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THEOREM $\{a_n\}_{n\in\mathbb{N}}$ SEQUENCE OF REAL NUMBERS

$\{b_n\}_{n\in\mathbb{N}}$ SEQUENCE OF REAL NUMBER

IF $\{a_n\}_{n\in\mathbb{N}}$ IS BOUNDED (OR CONVERGING) AND $\{b_n\}_{n\in\mathbb{N}}$ IS INFINITESIMAL, THAT IS, $\lim_n b_n = 0$, THEN $\{a_n b_n\}_{n\in\mathbb{N}}$ IS INFINITESIMAL, THAT IS, $\lim_n a_n b_n = 0$

PROOF

HRP

$\exists K > 0$ s.t. $|a_n| \le K \quad \forall n \in \mathbb{N}$

$0 \le |a_n b_n| \le |a_n| |b_n| \le K |b_n| \to 0$

$b_n \to 0 \Rightarrow |b_n| \to 0$

$\Rightarrow |a_n b_n| \to 0 \quad a_n b_n \to 0 \qquad \square$

EXAMPLES

$$\lim_n \frac{\sin(n)}{n} = 0$$

$$\lim \frac{2\cos(3n+5)}{n^2-1}$$

$n^2 \to +\infty \quad n^2-1 \to +\infty \quad \frac{1}{n^2-1} \to 0^+$

$$\lim_{n} \frac{n^2 + \sin(n)}{n^2+1} = 1$$

$$\frac{n^2+\sin(n)}{n^2+1} = \frac{n^2}{n^2+1} + \frac{\sin(n)}{n^2+1} \to 1+0 = 1$$

$$-1 \leq \sin(n) \leq 1$$

$$\frac{n^2-1}{n^2+1} \leq \frac{n^2+\sin(n)}{n^2+1} \leq \frac{n^2+1}{n^2+1} = 1$$

$$\downarrow \qquad\qquad \downarrow$$

$$1 \qquad\qquad 1$$

SANWICH THEOREM

THEOREM

$a_n \to \pm\infty$, $\{b_n\}_{n\in\mathbb{N}}$ BOUNDED, $b_n \neq 0$ DEFINITELY

THEN $\left|\frac{a_n}{b_n}\right| \to +\infty$

PROOF $|a_n| \to +\infty$

WE KNOW $\exists_{N_1} \in \mathbb{N}$ SUCH THAT $\forall n \geq N_1$

$0 \leq |b_n| \leq K$, SO

$\frac{1}{|b_n|} \geq \frac{1}{K} \quad \forall n \geq N_1$

So $\forall n \geq N_1 \quad \left|\frac{a_n}{b_n}\right| \geq \frac{1}{K}|a_n| \to +\infty$

THEN YOU CONCLUDE BY COMPARISON $\square$

EXAMPLE:

$\frac{a_n}{b_n}$ CAN BE IRREGULAR!

$a_n = n \to +\infty$; $b_n = (-1)^n$ $\quad \frac{a_n}{b_n} = (-1)^n n$ $\quad \underline{IRR}$

INDETERMINATE FORMS

$+\infty - \infty \qquad 0 \cdot \infty \qquad \frac{\infty}{\infty} \qquad \frac{0}{0}$

EXAMPLE

$+\infty - \infty$

- $a_n = n^2 \to +\infty \qquad b_n = -n \to -\infty$

$a_n + b_n = n^2 - n \to +\infty$

- $a_n = \sqrt{n^2+1} \to +\infty \qquad b_n = -n \to -\infty$

$a_n + b_n = \sqrt{n^2+1} - n = \sqrt{n^2+1} - \sqrt{n^2}$

$$= \frac{\left(\sqrt{n^2+1} - \sqrt{n^2}\right)\left(\sqrt{n^2+1} + \sqrt{n^2}\right)}{\left(\sqrt{n^2+1} + \sqrt{n^2}\right)}$$

$\boxed{n > 0}$

$(x^2 - y^2) = (x-y)(x+y)$

$$= \frac{(\sqrt{n^2+1})^2 - (\sqrt{n^2})^2}{\sqrt{n^2+1}+\sqrt{n^2}} = \frac{1}{\sqrt{n^2+1}+n} \to 0^+$$

