Exercises - Calculus
Academic Year 2021-2022

Sheet 7

1. Determine the behaviour of the following series (converging, diverging to $+\infty$, diverging to $-\infty$ or indeterminate). Determine also if the series is absolutely converging or not.

   Take into account that $\lim_n \arctan(x_n) = \pi/2$ for any sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \to +\infty$.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^4+1}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n \arctan n}{\sqrt{n^6+1}}$

   (d) $\displaystyle\sum_{n=3}^{\infty} \left(\frac{1}{n^3} - \frac{1}{n^2}\right)$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n!}$

   (f) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n(n^3+1)}}$

   (g) $\displaystyle\sum_{k=5}^{\infty} \frac{1-2k}{2^{k/2}}$

   (h) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3+(-1)^n}}$

   (i) $\displaystyle\sum_{n=2}^{\infty} n^{-n/2}$

   (j) $\displaystyle\sum_{k=1}^{\infty} \frac{2+(-1)^k}{k^2}$

   (k) $\displaystyle\sum_{k=3}^{\infty} (-1)^k \frac{\sqrt{k+1}-\sqrt{k}}{k}$

   (l) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n+\sqrt{n}}$

   (m) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}(\cos(n^2)+\sqrt{n})$

(n) $\displaystyle\sum_{n=3}^{\infty} \frac{\arctan n}{n\sqrt{n-1}}$

(o) $\displaystyle\sum_{n=1}^{\infty} 3^{-\sqrt{n}}$

(p) $\displaystyle\sum_{n=1}^{\infty} 2^n \cdot 3^{-\sqrt{n}}$

(q) $\displaystyle\sum_{n=5}^{\infty} (-1)^{n+2}(\sqrt[n]{3}-1)$

(r) $\displaystyle\sum_{n=2}^{\infty} \sqrt[3]{n+1}-\sqrt[3]{n}$

(s) $\displaystyle\sum_{n=2}^{\infty} \log\left(1+\frac{1}{n^2}\right)$

(t) $\displaystyle\sum_{n=1}^{\infty} n(1-e^{1/n^2})$

2. Determine for which values of $x \in \mathbb{R}$ the following series converges. When the series is converging, determine also if the series is absolutely converging or not.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

   (b) $\displaystyle\sum_{n=1}^{\infty} x^n \sin(1/n)$

   (c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{n}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(\sin x)^n}{n}$

   (e) $\displaystyle\sum_{n=1}^{\infty} (1-\cos(x/n))$

3. Verify that the series

$$1-\frac{1}{2\cdot 3}+\frac{1}{2}-\frac{1}{3\cdot 4}+\frac{1}{2^2}-\frac{1}{4\cdot 5}+\frac{1}{2^3}-\ldots-\frac{1}{n(n+1)}+\frac{1}{2^{n-1}}-\ldots$$

   is converging and compute its sum.

4. Determine the behaviour of the series

$$1-\frac{1}{3}+1-\frac{1}{9}+1-\frac{1}{27}+\ldots+1-\frac{1}{3^n}+1-\ldots$$

5. Determine, with respect to the parameter α, $\alpha > 0$, whether the following series converges

$$1+\frac{1}{1+\alpha}+\frac{1}{(1+\alpha)(1+2\alpha)}+\ldots+\frac{1}{(1+\alpha)(1+2\alpha)\cdot\ldots\cdot(1+n\alpha)}+\ldots$$

6. Determine, with respect to the parameter α, $\alpha > 0$, the behaviour of the series
$$\sum_{n=1}^{\infty} \left(\sqrt{1 + \frac{1}{n^{\alpha}}} - 1 \right).$$

7. Prove that, for any $a \neq 0, -1, -2, -3, \ldots$, we have
$$\sum_{n=0}^{\infty} \frac{1}{(a+n)(a+n+1)} = \frac{1}{a}.$$

8. Determine for which values of α, $0 < \alpha < \pi/2$, the following series converges
$$\sum_{n=0}^{\infty} 2^n (\sin \alpha)^{2n}.$$
When it is convergent, compute its sum.

9. Determine if the following series converge, specifying if the convergence is absolute or not
$$\text{a) } \sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{n(n+1)}; \quad \text{b) } \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n+1}-1}{n}; \quad \text{c) } \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{n}\right)^{\log n}.$$

10. Determine the behaviour of the following series. Determine also if the series is absolutely converging or not.
$$\text{a) } \sum_{n=0}^{+\infty} \frac{n^3 + \sin(n^2)}{2n^3+3}; \quad \text{b) } \sum_{n=1}^{+\infty} \frac{\sin(n^2)}{n^2}; \quad \text{c) } \sum_{n=1}^{+\infty} \cos(n\pi) \sin(n - \sqrt{n^2-1})$$

11. Determine whether the following sets are open, closed or not open and not closed. Determine their interior, closure, boundary, set of accumulation points and set of isolated points.

    (a) $A = (-5, 7) \setminus \{0, 4\}$
    (b) $A = (\mathbb{Z} \cap [5, 7]) \cup \{x \in \mathbb{R} : \ x^2 - 7x + 6 > 0\}$
    (c) $A = \{2n : \ n \in \mathbb{N}\}$
    (d) $A = \{x \in \mathbb{R} : \ \sqrt{x^2+1} > x + 3\}$
    (e) $A = \{x \in \mathbb{R} : \ \exp(3x^2 - 4) \geq 1/e\}$
    (f) $A = \{x \in \mathbb{R} : \ x^{11} - 3x^{10} + 5x^9 - 7x^4 + 3x^2 + 2 = 0\}$
    (g) $A = \{x \in \mathbb{R} : \ x^2 - 2x - 1 \geq 0\}$ and $B = \{x \in \mathbb{Q} : \ x^2 - 2x - 1 \geq 0\}$
    (h) $A = \{x = t^3 + 5 : \ t \in \mathbb{R} \cap [0, 2]\}$ and $B = \{x = q^3 + 5 : \ q \in \mathbb{Q} \cap [0, 2]\}$
    (i) $A = \bigcup_{n=1}^{\infty} \left(1 - \frac{1}{n}, 3n\right); \quad B = \bigcap_{n=1}^{\infty} \left(1 - \frac{1}{n}, 3n\right)$
    (j) $A = \bigcup_{n=1}^{\infty} \left(\frac{1}{2n+1}, \frac{1}{2n}\right]; \quad B = \bigcap_{n=1}^{\infty} \left(\frac{1}{2n+1}, \frac{1}{2n}\right]$

(k) $A = \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 5 + \frac{1}{n}\right]$; $B = \bigcap_{n=1}^{\infty} \left[\frac{1}{n}, 5 + \frac{1}{n}\right]$

(l) $A = \bigcap_{n\in\mathbb{N}} [n, n^2]$; $B = \bigcap_{n\in\mathbb{N}} [1/n^2, 1/n]$

12. Let A be a subset of $\mathbb{R}$. Assume that $\sup A$ is finite. Prove that $\sup A$ belongs to $\overline{A}$ and to ∂A.

    Hint: the exercise is not easy, try to use the characterizations with sequences.

13. Let A be a subset of $\mathbb{R}$. Let $B = \{x : \ x \text{ is an accumulation point of } A\}$ and $C = \{x : \ x \text{ is an isolated point of } A\}$. Prove that $B \cap C = \emptyset$ and $B \cup C = \overline{A}$.

1. Determine the behaviour of the following series (converging, diverging to $+\infty$, diverging to $-\infty$ or indeterminate). Determine also if the series is absolutely converging or not.

Take into account that $\lim_n \arctan(x_n) = \pi/2$ for any sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \to +\infty$.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^4+1}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n \arctan n}{\sqrt{n^6+1}}$

(d) $\displaystyle\sum_{n=3}^{\infty} \left(\frac{1}{n^3} - \frac{1}{n^2}\right)$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n!}$

(f) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n(n^3+1)}}$

(g) $\displaystyle\sum_{k=5}^{\infty} \frac{1-2k}{2^{k/2}}$

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3+(-1)^n}}$

(i) $\displaystyle\sum_{n=2}^{\infty} n^{-n/2}$

(j) $\displaystyle\sum_{k=1}^{\infty} \frac{2+(-1)^k}{k^2}$

(k) $\displaystyle\sum_{k=3}^{\infty} (-1)^k \frac{\sqrt{k+1}-\sqrt{k}}{k}$

(l) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n+\sqrt{n}}$

(m) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}(\cos(n^2)+\sqrt{n})$

1) $\displaystyle\sum_{N=1}^{\infty} \frac{N^2}{2^N}$ $\quad \dfrac{a_{N+1}}{a_N}$ $\quad \dfrac{(N+1)^2}{\cancel{2^N}^{+1}} \cdot \dfrac{\cancel{2^N}}{N^2} = \dfrac{N^2+1+2N}{2N^2} = \dfrac{\cancel{N^2}(1+1/N^2+2/N)}{2\cancel{N^2}} = \dfrac{1}{2}$ ABSOLUTELY CONVERGING

