$+$ $\times$

$-$ $\div$

- $\lim_{n} \sqrt[n]{n} = 1$

- $\lim_{n} \sqrt[3n^2]{2n} = 1$ $\quad (2n)^{1/3n^2} = e^{\log(2n)\frac{1}{3n^2}}$ $\quad e^0 = 1$

$$\frac{\log(2n)}{3n^2} = \frac{\log(2)+\log(n)}{3n^2} \to 0$$

- $(n^2+1)^{-1/\log(n)}$ $\quad [(+\infty)^0]$

$$e^{\log(n^2+1)\left(-\frac{1}{\log n}\right)}$$

$$\frac{-\log(n^2+1)}{\log(n)} = -\frac{\log\left(n^2\left(1+\frac{1}{n^2}\right)\right)}{\log(n)} = -\frac{\log(n^2)+\log\left(1+\frac{1}{n^2}\right)}{\log(n)}$$

$$-\left(\frac{2\cancel{\log}(n)}{\cancel{\log}(n)} + \frac{\overbrace{\log\left(1+\frac{1}{n^2}\right)}^{0}}{\underbrace{\log(n)}_{+\infty}}\right) \to -2$$

$$\Rightarrow \lim_{n} (n^2+1)^{-1/\log(n)} = e^{-2} = \frac{1}{e^2}$$

- $\lim_{n} \left(\frac{n^2-1}{n^2}\right)^{3n+1}$ $\quad [1^\infty]$

$$e^{\log\left(\frac{n^2-1}{n^2}\right)(3n+1)}$$

$$= \log\left(\frac{n^2-1}{n^2}\right)(3n+1) = \log\left(1-\frac{1}{n^2}\right)(3n+1) =$$

$$= \underbrace{\frac{\log\left(1-\frac{1}{n^2}\right)}{-\frac{1}{n^2}}}_{1} \underbrace{\left(-\frac{1}{n^2}\right)(3n+1)}_{0} \to 0$$

$$!!!\ \lim \frac{\log(1+x)}{x} = 1$$

$$\lim_n \left(\frac{n^2-1}{n^2}\right)^{3n+1} = e^0 = 1$$

GOES FASTER

- $\lim (2^n - n^3)$ $[\infty - \infty]$

$$2^n - n^3 = 2^n\left(1 - \overbrace{\frac{n^3}{2^n}}^{0}\right) \to \text{"}+\infty \cdot 1\text{"} = +\infty$$

$$\frac{n^3}{2^n} = \frac{n^3}{e^{(\log 2)n}} \to 0 \qquad b_1 = 3,\ \alpha = \log 2 > 0$$

- $\lim_n (2n^2 - \sqrt{n}) = \lim_n 2n^2\left(1 - \overbrace{\frac{\sqrt{n}}{2n^2}}^{0}\right) = +\infty$

- $\lim_n (n - \log(n)\sqrt{n}) = \lim_n n\left(1 - \frac{\log(n)\sqrt{n}}{n}\right) = \lim n\left(1 - \underbrace{\frac{\log n}{\sqrt{n}}}_{0}\right) = +\infty$

$\sqrt{n}\sqrt{n}$

$\sqrt{n}$ IS FASTER THAN $\log n$

- $\lim_{n} \left(\sqrt{n} - \cos(n)\sqrt{n}\right)$

$$\lim_{n} \underset{\downarrow\, +\infty}{\sqrt{n}} \underbrace{\left(1 - \cos(n)\right)}_{-\infty} = -\infty$$

- $\lim_{n} \dfrac{\sin^4(3/n)}{\sin(8/n^4)} = ?$

$$\frac{\sin^4\left(\frac{3}{n}\right)}{\left(\frac{3}{n}\right)^4} \to 1 \quad \left(\frac{\sin(3/n)}{3/n}\right)^4 = 1^4 = 1$$

$$\frac{\sin^4\left(\frac{3}{n}\right)}{\left(\frac{3}{n}\right)^4} \cdot \frac{\left(\frac{3}{\cancel{n}}\right)^4}{8/\cancel{n^4}} \cdot \frac{8/n^4}{\underset{\downarrow\, 1}{\sin(8/n^4)}} = 1 \cdot \frac{3^4}{8} \cdot 1 = \frac{3^4}{8}$$

$$\lim_{N} \underbrace{N^2}_{\infty} \underbrace{\sin\left(e^{1/N}-1\right)}_{0} \quad [\infty \cdot 0]$$

$$\lim_{N} \frac{\sin\left(e^{1/N}-1\right)}{1/N^2} \quad \frac{0}{0} \quad !!! \quad \boxed{\lim_{N} \frac{\sin x}{x} = 1}$$

$$\lim_{N} \underbrace{\frac{\sin\left(e^{1/N}-1\right)}{e^{1/N}-1}}_{1} \cdot \underbrace{\frac{e^{1/N}-1}{1/N}}_{1} \cdot \frac{1/N}{1/N^2} \to +\infty$$

