Problem 4. (10 points) Let $(X, Y) : \Omega \to \mathbb{R}^2$ be a random vector having Gaussian distribution with mean vector $(m_X, m_Y) = (2, -1)$ and covariance matrix

$$C = \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}.$$

Find:

4a) the moment generating function and the density of $Z := X + 3Y$ [2 points]

4b) $\mathsf{Cov}[X + 3Y, X - Y]$ [1 point]

4c) the joint density of the random vector $(U, V) := (2X - Y, X + Y)$ [3 points]

4d) the conditional density of U given V [3 points]

4e) the conditional expectation $\mathbb{E}[U \mid V = v]$ [1 point]

$\sigma_X^2 = VAR(X) = 4$

$\sigma_Y^2 = VAR(Y) = 1$

$\rho\sigma_X\sigma_Y = COV(X,Y) = 1$

$m_X = E[X] = 2$

$m_Y = E[Y] = -1$

1° MOMENT GENERATING FUNCTION $Z = X+3Y$

$E[Z] = E[X+3Y] = E[X] + 3E[Y] = 2+3(-1) = \boxed{-1}$

$VAR(Z) = VAR(X+3Y) = VAR(X) + 9VAR(Y) + 2\cdot 3COV(X,Y) = 4+9\cdot 1+6\cdot 1 = \boxed{19}$

FORMULA: $M_Z = \exp\left\{t\mu + \frac{1}{2}\sigma^2 t^2\right\} \Rightarrow M_Z = \exp\left\{-1\cdot t + \frac{1}{2}\cdot 19t^2\right\} = \exp\left\{-t + \frac{19}{2}t^2\right\}$

2° $COV[X+3Y, X-Y] = COV(X,X) + COV(X,-Y) + COV(3Y,X) + COV(3Y,-Y) = VAR(X) - COV(X,Y) + 3COV(X,Y) - 3VAR(Y) = VAR(X) + 2COV(X,Y) - 3VAR(Y) = 4+2\cdot 1-3\cdot 1 = \boxed{3}$

3° JOINT DENSITY OF $(U,V) = (2X-Y, X+Y)$

YOU NEED TO FIND

I MEAN VECTOR $\underline{m}$

II COV MATRIX $\underline{\underline{C}}$

I • TO FIND THE MEAN VECTOR FIRST YOU NEED TO FIND R

$R = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$

• THEN FIND $\underline{m}$

$m = R\begin{pmatrix} m_X \\ m_Y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4+1 \\ 2-1 \end{pmatrix} = \boxed{\begin{pmatrix} 5 \\ 1 \end{pmatrix}}$

II FIND $\underline{\underline{C}} = RCR^T$

$\underline{\underline{C}} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}^T = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2\cdot 4-1 & 2\cdot 1-1\cdot 1 \\ 1\cdot 4+1\cdot 1 & 1\cdot 1+1\cdot 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 7 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 7\cdot 2-1 & 7\cdot 1+1\cdot 1 \\ 5\cdot 2-2 & 5\cdot 1+2\cdot 1 \end{pmatrix} = \begin{pmatrix} 13 & 8 \\ 8 & 7 \end{pmatrix}$

FIND $f(U,V)$

$f_{(X,Y)}(x,y) = \frac{1}{2\pi}\frac{1}{\sqrt{\det \underline{\underline{C}}}} \exp\left\{-\frac{1}{2}\begin{pmatrix} x-m_X \\ y-m_Y \end{pmatrix}^T C^{-1}\begin{pmatrix} x-m_X \\ y-m_Y \end{pmatrix}\right\}$

$\Downarrow$

$f_{(U,V)}(u,v) = \frac{1}{2\pi}\frac{1}{\sqrt{\det \underline{\underline{C}}}} \exp\left\{-\frac{1}{2}\begin{pmatrix} U-m_U \\ V-m_V \end{pmatrix}^T C^{-1}\begin{pmatrix} U-m_U \\ V-m_V \end{pmatrix}\right\}$

⚠ USE x INSTEAD OF U, y INSTEAD V

• DET $\underline{\underline{C}} \Rightarrow \det(C) = (13\cdot 7) - (8\cdot 8) = \boxed{27}$

• $\begin{pmatrix} U-m_U \\ V-m_V \end{pmatrix}^T = \begin{pmatrix} U-5 \\ V-1 \end{pmatrix}^T = (U-5, V-1)$

• $C^{-1} = \begin{pmatrix} 13 & 8 \\ 8 & 7 \end{pmatrix}^{-1} = \frac{1}{27}\begin{pmatrix} 7 & -8 \\ -8 & 13 \end{pmatrix} = \begin{pmatrix} 7/27 & -8/27 \\ -8/27 & 13/27 \end{pmatrix}$

• $\begin{pmatrix} U-m_U \\ V-m_V \end{pmatrix} = \begin{pmatrix} U-5 \\ V-1 \end{pmatrix}$

$f_{(U,V)}(u,v) = \frac{1}{2\pi 3\sqrt{3}} \exp\left\{-\frac{1}{2}(U-5, V-1)\begin{pmatrix} 7/27 & -8/27 \\ -8/27 & 13/27 \end{pmatrix}\begin{pmatrix} U-5 \\ V-1 \end{pmatrix}\right\} = \frac{1}{2\pi 3\sqrt{3}} \exp\left\{\left(\frac{U-5}{2}, \frac{V-1}{2}\right)\begin{pmatrix} 7/27 & -8/27 \\ -8/27 & 13/27 \end{pmatrix}\begin{pmatrix} U-5 \\ V-1 \end{pmatrix}\right\} =$

$= \frac{1}{2\pi 3\sqrt{3}} \exp\left\{\left(\frac{U-5}{2}\cdot\frac{7}{27} + \frac{V-1}{2}\cdot\left(-\frac{8}{27}\right), \frac{U-5}{2}\cdot\left(-\frac{8}{27}\right) + \frac{V-1}{2}\cdot\left(\frac{13}{27}\right)\right)\begin{pmatrix} U-5 \\ V-1 \end{pmatrix}\right\} =$

$= \frac{1}{2\pi 3\sqrt{3}} \exp\left\{\left(\frac{U-5}{2}\cdot\frac{7}{27} + \frac{V-1}{\cancel{2}_1}\cdot\left(-\frac{\cancel{8}^4}{27}\right); \frac{U-5}{\cancel{2}_1}\cdot\left(-\frac{\cancel{8}^4}{27}\right) + \frac{V-1}{2}\cdot\left(\frac{13}{27}\right)\right)\begin{pmatrix} U-5 \\ V-1 \end{pmatrix}\right\} =$

$= \frac{1}{2\pi 3\sqrt{3}} \exp\left\{\left(\frac{7U-35-8V+8}{54}; \frac{-8U+40+13V-13}{54}\right)\begin{pmatrix} U-5 \\ V-1 \end{pmatrix}\right\} = \frac{1}{2\pi 3\sqrt{3}} \exp\left\{\left(\frac{7U-8V-27}{54}; \frac{-8U+13V+27}{54}\right)\begin{pmatrix} U-5 \\ V-1 \end{pmatrix}\right\} =$

$= \frac{1}{2\pi 3\sqrt{3}} \exp\left\{\frac{7U-8V-27}{54}(U-5) + \frac{-8U+13V+27}{54}(V-1)\right\} = \frac{1}{2\pi 3\sqrt{3}} \exp\left\{\frac{7U^2-8UV-27U-35U+40V+135}{54} + \frac{-8UV+13V^2+27V+8U-13V-27}{54}\right\}$

$f_{(U,V)}(U,V) = \frac{1}{2\pi 3\sqrt{3}} \exp\left\{\frac{7U^2+13V^2-54U+54V-16UV+108}{54}\right\} \stackrel{?}{=} \boxed{\frac{1}{2\pi\sqrt{27}} \exp\left\{-\frac{1}{2}\frac{7(x-5)^2+13(y-1)^2-16(x-5)(y-1)}{27}\right\}}$

4° CONDITIONAL DENSITY OF U GIVEN V

"GIVEN V"

$\downarrow$

II FIND $f_V(v)$

GENERAL FORMULA

$f_X(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}}$

$\Downarrow$

$f_V(v) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(v-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi\cdot 7}} e^{-\frac{(v-1)^2}{2\cdot 7}} = \frac{e^{-\frac{(v-1)^2}{14}}}{\sqrt{14\pi}} =$

