$A\underline{v} = \underline{w}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & & & & \\ a_{M1} & a_{M2} & a_{M3} & \cdots & a_{MN} \end{bmatrix}$$

$A \in \mathbb{R}^{M\times N}$

$\underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ $\quad \underline{b} \in \mathbb{R}^m$

SEARCH FOR $\underline{x} \in \mathbb{R}^N$ S.T. $A\underline{x} = \underline{b}$

THIS IS WHAT WE NEED TO FIND

$$\overset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & & & & \\ a_{M1} & a_{M2} & a_{M3} & \cdots & a_{MN} \end{bmatrix}} \overset{\underline{x}}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}} = \overset{\underline{b}}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}$$

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1N}x_N \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2N}x_N \\ \vdots \\ a_{M1}x_1 + a_{M1}x_1 + \dots + a_{MN}x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \Longrightarrow \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1N}x_N = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2N}x_N = b_2 \\ \vdots \\ a_{M1}x_1 + a_{M1}x_1 + \dots + a_{MN}x_N = b_m \end{cases}$$

$\downarrow$ AUGMENTED MATRIX

$[A \mid \underline{b}]$

$A \in \mathbb{R}^{M\times N}$ $\quad \underline{b} \in \mathbb{R}^m$

THEOREM THE PROBLEM SEARCH FOR $\underline{x} \in \mathbb{R}^N$ THAT SATISFY $A\underline{x} = \underline{b}$ IS EQUIVALENT (THE SOLUTION ARE THE SAME) TO THE LINEAR SYSTEM ASSOCIATED TO THE AUGMENTED MATRIX $[A|b]$

$A \in \mathbb{R}^{4\times 3}$

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 6 & 6 & 3 \\ -2 & 1 & -4 \\ 0 & 0 & 5 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 0 \\ -2 \\ 3 \\ 1 \end{bmatrix}$$

SEARCH FOR $\underline{x} \in \mathbb{R}^3$ S.T. $A\underline{x} = \underline{b}$

$\hookrightarrow \underline{x} = [x_1, x_2, x_3]$

$$\begin{bmatrix} 3x_1 + - x_2 + \cancel{0x_3} = 0 \\ 6x_1 + 6x_2 + 3x_3 = -2 \\ -2x_1 + x_2 - 4x_3 = 3 \\ \cancel{0x_1} + \cancel{0x_2} + 5x_3 = 1 \end{bmatrix}$$

AUGMENTED MATRIX:

$$\left[\begin{array}{ccc|c} 3 & -1 & 0 & 0 \\ 6 & 6 & 3 & -2 \\ -2 & 1 & -4 & 3 \\ 0 & 0 & 5 & 1 \end{array}\right]$$

SEARCH $\underline{x} \in \mathbb{R}^N$ $\boxed{A \in \mathbb{R}^{m \times n} \quad \underline{b} \in \mathbb{R}^m}$

$A\underline{x} = \underline{b}$

1 OR INFINITE

THIS LINEAR SYSTEM WILL HAVE (AT LEAST) A SOLUTION $\underline{x}$?

$\boxed{\text{IFF}}$ $\underline{b}$ IS A LINEAR COMBINATION OF THE COLUMNS OF A $\{\underline{a}_1, \underline{a}_2, \dots \underline{a}_N\}$ OF A

$\boxed{\text{IFF}}$ $\underline{b}$ IS THE SPAN $\{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_N\}$

$A = [\underline{a}_1, \underline{a}_2, \underline{a}_3, \dots \underline{a}_N]$ $\quad \underline{a}_1, \underline{a}_2, \dots \underline{a}_N \in \mathbb{R}^N$

$A\underline{x} = \underline{b}$

$A\underline{x} = \underline{a}_1 x_1 + \underline{a}_2 x_2 + \dots + \underline{a}_N x_N = \underline{b}$

$$\begin{bmatrix} 2 & 0 \\ 3 & -1 \\ 1 & 2 \end{bmatrix} = A \qquad \underline{b} \in \mathbb{R}^3 \quad \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 0 & 1 \\ 3 & -1 & 7 \\ 1 & 2 & 2 \end{array}\right] \qquad \text{SEARCH } \underline{x} \in \mathbb{R}^2 \qquad A\underline{x} = \underline{b}$$

NUMBER OF COLUMNS

??

$$\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

THEOREM LET $A \in \mathbb{R}^{m \times n}$ THEN THE FOLLOWING STATEMENTS ARE EQUIVALENT

