DOMENICO.CAFIERO@STUDENTI.UNIMI.IT

EX 1

$A := \{$ NATURAL NUMBERS DIVISIBLE BY 5 $\}$

$B := \{$ EVEN NUMBERS $\}$

$C := \{5, 10, 9, 20\}$

$(A \cap B) \cap C$

$(A \cap B) = \{ n : \in \mathbb{N} :$ DIVISIBLE BY 10 $\}$

$(A \cap B) \cap C = \{10, 20\}$

$$\sqrt{x-1} > -2;$$

$x - 1 \geq 0; \quad x \geq 1$

$$\sqrt{2x+1} \geq 5x + 3;$$

$2x + 1 \geq 0; \quad x \geq -\frac{1}{2}$

$B^2 - 4AC$

$2x + 1 \geq 25x^2 + 9 + 30x$

$25x^2 + 28x + 8 \leq 0$

IMP

$$\frac{2}{x}+\frac{2}{|x-1|}\geq 3;$$

$x-1\neq 0;\ x\neq 1 \vee x\neq 0$

$x-1>0;\ x>1$ $\quad \frac{2}{x}+\frac{2}{x-1}\geq 3;\quad \frac{2x-2+2x-3x^2+3x}{x(x-1)}\geq 0$

$x-1<0;\ x<1$

$\rightarrow \frac{2}{x}-\frac{2}{x-1}\geq 3$

$$\frac{2x-2-2x-3x^2+3x}{x(x-1)}\geq 0$$

$$\frac{-3x^2+3x-2}{x(x-1)}\geq 0$$

N: $9-24$

$\Delta<0$

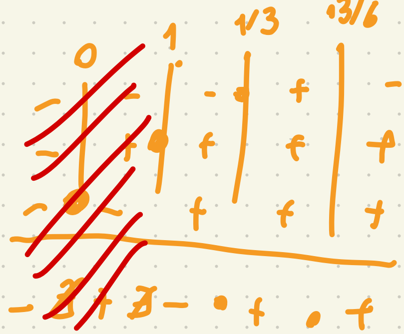
IMP

$$\frac{-3x^2+7x-2}{x(x-1)}\geq 0$$

$\frac{-7\pm 5}{-6} < \begin{matrix} 13/6 & 2{,}1 \\ 1/3 & 0{,}3 \end{matrix}$

$x>0$

$x-1>0;\ x>1$



$x\geq 1/3$

$$\frac{|x|-3}{\sqrt{x-2}} > \sqrt{x};$$

C.E. $x-2>0$; $x>2$

$x>0$; $x>0$

$x>2$

$$\frac{(|x|-3)^2}{x-2} > x; \qquad \frac{\cancel{x^2}+9-6|x|-\cancel{x^2}+2x}{x-2}>0$$

$$\frac{2x-6|x|+9}{x-2}>0$$

$x>0$

$x<0$ IMP

$$\frac{2x-6x+9}{x-2}>0$$

$$\frac{-4x+9}{x-2}>0$$

N: $-4x+9>0$; $x<\frac{9}{4}$

D: $x-2>0$; $x>2$

	2		$9/4$	
$+$		$+$	0	$-$
$-$	0	$+$		$+$
$-$	$\nexists$	$+$	0	$-$

$2 < x \leq \frac{9}{4}$

$$\frac{1}{2x}+|2x-1|<2;$$

$x \neq 0$

$$\frac{1}{2x}+|2x-1|-2<0$$

$2x-1 \geq 0;\ x \geq 1/2$

$2x-1<0;\ x<1/2$

$$\frac{1}{2x}+2x-1-2<0;$$

$$\frac{1+4x^2-6x}{2x}<0$$

$$\frac{4x^2-6x+1}{2x}<0$$

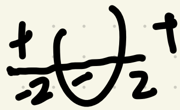
$N>0$ $\quad \frac{6 \pm 2\sqrt{5}}{8}<0 \quad \left\langle \begin{array}{l} \frac{3-\sqrt{5}}{4} \\ \frac{3+\sqrt{5}}{4} \end{array} \right.$

