$+ \quad \times$

$- \quad \div$

REMARK

THERE EXIST SEQUENCES WHICH ARE NOT CONVERGING

- $a_n = (-1)^n \quad \forall n \in \mathbb{N}$ $-1, 1, -1$

$\{a_n\}_{n \in \mathbb{N}}$ IS NOT CONVERGING

- $a_n = 3n \quad \forall n \in \mathbb{N}$ $3, 6, 9 \dots$

$\{a_n\}_{n \in \mathbb{N}}$ IS NOT CONVERGING, BUT IT ADMIST LIMIT IN THE FOLLOWING SET

DEFINITION

LET $\{a_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ BE A SEQUENCE OF REAL NUMBERS

$\{a_n\}_{n \in \mathbb{N}}$ IS DIVERGET TO $+\infty$ IF

$\forall M > 0 \ \exists n_0 \in \mathbb{N}$ SUCH THAT $\forall n \in \mathbb{N}$ WITH $n \geq n_0$ WE HAVE $a_n > M$

IN THIS CASE WE SAY THAT THE LIMIT OF $\{a_n\}_{n \in \mathbb{N}}$ IS $+\infty$ AS n GOES TO $+\infty$ OR THAT $\{a_n\}_{n \in \mathbb{N}}$ GOES OR DIVERGES $+\infty$ AS n GOES $+\infty$, AND WE WRITE

$\lim_{n \to +\infty} a_n = +\infty$ OR $a_n \to +\infty$ AS $n \to +\infty$

REMARK LET $C > 0$ BE A CONSTANT. WE CAN REPLACE $> M$ WITH $\geq M$, $> CM$, $\geq CM$

- $\{a_n\}_{n \in \mathbb{N}}$ IS DIVERGENT TO $-\infty$ IF $\forall M < 0$ $\exists n_0 \in \mathbb{N}$ SUCH THAT $\forall n \in \mathbb{N}$ WITH $n \geq n_0$ WE HAVE $a_n < M$

IN THIS CASE WE SAY THAT THE LIMIT OF $\{a_n\}_{n \in \mathbb{N}}$ IS $-\infty$ AS n GOES TO $+\infty$

OR THAT a_n GOES OR DIVERGES TO $-\infty$ AS n GOES TO $+\infty$, AND WE WRITE

$$\lim_{n \to +\infty} a_n = -\infty \text{ OR } a_n \to -\infty \text{ AS } n \to +\infty$$

REMARK LET $C > 0$ BE A CONSTANT. WE CAN REPLACE $< M$ WITH $\leq M$, $< CM$, $\leq CM$

DEFINITION $\{a_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$

- IF THE LIMIT ℓ EXISTS (FINITE, THAT IS, $\ell \in \mathbb{R}$ OR INFINITE, THAT IS, $\ell = +\infty$ OR $\ell = -\infty$) WE SAY THAT THE SEQUENCE ADMIT LIMIT OR IS REGULAR

- IF THE LIMIT l DOES NOT EXIST, WE SAY THAT THE SEQUENCE DOES NOT ADMIT LIMIT OR IS IRREGULAR

- IF THE LIMIT l EXIST AND IS FINITE, THAT IS

$$\exists \lim_{n \to +\infty} a_n = l \in \mathbb{R}$$ THE SEQUENCE IS CONVERGENT

- IF THE LIMIT l EXIST AND IS INFINITE, THAT IS,

$$\exists \lim_{n \to +\infty} a_n = +\infty \text{ OR } \exists \lim_{n \to +\infty} a_n = -\infty$$

THE SEQUENCE IS DIVERGENT OR INFINITE

- IF THE SEQUENCE IS CONVERGING TO 0, THAT IS,

$$\exists \lim_{n \to +\infty} a_n = 0$$

THE SEQUENCE IS INFINITESIMAL

REMARK IF A SEQUENCE ADMITS A LIMIT $l \in \mathbb{R} \cup \{-\infty, +\infty\} = [-\infty, +\infty]$, THEN SUCH A LIMIT IS UNIQUE

WE NEED TO CHECK THE FOLLOWING CASES

- $l_1 = \pm\infty$ AND $l_2 \in \mathbb{R} \Rightarrow l_1 = l_2$
- $l_1 = +\infty$ AND $l_2 = -\infty \Rightarrow l_1 = l_2$

$\lim\limits_{n} a_n = +\infty = l_1$ AND $\lim\limits_{n} a_n = l_2 \in \mathbb{R}$ $\qquad \lim\limits_{n} = \lim\limits_{n \to +\infty}$

FIX $\varepsilon = 1$ $\exists n_0 \in \mathbb{N}$ S.T. $\forall n \geq n_0$

$l_2 - 1 < a_n < l_2 + 1 \leq |l_2| + 1 = M$

FIX SUCH M $\exists n_1 \in \mathbb{N}$ S.T. $\forall n \geq n_1$ WE HAVE $a_n > M$

$\Rightarrow \forall n \geq \max\{n_0, n_1\}$ WE HAVE $a_n < M < a_n$

CONTRADICTION!