- THE LINEAR SYSTEM HAS AT LEAST ONE SOLUTION FOR EACH $b \in \mathbb{R}^N$
- ANY $b \in \mathbb{R}^N$ IS A LINEAR COMBINATION OF THE COLUMNS OF A
- THE SPAN $\{\underline{a}_1, \underline{a}_2, \dots \underline{a}_N\}$ OF THE COLUMNS OF A IS EQUAL TO THE WHOLE $\mathbb{R}^m$
- EVERY ROW OF A HAS A PIVOT POSITION (OR IS A ROW OF 0)

IF ALSO b IS 0 I CAN NOT CONSIDER THAT ROW, BUT IF $b \neq 0$ HAVE NO SOLUTION

$$A\underline{x} = \underline{b} \qquad [A|b] \xrightarrow[\text{ROW REDUCTION}]{\text{START ECHELON FORM}} [A|b] \quad \left[\begin{array}{ccccc|c} \boxed{*} & * & * & * & * & * \\ 0 & 0 & \boxed{*} & * & * & * \\ 0 & 0 & 0 & \boxed{*} & * & * \\ 0 & 0 & 0 & 0 & \boxed{*} & * \end{array}\right]$$

HOMOGENEUS LINEAR SYSTEM

$A \in \mathbb{R}^{m \times n}$ $\quad \underline{0} \in \mathbb{R}^{m}$

SEARCH $x \in \mathbb{R}^{n}$ $\quad A\underline{x} = \underline{0}$ $\quad \underline{0} = [0 \;\; 0 \;\; 0 \dots 0]$

AUGMENTED MATRIX

$$\left[\begin{array}{c|c} A & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right] \xrightarrow[\text{ALGORITHM}]{\text{ROW REDUCTION}} \left[\begin{array}{ccccc|c} \boxed{*} & * & * & * & * & 0 \\ 0 & \boxed{*} & * & * & * & 0 \\ 0 & 0 & 0 & \boxed{*} & * & 0 \\ 0 & 0 & 0 & 0 & \boxed{*} & 0 \end{array}\right]$$

FREE VARIABLE, INFINITE SOLUTION

NO FREE VARIABLES

IT ALWAYS EXISTS AT LEAST A SOLUTION

THE VECTOR $\underline{x} = 0$ IS ALWAYS A SOLUTION
BUT IS THE ONLY ONE?

$$\left[\begin{array}{ccccc|c} \boxed{*} & * & * & * & * & 0 \\ 0 & \boxed{*} & * & * & * & 0 \\ 0 & 0 & \boxed{*} & * & * & 0 \\ 0 & 0 & 0 & \boxed{*} & * & 0 \\ 0 & 0 & 0 & 0 & \boxed{*} & 0 \end{array}\right]$$

IF EVERY COLUM OF A HAS A PIVOT POSITION THEN THE SOLUTION IS UNIQUE (ONLY $\underline{x} = \underline{0}$)

OTHERWISE IF AT LEAST ONE COLUM OF A DOES NOT HAVE A PIVOUT POSITION THEN THERE ARE INFINITE SOLUTION (INCLUDING $\underline{x} = \underline{0}$)

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & 6 \\ 2 & -4 & 2 \end{bmatrix}$$

WHAT ARE THE SOLUTION $\underline{x} \in \mathbb{R}^3$ OF $A\underline{x} = \underline{0}$?

AUGMENTED MATRIX $\qquad$ IN ECHELON FORM

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 2 & -4 & 2 & 0 \end{array}\right] \; 2 \qquad \left[\begin{array}{ccc|c} \boxed{1} & -2 & 1 & 0 \\ 0 & \boxed{3} & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

ONE OR INFINITE SOLUTIONS?

NO PIVOT POSITION ON X_3 $\Rightarrow$ X_3 IS A FREE VARIABLE $\quad x_3 \in \mathbb{R}$

$\Downarrow$

INFINITE SOLUTION

$$\begin{cases} 3x_2 + 6x_3 = 0 \; ; \quad x_2 = -2x_3 \\ x_1 - 2x_2 + x_3 = 0 \; ; \quad x_1 = 2x_2 - x_3 \; ; \quad x_1 = -5x_3 \end{cases}$$

THE SOLUTION:

$$\begin{cases} x_1 = -5x_3 \\ x_2 = -2x_3 \\ x_3 \in \mathbb{R} \end{cases} = \begin{cases} x_1 = -5x_3 \\ x_2 = -2x_3 \\ x_3 = x_3 \end{cases} \in \mathbb{R}$$

SOLUTION AS A VECTOR FORM

$$\Rightarrow \underline{x} = \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} x_3$$

