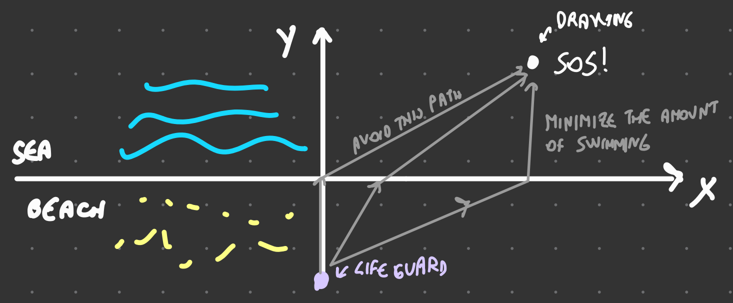
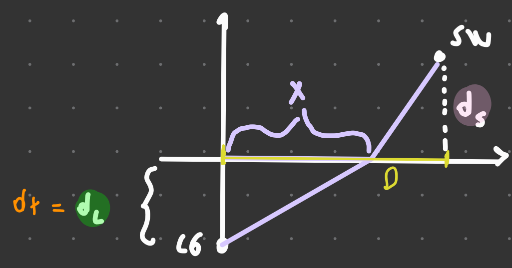
$+$ $\times$

$-$ $\div$



- BEST PATH: TWO STRAIGHT LINES
- ABLE TO PARAMETRIZE A WHOLE TRAJECTORY WITH ONE PARAMETER (X)

v_s: SPEED SWIMMING
v_r: SPEED RUNNING



WANT TO MINIMIZE $T(x)$

$$\Downarrow$$

$$\frac{d}{dx} T(x) = 0$$

A POINT THAT SATISFIES THIS CONDITION IS CALLED FIXED/STATIONARY POINT

$$\frac{1}{v_r} \frac{x}{\sqrt{d_L^2 + x^2}} - \frac{1}{v_s} \frac{D-x}{\sqrt{d_s^2 + (D-x)^2}} = 0$$

$$\frac{x}{\sqrt{d_L^2 + x^2}} - \frac{v_r}{v_s} \frac{D-x}{\sqrt{d_s^2 + (D-x)^2}} = 0$$

RESCALE OF THE PARAMETERS, TRY TO DISCOVER COMBINATION OF PARAMETERS S.T. DEPENDS ON LESS

ONLY MOMENT WHERE VELOCITY APPEAR IS THIS RATIO

AKA MULTIPLY BY A FACTOR

RESCALING v_r AND v_s

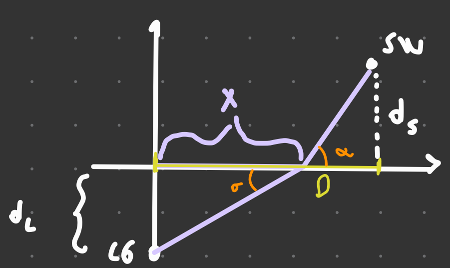
$L \rightsquigarrow \lambda L$

$$\frac{\lambda x}{\sqrt{\lambda^2 x^2 + \lambda^2 d_r^2}} = \frac{\cancel{\lambda} v_r}{\cancel{\lambda} v_s} \frac{\lambda D - \lambda x}{\sqrt{\lambda^2 d_s^2 + (\lambda D - \lambda x)^2}} \Rightarrow \frac{\cancel{\lambda} x}{\cancel{\lambda}\sqrt{x^2 + d_r^2}} = \frac{v_r}{v_s} \frac{\cancel{\lambda}(D-x)}{\cancel{\lambda}\sqrt{d_s^2 + (D-x)^2}}$$

SO WE CONCLUDE THAT

$$\lambda x(v_r, v_s, d_r, d_s, D) = x(\lambda v_r, \lambda v_s, \lambda d_r, \lambda d_s, \lambda D)$$

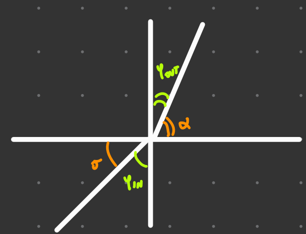
CHANGE HOW I MEASURE X DOES NOT CHANGE THE NATURE OF THE SOLUTION AND HOW IT DEPENDS ON THE PARAMETERS



INSTEAD OF LENGTHS

USE ANGLE TO PARAMETRIZE THE SOLUTION

$$\frac{1}{v_r} \cos(\sigma) = \frac{1}{v_s} \cos(\alpha)$$



SNELL'S LAW →

THE ONE FOR THE LIGTH PASSING THROUGH DIFFERENT OBJECT (MEDIUM)

