$+$ $\times$

$-$ $\div$

TRANSFORMATION OF R.V.'S

X CONTINOUS RANDOM VARIABLE

$h: \mathbb{R} \to \mathbb{R}$
- h IS 1-TO-1
- DIFF. BLE
- INVERSE ALSO DIFF. BLE (h^{-1})

Y NEW RANDOM VARIABLE BASED ON X

$Y = h(X)$

PROBLEM: GIVEN f_X FIND f_Y $\quad y \in \mathbb{R}$

SOLUTION: IF h STRICTLY INCREASING THEN $f_Y(y) = f_X(h^{-1}(y)) \cdot \underbrace{\left[\frac{d}{dy} h^{-1}(y)\right]}_{>0}$

>0 BECAUSE WE ASSUME THE FUNCTION IS INCREASING

NOTICE THAT IF h STRICTLY INCREASING ALSO h^{-1} IS STRICTLY INCREASING.

IF h IS STRICTLY DECREASING THEN

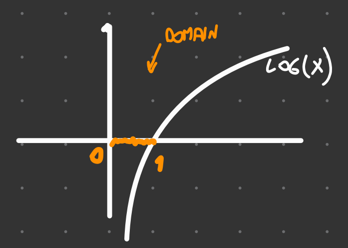
$$f_Y(y) = f_X(h^{-1}(y)) \cdot \left|\frac{d}{dy} h^{-1}(y)\right|$$

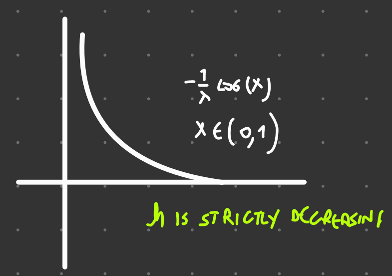
ABSOLUTE VALUE TO "REMOVE" THE MINUS SIGN

- EXERCISE: LET X BE UNIFORM (CONTINOUS) ON $(0,1)$ PUT $Y = -\frac{1}{\lambda}\log(X)$ FIND THE DENSITY OF Y (AKA f_Y)

  $\lambda > 0$ FIXED

- SOLUTION: $h(x) = -\frac{1}{\lambda}\log(x) \quad x \in (0,1)$

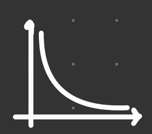




h^{-1} IS GIVEN BY

$-\frac{1}{\lambda}\log(x) = y; \quad x = e^{-\lambda y} = h^{-1}(y)$

$y \in (0, +\infty)$ CODOMAIN OF X



$$\boxed{\frac{d}{dy} h^{-1}(y) = -\lambda e^{-\lambda y}}$$

IF f IS UNIFORM ON (a, b) THEN

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{IF } x \in (a,b) \\ 0 & \text{OTHERWISE} \end{cases}$$

IN OUR CASE $a = 0, b = 1 \Rightarrow \begin{cases} 1 & \text{IF } x \in (0,1) \\ 0 & \text{OTHERWISE} \end{cases}$

FINAL SOLUTION

$$f_Y(y) = \underbrace{f_X(e^{-\lambda y})}_{\text{ALWAYS EQUAL TO 1 BECAUSE BETWEEN } (0,1)} \left|-\lambda e^{-\lambda y}\right| = \boxed{\lambda e^{-\lambda y} \text{ IF } y \in (0, +\infty)} \Rightarrow f_Y(y) = \begin{cases} \boxed{\lambda e^{-\lambda y}} & \text{IF } y \in (0,+\infty) \\ 0 & \text{OTHERWISE} \end{cases}$$

FINAL RESULT

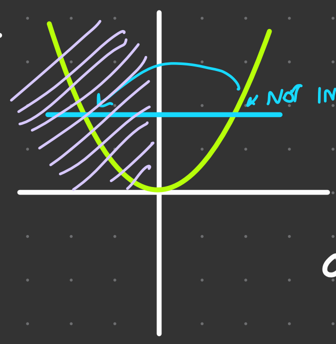
THIS IS THE EXPONENTIAL

IF $y \in (0, +\infty)$ THEN $e^{-\lambda y} \in (0,1)$

$Y = -\frac{1}{\lambda} \ln(X) \quad X \sim \text{UNIF}(0,1)$

SO, YOU CAN SIMULATE AN EXPONENTIAL RANDOM VARIABLE (Y) BY SIMULATING X (WHICH IS UNIFORM) AND THEN TRANSFORMING BY h THE RESULT OF THE SIMULATION OF X

