PRODUCT OF CONVERGENT SEQUENCES

GIVEN THAT $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ AND $\{b_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ PROVE THAT

IF $a_n \to a$ AND $b_n \to b$ THEN $\{a_n \cdot b_n\} \to a \cdot b$

LET $\varepsilon > 0$ WE WANT TO PROVE $|a_n b_n - ab| < \varepsilon$

SINCE THE CONVERGING SEQUENCES $a_n \to a$ AND $b_n \to b$

$|a_n - a|$ IS ARBITRARY SMALL

$|b_n - b|$ IS ARBITRALY SMALL

REWRITE $|a_n b_n - ab|$ AS

$$|(a_n b_n - a b_n) + (a b_n - ab)|$$

USING THE TRIANGLE INEQUALITY WE KNOW THAT $|x+y| \leq |x| + |y|$

THAT IS

$$|(a_n b_n - a b_n) + (a b_n - ab)| \leq |a_n b_n - a b_n| + |a b_n - ab|$$

$$= |b_n||a_n - a| + |a||b_n - b|$$

WE WANT TO PROVE THAT $|b_n||a_n - a| + |a||b_n - b| < \varepsilon$

TO INSURE SO WE WANT TO BE SURE THAT

$|b_n||a_n - a| < \frac{\varepsilon}{2}$ AND $|a||b_n - b| < \frac{\varepsilon}{2}$

THAT IS

$b_N - b < \dfrac{\varepsilon}{2|a|+1}$

ADD 1 TO AVOID THE POSSIBILITY OF $|a|=0$

$\hookrightarrow$ a IS A CONSTANT

$|b_N||a_N - a| < \dfrac{\varepsilon}{2}$

SINCE $\{b_N\}$ IS BOUNDED

$\exists C \in \mathbb{R}$, $C \geq |b_N|$ $\forall n > N$

$|a_N - a| < \dfrac{\varepsilon}{2C+1}$

TO ENSURE WE ARE NOT DIVIDING BY 0 ADD +1

BY DEFINITION OF CONVERGENT SEQUENCE

$\exists N_1 > N$, FOR ALL $n > N_1$

$|a_N - a| < \dfrac{\varepsilon}{2C+1}$

$\exists N_2 > N$, FOR ALL $n > N_2$

$|b_N - b| < \dfrac{\varepsilon}{2|a|+1}$

$N = \max\{N_1, N_2\}$

$|a_N b_N - ab| \leq |b_N||a_N - a| + |a||b_N - b|$

SINCE $n > N$

$|a_N - a| < \dfrac{\varepsilon}{2C+1}$ AND $|b_N - b| < \dfrac{\varepsilon}{2|a|+1}$

$|a_N b_N - ab| \leq |b_N| \dfrac{\varepsilon}{2C+1} + |a| \dfrac{\varepsilon}{2|a|+1}$ AND SINCE $C \geq |b_N|$

$$|a_N b_N - ab| \leq \frac{c\,\varepsilon}{2c+1} + |a|\frac{\varepsilon}{2|a|+1} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

FOR AN ARBITRARY $\varepsilon > 0$ WE CAN FIND A NUMBER N SO THAT FOR EVERY TERM OF OUR SEQUENCE AFTER N IS WITHIN ε OF ab

THUS IF a_N CONVERGES TO a AND b_N CONVERGES TO b THAN $a_N b_N$ CONVERGES TO ab

THE PRODUCT OF CONVERGING SEQUENCES CONVERGES TO THE PRODUCT OF THEIR LIMITS