Knowledge Representation and Reasoning

Exercise Session 4

Exercise 1. NNF (∗)

Transform the following $\mathcal{ALC}$ concepts to negation normal form.

1. $\neg(A \sqcup \exists r.A) \sqcup \forall r.\neg B$ $\neg A \sqcap \forall r.\neg A \sqcup \forall r.\neg B$
2. $\exists r.\neg(\forall s.B \sqcap B) \sqcap (\neg B \sqcup A)$ $\exists r.(\exists s.\neg B \sqcup \neg B) \sqcap (\neg B \sqcup A)$
3. $\neg(\exists r.\neg A \sqcup \forall s.\neg(\neg A \sqcup B) \sqcup \neg A)$ $\forall r.A \sqcap \exists s.(\neg A \sqcup B) \sqcap A$

Exercise 2. Satisfiability (∗)

Decide whether the following concepts are satisfiable or not. If they are, provide a satisfying interpretation.

1. $A \sqcap \neg\forall r.(A \sqcup B) \sqcap \neg\exists r.\neg B$
2. $B \sqcap (\neg B \sqcup \exists r.\bot) \sqcup \forall r.\bot$

Exercise 3. Disjunctions (∗∗)

Let $\mathcal{ELU}_\bot$ be the extension of $\mathcal{EL}_\bot$ which allows also for the disjunction constructor ($\sqcup$). Show that $\mathcal{ELU}_\bot$ and $\mathcal{ALC}$ are equivalent.

Exercise 4. Domain size (∗∗)

Construct concepts C and D such that for any interpretation $\mathcal{I}$ it holds that:

1. if $C^\mathcal{I} \neq \emptyset$, then $\Delta^\mathcal{I}$ must have at least two elements
2. if $D^\mathcal{I} \neq \emptyset$, then $\Delta^\mathcal{I}$ must have at least 7 elements

Exercise 5. Disjoint Unions (∗∗)

Let $\mathcal{T}$ be a consistent $\mathcal{ALC}$ TBox. Show that the disjoint union of two models of $\mathcal{T}$ is also a model of $\mathcal{T}$.

Exercise 6. TBox Consistency $(*)$

Check whether the following TBoxes are consistent. If they are, provide a model.

1. $\mathcal{T}_1 = \{A \sqsubseteq \exists r.A,\ A \sqsubseteq \forall r.\neg A\}$
2. $\mathcal{T}_2 = \{A \sqsubseteq \exists r.A,\ \forall r.\neg A \sqsubseteq A\}$
3. $\mathcal{T}_3 = \{A \sqsubseteq \exists r.\neg A,\ \forall s.\neg A \sqsubseteq A, \top \sqsubseteq \forall r.\forall s.A\}$

Exercise 7. Satisfiability $(*)$

Decide whether the following concepts are satisfiable w.r.t. the TBox $\mathcal{T}_1$ from Exercise 6.

1. $A \sqcup \forall r.A$
2. $\neg A \sqcup \exists r.\bot$
3. $\forall r.\exists r.A$

Decide whether $\forall r.\bot$ is satisfiable w.r.t. the TBox $\mathcal{T}_2$ from Exercise 6.

Exercise 8. Subsumption $(*)$

Check whether the following subsumption relations hold w.r.t. the TBox $\mathcal{T}_1$ from Exercise 6.

1. $\exists r.(A \sqcap B) \sqsubseteq \forall r.B$
2. $B \sqcup C \sqsubseteq \forall r.A$
3. $\exists s.\neg A \sqsubseteq \exists r.\neg A$

Exercise 9. Knowledge Base Consistency $(*)$

Check whether the following ABoxes are consistent w.r.t. the TBox $\mathcal{T}_1$ from Exercise 6.

1. $\{r(a,b), \forall r.A(a)\}$
2. $\{\exists r.(B \sqcup A)(a), s(b,a), \forall s.\forall r.\neg B(b)\}$
3. $\{r(a,b), r(b,c), r(c,a)\}$

Exercise 10. Number Restrictions $(***)$

Let $\mathcal{ALCQ}$ be the logic that extends $\mathcal{ALC}$ with *qualified number restrictions* $\geq n\ r.C$ expressing the class of objects that have at least n r-successors belonging to the class C. For example,

$$Person \sqcap \geq 2 hasChild.Female$$

is the class of people having at least two daughters.

