Knowledge Representation and Reasoning

Exercise Session 4

Exercise 1. NNF (∗)

Transform the following $\mathcal{ALC}$ concepts to negation normal form.

1. $\neg(A \sqcup \exists r.A) \sqcup \forall r.\neg B$
2. $\exists r.\neg(\forall s.B \sqcap B) \sqcap (\neg B \sqcup A)$
3. $\neg(\exists r.\neg A \sqcup \forall s.\neg(\neg A \sqcup B) \sqcup \neg A)$

Exercise 2. Satisfiability (∗)

Decide whether the following concepts are satisfiable or not. If they are, provide a satisfying interpretation.

1. $A \sqcap \neg\forall r.(A \sqcup B) \sqcap \neg\exists r.\neg B$
2. $B \sqcap (\neg B \sqcup \exists r.\bot) \sqcup \forall r.\bot$

Exercise 3. Disjunctions (∗∗)

Let $\mathcal{ELU}_\bot$ be the extension of $\mathcal{EL}_\bot$ which allows also for the disjunction constructor ($\sqcup$). Show that $\mathcal{ELU}_\bot$ and $\mathcal{ALC}$ are equivalent.

Exercise 4. Domain size (∗∗)

Construct concepts C and D such that for any interpretation $\mathcal{I}$ it holds that:

1. if $C^\mathcal{I} \neq \emptyset$, then $\Delta^\mathcal{I}$ must have at least two elements
2. if $D^\mathcal{I} \neq \emptyset$, then $\Delta^\mathcal{I}$ must have at least 7 elements

Exercise 5. Disjoint Unions (∗∗)

Let $\mathcal{T}$ be a consistent $\mathcal{ALC}$ TBox. Show that the disjoint union of two models of $\mathcal{T}$ is also a model of $\mathcal{T}$.

Exercise 6. TBox Consistency (∗)

Check whether the following TBoxes are consistent. If they are, provide a model.

1. $\mathcal{T}_1 = \{A \sqsubseteq \exists r.A,\ A \sqsubseteq \forall r.\neg A\}$
2. $\mathcal{T}_2 = \{A \sqsubseteq \exists r.A,\ \forall r.\neg A \sqsubseteq A\}$
3. $\mathcal{T}_3 = \{A \sqsubseteq \exists r.\neg A,\ \forall s.\neg A \sqsubseteq A, \top \sqsubseteq \forall r.\forall s.A\}$

Exercise 7. Satisfiability (∗)

Decide whether the following concepts are satisfiable w.r.t. the TBox $\mathcal{T}_1$ from Exercise 6.

1. $A \sqcup \forall r.A$
2. $\neg A \sqcup \exists r.\bot$
3. $\forall r.\exists r.A$

Decide whether $\forall r.\bot$ is satisfiable w.r.t. the TBox $\mathcal{T}_2$ from Exercise 6.

Exercise 8. Subsumption (∗)

Check whether the following subsumption relations hold w.r.t. the TBox $\mathcal{T}_1$ from Exercise 6.

1. $\exists r.(A \sqcap B) \sqsubseteq \forall r.B$
2. $B \sqcup C \sqsubseteq \forall r.A$
3. $\exists s.\neg A \sqsubseteq \exists r.\neg A$

Exercise 9. Knowledge Base Consistency (∗)

Check whether the following ABoxes are consistent w.r.t. the TBox $\mathcal{T}_1$ from Exercise 6.

1. $\{r(a, b), \forall r.A(a)\}$
2. $\{\exists r.(B \sqcup A)(a), s(b, a), \forall s.\forall r.\neg B(b)\}$
3. $\{r(a, b), r(b, c), r(c, a)\}$

Exercise 10. Number Restrictions (∗∗∗)

Let $\mathcal{ALCQ}$ be the logic that extends $\mathcal{ALC}$ with *qualified number restrictions* $\geq n\ r.C$ expressing the class of objects that have at least n r-successors belonging to the class C. For example,

$$Person \sqcap \geq 2 hasChild.Female$$

is the class of people having at least two daughters.

