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$-$ $\div$

EXERCISE I am at a supermarket, queuing at the checkout. I am at queue number 0, which is completely stuck. I estimate that the speed of the other queues is proportional to their distance from mine, namely $v(x) = \alpha x$, where $v(x)$ is the velocity (number of persons per unit time) of the x-th queue, and α is a positive parameter. I want to move to another queue; my walking speed is 1 queues per second. All queues are d persons long, and always stay d persons long, due to new people arriving. Which queue should I choose if I am very hungry?

HINT: pretend that all variables are continuous, and solve the appropriate variational principle.

NUMBER CHECK

NUMBER of PEOPLE in QUEUE

$$T(x) = x + \frac{d}{V(x)}$$

DEQUEUING TIME

$$T'(x) = \frac{d}{dx}\left(x + \frac{d}{V(x)}\right) = 1 - \frac{d}{\alpha x^2} = \frac{\alpha x^2 - d}{\alpha x^2}$$

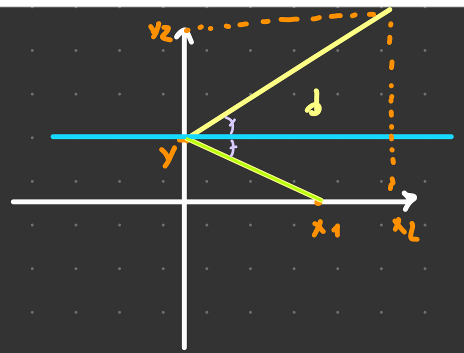
$V(x) \to \alpha x$

STATIONARY POINT

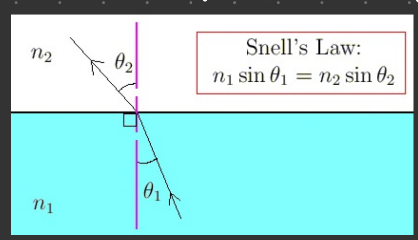
$$T'(x) = 0 \Rightarrow \alpha x^2 - d = 0 \Rightarrow x = \pm\sqrt{\frac{d}{\alpha}}$$

SOLUTION: EITHER $\lfloor x \rfloor$ or $\lceil x \rceil$

EXERCISE [difficult] We saw in class how Snell's law of refraction can be obtained from Fermat's principle of minimum time. From the same principle, obtain the law of reflection, namely the fact that the incident and reflected rays form the same angle with respect to the surface. Consider the 2-dimensional case, where a ray of light starting at position $(x_1, 0)$ needs to reach the point at position (x_2, y_2) in the least amount of time, by reflecting off the y axis at the point $(0, y)$.



SNELL LAW $n_1 \sin\theta_1 = n_2 \sin\theta_2$



$$d = \sqrt{x_1^2 + y^2} + \sqrt{x_2^2 + (y_2 - y)^2}$$

MINIMIZE WITH RESPECT TO Y

$$d' = \frac{-\frac{1}{\cancel{2}}\cancel{2}y}{\sqrt{x_1^2+y^2}} + \frac{-\frac{1}{\cancel{2}}\cancel{2}(y_2-y)}{\sqrt{x_2^2+(y_2-y)^2}} = \frac{-y}{\sqrt{x_1^2+y^2}} - \frac{y_2 - y}{\sqrt{x_2^2+(y_2-y)^2}}$$

$$\frac{y}{\sqrt{x_1^2+y^2}} = \frac{y_2 - y}{\sqrt{x_2^2+(y_2-y)^2}}$$

EXERCISE An optimization principle requires the minimization of the function $f(x) = x(x - \alpha)$, where the variable $x \geq 0$ is non-negative and α is a real parameter. What is the minimum, $\bar{x}$, as a function of α?

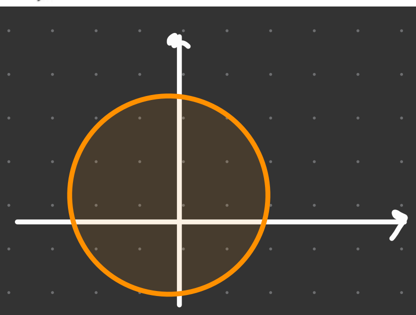
$f(x) = x(x - \alpha)$ $x \geq 0$ MINIMUM

$f(x) = x^2 - \alpha x$

$f'(x) = 2x - \alpha = 0$; $\alpha = 2x$; $\boxed{\bar{x} = \frac{\alpha}{2}}$ MIN

EXERCISE A firm can allocate between 0 and 1M€ to research and development this year. The expected additional revenue, as a function of this investment x, is estimated to be βx^2, where $\beta \geq 0$ is a real parameter that depends on market conditions (its dimensions are 1/M€). What capital should be invested to maximize cash flow this year? Is there a critical value of β above which the optimal strategy changes abruptly? (Define cash flow simply as revenue minus expenses, and suppose that the only expenses are those for R&D.)

0 - 1M€ $\qquad \beta x^2 \downarrow \qquad \frac{1}{M€}$



$y = 6x^2$

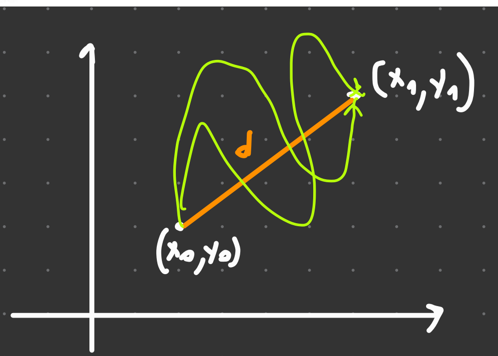
$0 < x < 1000000$

INVESTED

CASH FLOW: $y = 6x^2 - x$

MAXIMISE CASH FLOW: $y' = 26x - 1 = 0; \quad x = \frac{1}{26};$

EXERCISE [partially seen in class] Write the Euler-Lagrange equation expressing the minimization of the length of a function whose graph connects two points (x_0, y_0), (x_1, y_1). Solve the equation to find that the shortest path between the two is a straight line.



EULER EQUATION:

$d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$

$L(x, y, \dot{x}, \dot{y})$