GENERAL FACTS ABOUT CONTINUOUS R.V.'S

$f:\mathbb{R}\to\mathbb{R}$ is the probability density function

Recall that:

a) $f(x)\geq 0 \quad \forall x\in\mathbb{R}$

b) f is integrable on any interval.

c) $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

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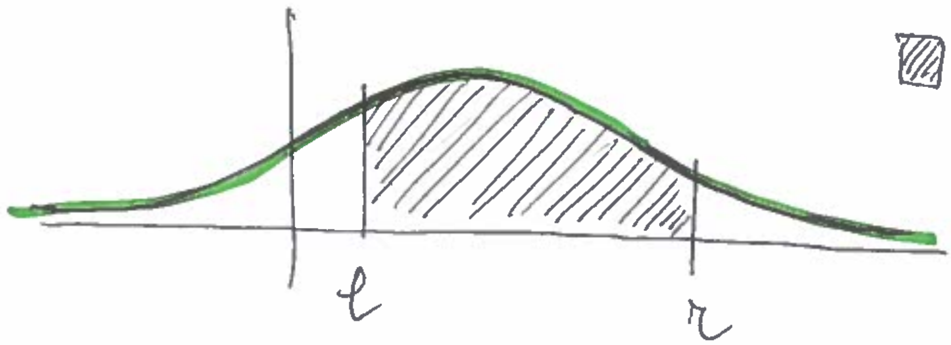
1) CONNECTION BETWEEN f and the PROBABILITY

$$\boxed{P[X\in(\ell,r)] = \int_{\ell}^{r} f(x)\,dx} \qquad \ell<r.$$



▨ $= P[X\in(\ell,r)]$

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2) EXPECTATION

$$\boxed{\mathbb{E}[X] \overset{\text{Def}}{=} \int_{-\infty}^{+\infty} x f(x)\,dx}$$

(Whenever the integral makes sense)

If $f(x)=0$ outside $\underbrace{[a,b]}_{\text{SUPPORT of } f}$, then

$$\mathbb{E}[X] = \int_a^b x f(x)\,dx$$

3) EXPECTATION OF A TRANSFORMED R.V.'S

Let $h:\mathbb{R}\to\mathbb{R}$ be a given function.

Then
$$\boxed{\mathbb{E}[h(X)] = \int_{-\infty}^{+\infty} h(x) f(x)\, dx.}$$
whenever the integral makes sense.

In particular, we have

$h(x)=x^2 \implies$ SECOND MOMENT
$$\boxed{\mathbb{E}[X^2] = \int_{-\infty}^{+\infty} x^2 f(x)\, dx}$$

$h(x)=x^n \implies$ n-th MOMENT
$$\boxed{\mathbb{E}[X^n] = \int_{-\infty}^{+\infty} x^n f(x)\, dx}$$

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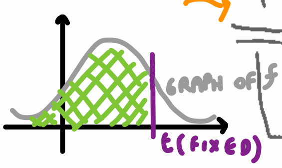
4) DISTRIBUTION FUNCTION

FIXED t CONSIDER $X\le t$

$$\boxed{F(t)} \stackrel{!}{=} P[X\le t] = P[X\in(-\infty,t]] =$$

FOR CONTINOUS CASE

$$\stackrel{!}{=} \boxed{\int_{-\infty}^{t} f(x)\,dx}, \quad t\in\mathbb{R}.$$



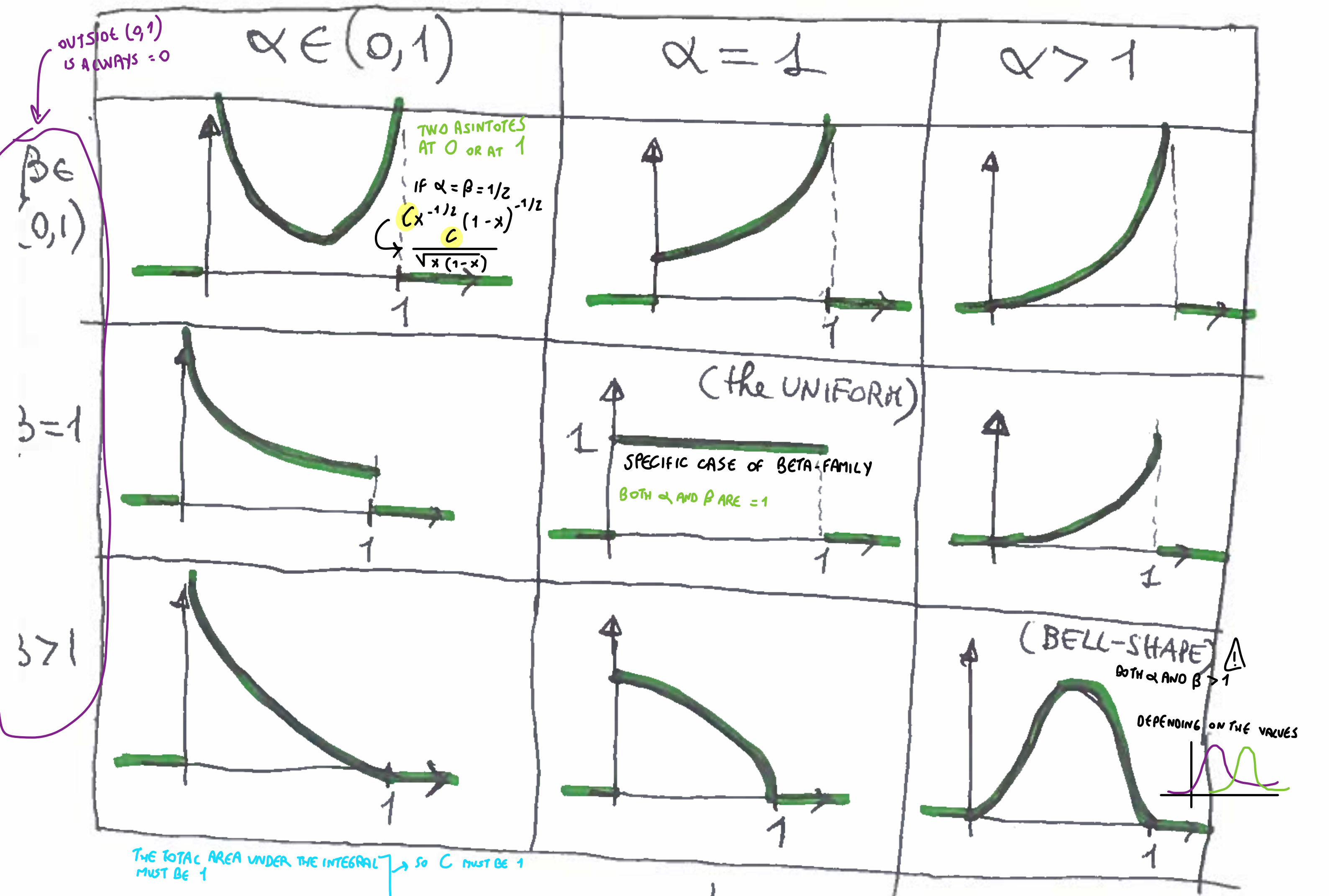
Viceversa, $\boxed{f(x)=F'(x)}$, $\forall x\in\mathbb{R}$