$x>0$

	$\frac{3-\sqrt{5}}{4}$		0		$\frac{3+\sqrt{5}}{4}$	
$+$		$-$		$-$		$+$
$-$		$-$		$+$		$+$
$-$	$\bullet$	$+$	$\cancel{0}$	$-$	$\bullet$	$+$
			$1/2$			

34) $\dfrac{|x-1|^2-1}{\sqrt{(x-1)^2-4}} \geq |x-1|;$

$t = x-1$

$t^2-4>0$ 

$$\frac{t^2-1}{\sqrt{t^2-4}} \geq |t|$$

$t<-2 \vee t>2$

$t^2-1 \geq |t|\sqrt{t^2-4}$

$t>0;\quad t^2-1 \geq t\sqrt{t^2-4}$

$t<0;\quad t^2-1 \geq -t\sqrt{t^2-4}$

CIAO

$x^2 + 2x + 2y^2 - 8y = 0$

ELLIPSE

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1$$

$x^2 + 2x + (-1)^2 - (-1)^2$

$\underbrace{x^2 + 2x + 1} - 1$

$(x + 1)^2 - 1$

$2y^2 - 8y = 2(y^2 - 4y)$

TUTORING 12/10

EX 1

$\forall n \in N$

$n < 10$

3 STEPS

I) BASE OF INDUCTION $n=1$

$1 < 10$ ✓

II) INDUCTION HYPOTESIS

WE SUPPOSED THAT (*) HOLDS FOR A FIXED $\bar{n}$ in $\mathbb{N}$

III) PROVE $n \Rightarrow n+1$

$$\underbrace{\bar{n}+1}_{n<10^n} < 10^n + 1 < 10^n + 10^n < \dots < 10 \cdot 10^n = 10^{n+1}$$

$10^n > 1$ BECAUSE $n > 1$

BY INDUCTION

$n < 10^n \quad \forall n \in \mathbb{N}$

EX 1.B $\forall n \in \mathbb{N} \quad 2^{n-1} \leq n! \quad \left(2^{n-1} < n! \ \forall n \geq 3\right)$

$n_0 = 1 \qquad 2^{1-1} = 2^0 = 1 \leq 1! = 1 \qquad 1 = 1$

WE SUPPOSE

$2^{\bar{n}-1} \leq \bar{n}! \quad \bar{n} \in \mathbb{N}$ FIXED $\bar{n} > n_0$

$$2^{(\bar{N}+1)-1} = 2^{\bar{N}} = 2\cdot 2^{\bar{N}-1} \leq 2\cdot \bar{N}!$$

TWO CASES

- $\bar{N}=1$ $\quad 2\bar{N}! = 2\cdot 1! = 2! = (\bar{N}+1)!$ ✓
- $\bar{N}\geq 2$ $\quad 2\bar{N}! < (\bar{N}+1)\cdot \bar{N}!$ ✓

$$\wedge$$

$$\underbrace{\bar{N}!\cdot\bar{N}!+\ldots+\bar{N}!}_{(\bar{N}+1)\text{ TIMES}}$$

BY INDUCTION $2^{N-1}\leq N!$

FOR $2^{N-1}< N!$ $\forall_{N\geq 3}$

FOR $N_0=3$ $\quad 2^{3-1}=2^2=4 \quad 3! = 3\cdot 2\cdot 1 = 6$

$4<6$ ✓

THE REMAINING PARTS ARE DONE IN THE SAME WAY

2ω

(*) $$\sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6} \quad \forall_{N\in\mathbb{N}}$$