EXAMPLE

- $a_n = (-1)^n$ IS IRREGULAR $\nexists \lim\limits_{n} (-1)^n$
- $a_n = 3n$ THEN $\lim\limits_{n} a_n = \lim\limits_{n} 3n = +\infty$

FIX $M > 0$ LET $n_0 \in \mathbb{N}$ SUCH THAT $n_0 > M/3$

THEN $\forall n \geq n_0$ WE HAVE

$$a_n = 3n > 3n_0 > \frac{3M}{3} = M$$

- $a_n = -n^2$ THEN $\lim\limits_{n} a_n = \lim\limits_{n} (-n^2) = -\infty$

FIX $M < 0$ LET $n_0 \in \mathbb{N}$ SUCH THAT $n_0 > \sqrt{|M|}$ THE $\forall n \geq n_0$ WE HAVE

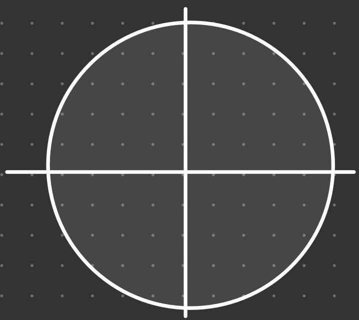
$$a_n = -n^2 \leq -n_0^2 < -|M| = M$$

- $a_n = (-1)^n n$ IS IRREGULAR, THAT IS, $\nexists \lim\limits_{n} ((-1)^n n)$

$-1, 2, -3, 4 \ldots$

MOTIVATION

1) LENTH OF A CIRCUIT



HOW TO COMPUTE THE LENTH OF THE UNIT CIRCLE

TAKE $n \in \mathbb{N}$ AND LET a_n THE PERIMETER OF THE REGULAR POLYGON WITH N SIDES ($n \geq 3$) INSCRIBED IN THE CIRCLE

ONE CAN SHOW THAT $\exists \lim_n a_n = 2\pi$

AND LIM IS THE LENTH OF THE UNIT CIRCLE

2)

$+$ $\times$

$-$ $\div$

REMARK: THERE EXIST SEQUENCES WHICH ARE **NOT** CONVERGING

- $a_n = (-1)^n \quad \forall n \in \mathbb{N} \quad -1, 1, -1, 1 \dots$

  a_n IS **NOT** CONVERGING

- $a_n = 3n \quad \forall n \in \mathbb{N} \quad 3, 6, 9, 12 \dots$

  a_n IS **NOT** CONVERGING BUT IT ADMITS LIMIT IN THE FOLLOWING SENSE

DEFINITION LET $\{a_n\} \subseteq \mathbb{R}$ BE A SEQUENCE OF REAL NUMBERS

- a_n IS **DIVERGENT** TO $+\infty$ IF

  $\forall M > 0 \quad \exists n_0 \in \mathbb{N}$ SUCH THAT $\forall n \in \mathbb{N}$ WITH $n \geq n_0$ WE HAVE $a_n > M$

  IN THIS CASE, WE SAY THAT THE **LIMIT** OF $\{a_n\}_{n\in\mathbb{N}}$ IS $+\infty$ AS n GOES TO $+\infty$ OR THAT $\{a_n\}_{n\in\mathbb{N}}$ GOES OR DIVERGES $+\infty$ AS n GOES $+\infty$ AND WE WRITE

  $$\lim_{n\to+\infty} a_n = +\infty \quad \text{OR} \quad a_n \to +\infty \text{ AS } n \to +\infty$$

REMARK LET $C > 0$ BE A CONSTANT. WE CAN REPLACE **$> M$** WITH $\geq M$, $> CM$, $\geq CM$

• $\{a_n\}_{n\in\mathbb{N}}$ IS DIVERGENT TO $-\infty$ IF

$\forall M<0 \ \exists n_0\in\mathbb{N}$ SUCH THAT $\forall n\in\mathbb{N}$ WITH $n\geq n_0$ WE HAVE $a_n<M$

IN THIS CASE WE SAY THAT THE LIMIT OF $\{a_n\}_{n\in\mathbb{N}}$ IS $-\infty$ AS n GOES TO $+\infty$

OR THAT a_n GOES OR DIVERGES TO $-\infty$ AS n GOES TO $+\infty$, AND WE WRITE

$$\lim_{n\to+\infty} a_n=-\infty \quad \text{OR} \quad a_n\longrightarrow-\infty \text{ AS } n\longrightarrow+\infty$$

REMARK:

LET $C>0$ BE A CONSTANT. WE CAN REPLACE $<M$ WITH $\leq M$, $<CM$, $\leq CM$

DEFINITION $\{a_n\}_{n\in\mathbb{N}}\subseteq\mathbb{R}$

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- IF THE LIMIT l EXISTS AND IS INFINITE, THAT IS,

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THE SEQUENCE IS <u>DIVERGENT OR INFINITE</u>

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REMARK

IF A SEQUENCE ADMITS A LIMIT $l \in \mathbb{R} \cup \{-\infty, +\infty\} = [-\infty, +\infty]$, THEN SUCH A LIMIT IS UNIQUE