FIND SOLUTION $\underline{x} \in \mathbb{R}^4$ S.T. $A\underline{x} = \underline{0}$

$$
\text{SAME ROW}\quad
\overbrace{\left[\begin{array}{cccc|c}
\boxed{-2} & 1 & 0 & 3 & 0 \\
4 & -2 & 4 & -5 & 0 \\
-2 & 1 & 0 & 2 & 0 \\
4 & -2 & 4 & -5 & 0
\end{array}\right]}^{A}
\;
\begin{matrix} -2 \\ 1 \\ -2 \end{matrix}
=
\left[\begin{array}{cccc|c}
\boxed{-2} & 1 & 0 & 3 & 0 \\
0 & 0 & \boxed{4} & 1 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 4 & 1 & 0
\end{array}\right]
\; 1 \;
\begin{array}{c}
\left[\begin{array}{cccc|c}
\boxed{-2} & 1 & 0 & 3 & 0 \\
0 & 0 & \boxed{4} & 1 & 0 \\
0 & 0 & 0 & \boxed{-1} & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
\begin{array}{l} 1^{ST} \\ 2^{ND} \\ 3^{RD} \\ \end{array}
$$

$\uparrow$ x_2 FREE VARIABLE

$$
\begin{array}{l}
3^{RD} \\ 2^{ND} \\ 1^{ST}
\end{array}
\begin{cases}
-x_4 = 0; & x_4 = 0 \\
4x_3 + x_4 = 0; \quad x_3 = -\frac{1}{4}x_4; & x_3 = 0 \\
-2x_1 + x_2 + 3x_4 = 0; \quad x_1 = \frac{1}{2}x_2 + \frac{3}{2}x_4; & x_1 = \frac{1}{2}x_2
\end{cases}
$$

$$
\begin{cases}
x_1 = \frac{1}{2}x_2 \\
x_2 \in \mathbb{R} \\
x_3 = 0 \\
x_4 = 0
\end{cases}
\qquad
\underline{x} = \begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 \qquad x_2 \in \mathbb{R}
$$

$$
\begin{array}{c}
1^{ST} \\ 2^{ND} \\
\end{array}
\overbrace{\left[\begin{array}{cccc|c}
\boxed{1} & 2 & 3 & 1 & 0 \\
0 & 0 & \boxed{2} & -1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right]}^{A}
\qquad A\underline{x} = \underline{0} \qquad \underline{x} \in \mathbb{R}^4
$$

$\uparrow x_2$ $\uparrow x_4$

FREE VARIABLES

$$
\begin{array}{l}
2^{ND} \\ 1^{ST}
\end{array}
\begin{cases}
2x_3 - x_4 = 0; & x_3 = \frac{1}{2}x_4 \\
x_1 + 2x_2 + 3x_3 + x_4 = 0; \quad x_1 = -2x_2 - 3x_3 - x_4; \quad x_1 = -2x_2 - \frac{3}{2}x_4 - x_4; & x_1 = -2x_2 - \frac{5}{2}x_4
\end{cases}
$$

$$
\begin{cases}
x_1 = -2x_2 - \frac{5}{2}x_4 \\
x_2 \in \mathbb{R} \\
x_3 = \frac{1}{2}x_4 \\
x_4 \in \mathbb{R}
\end{cases}
\qquad
\underline{x} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -5/2 \\ 0 \\ 1/2 \\ 1 \end{bmatrix} x_4
$$

SEARCH FOR $\underline{x} \in \mathbb{R}^N$ S.T. $A\underline{x} = \underline{b}$

$A \in \mathbb{R}^{m \times N}$ $\quad \underline{b} \in \mathbb{R}^m$ $\quad \underline{y} \in \mathbb{R}^N$ $\quad A\underline{y} = \underline{b}$

$\underline{x} \in \mathbb{R}^N$ $\quad \underline{x}$ SOLUTION OF $A\underline{x} = \underline{b}$

TAKE ANY $\underline{y} \in \mathbb{R}^N$ SOLUTION OF $A\underline{y} = \underline{0}$ (THE HOMOGENEOUS LINEAR SYSTEM ASSOCIATED TO A)

$$A(\underline{x} + \underline{y}) = A\underline{x} + A\underline{y} = \underline{b} + \underline{0} = \underline{b}$$

$$A \in \mathbb{R}^{m \times n} \qquad \underline{b} \in \mathbb{R}^m$$

$\boxed{\text{THEOREM}}$ ALL SOLUTIONS $\underline{x} \in \mathbb{R}$ OF $A\underline{x} = \underline{b}$ CAN BE EXPRESSED AS $\underline{v} + \underline{y}$ WHERE $\underline{v}$ IS ANY ONE PARTICULAR SOLUTION OF THE SYSTEM $(A\underline{v} = \underline{b})$ AND $\underline{y} \in \mathbb{R}^n$ IS ANY SOLUTION OF THE HOMOGENEOUS LINEAR SYSTEM $A\underline{y} = \underline{0}$

EX1.

