Written exam of Calculus - Part 1 - Sample 1

IT IS FORBIDDEN TO USE CALCULATORS, BOOKS, NOTES, AND SO ON

TIME ALLOWED: 90 MINUTES

NAME:.................................... **ID NUMBER:**....................................

SURNAME:..

PART A Write **only** the answer.

1.1 (3 POINTS)

How many are the natural numbers with 3 digits such that the first digit is 2 and they are divisible by 5?

ANSWER: 20

_ _ _
2   5
    0

1.2 (3 POINTS)

State the definition of limit of a converging sequence.

DEFINITION: GIVEN A SEQUENCE a_n IT IS CONVERGENT IFF $\exists \lim_{n\to+\infty} a_n = l \in (-\infty, +\infty)$

1.3 (3 POINTS)

Compute, if it exists, $\lim_{x\to 0} \dfrac{\sin(x^2)}{(e^{2x}-1)\sqrt{1-\cos(x)}}$.

ANSWER:

$$\lim_{x\to 0} \frac{1-\cos(x)}{x} = 0$$

$$\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$$

$$\lim_{x\to 0} \frac{e^x-1}{x} = 1$$

$$\lim_{x\to 0} \frac{\sin(x^2)}{(e^{2x}-1)\sqrt{1-\cos(x)}} \cdot \frac{2x}{2x} \quad \to 1$$

$$\lim_{x\to 0} \frac{x}{e^x-1} = 1$$

$$\lim_{x\to 0} \frac{\sin(x^2)}{2x\sqrt{1-\cos(x)}} \cdot \frac{x^2}{x^2} \quad \to 1$$

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

$$\lim_{x\to 0} \frac{x}{2\sqrt{1-\cos(x)}} \quad \left[\frac{0}{0}\right] \text{APPLY HOPITAL}$$

$$\lim_{x\to 0} \frac{1}{2\dfrac{}{2\sqrt{1-\cos(x)}}\cdot(+\sin(x))} = \lim_{x\to 0} \frac{1}{\dfrac{\sin(x)}{\sqrt{1-\cos(x)}}} = \lim_{x\to 0} \frac{\sqrt{1-\cos(x)}}{\sin(x)} = \lim_{x\to 0} \frac{1-\cos(x)}{}$$

PART B Write a **complete** solution.

1.4 (8 POINTS)

Write the Taylor polynomial of order 3 of the function

$$f(x) = \log(e+1)(x-1)$$

at the point $x_0 = 1$.

Anther possible example: write the Taylor formula of order 2 with Lagrange remainder for the function $f(x) = \sin(\pi x) - x^2$ at the point $x_0 = 1$.

SOLUTION:

$$f(x) = \cos(e+1)(x-1)$$

$$f'(x) = \cos(e+1)$$

$$f''(x) = 0$$
$$f'''(x) = 0$$

$$f(x) + f'(x)(x-1) + \frac{f''(x)(x-1)^2}{2} + \frac{f'''(x)(x-1)^3}{3!}$$

$$\cos(e+1)(x-1) + \cos(e+1)(x-1) + 0 + 0$$

1.5 (8 POINTS)

Study the following function

$$f(x) = \frac{x-2}{x^2-1}$$

and draw its graph.

Another possible example: $f(x) = \cancel{\frac{x-2}{x^2-1}}$.

SOLUTION:

$D:\ x \neq \pm 1$

SIGN

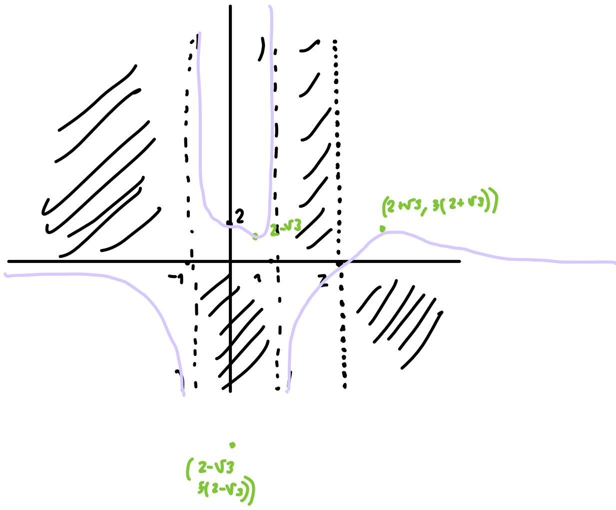
$f(0) = 2$

$x - 2 > 0;\ x > 2$

$x^2 - 1 > 0$

$x < -1 \lor x > 1$

	-1		1		2	
$-$		$-$		$-$		$+$
$+$		$-$		$+$		$+$
$-$		$+$		$-$		$+$



$$\lim_{x \to +\infty} f(x) = 0^+$$

$$\lim_{x \to -\infty} f(x) = 0^-$$

$$f(x) = \frac{x-2}{x^2-1}$$

$$\lim_{x \to -1^+} \frac{1-2/x}{x-1/x} = \frac{-}{-1^+ + 1^-} = +\infty$$

$$\lim_{x \to -1^-} \frac{1-2/x}{x-1/x} = -\infty$$

$$\lim_{x \to 1^+} \frac{1-2/x}{x-1/x} = \frac{-1^-}{1^+-1^-} = -\infty$$

$$\lim_{x \to 1^-} \frac{1-2/x}{x-1/x} = \frac{-1^+}{1^- - 1^+} = +\infty$$

ASYNDOTES:

HORIZONTAL: $y = 0$

VERTICAL: $x = -1 \land x = 1$

DERIVATES

$f(x) = \dfrac{x-2}{x^2-1}$ $\quad f'(x) = \dfrac{x^2-1-(x-2)(2x)}{(x^2-1)^2}$

$f'(x) = \dfrac{x^2-1-2x^2+4x}{(x^2-1)^2} = \dfrac{-x^2+4x-1}{(x^2-1)^2}$

SIGN OF $f'(x)$:

$D:\ (x^2-1)^2 > 0\ \forall x \neq \pm 1$

$N:\ -x^2+4x-1 > 0$

$16 - 4(-1)(-1) = 12$

$\dfrac{-4 \pm 2\sqrt{3}}{-2} = 2 \mp \sqrt{3}$ → $2-\sqrt{3}$, $2+\sqrt{3}$

$2-\sqrt{3}$ $\quad 2+\sqrt{3}$

MIN MAX

$(2-\sqrt{3}, f(2-\sqrt{3}))$ MIN

$(2+\sqrt{3}, f(2+\sqrt{3}))$ MAX

1.6 (7 POINTS)

State and prove the Lagrange Theorem.

STATEMENT:

PROOF: