$+$ $\times$

$-$ $\div$

$$\lim_{x \to \pm\infty} \frac{-2x^3+5x-3}{x^5+2x} = 0 \qquad \frac{-2x^3+5x-3}{x^5+2x} = \underbrace{\frac{x^3}{x^5}}_{0} \overbrace{\left(\frac{-2+5/x^2-3/x^3}{1+\frac{2}{x^3}} \right)}^{-2}$$

$$\lim_{x \to \pm\infty} \frac{-2x^5+5x-3}{x^5+2x} = -2 \qquad \frac{-2x^5+\frac{5}{x}-3}{x^5-2x} = \cancel{\frac{x^5}{x^5}} \overbrace{\left(\frac{-2+5/x^4-3}{1+2/x^4} \right)}^{-2}$$

$$\lim_{x \to \pm\infty} \frac{x^5+2x}{-2x^4+5x-3} = \mp\infty \qquad \frac{x^5+2x}{-2x^4+5x-3} = \underbrace{\frac{x^5}{x^4}}_{x = \pm\infty} \underbrace{\frac{(1+2/x^4)}{(-2+5/x^3-3/x^4)}}_{-\frac{1}{2}}$$

$$\lim_{x \to \pm\infty} \frac{x^5+2x}{-2x^3+5x-3} = -\infty \qquad \frac{x^5+2x}{-2x^3+5x-3} = \underbrace{\frac{x^5}{x^3}}_{x^2 = +\infty} \underbrace{\frac{(1+2/x^4)}{(-2+5/x^2-3/x^3)}}_{-\frac{1}{2}}$$

$$\frac{P_N(x)}{Q_N(x)} \qquad A = \{x \in \mathbb{R} : Q_N(x) = 0\}$$

$x_0 \in A$ SO $Q_N(x_0) = 0$ x_0 IS AN ACC. POINT OF $\mathbb{R} \setminus A$

$$\lim_{x \to x_0} \frac{P_N(x)}{Q_N(x)} \; ?$$

LET $1 \leq \ell \leq M$ BE THE MULTIPLICITY OF x_0 AS A ZERO OF Q_M, THAT IS

$$Q_M(x) = (x - x_0)^{\ell} \, \tilde{Q}_{M-\ell}(x)$$

$\tilde{Q}_{M-\ell}$ POLY. OF DEGREE $M-\ell$ SUCH THAT $\tilde{Q}_{M-\ell}(x_0) \neq 0$

LET $0 \leq i \leq N$ BE THE MULTIPLICITY OF x_0 AS A ZERO OF P_N THAT IS $P_N(x) = (x - x_0)^i \, \tilde{P}_{N-i}(x)$

$\tilde{P}_{N-i}$ POLY OF DEGREE $N-i$ SUCH THAT $\tilde{P}_{N-i}(x_0) \neq 0$

$i = 0 \iff P_N(x_0) \neq 0$ AND $\tilde{P}_{N-0} = P_N$

$$\frac{P_N(x)}{Q_M(x)} = \frac{(x - x_0)^i}{(x - x_0)^{\ell}} \, \frac{\tilde{P}_{N-i}(x)}{\tilde{Q}_{M-\ell}(x)}$$

$x \to x_0$ $\dfrac{\tilde{P}_{N-i}(x_0)}{\tilde{Q}_{M-\ell}(x_0)} \neq 0$

WHAT ABOUT

$$\lim_{x \to x_0} \frac{(x - x_0)^i}{(x - x_0)^{\ell}}$$

$k = i - \ell \in \mathbb{Z}$

$\downarrow$

$$\lim_{x \to x_0} (x - x_0)^k = ?$$

$k > 0$ $\lim_{x \to x_0} (x - x_0)^k = 0$

$k = 0$ $\lim_{x \to x_0} (x - x_0)^0 = 1$

$k < 0$ $\lim_{x \to x_0} (x - x_0)^k$ IT DEPENDS IF $-k \in \mathbb{N}$ IS EVEN OR ODD

$$\lim_{x \to x_0} \frac{1}{(x - x_0)^n} = +\infty \text{ IF } n \text{ IS EVEN}$$

$$\lim_{x \to x_0} \frac{1}{(x - x_0)^n} = \pm\infty \text{ IF } n \text{ IS ODD}$$

