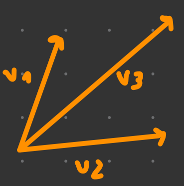
$H = \text{SPAN}\{\underline{v_1}, \underline{v_2}, \underline{v_3}, \dots \underline{v_p}\}\ \mathbb{R}^N$ LINEARLY DEPENDENTED → AT LEAST ONE OF THEM CAN BE WRITTEN AS A COMBINATION OF THE OTHERS

$c_1\underline{v_1} + c_2\underline{v_2} + c_3\underline{v_3} + \dots + c_p\underline{v_p} = \underline{0}$ $\quad c_1, c_2, c_3 \in \mathbb{R}$

EX: V_3 IS A LINEAR COMBINATION OF THE OTHERS VECTORS

$c_1\underline{v_1} + c_2\underline{v_2} + \dots + c_p v_p = -c_3\underline{v_3}$

MOVE IT TO THE OTHER SIDE AND DIVIDE BY C3



v_3 (THAT IS $v_1 + v_2$) IS NOT USEFUL AND I CAN "KILL IT"

DEFINITION: LET H BE A LINEAR SUBSPACE IN $\mathbb{R}^N$. THE $\{\underline{v_1}, \underline{v_2}, \dots \underline{v_p}\} \in \mathbb{R}^N$ ARE A BASIS FOR H IF

1. $H = \text{SPAN}\{\underline{v_1}, \underline{v_2}, \dots \underline{v_p}\}$
2. $\underline{v_1}, \underline{v_2}, \dots \underline{v_p}$ ARE LINEARLY INDIPENDENT

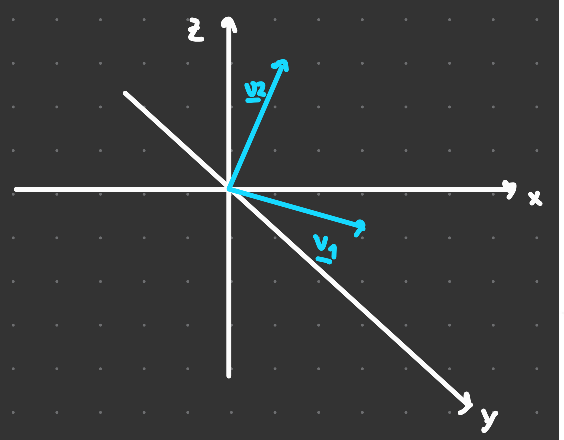
THEOREM LET $\{\underline{v_1}, \underline{v_2}, \dots \underline{v_p}\}$ BE A BASIS FOR H ($H = \text{SPAN}\{\underline{v_1}, \underline{v_2}, \dots \underline{v_p}\}$) THEN ANY VECTOR $\underline{w} \in H$ CAN BE WRITTEN IN A UNIQUE WAY AS $\underline{w} = c_1\underline{v_1} + c_2\underline{v_2} + \dots c_p\underline{v_p}$ $\quad c_1, c_2, c_3, \dots c_p \in \mathbb{R}$

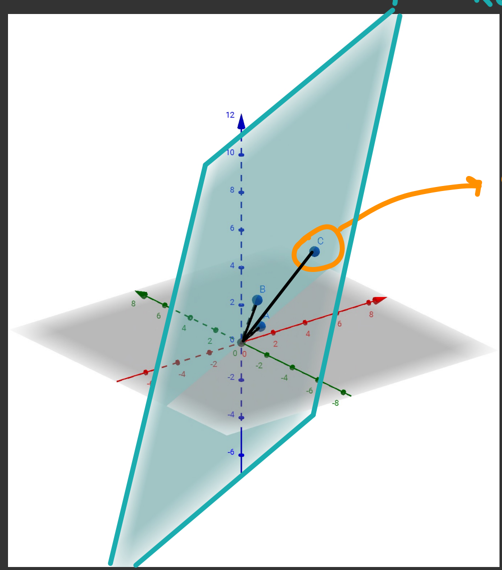
THE VALUES $c_1, c_2, \dots, c_p$ ARE CALLED THE COORDINATES OF $\underline{w}$ IN THE BASIS $\{\underline{v_1}, \underline{v_2}, \dots \underline{v_p}\}$