$= \frac{1}{\sqrt{14\pi}} \exp\left\{\frac{-(v-1)^2}{14}\right\}$

III $f(U=u|V=v) = \frac{f(U=u, V=v)}{f(V=v)} = \dfrac{\frac{1}{2\pi 3\sqrt{3}} \exp\left\{\frac{7U^2+13V^2-54U+54V-16UV+108}{54}\right\}}{\frac{1}{\sqrt{14\pi}} \exp\left\{\frac{-(V-1)^2}{14}\right\}} =$

$= \frac{\sqrt{14\pi}}{6\pi\sqrt{3}} \exp\left\{-\frac{1}{2}\frac{7(x-5)^2+13(y-1)^2-16(x-5)(y-1)}{27} + \frac{(v-1)^2}{14}\right\} = \boxed{\frac{1}{\sqrt{2\pi\frac{27}{7}}} \exp\left\{-\frac{1}{2}\frac{\left(x-\frac{8}{7}y-\frac{27}{7}\right)^2}{27/7}\right\}}$

$f(U|V)(x|y)$

4e) $E[U|V=v] = \frac{8}{7}v + \frac{27}{7}$

INVERT SIGN

UTTO STRATEGY

$$\underline{m} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$$\underline{\underline{C}}^{-1} = \begin{pmatrix} 7/27 & -8/27 \\ -8/27 & 13/27 \end{pmatrix} \quad f(u,v)(u,v) = \frac{1}{2\pi} \frac{1}{\sqrt{\det \underline{\underline{C}}^{-1}}} \exp\left\{ -\frac{1}{2} \frac{7}{27}(u-5)^2 + \left(-\frac{8}{27} \cdot -\frac{8}{27}\right)(u-5)(v-1) + \frac{13}{27}(v-1)^2 \right\} = \boxed{\frac{1}{2\pi\sqrt{27}} \exp\left\{ -\frac{1}{2} \frac{7(u-5)^2 - 16(u-5)(v-1) + 13(v-1)^2}{27} \right\}}$$

$$\det \underline{\underline{C}}^{-1} = \frac{91-64}{729} = \frac{27}{729} = 27$$

- mean vector : $R \begin{pmatrix} m_x \\ m_y \end{pmatrix}$

- covariance : $R C^t R$

- To find density we use mean and covariance that we have found:

$$\begin{pmatrix} U \\ V \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}}_{R} \begin{pmatrix} X \\ Y \end{pmatrix}$$

- mean : $\overset{R}{\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}} \overset{\binom{m_x}{m_y}}{\begin{pmatrix} 2 \\ 0 \end{pmatrix}} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} m_u \\ m_v \end{pmatrix}$

- Covariance : $\overset{{}^tR}{\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}} \overset{C}{\begin{pmatrix} 1 & 1 \\ 1 & 9 \end{pmatrix}} \overset{R}{\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}} = \begin{pmatrix} 3 & 11 \\ 2 & 26 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 2 & 26^3 \end{pmatrix} =$

$$= \begin{pmatrix} 28 & 30 \\ 56 & 76 \end{pmatrix} = C^{-1}$$

$$\left[f_{U,V} = \frac{1}{2\pi} \cdot \frac{1}{\sqrt{\det C}} \cdot \exp\left\{ -\frac{1}{2} {}^t\begin{pmatrix} u - m_u \\ v - m_v \end{pmatrix} C^{-1} \begin{pmatrix} u - m_u \\ v - m_v \end{pmatrix} \right\} \right]$$

$$= \frac{1}{2\pi \cdot \sqrt{448}} \exp\left\{ -\frac{28(u-4)^2 + \underset{\llcorner 56+30}{86}(u-4)(v+2) + 76(v+2)^2}{2 \cdot 448} \right\}$$

$\det C = 448$

$$C^{-1} = \frac{1}{\det C} \begin{pmatrix} 28 & -30 \\ -56 & 76 \end{pmatrix}$$

- To find conditional density:

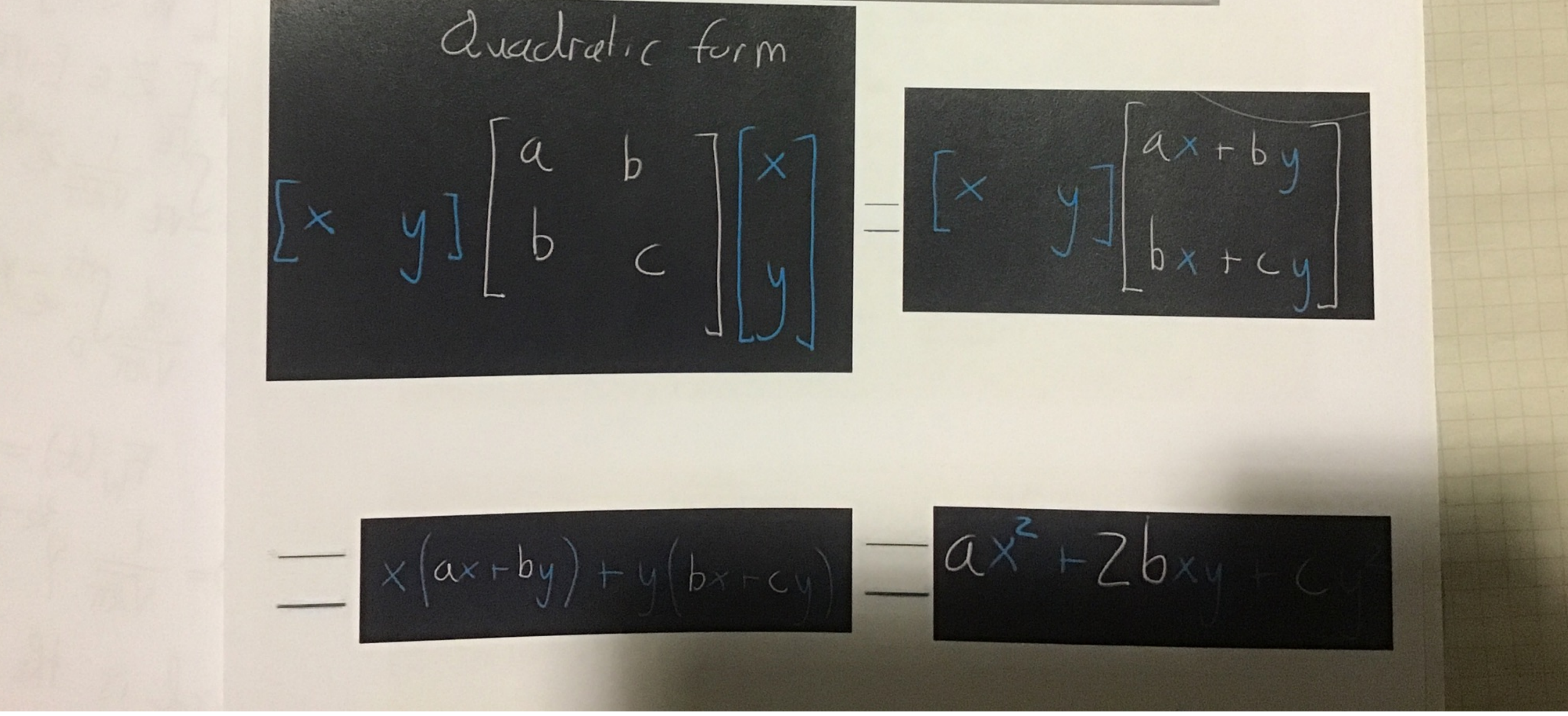
$$\cancel{f_{X|Y}} \quad f_{U|V}(u|v) = \frac{f_{U,V}(u,v)}{f_V(v)}$$

$$f_V(v) \longrightarrow \text{M.G.F} : M_{U,V}(z,w) = \exp\left\{ 4z - 2w + \frac{1}{2}\left[28z^2 + 86zw + 76w^2 \right] \right\}$$

