = 1$

$$\left[\begin{array}{cccc|c} \boxed{1} & -1 & 0 & -1 & 0 \\ 2 & 3 & 1 & 2 & 1 \\ -3 & -1 & -2 & -1 & 4 \end{array}\right] \begin{matrix} 2 \\ -3 \end{matrix} = \left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 0 \\ 0 & 5 & 1 & 4 & 1 \\ 0 & -4 & -2 & -4 & 4 \end{array}\right] \cdot \tfrac{1}{2} = \left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 0 \\ 0 & \boxed{-2} & -1 & -2 & 2 \\ 0 & 5 & 1 & 4 & 1 \end{array}\right] -\tfrac{5}{2} =$$

$$= \left[\begin{array}{cccc|c} \boxed{1} & -1 & 0 & -1 & 0 \\ 0 & \boxed{-2} & -1 & -2 & 2 \\ 0 & 0 & \boxed{-3/2} & -1 & 6 \end{array}\right] \longrightarrow -\tfrac{3}{2}x_3 - x_4 = 6 \;;\quad x_3 = -4 - \tfrac{2}{3}x_4$$

$x_1 \quad x_2 \quad x_3 \quad x_4$

↳ x_4 HAS NO PIVOT POSITION

⇓

INFINITE SOLUTIONS

3RD ROW: $x_3 = -4 - \frac{2}{3}x_4$

2ND ROW: $-x_2 - \frac{1}{2}x_3 - x_4 = 1$; $x_2 = -\frac{1}{2}x_3 - x_4 - 1$; $x_2 = 2 + \frac{1}{3}x_4 - x_4 - 1$; $x_2 = 1 - \frac{2}{3}x_4$

1ST ROW: $x_1 - x_2 - x_4 = 0$; $x_1 = x_2 + x_4$; $x_1 = 1 - \frac{2}{3}x_4 + x_4$; $x_1 = \frac{1}{3}x_4 + 1$

SOLUTION SET

$$\begin{cases} x_1 = \frac{1}{3}x_4 + 1 \\ x_2 = -\frac{2}{3}x_4 + 1 \\ x_3 = -4 - \frac{2}{3}x_4 \\ x_4 \in \mathbb{R} \end{cases}$$

$$\left[\begin{array}{cccc|c} \boxed{1} & -2 & 1 & 4 & -1 \\ 2 & -4 & 5 & 7 & 0 \\ 1 & -2 & 4 & 3 & 1 \end{array}\right] \begin{matrix} 2 \\ 1 \end{matrix} = \left[\begin{array}{cccc|c} \boxed{1} & -2 & 1 & 4 & -1 \\ 0 & 0 & \boxed{3} & -1 & 2 \\ 0 & 0 & 3 & -1 & 2 \end{array}\right] 1 = \left[\begin{array}{cccc|c} \boxed{1} & -2 & 1 & 4 & -1 \\ 0 & 0 & \boxed{3} & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = \text{INFINITE SOLUTION}$$

2 ROWS ARE THE SAME

IS IN EF

x_2 AND x_4 ARE FREE VARIABLES

2ND ROW: $3x_3 - 1x_4 = 2$; $x_3 = \frac{2}{3} + \frac{1}{3}x_4$

1ST ROW: $x_1 - 2x_2 + x_3 + 4x_4 = -1$; $x_1 = 2x_2 - x_3 - 4x_4 - 1$; $x_1 = 2x_2 - \frac{2}{3} - \frac{1}{3}x_4 - 4x_4 - 1$; $x_1 = 2x_2 - \frac{13}{3}x_4 - \frac{5}{3}$

$$\begin{cases} x_1 = -5/3 + 2x_2 - \frac{13}{3}x_4 \\ x_2 \in \mathbb{R} \text{ (FREE VARIABLE)} \\ x_3 = \frac{2}{3} + \frac{1}{3}x_4 \\ x_4 \in \mathbb{R} \text{ (FREE VARIABLE)} \end{cases}$$

VECTOR: A VECTOR $\underline{V} \in \mathbb{R}^N$ IS A COLLECTION $[V_1, V_2, V_3 \dots, V_N]$ WITH $V_1, V_2, V_3 \dots, V_N \in \mathbb{R}$

SOMETIMES THEY ARE WRITTEN AS $\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_N \end{bmatrix} = \underline{V}$

SIMPLE OPERATIONS W/ VECTORS

CAN'T SUM OF VECTORS W/ DIFFERENT LENGTH

- SUM OR DIFFERENCE OF VECTORS

$\underline{V}, \underline{W} \in \mathbb{R}^N$ $\quad \underline{V} + \underline{W} \Rightarrow \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_N \end{bmatrix} + \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ \vdots \\ W_N \end{bmatrix} = \begin{bmatrix} V_1 + W_1 \\ V_2 + W_2 \\ V_3 + W_3 \\ \vdots \\ V_N + W_N \end{bmatrix}$

$$\begin{bmatrix} 3 \\ 2 \\ -6 \end{bmatrix} + \begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 3-1 \\ 2+7 \\ -6+2 \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \\ -4 \end{bmatrix}$$