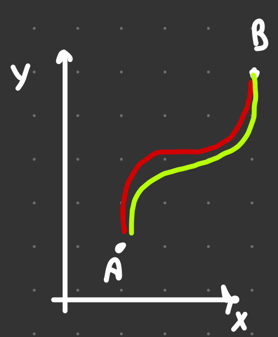
$$\boxed{\frac{\sin \varphi_{in}}{v_r} = \frac{\sin(\varphi_{out})}{v_s}}$$

FERMAT PRINCIPLE OF LEAST TIME

STARTING FROM THE LIFEGUARD EXAMPLE

$$\Downarrow$$

GENERAL PRINCIPLE: IF I WANT TO GO FROM A TO B THE TRAJECTORY SHOULD BE THE ONE THAT MINIMIZE TIME

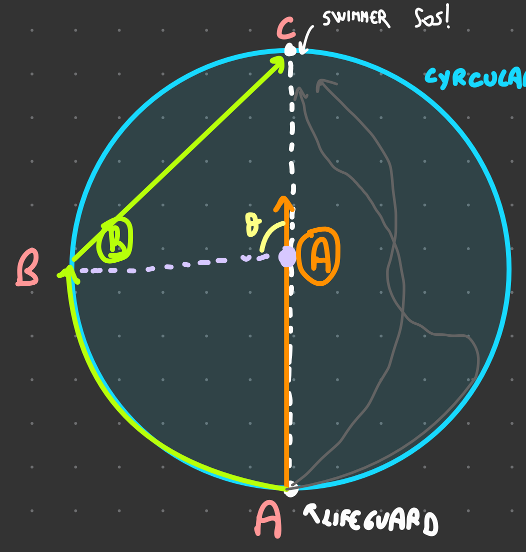


IF I CHANGE THE TRAJECTORY A LITTLE BIT THE NEW ONE WOULD TAKE MORE.

NEED A WHOLE FUNCTION $Y(x)$ $\quad x \in (A_x, B_x)$

FROM SINGLE VALUE/VECTOR TO A FUNCTION TO DEFINE A TRAJECTORY. IN MOST GENERAL CASE WE NEED A WHOLE FUNCTION

Ex. of OPTIMIZATION PRINCIPLE



CYRCULAR SWIMMING POOL RADIUS = 1 (NON-IMPORTANT)

Ⓐ: SWIM STRAIGTH

Ⓑ: RUNNING AND THEN SWIM

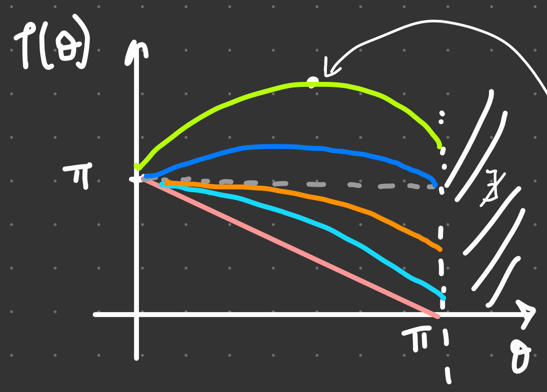
• OTHER TRAJECTORY NOT CONSIDERED (TAKE DEFINETLY TOO MUCH)

v_r = VELOCITY OF RUNNING = 1 — IT STILL GENERALIZE, DON'T ADD NEW CONSTRAINT

$$T(\theta) = \underbrace{\pi - \theta}_{\overset{\frown}{AB}\ (\text{FROM } A \text{ TO } B)} + \underbrace{\frac{2}{v_s} \sin\left(\frac{\theta}{2}\right)}_{\overline{BC}\ (\text{FROM } B \text{ TO } C)} \Rightarrow \text{MINIMIZE } T(\theta) \quad \theta \in [0, \pi]$$

$=0$ $A \neq B$, RUN AND THEN SWIM

$=\pi \Rightarrow$ A AND B ON SAME PLACE (JUST SWIM)



• $v_s \to \infty \Rightarrow T(\theta) = \pi - \theta + 0$

AFTER A WHILE YOU GET THIS KIND OF POINT THOSE ARE STATIONARY POINT BUT ARE ALSO THE MAXIMUM (:/) NOT INTERESTING FOR US, WE NEED THE MIN

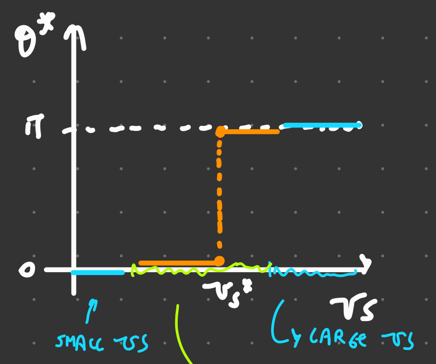
⚠ THERE MUST A POSSIBILITY THAT IS THE BEST (MINIMUM TIME) WE NEED TO FIND (OR ARE ALL THE SAME (BUT IS NOT OUR CASE))

WE CALL IT θ^*

$\boxed{\theta^* = \pi}$ ↙ IN OUR CASE

PLOT WHERE THE MINIMUM LIES

$$\theta^* = \arg\min_{\theta} T(\theta)$$



BUT WHAT BETWEEN?

