17/MAY/2022

THEOREM $A \in \mathbb{R}^{N \times N}$. LET $\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_N}\} \in \mathbb{R}^N$ BE EIGENVECTORS OF A ASSOCIATED TO DIFFERENT EIGENVALUES $\{\lambda_1, \lambda_2, \dots \lambda_N\} \in \mathbb{R}$ THEN $\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_N}\}$ ARE LINEARLY INDIPENDENT

DEFINITION TWO MATRIXES $A, B \in \mathbb{R}^{N \times N}$ ARE SAID TO BE SIMILAR IF IT EXIST $P \in \mathbb{R}^{N \times N}$ INVERTIBLE SUCH THAT

$\boxed{B = PAP^{-1}}$ (OR EQUIVALENTLY $A = P^{-1} B P$)

$$P^{-1}B = \underbrace{\cancel{P^{-1}P}}_{I}AP^{-1}$$

$$P^{-1}BP = A\underbrace{P^{-1}P}_{I}$$

$$P^{-1}BP = A$$

THEOREM IF A AND $B \in \mathbb{R}^{N \times N}$ ARE SIMILAR, THEN THEY HAVE THE SAME EIGENVALUES (AND THE SAME CHARACTERISTIC POLYNOMIA)

PROOF $\det(A - \lambda I) = 0$ $\quad \det(B - \lambda I) = 0$ $\quad$ $B = PAP^{-1}$, $P \in \mathbb{R}^{N \times N}$

$$B - \lambda I = PAP^{-1} - \lambda I = PAP^{-1} - \lambda PP^{-1} = P(A - \lambda I)P^{-1}$$

$$\det(P)\det(P^{-1}) = \det(PP^{-1}) = \det(I) = 1$$

$$\hookrightarrow \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \cdot 1 \cdot \ldots 1$$

$$\det(B - \lambda I) = \det(P(A - \lambda I)P^{-1}) = \det(P) = \det(A - \lambda I)\det(P^{-1}) = \det(A - \lambda I)$$

JUST NUMBER

$D \in \mathbb{R}^{N \times N}$ DIAGONAL

$\underline{x} \in \mathbb{R}^N$

$$D\underline{x} = \overbrace{\begin{bmatrix} d_{11} & & & 0 \\ & d_{22} & & \\ & & d_{33} \ddots & \\ 0 & & & d_{NN} \end{bmatrix}}^{D} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} d_{11}x_1 \\ d_{22}x_2 \\ d_{33}x_3 \\ \vdots \\ d_{NN}x_N \end{bmatrix}$$

$A \in \mathbb{R}^{N \times N}$ $\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_N}\}$ EIGENVECTOR LINEARLY INDIPENDENT

$\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_N}\}$ ARE A BASIS FOR $\mathbb{R}^N$

$\underline{w} = c_1\underline{v_1} + c_2\underline{v_2} + \dots + c_N\underline{v_N}$ $\quad \{c_1, c_2, \dots, c_N\} \in \mathbb{R}$

COORDINATES

$$A\underline{w} = A(c_1\underline{v_1} + c_2\underline{v_2} + \dots + c_N\underline{v_N}) = c_1A\underline{v_1} + c_2A\underline{v_2} + \dots c_NA\underline{v_N} = c_1\lambda_1\underline{v_1} + c_2\lambda_2\underline{v_2} + \dots + c_N\lambda_N\underline{v_N}$$

DEFINITION $A \in \mathbb{R}^{N \times N}$ IS SAID TO DIAGONALIZABLE IF IT IS SIMILAR TO A DIAGONAL MATRIX, IN OTHER WORDS IF IT EXISTS $D \in \mathbb{R}^{N \times N}$ DIAGONAL, AND $P \in \mathbb{R}^{N \times N}$ INVERTIBLE SUCH THAT $A = PDP^{-1}$

$$A = PDP^{-1} \quad (D = P^{-1}AP)$$

$P = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ $D = \begin{bmatrix} 7 & 0 \\ 0 & -2 \end{bmatrix}$

$P^{-1} \frac{1}{6} \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$ $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & -2 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 21 & -2 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 42 & -27 \\ 0 & -12 \end{bmatrix} = \begin{bmatrix} 7 & -9/2 \\ 0 & -2 \end{bmatrix}$

THEOREM $A \in \mathbb{R}^{n \times n}$ IS DIAGONIZABLE IF AND ONLY IF ONE CAN FIND $\{\underline{v_1}, \underline{v_2}, \dots \underline{v_n}\}$ EIGENVECTORS OF A WHICH ARE LINEARLY INDEPENDENT (WHICH IS LIKE SAYING THAT WE CAN FIND A BASIS OF $\mathbb{R}^n$ MADE OF EIGENVECTORS OF A)

MORE PRECISELY WE CAN WRITE $A = PDP^{-1}$ IF AND ONLY IF $P = [\underline{v_1}, \underline{v_2}, \dots \underline{v_n}] \in \mathbb{R}^{n \times n}$ AND $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$ (ORDERED IN THE SAME WAY)