Device adequate tableau rules to handle number restrictions.

ALC: $C ::= A \mid C \sqcap C \mid \neg C \mid \exists r.C \mid C \sqcup D \mid \forall r.C$

$C \sqcup D := \neg(\neg C \sqcap \neg D)$

$\forall r.C := \neg \exists r.\neg C$

Exercise 1. NNF (∗)

Transform the following $\mathcal{ALC}$ concepts to negation normal form.

1. $\neg(A \sqcup \exists r.A) \sqcup \forall r.\neg B$
2. $\exists r.\neg(\forall s.B \sqcap B) \sqcap (\neg B \sqcup A)$
3. $\neg(\exists r.\neg A \sqcup \forall s.\neg(\neg A \sqcup B) \sqcup \neg A)$

NORMAL FORM: NEGATION APPLY ONLY TO CONCEPT NAME

1. $(\neg A \sqcap \forall r\ \neg A) \sqcup \forall r.\neg B$

2. $\exists r.(\exists s.\neg B \sqcup \neg B) \sqcap (\neg B \sqcup A)$

Exercise 2. Satisfiability (*)

Decide whether the following concepts are satisfiable or not. If they are, provide a satisfying interpretation.

1. $A \sqcap \neg\forall r.(A \sqcup B) \sqcap \neg\exists r.\neg B$

2. $B \sqcap (\neg B \sqcup \exists r.\bot) \sqcup \forall r.\bot$

CHECK SATISFIABILITY

1. $A \sqcap \neg\forall r.(A \sqcup B) \sqcap \neg\exists r.\neg B\ (a)$

FIRST IT NEED TO BE IN NNF THAT IS

$A \sqcap \exists r.(\neg A \sqcap \neg B) \sqcap \forall r.B\ (a)$

$A(a) \quad \exists r.(\neg A \sqcap \neg B)(a) \quad \forall r.B(a)$

$\exists r(\neg A \sqcap \neg B)(a)$

$r(a,b) \quad (\neg A \sqcap \neg B)(b) \quad \forall r.B(a)$

$\downarrow$

$B(b)$

$A\ a \xrightarrow{r} b\ \ \neg A, \neg B, B$

CLASH

2. $B(a) \sqcap (\neg B \sqcup \exists r.\bot)(a) \sqcup \forall r.\bot(a)$

$\neg B(a)$ $B(a)$ CLASH	$B(a)\quad \exists r.\bot(a)$ $r(a,b)$ $a\ B \xrightarrow{r} b$ CLASH	$\forall r.\bot(a)$ WE CAN NOT APPLY THE $\forall$ RULE BECAUSE IT REQUIRES AN r. SUCCESSOR THAT a DOES NOT HAVE THIS IS A SATURATED-OPEN SET OF ASSERTION WITHOUT A CLASH SAT

Exercise 3. Disjunctions (**)

Let $\mathcal{ELU}_\bot$ be the extension of $\mathcal{EL}_\bot$ which allows also for the disjunction constructor ($\sqcup$). Show that $\mathcal{ELU}_\bot$ and $\mathcal{ALC}$ are equivalent.

ALC: $C ::= A \mid C \sqcap C \mid \exists r.C \mid \neg C$

$ELU_\bot$: $C ::= A \mid \bot \mid \top \mid C \sqcap C \mid \exists r.C \mid C \sqcup C$

WE HAVE TO SHOW THAT ALC AND $ELU_\bot$ ARE EQUIVALENT, SO THAT EVERYTHING THAT WE CAN EXPRESS IN $ELU_\bot$ CAN BE ALSO EXPRESSED IN ALC AND VICEVERSA

$\bot = A \sqcap \neg A$

$\top = A \sqcup \neg A$

HOW DO WE EXPRESS THE DISJUNCTION $C \sqcup D$? $\neg(\neg C \sqcap \neg D)$

SO EVERYTHING THAT IS EXPRESSED IN $ELU_\bot$ CAN BE EXPRESSED IN ALC. NOW CHECK THE OPPOSITE

$\mathcal{K}$: $A \sqcap X_A \sqsubseteq \bot$ A CONCEPT NAME AND ITS NEGATION SHOULD BE DISJOINT (NO OBJECT BELONG TO BOTH)