2) $\displaystyle\sum_{N=1}^{\infty} \frac{N}{\sqrt{N^4+1}}$ $\quad \dfrac{a_{N+1}}{a_N}$ $\quad \dfrac{N+1}{\sqrt{(N+1)^4+1}} \cdot \dfrac{\sqrt{N^4+1}}{N} = \dfrac{N+1}{\sqrt{N^4+1+4N^3+4N+6N^2+1}} \cdot \dfrac{\sqrt{N^4(1+\frac{1}{N^4})}}{N} =$

$= \dfrac{N+1}{\sqrt{N^4+4N^3+6N^2+4N+2}} \cdot \dfrac{N^{\cancel{2}}\sqrt{1+1/N^4}}{\cancel{N}} = \dfrac{(N+1)N\sqrt{1+1/N^4}}{\sqrt{N^4\left(1+\frac{4}{N}+\frac{6}{N^2}+\frac{4}{N^3}+\frac{2}{N^4}\right)}}$

$= \dfrac{(N+1)\cancel{N}\sqrt{1+1/N^4}}{N^{\cancel{2}}\sqrt{1+\frac{4}{N}+\frac{6}{N^2}+\frac{4}{N^3}+\frac{2}{N^4}}} = \dfrac{(N+1)\sqrt{1+1/N^4}}{N\sqrt{1+\frac{4}{N}+\frac{6}{N^2}+\frac{4}{N^3}+\frac{2}{N^4}}}$ *

$$\frac{n+1}{\sqrt{(n+1)^4+1}} \cdot \frac{\sqrt{n^4+1}}{n} \qquad \frac{n\left(1+\cancel{1/n}\right)\sqrt{n^4\left(1+\cancel{\frac{1}{n^4}}\right)}}{\sqrt{n^4\left(1+\cancel{\frac{4}{n}}+\cancel{\frac{6}{n^2}}+\cancel{\frac{4}{n^3}}+\cancel{\frac{2}{n^4}}\right)} \quad n}$$

$$= \frac{n^2}{n^2} = 1$$

$$\lim_n \sqrt[n]{a_n} = \lim_n \left(\frac{n}{\sqrt{n^4+1}}\right)^{1/n} = \left(\frac{n}{\sqrt{n^4(1+1/n^4)}}\right)^{1/n} \quad \frac{\cancel{n}}{n^{\cancel{2}}} \quad \left(\frac{1}{n}\right)^{1/n} = \frac{1}{\sqrt[n]{n}}$$

$$= \frac{1}{\sqrt[n]{n}} \; \frac{\sqrt[n]{n}}{\sqrt[n]{n}} = \frac{\sqrt[n]{n}}{n} = \frac{n^{1/n}}{n} = n^{1/n-1} = \infty^{-1} = 0$$

d) $\sum_{n=3}^{\infty} \left(\frac{1}{n^3} - \frac{1}{n^2}\right)$

e) $\sum_{n=1}^{\infty} \frac{n^2}{n!}$

$$\frac{a_{n+1}}{a_n} \quad \frac{(n+1)^2}{(n+1)!} \cdot \frac{n!}{n^2} = \frac{n^2+1+2n}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{n^2} = \frac{\cancel{n^2}\left(1+1/n^2+2/n\right)}{n+1} \cdot \frac{1}{\cancel{n^2}}$$

$$\frac{1+0+0}{n} = 0 \qquad l<1$$

f) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n(n^3+1)}}$

$$\frac{1}{\sqrt{n^4(1+1/n^4)}} \sim \frac{1}{n^2}$$

con $2>1$

$\sum^{\infty} a_n$ converge

g) $\sum_{k=5}^{\infty} \frac{1-2k}{2^{k/2}} \sim -\frac{1}{2^{k/2}} = 0$

converge

converge

H) $\sum_{N=1}^{\infty} \frac{1}{N^{3+(-1)^N}}$

$(a_N \to 0$ DEFINETLY$)$

I) $\sum_{N=2}^{\infty} N^{-N/2}$ $\quad \sqrt[N]{N^{-N/2}}$

$N^{-\frac{N}{2}\cdot\frac{1}{N}} = N^{-1/2} = \frac{1}{N^{1/2}}$ $\quad \frac{1}{2} < 1$

$\boxed{\text{DIVERGE}}$

J) $\sum_{K=1}^{\infty} \frac{2+(-1)^K}{K^2}$ $\quad \frac{2}{K^2} + \frac{(-1)^K}{K^2}$