Device adequate tableau rules to handle number restrictions.

Exercise 2. Satisfiability * (*)

Decide whether the following concepts are satisfiable or not. If they are, provide a satisfying interpretation.

1. $A \sqcap \neg\forall r.(A \sqcup B) \sqcap \neg\exists r.\neg B$
2. $B \sqcap (\neg B \sqcup \exists r.\bot) \sqcup \forall r.\bot$ *

1. $\underset{①}{A} \sqcap \underset{②}{\exists r.\neg A \sqcap \neg B} \sqcap \underset{③}{\forall r.B}$ $\longrightarrow$ $A(a)$, $\exists r(\neg A \sqcap \neg B)(a)$, $\forall r.B(a)$

CONJUNCTION RULE

$A \xrightarrow{r} \neg A, \neg B, B$

USAT

CREATE A NEW OBJECT

$r(a,b)$, $\neg A \sqcap \neg B(b)$

CONJUNCTION RULE

$\neg A(b)$ $\neg B(b)$

$B(b)$

CRASH

FOR ALL RULE

$\boxed{\text{UNSAT}}$

2. $B \sqcap (\neg B \sqcup \exists r.\bot) \sqcup \forall r.\bot$

$(B \sqcap (\neg B \sqcup \bot)) \sqcup \forall r.\bot$

$B, \neg B$ UNSAT

$B, \bot$ UNSAT(?) *

$B \to \bot$

$\forall r.\bot$ ~~UN~~SAT ? *

$B \sqcap (\neg B \sqcup \exists r.\bot)(a)$

$B(a)$, $\neg B \sqcup \exists r.\bot(a)$

$B(a)$, $\neg B(a)$

CRASH

$B(a)$, $\exists r.\bot(a)$

EXISTENTIAL RULE

$r(a,b)$ $\bot(b)$

CRASH

$\forall r.\bot(a)$

SATURATED AND OPEN SET OF ASSERTION

SAT

DOMAIN: $\Delta^I = \{a\}$

$B^I = \{\delta \mid B(\delta) \in A\} = \emptyset$

NO OBJECT BELONG TO B

$r^I = \{(\delta, y) \mid r(\delta, y) \in A\}$

a — MODEL

Exercise 3. Disjunctions (**)

Let $\mathcal{ELU}_\bot$ be the extension of $\mathcal{EL}_\bot$ which allows also for the disjunction constructor ($\sqcup$). Show that $\mathcal{ELU}_\bot$ and $\mathcal{ALC}$ are equivalent.

(USING TBOXES)

ALC $\quad C ::= A \mid C \sqcap C \mid \exists r.C \mid \neg C$

$\mathcal{ELU}_\bot \quad C ::= A \mid \bot \mid \top \mid C \sqcap C \mid \exists r.C \mid C \sqcup C$

$\bot := A \sqcap \neg A$

$\top := \neg \top$

$C \sqcup D := \neg(\neg C \sqcap \neg D)$

$\boxed{\begin{array}{l} \mathcal{EL}_\bot \quad A \sqcap B \sqsubseteq \bot \\ A \sqsubseteq \neg B \\ B \sqsubseteq \neg A \end{array}}$

EXPRESS NEGATIONS

CREATE A NEW CONCEPT THAT HAS TO BE INTERPRETED AS THE COMPLEMENT OF C^I

$C \longrightarrow$ "$\neg C$" $\quad D^I = \Delta^I \setminus C^I$

$A \quad X_A$ NEW CONCEPT NAME HAS TO BE INTERPRETED AS COMPLEMENT OF A

$A \sqcap X_A \sqsubseteq \bot$ NO INTERSECTION

$\top \sqsubseteq A \sqcup X_A$ AT LEAST IN A OR X_A

$X_A^I = (\neg A)^I$ SAME INTERPRETATION

ALC COMPONENTS CAN BE TRANSFORMED TO NNF $\rightarrow$ SUFFICES TO LOOK AT NEGATIONS OF CONCEPT NAMES

Exercise 4. Domain size (**)