IF WE CONSIDER $h(x) = x^2$



$h: \mathbb{R} \to [0, +\infty]$

I CAN SPLIT THE DOMAIN IN 2 PARTS IF I RESTRICT THE DOMAIN

CONSIDERING ONLY $h(x) = x^2$ DEFINED IN $[0, +\infty]$ THEN THE FUNCTION IS 1-TO-1

THE ALGORITHM BASED ON THE DISTRIBUTION FUNCTION

EXERCISE LET Z BE STANDARD NORMAL/GAUSSIAN $\;\; m=0,\ \sigma^2 = 1$

$Y = Z^2$ FIND f_Y

$h(x) = x^2$

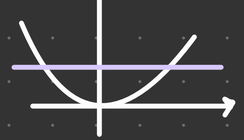
$F_Y(t) = P[Y \le t]$ (DEFINITION OF THE DISTRIBUTION FUNCTION)

$F_Y(t) = P[Z^2 \le t]$ FIRST SOLVE THE INEQUALITY $Z^2 \le t$

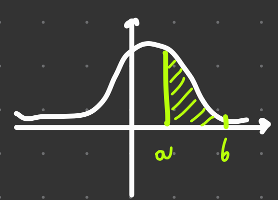
1) IF $t < 0$, $Z^2 \le t$ EMPTY SET (ALWAYS FALSE)

SO $F_Y(t) = P[\emptyset] = 0$

2) IF $t \ge 0$, $z^2 \le t$; $-\sqrt{t} \le Z \le \sqrt{t} \iff Z \in [-\sqrt{t}, \sqrt{t}]$



$F_Y(t) = P[Z \in [-\sqrt{t}, \sqrt{t}]] = P(Z \in (a,b)) = \int_a^b f_Z(x)\,dx$



$$\boxed{f_Z(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}}$$

EVEN FUNCTION / DOMAIN SYMMETRIC W.R.T. TO THE ORIGIN

CHANGE OF VARIABLE $h(x) = x^2 = y$

$$= \int_{-\sqrt{t}}^{\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = 2 \int_0^{\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx \to \cancel{2} \int_0^t \frac{1}{\sqrt{2\pi}} e^{-y/2} \frac{1}{\cancel{2}\sqrt{y}}\,dy = \frac{1}{\sqrt{2\pi}} \int_0^t e^{-y/2} \frac{1}{\sqrt{y}}\,dy = \boxed{\frac{1}{\sqrt{2\pi}} \int_0^t e^{-y/2}\, y^{1/2-1}\,dy}$$

REDUCE THE DOMAIN (IS EVEN)

$x = \sqrt{y}$

$dx = \frac{1}{2\sqrt{y}}\,dy$

GAMMA DISTRIBUTION

$$= \int_0^t \frac{1}{\sqrt{2\pi}} y^{1/2-1} e^{-y/2}\,dy$$

$$\boxed{F_Y(t) = \int_{-\infty}^t f_Y(y)\,dy \quad \text{TRUE IN GENERAL}}$$

GAMMA (τ, λ): $\frac{\lambda^\tau}{\Gamma(\tau)} y^{\tau-1} e^{-\lambda y}$

CHOOSE λ AND τ

$\tau = 1/2 \quad \lambda = 1/2$

$$\frac{\left(\frac{1}{2}\right)^{1/2}}{\Gamma(1/2)} y^{1/2-1} e^{-y/2}$$

$\Gamma(1/2) = \sqrt{\pi} \quad (1/2)^{1/2} = \frac{1}{\sqrt{2}}$

FINAL SOLUTION

$$f(y) = \begin{cases} \frac{1}{\sqrt{2\pi}} y^{1/2-1} e^{-y/2} & \text{IF } y > 0 \\ 0 & \text{OTHERWISE} \end{cases} \to \text{GAMMA}(1/2, 1/2)$$

→ "CHI" SQUARE DISTRIBUTION WITH 1 DEGREE OF FREEDOM

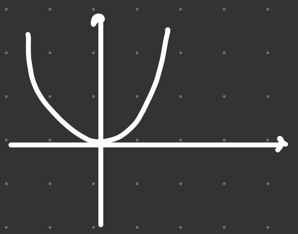
$\chi(1)$

· EXERCISE: LET X BE UNIFORM IN $[-1,1]$

$Y = X^{2N}$ $\quad N \in \mathbb{N}$ $\quad$ FIND f_Y

$h(x) = x^{2N}$ $\quad h: [-1,1] \to [0,+\infty)$

$x^2 / x^4 / x^6$ $\quad$ NOT INJECTIVE



USE AGAIN THE ALGORITHM OF THE DISTRIBUTION FUNCTION

$$F_Y(t) = P[Y \le t] = P(X^{2N} \le t]$$

WE CONSIDER THE INEQUALITY $X^{2N} \le t$

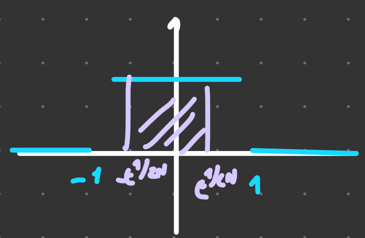
- IF $t < 0$ THE SOLUTION IS $\emptyset$ $\quad F_Y(t) = P[\emptyset] = 0$
- IF $t \ge 0$ THE SOLUTION IS $-t^{1/2N} \le X \le t^{1/2N} = \sqrt[2N]{t}$