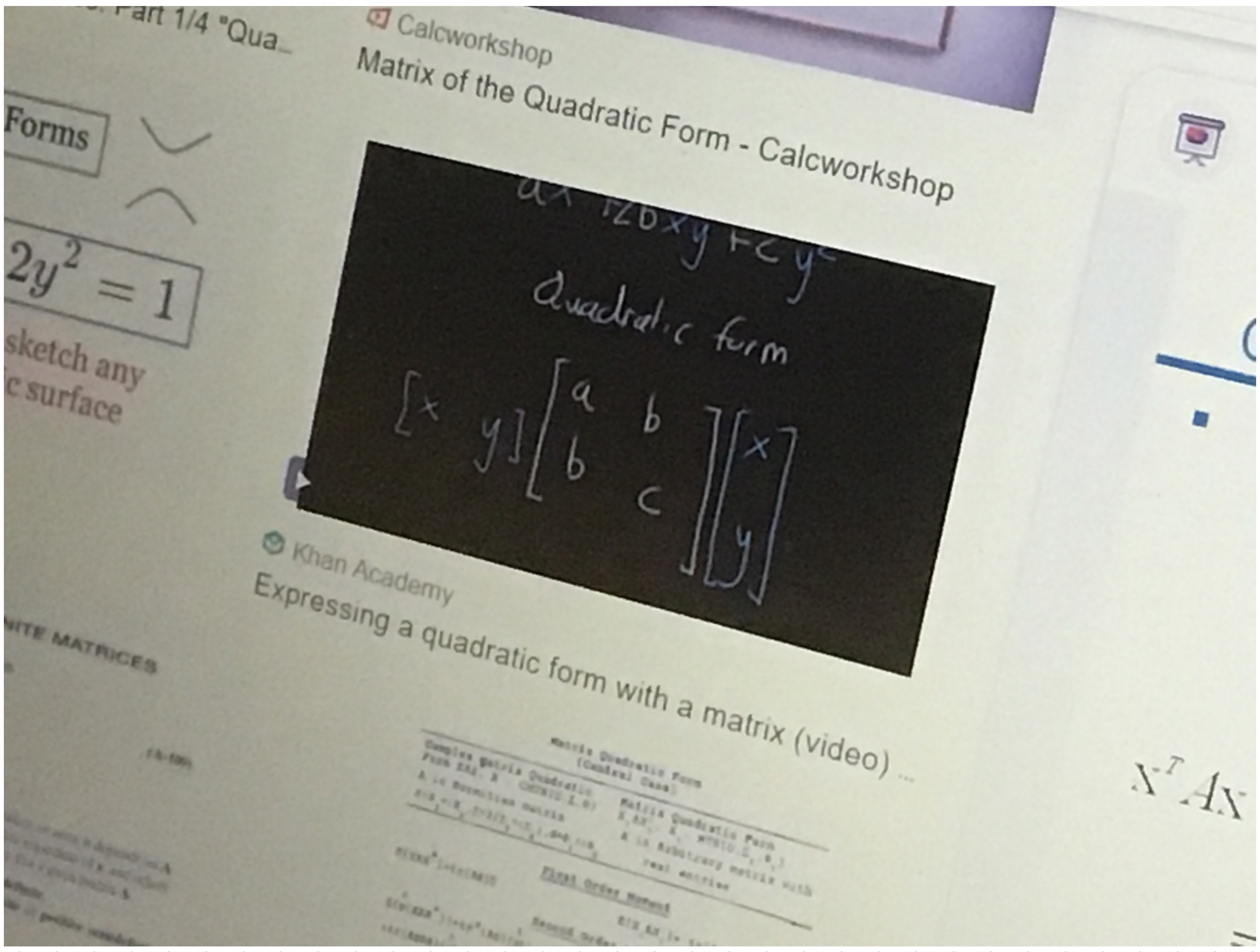
We put $z = 0$ and find $\left[M_V(w) = M_{U,V}(0,w) = \exp\left\{ \underset{\text{mean}}{-2w} + \frac{\overset{\text{variance}}{76}}{2} w^2 \right\} \right]$

Hence $V \sim \mathcal{N}(-2, 76) \longrightarrow f_V(w) = \frac{1}{\sqrt{2\pi 76}} \cdot e^{-\frac{(w+2)^2}{2 \cdot 76}}$

No[illegible] we can find $f_{U|V}(u|v)$.



