$+$ $\times$

$-$ $\div$

DISCRETE RANDOM VARIABLE

1) BERNOULLI (0/1 EXPERIMENTS)
2) BINOMIAL (K SUCCESSES IN N EXPERIMENTS)
3) HYPERGEOMETRIC (DRAWING BALLS W/OUT REPLACEMENT)
4) POISSON (COUNTING PROBLEMS LIKE QUEUES)
5) GEOMETRIC (WAITING TIMES)

DISCRETE R.V.'S: THE GEOMETRIC

EXAMPLE: 1 2 3 ...

STUDENT PASS EXAM BY ANSWERING AT RANDOM. EVALUATE THE PROBABILITY OF PASSING THE EXAM AS EQUAL TO $p \in (0,1)$

p IS FIXED NUMBER

$X_1, X_2, X_3, \dots$ BERNULLI R.V.'S

$$X_i = \begin{cases} 1 & \text{PASS EXAM } i\text{-TH SESSION} \\ 0 & \text{FAIL EXAM } i\text{-TH SESSION} \end{cases}$$

$T = \min\{n \in \mathbb{N} \mid X_n = 1\}$ → THE FIRST INDEX n IN WHICH THE STUDENT PASS THE EXAM

$\{T=1\}$ ← THE STUDENT PASS THE EXAM AT FIRST SESSION

$\{T=2\}$ ← FAIL FIRST EXAM BUT PASS THE SECOND

⚠ MAIN PROBLEM: $P[T=n] = ?$ $n \in \mathbb{N}$ (AT A GENERIC n)

1) REWRITE $\{T=n\}$ AS BERNULLI VAR.

$\{T=n\}$

- IF $n=1$, $\{T=1\} = \{X_1 = 1\}$
- IF $n \geq 2$, $\{T=n\} = \{X_1 = 0, X_2 = 0, \dots X_{n-1} = 0, X_n = 1\}$

COMMAS, MEANS INTERSECTION $\cap$

- ASSUMPTION
  - BERNULLI R.V.'S TRIALS $X_1, X_2, \dots$ ARE IDENTICALLY DISTRIBUTED → $P[X_1=1] = P[X_2=1] = \dots = p \in (0,1)$
    ↘ $P[X_1=0] = P[X_2=0] = \dots = 1-p \in (0,1)$
  - BERNOULLI RANDOM VARIABLE ($X_1, X_2, \dots$) ARE INDEPENDENT

$$P[X_1 = x_1, X_2 = x_2, \dots X_n = x_n] = P[X_1 = x_1] \cdot P[X_2 = x_2] \cdot \dots \cdot P[X_n = x_n] = \prod_{i=1}^{n} P[X_i = x_i] \quad \forall n \in \mathbb{N} \quad \forall x_1, x_2, \dots x_n \in \{0,1\}$$

(BIG PIE)

- SOLUTION
  - IF $n=1$, $P[T=1] = P[X_1 = 1] = p$
  - IF $n \geq 2$ $P[T=n] = P[X_1 = 0, \dots, X_{n-1} = 0, X_n = 1] \overset{\text{INDEPENDENT}}{=} P[X_1 = 0] \cdot \dots \cdot P[X_{n-1} = 0] \cdot P[X_n = 1]$

    BY IDENTICAL DISTRIBUTION $(1-p) \cdot \dots \cdot (1-p) \cdot p = (1-p)^{n-1} p$

- CONCLUSION

$$P[T=n] = (1-p)^{n-1} \cdot p = p_T(n) = \text{PROBABILITY MASS FUNCTION}$$

- EXPECTATION

$$E[T] = \underbrace{\sum_{n=1}^{+\infty} n \, p_T(n)}_{\text{DEF OF EXPECTATION OF DESCRETE RANDOM VAR}} = \sum_{n=1}^{+\infty} n (1-p)^{n-1} p = \frac{1}{p}$$

TO PROVE THIS FACT REMEMBER THE GEOMETRIC SERIES

$$\sum_{n=0}^{+\infty} z^n = 1 + z + z^2 + z^3 + \dots = \frac{1}{1-z} \qquad \forall z \in (-1,1)$$

→ IF NOT RESTRICTED IS PARADOX (SUM POSITIVE NUMBER THAT GIVES -1)

FIRST DERIVATIVE

$$\frac{d}{dz}\left(\sum_{n=0}^{+\infty} z^n\right) = \frac{d}{dz}\left(1+z+z^2+\dots\right) = \left(\cancel{0}+1+2z+3z^2+\dots\right) = \sum_{n=1}^{+\infty} n z^{n-1}$$