LIKE $\lim_{x \to 0} \frac{1}{x}$; $\lim_{x \to 0} \frac{1}{x^2}$

EXAMPLE

$$\lim_{x \to 1^\pm} \frac{x^3 - 4x^2 + 3x}{x^2 + x - 2} = \lim_{x \to 1^\pm} \frac{\cancel{(x-1)}(x-3)x}{\cancel{(x-1)}(x+2)} \quad i = 1 \quad l = 1$$

$$\lim_{x \to 1^\pm} \frac{(x-x)\,x}{(x+c)} = -\frac{2}{3}$$

$$f(x) = \frac{x^3 - 4x^2 + 3x}{x^2 + x - 2}$$

f CAN BE EXTENDED BY CONTINUITY IN 1 BY CONVERGING

$$\tilde{f}(x) = \begin{cases} \dfrac{x^3 - 4x^2 + 3x}{x^2 + x - 2} & x \neq \pm 1 \quad \boxed{x \neq -2} \\ -\dfrac{2}{3} & \text{if } x = 1 \end{cases}$$

ACTUALLY

$$\tilde{f}(x) = \frac{(x-3)x}{x+2} \qquad x \neq -2$$

- $$\lim_{x \to 1^\pm} \frac{x^3 - 4x^2 + 3x}{x^3 - 3x + 2} = \lim_{x \to 1^\pm} \frac{(x-1)}{(x-1)^2} \frac{(x-3)x}{x-2}$$

$x \to 1 \quad -\frac{2}{3}$

$$\lim_{x \to 1^\pm} \frac{1}{x-1} = \pm\infty$$

$$\lim_{x \to 1^\pm} \frac{x^3 - 4x^2 + 3x}{x^3 - 3x + 2} = \mp\infty$$

$i = 1 \quad l = 2$

$k = -1$

$k = 1$ ODD

- $$\lim_{x \to 1^\pm} \frac{(x-1)^2 (x-3) x}{(x-1)(x+2)} = 0$$ [CAN BE EXTENDED BY CONTINUITY IN 0]

$$\tilde{f}(x) = \frac{(x-1)(x-3)x}{x+2}$$

$i = 2 > l = 1 \qquad k = i - l > 0$

- $\lim_{x \to 1^{\pm}} \frac{(x-1)(x-3)x}{(x-1)^2(x+2)} = -\infty$ $\quad i=2 \quad l=3 \quad k=-2 \quad n=-k=2$ EVEN

$x \to 1^{\pm} \to +\infty$

$x \to 1^{\pm} = \frac{2}{3}$

LIMITS AND CONTINUITY FOR THE COMPOSITION OF FUNCTION

THEOREM $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m \quad g : B \in \mathbb{R}^m \to \mathbb{R}^l$

ASSUME $f(A) \subseteq B$ SO WE CAN DEFINE

$h = g \circ f : A \subseteq \mathbb{R}^n \to \mathbb{R}^l$

$x^0 \in A$ AND $y^0 = f(x^0) \in B$

THEN

- IF f IS CONTINUOUS IN x^0 AND g IS CONTINUOS IN y^0 THEN $h = g \circ f$ IS CONTINUOUS IN x^0
- IF f IS CONTINUOUS IN A AND g IS CONTINUOUS IN B THEN $h = g \circ f$ IS CONTINUOUS IN A

IN OTHER WORDS, THE COMPOSITION OF CONTINUOUS FUNCTIONS IS A CONTINUOUS FUNCTION

PROOF x^0 ISOLATED OK

- x^0 IS ACC. POINT OF A

LET $\{x_N\}_{N \in \mathbb{N}} \subseteq A$ SUCH THAT $x^N \to x^0 \Rightarrow$

✓CONT of f

✓CONT of g

$$y^N = f(x^N) \to y^0 = f(x^0) \Rightarrow g(y^N) \to g(y^0)$$

$$g(y^N) = g(f(x^N)) = (g \circ f)(x^N) \to g(y^0) = g(f(x^0)) =$$

$$(g \circ f)(x^0) \Rightarrow g \circ f \text{ IS CONT. IN } x_0$$

IMPORTANT REMARK

A SIMILAR RESULT HOLDS FOR THE LIMIT OF THE COMPOSITION OF A FUNCTION

x^0 BE AN ACC. POINT FOR A

LET $\lim_{x \to x^0} f(x) = y^0 \in \mathbb{R}^M$

IF $y^0 \in B$ AND g IS CONTINUOUS IN y^0, THEN

$$\lim_{x \to x^0} (g \circ f)(x) = g(y^0) = g\left(\lim_{x \to x^0} f(x)\right)$$

ASSUME y^0 ACC. POINT FOR B AND

$$\exists \lim_{y \to y^0} g(y) = z^0 \in \mathbb{R}^l$$

IT IS STILL TRUE THAT

$$\exists \lim_{x \to x^0} (g \circ f)(x) = \lim_{y \to y^0} g(y) = \lim_{y \to \lim_{x \to x^0} f(x)} g(y)$$

TRUE PROVIDED $f(x) \neq y^0 \; \forall x \in D$

FOR $\forall x \in A \cap B_{r_0}(x^0)$ FOR SOME $r_0 > 0$)

EXAMPLE

$$f(x) = 0 \;\; \forall x \, ; \quad g(y) = \begin{cases} 2 & y \neq 0 \\ 1 & y = 0 \end{cases}$$

$$(g \circ f)(x) = 1 \Rightarrow \lim_{x \to 0} (g \circ f)(x) = 1$$

BUT $\lim_{y \to \lim_{x \to 0} f(x) = 0} g(y) = 2$

REMARK $M = 1$

SAME RESULT FOR $x \to x_0^{\pm}$, $x \to \pm\infty$

$\boxed{M = 1}$ $\lim_{x \to x^0} f(x) = l \in [-\infty, +\infty]$

EXAMPLE

- $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{y \to 0^+} f(y) = \lim\limits_{x \to 0^-} f(-x)$

$y = f(x) = -x$

- $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{y \to +\infty} f(y) = \lim\limits_{x \to -\infty} f(-x) \quad y = f(x) = -x$

- $\lim\limits_{x \to \pm\infty} f(x) = \lim\limits_{y \to \pm\infty} f(y) = \lim\limits_{x \to 0^\pm} f\left(\frac{1}{x}\right) \quad y = f(x) = \frac{1}{x}$

- $\lim\limits_{x \to 0} f(x) = \lim\limits_{y \to 0} f(y) = \lim\limits_{x \to x_0} f(x - x_0) \quad y = f(x) = x - x_0$

ADJOIN $\mathbb{R}^N$

SO $\lim\limits_{x \to x_0} f(x) = \lim\limits_{y \to 0} f(y + x_0) = \lim\limits_{x \to 0} f(x + x_0)$

$$\lim_{x \to x_0} \frac{1}{x - x_0} = \lim_{x \to 0} \frac{1}{x}$$

LIMITS OF ELEMENTARY FUNCTION

POWERS $\alpha > 0$

- $[0, +\infty) \ni x \to x^{\alpha}$ IS CONTINUOUS AND

$$\lim_{x \to +\infty} x^{\alpha} = +\infty$$

- $(0, +\infty) \ni x \to x^{-\alpha}$ IS CONTINUOUS AND

$$\lim_{x \to +\infty} x^{-\alpha} = 0 \text{ AND } \lim_{x \to 0^+} x^{-\alpha} = +\infty$$

- $\alpha = \frac{m}{n}$ $m, n \in \mathbb{N}$ WITH NO COMMON FACTOR; n **ODD**

$\mathbb{R} \ni x \to x^{m/n}$ IS CONTINUOUS AND $\lim_{x \to +\infty} x^{m/n} = +\infty$

$$\lim_{x \to -\infty} x^{n/m} = \begin{cases} +\infty & m \text{ EVEN} \\ -\infty & m \text{ ODD} \end{cases}$$

- m, n AS BEFORE

$\mathbb{R} - \{0\} \ni x \to x^{-m/n}$ IS CONTINUOUS AND $\lim_{x \to \pm\infty} x^{-m/n} = 0$

$$\lim_{x \to 0^+} x^{-m/n} = +\infty \text{ AND}$$

$$\lim_{x \to 0^-} x^{-m/n} = \begin{cases} +\infty & m \text{ EVEN} \\ -\infty & m \text{ ODD} \end{cases}$$