WE NEED TO CHECK THE FOLLOWING CASES

$$l_1 = \pm\infty \quad \text{AND} \quad l_2 \in \mathbb{R} \Rightarrow l_1 = l_2$$

$$l_1 = +\infty \quad \text{AND} \quad l_2 = -\infty \Rightarrow l_1 = l_2$$

$\lim_{n \to +\infty} a_n = +\infty = l_1$ AND $\lim_n a_n = l_2 \in \mathbb{R}$

$\boxed{\lim_n = \lim_{n \to +\infty}}$

FIX $\varepsilon = 1$ $\exists n_0 \in \mathbb{N}$ S.T. $\forall n \geq n_0$

$l_2 - 1 < a_n < l_2 + 1 \leq |l_2| + 1 = M$

FIX SUCH AN M $\exists n_1 \in \mathbb{N}$ S.T. $\forall n \geq n_1$ WE HAVE

$$a_n > M$$

$\Rightarrow$ $\forall n \geq \max\{n_0, n_1\}$ WE HAVE $a_n < M < a_n$

CONTRADICTION!

EXAMPLE

- $a_n = (-1)^n$ IS IRREGULAR $\nexists \lim_n (-1)^n$

- $a_n = 3n$ THEN $\lim_n a_n = \lim_n (3n) = +\infty$

FIX $M > 0$. LET $n_0 \in \mathbb{N}$ SUCH THAT $n_0 > M/3$

THEN $\forall n \geq n_0$ WE HAVE

$$a_n = 3n \geq 3n_0 > \frac{3M}{3} = M$$

- $a_n = -n^2$ THEN $\lim_n (a_n) = \lim_n (-n^2) = -\infty$

FIX $M < 0$ LET $n_0 \in \mathbb{N}$ SUCH THAT $n_0 > \sqrt{|M|}$ THEN

$\forall n \geq n_0$ WE HAVE

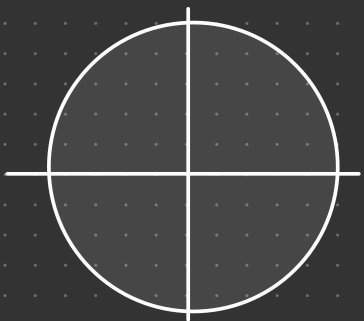
$$a_n = -n^2 \leq -n_0^2 < -|M| = M$$

- $a_n = (-1)^n n$ IS IRREGULAR, THAT IS, $\nexists \lim_n \left((-1)^n n\right)$

$-1, 2, -3, 4 \ldots$

MOTIVATION

1) LENGTH OF A CIRCLE



HOW TO COMPUTE THE LENTH OF THE UNIT CIRCLE?

TAKE $n \in \mathbb{N}$ AND LET a_n THE PERIMETER OF THE REGULAR POLYGON WITH n SIDES $(n \geq 3)$ INSCRIBED IN THE CIRCLE

ONE CAN SHOW THAT $\exists \lim_n a_n = 2\pi$ AND 2π IS THE LENGTH OF THE UNIT CIRCLE

2) ZENONE'S PARADOX

ZENONE FROM ELEA (V CENTURY B.C.)

ACHILLES (A.) AND THE TURTLE (T.)

A. RUNS AT A SPEED OF $10\,m/s$

T. RUNS AT A SPEED OF $1\,m/s$

ZENONE'S PARADOX:

IF T. STARTS 1m AHEAD OF A., A NEVER REACHES T.

REASONING BY ZENONE:

$A(t)$, $T(t)$ POSITION OF A. AND T. AT TIME t

(t TIME IN SECONDS, POSITION IN m)

$t_0 = 0\ s$ $\quad A(0) = A(t_0) = 0\ m$

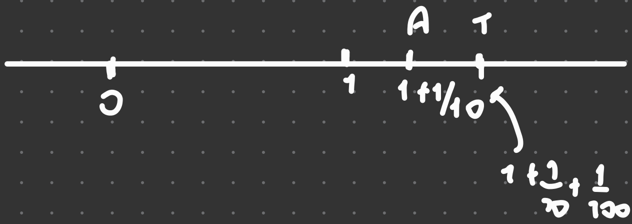
$T(0) = T(t_0) = 1\ m$

$t_1 = \frac{1}{10}\ s$ $\quad A(t_1) = 1$

$T(t_1) = \left(1 + \frac{1}{10}\right) m$

$t_2 = \left(\frac{1}{10} + \frac{1}{100}\right) s$ $\quad A(t_2) = \left(1 + \frac{1}{10}\right) s$

$T(t_2) = \left(1 + \frac{1}{10} + \frac{1}{100}\right) s$



$\forall n \in \mathbb{R} \quad \varepsilon_n = \left(1 + \frac{1}{10} + \frac{1}{100} + \dots + \frac{1}{10n}\right)$

$$A(\varepsilon_n) = T(\varepsilon_{n-1}) = \left(1 + \frac{1}{10} + \dots + \frac{1}{10^{n-1}}\right) m$$

$\boxed{\text{BUT}}$ $\quad T(\varepsilon_n) = \left(1 + \frac{1}{10} + \dots + \frac{1}{10^{n-1}} + \frac{1}{10^n}\right) m$