CONTINUOUS RANDOM VARIABLE : THE BETA

Let α, β be fixed >0.
Put
$$f(x) = \begin{cases} C(\alpha,\beta)\, x^{\alpha-1}(1-x)^{\beta-1}, & \text{if } x \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$

$C(\alpha,\beta)$ is just a constant (>0). We will see later its characterization. α, β FIXED PARAMETER

The graph of f is as follows :



THE TOTAL AREA UNDER THE INTEGRAL MUST BE 1 → SO C MUST BE 1

Proposition 1. $\forall \alpha, \beta > 0$, $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx < +\infty$.

THE INTEGRAL IS ALWAYS FINITE
IF YOU HAVE 1 OR 2 ASINDOTES IS NOT CLEAR

Thus, $C(\alpha,\beta) := \dfrac{1}{\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx}$

$\int_0^1 \frac{1}{\sqrt{x}}\,dx$ ⇒ CONSIDER ε ⇒ $\int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_\varepsilon^1 = 2 - \sqrt{\varepsilon}$

"IMPROPER INTEGRAL"

NOW LET $\varepsilon \to 0$: $\lim_{\varepsilon \to 0^+} 2 - \sqrt{\varepsilon} = 2$ → TOTAL AREA BELOW GRAPH

With this choice of $C(\alpha,\beta)$, f is a probability density function that generalizes the uniform case.

A further characterization of $C(\alpha,\beta)$ can be given in terms of a special function, called GAMMA FUNCTION.

$$\Gamma(z) := \int_0^{+\infty} e^{-t} \cdot t^{z-1}\, dt \quad , \; z > 0$$

thus $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$. The following fact about the gamma function are worth noticing:

1) $\Gamma(1) = \int_0^{+\infty} e^{-t} dt = 1$

2) $\Gamma(z+1) = z \cdot \Gamma(z)$ [by integration by parts]

3) $\Gamma(2) = 1 \cdot \Gamma(1) = 1 = 1!$

$\Gamma(3) = 2 \cdot \Gamma(2) = 2 = 2!$

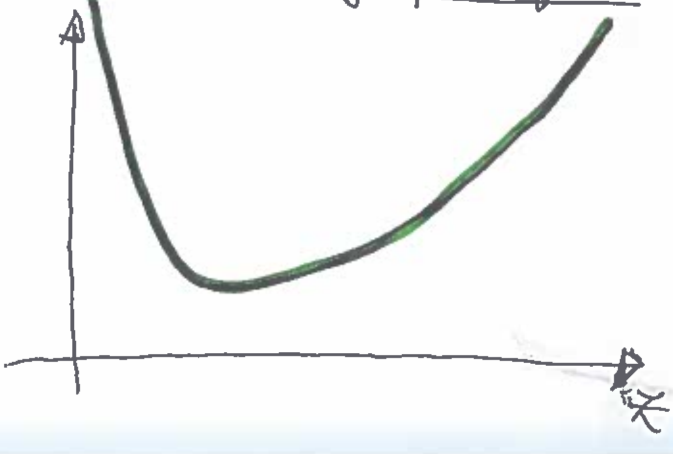
$\Gamma(4) = 3 \cdot \Gamma(3) = 3 \cdot 2 = 3!$

$\Gamma(5) = 4 \cdot \Gamma(4) = 4 \cdot 3 \cdot 2 = 4!$

$\vdots$

$$\boxed{\Gamma(n+1) = n!}$$

graph of Γ



4) $\Gamma(1/2) = \sqrt{\pi}$

5) $\lim_{z \to 0^+} \Gamma(z) = +\infty$

The interest for the gamma function comes from this very nice:

EULER'S THEOREM

(After L. Euler (1707–1783)):

$$\boxed{\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}}$$

Example: Solve $\int_0^1 x^{6}(1-x)^{4}dx$. [very lengthy by hands]