- $a_n = \sqrt{n^2+1} \to +\infty \quad b_n = -n + (-1)^n \to -\infty$

$$a_n + b_n = \frac{1}{\sqrt{n^2+1}+n} + (-1)^n$$ IRREGULAR

WHY

$0 \cdot \infty$

- $a_n = \frac{1}{n} \to 0 \quad b_n = n^2 \to +\infty$

$$a_n b_n = n \to +\infty$$

- $a_n = \frac{1}{n^2} \to 0 \quad b_n = n \to +\infty$

$$a_n b_n = \frac{1}{n} \to 0$$

- $a_n = \frac{2}{n} \to 0 \quad b_n = 3(n+1) \to +\infty$

$$a_n b_n = \frac{2}{n} \cdot 3(n+1) \to 6$$

- $a_n = \frac{(-1)^n}{n} \to 0; \quad b_n = 3(n+1) \to +\infty$

DA QUE ARRIVA

$$a_n b_n = (-1)^n 6 \frac{n+1}{n}$$ IRREGULAR

$$6(-1)^n \frac{n+1}{n} = 6\left[(-1)^n + \frac{(-1)^n}{n}\right]$$

$\frac{\infty}{\infty}$ AND $\frac{0}{0}$

- $a_n = \frac{1}{n} \to 0$ $\quad b_n = \frac{1}{n^2} \to 0$

$$\frac{a_n}{b_n} = \frac{1/n}{1/n^2} \quad 0 \Rightarrow \frac{a_n}{b_n} = n \to +\infty$$

- $a_n = n/2 \to +\infty$ $\quad b_n = 3(n+1) \to +\infty$

$$\frac{a_n}{b_n} \stackrel{\infty}{=} \frac{n}{6(n+1)} \to \frac{1}{6}$$

- $a_n = \frac{(-1)^n}{n} \to 0$ $\quad b_n = \frac{1}{3(n+1)} \to 0$

$$\frac{a_n}{b_n} = (-1)^n \cdot \frac{3(n+1)}{n}$$ IRREGULAR $\frac{0}{0}$

- $a_n = (3 + \sin(n))\, n \to +\infty$ $\quad b_n = n \to +\infty$

$$\frac{a_n}{b_n} = 3 + \sin(n) \to +\infty$$ IRREGULAR

$$1 \le \sin(n) \le 1$$

$$2 \le 3 + \sin(n) \le 4$$

SUBSEQUENCE

$\{a_n\}_{n \in \mathbb{N}}$ SEQUENCE OF REAL NUMBERS

$f: \mathbb{N} \to \mathbb{R}$

$\mathbb{N} \ni n \to f(n) = a_n \in \mathbb{R}$

LET $\gamma: \mathbb{N} \to \mathbb{N}$

$\mathbb{N} \ni k \to \gamma(k) = n_k \in \mathbb{N}$

BE STRICTLY INCREASING, THAT IS, $\forall k_1, k_2 \in \mathbb{N}$

$k_1 < k_2 \Rightarrow \gamma(k_1) = n_{k_1} < n_{k_2} = \gamma(k_2)$

THE SEQUENCE $h = f \circ \gamma \quad \mathbb{N} \to \mathbb{R}$

$\mathbb{N} \ni k \to a_{n_k} \in \mathbb{R}$

IS A SUBSEQUENCE OF THE SEQUENCE $\{a_n\}_{n \in \mathbb{N}}$ AND IS DENOTED

$\{a_{n_k}\}_{k \in \mathbb{N}}$

EXERCISE

$g: \mathbb{N} \to \mathbb{N}$ STRICTLY INCREAS PROVE THAT

$g(k) = n_k \geq k \quad \forall k \in \mathbb{N}$ IN PARTICULAR

$\lim_{k \to \infty} g(k) = +\infty$

EXERCISE

A SUBSEQUENCE OF $\{a_{n_k}\}_{n \in \mathbb{N}}$ IS ALSO A SUBSEQUENCE OF $\{a_n\}_{n \in \mathbb{N}}$

- $f: \mathbb{N} \to \mathbb{R}$
- $h = f \circ g: \mathbb{N} \to \mathbb{R} \quad g: \mathbb{N} \to \mathbb{N}$ STRICTLY INCREASING
- $h \circ g_1: \mathbb{N} \to \mathbb{R} \quad g_1: \mathbb{N} \to \mathbb{N}$ STRICTLY INCREASING

$h \circ g_1 = f \circ (g \circ g_1)$

EXAMPLE: $g(k) = n_k = k$

$\{a_{n_k}\}_{n \in \mathbb{N}} = \{a_k\}_{n \in \mathbb{N}} = \{a_n\}_{n \in \mathbb{N}}$ IS A SUBSEQUENCE