IT'S CLEAR THAT A. REACHES T. AT TIMES S

$S = \frac{1}{9} s$ BY SOLVING

$$10\varepsilon = 1 + 1\varepsilon : \varepsilon = \frac{1}{9}$$

AND $A(S) = T(S) = \frac{10}{9} m$

I CLAIM THAT $a_n = \left(1 + \frac{1}{10} + \dots + \frac{1}{10^{n-1}}\right) \ \forall n \in \mathbb{N}$

IS A SEQUENCE COVERING TO $\frac{10}{9}$

$$\left| a_n - \frac{10}{9} \right|$$

$$a_n = 1 + \frac{1}{10} + \dots + \frac{1}{10^{n-1}} = \frac{1 - \left(\frac{1}{10}\right)^n}{1 - \frac{1}{10}} : 1 + x + \dots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

$x \neq 1$

$x = 1/10$

$$= \frac{10}{9}\left(1-\frac{1}{10^n}\right)$$

$$\left|\omega_n - \frac{10}{9}\right| = \frac{10}{9}\left|1-\left(1-\frac{1}{10^n}\right)\right| = \frac{1}{9\cdot 10^{n-1}} < \frac{1}{9}\,\frac{1}{(n-1)} \leq \frac{1}{n-1}$$

$10^n > n$

FIX $\varepsilon > 0$ LET $n_0 \in \mathbb{N}$ S.T. $n_0 > 1 + \frac{1}{\varepsilon}$ THEN

$\forall n \geq n_0$ WE HAVE

$$\left|\omega_n - \frac{10}{9}\right| \leq \frac{1}{n-1} \leq \frac{1}{n_0 - 1} < \varepsilon \quad \text{SO} \quad \lim_n \omega_n = \frac{10}{9}$$

ANALOGOUSLY, ONE CAN SHOW $\lim_n t_n = \frac{1}{9} = S$

$$\frac{1}{9} = 0.\bar{1} = 0.111\ldots$$

$$= \frac{1}{10} + \frac{1}{100} + \ldots \frac{1}{10^n} + \ldots$$

DEFINITION

$A \subseteq \mathbb{R}$ IS **BOUNDED** IF AND ONLY IF IT IS BOUNDED FROM ABOVE AND FROM BELOW

PROPOSITION A IS BOUNDED $\Leftrightarrow$ $\exists M \in \mathbb{R}$, WITH $M > 0$, SUCH THAT $|x| \leq M$ $\forall x \in A$, THAT IS, $\underbrace{\sup\{|x| : x \in A\}}_{< +\infty} \leq M$

PROOF "$\Rightarrow$" A BOUNDED MEANS

$\exists \ell, C \in \mathbb{R}$ SUCH THAT

$$l \leq x \leq L \quad \forall x \in A$$

LET $M = \max\{|l|, |L|\}$ THEN

$$-M \leq l \leq x \leq M \quad \forall x \in A \text{ SO } |x| \leq M \quad \forall x \in A$$

"$\Leftarrow$" IF $|x| \leq M \quad \forall x \in A$ THEN $-M \leq x \leq M \quad \forall x \in A$

SO $-M$ LOWER BOUND AND M UPPER BOUND SO A IS BOUNDED

DEFINITION: A SEQUENCE $\{a_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ IS BOUNDED IF $\exists M \in \mathbb{R}$ WITH $M > 0$ SUCH THAT

$$|a_n| \leq M \quad \forall n \in \mathbb{N}$$

THEOREM: IF $\{a_n\}_{n \in \mathbb{N}}$ IS **CONVERGING**, THEN IT IS BOUNDED

PROOF: $\{a_n\}_{n \in \mathbb{N}}$ CONV $\Rightarrow \exists a \in \mathbb{R}$ SUCH THAT $\lim_n a_n = a$

FIX $\varepsilon = 1$ $\exists n_0 \in \mathbb{N}$ SUCH THAT $\forall n \geq n_0$ WE HAVE

$$a - 1 < a_n < a + 1$$

$$|a_n| < |a| + 1$$

SO

$$d(a_n, a) < 1 \Rightarrow d(a_n, 0) \leq d(a_n, a) + d(a, 0)$$
$$< 1 + |a|$$

TAKE $M = \max\{|a_1|, \dots, |a_{n_0 - 1}|, |a| + 1\}$

IT IS EASY TO SHOW THAT

$$|a_n| \leq M \quad \forall n \in \mathbb{N} \quad \square$$

REMARKS:

- $\{a_n\}_{n \in \mathbb{N}}$ BOUNDED $\not\Rightarrow$ $\{a_n\}_{n \in \mathbb{N}}$ CONVERGENT

  $(-1)^n$ BDD BUT IRREGULAR

- $\{a_n\}_{n \in \mathbb{N}}$ CONVERGENT $\Rightarrow$ $\{a_n\}_{n \in \mathbb{N}}$ BDD

- $\{a_n\}_{n \in \mathbb{N}}$ DIVERGENT TO $+\infty$ $\Rightarrow$ BOUNDED FROM BELOW
  UNBOUNDED FROM ABOVE

- $\{a_n\}_{n \in \mathbb{N}}$ DIVERGENT TO $-\infty$ $\Rightarrow$ BOUNDED FROM ABOVE
  UNBOUNDED FROM BELOW

- $\{a_n\}_{n \in \mathbb{N}}$ UNBOUNDED FROM ABOVE (BELOW) $\not\Rightarrow$ $\lim_n a_n = +\infty$ $(-\infty)$

$$a_n = \begin{cases} 1 & n \text{ ODD} \\ n & n \text{ EVEN} \end{cases}$$

BOUNDED FROM BELOW
UNBOUNDED FROM ABOVE
BUT IT IS IRREGULAR!