EXPONENTIALS

- $\mathbb{R} \ni x \to \exp(x) = e^x$ IS CONTINUOUS AND

$\lim_{x \to +\infty} e^x = +\infty$ AND $\lim_{x \to -\infty} e^x = 0$

$$\lim_{x \to -\infty} e^x = \lim_{x \to +\infty} e^{-x} = \lim_{x \to +\infty} \frac{1}{e^x} = 0$$

REMARK

$a > 0 \quad a \neq 1 \quad a^x = e^{(\log a)x} \qquad \log a \neq 0$

LOGARITHM

- $\lim_{x \to \pm\infty} \left(1 + \frac{1}{x}\right)^x = e$

- $(0, +\infty) \ni x = \log(x)$ IS CONTINUOUS AND

$\lim_{x \to +\infty} \log(x) = +\infty$ AND $\lim_{x \to 0^+} \log(x) = -\infty$

$$\lim_{x \to 0^+} \log(x) = \lim_{x \to +\infty} \log\left(\frac{1}{x}\right) = \lim_{x \to +\infty} (-\log(x)) = -\infty$$

REMARK $a>0$ $a\neq 1$ $\log_a(x)=\dfrac{\log(x)}{\log(a)}$ $\log a\neq 0$

- $\lim\limits_{x\to 0} \dfrac{e^x-1}{x}=1$ $\lim\limits_{x\to 0} \dfrac{\log(1+x)}{x}=1$

TRIGONOMETRIC FUNCTION

$\cos:\mathbb{R}\to\mathbb{R}$ AND $\sin:\mathbb{R}\to\mathbb{R}$ ARE CONTINOUS

$\nexists \lim\limits_{x\to\pm\infty} \cos(x)$ $\nexists \lim\limits_{x\to\pm\infty} \sin(x)$

EXAMPLE

$x_n=n\pi$ $\cos(n\pi)=(-1)^n$ DOES NOT ADMIT LIMIT

- $\tan \mathbb{R}\setminus\left\{\frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}\right\}\to\mathbb{R}$ IS CONTINOUS

AND $\lim\limits_{x\to -\frac{\pi}{2}^{\pm}} = \mp\infty$ AND $\lim\limits_{x\to\frac{\pi}{2}^{\pm}} \tan(x)=\mp\infty$

AND $\nexists \lim\limits_{x\to\pm\infty} \tan(x)$

$x\to\frac{\pi}{2}^{\pm}$ $\sin(x)=1$ AND $\cos(x)\to 0^{\mp}$

$x\to-\frac{\pi}{2}$ $\sin(x)\to -1$ AND $\cos(x)\to 0^{\pm}$

- $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ AND $\lim_{x \to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$

EXAMPLE

$f(x) = \frac{\sin(x)}{x}$ $x \neq 0$ CAN BE EXTENDED TO $\tilde{f}(x) = \begin{cases} \frac{\sin(x)}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

WHICH IS CONTINUOS ON $\mathbb{R}$

FINAL REMARK

$$f(x) = \frac{\cos^2(x) - \cos(1+x^2) - 3xe^{+x^2} - 5}{3\sin(x^3)+5}$$

IS CONTINUOUS WHENEVER IT IS DEFINE (IN THIS CASE IN $\mathbb{R}$)

ORDER ON INFINITES AND INFINITESIMAL

ORDER OF INFINITES

WE HAVE THE FOLLOWING ORDER (FROM THE SMALLER TO THE BIGGER) AMONG THE INFINITE

$\forall a > 0$, $a < b < b_1$, $\alpha > 0$

AS $x \to +\infty$

$(\log x)^a$, x^b, x^{b_1}, $e^{\alpha x}$, x^x

THAT IS

$$\lim_{x \to +\infty} \frac{(\log x)^a}{x^b} = \lim_{x \to +\infty} \frac{x^b}{x^{b_1}} = \lim_{x \to +\infty} \frac{x^{b_1}}{e^{\alpha x}} = \lim_{x \to +\infty} \frac{e^{\alpha x}}{x^x} = 0$$