$v_s^* = \frac{2}{\pi}$

THERE ARE CASES FOR WHICH $\theta^* = \pi$ IS NOT THE MINIMUM

• FOR LARGE $v_s \Rightarrow \theta^* = \pi$

• FOR SMALL $v_s \Rightarrow \theta^* = 0$

CHANGE IN PHYSICAL BEHAVE OF THE SYSTEM

CALLED (FIRST-ORDER) PHASE TRANSITION

FERMAT'S PRINCIPLE

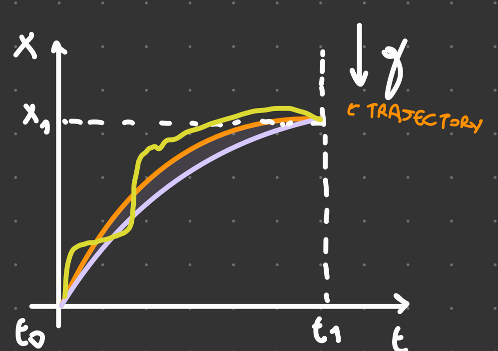
• MINIMIZE

• MAXIMIZE

• FIND STATIONARY POINTS

NEWTONIAN PHYSIC

$F = m\ddot{x}$ — SECOND DERIVATIVE AKA ACCELERATION



IS COMPATIBLE W/ NEWTONIAN DINAMICS?

↓

COMPUTE $\ddot{x}$

CHECK $\underbrace{-\gamma = \ddot{x}}_{\text{SATISFIED } \forall t}$ FOR ALL TIMES IF TRUE: YES

$x(t) \quad x \in (t_0, t_1)$

$\mathbb{R} \to S[x(\cdot)]$

"$S\left((x(t))_{t=t_0,\dots,t_1}\right)$" IS A FUNCTION OF THE WHOLE TRAJECTORY

BOUNDARY CONDITION

$$\begin{cases} x(t_0) = x_0 \\ x(t_1) = x_1 \end{cases}$$

IF THE RED CURVE IS COMPATIBLE W/ N. DYNAMICS A LITTLE BIT
THEN LARGER ON S THEN

EVERY TRAJECTORY CLOSE TO THE RED SHOULD HAVE A VALUE FOR S EVEN
LARGER THEN

SMALL CHANGE/DISPLACEMENT

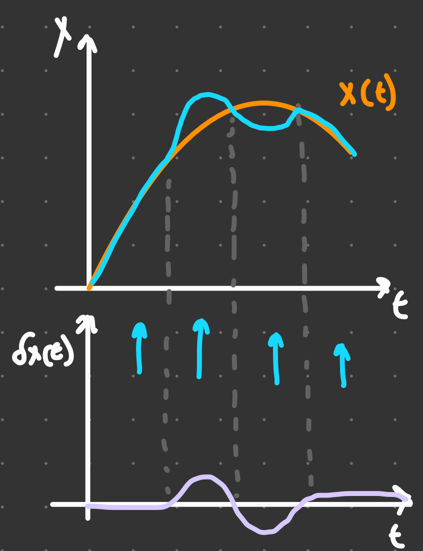
$x(t) + \delta x(t)$

ADD DISPLACEMENT ADD TO THE WHOLE TRAJECTORY.
THE DISPLACEMENT DEPENDS ON t

$\vec{x} = (x_1, x_2, \dots, x_N)$

$\vec{x} + \vec{\delta}_x = (x_1 + dx_1, x_2 + dx_2, \dots)$

POTENTIALLY DIFFERENT



$\frac{d}{dx} S(x) = 0$

$S(x + dx) = S(x)$

$$\boxed{S\left[x(\cdot) + \delta x(\cdot)\right] = S\left[x(\cdot)\right]}$$

$\frac{\delta S}{\delta x(\cdot)}[x(\cdot)] = 0$

"ACTION" = S

$S[x(\cdot)] = \int_{t_0}^{t_1} L(x(t), \dot{x}(t))\, dt$ $\in \mathbb{R}$

INITIAL AND FINAL TIME

L = "LAGRANGIAN"