COMPARISON THEOREMS

PERMANENCE OF THE SIGN THEOREM

LET $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ SEQUENCE OF REAL NUMBERS

SUCH THAT $\exists \lim_n a_n = l \in [-\infty, +\infty]$

a) IF $l>0$ (INCLUDING $l=+\infty$), THEN $a_n>0$
DEFINITELY, THAT IS

$\exists n_0 \in \mathbb{N}$ SUCH THAT $\forall n \geq n_0$ WE HAVE
$a_n > 0$

b) IF $l<0$ (INCLUDING $l=-\infty$), THEN $a_n<0$
DEFINITELY THAT IS

$\exists n_0 \in \mathbb{N}$ SUCH THAT $\forall n \geq n_0$ WE HAVE $a_n<0$

PROOF: a) $l \in \mathbb{R}$ $l>0$

HRP $\forall \varepsilon>0$ $\exists n_0 \in \mathbb{N}$ S.T. $\forall n \geq n_0$ WE HAVE

$l-\varepsilon < a_n < l+\varepsilon$

PICK $\varepsilon>0$ SUCH THAT $l-\varepsilon>0$ $(\varepsilon = l/2)$

THEN FOR $n_0 \in \mathbb{N}$ AS HRP, $\forall n \geq n_0$ WE HAVE
$0 < l-\varepsilon < a_n$ SO $a_n > 0$

IN PARTICULAR $\exists n_0 \in \mathbb{N}$ SUCH THAT $\forall n \geq n_0$ $a_n > \frac{l}{2}$

EXERCISE: PROVE THE CASE $l = +\infty$ AND b)

COROLLARY:

c) IF $a_n \geq 0$ DEFINITELY, THEN $l \geq 0$ (POSSIBLY $+\infty$)

d) IF $a_n \leq 0$ DEFINITELY, THEN $l \leq 0$ (POSSIBLY $-\infty$)

PROOF:

c) BY CONTRADICTION ASSUME $l < 0$ THEN BY b) $a_n < 0$ DEFINITELY

CONTRADICTION!

REMARK: $a_n > 0$ $\forall n \in \mathbb{N}$ $\Rightarrow$ $l > 0$? $\boxed{\text{NO}}$

YOU CAN ONLY SAY THAT $l \geq 0$

$a_n = \frac{1}{n} > 0$ BUT $\lim_n a_n = 0$

REMARK: WHEN DEALING WITH LIMITS, IT IS ALWAYS ENOUGH TO ASSUME THAT A CERTAIN PROPERTY OF THE SEQUENCE HOLDS DEFINETELY. THE LIMIT DOES NOT DEPEND ON THE BEHAVIOUR OF THE FIRST $\tilde{N}$ TERMS NO MATTER WHO THE FIXED $\tilde{N} \in \mathbb{N}$ IS.

COMPARISON THEOREM:

$\{a_n\}_{n\in\mathbb{N}}$ $\{b_n\}_{n\in\mathbb{N}}$ SEQUENCES

OF REAL NUMBERS, SUCH THAT

$a_n \leq b_n$ $\forall n\in\mathbb{N}$ OR DEFINITELY

THEN

a) $\exists \lim_n a_n = +\infty \Rightarrow \exists \lim_n b_n = +\infty$

b) $\exists \lim_n b_n = -\infty \Rightarrow \exists \lim_n a_n = -\infty$

c) $\exists \lim_n a_n = a \in [-\infty, +\infty]$ AND $\exists \lim_n b_n = b \in [-\infty, +\infty]$
$\Rightarrow a \leq b$

PROOF:

a) HYP $\forall M > 0$ $\exists n_0 \in \mathbb{N}$ S.T. $\forall n \geq n_0$ WE HAVE
$a_n > M$

THESIS: $\forall M > 0$ $\exists n_1 \in \mathbb{N}$ S.T. $\forall n \geq n_1$ WE HAVE $b_n > M$