N

$$\lim_{N} \frac{-\cos(1/N) + e^{2/N^2}}{\sin(3/N^2)}$$

$$\lim_{N} \frac{-\cos\left(\frac{1}{N}\right) +1 - 1 + e^{2/N^2}}{\sin(3/N^2)} = \frac{1-\cos\left(\frac{1}{N}\right)}{\sin\left(\frac{3}{N^2}\right)} + \frac{e^{3/N^2}-1}{\sin(3/N^2)}$$

$$\underbrace{\frac{1-\cos(1/N)}{1/N^2}}_{1/2} \cdot \underbrace{\frac{1/N^2}{3/N^2}}_{1/3} \cdot \underbrace{\frac{3/N^2}{\sin(3/N^2)}}_{1} + \frac{e^{3/N^2}-1}{3/N^2} \cdot \underbrace{\frac{3/N^2}{\sin(3/N^2)}}_{1} \to$$

$$\to \frac{1}{6} + 1 = \frac{7}{6} \qquad \frac{e^x-1}{x}$$

$$\lim_{n} \frac{-\cos\left(\frac{1}{n}\right) + e^{-1/2n^2}}{\sin\left(3/n^2\right)} = \frac{1}{2} \cdot \frac{1}{3} - \frac{1}{2} \cdot \frac{1}{3} = 0$$

WHAT ABOUT

$$\lim_{n} \frac{-\cos\left(\frac{1}{n}\right) + e^{-1/2n^2}}{\sin\left(3/n^3\right)} = [\infty - \infty]$$

HOW TO SOLVE THIS?

$\boxed{\text{SERIES}}$

DEFINITION LET $\{a_n\}_{n \in \mathbb{N}}$ BE A SEQUENCE OF REAL NUMBERS. FOR ANY $n \in \mathbb{N}$ LET $s_n \sum_{i=1}^{n} a_i = a_1 + a_2 \ldots + a_n$

THE PAIR OF SEQUENCES $\left(\{a_n\}_{n \in \mathbb{N}}, \{s_n\}_{n \in \mathbb{N}}\right)$ IS CALLED SERIES W/ GENERAL TERM a_n AND IS DENOTED AS:

$$\sum_{n=1}^{\infty} a_n \text{ OR } \sum_{n} a_n$$

a_n GENERAL TERMS OF THE SERIES

S_n N-TH PARTIAL SUMS OF THE SERIES

LET US STUDY $\lim_n S_n$, IF IT EXISTS. WE HAVE THE FOLLOWING SITUATIONS

- IF $\exists \lim_n S_n = S$, THEN $S = \sum_{n=1}^{\infty} a_n$ IS THE SUM OF THE SERIES
- IF $S \in \mathbb{R}$ THEN THE SERIES IS CONVERGENT IF

  $S = \pm\infty$ THEN THE SERIES IS DIVERGENT TO $+\infty$ OR $-\infty$ RESPECTIVELY
- IF $\nexists \lim_n S_n$, THEN THE SERIES IS INDETERMINATE OR IRREGULAR

REMARKS

1) IF $\{a_n\}_{n \in \mathbb{N}}$ IS DEFINITELY 0, THAT IS

$\exists n_0 \in \mathbb{N}$ SUCH THAT $a_n = 0$ $\forall n > n_0$, THEN

$\forall n \geq n_0 \quad S_n = S_{n_0}$

$n > n_0 \quad S_n = \underbrace{a_1 + a_2 + \dots a_{n_0}}_{= S_{n_0}} + a_{n_0+1} + \dots a_n$

SO $\lim_{n} S_n = S_{n_0}$ THAT IS $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{n_0} a_n$

2) $\sum_{n=1}^{\infty} a_n$ DENOTES BOTH THE SERIES AND WHEN IT IS WELL DEFINED THE SUM OF THE SERIES

EXAMPLE

1) $\sum_{n=1}^{+\infty} n = +\infty \qquad a_n = n$

$$S_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \geq n \to +\infty$$

2) $\sum_{n=0}^{+\infty} x^n = 1 + x + x^2 + \dots + x^n + \dots$

GEOMETRIC SERIES WITH RATIO $x \in \mathbb{R}$

- $x = 0 \quad 1 + 0 + 0 \dots + 0 + \dots \quad \sum_{n=1}^{+\infty} 0^n = 1$

- $x = 1 \quad 1 + 1^1 + 1^2 + \dots 1^n \quad \sum_{n=0}^{+\infty} a_n \quad a_n = 1 \ \forall n \geq 0$

$$S_n = \sum_{i=0}^{n} a_n = \sum_{i=0}^{n} 1 = n+1 \underset{+\infty}{\downarrow} \quad \forall n \geq 0$$

$$\sum_{n=0}^{+\infty} 1^n = \sum_{n=0}^{+\infty} 1 = +\infty$$

• $x = -1 \quad \sum_{n=0}^{+\infty} (-1)^n = 1 - 1 + 1 - 1 \ldots$ INDETERMINATED