Construct concepts C and D such that for any interpretation $\mathcal{I}$ it holds that:

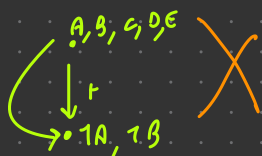
1. if $C^{\mathcal{I}} \neq \emptyset$, then $\Delta^{\mathcal{I}}$ must have at least two elements
2. if $D^{\mathcal{I}} \neq \emptyset$, then $\Delta^{\mathcal{I}}$ must have at least 7 elements

1)

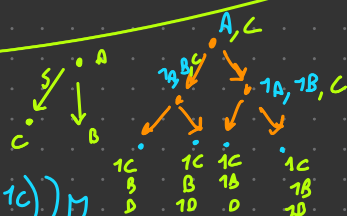
$A \wedge (\exists r. \neg A, B)$

$C := A \sqcap \exists r. \neg A$

2) $\cancel{(A \sqcap B \sqcap C \sqcap D \sqcap E)} \sqcap \exists r. \neg A \sqcap \exists s. \neg B \sqcap$



$A \sqcap \exists r. (B \sqcap \forall r. \neg A) \sqcap \exists s. (C \sqcap \forall r. \neg B)$



$A \sqcap C \sqcap (\exists r. B \sqcap (\exists r. D \sqcap \exists r. \neg D \sqcap \forall r. (B \sqcap \neg C)) \sqcap$

$\exists r. \neg B \; (\exists r. D \sqcap \exists r. \neg D \sqcap \forall r. (B \sqcap \neg C) \sqcap$

$\forall r. (\neg A \sqcap C)$

Exercise 5. Disjoint Unions (**)

Let $\mathcal{T}$ be a consistent $\mathcal{ALC}$ TBox. Show that the disjoint union of two models of $\mathcal{T}$ is also a model of $\mathcal{T}$.

$\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$

$\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$ $\quad \Delta^{\mathcal{I}} \cap \Delta^{\mathcal{J}} = \emptyset$

$\mathcal{I} \uplus \mathcal{J} = (\Delta^{\mathcal{IJ}}, \cdot^{\mathcal{IJ}})$

$\Delta^{\mathcal{IJ}} = \Delta^{\mathcal{I}} \cup \Delta^{\mathcal{J}}$

$\Delta^{\mathcal{IJ}} = \Delta^{\mathcal{I}} \cup \Delta^{\mathcal{J}}$

$r^{\mathcal{IJ}} = r^{\mathcal{I}} \cup r^{\mathcal{J}}$

$\mathcal{EL} ::= A \mid C \sqcap C \mid \exists r. C$

FOR ALL EL CONCEPTS C

$C^{\mathcal{IJ}} = C^{\mathcal{I}} \cup C^{\mathcal{J}}$

$\mathcal{ALC}: \quad A \mid C \sqcap C \mid \exists r. C \mid \neg C$

BY INDUCTION: ASSUME $C^{\mathcal{IJ}} = C^{\mathcal{I}} \cup C^{\mathcal{J}}$

TO SHOW $(\neg C)^{\mathcal{IJ}} = (\neg C)^{\mathcal{I}} \cup (\neg C)^{\mathcal{J}}$

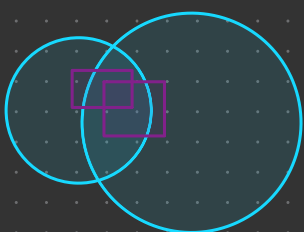
$(\neg C)^{\mathcal{IJ}} = \Delta^{\mathcal{IJ}} \setminus C^{\mathcal{IJ}}$

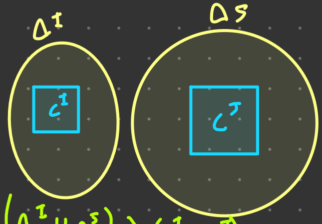
$= \Delta^{\mathcal{I}} \cup \Delta^{\mathcal{J}} \setminus C^{\mathcal{I}} \cup C^{\mathcal{J}}$