$$t \ge 0 \quad F_Y(t) = P[X \in (-t^{1/2N}, t^{1/2N}]]$$

$$P[X \in (a,b]] = \int_a^b f_X(x)\,dx \quad \text{(ALWAYS TRUE FOR CONTINUOUS R.V.'S)}$$

$$= \int_{-t^{1/2N}}^{t^{1/2N}} f_X(x)\,dx$$

$$f_X(x) = \begin{cases} \frac{1}{b-a} = \frac{1}{2} & \text{IF } x \in (-1,1) \\ 0 & \text{OTHERWISE} \end{cases}$$

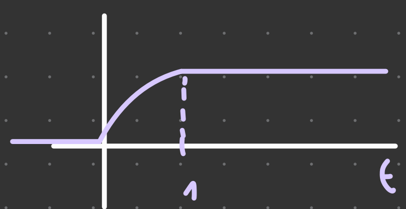


IF $t \in (0,1)$ ALSO $t^{1/2N} \in (0,1)$

IF $t > 1$ ALSO $t^{1/2N} > 1$ SO

$$\int_{t^{1/2N}}^{t^{1/2N}} f_X(x)\,dx = 1$$

$$F_Y(t) = \begin{cases} 0 & \text{IF } t < 0 \\ t & \text{IF } t \in [0,1] \\ 1 & \text{IF } t > 1 \end{cases}$$



$$f_Y(y) = \begin{cases} 0 & \text{IF } y \notin [0,1] \\ \frac{1}{2N} y^{1/2N - 1} & \text{IF } y \in (0,1) \end{cases}$$

NOTICE THAT $\frac{1}{2N}^{1/2N-1} = \dfrac{\Gamma(\frac{1}{2N}+1)}{\Gamma(1/2N)\Gamma(1)} y^{1/2N-1} (1-y)^{1-1}$

BETA WITH $\alpha = \frac{1}{2N}$ $\quad \beta = 1$

$$Y = X^{2N}$$

SO $Y \sim \text{BETA}\left(\frac{1}{2N}, 1\right)$ CAN BE SIMULATED STARTING FROM THE SIMULATION OF $X \sim \text{UNIF}(-1,1)$

TRANSFORMATION OF DISCRETE R.V.'S

LET X BE A DISCRETE R.V. WITH PROB. MASS FUNCTION P_X LET WE SUPPOSE THAT $X: \Omega \to \{1, 2, \dots, k\}$

$$P_X(i) = P[X = i] \quad i \in \{1, 2, \dots k\}$$

$h: \mathbb{R} \to \mathbb{R}$ IS ANY FUNCTION

$$cod(h) = \{y \in \mathbb{R} \mid y = h(i) \text{ FOR SOME } i \in \{1, \dots, k\}\}$$

$$\{y_1, y_2, \dots y_n\} \quad 1 \le n \le k$$

LET $Y = h(X)$ PROBLEM FIND P_Y

SOLUTION

$$P_Y(y_x) = P(Y = y_x) = P[h(X) = y_x]$$

$$x \in \{1, \dots m\} = P[X \in h^{-1}(\{y_x\})]$$

$$P_Y(y_r) = P[X \in h^{-1}(\{y_r\})] =$$

$$= \sum_{\substack{i \in \{1 \dots k\} \\ h(i) = y_r}} P_X(i)$$

$k = 3$

$X: \Omega \to \{1, 2, 3\}$

$P_X(1) = 0{,}2$

$P_X(2) = 0{,}4$

$P_X(3) = 0{,}4$

$h(1) = 0, \ h(2) = \sqrt{\pi}, \ h(3) = 0$

$cod(h) = \{0, \sqrt{\pi}\} = \{y_1, y_2\} \qquad m = 2$

$y_1 = 0, \quad y_2 = \sqrt{\pi}$

$Y = h(X) \qquad P_Y: \{0, \sqrt{\pi}\} \to [0, 1]$

$P_Y(0) = P_X(1) + P_X(3) = 0{,}2 + 0{,}4 = 0{,}6$