$S_0 = 1 \quad S_1 = 0 \quad S_2 = 1 \quad S_3 = 0$

$S_{2n} = 1 \quad \forall n \geq 0$

$\Rightarrow S_n$ is IRREGULAR

$S_{2n+1} = 0 \quad \forall n \geq 0$

$\sum_{n=0}^{+\infty} (-1)^n$ is INDETERMINATE

• $x \neq 0, 1$

$$\underbrace{1 + x + \ldots + x^n}_{S_n} = \frac{1 - x^{n-1}}{1 - x} \quad \forall n \geq 0$$

• $-1 < x < 1$ then $\lim_n x^{n+1} = 0$

$$\sum_{n=0}^{+\infty} x^n = \lim_n \frac{1 - x^{n+1}}{1 - x} = \frac{1}{1 - x} \quad \text{CONVERGING}$$

$x > 1$ then $\lim_n x^{n+1} = +\infty$

so $\lim_n \frac{1 - x^{n+1}}{1 - x} = \lim_n \frac{x^{n+1} - 1}{x - 1 > 0} + \infty$

$$\Rightarrow \sum_{n=0}^{+\infty} x^n = +\infty$$

• $x < -1 \quad x^{n+1} (-1)^{n+1} |x|^{n+1} \to +\infty$

$\Rightarrow \nexists \lim_n S_n = \lim_n \frac{1 - x^{n+1}}{1 - x}$ $+\infty$

$\Rightarrow \sum_{n=0} x^n$ INDETERMINATE

SUMMARY

$\boxed{x \geq 1}$ $\sum_{n=0}^{+\infty} x^n = +\infty$

$\boxed{-1 < x < 1}$ $\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}$ CONVERGING

$\boxed{x \leq -1}$ $\sum_{n=0}^{+\infty} x^n$ INDETERMINATE

ZENONE'S PARADOX

$$1 + \frac{1}{10} + \frac{1}{100} + \dots \frac{1}{10^n} \dots = \sum_{n=0}^{+\infty} \left(\frac{1}{10}\right)^n = \frac{1}{1 - \frac{1}{10}} = \frac{10}{9}$$

$x = \frac{1}{10}$

REMARK

LET $\sum_{n=1}^{\infty} a_n$ AND $\sum_{n=1}^{\infty} b_n$ BE TWO SERIES

$s_n = \sum_{i=1}^{n} a_i$; $t_n = \sum_{i=1}^{n} b_i$

IF $a_n = b_n$ DEFINETELY THEN THE 2 SERIES HAVE THE SAME BEHAVIOUR OR CHARACTHER

(CONVERGING, DIVERGENT TO $+\infty$ OR TO $-\infty$, INDETERMINATE)

$a_n = b_n \quad \forall n \geq n_0 + 1$ FOR SOME $n_0 \in \mathbb{N}$

$\forall n > n_0$

$S_n = S_{n_0} + a_{n_0+1} + \dots + a_n$

$$\Rightarrow t_n = S_n - S_{n_0} + t_{n_0}$$

$t_n = t_{n_0} + b_{n_0+1} + \dots + b_n$

THE SUM, IF FINITE, CAN BE DIFFERENT!

IN PARTICULAR THE "REMAINDER SERIES"

$\sum_{n=n_0+1}^{+\infty} a_n$ HAS THE SAME BEHAVIOUR OF $\sum_{n=1}^{+\infty} a_n \quad \forall n_0 \in \mathbb{N}$

PROPOSITION

LET $\sum_{n=1}^{+\infty} a_n$ AND $\sum_{n=1}^{+\infty} b_n$ BE TWO CONVERGING SERIES

THEN $\forall c \in \mathbb{R}$ WE HAVE

$\sum_{n=1}^{+\infty} (c a_n)$ IS CONVERGING TO $c \sum_{n=1}^{\infty} a_n$

$$\sum_{i=1}^{n} (c a_i) = c \sum_{i=1}^{n} a_i = c S_n \to c \sum_{n=1}^{+\infty} a_n$$