$=$

WE KNOW NO OBJECT BELONG TO BOTH

$\Delta^{\mathcal{I}} \cap \Delta^{\mathcal{J}} = \emptyset$





$(\Delta^{\mathcal{I}} \cup \Delta^{\mathcal{J}}) \setminus (C^{\mathcal{I}} \cup C^{\mathcal{J}})$

$\Delta^{\mathcal{I}} \cap \Delta^{\mathcal{J}} = \emptyset$

$C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$

$C^{\mathcal{J}} \subseteq \Delta^{\mathcal{J}}$

TAKE $C \sqsubseteq D \in \mathcal{J}$ $\quad C^{\mathcal{IJ}} = C^{\mathcal{I}} \cup C^{\mathcal{J}}$

$\overset{\text{MODELS}}{\subseteq} D^{\mathcal{I}} \cup D^{\mathcal{J}}$

$D^{\mathcal{IJ}}$

Exercise 6. TBox Consistency (*)

Check whether the following TBoxes are consistent. If they are, provide a model.

1. $\mathcal{T}_1 = \{A \sqsubseteq \exists r.A,\ A \sqsubseteq \forall r.\neg A\}$
2. $\mathcal{T}_2 = \{A \sqsubseteq \exists r.A,\ \forall r.\neg A \sqsubseteq A\}$
3. $\mathcal{T}_3 = \{A \sqsubseteq \exists r.\neg A,\ \forall s.\neg A \sqsubseteq A, \top \sqsubseteq \forall r.\forall s.A\}$

1) $\top \sqsubseteq \neg A \sqcup \exists r.A$ $\quad \top \sqsubseteq \neg A \sqcup \forall r.\neg A$

$\neg A \sqcup \exists r.A$ $\quad \neg A \sqcup \forall r.\neg A$

$\boxed{A \to B \quad \neg A \vee B}$

$\neg A$ TRUE

$\bullet_{\neg A}$

$\neg A$ TRUE

$\bullet_{\neg A}$

$\boxed{\text{MODEL}}$ ✓

$\forall r.\neg A$ TRUE

$\bullet_{\neg A} \xrightarrow{r} \neg A$?

$\boxed{\text{MODEL}}$ ✓

$\exists r.A$ TRUE

$\bullet \xrightarrow{r} A$

$\neg A$ TRUE

$\bullet_{\neg A} \xrightarrow{r} A \xrightarrow{r} A$

$\forall r.\neg A$ TRUE

$\bullet \xrightarrow{r} A, \neg A$

$\boxed{\text{NOT MODEL}}$

2) $\top \sqsubseteq \neg A \sqcup \exists r.A$ $\quad \top \sqsubseteq \forall r.\neg A \sqcup A$

$\neg A$ TRUE

$\forall r.\neg A$ TRUE

$\bullet \xrightarrow{r} \neg A$?

A TRUE

X

$\exists r.A$ TRUE

$\forall r.\neg A$ TRUE

$\xrightarrow{r} A, \neg A$

X

A TRUE

$\xrightarrow{r} A$?

FARE 3

Exercise 7. Satisfiability (*)

Decide whether the following concepts are satisfiable w.r.t. the TBox $\mathcal{T}_1$ from Exercise 6.

1. $A \sqcup \forall r.A$
2. $\neg A \sqcup \exists r.\bot$
3. $\forall r.\exists r.A$

Decide whether $\forall r.\bot$ is satisfiable w.r.t. the TBox $\mathcal{T}_2$ from Exercise 6.