PROOF $(\Rightarrow)$ $\{\underline{v_1}, \underline{v_2}, \dots \underline{v_n}\} \in \mathbb{R}^n$ EIGENVECTORS OF A, LINEARLY INDIPENDENT (BASIS OF $\mathbb{R}^n$)

$P = [\underline{v_1}, \underline{v_2}, \dots \underline{v_n}] \in \mathbb{R}^{n \times n}$ IS INVERTIBLE $\quad P\underline{x} = \underline{0}$ HAS ONLY THE TRIVIAL SOLUTION $\underline{x} = \underline{0} \iff P$ INVERTIBLE

$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$ $\quad A \overset{?}{=} PDP^{-1}$ IS THE SAME AS $AP \overset{?}{=} PD$

$AP = A[\underline{v_1}, \underline{v_2}, \dots \underline{v_n}] = [A\underline{v_1}, A\underline{v_2}, \dots A\underline{v_n}] = [\lambda_1\underline{v_1}, \lambda_2\underline{v_2}, \dots, \lambda_n\underline{v_n}]$

$PD = [\underline{v_1}, \underline{v_2}, \dots \underline{v_n}] \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix} = [\lambda_1\underline{v_1}, \lambda_2\underline{v_2}, \dots, \lambda_n\underline{v_n}]$ $\quad AP = PD$

OBSERVATION

$\{\underline{v_1}, \underline{v_2}, \dots \underline{v_n}\}$ LIN. INDEP. EIGENVALUE OF A

$\underline{x} \in \mathbb{R}^n$ $\quad \underline{x} = c_1\underline{v_1} + c_2\underline{v_2} + c_3\underline{v_3} + \dots + c_n\underline{v_n} = P\underline{c}$ $\quad \underline{c} \in \mathbb{R}^n = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

$\underline{c} = P^{-1}\underline{x}$ $\quad A = PDP^{-1}$

EXAMPLE

$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$
1) IS DIAGONIZABLE
2) IF YES WRITE IT IN THE DIAGONAL FORM (FIND D, P)

THE MATRIX IS DIAGONIZABLE IF AND ONLY IF I CAN FIND $\{\underline{v_1}, \underline{v_2}\}$ EIGENVECTORS AND LIN. INDIPENDENDENT

FIND THE EIGENVALUES

$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda) - 30 = \lambda^2 - 3\lambda + 28$ $\quad \lambda = 7$, $\lambda = -4$

IS IT DIAGONIZABLE
↓
THERE ARE 2 EIGENVECTOR THAT ARE LINEARLY INDIPENDENT (SEE BEFORE)

$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix}$ $\quad P = [\underline{v_1}, \underline{v_2}]$

- FIND $\underline{v_1}$ AND $\underline{v_2}$ BY SOLVING $(A - \lambda I)\underline{x} = \underline{0}$

$\boxed{v_1}$ $(A-\lambda I)=(A-7I)=\begin{pmatrix}-6 & 6\\ 5 & -5\end{pmatrix}\underline{x}=0$ $\quad \begin{pmatrix}-6 & 6\\ 5 & -5\end{pmatrix}\underline{x}\Big) -5/6 = \begin{bmatrix}-6 & 6\\ 0 & 0\end{bmatrix}\underline{x}=0$ $\quad -6x_1+6x_2;\ x_1=x_2$ $\quad \begin{cases} x_1=x_2\\ x_2=x_2\in\mathbb{R}\end{cases}$ $\quad \underline{x}=\begin{bmatrix}1\\1\end{bmatrix}x_2 \quad x_2\in\mathbb{R}$

FREE VARIABLE

$\boxed{v_2}$ $(A-\lambda I)=(A+4I)=\begin{pmatrix}5 & 6\\ 5 & 6\end{pmatrix}\underline{x}=0$ $\quad \begin{pmatrix}5 & 6\\ 5 & 6\end{pmatrix}\underline{x}\Big) 1 = \begin{bmatrix}5 & 6\\ 0 & 0\end{bmatrix}\underline{x}=0$ $\quad 5x_1+6x_2=0;\ x_1=-\frac{6}{5}x_2$ $\quad \begin{cases} x_1=-\frac{6}{5}x_2\\ x_2=x_2\in\mathbb{R}\end{cases}$ $\quad \underline{x}=\begin{bmatrix}-6/5\\1\end{bmatrix}x_2 \quad x_2\in\mathbb{R}$

FREE VARIABLE

SO $P=[\underline{v_1},\underline{v_2}]=\begin{bmatrix}1 & -6/5\\ 1 & 1\end{bmatrix}$ $\qquad A=PDP^{-1}$

---

① FIND THE EIGENVALUES OF A ($\det(A-\lambda I)=0$ FIND OF ROOTS)