FIX $M > 0$ PICK $n_0 \in \mathbb{N}$ FROM HYP THEN

$\forall n \geq n_0$ WE HAVE $M < a_n \leq b_n$ $(n_1 = n_0)$

b) EXERCISE:

c) • $b = -\infty \overset{b)}{\Rightarrow} a = -\infty$ so $a \leq b$

• $b = +\infty \overset{b)}{\Rightarrow} a \leq b$

$\ulcorner -\infty < c < +\infty \quad \forall c \in \mathbb{R} \lrcorner$

• $a = -\infty \Rightarrow a \leq b$

• $a = +\infty \overset{a)}{\Rightarrow} b = +\infty$ so $a \leq b$

• $a \in \mathbb{R}$, $b \in \mathbb{R}$

BY CONTRADICTION, ASSUME $b < a$

HRP $\quad \forall \varepsilon > 0 \ \exists N_1 \in \mathbb{N}$ S.T. $\forall n \geq N_1$ WE HAVE

$$a - \varepsilon < a_n < a + \varepsilon$$

$\forall \varepsilon > 0 \ \exists N_2 \in \mathbb{N}$ S.T. $\forall n \geq N_2$ WE HAVE

$$b - \varepsilon < b_n < b + \varepsilon$$

IF $b < a$, WE CAN FIND $\varepsilon > 0$ SUCH THAT

$$b + \varepsilon < a - \varepsilon \quad (0 < 2\varepsilon < a - b)$$

FOR THIS $\varepsilon>0$, LET $N \geq \max\{N_1, N_2\}$ THEN

$$b_N < b + \varepsilon < a - \varepsilon < a_N \leq b_N$$ CONTRADICTION!

SQUEEZE OR SANDWICH THEOREM (TEOREMA DEI CARABINIERI)

LET $\{a_n\}_{n\in\mathbb{N}}$ $\{b_n\}_{n\in\mathbb{N}}$ $\{c_n\}_{n\in\mathbb{N}}$ SEQ. OF REAL NUMBERS S.T.

$$a_n \leq c_n \leq b_n \quad \forall n\in\mathbb{N} \text{ OR DEFINITELY}$$

ASSUME THAT FOR SOME $l \in [-\infty, +\infty]$ WE HAVE

$$\lim_n a_n = \lim_n b_n = l$$

THEN $\exists \lim_n c_n = l$

PROOF • $l = +\infty$ TRUE FOR a) COMPARISON $a_n \leq c_n$

• $l = -\infty$ TRUE FOR b) COMPARISON $c_n \leq b_n$

• $l \in \mathbb{R}$ FIX $\varepsilon > 0$ WE KNOW

- $\exists N_1 \in \mathbb{N}$ SUCH THAT $\forall n \geq N_1$ $l - \varepsilon < a_n < l + \varepsilon$
- $\exists N_2 \in \mathbb{N}$ SUCH THAT $\forall n \geq N_2$ $l - \varepsilon < b_n < l + \varepsilon$

$\Rightarrow \forall n \geq n_0 = \max\{n_1, n_2\}$ WE HAVE

$$l - \varepsilon < a_n \leq c_n \leq b_n < l + \varepsilon$$

THEREFORE

$$\exists n_0 \in \mathbb{N} \text{ S.T. } b_n < l + \varepsilon$$

$$l - \varepsilon < c_n < l + \varepsilon$$

$$\Rightarrow \lim_n c_n = l \quad \square$$

CALCULUS RULES FOR CONVERGING SEQUENCES

THEOREM: LET $\{a_n\}_{n \in \mathbb{N}}$ $\{b_n\}_{n \in \mathbb{N}}$ BE TWO CONVERGING SEQUENCES, THAT IS

$$\exists \lim_n a_n = a \in \mathbb{R} \text{ AND } \exists \lim_n b_n = b \in \mathbb{R}$$

THEN

1) $\forall c \in \mathbb{R}$ $\lim_n (c a_n) = c a$ $\left[\lim_n (-a_n) = -a ;\ c = -1 \right]$

2) $\lim_n (a_n + b_n) = a + b$ $\left[\lim_n (a_n - b_n) = a - b \right.$

$$\begin{array}{cc} \| & \| \\ a + (-b) & a + (-b) \end{array} \Big]$$

3) $\lim_n (a_n b_n) = ab$

4) IF $b \neq 0$ THEN $\lim_n \frac{1}{b_n} = \frac{1}{b}$

REMARK: IF $b \neq 0$ THEN $b_n \neq 0$ DEFINITELY, SO AT LEAST DEFINITELY $1/b_n$ IS WELL DEFINED

5) IF $b \neq 0$ THEN $\lim_n \frac{a_n}{b_n} = \frac{a}{b}$ $\left[\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n} \overset{4)}{\to} a \cdot \frac{1}{b} = \frac{a}{b} \right]$

PROOF

HRP • $\forall \varepsilon > 0$ $\exists n_1 \in \mathbb{N}$ S.T. $\forall n \geq n_1$ WE HAVE

$$a - \varepsilon < a_n < a + \varepsilon$$

• $\forall \varepsilon > 0$ $\exists n_2 \in \mathbb{N}$ S.T. $\forall n \geq n_2$ WE HAVE

$$b - \varepsilon < b_n < b + \varepsilon$$

1) $c = 0$ $\quad 0 \cdot a_n = 0 \to 0 = 0a$

$c \neq 0$ $\quad |ca_n - ca| = |c(a_n - a)| = |c|\,|a_n - a|$

FIX $\varepsilon > 0$ $\exists n_1 \in \mathbb{N}$ S.T. $\forall n \geq n_1$ WE HAVE