$(\alpha=7, \beta=5)$

By Euler's theorem $= \dfrac{\Gamma(7)\Gamma(5)}{\Gamma(12)} \overset{\text{FROM BEFORE}}{=} \dfrac{6!\,4!}{11!}$

$$= \frac{\cancel{6!}\;\cancel{4}\cdot\cancel{3}\cdot\cancel{2}}{11\cdot 10\cdot \underset{3}{\cancel{9}}\cdot\cancel{8}\cdot 7\cdot \cancel{6!}} = \frac{1}{11\cdot 10\cdot 21}$$

$$= \frac{1}{2310}$$

IS THE INVERSE BECAUSE SHOULD BE $1/\text{AREA} \Rightarrow \dfrac{1}{\frac{\Gamma(\alpha)+\Gamma(\beta)}{\Gamma(\alpha+\beta)}} \Rightarrow \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)+\Gamma(\beta)}$

Thus, the beta density becomes

$$f(x) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}dx & \text{if } x\in(0,1) \\ 0 \longrightarrow & \text{otherwise} \end{cases}$$

It is a probability density function because

$f(x)\geq 0 \;\forall x\in\mathbb{R}$, and $\int_{-\infty}^{+\infty} f(x)dx = \int_0^1 f(x)dx = 1$.

MEAN VALUE

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x\, f(x)\, dx \quad [\text{general formula}]$$

SUBSTITUTE

$$= \int_0^1 x \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} dx$$

IS A CONSTANT

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{(\alpha+1)-1} (1-x)^{\beta-1} dx$$

$$\underset{(\text{Euler's th.})}{=} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cancel{\Gamma(\beta)}} \frac{\Gamma(\alpha+1)\cancel{\Gamma(\beta)}}{\Gamma(\alpha+\beta+1)}$$

$$\underset{(\text{Prop 2) of } \Gamma)}{=} \frac{\cancel{\Gamma(\alpha+\beta)}}{\cancel{\Gamma(\alpha)}} \frac{\alpha \cancel{\Gamma(\alpha)}}{(\alpha+\beta)\cancel{\Gamma(\alpha+\beta)}} = \boxed{\frac{\alpha}{\alpha+\beta}}$$

UNIFORM: α AND $\beta = 1$
$\hookrightarrow = 1/2$ (EXACTLY THE MIDDLE POINT)

SECOND MOMENT

$$\mathbb{E}[X^2] = \int_{-\infty}^{+\infty} x^2 f(x)\, dx \quad [\text{general formula}]$$

$$= \int_0^1 x^2 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{(\alpha+2)-1} (1-x)^{\beta-1} dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cancel{\Gamma(\beta)}} \frac{\Gamma(\alpha+2)\cancel{\Gamma(\beta)}}{\Gamma(\alpha+\beta+2)}$$

$$= \frac{\cancel{\Gamma(\alpha+\beta)}}{\cancel{\Gamma(\alpha)}} \cdot \frac{\alpha(\alpha+1)\cancel{\Gamma(\alpha)}}{(\alpha+\beta)(\alpha+\beta+1)\cancel{\Gamma(\alpha+\beta)}} = \boxed{\frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}}$$

VARIANCE

$$Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \frac{\alpha^2}{(\alpha+\beta)^2}$$

$$= \frac{\alpha}{\alpha+\beta}\left[\frac{\alpha+1}{\alpha+\beta+1} - \frac{\alpha}{\alpha+\beta}\right]$$

$$= \frac{\alpha\left[\cancel{\alpha^2} + \cancel{\alpha\beta} + \cancel{\alpha} + \beta - \cancel{\alpha^2} - \cancel{\alpha\beta} - \cancel{\alpha}\right]}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$= \boxed{\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}$$

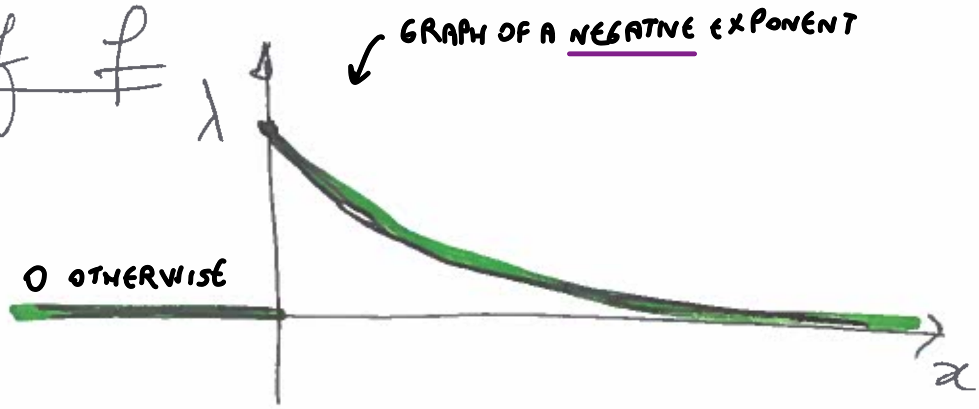
CONTINUOUS RANDOM VARIABLES: THE EXPONENTIAL

ONLY 1 PARAMETER: λ

Let $\lambda > 0$ be fixed.

Put $f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$

Graph of f



f is a probability density function, because

1) $f(x) \geq 0 \quad \forall x \in \mathbb{R}$

2) $\underbrace{\int_{-\infty}^{+\infty} f(x)\,dx = \int_0^{+\infty}}_{\text{0 IF } f(x) \text{ IS NEGATIVE}} \lambda e^{-\lambda x}\,dx = \left[-e^{-\lambda x}\right]_{x=0}^{x \to +\infty}$

$= \underset{x \to +\infty}{0} - \underset{x \to 0}{(-1)} = \boxed{+1}$ TOTAL AREA

[observe that $-e^{-\lambda x}$ is a primitive of $\lambda e^{-\lambda x}$, $\forall x > 0$, meaning that

$\frac{d}{dx}\left(-e^{-\lambda x}\right) = \lambda e^{-\lambda x}$.