1)

$A \sqcup \forall r.A$

$\neg A$ | $\neg A, \forall r.\neg A$

$\neg A, A$ ✗ | $\neg A\ \forall r.A$

$\neg A$ • √

$\neg A$

$\neg A, A$ • ✗ | $\forall r.\neg A, \forall r.A,$

SAT

$A \sqcup \forall r.A(a)$, $\neg A \sqcup \exists r.A(a)$, $\neg A \sqcup \forall r.\neg A(a)$

$A(a)$

HAVE A CLASH

$A(a), \exists r.A(a), \forall r.\neg A(a)$

$r(a,b), A(b)$

$\neg A(b)$

CLASH

$\forall r.A(a)$

$\neg A(a)$

SATURATED AND OPEN SET OF ASSERTIONS

$\boxed{\text{SAT}}$

$\neg A \sqsubseteq \exists r.\bot$

$\neg A$ | $\neg A, \forall r.\neg A$

$\neg A, \neg A$ | $\neg A, \exists r.\bot$ | $\forall r.\neg A$ $\neg A, \neg A$ | $\neg A, \forall r.\neg A$ $\exists r.A$

$\neg A$ • √

$\forall r.\exists r.A$

$\neg A$ | $\neg A, \forall r\neg A$

√

Exercise 8. Subsumption (*)

Check whether the following subsumption relations hold w.r.t. the TBox $\mathcal{T}_1$ from Exercise 6.

1. $\exists r.(A \sqcap B) \sqsubseteq \forall r.B$
2. $B \sqcup C \sqsubseteq \forall r.A$
3. $\exists s.\neg A \sqsubseteq \exists r.\neg A$

EVERY OBJECT ON THE LEFT MUST BELONG TO THE CONCEPT ON THE RIGHT

SHOW THAT IS IMPOSSIBLE TO HAVE AN ELEMENT ON THE LEFT THAT DOES NOT BELONG ALSO TO RIGHT

$C \sqsubseteq D$ IFF $\boxed{C \sqcap \neg D \text{ IS UNSAT}}$

1. $\exists r.(A \sqcap B) \sqcap \exists r.\neg B$

$\exists r.(A \sqcap B)(a)$ $\exists r.\neg B(a)$

FROM 6 $\neg A \sqcup \exists r.A(b)$ $A \sqcup \forall r.A(b)$

$r(a,b)$ $A \sqcap B(b)$ $A(b)$ $B(b)$

$\exists r.A(b)$ $\forall r.\neg A(b)$

$r(b,c)$ $A(c)$ $\neg A(c)$

CLASH

NO OTHER POSSIBILITY (CLASH BEFORE CHOOSING)

$\boxed{\text{THE CONCEPT IS UNSAT SO THE SUBSUMPTION HOLDS}}$

Exercise 9. Knowledge Base Consistency (∗)

Check whether the following ABoxes are consistent w.r.t. the TBox $\mathcal{T}_1$ from Exercise 6.

1. $\{r(a,b), \forall r.A(a)\}$ $\rightarrow$ $A(b)$ CLASH
2. $\{\exists r.(B \sqcup A)(a), s(b,a), \forall s.\forall r.\neg B(b)\}$
3. $\{r(a,b), r(b,c), r(c,a)\}$

Exercise 10. Number Restrictions (***)

Let $\mathcal{ALCQ}$ be the logic that extends $\mathcal{ALC}$ with *qualified number restrictions* $\geq n\ r.C$ expressing the class of objects that have at least n r-successors belonging to the class C. For example,

$$Person \sqcap \geq 2 hasChild.Female$$

is the class of people having at least two daughters.

Device adequate tableau rules to handle number restrictions.

AT LEAST N R-SUCCESSOR

$(\geq N.\ R.C)(a)$ $\qquad \exists r.C \equiv \geq 1 r.C$

THEN ADD $\{ r(a,b)\ \ C(b_i) \mid 1 \leq j \leq N \}$ TO A

$\neg(\geq N.\ R.C) \equiv (\leq N-1\ r.C)$ $\qquad \forall r.C = \neg\exists r.\neg C$

$= \neg(\geq 1\ r.\neg C)$

$= (\leq 0\ r.\neg C)$

≤ 2 HAS CHILD. FEMALE (RAFEC)

$\exists$. HAS CHILD (FEMALE $\sqcap$ STUDENT)

" " $\sqcap$ MUSICIAN)

" " $\sqcap$ SCIENTIST)

$\neg P \leq N.\ R.C(a) \in A$

$r(a,b)\ldots r(a,b_{N+1}),\ C(b_i)$ THEN CLASH