ALSO THERE SHOULD NOT BE A CONCEPT THAT BELONGS TO NONE

$\top \sqsubseteq A \sqcup X_A$

SO $X_A^I = (\neg A)^I$

Exercise 4. Domain size

(**)

Construct concepts C and D such that for any interpretation $\mathcal{I}$ it holds that:

1. if $C^{\mathcal{I}} \neq \emptyset$, then $\Delta^{\mathcal{I}}$ must have at least two elements
2. if $D^{\mathcal{I}} \neq \emptyset$, then $\Delta^{\mathcal{I}}$ must have at least 7 elements

$$
\begin{tikzcd}
A, B \arrow[d, "r"] \\
\neg A, B \arrow[d, "r"] \\
\neg B, A
\end{tikzcd}
$$

$$
\begin{tikzcd}
& & & A \arrow[dll] \arrow[drr] & & & \\
& \neg A, B, C \arrow[dl] \arrow[dr] & & & & \neg A, \neg B, \neg C \arrow[dl] \arrow[dr] & \\
\neg A, B, \neg C, D & & \neg A, B, \neg C, \neg D & & \neg A, \neg B, \neg C, D & & \neg A, \neg B, \neg C, \neg D
\end{tikzcd}
$$

Exercise 5. Disjoint Unions (∗∗)

Let $\mathcal{T}$ be a consistent $\mathcal{ALC}$ TBox. Show that the disjoint union of two models of $\mathcal{T}$ is also a model of $\mathcal{T}$.

BORING

Exercise 6. TBox Consistency (*)

Check whether the following TBoxes are consistent. If they are, provide a model.

1. $\mathcal{T}_1 = \{A \sqsubseteq \exists r.A,\ A \sqsubseteq \forall r.\neg A\}$
2. $\mathcal{T}_2 = \{A \sqsubseteq \exists r.A,\ \forall r.\neg A \sqsubseteq A\}$
3. $\mathcal{T}_3 = \{A \sqsubseteq \exists r.\neg A,\ \forall s.\neg A \sqsubseteq A, \top \sqsubseteq \forall r.\forall s.A\}$

CHECK CONSISTENCY

REMEMBER!! FOR THE TBox CONSISTENCY WE MUST HAVE THE FORM $\top \sqsubseteq A \sqcup B$

THE $\sqsubseteq$ IS LIKE AN IMPLIES ($\rightarrow$)

$A \sqsubseteq \exists r.A$ $\quad$ $A \sqsubseteq \forall r.\neg A$

$\top \sqsubseteq \neg A \sqcup \exists r.A$ $\quad$ $\top \sqsubseteq \neg A \sqcup \forall r.\neg A$

$\top(a)$ $\quad$ $(\neg A \sqcup \exists r.A)(a)$ $\quad$ $(\neg A \sqcup \forall r.\neg A)(a)$

$\neg A(a)$

SAT NO OTHER SPLIT NEEDED BECAUSE $\neg A$ TRUE ALSO IN THE OTHER

SAT

$\exists r.A(a)$ $\quad$ $r(a,b)$ $\quad$ $A(b)$

$\neg A(a)$

NEW OBJECT b
USE SUBSUMPTION RULE AGAIN

$\forall r.\neg A(a)$

$\neg A(b)$

CLASH

$(\neg A \sqcup \exists r.A)(b)$ $\quad$ $\neg A \sqcup \forall r.\neg A(b)$

$\neg A(b)$

CLASH

$\exists r.A(b)$ $\quad$ $\forall r.\neg A(b)$

$r.(b,c)$ $\quad$ $A(c)$ $\quad$ $\neg A(c)$

CLASH

Exercise 7. Satisfiability (*)

Decide whether the following concepts are satisfiable w.r.t. the TBox $\mathcal{T}_1$ from Exercise 6.

