$+$ $\times$

$-$ $\div$

p_x NOTATION FOR PROB. MASS FUNCTION

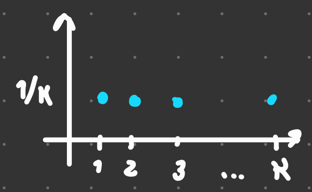
$\mathbb{P}$ CAPITAL P NOTATION FOR PROBABILITY

UNIFORM RANDOM VARIABLE

DISCRETE

$X : \Omega \to \{1, 2, 3, \dots K\}$ $\quad K \in \mathbb{N}$ $\boxed{\text{FIXED}}$

$p_x(i) = \mathbb{P}[X = i] = \frac{1}{K}$



CONSTANT

(DOMAIN DESCRETE)

FAIR DICE → OUTCOME EQUALLY PROB.

$K = 6 \qquad p_x : \{1, 2, \dots 6\} \to [0, 1]$

$p_x(i) = \frac{1}{6}$ FOR $i = 1, 2, 3, 4, 5, 6$

- EXPECTATION OF RANDOM VARIABLE

$E[X] = \sum_{i=1}^{K} i \, p_x(i) = \underbrace{\frac{1}{K}}_{\text{IS CONSTANT}} \sum_{i=1}^{K} i$

$1 + 2 + 3 + \dots + K = \frac{K(K+1)}{2}$

$E[X] = \frac{1}{\cancel{K}} \cdot \frac{\cancel{K}(K+1)}{2} = \frac{K+1}{2}$

FOR THE DICE

$E[X] = \frac{K+1}{2} = \frac{7}{2} = 3.5$

- VARIANCE VAR(X)

$VAR(X) = E[X^2] - (E[X])^2$

$(E[X])^2 = \left(\frac{K+1}{2}\right)^2$

???

$E[X^2] = \sum_{i=1}^{K} i^2 p_x(i) = \frac{1}{K} \sum_{i=1}^{K} i^2 = \frac{K(K+1)(2K+1)}{6}$

$VAR(X) = \frac{K(K+1)(2K+1)}{6} - \frac{(K+1)^2}{4} = \frac{K^2 - 1}{12}$

EXERCISE:

2 (FAIR) DICE

- 1ST DICE → 4€
- THROW 2ND AND TAKE ?€

$p_x(i) = 1/K = 1/6$

BETTER TAKE 4€ BECAUSE GREATER THEN THE EXPECTATION $4 > E[X]$ ($= 3.5$)

$E[X] = \frac{1}{K} \sum_{i=1}^{K} i = \frac{1}{K} \frac{K(K+1)}{2} = \frac{K+1}{2} = \frac{6+1}{2} = 3.5$

DISCRETE RANDOM VARIABLE: THE BINOMIAL VARIABLE

EXAMPLE: EXAM 4 QUESTIONS. PROB OF EXACTLY 2 CORRECT

QUESTION i
- A1
- A2
- A3
- A4

$$X_i = \begin{cases} 1 & \text{IF ANSWER TO } i\text{-TH QUESTION IS CORRECT} \\ 0 & \text{IF ANSWER TO } i\text{-TH QUESTION IS WRONG} \end{cases}$$

⚠ COMMAS ARE USED FOR $\cap$ INTERSECTION

$$P[X_1 = x_1, X_2 = x_2, X_3 = x_3, X_4 = x_4]$$
$$\Downarrow$$
$$P[\{X_1 = x_1\} \cap \{X_2 = x_2\} \cap \dots]$$

$$P[X_1 = 1] = P[X_2 = 1] = P[X_3 = 1] = P[X_4 = 1] = 1/4$$
$$P[X_1 = 0] = P[X_2 = 0] = P[X_3 = 0] = P[X_4 = 0] = 3/4$$

$$P[S_4 = 2] \Rightarrow P[X_1 + X_2 + X_3 + X_4 = 2]$$

1 CORRECT
0 FAIL

REARRANGE COMBINATIONS

(1, 1, 0, 0) (1, 0, 1, 0)
(1, 0, 0, 1) (0, 1, 1, 0)
(0, 1, 0, 1) (0, 0, 1, 1)

$$E = \{X_1 = 0, X_2 = 0, X_3 = 1, X_4 = 1\} \cup \{X_1 = 1, X_2 = 1, X_3 = 0, X_4 = 0\} \cup \dots$$

$$P(E) = 6 \cdot \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^2 = \frac{54}{256}$$

