# LIMIT OF A MONOTONE SEQUENCE

IF A SEQUENCE IS (EVENTUALLY) MONOTONE AND BOUNDED

THEN IT IS CONVERGENT

PROVE THAT: IF A SEQUENCE IS INCREASING AND BOUNDED ABOVE

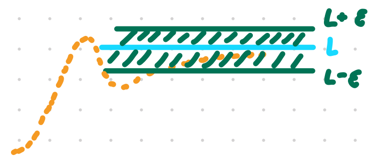
THEN IT IS CONVERGENT

- ASSUME $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ IS AN INCREASING SEQUENCE AND BOUNDED ABOVE

PROVE IT IS CONVERGENT

$$\forall \varepsilon>0,\ \exists N_0\in\mathbb{N} \text{ s.t. } \quad \forall n\in\mathbb{N},\ n\geq N_0 \Rightarrow L-\varepsilon < a_n < L+\varepsilon$$

BUT $L$ IS NOT GIVEN TO ME AS PART OF THE STATEMENT OF THE PROOF




THE DEFINITION THAT A SEQUENCE IS CONVERGENT IS THAT THERE $\exists$ A REAL NUMBER $L$ SUCH THAT...

$$\forall \varepsilon>0,\ \exists N_0\in\mathbb{N} \text{ s.t. } \quad \forall n\in\mathbb{N},\ n> N_0 \Rightarrow L-\varepsilon < a_n < L+\varepsilon$$

ALL THE TERMS OF $a_n$

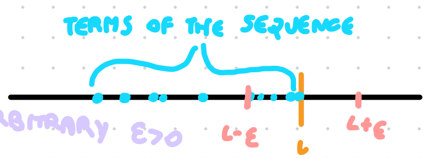
CONSIDER THE SET $A=\{a_n \mid n\in\mathbb{N}\}$

BY THE LEAST UPPER BOUND AXIOM

IT IS NON-EMPTY AND BOUNDED ABOVE, SO IT HAS A SUPREMUM

$L=$ SUP OF THE SEQUENCE ($L=\sup A$)

WE WANT TO PROVE THAT $L=\lim\limits_{n\to\infty} a_n$



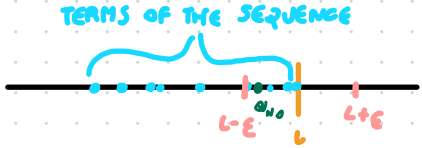

FIX AN ARBITRARY $\varepsilon>0$

TAKE AN INTERVAL OF RADIUS $\varepsilon$ (WITH $\varepsilon>0$) AROUND $L$

$L-\varepsilon$ IS NOT AN UPPER BOUND OF THE SEQUENCE

BY DEFINITION OF SUPREMUM

SO $\boxed{\exists N_0 \in \mathbb{N} \text{ s.t. } L-\varepsilon < a_{N_0}} \leq L$




FIX $N \in \mathbb{N}$. ASSUME $N \geq N_0$ SO $L-\varepsilon < a_N < L+\varepsilon$

SO THERE MUST EXIST ONE TERM OF THE SEQUENCE IN GREEN BEWEEN $L-\varepsilon$ AND $L$, BUT BECAUSE THE SEQUENCE IS INCREASING ALL TERMS AFTER $a_{N_0}$ MUST BE GREATER THAN $a_{N_0}$ AND LESS THAN $L$

$$\Downarrow$$

$$\forall N \geq N_0 \quad a_{N_0} \leq a_N \leq L$$

- WE KNOW $L-\varepsilon < a_{N_0}$
- BECAUSE THE SEQUENCE IS INCREASING $a_{N_0} \leq a_N$
- BY DEFINITION OF SUPREMUM $a_N \leq L$

THUS

$$L-\varepsilon < a_{N_0} \leq a_N < L < L+\varepsilon$$

$$\Downarrow$$

$$L-\varepsilon < a_N < L+\varepsilon$$

**Proof.**

- Let $\{a_n\}_{n=0}^{\infty}$ be an increasing, bounded above sequence.
- Consider the set $\mathcal{A} = \{ a_n \mid n \in \mathbb{N} \}$.
  It is non-empty and bounded above, so it has a supremum.
- I take $L = \sup \mathcal{A}$. I will prove that $L = \lim_{n \to \infty} a_n$.
- Fix an arbitrary $\varepsilon > 0$.
- By definition of supremum, $\exists n_0 \in \mathbb{N}$ s.t. $L - \varepsilon < a_{n_0}$.
  I take that value of $n_0$.
- Fix $n \in \mathbb{N}$. Assume $n \geq n_0$. WTS $L - \varepsilon < a_n < L + \varepsilon$.
  - We know $L - \varepsilon < a_{n_0}$
  - Because the sequence is increasing, $a_{n_0} \leq a_n$.
  - By definition of supremum, $a_n \leq L$

  Thus

$$L - \varepsilon < a_{n_0} \leq a_n \leq L$$