$$|ca_n - ca| = |c|\,|a_n - a| < |c|\varepsilon \Rightarrow \lim_n ca_n = ca$$

2) $|(a_n+b_n)-(a+b)| = |(a_n-a)+(b_n-b)| \leq |a_n-a|+|b_n-b|$

FIX $\varepsilon>0$ $\forall n \geq n_0 = \max(n_1, n_2)$ WE HAVE

$$|(a_n+b_n)-(a+b)| \leq |a_n-a|+|b_n-b| < \varepsilon+\varepsilon = 2\varepsilon$$

$$\Rightarrow \lim_n (a_n+b_n) = a+b$$

3) $|(a_n b_n)-(ab)| = |\underbrace{a_n b_n - a_n b} + \underbrace{a_n b - ab}| =$

$= |a_n(b_n-b)+b(a_n-a)| \leq |a_n(b_n-b)|+|b(a_n-a)|$

$= |a_n|\,|b_n-b| + |b|\,|a_n-a|$

WE KNOW $\exists M>0$ SUCH THAT $|a_n| \leq M$ $\forall n$

FIX $\varepsilon>0$ $\forall n \geq n_0 = \max\{n_1, n_2\}$ WE HAVE

$|a_n b_n - ab| \leq |a_n|\,|b_n-b| + |b|\,|a_n-a| <$

$M\varepsilon + |b|\varepsilon = \underbrace{(M+|b|)}_{c}\varepsilon \Rightarrow \lim_n a_n b_n = ab$

4) $b \neq 0$

- $b>0$ $\exists \tilde{n} \in \mathbb{N}$ S.T. $\forall n \geq \tilde{n}$ $b_n \geq b/2 > 0$
- $b<0$ $\exists \tilde{n} \in \mathbb{N}$ S.T. $\forall n \geq \tilde{n}$ $b_n \leq b/2 < 0$

IN BOTH CASES $\forall n \geq \tilde{n}$ WE HAVE

$$\frac{1}{|b_n|} \le \frac{2}{|b|} \quad \text{TAKE } n \ge \tilde{n}$$

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b - b_n}{b_n b}\right| = \frac{|b_n - b|}{|b_n||b|} \le \frac{2|b_n - b|}{|b|^2}$$

FIX $\varepsilon > 0$ THEN $\forall n \ge n_0 = \max\{\tilde{n}, n_2\}$ WE HAVE

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| \le \frac{2}{|b|^2}|b_n - b| < \underbrace{\frac{2}{|b|^2}}_{c}\varepsilon \Rightarrow \lim_n \frac{1}{b_n} = \frac{1}{b}$$

EXAMPLE:

- $\lim_n \frac{1}{n} = 0 \Rightarrow \lim_n \left(\frac{1}{n}\cdot\frac{1}{n}\right) \lim_n \frac{1}{n^2} = 0$

BY INDUCTION $\lim_n \frac{1}{n^k} = 0 \quad \forall k \in \mathbb{N}$

- $\lim_n \frac{1 - 3/n}{5 + 2/n^3} = \frac{1}{5}$ $\quad \frac{3}{n} \to 0 \quad \frac{2}{n^3} \to 0$

$1 - \frac{3}{n} \to 1 \quad 5 + \frac{2}{n^2} \to 5 \ne 0$

- $\lim_n \frac{n^3 - 3n + 1}{n^3 + n^2} = 1$

$$\frac{\cancel{n^3}(\overbrace{1 - 3/n^2 + 1/n^3}^{\to 1})}{\cancel{n^3}\underbrace{(1 + 1/n)}_{1}}$$

$$\bullet \lim_{n} \frac{-2n^4-5}{5n^3+2} = 0$$

$$\frac{n^4\left(-2-5/n^4\right)}{n^3\left(5+2/n^3\right)} = \underbrace{\frac{1}{n}}_{\overset{\shortparallel}{0}} \frac{\overbrace{\left(-2-5/n^4\right)}^{-2}}{\underbrace{\left(5+2/n^3\right)}_{5}} = 0\,\frac{(-2)}{5} = 0$$

$$\lim_{n} \frac{-2n^4+5}{n^3+3n} = ?$$

$$\frac{\cancel{n^4}\overbrace{\left(-2+5/n^4\right)}^{-2}}{\cancel{n^3}\underbrace{\left(1+3/n^2\right)}_{1}} = -\infty$$

OTHER CALCULUS RULES FOR SEQUENCES

DEF: $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$; $a \in \mathbb{R}$

- a_n CONVERGES TO a FROM THE RIGHT, WRITE IT

  $a_n \to a^+$ AS $n \to +\infty$, IF

  $a \to a$ AND $A_n > a$ DEFINITELY

- a_n CONVERGES TO a FROM THE LEFT, WRITE IT

  $a_n \to a^-$ AS $n \to +\infty$, IF

  $a_n \to a$ AND $a_n < a$ DEFINITELY

THEOREM $\{a_n\}_{n\in\mathbb{N}}$ $\{b_n\}_{n\in\mathbb{N}}$ SEQ. OF REAL NUMBERS