[IF THE SUM OF THE ROOT MULTIPLICITY IS $< N$ THEN A IS NOT DIAGONIZABLE]

② FOR EVERY EIGENVALUE FIND A BASIS FOR THE ASSOCIATED EIGENSPACE

(WHICH MEANS FINDING A BASIS FOR $(A-\lambda I)\underline{x}=\underline{0}$). THE DIMENSION OF THE EIGENSPACE IS EQUAL TO THE NUMBER OF VECTOR IN THE BASIS

③ IF THE SUM OF THE DIMENSION OF ALL THE EIGENSPACE IS N THEN A IS DIAGONIZABLE, OTHERWISE IS NOT

④ IF YES $\quad A=PDP^{-1} \quad P=[\underline{v_1},\underline{v_2},\dots \underline{v_n}] \quad D=\begin{bmatrix}\lambda_1 & & 0\\ & \lambda_2 & \\ 0 & & \ddots\ \lambda_n\end{bmatrix}$

$A=\begin{bmatrix}2 & 2 & -1\\ 1 & 3 & -1\\ -1 & -2 & 2\end{bmatrix}$ $\quad \lambda=\{1,5\}$ ARE ALL EIGENVALUES OF A

- IS A DIAGONIZABLE?
- IF YES, FIND P, D

$\boxed{\lambda=1}$ $\begin{bmatrix}1 & 2 & -1\\ 1 & 2 & -1\\ -1 & -2 & 1\end{bmatrix}\underline{x}=0 \quad \Big)1\Big)_{-1} = \begin{bmatrix}1 & 2 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\underline{x}=0$ $\quad x_1+2x_2-x_3=0;\ x_1=-2x_2+x_3$ $\quad \begin{cases}x_1=-2x_2+x_3\\ x_2=x_2\in\mathbb{R}\\ x_3=x_3\in\mathbb{R}\end{cases}$ $\quad \underline{x}=\begin{bmatrix}-2\\1\\0\end{bmatrix}x_2+\begin{bmatrix}1\\0\\1\end{bmatrix}x_3$

FREE VARIABLES $\Rightarrow$ DIM = 2

$\lambda=1$ $\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$ DIM = 2

$\boxed{\lambda=5}$ $\begin{bmatrix}-3 & 2 & -1\\ 1 & -2 & -1\\ -1 & -2 & -3\end{bmatrix}\underline{x}=0 \quad \begin{bmatrix}1 & -2 & -1\\ -3 & 2 & -1\\ -1 & -2 & -3\end{bmatrix}\underline{x}=0 \quad \Big)\Big)3\Big)_{-1} \quad \begin{bmatrix}1 & -2 & -1\\ 0 & -4 & -4\\ 0 & 0 & 0\end{bmatrix}\underline{x}=0$

$x_1-2x_2-x_3=0;\ x_1=-x_3$

$-4x_2-4x_3=0;\ x_2=-x_3$

$\begin{cases}x_1=-x_3\\ x_2=-x_3\\ x_3=x_3\in\mathbb{R}\end{cases}$ $\quad \underline{x}=\begin{bmatrix}-1\\-1\\1\end{bmatrix}x_3$

ONE FREE VARIABLE $\Longrightarrow$ DIM = 1

$\lambda=5$ $\left\{\begin{bmatrix}-1\\-1\\1\end{bmatrix}\right\}$ DIM = 1

SINCE $\lambda=1$ HAVE DIM = 2

THERE IS DOUBLE MULTIPLICITY

$D=\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 5\end{bmatrix}$

$P=[\underline{v_1},\underline{v_2},\underline{v_3}]$ $\begin{bmatrix}-2 & 1 & -1\\ 1 & 0 & -1\\ 0 & 1 & 1\end{bmatrix}$

THE ORDER MUST BE THE SAME ($\lambda=5$ CAN BE THE FIRST COLUMN IN D, BUT MUST BE SAME COLUMN ALSO IN P)

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

- IS A DIAGONIZABLE?
- IF YES, FIND P, D

FIRST FIND THE EIGENVALUES

$\boxed{2, -3}$ BECAUSE IS AN UPPER TRIANGULAR, LOOK AT JUST THE DIAGONAL

$\boxed{-3}$

$$\begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

ALREADY IN EF

$5x_1 + x_2 = 0; \; x_1 = 0$

$5x_2 = 0; \; x_2 = 0$

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = x_3 \in \mathbb{R} \end{cases}$$

$$\underline{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x_3 \quad x_3 \in \mathbb{R}$$

FREE VARIABLE $\Rightarrow$ DIM $= 1$

$\boxed{2}$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

FREE VARIABLE $\Rightarrow$ DIM $= 1$

$1 + 1 = 2$ THAT IS LESS THAN 3

$\Downarrow$

A IS NOT DIAGONIZABLE

EX FOR TRAINING

$$\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$$

- FIND EIGENVALUES ($\lambda = 1, 2, 3$)
- IS DIAGONIZABLE?
- IF YES FIND P, D