Quadratic form
$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} ax+by \\ bx+cy \end{bmatrix}$
$= x(ax+by)+y(bx+cy) = ax^2+2bxy+cy^2$



Part 1/4 "Qua...
Calcworkshop
Matrix of the Quadratic Form - Calcworkshop
Forms
$2y^2 = 1$
sketch any
c surface
Quadratic form
$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$
Khan Academy
Expressing a quadratic form with a matrix (video) ...
NITE MATRICES
$X^T A X$

Problem 4 (10 Points) Let (X,Y) be a continuous random vector having bivariate GAUSSIAN distribution with mean vector $(m_X, m_Y) = (2,0)$ and covariance matrix $\begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 9 \end{pmatrix}$

4a) $\mathbb{E}[3X-4Y]$

4b) $\mathrm{Cov}(2X+Y, -X+3Y)$

3c) the distribution of the random vector $(U,V) = (2X+Y, -X+3Y)$

3d) the conditional distribution of X given Y.

$\sigma_x^2 = VAR(x) = 1$

$\sigma_y^2 = VAR(y) = 9$

$\rho\sigma_x\sigma_y = COV(x,y) = 1$

$m_x = E[x] = 2$

$m_y = E[y] = 0$

$\boxed{1}$ $E[3x-4y] = 3E[x] - 4E[y] = 3\cdot 2 - 4\cdot 0 = \boxed{6}$

$\boxed{2}$ $cov(2x+y, -x+3y) = cov(2x,-x) + cov(2x,3y) + cov(y,-x) + cov(y,3y) =$

$= -2VAR(x) + 6cov(x,y) - cov(x,y) + 3VAR(y) = -2VAR(x) + 5cov(x,y) + 3VAR(y) =$

$= -2\cdot 1 + 5\cdot 1 + 3\cdot 9 = \boxed{30}$

$\boxed{3}$ DISTRIBUTION OF $(u,v) = (2x+y, -x+3y)$

$\boxed{I}$ FIND MEAN VECTOR $\underline{m}$

$\boxed{II}$ FIND COV MATRIX $\underline{\underline{C}}$

$\boxed{I}$ $R = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$

$$\underline{m} = R\begin{pmatrix} m_x \\ m_y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

$\boxed{II}$ $\underline{\underline{C}} = RCR^T$

$$\underline{\underline{C}} = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 9 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}^T = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 9 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2+1 & 2+9 \\ -1+3 & -1+27 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 11 \\ 2 & 26 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 6+11 & -3+33 \\ 4+26 & -2+78 \end{pmatrix} = \begin{pmatrix} 17 & 30 \\ 30 & 76 \end{pmatrix}$$

$\boxed{4}$ CONDITIONAL DISTRIBUTION OF X GIVEN Y

$$f(x=x|y=y) = \frac{f_{(x,y)}(x,y)}{f_y(y)}$$

$$f_{(x,y)}(x,y) = \frac{1}{2\pi}\frac{1}{\sqrt{\det \underline{\underline{C}}}} \exp\left\{-\frac{1}{2}\begin{pmatrix} x-m_x \\ y-m_y \end{pmatrix}^T C^{-1}\begin{pmatrix} x-m_x \\ y-m_y \end{pmatrix}\right\} =$$

- $\det(C) = (1\cdot 9) - (1\cdot 1) = 8$
- $\begin{pmatrix} x-m_x \\ y-m_y \end{pmatrix}^T = \begin{pmatrix} x-2 \\ y-0 \end{pmatrix}^T = (x-2; y)$
- $C^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 9 \end{pmatrix}^{-1} = \frac{1}{8}\begin{pmatrix} 9 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 9/8 & -1/8 \\ -1/8 & 1/8 \end{pmatrix}$
- $\begin{pmatrix} x-m_x \\ y-m_y \end{pmatrix} = \begin{pmatrix} x-2 \\ y-0 \end{pmatrix}$