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & 6 \\ 2 & -4 & 2 \end{bmatrix} = A \qquad \underline{b} = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$$

AUGMENTED MATRIX

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 3 & 6 & 6 \\ 2 & -4 & 2 & 4 \end{array}\right] \xrightarrow{2} \begin{array}{c} 1^{ST} \\ 2^{ND} \\ \\ \end{array} \left[\begin{array}{ccc|c} \boxed{1} & -2 & 1 & 2 \\ 0 & \boxed{3} & 6 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

X_3 FREE VARIABLE
INFINITE SOLUTIONS

SOLUTION OF THE SYSTEM

$$\begin{cases} 2^{ND} & 3x_2 + 6x_3 = 6; \quad x_2 = 2 - 2x_3 \\ 1^{ST} & x_1 - 2x_2 + x_3 = 2; \quad x_1 = 2 + 2x_2 - x_3; \quad x_1 = 2 + 4 - 4x_3 - x_3; \quad x_1 = 6 - 5x_3 \end{cases}$$

$$\begin{cases} x_1 = 6 - 5x_3 \\ x_2 = 2 - 2x_3 \\ x_3 \in \mathbb{R} \end{cases} \qquad \underline{x} = \underbrace{\begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix}}_{v} + \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} x_3$$

SOLUTION OF $A\underline{x} = \underline{0}$

EX2:

$$\left[\begin{array}{cccc|c} \boxed{1} & 2 & 3 & 1 & 7 \\ 0 & 0 & \boxed{2} & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

X_2 AND X_4 FREE VARIABLES

$$\begin{cases} 2x_3 - x_4 = 2; \quad x_3 = 1 + \frac{1}{2}x_4 \\ x_1 + 2x_2 + 3x_3 + x_4 = 7; \quad x_1 = 7 - 2x_2 - 3x_3 - x_4; \quad x_1 = 7 - 2x_2 - 3 - 3/2\, x_4 - x_4; \quad x_1 = 4 - 2x_2 - 5/2\, x_4 \end{cases}$$

$$\begin{cases} x_1 = 4 - 2x_2 - 5/2\, x_4 \\ x_2 = x_2 \in \mathbb{R} \\ x_3 = 1 + 1/2\, x_4 \\ x_4 = x_4 \in \mathbb{R} \end{cases} \qquad \underline{x} = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -5/2 \\ 0 \\ 1/2 \\ 1 \end{bmatrix} x_4 \qquad \begin{array}{l} x_2 \in \mathbb{R} \\ x_4 \in \mathbb{R} \end{array}$$

EX3:

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & 0 & 1 & 3 \\ 2 & -5 & 2 & 3 & 3 & 2 \\ 1 & -3 & 1 & 4 & 4 & 0 \end{array}\right] = \left[\begin{array}{ccccc|c} \boxed{1} & -2 & 1 & 0 & 1 & 3 \\ 0 & \boxed{-1} & 0 & 3 & 1 & -4 \\ 0 & -1 & 0 & 4 & 3 & -3 \end{array}\right] = \left[\begin{array}{ccccc|c} 1 & -2 & 1 & 0 & 1 & 3 \\ 0 & -1 & 0 & 3 & 1 & -4 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{array}\right]$$

x_3 x_5

FREE VARIABLES, INFINITE SOLUTIONS

$$\begin{cases} x_4 + 2x_5 = 1; \quad x_4 = 1 - 2x_5 \\ -x_2 + 3x_4 + x_5 = -4; \quad x_2 = 4 + 3x_4 + x_5; \quad x_2 = 4 + 3 - 6x_5 + x_5; \quad x_2 = 7 - 5x_5 \\ x_1 - 2x_2 + x_3 + x_5 = 3; \quad x_1 = 3 + 2x_2 - x_3 - x_5; \quad x_1 = 3 + 14 - 10x_5 - x_3 - x_5; \quad x_1 = 17 - x_3 - 11x_5 \end{cases}$$

$$\begin{cases} x_1 = 17 - x_3 - 11x_5 \\ x_2 = 7 - 5x_5 \\ x_3 = x_3 \in \mathbb{R} \\ x_4 = 1 - 2x_5 \\ x_5 = x_5 \in \mathbb{R} \end{cases} \qquad \underline{x} = \begin{bmatrix} 17 \\ 7 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -11 \\ -5 \\ 0 \\ -2 \\ 1 \end{bmatrix} x_5$$