$\mathbb{R}^3$

$$\begin{array}{c} \quad x\ \ y\ \ z \\ \underline{v_1} = [2, 1, 0] \end{array} \qquad \underline{v_2} = [1\ \ 0\ \ 2] \qquad \underline{w} = [7\ \ 3\ \ 2]$$

$H = \text{SPAN}\{\underline{v_1}, \underline{v_2}\}$





ALL THE VECTORS HERE CAN BE REPRESENTED WITH RESPECT TO THE B VECTOR AND THE A VECTOR BECAUSE THEY ARE IN THE SAME PLANE

C IS IN THE

$\underline{w} \in H$?

$$\begin{array}{cc} v_1 & v_2 \end{array}$$
$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 7 \\ 1 & 0 & 3 \\ 0 & 2 & 2 \end{array}\right] \xrightarrow{-1/2} \left[\begin{array}{cc|c} 2 & 1 & 7 \\ 0 & -1/2 & -1/2 \\ 0 & 2 & 2 \end{array}\right] \xrightarrow{\cdot 4} \left[\begin{array}{cc|c} 2 & 1 & 7 \\ 0 & -1/2 & -1/2 \\ 0 & 0 & 0 \end{array}\right] \Longrightarrow \begin{array}{l} 2^{ND} \\ 1^{ST} \end{array} \begin{cases} -1/2\, x_2 = -1/2;\ x_2 = 1 \\ 2x_1 + x_2 = 7;\ x_1 = 3 \end{cases}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ x_1 & x_2 \end{array}$$

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} x_2 = 3\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 2 \end{bmatrix}$$

EX 2: $\underline{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\quad \underline{v_2} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$ $\quad v_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

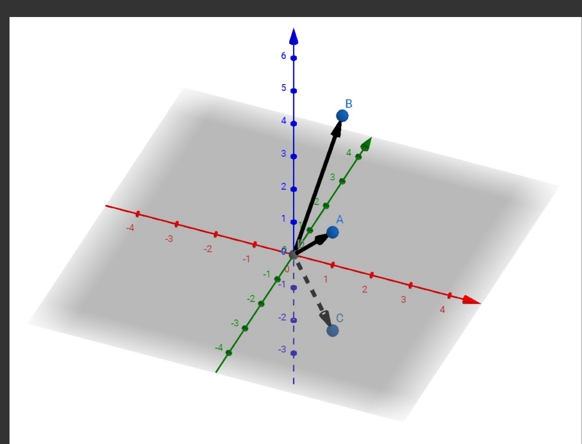
- TELL IF $\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ ARE A BASIS FOR $\mathbb{R}^3$
- IF YES, THEN EXPRESS $\underline{w} = [3, 3, -6]$ IN TERMS OF ITS COORDINATES IN SUCH BASIS

- SPAN $\{\underline{v_1}, \underline{v_2}, \underline{v_3}\} \in \mathbb{R}^3$
- LIN INDIPENDENT

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & -2 \end{bmatrix} \xrightarrow{\cdot 1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

YES ITS A BASIS FOR IR

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -6 \end{bmatrix}$$



$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 3 & 0 & 3 \\ 1 & 0 & -2 & -6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & -3 & -9 \end{array}\right]$$

$-3x_3 = -3;\ x_3 = 3$

$3x_2 + 2x_3 = 3;\ x_2 = 1$

$x_1 + x_3 = 3;\ x_1 = 0$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} 0 + \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} 1 + \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} 3 = \begin{bmatrix} 3 \\ 3 \\ -6 \end{bmatrix}$$

Ex 3: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

- Tell if $\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ are a basis for $\mathbb{R}^3$
- If yes, then express $\underline{v} = [3, 3, -6]$ in terms of its coordinates in such basis

IDENTITY MATRIX

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -6 \end{bmatrix}$$

$A \in \mathbb{R}^{m \times n}$

$A = [\underline{a_1}, \underline{a_2}, \dots \underline{a_n}]$

$Col(A) = span\{\underline{a_1}, \underline{a_2}, \dots, \underline{a_n}\} \subseteq \mathbb{R}^m$

$Nul(A) = \{\underline{x} \in \mathbb{R}^n \text{ such that } A\underline{x} = \underline{0}\} \subseteq \mathbb{R}^n$

BASIS FOR $Col(A)$ AND $Null(A)$

$$A = \begin{bmatrix} 1 & 2 & -2 & 2 \\ 0 & 0 & 1 & -1 \\ -1 & -2 & 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -2 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 2 & -2 & 2 \\ 0 & 0 & \boxed{1} & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\uparrow x_1 \quad \uparrow x_3$

NOT UNIQUE THE EF. CAN FIND OTHER BASIS

JUST FIND THE NON-FREE VARIABLES AND TAKE THEIR RESPECTIVE VECTORS

THOSE TWO VECTORS ARE BASIS THE OTHER ARE LINEARLY DEPENDENT

$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right\}$ ARE A BASIS FOR $Col(A)$

$$Col(A) = span\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} \right\} = span\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right\}$$

$A\underline{x} = \underline{0}$

$$\begin{bmatrix} \boxed{1} & 2 & -2 & 2 \\ 0 & 0 & \boxed{1} & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

OR EQUIVALENTLY

$$\left[\begin{array}{cccc|c} \boxed{1} & 2 & -2 & 2 & 0 \\ 0 & 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

2^{nd} $\{ x_3 = x_4$

3^{rd} $\{ x_1 + 2x_2 - 2x_3 + 2x_4 = 0;\ x_1 = -2x_2$

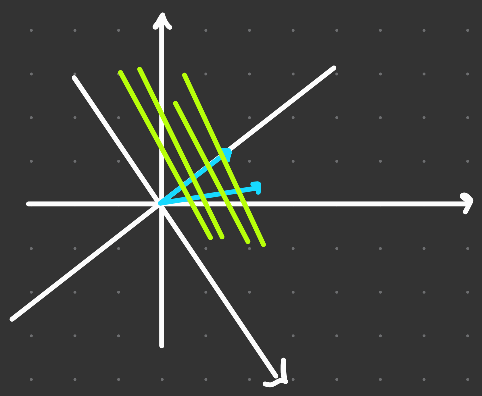
$$\begin{cases} x_1 = -2x_2 \\ x_2 = x_2 \in \mathbb{R} \\ x_3 = x_4 \\ x_4 = x_4 \in \mathbb{R} \end{cases} \Longrightarrow$$

PARAMETRIC EXPRESSION

$$\underline{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} x_4 \quad x_2, x_4 \in \mathbb{R}$$

THEOREM H A LINEAR SUBSPACE IN $\mathbb{R}^N$. THEN IF $\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_p}\}$ IS A BASIS FOR H THEN ANY OTHER BASIS OF H HAS EXACTLY P VECTORS. SUCH NUMBER P IS CALLED THE **DIMENSION** OF H

$\mathbb{R}^3$



$\mathbb{R}^N$ $\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix} \begin{bmatrix}0\\1\\ \vdots\\0\end{bmatrix} \begin{bmatrix}0\\0\\ \vdots\\0\end{bmatrix} \cdots \begin{bmatrix}0\\ \vdots\\0\\1\end{bmatrix}$

$\{\underline{0}\} \in \mathbb{R}^N$ A LINEAR SUBSPACE $\dim\{\underline{0}\} = 0$

$H = \text{SPAN}\{\underline{v_1}, \underline{v_2}, \dots \underline{v_p}\}$ $\underline{v_1}, \underline{v_2}, \dots, \underline{v_p} \in \mathbb{R}^N$

IF $\{\underline{v_1}, \underline{v_2}, \dots \underline{v_p}\}$ ARE LIN. INDEP THEN $\dim(H) = P$ OTHERWISE $\dim(H) < P$.

$A \in \mathbb{R}^{M \times N}$ $\quad \text{Col}(A) \quad \dim \text{Col}(A) \overset{???}{=}$

$\text{Nul}(A) \quad \dim \text{Nul}(A)$

$[A] \xrightarrow{EF} \begin{bmatrix} \boxed{*} & * & * & * & * & * \\ 0 & 0 & \boxed{*} & * & * & * \\ 0 & 0 & 0 & \boxed{*} & * & * \\ 0 & 0 & 0 & 0 & 0 & \boxed{*} \end{bmatrix}$

COLUMNS WITH PIVOT POSITION $\Rightarrow$ 4 VECTORS $\Rightarrow$ A BASIS FOR $\text{Col}(A)$ HAS ONE VECTOR FOR EVERY COLUMN WITH A PIVOT POSITION.