$$f_{(x,y)}(x,y) = \frac{1}{4\pi\sqrt{2}} \exp\left\{-\frac{1}{2}(x-2;y)\begin{pmatrix} 9/8 & -1/8 \\ -1/8 & 1/8 \end{pmatrix}\begin{pmatrix} x-2 \\ y \end{pmatrix}\right\} = \frac{1}{4\pi\sqrt{2}} \exp\left\{\left(\frac{-x+2}{2}; \frac{-y}{2}\right)\begin{pmatrix} 9/8 & -1/8 \\ -1/8 & 1/8 \end{pmatrix}\begin{pmatrix} x-2 \\ y \end{pmatrix}\right\} = \frac{1}{4\pi\sqrt{2}} \exp\left\{\left(\frac{-x+2}{2}\cdot\frac{9}{8} + \frac{y}{2}\left(\frac{1}{8}\right); \frac{-x+2}{2}\left(-\frac{1}{8}\right) + \frac{-y}{2}\left(\frac{1}{8}\right)\right)\begin{pmatrix} x-2 \\ y \end{pmatrix}\right\} =$$

$$= \frac{1}{4\pi\sqrt{2}} \exp\left\{\left(\frac{-9x+18}{16} + \frac{y}{16}; \frac{x-2}{16} - \frac{y}{16}\right)\begin{pmatrix} x-2 \\ y \end{pmatrix}\right\} = \frac{1}{4\pi\sqrt{2}} \exp\left\{\left(\frac{-9x+18+y}{16}; \frac{x-2-y}{16}\right)\begin{pmatrix} x-2 \\ y \end{pmatrix}\right\} = \frac{1}{4\pi\sqrt{2}} \exp\left\{\frac{-9x^2+18x+yx+18x-36-2y+9xy-18y-y^2}{16} + \frac{xy-2y-y^2}{16}\right\} =$$

$$= \frac{1}{4\pi\sqrt{2}} \exp\left\{\frac{-9x^2-2y^2+36x-22y+11xy-36}{16}\right\} \ldots$$

$$f_y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi 9}} e^{-\frac{(y-0)^2}{18}} = \frac{1}{3\sqrt{2\pi}} e^{-\frac{y^2}{18}}$$

$$f(x=x|y=y) = \frac{f_{(x,y)}(x,y)}{f_y(y)} = \frac{\frac{1}{4\pi\sqrt{2}} \exp\left\{\frac{-9x^2-2y^2+36x-22y+11xy-36}{16}\right\}}{\frac{1}{3\sqrt{2\pi}} e^{-\frac{y^2}{18}}}$$

OLD

Risolvi:
Let (X,Y): Ω -> IR^2 be a random vector having Gaussian distribution with mean vector (mx, my) = (2, 0) and covariance matrix

$$C = \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 9 \end{pmatrix} \qquad m_x = 2 \quad m_y = 0$$

find
1) expectation of Z = 3X-4Y
2) the moment generating function of Z = 3X-4Y
3) the density of Z = 3X-4Y
4) Cov(2X+Y, -X+3Y)
5) The distribution of (U,V) = (2X+Y, -X+3Y)
6) The joint density of the random vector (U,V) = (2X+Y, -X+3Y)
7) The conditional distribution of X given Y
8) The conditional expectation E[U | V = v]
9) The conditional density of U given V

I KNOW:
- $E[X] = m_x = 2$
- $E[Y] = m_y = 0$
- $VAR(X) = \sigma_x^2 = 1$
- $VAR(Y) = \sigma_y^2 = 9$
- $COV(X,Y) = \rho\sigma_x\sigma_y = 1$

1) EXPECTATION OF $Z = 3X + 4Y$

$$E[Z] = E[3X4Y] = 3E[X] + 4E[Y] = 3\cdot 2 + 4\cdot 0 = \boxed{6}$$

2) MOMENT GENERATING FUNCTION $Z = 3X - 4Y$

$$COV(X,Y) = \rho\sigma_x\sigma_y = 1$$

$$VAR(Z) = VAR(3X-4Y) = 3^2 VAR(X) + (-4)^2 VAR(Y) + \overbrace{2\cdot 3(-4)\cdot COV(X,Y)}^{2COV(X,Y)} = 153 - 12\cdot 2 = 129$$

$$M_X(t) = EXP(\mu t$$

3) DENSITY OF $Z = 3X - 4Y$

I KNOW IT IS A GAUSSIAN: $\longrightarrow$ I KNOW: $f(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$$f(z) = \frac{1}{\sqrt{2\pi\sigma_z^2}} e^{-\frac{(x-\mu_z)^2}{2\cdot\sigma_z^2}} = \frac{1}{\sqrt{2\pi\cdot 129}} e^{-\frac{(x-6)^2}{2\cdot 129}}$$