CONV

$\left|\frac{1}{K^2}\right|$ $\quad 2>1$ CON

K) $\sum_{K=3}^{\infty} \left|(-1)^K \frac{\sqrt{K+1}-\sqrt{K}}{K}\right|$

$\frac{\sqrt{K+1}-\sqrt{K}}{K}$ $\quad \frac{(\sqrt{K+1}+\sqrt{K})}{(\sqrt{K+1}+\sqrt{K})}$

$\frac{K+1-K}{K\sqrt{K+1}+K\sqrt{K}} \sim \frac{1}{K(\sqrt{K+1}+\sqrt{K})}$

$a_{N+1} \le a_N$

POSIT (1)

INFINITE (2)

$\frac{1}{K}$ DECRESCE (3)

LIEBNIZ

(f) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n(n^3+1)}}$

(g) $\sum_{k=5}^{\infty} \frac{1-2k}{2^{k/2}}$

(h) $\sum_{n=1}^{\infty} \frac{1}{n^{3+(-1)^n}}$

(i) $\sum_{n=2}^{\infty} n^{-n/2}$

(j) $\sum_{k=1}^{\infty} \frac{2+(-1)^k}{k^2}$

(k) $\sum_{k=3}^{\infty} (-1)^k \frac{\sqrt{k+1}-\sqrt{k}}{k}$

(l) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n+\sqrt{n}}$

(m) $\sum_{n=1}^{\infty} \frac{1}{n}(\cos(n^2)+\sqrt{n})$

$$\sum_{n=2}^{\infty} \frac{n^2-1}{n^4\cos(n)} \sim \frac{1}{n^2} \qquad \boxed{\frac{1}{n^2}} \quad \alpha > 1$$

$\downarrow$ CONVERGE

$\uparrow$ CONVERGE

l) $$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n+\sqrt{n}}$$

$\downarrow$

$$\frac{1}{n+\sqrt{n}}$$

POSITIVO
INFINITESIMO
DECRESCE
QUINDI CONVERGE

$a_{n+1} \le a_n$

m) $$\sum_{n=1}^{\infty} \frac{1}{n}\left(\cos(n^2)+\sqrt{n}\right)$$

$\downarrow$

$$\frac{1}{n}\left(\cos(n^2)+\sqrt{n}\right) < \frac{1}{n}\cdot\left(\cancel{1}+\sqrt{n}\right) \sim \frac{1}{\sqrt{n}} \qquad \frac{1}{n^{1/2}} \text{ DIVER}$$

$$\frac{1}{2}\left(\cos(n^2)+\sqrt{n}\right)$$

(n) $$\sum_{n=3}^{\infty} \frac{\arctan n}{n\sqrt{n-1}}$$

(o) $$\sum_{n=1}^{\infty} 3^{-\sqrt{n}}$$

N) $$\sum_{N=3}^{\infty} \frac{\arctan N}{N\sqrt{N-1}} = \frac{\pi/2}{N\sqrt{N-1}} = \frac{\pi}{2N\sqrt{N-1}} =$$

$$\lim_{N} \sqrt[N]{a_N} = l$$

O) $$\sum_{N=1}^{\infty} 3^{-\sqrt{N}} = 3^{-\sqrt{N}\cdot\frac{1}{N}} = 3^{-\cancel{\sqrt{N}}\cdot\frac{1}{\cancel{\sqrt{N}}\sqrt{N}}} \quad 3^{-\frac{1}{\sqrt{N}}} = \left(\frac{1}{3}\right)^{1/\sqrt{N}} = 0$$

CONVERGE

10. Determine the behaviour of the following series. Determine also if the series is absolutely converging or not.

a) $\displaystyle\sum_{n=0}^{+\infty} \frac{n^3+\sin(n^2)}{2n^3+3}$; b) $\displaystyle\sum_{n=1}^{+\infty} \frac{\sin(n^2)}{n^2}$; c) $\displaystyle\sum_{n=1}^{+\infty} \cos(n\pi)\sin(n-\sqrt{n^2-1})$

$(a_N \text{ TO DEFINETLY})$

a) $$\sum_{N=0}^{+\infty} \frac{N^3+\sin(N^2)}{2N^3+3}$$

$$\frac{a_{N+1}}{a_N} \qquad \frac{(N+1)^3+\sin((N+1)^2)}{2(N+1)^3+3} \cdot \frac{2N^3+3}{N^3+\sin(N^2)}$$

$$\frac{N^3+1+3N^2+3N+\sin(N+1)^2}{2N^3+2+6N^2+6N+3} \cdot \frac{2N^3+3}{N^3+\sin(N^2)} = 1$$