6 POSSIBLE 0-1 COMBINATION

$$\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{256} \Rightarrow \text{MULTIPLY BY } 6 = \frac{54}{256}$$

$$\binom{N}{K} \left(\frac{1}{4}\right)^K \left(\frac{3}{4}\right)^{N-K}$$

THEOREM

IF $X_1, X_2, \dots X_N$ BERNULLI RANDOM VARIABLE S.T.

- IDENTICAL DISTRIBUTION
- INDIPENDENT

THEN $S_N = X_1 + X_2 + \dots + X_N$ HAVE PMF (PROBABILITY MASS FUNCTION)

$$P_{S_N}(K) = \binom{N}{K} p^K (1-p)^{N-K} \qquad K \in \{0, 1, \dots N\}$$

$N = 20$ QUESTION
$P = 1/4$ (CORRECT ANSWER)
$E[S_N] = 20 \cdot \frac{1}{4} = 5$

- EXPECTATION

$$E[S_N] = \sum_{K=0}^{N} K P_{S_N}(K) = \boxed{NP}$$

- VARIANCE

$$VAR(S_N) = E[S_N^2] - (E[S_N])^2 = E[S_N(S_N - 1)] + NP$$

DISCRETE RANDOM VARIABLE: THE HYPERGEOMETRIC VARIABLE

URN N BALLS INSIDE
- $1 \to N$ NUMBERED
- BLACK OR WHITE

$\downarrow$

1 TO W WHITE
$W+1$ TO N BLACK

$N = 10$
4 WHITE
6 BLACK

PICK 3 W/OUT REPLACEMENT
$\downarrow$
0,1,2,3 WHITES

$$\frac{\binom{4}{2}\binom{6}{1}}{\binom{10}{3}}$$

REPETITION SAME EXPERIMENT
$\Downarrow$
BINOMIAL

NON-REPLACEMENT, CHANGE URN
$\Downarrow$
HYPERGEOMETRIC

LOTTERY TICKET

5 NUMBERS TO WIN $\boxed{1, 2, 3 \to 90}$

CHOOSE
4, 12, 34, 30, 21, 71, 88

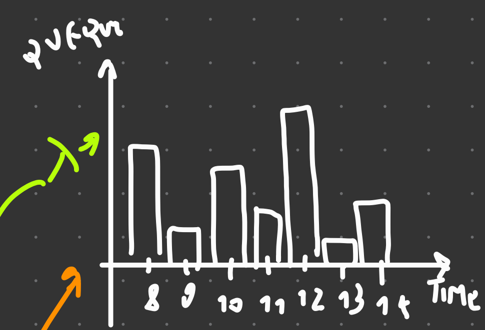
EVENT: ALL 5 NUMBERS OF THE RESULT ARE ON MY LIST

$$\frac{\binom{7}{5}\binom{7}{2}}{\binom{90}{7}}$$

DISCRETE RANDOM VARIABLES: THE POISSON

$$X : \Omega \to N_0 = \{0, 1, 2, 3, \dots\}$$

$$p_X(i) = P[X=i] = e^{-\lambda}\frac{\lambda^i}{i!} \quad \text{FOR } i = \{0, 1, 2, \dots\}$$



AVERAGE PERSON IN QUEUE IN POST OFFICE

$\lambda = E[X]$

IS THE EXPECTED VALUE

$$e^x = \sum_{n=0}^{+\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

INFINITE POLYNOMIAL

IS TAYLOR EXPANSION