MEAN VALUE

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x \boxed{f(x)}\, dx$$ [General formula

CHANGE OF VARIABLE $\boxed{[\lambda x = y]}$

$$= \frac{1}{\lambda}\int_0^{+\infty} \underbrace{\lambda x}_{y} \boxed{\lambda e^{-\lambda x}}\, dx = \frac{1}{\lambda}\int_0^{+\infty} y e^{-y}\, dy$$

OR: $y = \lambda x$

$x = \frac{y}{\lambda}$

$dx = \frac{1}{y} dy$

FIND z

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt$$

$z = 2$ $\quad \Gamma(2) = \int_0^{+\infty} e^{-t} t^1 dt$

$$= \frac{1}{\lambda}\Gamma(2) = \frac{1}{\lambda}$$

SECOND MOMENT

$$\mathbb{E}[X^2] = \int_{\infty}^{+\infty} x^2 f(x)\, dx$$ [General]

$$= \int_0^{+\infty} x^2 \lambda e^{-\lambda x}\, dx \overset{[\lambda x = y]}{=} \frac{1}{\lambda^2}\int_0^{+\infty} y^2 e^{-y}\, dy$$

$$= \frac{1}{\lambda^2}\Gamma(3) = \frac{2}{\lambda^2}$$

VARIANCE :

$$Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{2-1}{\lambda^2}$$

$$= \frac{1}{\lambda^2}$$

CONTINUOUS RANDOM VARIABLES: THE GAMM

2 PARAMETERS: λ, τ

Let $\lambda, \tau > 0$ be fixed. TAO

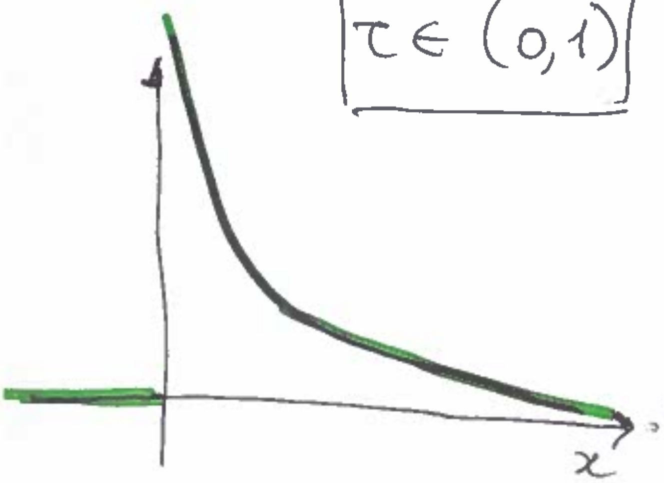
Put

$$f(x) = \begin{cases} \underbrace{\frac{\lambda^\tau}{\Gamma(\tau)}}_{\text{CONSTANT}} \underbrace{x^{\tau-1}}_{\text{POWER}} \underbrace{e^{-\lambda x}}_{\text{EXPONENTIAL}}, & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

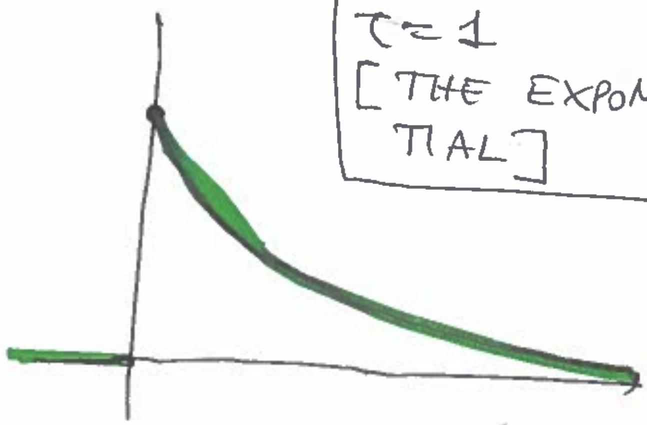
- λ is the SCALE PARAMETER
- τ is the SHAPE PARAMETER

Indeed, the graph of f changes its form according to the value of τ.

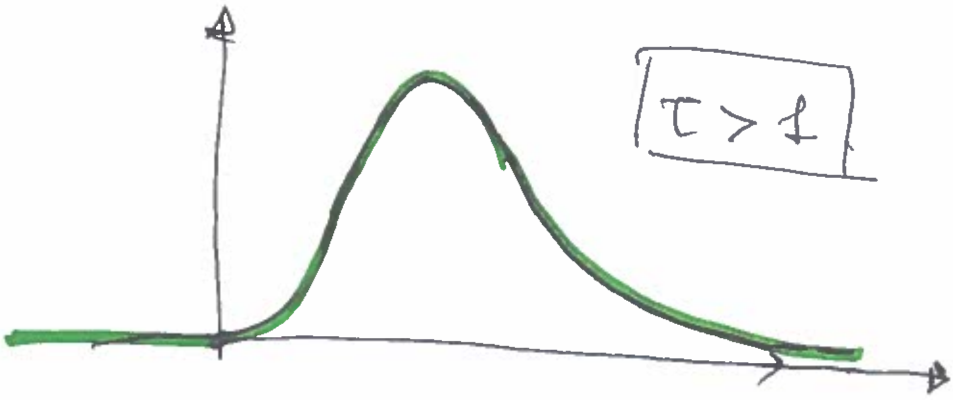
$\boxed{\tau \in (0,1)}$



$\tau = 1$
[THE EXPONENTIAL]



$\boxed{\tau > 1}$



SPECIAL CASE [VERY, VERY IMPORTANT IN STATISTICS]