ORDER OF INFINTESIMAL

WE HAVE THE FOLLOWING ORDERING (FROM THE SMALLER TO THE BIGGER) AMONG THE INFINTESIMAL

$\forall a > 0, \ 0 < b < b_1, \ \alpha > 0$

- AS $x \to +\infty$

$(\log x)^{-a} \quad \frac{1}{x^b} \quad \frac{1}{x^{b_1}} \quad e^{-\alpha x} \quad x^{-x}$

THAT IS

$$\lim_{x \to +\infty} \frac{\frac{1}{x^b}}{\log(x)^{-a}} = \lim_{x \to +\infty} \frac{1/x^{b_1}}{1/x^b} = \lim_{x \to +\infty} \frac{e^{-\alpha x}}{1/x^{b_1}} =$$

$$\lim \frac{x^{-x}}{e^{-\alpha x}} = 0$$

- AS $y \to 0^+$ $\quad \left(\log \frac{1}{y}\right)^{a} \quad y^b, \ y^{b_1}, \ e^{-\alpha/y}, \ \left(\frac{1}{y}\right)^{-1/y}$

$$\lim_{y \to 0} \frac{y^b}{\log(1/y)^a} = \lim_{y \to 0^+} \frac{y^{b_1}}{y^b} = \lim_{y \to 0^+} \frac{e^{-\alpha/y}}{y^{b_1}}$$

$$= \lim_{y \to 0^+} \frac{(1/y)^{-1/y}}{e^{-\alpha/y}} = 0 \qquad x = 1/y$$

REMARK

$a > 0,\ b > 0\ \alpha > 0$

- $\lim_{x \to +\infty} x^b e^{-a x} = 0$ $[\infty \cdot 0]$
- $\lim_{x \to 0^+} y^b |\log y|^\alpha = 0$ $[0 \cdot \infty]$

SINCE $|\log y| = -\log(y) = \log\left(\frac{1}{y}\right)$ $0 < y \leq 1$

EXAMPLE

$\lim_{x \to \pm\infty} \arctan(x) = \pm \frac{\pi}{2}$

ARCTAN IS ODD, ENOUGH TO CONSIDER $x \to +\infty$

ARCTAN IS INCREASING AND $|\arctan(x)| < \pi/2$ $\forall x \in \mathbb{R}$

SO $\exists \lim_{x \to +\infty} \arctan(x) = l \leq \pi/2$

BY CONTRADICTION ASSUME $l < \pi/2$ THEN

$+\infty = \lim_{x \to +\infty} x = \lim_{x \to +\infty} \tan(\arctan(x)) = \tan l < +\infty$

CONTRADICTION

- $\lim_{x \to \frac{\pi}{2}} \dfrac{\cos(x)}{\frac{\pi}{2} - x}$ $\quad y = f(x) = \frac{\pi}{2} - x$

$x \to \frac{\pi}{2}^-$ THEN $y = f(x) \to 0^+$

$g(y) = \dfrac{\cos\left(\frac{\pi}{2} - y\right)}{y} = \dfrac{\sin(y)}{y} \xrightarrow{y \to 0} 1$

$$\frac{\cos(x)}{\frac{\pi}{2}-x} = \cancel{g}(f(x)) = \frac{\cos\left(\frac{\pi}{2}-y\right)}{y} \to 1$$

- $$\lim_{x \to \frac{\pi}{2}^-} \left(\frac{\pi}{2}-x\right)\tan(x) = 1$$

FROM THE PREVIOUS ONE

$$\hookrightarrow \frac{\sin(x)}{\cos(x)}$$

$$\frac{\pi/2 - x}{\cos(x)} \cdot \sin(x)$$

- $$\lim_{x \to +\infty} x\left(\frac{\pi}{2} - \arctan(x)\right) = 1 \qquad \infty \cdot 0$$

$$x \to +\infty \Rightarrow y = \arctan(x) \to \frac{\pi}{2}^-$$

$$x\left(\frac{\pi}{2} - \arctan(x)\right) = \tan(y)\left(\frac{\pi}{2} - y\right) \to 1$$

$$y \to \frac{\pi}{2}^-$$