- $\sum_{n=1}^{+\infty}(a_n+b_n)$ IS CONVERGING TO $\sum_{n=1}^{+\infty} a_n + \sum_{n=1}^{+\infty} b_n$

$$\sum_{i=1}^{N}(a_i+b_i) = \sum_{i=1}^{N} a_i + \sum_{i=1}^{N} b_i = S_N + t_N \to \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{+\infty} b_n$$

REMARK

IF $\sum_{n=1}^{+\infty} a_n = \pm\infty$ AND $c \neq 0$ THEN

$$\sum_{n=1}^{+\infty}(c\,a_n) = \begin{cases} \pm\infty & \text{IF } c>0 \\ \mp\infty & \text{IF } c<0 \end{cases}$$

- IF $\sum_{n=1}^{+\infty} a_n = \pm\infty$ AND $\sum_{n=1}^{+\infty} b_n = S \in \mathbb{R}$, THEN

$$\sum_{n=1}^{+\infty}(a_n+b_n) = \pm\infty$$

ISSUME LET $\sum_{n=1}^{+\infty} a_n$ BE A SERIES

FIND THE BEHAVIOUR OF THE SERIES

THEOREM IF $\sum_{N=1}^{+\infty} a_N$ IS CONVERGING THEN $\lim_N a_N = 0$

PROOF LET $S_N = \sum_{i=1}^{N} a_i$ IF $\exists \lim_N S_N = S \in \mathbb{R}$

THEN $\exists \lim_N S_{N+1} = S \in \mathbb{R}$

THEN $\lim_N (S_{N+1} - S_N) = S - S = 0$

$$S_{N+1} = \sum_{i=1}^{N+1} a_i = \sum_{i=1}^{N} a_i + a_{N+1} = S_N + a_{N+1}$$

$$\Rightarrow \lim_N a_{N+1} = 0 \Rightarrow \lim_N a_N = 0$$

REMARK: $\lim_N a_N = 0$ IS A NECESSARY CONDITION FOR THE SERIES TO BE CONVERGENT

UNFORTUNATELY IT IS NOT A SUFFICENT CONDITION FOR THE SERIES TO BE CONVERGING

EXAMPLE $\sum_{N=1}^{+\infty} \frac{1}{N}$ HARMONIC SERIES

WE SHOW THAT THE SUM $\sum_{N=}^{+\infty} \frac{1}{N} = +\infty$

$S_1 = 1 \quad S_2 = 1 + \frac{1}{2} \; ; \; S_4 = 1 + \frac{1}{2} \left(\frac{1}{3} + \frac{1}{4}\right) \geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right)$

$= 1 + \frac{1}{2} + \frac{1}{2}$

$$S_8 = 1 + \frac{1}{2}\left(\frac{1}{3} + \frac{1}{4}\right)\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

$$S_{2k} \geq 1 + \frac{k}{2} \Rightarrow \lim_{k} S_{2k} = +\infty \Rightarrow \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$$

WHY?

PROPOSITION LET $\sum_{n=1}^{+\infty} a_n$ BE A SERIES WITH NON NEGATIVE TERMS (THAT IS $a_n \geq 0 \ \forall n \in \mathbb{N}$) OR WITH POSITIVE TERMS (THAT IS $a_n > 0 \ \forall n \in \mathbb{N}$)

THEN EITHER $\sum_{n=1}^{+\infty} a_n$ CONVERGES OR $\sum_{n=1}^{+\infty} a_n = +\infty$

(IT IS NEVER INDETERMINATE)

A SERIES WITH NON-NEGATIVE NUMBERS IS NEVER INDETERMINATE!

NOTATION $\sum_{n=1}^{+\infty} a_n$ SERIES W/ NON-NEGATIVE NUMBERS

WE WRITE $\sum_{n=1}^{+\infty} a_n < +\infty$ TO SAY IT IS CONVERGING

PROPOSITION

$$S_N = \sum_{i=1}^{N} a_i \qquad S_{N+1} = S_N + \underbrace{a_{N+1}}_{\geq 0} \geq S_N$$

$\Rightarrow \{S_N\}_{N \in \mathbb{N}}$ IS ≥ 0 IS INCREASING $\Rightarrow$

- IF $\sup_N S_N < +\infty$ (IS FINITE) THE $\{S_N\}_N$ AND $\sum_{N=1}^{+\infty} a_N$ ARE CONVERGING

- IF $\sup_N S_N = +\infty$ THEN $\{S_N\}_N$ AND $\sum_{N=1}^{+\infty} a_N$ ARE CONVERGING TO $+\infty$

— · - - - X

$$S_{2^K} = \sum_{N=1}^{} \frac{1}{X} = 1 + \frac{K}{2} \Rightarrow \sup_N S_N = +\infty \Rightarrow \lim_N S_N = +\infty$$