1) $a_n \to +\infty \iff -a_n \to -\infty$

2) $a_n \to \pm\infty \Rightarrow |a_n| \to +\infty$

3) $a_n \to 0^+ \iff -a_n \to 0^-$

4) $a_n \to 0 \iff |a_n| \to 0$ $(d(a_n, 0) \to 0)$

5) $\{a_n\}$ BOUNDED (OR CONVERGING), $b_n \to \pm\infty$

$\Rightarrow a_n + b_n \to \pm\infty$

PROOF WE KNOW $\exists K > 0$ S.T. $|a_n| \leq K$ $\forall n$

$\forall M > 0$ $\exists n_0 \in \mathbb{N}$ S.T. $\forall n \geq n_0$ $b_n > M$ $(+\infty)$

WE WANT TO SHOW THAT

$\forall M_1 > 0$ $\exists n_1 \in \mathbb{N}$ S.T. $\forall n \geq n_1$ $a_n + b_n > M_1$

PICK $M = M_1 + K$ THEN $\forall n \geq n_0$

$a_n + b_n > -K + M = -K + M_1 + K = M_1$ $(n_1 = n_0)$

EXAMPLE NOTE THAT $\nexists \lim_n \sin(n)$; $\nexists \lim_n \cos(n)$

$a_n = \sin n$ $b_n = n$ $|\sin n| \leq 1$ $\forall n \in \mathbb{N}$ HENCE

$$\lim_n (\sin(n) \pm n) = \pm\infty$$

$$\lim_n (n^2 + 3\sin(n)) = +\infty; \lim_n (-n + 2\cos^2(n)) = -\infty$$

6) $\{a_n\}_{n\in\mathbb{N}}$ CONVERGING AND $\{b_n\}_{n\in\mathbb{N}}$ IRREGULAR

$\Rightarrow a_n + b_n$ IS IRREGULAR

PROOF: IF $a_n + b_n$ IS REGULAR, THEN

$b_n = (a_n + b_n) - a_n$ WOULD BE REGULAR AS WELL

EXAMPLE: $\nexists \lim_n (3 + \sin(n))$; $\nexists \lim_n \left(\cos(n) - \frac{1}{n^2+1}\right)$

7) $a_n \to \pm\infty$ AND $b_n \to \pm\infty \Rightarrow a_n + b_n \to \pm\infty$

EXAMPLE $\lim_n (n^3 + 3n + 2) = +\infty$

$$\lim_n (-\sqrt{n^2+1} - n) = -\infty$$

NOTE THAT $\sqrt{n^2+1} \geq n = \sqrt{n^2} \to +\infty$

BY COMPARISON $\sqrt{n^2+1} \to +\infty$

8) $a_n \to l \in [-\infty, +\infty]$ $l \neq 0$ $b_n \to \pm\infty$

$$\Rightarrow a_n b_n \to \begin{cases} \pm\infty & \text{IF } l>0 \\ \mp\infty & \text{IF } l<0 \end{cases}$$

EXAMPLE: $\lim_n n = +\infty \Rightarrow \lim_n n \cdot n = \lim_n n^2 = +\infty$

BY INDUCTION $\lim_n n^k = +\infty$ $\forall k \in \mathbb{N}$

- $P_k(n) = a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n + a_0$ $k \in \mathbb{N}$ $k \geq 1$ $a_k \neq 0$

$$\lim_{n \to +\infty} P_k(n) = \begin{cases} +\infty & \text{IF } a_k > 0 \\ -\infty & \text{IF } a_k < 0 \end{cases}$$

IN FACT

$$P_k(n) = \underset{\downarrow\, a_k}{a_k} \underset{\downarrow\, +\infty}{n^k} \underbrace{\left(1 + \frac{a_{k-1}}{a_k} \frac{1}{n} + \dots + \frac{a_0}{a_k} \overset{\nearrow 0}{\frac{1}{n^k}}\right)}_{\downarrow\, 1}$$

- $\lim_n \left(\dfrac{n - 2 + 3/n^2}{1 + 3/n^2} \right) = -\infty$

- $\lim_n \underset{\downarrow\, -\infty}{(-n^3)} \underset{\downarrow\, -1}{\left(-1 + \frac{4}{n^3}\right)} = +\infty$

9) $a_n \to \pm\infty$ $\Rightarrow \frac{1}{a_n} \to 0^{\pm}$
(INFINITE) (INFINTESIMAL)

PROOF:

FIX $\varepsilon > 0$ $\exists n_0 \in \mathbb{N}$ S.T. $\forall n \geq n_0$ $a_n > 1/\varepsilon = M$

SO $0 < \frac{1}{a_n} < \varepsilon$

10) $a_n \neq 0$ DEFINITELY AND $a_n \to 0 \Rightarrow \frac{1}{|a_n|} \to +\infty$

IN PARTICULAR

$a_n \to 0^{\pm} \Rightarrow \frac{1}{a_n} \to \pm\infty$

EXAMPLE:

$a_n = (-1)^n \cdot \frac{1}{n} \to 0$ BUT $\frac{1}{a_n} = (-1)^n \cdot n$

IS IRREGULAR!