When $\lambda = \frac{1}{2}$, $\tau = \frac{n}{2}$ $(n \in \mathbb{N})$ the density

$$f(x) = \begin{cases} \dfrac{1}{2^{n/2}\Gamma(n/2)} x^{n/2 - 1} e^{-x/2}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

is named **CHI-SQUARED DENSITY WITH n DEGREES OF FREEDOM.**

CHECK THAT $\int_0^{+\infty} \frac{\lambda^\tau}{\Gamma(\tau)} x^{\tau-1} e^{-\lambda x} dx = 1$

For generic $\lambda, \tau > 0$ f is a probability density function! In fact,

1) $f(x) \geq 0 \quad \forall x \in \mathbb{R}$

2) $\int_{-\infty}^{+\infty} f(x) dx = \int_0^{+\infty} \frac{\lambda^\tau}{\Gamma(\tau)} x^{\tau-1} e^{-\lambda x} dx =$

CONSTANT

SUBSTITUTE $[\lambda x = y]$

$\Gamma(z) = \int_0^{+\infty} e^{-y} y^{z-1} dy$

$$= \frac{1}{\Gamma(\tau)} \int_0^{+\infty} y^{\tau-1} e^{-y} dy = \frac{\Gamma(\tau)}{\Gamma(\tau)} = 1.$$

	ILL	NOT-ILL
SMOKER	30	10
NOT SMOKER	20	40

NUMBER OF DEGREES = (NUMBER OF ROWS - 1)(NUMBER OF COLUMNS - 1)

$\boxed{N = 100}$

95% $\chi^2(1)$

THE MEAN

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x f(x)\,dx \qquad [\text{General}]$$

$$= \int_0^{+\infty} x \cdot \frac{\lambda^\tau}{\Gamma(\tau)} x^{\tau-1} e^{-\lambda x}\,dx \qquad [\lambda x = y]$$

$$= \frac{1}{\Gamma(\tau)} \frac{1}{\lambda} \int_0^{+\infty} y^{(\tau+1)-1} e^{-y}\,dy = \frac{\Gamma(\tau+1)}{\lambda\Gamma(\tau)}$$

$$= \frac{\tau\Gamma(\tau)}{\lambda\Gamma(\tau)} = \boxed{\left(\frac{\tau}{\lambda}\right)}$$

THE SECOND MOMENT

$$\mathbb{E}[X^2] = \int_{-\infty}^{+\infty} x^2 f(x)\,dx \qquad [\text{General}]$$

$$= \int_0^{+\infty} x^2 \frac{\lambda^\tau}{\Gamma(\tau)} x^{\tau-1} e^{-\lambda x}\,dx \qquad [\lambda x = y]$$

$$= \frac{1}{\Gamma(\tau)} \frac{1}{\lambda^2} \int_0^{+\infty} y^{(\tau+2)-1} e^{-y}\,dy = \frac{\Gamma(\tau+2)}{\lambda^2\Gamma(\tau)}$$

$$= \frac{\tau(\tau+1)\cancel{\Gamma(\tau)}}{\lambda^2\cancel{\Gamma(\tau)}} = \boxed{\frac{\tau(\tau+1)}{\lambda^2}}$$

VARIANCE: $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$$= \frac{\tau(\tau+1)}{\lambda^2} - \frac{\tau^2}{\lambda^2} = \frac{\tau}{\lambda^2}$$

CONTINUOUS RANDOM VARIABLES: THE GAUSSIAN

Let $m \in \mathbb{R}$, $\sigma^2 > 0$ be fixed.

Put
$$\boxed{f(x) = \frac{1}{\sqrt{2\pi\boxed{\sigma^2}}} \exp\left\{-\frac{(x-\boxed{m})^2}{2\boxed{\sigma^2}}\right\} \qquad x \in \mathbb{R}}$$

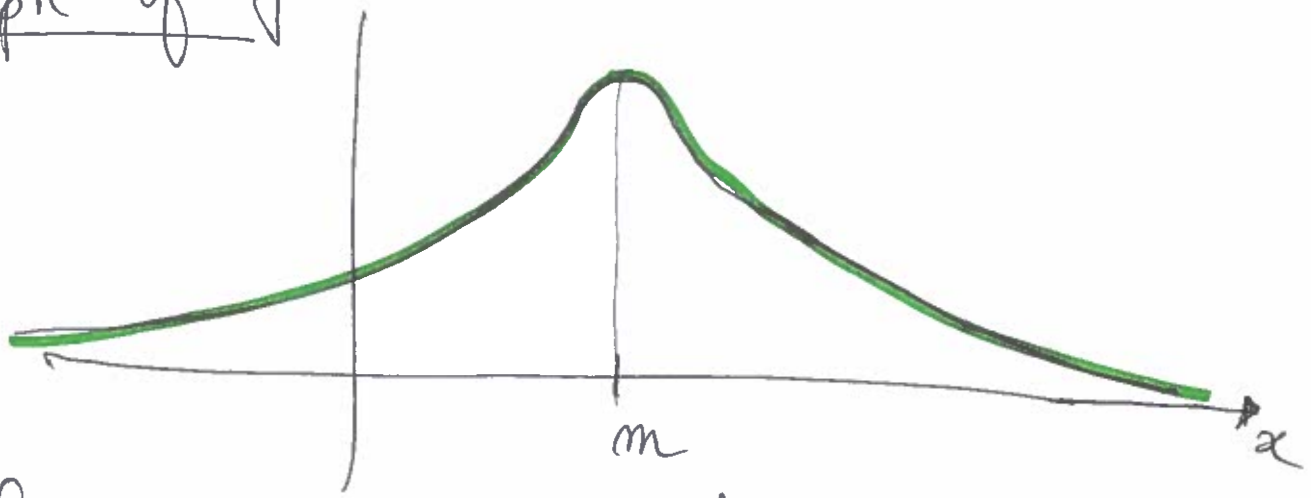
CONSTANT

$\boxed{m}$ is called MEAN

$\boxed{\sigma^2}$ is called VARIANCE

As we shall see, it is actually true that $\mathbb{E}[X] = \boxed{m}$, $\mathrm{Var}(X) = \boxed{\sigma^2}$.