$g(k) = n_k = k+1$

$\{a_{n_k}\}_{k\in\mathbb{N}} = \{a_{k+1}\}_{k\in\mathbb{N}} = \{a_n\}_{n\geq 2}$ IS A SUBSEQUENCE

$\forall n_0 \in \mathbb{N},\ n_0 > 1 \quad \{a_n\}_{n\geq n_0}$ IS A SUBSEQUENCE

$a_4, a_5, a_6 \dots a_n, \dots \qquad n_0 = 4$

$n_k = k + n_0 - 1$

$g(k) = 2k \quad \{a_{n_k}\}_{k\in\mathbb{N}} = \{a_{2k}\}_{k\in\mathbb{N}} = \{a_{2n}\}_{n\in\mathbb{N}}$

SUBSEQUENCE OF EVEN INDEXES

$g(k) = 2k-1 \quad (a_{n_k})_{k\in\mathbb{N}} = \{a_{2k-1}\}_{k\in\mathbb{N}} = \{a_{2n-1}\}_{n\in\mathbb{N}}$

SUBSEQUENCE OF ODD INDEXES

$a_1, a_3, a_5 \dots a_{2n-1}$

$g(k) = 3k \quad \{a_{n_k}\}_{k\in\mathbb{N}} = \{a_{3k}\}_{k\in\mathbb{N}} \quad \{a_{3n}\}_{n\in\mathbb{N}}$

$a_3, a_6, a_9 \dots a_{3n}, \dots$

$g(k) = k^2 \quad \{a_{n_k}\}_{k\in\mathbb{N}} = \{a_{k^2}\}_{k\in\mathbb{N}} = \{a_{n^2}\}_{n\in\mathbb{N}}$

$a_1, a_4, a_9, a_{16} \dots a_{n^2} \dots$

PROPOSITION

LET $\{n_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ SUCH THAT $\lim_k n_k = +\infty$ AS $k \to +\infty$

LET $\{w_n\}_{n\in\mathbb{N}}$ BE A SEQUENCE. IF $\exists \lim_n w_n = \lambda \in [-\infty, +\infty]$

THEN $\exists \lim_k w_{n_k} = \lambda$

IN PARTICULAR

ANY SUBSEQUENCE OF A REGULAR SEQUENCE IS REGULAR AND GOES TO THE SAME LIMIT!

PROOF $l \in \mathbb{R}$

HYP $\forall N > 0 \; \exists k_0 \in \mathbb{N}$ SUCH THAT $\forall k \geq k_0$ WE HAVE $n_k \geq N$

- $\forall \varepsilon > 0 \; \exists n_0 \in \mathbb{N}$ SUCH THAT $\forall n \geq n_0$ WE HAVE

$$l - \varepsilon < w_n < l + \varepsilon$$

THESIS $\forall \varepsilon > 0 \; \exists k_1 \in \mathbb{N}$ SUCH THAT $\forall k \geq k_1$ WE HAVE

$$l - \varepsilon < w_{n_k} < l + \varepsilon$$

FIX $\varepsilon > 0$ FIND $N_0 \in \mathbb{N}$ FOR $M = N_0$ $\exists K_0 \in \mathbb{N}$

SUCH THAT $\forall k \geq K_0$ WE HAVE $N_k > M = N_0$ SO

PROOF IS CONCLUDED BY TAKING $K_1 = K_0$ $\square$

CROLLARY

$\{a_n\}_{n \in \mathbb{N}}$ SEQ. OF REAL NUMBERS

- IF $\exists$ A SUBSEQUENCE WHICH IS IRREGULAR THEN $\{a_n\}_{n \in \mathbb{N}}$ IS IRREGULAR

- IF THERE $\exists$ TWO DIFFERENT SUBSEQUENCE WITH DIFFERENT LIMITS THAN $\{a_n\}_{n \in \mathbb{N}}$ IS IRREGULAR