$N_0=1$

$$\sum_{i=1}^{1} i^2 = 1 = \frac{1\,(1+1)(2+1+1)}{6} = \frac{2\cdot 3}{6} = 1$$

WE SUPPOSE TRUE (#) FOR A FIXED $N \in \mathbb{N}$

$N \Rightarrow N+1$

$$\sum_{i=1}^{N+1} i^2 = \sum_{i=2}^{N} i^2 + (N+1)^2 = \frac{N(N+1)(2N+1)}{6} + (N+1)^2$$

$$= \frac{(N^2+N)(2N+1) + 6(N^2+1+2N)}{6}$$

$$= \frac{2N^3+N^2+2N^2+N+6N^2+12N+6}{6}$$

$$= \frac{2N^3+9N^2+13N+6}{6}$$

TWO WAYS

TRY TO RECONSTRUCT $(N+1)(N+2)((2(N+1)+1)$

PROVE THAT THE NUMERATOR $=(N+1)(N+2)(2(N+1)+1)$ BY CALCULATION

$$(N+1)(N+2)(2(N+1)+1) = (N^2+\underbrace{2N+N}_{3N}+2)(2N+3)$$

$$= 2N^3+6N^2+4N+3N^2+9N+6$$

$$= 2N^3+9N^2+13N+6 \quad \checkmark$$

3. Let us consider the following subsets of $\mathbb{R}$

$$A = \{n \in \mathbb{N} : n \text{ is even}\}; \quad B = \{n \in \mathbb{N} : n < 12\}; \quad C = \{n \in \mathbb{N} : n \leq 12\}.$$

Determine the supremum and the infimum and establish if they are the maximum and the minimum, respectively, for the following sets: A, B, C, $A \cap B$, $A \cap C$, $A \cup C$, $A \setminus C$.

$\omega = \sup A$ IF $\forall \alpha \in A \;\; \alpha \leq \omega$ $\quad$ ω IS THE MIN OF THE MAJORANT

A) $\sup A = +\infty$ → BECAUSE IF I FIX $\omega \in A$ I CAN SAY $2\omega > \omega$ AND $2\omega \in A$ (THIS HOLDS $\forall \omega \in A$)

$\nexists \max A$

$\inf A = 2 = \min A$

B) $\sup B = \max B = 11$

$\inf B = \min B = 1$

C) $\sup C = 12 = \max C$

$\inf C = 1 = \min C$

$A \cap B$) $\inf(A \cap B) = 2 = \min(A \cap B)$

$\sup(A \cap B) = 10 = \max(A \cap B)$ $\quad \{2, 4, 6, 8, 10\}$

$A \cap C$) $\inf(A \cap C) = 2 = \min(A \cap C)$

$\sup(A \cap C) = 12 = \max(A \cap C)$

$A \cup C)$

$\sup(A \cup C) = +\infty \quad \nexists \max$

$\inf(A \cup C) = 1 = \min(A \cup C)$

$A \setminus C)$

$\sup(A \setminus C) = +\infty \qquad \{14, 16, 18 \dots\}$

$\inf(A \setminus C) = 14 = \min(A \setminus C)$

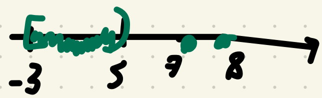
ex 4 $\quad A = (-3,5) \cup \{7,8\}$

$B = (0, \sqrt{2})$

$C = (0, \sqrt{2}) \cap \mathbb{Q}$

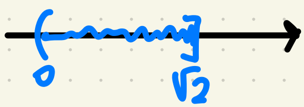
$\sup A = 8 = \max A$

$\inf A = -3 = \min A$



$\sup B = \sqrt{2} = \max B$

$\inf B = 0 \quad \nexists \min$



$\varepsilon > 0$ "small enough" I can always find an $\bar{\varepsilon}$ $0 < \bar{\varepsilon} < \varepsilon$ and this hold $\forall \varepsilon > 0$

$C)$

$\inf C = 0 \quad \nexists \min C$

$\sup C = \sqrt{2}$

$\sqrt{2} \notin \mathbb{Q} \qquad \forall q \in \mathbb{Q} \ \exists p \in \mathbb{Q} : q < p < \sqrt{2}$

$\nexists \max C$

EX 5

$$A = \{ x \in \mathbb{R} : (1-x)(x^2-3) \geq 0 \text{ AND } (x+2)(x-5) < 0 \}$$

$F_1 \geq 0 \iff 1-x \geq 0 \iff x \leq 1$

$F_2 \geq 0 \iff x^2-3 \geq 0 \iff x \leq -\sqrt{3} \vee x \geq \sqrt{3}$

		$-\sqrt{3}$		1		$\sqrt{3}$	
	$+$		$+$	0	$-$		$-$
	$+$	0	$-$		$-$	0	$+$
	$+$	0	$-$	0	$+$	0	$-$