Graph of f

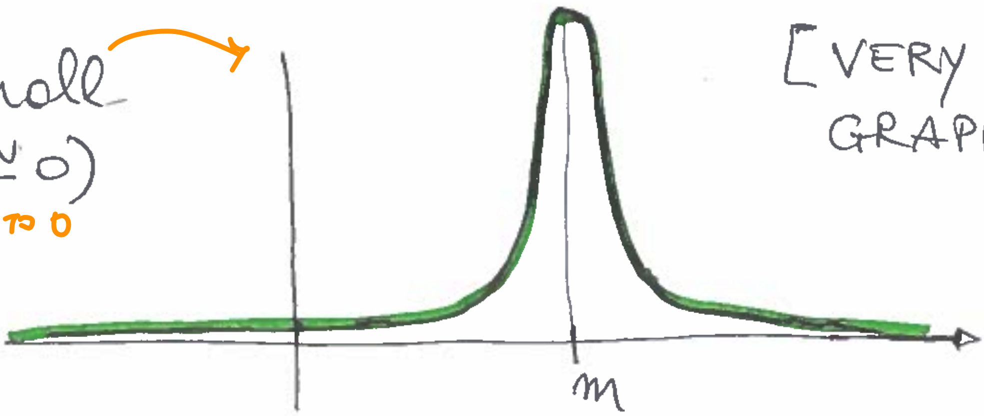


- f is maximum at $x = m$
- $f(x) > 0 \quad \forall x \in \mathbb{R}$
- $\lim_{x \to \pm\infty} f(x) = 0$

Thus, the parameter is the <u>abscissa</u> of the (unique) maximum point of f.

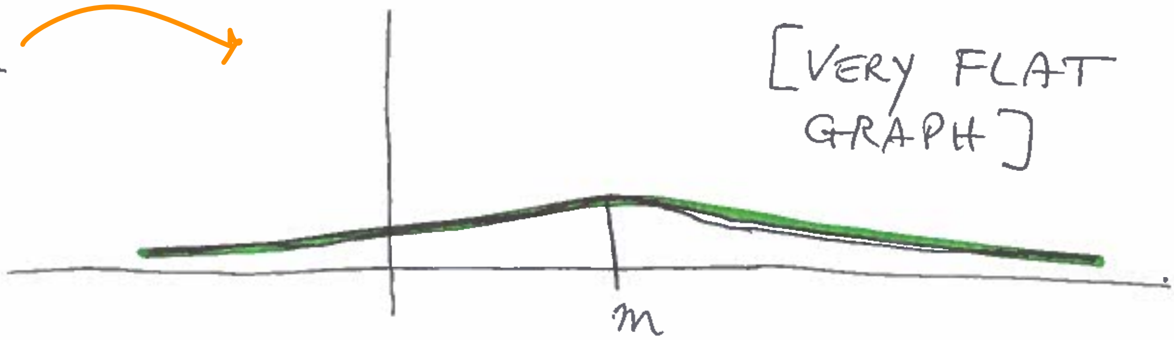
The geometric meaning of σ^2 is to regulate the openness of the bell-shape

σ^2 small
$(\sigma^2 \simeq 0)$
CLOSER TO 0



[VERY SPIKY GRAPH]

σ^2 large



[VERY FLAT GRAPH]

We see now that f is a <u>density</u> because $\int_{-\infty}^{+\infty} f(x)dx = 1$. Indeed,

CHANGE OF VARIABLE
$[x-m=y]$

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-m)^2}{2\sigma^2}\right\} dx =$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{y^2}{2\sigma^2}\right\} dy = \left[\frac{y}{\sigma} = z\right]$$

where $\sigma := \sqrt{\sigma^2}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-z^2/2} dz = \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} e^{-z^2/2} dz$$

$z^2/2 = w \Rightarrow z = \sqrt{2w}$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} e^{-w} \frac{\sqrt{2}\, dw}{2\sqrt{w}} = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} e^{-w} \frac{dw}{\sqrt{w}} =$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{+\infty} e^{-w} \cdot w^{1/2-1} dw = \frac{1}{\sqrt{\pi}} \cdot \Gamma(1/2) = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1$$

THE MEAN

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x f(x) dx = \int_{-\infty}^{+\infty} x \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-m)^2}{2\sigma^2}\right\} dx$$

$$\overset{x-m=y}{=} \int_{-\infty}^{+\infty} (y+m) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{y^2}{2\sigma^2}\right\} dy$$

$$\overset{[\text{linearity}]}{=} \underbrace{\int_{-\infty}^{+\infty} y \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{y^2}{2\sigma^2}\right\} dy}_{=0} + m$$

$$= m$$

THE VARIANCE.

$$\mathrm{Var}(X) = \mathbb{E}\left[(X-m)^2\right] =$$

$$\int_{-\infty}^{+\infty} (x-m)^2 \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-m)^2}{2\sigma^2}\right\} dx \quad [x-m=y]$$

$$= \int_{-\infty}^{+\infty} y^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-y^2/2\sigma^2} dy = \sigma^2 \int_{-\infty}^{+\infty} z^2 \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$\left[\frac{y}{\sigma} = z\right]$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^{+\infty} z^2 e^{-z^2/2} dz = \frac{2\sigma^2}{\cancel{\sqrt{2}}\sqrt{\pi}} \int_0^{+\infty} (\cancel{2}w) e^{-w} \frac{\cancel{\sqrt{2}}}{\cancel{2}\sqrt{w}} dw$$

$$z^2/2 = w$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{+\infty} w^{3/2-1} e^{-w} dw = \frac{2\sigma^2}{\sqrt{\pi}} \Gamma(3/2) =$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{2} \Gamma(1/2) = \sigma^2 .$$

