QR FACTORIZATION

$A \in \mathbb{R}^{m \times n}$ $\quad$ A LIN. INDEP. COLUMNS

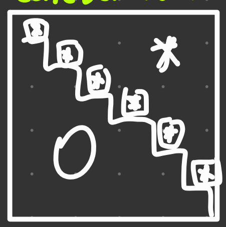
$A = QR$ $\quad$ $Q \in \mathbb{R}^{m \times n}$, $R \in \mathbb{R}^{n \times n}$

COLUMNS OF Q FORM AN ORTHONORMAL BASIS $Col(A)$ $\quad$ $Q^T Q = I_n$

R UPPER TRIANGULAR MATRIX, WITH DIAGONAL ENTRIES STRICLY POSITIVE

R IS INVERTIBLE

ECHELON FORM



LEAST SQUARE PROBLEM

$A \in \mathbb{R}^{m \times n}$ $(m \geq n)$

COLUMNS OF A ARE LIN. INDEPENDENT $\quad$ $\underline{b} \in \mathbb{R}^m$

L.S.P. (LEAST SQUARE PROBLEM) ASSOCIATED TO $A\underline{x} = \underline{b}$

NORMAL EQUATIONS: $A^T A \hat{\underline{x}} = A^T \underline{b}$ $\quad$ $A = QR$

$(QR)^T QR \hat{\underline{x}} = (QR)^T \underline{b}$ $\quad$ REMEMBER: $(QR)^T = R^T Q^T$

$\downarrow$

$R^T \underbrace{Q^T Q}_{I} R \hat{\underline{x}} = R^T Q^T \underline{b}$

$\underbrace{(R^T)^{-1} R^T}_{I} R \hat{\underline{x}} = (R^T)^{-1} R^T Q^T \underline{b}$

$\Downarrow$

$\boxed{R \hat{\underline{x}} = Q^T \underline{b}}$ $\longrightarrow$ VERY EASY TO SOLVE

$\Downarrow$ R IS INVERTIBLE MULTIPLY BOTH SIDE BY R^{-1}

$\hat{\underline{x}} = R^{-1} Q^T \underline{b}$

THEOREM $A \in \mathbb{R}^{m \times n}$ WITH LIN. INDEP. COLUMNS, $\underline{b} \in \mathbb{R}^m$. THEN THE LEAST SQUARE SOLUTION ASSOCIATED TO $A\underline{x} = \underline{b}$ IS GIVEN BY $\hat{\underline{x}} = R^{-1} Q^T \underline{b}$ WHERE $A = QR$ IS THE QR FACTORIZATION OF A

OBSERVATION $\hookrightarrow$ IS EQUAL TO SAY THAT $R \hat{\underline{x}} = Q^T \underline{b}$

$A^T A \hat{\underline{x}} = A^T \underline{b}$

$R \hat{\underline{x}} = Q^T \underline{b}$

EXAMPLE

$$A = \begin{bmatrix} 1 & 1 & 4 \\ -1 & -1 & -2 \\ 1 & -2 & -3 \\ 0 & 0 & 1 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

① WRITE THE QR FACTORIZATION OF A

② BUILD THE LINEAR SYSTEM ABOVE $R\hat{\underline{x}} = Q^T \underline{b}$

①
$$\underline{v_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \quad \underline{v_2} = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 0 \end{bmatrix} \quad \underline{v_3} = \begin{bmatrix} 4 \\ -2 \\ -3 \\ 1 \end{bmatrix}$$

$$\underline{\tilde{w}_1} = \underline{v_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\underline{\tilde{w}_2} = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 0 \end{bmatrix} - \frac{(\underline{\tilde{w}_1}, \underline{v_2})}{\|\underline{\tilde{w}_1}\|^2} \underline{\tilde{w}_1} = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 0 \end{bmatrix} - \frac{0}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 0 \end{bmatrix}$$

$$\underline{\tilde{w}_3} = \begin{bmatrix} 4 \\ -2 \\ -3 \\ 1 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \frac{12}{6}\begin{bmatrix} 1 \\ -1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$(1\cdot 4) + (-1\cdot(-2)) + (1\cdot(-3)) + (0\cdot 1)$$

$$\underline{\tilde{w}_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \quad \underline{\tilde{w}_2} = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 0 \end{bmatrix} \quad \underline{\tilde{w}_3} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\|\tilde{w}_1\| = \sqrt{3} \quad \|\underline{\tilde{w}_2}\| = \sqrt{6} \quad \|\underline{\tilde{w}_3}\| = \sqrt{3}$$

$$1^2+(-1)^2+(1)^2+(0)^2$$

$$\underline{w_1} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \quad \underline{w_2} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -1 \\ -2 \\ 0 \end{bmatrix} \quad \underline{w_3} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 0 & 0 & 1/\sqrt{3} \end{bmatrix}$$

$$R = Q^T A$$

$$R = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} & 0 \\ 1/\sqrt{6} & -1/\sqrt{6} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & 0 & 1/\sqrt{3} \end{bmatrix}\begin{bmatrix} 1 & 1 & 4 \\ -1 & -1 & -2 \\ 1 & -2 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{3} & 0 & 3/\sqrt{3} \\ 0 & 6/\sqrt{6} & 12/\sqrt{6} \\ 0 & 0 & 3/\sqrt{3} \end{bmatrix}$$

ALREADY FOUND

② $R\hat{\underline{x}} = Q^T \underline{b}$

$$Q^T \underline{b} = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} & 0 \\ 1/\sqrt{6} & -1/\sqrt{6} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & 0 & 1/\sqrt{3} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{3} \\ -1/\sqrt{6} \\ 1/\sqrt{3} \end{bmatrix}$$

$\cdot 1 \quad \cdot 0 \quad \cdot 1 \quad \cdot 0$

$$\begin{bmatrix} 3/\sqrt{3} & 0 & 3/\sqrt{3} \\ 0 & 6/\sqrt{6} & 12/\sqrt{6} \\ 0 & 0 & 3/\sqrt{3} \end{bmatrix}\begin{bmatrix} \underline{\hat{x}_1} \\ \underline{\hat{x}_2} \\ \underline{\hat{x}_3} \\ \underline{\hat{x}_3} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{3} \\ -1/\sqrt{6} \\ 1/\sqrt{3} \end{bmatrix}$$