$\dim \text{Col}(A) =$ "THE NUMBER OF COLUMNS WITH A PIVOT POSITION"

$A\underline{x} = \underline{0}$ $\quad \text{Nul}(A)$ FOR EVERY FREE VARIABLE WE FIND A CORRESPONDING VECTOR IN THE **PARAMETRIC VECTOR FORM** OF $A\underline{x} = \underline{0}$

$\Downarrow$

THEREFOR YOU HAVE AS MANY VECTORS AS THE NUMBER OF FREE VARIABLES, WHICH IS NOTHING BUT THE COLUMNS WITHOUT A PIVOT POSITION, SUCH VECTORS ARE A BASIS FOR $\text{Nul}(A)$

$\dim \text{Nul}(A) =$ "THE NUMBERS OF COLUMNS WITHOUT A PIVOT POSITION"

$\boxed{\text{DEF}}$ THE RANK OF A MATRIX $A \in \mathbb{R}^{M \times N}$ IS $\text{RANK}(A) = \dim \text{Col}(A)$

$A \in \mathbb{R}^{M \times N}$ $\quad$ A HAS N COLUMNS

$\dim \text{Col}(A) + \dim \text{Nul}(A) = N$

THEOREM $A \in \mathbb{R}^{M \times N}$. THEN

$\dim \text{Col}(A) + \dim \text{Nul}(A) = N$ $\quad$ [OR EQUIVALENTLY $\text{RANK}(A) + \dim \text{Nul}(A) = N$]

THEOREM Let H be a linear subspace in $\mathbb{R}^n$ of dimension p. Let $\underline{v_1}, \underline{v_2}, \dots \underline{v_p}$ in H

i) If $\text{span}\{\underline{v_1}, \underline{v_2}, \dots \underline{v_p}\} = H$ then $\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_p}\}$ are a basis for H

ii) If $\{\underline{v_1}, \underline{v_2}, \dots \underline{v_p}\}$ are linearly indipendent then $\{\underline{v_1}, \underline{v_2}, \dots \underline{v_p}\}$ are a basis for H

① span $\{\underline{v_1}, \underline{v_2}, \dots \underline{v_p}\} = H$

② linear indipendence

i) $A = \underbrace{[\underline{v_1}, \underline{v_2}, \dots \underline{v_p}]}_{p}$

$$\dim \text{Col}(A) = \dim \text{Span}\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_p}\} = \underbrace{\dim H}_{\text{1'st theo}} = p$$

$\Longrightarrow \dim \text{Nul}(A) + \dim \text{Col}(A) = p \Rightarrow \dim \text{Nul}(A) = 0 \Longrightarrow A\underline{x} = \underline{0}$ has only solution $\underline{x} = \underline{0}$ $\Rightarrow$ columns of A are linearly dependent $\Rightarrow \{\underline{v_1}, \underline{v_2}, \dots, \underline{v_p}\}$ are linearly indep.

$\mathbb{R}^n$ $\underline{v_1}, \underline{v_2}, \dots \underline{v_m}$

THEOREM $A \in \mathbb{R}^{n \times n}$ (square matrix)

The following statements are equivalent:

① A is invertible

② $\text{Col} A = \mathbb{R}^n$

③ $\dim \text{Col}(A) = n$ $\quad (\text{rank}(A) = n)$

④ The colums of A are a basis for $\mathbb{R}^n$

⑤ $\text{Nul}(A) = \{\underline{0}\}$

⑥ $\dim \text{Nul}(A) = 0$

$A \in \mathbb{R}^{n \times n}$

$$A \xrightarrow{EF} \begin{bmatrix} \boxed{*} & & * \\ & \boxed{*} & \\ 0 & & \ddots & \boxed{*} \end{bmatrix}$$

then A is invertible

③ + ⑥ $\underbrace{\dim \text{Col}(A)}_{n} + \underbrace{\dim \text{Nul}(A)}_{0} = n$

$$\underline{v_1} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 0 \end{bmatrix} \quad \underline{v_2} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix} \quad \underline{v_3} = \begin{bmatrix} 1 \\ -3 \\ 3 \\ 1 \end{bmatrix} \qquad \underline{w} = \begin{bmatrix} 2 \\ -2 \\ 4 \\ 3 \end{bmatrix}$$

- are $\underline{v_1}, \underline{v_2}, \underline{v_3}$ a basis for H?
- is $\underline{w} \in H$?