THE STANDARD NORMAL (GAUSSIAN)

(OR GAUSSIAN DISTRIBUTION W/ $m=0$; $\sigma^2=1$)

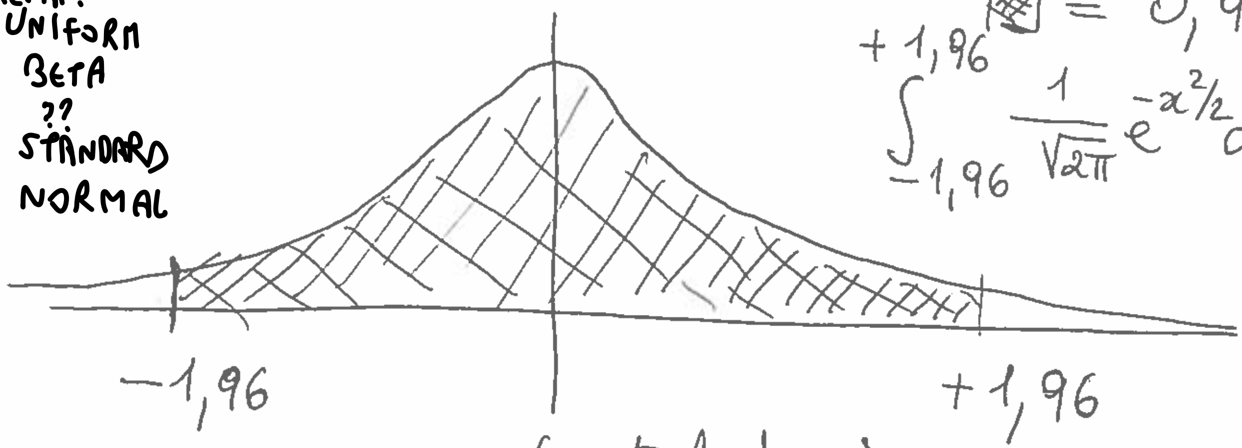
It is obtained by putting $m=0$ and $\sigma^2 = 1$. That is,

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad , x \in \mathbb{R}}$$

$Z \sim N(m, \sigma^2)$ SHORTER TO DENOTE THAT Z HAS GAUSSIAN DENSITY WITH MEAN m AND VARIANCE σ^2

In particular

RECAP:
UNIFORM
BETA
??
STANDARD
NORMAL



$$\text{(shaded area)} = 0,95$$

$$\int_{-1,96}^{+1,96} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 0,95$$

95%

Usually, the r.v. is denoted by Z. → FOR STANDARD NORMAL USE Z FOR RANDOM VARIABLE

TRANSFORMATION OF R.V.'S. [Chapt. 5]

Let $X : \Omega \to \mathbb{R}$ be a r.v. of continuous type, with density f_X.
(SINCE WE ARE CONTINOUS)

Let $h : \mathbb{R} \to \mathbb{R}$ be a function such that

a) h is one-to-one (injective and surjective)
THE GRAPH INTERSECT AND HORIZONTAL LINE IN EXACTLY ONE POINT
EVERY ORIZONTAL LINE HAVE AT LEAST ONE INTERSECTION

b) h is differentiable

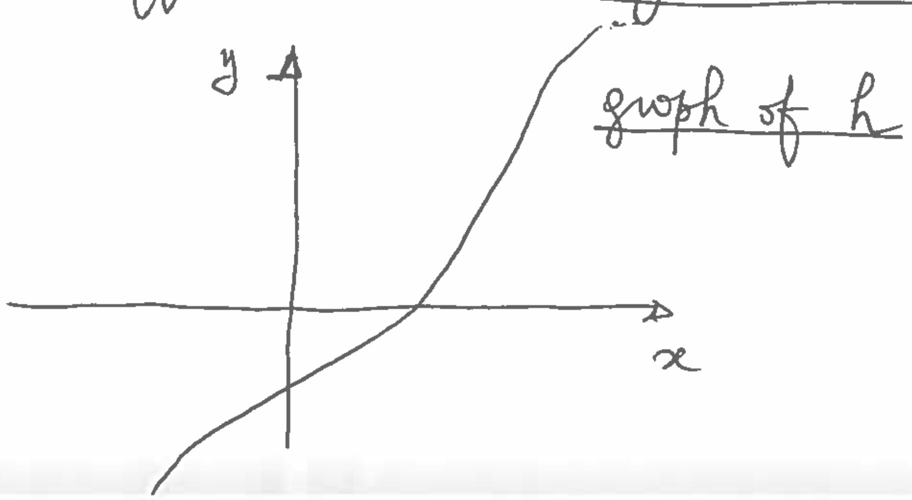
c) h^{-1} is differentiable

REVERSE
$y = h(x) \longrightarrow x = h^{-1}(y)$

We put $\boxed{Y = h(X)}$. We find that Y is again a continuous random variable.

Problem. Find the density of Y.

Solution. If h fulfills a)-b)-c), h must be strictly monotonic. With loss of generality, we can suppose h strictly increasing.