EXAMPLE

$a_n = (-1)^n$ IRR $\quad \nexists \lim_n (-1)^n$

$$\lim_n a_{2n} = \lim_n (-1)^{2n} = \lim_n 1 = 1$$

$$\lim_n a_{2-1} = \lim_n (-1)^{2n-1} = \lim_n (-1) = -1$$

• $a_n = (-1)^n n$ IRR $\quad \nexists \lim_n (-1)^n n$

$$\lim a_{2n} = \lim_n (-1)^{2n}(2n) = \lim_n 2n = +\infty$$

$$\lim_n a_{2n-1} = \lim_n (-1)^{2n-1}(2n-1) = \lim_n (-(2n-1)) = -\infty$$

EXERCISE

$\{a_n\}_{n\in\mathbb{N}}$ IF $\exists\, l \in [-\infty, +\infty]$ SUCH THAT

$\lim_n a_{2n} = \lim_n a_{2n-1} = l$, THEN $\exists \lim_n a_n = l$

BOLZANO-WEIERSTRASS THEOREM

$\{a_n\}_{n\in\mathbb{N}}$ SEQUENCE OF REAL NUMBERS

IF $\{a_n\}_{n\in\mathbb{N}}$ IS **BOUNDED** IT HAS A **CONVERGING** SUBSEQUENCE, THAT IS $\exists \{a_{n_k}\}_{k\in\mathbb{N}}$ AND $a\in\mathbb{R}$ SUCH THAT $\lim_k a_{n_k} = a$

MONOTONE SEQUENCE

A SEQUENCE $\{a_n\}_{n\in\mathbb{N}}$ IS A FUNCTION $f : \mathbb{N} \to \mathbb{R}$

$n \mapsto f(n) = a_n$

$\{a_n\}_{n\in\mathbb{N}}$ IS INCREASING DECREASING AND MONOTONE

STRICTLY INCREASING, STRICTLY DECREASING,

STRICTLY MONOTONE IFF f IS

EXERCISE $\{a_n\}_{n\in\mathbb{N}}$ IS INCREASING $\Leftrightarrow$ $a_n \le a_{n+1}$ $\forall n \in \mathbb{N}$

EXAMPLE $a_n = \frac{1}{n}$ STRICTLY DECREASING $a_{n+1} < a_n$ $\forall n$

- $a_n = n^2$ STRICTLY INCREASING $a_{n+1} > a_n$ $\forall n$
- $a_n = 3$ BOTH INC AND DECREASING $a_{n+1} \ge a_n$ $\forall n$ AND $a_{n+1} \le a_n$ $\forall n$

THEOREM

LET $\{a_n\}_{n\in\mathbb{N}}$ BE A MONOTONE SEQUENCE

THEN $\{a_n\}_{n\in\mathbb{N}}$ IS REGULAR IN FACT WE HAVE

- IF $\{a_n\}_{n\in\mathbb{N}}$ IS INCREASING, THEN

$$\exists \lim_n a_n = \sup_{n\in\mathbb{N}} a_n = \sup\{a_n : n\in\mathbb{N}\}$$

IN PARTICULAR, IF $\{a_n\}_{n\in\mathbb{N}}$ IS INCREASING AND BOUNDED FROM ABOVE THEN IT IS CONVERGING

- IF $\{a_n\}_{n\in\mathbb{N}}$ IS DECREASING THEN

$$\exists \lim_n a_n = \inf_{n\in\mathbb{N}} a_n = \inf\{a_n : n\in\mathbb{N}\}$$

IN PARTICULAR IF $\{a_n\}_{n\in\mathbb{N}}$ IS DECREASING AND BOUNDED FROM BELOW THEN IT IS CONVERGING

NOTATION

$f: X \to \mathbb{R}$ X ARBITRARY SET, $X \neq \emptyset$

$$\sup_X f = \sup_{x \in X} f(x) = \sup \{ f(x) : x \in X \} = \sup f(X)$$

SAME NOTATION FOR INF AND IF THEY EXIST MIN AND MAX

PROOF ONLY FOR INCREASING $\{a_n\}_{n \in \mathbb{N}}$ INCREASING

- $\sup_{n \in \mathbb{N}} a_n = +\infty$

THAT IS $\forall N > 0 \ \exists n_0 \in \mathbb{N}$ SUCH THAT $a_{n_0} > N$

FOR ANY $n \geq n_0$ WE HAVE $a_n \geq a_{n_0}$ THERFORE $\forall n \geq n_0$ WE HAVE $N < a_{n_0} \leq a_n \Rightarrow \lim_{n} a_n = +\infty$

- $\sup_{n \in \mathbb{N}} a_n = l \in \mathbb{R}$

THAT IS, $a_n \leq l \ \forall n \in \mathbb{N}$ AND

$\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N}$ SUCH THAT $l - \varepsilon < a_{n_0}$

FOR ANY $n \geq n_0$ WE HAVE $a_n \geq a_{n_0}$ THERFORE

$\forall n \geq n_0$ WE HAVE $l - \varepsilon < a_{n_0} \leq a_n \leq l \leq l + \varepsilon \Rightarrow \lim_n a_n = l$