1. $A \sqcup \forall r.A$
2. $\neg A \sqcup \exists r.\bot$
3. $\forall r.\exists r.A$

Decide whether $\forall r.\bot$ is satisfiable w.r.t. the TBox $\mathcal{T}_2$ from Exercise 6.

FIRST WE SHOULD TRANSFORM IN NORMAL FORM BUT ALL ALREADY ARE

TBox:

$(A \sqcup \forall r.A)(a)$ 1

$\top \sqsubseteq \neg A \sqcup \exists r.A \quad \top \sqsubseteq \neg A \sqcup \forall r.\neg A$

$(\neg A \sqcup \exists r.A)(a)$ 2 $\quad (\neg A \sqcup \forall r.\neg A)(a)$ 3

$A(a)$

FORCED TO HAVE

$\exists r.A(a)$ AND $\forall r.\neg A(a)$

$r(a,b)$ $A(b)$ $\neg A(b)$

CLASH

$\forall r.A(a)$

$\neg A(a)$

THE 3 IS ALREADY TRUE, SATURATED AND OPEN SET OF ASSERTION

SAT

$\exists r.A(a)$

$r(a,b)$ $A(b)$

$\neg A(a)$

...

??

$\forall r.\neg A(a)$

$\neg A(b)$

CLASH

Exercise 8. Subsumption (*)

Check whether the following subsumption relations hold w.r.t. the TBox $\mathcal{T}_1$ from Exercise 6.

1. $\exists r.(A \sqcap B) \sqsubseteq \forall r.B$
2. $B \sqcup C \sqsubseteq \forall r.A$
3. $\exists s.\neg A \sqsubseteq \exists r.\neg A$

CHECK SUBSUMPTION

REMEMBER!! TO CHECK SUBSUMPTION WE HAVE TO CHECK THAT THERE IS NO OBJECT THAT BELONGS TO THE LEFT BUT NOT TO THE RIGTH

WE HAVE TO CHECK THAT THE NEGATION IS UNSATISFIABLE

$\exists r.(A \sqcap B) \sqsubseteq \forall r.B$ BECOMES $\neg\exists r.(A \sqcap B) \sqcup \forall r.B$ AND NEGATED

IS $\exists r.(A \sqcap B) \sqcap \exists r.\neg B$ TBox: $\top \sqsubseteq \neg A \sqcup \exists r.A$ $\top \sqsubseteq \neg A \sqcup \forall r.\neg A$

$\exists r(A \sqcap B)(a)$ $\exists r.\neg B(a)$ $(\neg A \sqcup \exists r.A)(a)$ $(\neg A \sqcup \forall r.\neg A)(a)$

$r(a,b)$ $A(b)$ $B(b)$ $r(a,b)$ $\neg B(b)$

CLASH → THE CLASH WAS FINDED BEFORE ANY CHOICE RULE SO THERE IS NO OTHER POSSIBILITY

SINCE IT IS UNSAT THEN THE SUBSUMPTION HOLDS

Exercise 9. Knowledge Base Consistency (*)

Check whether the following ABoxes are consistent w.r.t. the TBox $\mathcal{T}_1$ from Exercise 6.

1. $\{r(a,b), \forall r.A(a)\}$
2. $\{\exists r.(B \sqcup A)(a), s(b,a), \forall s.\forall r.\neg B(b)\}$
3. $\{r(a,b), r(b,c), r(c,a)\}$

1. $r(a,b)$ $\forall r.A(a)$ MEANS $A(b)$

WE ALREADY KNOW THAT $A(b)$ WILL LEAD TO A CRASH IN TBOX1

SO THE ABOX IS NON CONSISTEN WITH THE TBOX

Exercise 10. Number Restrictions (∗∗∗)

Let $\mathcal{ALCQ}$ be the logic that extends $\mathcal{ALC}$ with *qualified number restrictions* $\geq n\ r.C$ expressing the class of objects that have at least n r-successors belonging to the class C. For example,

$$Person \sqcap \geq 2 hasChild.Female$$

is the class of people having at least two daughters.

Device adequate tableau rules to handle number restrictions.