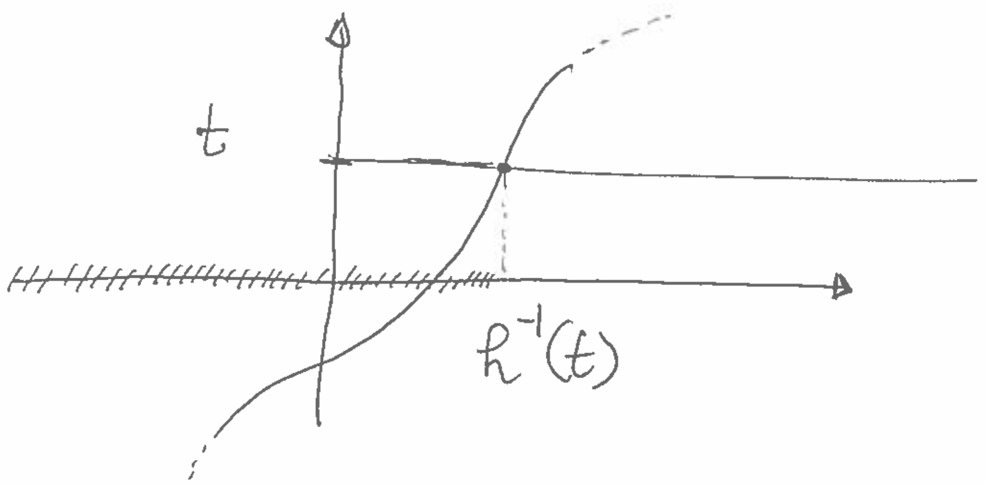
Let us consider the <u>distribution function</u> of Y, that is

CONNECT THE NOTION OF DENSITY W/ THE NOTION OF PROBABILITY

$$F_Y(t) \overset{\text{Def}}{=} P[Y \leq t] = P[h(X) \leq t]$$

$\forall t \in \mathbb{R}$. Observe that the inequality

$$\boxed{h(x) \leq t} \quad \text{is equivalent to} \quad \boxed{x \leq h^{-1}(t)}$$



Thus, $F_Y(t) = P[h(X) \leq t] = P[X \leq h^{-1}(t)]$

$$= F_X(h^{-1}(t)) = \int_{-\infty}^{h^{-1}(t)} f_X(x)\,dx$$

Now, <u>we change the variable in the integral</u> putting $\boxed{y = h(x)}$. $\Rightarrow$ $\boxed{x = h^{-1}(y)}$

- CHANGE OF THE DOMAIN: $\underset{x}{(-\infty, h^{-1}(t)]} \longrightarrow \underset{y}{(-y, t]}$
- CHANGE OF THE INTEGRAND: $f_X(x) \to f_X(h^{-1}(y))$
- CHANGE OF THE DIFFERENTIAL: $dx \longrightarrow \frac{d}{dy} h^{-1}(y)\,dy$

Whence, $\displaystyle\int_{-\infty}^{h^{-1}(t)} f_X(x)\,dx = \int_{-\infty}^{t} f_X(h^{-1}(y))\left[\frac{d}{dy}h^{-1}(y)\right]dy$

Since $F_Y(t) = \displaystyle\int_{-\infty}^{t} f_Y(y)\,dy$, we deduce

$$\boxed{f_Y(y) = f_X(h^{-1}(y))\cdot\left[\frac{d}{dy}h^{-1}(y)\right]} \quad \boxed{y \in \mathbb{R}}$$

<u>Remark</u>: The same formula works if $h : (a,b) \to (c,d)$ such that

a) h is one-to-one
b) h is differentiable
c) h^{-1} is differentiable

and $f_X(x) = 0 \quad \forall x \in [a,b]$.

In this case,

$$\boxed{f_Y(y) = f_X(h^{-1}(y))\cdot\left[\frac{d}{dy}h^{-1}(y)\right]} \quad \boxed{y \in (c,d)}$$

EXAMPLE/EXERCISE: THE LOG-NORMAL R.V.

Let Z be a r.v. with density equal to the standard gaussian (standard normal) that is

$$\boxed{f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad x \in \mathbb{R}}$$

We take $h(x) = e^x$, so $h: \mathbb{R} \to (0,+\infty)$ is one-to-one and both h and h^{-1} are differentiable. $h^{-1}(y) = \log y$, $h^{-1}: (0,+\infty) \to \mathbb{R}$

We put $Y = h(Z)$. Thus,

$$f_Y(y) = f_X(h^{-1}(y)) \cdot \left[\frac{d}{dy} h^{-1}(y)\right], \quad y \in (0,+\infty)$$

$$= f_X(\log y) \cdot \frac{1}{y}$$

$$= \boxed{\frac{1}{\sqrt{2\pi} \cdot y} \exp\left\{-\frac{(\log y)^2}{2}\right\}} \quad y \in (0,+\infty)$$

EXERCISE. Let $Z \sim \mathcal{N}(0,1)$. [Standard Normal].

Then, $W = Z^2 \sim \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right) = \chi^2(1)$

SOLUTION. We need to find the density of W.

Let us first consider the distribution function F_W of W, that is: $(t>0)$

$$F_W(t) = P[W \leq t] = P[Z^2 \leq t] =$$

$$= P[Z \in [-\sqrt{t}, \sqrt{t}]]$$

$$= \int_{-\sqrt{t}}^{\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 2\int_0^{\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$\underset{\substack{\Downarrow \\ x=\sqrt{y}}}{[x^2=y]} = \frac{\cancel{2}}{\sqrt{2\pi}} \int_0^t e^{-y/2} \frac{dy}{\cancel{2}\sqrt{y}} = \frac{1}{\sqrt{2\pi}} \int_0^t y^{1/2-1} e^{-y/2} dy.$$

Since $F_W(t) = \int_0^t f_W(y)dy$, we have

$$f_W(y) = \frac{1}{\sqrt{2\pi}} y^{1/2-1} e^{-y/2} = \boxed{\frac{(1/2)^{1/2}}{\Gamma(1/2)} y^{1/2-1} e^{-y/2}}$$

which is the Gamma density with $\lambda = 1/2$ and $\tau = 1/2$.