$f_1 \geq 0 \quad x \geq -2$

$x-5 \geq 0 \quad x \geq 5$

	-2		5	
$-$	0	$+$	0	$+$
$-$	0	$-$		$+$
$+$		$-$	0	$+$

$A_1 = (-\infty, -\sqrt{3}) \cup (\sqrt{3}, +\infty)$

$A_2 = (-2, 5)$

$$A = A_1 \cap A_2 \quad \underbrace{(-\infty, -\sqrt{3}) \cup (-\sqrt{3}, -2) \cup (-2, 5)}_{-2,\, -\sqrt{3}}$$

$$B = \{ x \in \mathbb{R} : |x-2| > 1 \text{ AND } |x+1| - 2 < 0 \}$$

$$|x-2| > 1 \qquad |f(x)| = \begin{cases} f(x) & \text{IF } f(x) \geq 0 \\ -f(x) & \text{IF } f(x) < 0 \end{cases}$$

$$\begin{cases} x-2 > 1 & \text{IF } x-2 \geq 0 \\ -(x-2) > 1 & \text{IF } x-2 < 0 \end{cases}$$

$$\begin{cases} x-2 > 1 \\ x \geq 2 \end{cases} \Rightarrow \begin{cases} x > 3 \\ x \geq 2 \end{cases} \Rightarrow x > 3$$

$$\begin{cases} -(x-2) > 1 \\ x-2 < 0 \end{cases} \Rightarrow \begin{cases} x-2 < -1 \\ x < 2 \end{cases} \Rightarrow \begin{cases} x < 1 \\ x < 2 \end{cases} \Rightarrow x < 1$$

$$x > 3 \vee x < 1 \qquad (-\infty, 1) \cup (3, +\infty)$$

② $|x+1| - 2 < 0$

$$|x+1| < 2 \Rightarrow -2 < x+1 < 2$$

$$\begin{cases} x+1 > -2 \\ x+1 < 2 \end{cases} \Rightarrow \begin{cases} x > -3 \\ x < 1 \end{cases} \quad (-3, 1) = B_2$$

$$B = B_1 \cap B_2$$

$$(-3, 1)$$

INF $B = -3$

SUP $B = 1$

ALWAYS EXIST IF THE SET IS NOT $\emptyset$

$\nexists$ MAX

$\nexists$ MIN

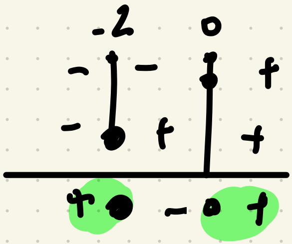
ex. 6)

$$A \cap \mathbb{N} \cap \{x \in \mathbb{R} : x^2 + 8x \geq 6x\}$$

$x^2 + 8x - 6x \geq 0 \quad x(x+2) \geq 0$

$x \geq 0$

$x + 2 \geq 0 \quad x \geq -2$



$(-\infty, -2] \cup [0, +\infty)$

NO

$A = \mathbb{N} \cap A_1$ NO 0 (NOT IN $\mathbb{N}$)

$A = \mathbb{N} \cap A_1 = \mathbb{N}$

$\inf A = \inf \mathbb{N} = 1 = \min A$

$\sup A = \sup \mathbb{N} = +\infty \quad \nexists \max A$

c) $* \{x \in \mathbb{R} : \sqrt{x+5} > 3-x\}$

$\sqrt{x+5} > 3-x \qquad x \geq -5$

IF $3-x < 0$ $*$ HOLDS $\forall x \geq -5$ BECAUSE $\sqrt{f(x)} \geq 0$

$3-x \geq 0 \quad x \leq 3 \qquad C \subseteq [-5,3]$

$x+5 > (3-x)^2$

$x+5 > 9+x^2-6x$

$x^2-7x+4 < 0 \qquad x_{1,2}\ \frac{7 \pm \sqrt{33}}{2}$