4) $COV(2X+Y, -X+3Y) = COV(2X,-X) + COV(2X,3Y) + COV(Y,-X) + COV(Y,3Y) =$

$$= -2VAR(X) + 6COV(X,Y) - COV(X,Y) + 3VAR(Y) = -2\cdot 1 + 5\,COV(X,Y) + 3\cdot 9 = -2+5\cdot 1+27 = 30$$

5) DISTRIBUTION OF $(U,V) = (2X+Y, -X+3Y)$

YOU NEED TO STATE
- MEAN VECTOR $\underline{m}$
- COV MATRIX $\underline{C}$

① FIND R: $R = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} U \\ V \end{pmatrix}$

② FIND $\underline{m} = R\begin{pmatrix} m_x \\ m_y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2\cdot 2 + 1\cdot 0 \\ -1\cdot 2 + 3\cdot 0 \end{pmatrix} = \boxed{\begin{pmatrix} 4 \\ -2 \end{pmatrix}}$ $\underline{m}$

FROM TEXT

T $[\,=\,]^T = [\,||\,]$

③ FIND $\underline{C} = RCR^T = \underbrace{\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 9 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}}_{\text{THE ORDER MATTERS!}} = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 9 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2\cdot 1 + 1\cdot 1 & 2\cdot 1 + 1\cdot 9 \\ -1\cdot 1 + 3\cdot 1 & -1\cdot 1 + 3\cdot 9 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 11 \\ 2 & 26 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3\cdot 2 + 11\cdot 1 & 3\cdot(-1) + 11\cdot 3 \\ 2\cdot 2 + 26\cdot 1 & 2\cdot(-1) + 26\cdot 3 \end{pmatrix} = \boxed{\begin{pmatrix} 17 & 30 \\ 30 & 76 \end{pmatrix}}$ $\underline{C}$

6) JOINT DENSITY OF $(U,V) = (2X+Y, -X+3Y)$

REMEMBER: $f_{(X,Y)}(x,y) = \frac{1}{2\pi}\frac{1}{\sqrt{DET\,\underline{C}}} \exp\left\{-\frac{1}{2}\begin{pmatrix} x-m_x \\ y-m_y \end{pmatrix}^T C^{-1}\begin{pmatrix} x-m_x \\ y-m_y \end{pmatrix}\right\}$

$$f_{(U,V)}(u,v) = \frac{1}{2\pi}\frac{1}{\sqrt{DET\,\underline{C}}} \exp\left\{-\frac{1}{2}\begin{pmatrix} u-m_u \\ v-m_v \end{pmatrix}^T C^{-1}\begin{pmatrix} u-m_u \\ v-m_v \end{pmatrix}\right\} = \frac{1}{2\pi}\frac{1}{\sqrt{8}} \exp\left\{-\frac{1}{2}\begin{pmatrix} u-4 \\ v-(-2) \end{pmatrix}^T \begin{pmatrix} 9/8 & -1/8 \\ -1/8 & 1/8 \end{pmatrix}\begin{pmatrix} u-4 \\ v-(-2) \end{pmatrix}\right\}$$

$$= \frac{1}{4\pi\sqrt{2}} \exp\left\{\frac{-1}{2}(u-4, v+2)\begin{pmatrix} 9/8 & -1/8 \\ -1/8 & 1/8 \end{pmatrix}\begin{pmatrix} u-4 \\ v+2 \end{pmatrix}\right\}$$

$$= \frac{1}{4\pi\sqrt{2}} \exp\left\{\frac{-1}{2}\left((u-4)\cdot 9/8\right.\right.$$

- $DET(C) = 1\cdot 9 - (1\cdot 1) = 8$
- $C^{-1} = \begin{pmatrix} 9 & -1 \\ -1 & 1 \end{pmatrix}\cdot\frac{1}{DET(C)} = \frac{1}{8}\begin{pmatrix} 9 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 9/8 & -1/8 \\ -1/8 & 1/8 \end{pmatrix}$
- $\boxed{m_u = 4;\ m_v = -2}$ $\underline{m}$ (BOTH FROM BEFORE)

7) CONDITIONAL DISTRIBUTION OF X GIVEN Y