- MULTIPLICATION OF A VECTOR TIMES A REAL NUMBER (SCALAR)

$\underline{V} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix}$ $\quad \alpha \in \mathbb{R}$ $\quad \alpha \underline{V} = \begin{bmatrix} \alpha V_1 \\ \alpha V_2 \\ \vdots \\ \alpha V_N \end{bmatrix}$ $\qquad 5 \begin{bmatrix} 3 \\ 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \\ -30 \end{bmatrix}$

SIMPLE PROPERTIES OF VECTOR OPERATION

$\underline{v}, \underline{w}, \underline{u} \in \mathbb{R}^N$ $\quad \alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$

- $\underline{v} + \underline{w} = \underline{w} + \underline{v}$
- $\underline{v} + \underline{w} + \underline{u} = \underline{v} + (\underline{w} + \underline{u})$
- $\alpha(\underline{v} + \underline{w}) = \alpha \underline{v} + \alpha \underline{w}$
- $(\alpha + \beta)\underline{v} = \alpha \underline{v} + \beta \underline{v}$
- $(\alpha\beta)\underline{v} = \alpha(\beta \underline{v})$

MULTIPLICATION OF A MATRIX AND A VECTOR

$A \in \mathbb{R}^{M \times N}$ $\quad \underline{v} \in \mathbb{R}^N$

MATRIX

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1N} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2N} \\ \vdots & & & & \vdots \\ a_{M1} & a_{M1} & a_{M3} & \dots & a_{MN} \end{bmatrix}$

$\underline{v} = [v_1, v_2, v_3]$

$A \; [\underline{a}_1 \; \underline{a}_2 \; \dots \; \underline{a}_N]$ ← LIKE VECTORS

$$\boxed{A\underline{v} = \underline{a}_1 v_1 + \underline{a}_2 v_2 + a_3 v_3 + \dots + a_N v_M}$$

$$\underbrace{\begin{bmatrix} 3 & 4 & 2 \\ 2 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}}_{\underline{A}} \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{\underline{v}} = 1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 8 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 6 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 17 \\ 8 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{33} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} v_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} v_2 + \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} v_3 = \begin{bmatrix} a_{11} v_1 \\ a_{21} v_1 \\ a_{31} v_1 \end{bmatrix} + \begin{bmatrix} a_{12} v_2 \\ a_{22} v_2 \\ a_{32} v_2 \end{bmatrix} + \begin{bmatrix} a_{13} v_3 \\ a_{23} v_3 \\ a_{33} v_3 \end{bmatrix} =$$

$\underline{A} \in \mathbb{R}^{3 \times 2}$ $\qquad$ $\underline{v} \in \mathbb{R}^3$

$$= \begin{bmatrix} a_{11} v_1 + a_{12} v_2 + a_{13} v_3 \\ a_{21} v_1 + a_{22} v_2 + a_{23} v_3 \\ a_{31} v_1 + a_{32} v_2 + a_{33} v_3 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & & & \vdots \\ a_{M1} & a_{M2} & \dots & a_{MN} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_M \end{bmatrix} = \begin{bmatrix} a_{11} v_1 + a_{12} v_2 + \dots + a_{1N} v_N \\ a_{21} v_1 + a_{22} v_2 + \dots + a_{2N} v_N \\ \vdots \qquad\qquad \vdots \\ a_{M1} v_1 + a_{M2} v_2 + \dots a_{MN} v_N \end{bmatrix}$$

DEFINITION: LINEAR COMBINATION OF VECTORS

A LINEAR COMBINATION OF VECTORS $\{\underline{v_1}, \underline{v_2}, \underline{v_3} \dots \underline{v_p}\}$ EACH IN $\mathbb{R}^N$ IS ANY VECTOR $\underline{w}$ THAT CAN BE WRITTEN AS $\underline{w} = c_1 \underline{v_1} + c_2 \underline{v_2} + c_3 \underline{v_3} + \dots + c_p \underline{v_p} \in \mathbb{R}^N$

WITH $c_1, c_2 \dots c_p \in \mathbb{R}$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 0 \\ 6 \\ 0 \\ 6 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \\ 0 \end{bmatrix} \qquad c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 6 \\ 0 \\ 5 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -1 \\ 2 \\ 0 \end{bmatrix}$$

$c_1, c_2, c_3 \in \mathbb{R}$

AS AN EXAMPLE:

$c_1 = 2 \quad c_2 = 0 \quad c_3 = 3$

$$\begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 12 \\ 8 \end{bmatrix}$$

$\{\underline{v_1}, \underline{v_2}, \dots \underline{v_p}\}$ VECTORS IN $\mathbb{R}^N$

DEFINITION THE SPAN $\{\underline{v_1}, \underline{v_2}, \dots \underline{v_p}\}$ IS THE SET OF ALL LINEAR COMBINATIONS OF SUCH VECTORS

$\mathbb{R}^3$ ← 3 ENTRIES

$$\text{SPAN} \left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\underline{v_1}}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\underline{v_2}} \right\}$$

$c_1, c_2 \in \mathbb{R}$

INFINITE NUMBER OF VECTORS

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ 0 \end{bmatrix}$$