⋮

CONCLUSION

$$\sum_{n=1}^{+\infty} n z^{n-1} = \left(\frac{1}{1-z}\right)^2$$

$= \frac{1}{p}$

p SMALL $\Rightarrow \frac{1}{p}$ LARGE

THE LESS PROBABILITY p THE LONGER THE STUDENT HAVE TO WAIT

CONTINOUS RANDOM VARIABLE

MEASURE PHYSICAL QUANTITIES THAT ARE CONTINOUS BY NATURE

$X: \Omega \to \mathbb{R}$

$P: [X \in (a,b)] = ?$

SYMBOL P_X

IN THIS CASE THE PROB MASS FUNCTION IS REPLACED BY THE <u>PROBABILITY DENSITY FUNCTION</u>

↳ f_X (SOMETIMES ONLY f)

• DEFINITION

WE SAY THAT A FUNCTION $f: \mathbb{R} \to \mathbb{R}$ IS A PROB. DENSITY IF

1) $f(x) \geq 0 \quad \forall x \in \mathbb{R}$

2) f MUST BE INTEGRABLE (FOR EXAMPLE THIS IS TRUE IF f IS PIECEWISE CONTINOUS)

3) $\int_{-\infty}^{+\infty} f(x)\,dx = 1$ [THE TOTAL AREA BELOW $f(x)$ AND THE X-AXIS MUST BE EQUAL TO 1]

• EXAMPLES

1) UNIFORM (SIMPLEST)

2) BETA (GENERALIZE UNIFORM)

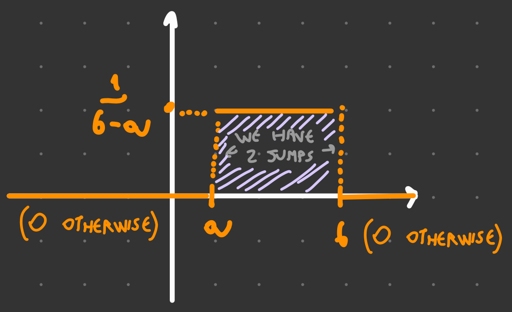
3) EXPONENTIAL

4) GAMMA (GENERALIZE EXPONENTIAL)

5) GAUSSIAN

UNIFORM R.V.

FIX AN INTERVAL $[a,b]$ THEN $f(x) = \begin{cases} \frac{1}{b-a} & \text{IF } x \in [a,b] \\ 0 & \text{OTHERWISE} \end{cases}$



1) $f(x) \geq 0$ YES BY GRAFICAL INSPECTION

2) f IS INTEGRABLE

SUFFICIENT TO CALCULATE THE AREA OF RECTANGLE

3) $\int_{-\infty}^{+\infty} f(x)dx = 1 \Rightarrow \int_a^b f(x)dx = \cancel{(b-a)} \cdot \frac{1}{\cancel{(b-a)}} = 1$

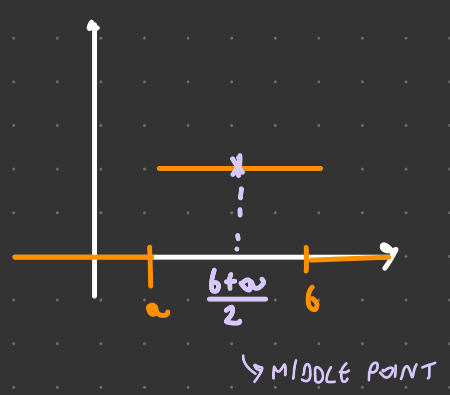
> **GENERAL RULE ABOUT EXPECTATION**
>
> CONTINOUS RANDOM VARIABLE
>
> $$E[X] = \int_{-\infty}^{+\infty} x f(x) dx$$

IF $f(x) > 0$ IN $[a,b]$ AND $f(x) = 0$ IF $x \notin [a,b]$ THEN

$$\int_{-\infty}^{+\infty} x f(x) dx = \int_a^b x f(x) dx$$

SO FOR THE UNIFORM

$$E[X] = \int_{-\infty}^{+\infty} x f(x)\, dx = \int_a^b x \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b x\, dx = \frac{1}{b-a} \left[\frac{x^2}{2}\right]_a^b = \frac{1}{b-a}\left(\frac{b^2}{2} - \frac{a^2}{2}\right) = \frac{b+a}{2}$$



MOMENTS

N-TH MOMENT OF X IS DEFINED AS $E[X^N]$ 2ND MOMENT $E[X^2]$

$E[X] \overset{DEF}{=} \int_{-\infty}^{+\infty} x f(x) dx$ $\quad$ $VAR(X) = E[X^2] - (E[X])^2$ $\quad$ $E[X^2] = \int_{-\infty}^{+\infty} x^2 f(x) dx$

$$E[X^2] = \frac{a^2 + b^2 + ab}{3}$$

$$VAR(X) = E[X^2] - (E[X])^2 = \frac{(b-a)^2}{12}$$