$a \in \mathbb{R}$ $\quad 1 \cdot a = 1 \cdot a = a$ $\quad I_m \in \mathbb{R}^{m \times m}$ IDENTITY MATRIX

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & & & 0 \\ 0 & & 1 & & 0 \\ 0 & & & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$A \in \mathbb{R}^{m \times n}$ $\quad \underbrace{I_m A = A I_n = A}_{\text{BUT PRODUCT OF MATRIXES IS NOT SYMMETRIC}}$

$a \in \mathbb{R}$, $a \neq 0$ $\quad a^{-1} \cdot a = a \cdot a^{-1} = 1$

$$a^{-1} = \frac{1}{a}$$

ONLY FOR SQUARE MATRIX

DEFINITION: $A \in \mathbb{R}^{n \times n}$ A IS CALLED INVERTIBLE IF IT EXISTS $C \in \mathbb{R}^{n \times n}$ SUCH THAT $CA = AC = I_n$
OTHERWISE A IS SAID NON INVERTIBLE

OBSERVATION: IF IT EXISTS, THE MATRIX C IS UNIQUE

$CA = AC = I_n$ $\quad BA = AB = I_n$

$C \in \mathbb{R}^{n \times n}$ $\quad B \in \mathbb{R}^{n \times n}$

PROPERTY OF THE IDENTITY

$$B = BI_n = BAC = I_n C = C \quad \text{so } B = C$$

- THE MATRIX C (IF IT EXISTS) IS UNIQUE AND IT IS CALLED THE INVERSE MATRIX OF A.
  IT IS INDICATED WITH $C = A^{-1}$

$A \in \mathbb{R}^{n \times n}$ INVERTIBLE

$$A^{-1} A = I_n \quad A^{-1} \in \mathbb{R}^{n \times n}$$

INVERSE OF A

$n = 2$ $\quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\quad A \in \mathbb{R}^{2 \times 2}$

CHECK THAT IF $(a_{11}a_{22} - a_{12}a_{21}) \neq 0$
THEN A IS INVERTIBLE, ITS INVERSE BEING $A^{-1} = \frac{1}{(a_{11}a_{22} - a_{12}a_{21})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

$A^{-1} A = I_n$

$$A^{-1}A = \underbrace{\frac{1}{(a_{11}a_{22} - a_{12}a_{21})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}}_{A^{-1}} \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} = \frac{1}{(a_{11}a_{22} - a_{12}a_{21})} \begin{bmatrix} \overbrace{a_{22}a_{11} - a_{12}a_{21}}^{1} & \overbrace{a_{22}a_{12} - a_{12}a_{22}}^{0} \\ \underbrace{-a_{21}a_{11} + a_{11}a_{21}}_{0} & \underbrace{-a_{21}a_{12} + a_{11}a_{22}}_{1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow AA^{-1} = I_n$$

THEOREM: IF $A \in \mathbb{R}^{N \times N}$ IS AN INVERTIBLE MATRIX, THEN THE LINEAR SYSTEM $A\underline{x} = \underline{b}$ $(\underline{b} \in \mathbb{R}^N, \underline{x} \in \mathbb{R}^N)$ ALWAYS HAS ONE UNIQUE SOLUTION $\underline{x}$ FOR ANY $\underline{b} \in \mathbb{R}^N$

$A\underline{x} = \underline{b}$ $\quad \underline{x} = A^{-1}\underline{b}$ IS A SOLUTION CHECK $A(A^{-1}\underline{b}) \overset{?}{=} \underline{b}$

$$\Downarrow$$

$$AA^{-1}\underline{b} = I_N\underline{b} = \underline{b}$$

UNIQUENESS OF $\underline{x}$

$A\underline{y} = \underline{b}$

$\underline{y} \in \mathbb{R}^N$ $\quad \underline{y} = I_N\underline{y} = A^{-1}A\underline{y} = A^{-1}(A\underline{y}) = A^{-1}\underline{b} = \underline{x}$ $\quad$ SO $\underline{y} = \underline{x}$

THEOREM

1. $A \in \mathbb{R}^{N \times N}$ INVERTIBLE. THEN ALSO A^{-1} IS INVERTIBLE AND ITS INVERSE IS A
2. $A, B \in \mathbb{R}^{N \times N}$ INVERTIBLE. THEN AB IS ALSO INVERTIBLE, WITH $(AB)^{-1} = A^{-1}B^{-1}$
3. $A \in \mathbb{R}^{N \times N}$ INVERTIBLE. THEN A^T IS INVERTIBLE, WITH $(A^T)^{-1} = (A^{-1})^T$

① $AA^{-1} = A^{-1}A = I_N$

A^{-1} INVERTIBLE $\quad CA^{-1} = A^{-1}C = I_N$? $\quad C \in \mathbb{R}^{N \times N}$

$C = A$

② IS AB INVERTIBLE?

$C(AB) = (AB)C = I_N$

$C = B^{-1}A^{-1}$ $\quad C(AB) = (B^{-1}A^{-1})(AB) = B^{-1}\underbrace{A^{-1}A}_{I_N}B = B^{-1}I_NB = B^{-1}B = I_N$

OBSERVATION $A_1, A_2, \ldots A_p \in \mathbb{R}^{N \times N}$ ALL INVERTIBLE

THEN $(A_1A_2A_3 \ldots A_p) \in \mathbb{R}^{N \times N}$ IS INVERTIBLE, AND THE INVERSE IS $(A_p^{-1} \ldots A_3^{-1}A_2^{-1}A_1^{-1})$

ELEMENTARY MATRIX AN ELEMENTARY MATRIX $E \in \mathbb{R}^{N \times N}$ IS A MATRIX OBTAINED BY APPLYING AN ELEMENTARY ROW OPERATION TO I_N

$N = 3$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ELEMENTARY ROW OPERATION

- SWITCH THE FIRST AND THIRD ROWS $\longrightarrow E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
- MULTIPLY THE SECOND ROW BY 3 $\longrightarrow E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- SUBTRACT THE FIRST ROW MULTIPLIED BY 2 TO THE SECOND ROW $\longrightarrow E_3 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$A=\begin{bmatrix}1&4\\2&5\\3&6\end{bmatrix}\in\mathbb{R}^{3\times 2}\qquad E_1A=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}1&4\\2&5\\3&6\end{bmatrix}=\begin{bmatrix}3&6\\2&5\\1&4\end{bmatrix}$$

$$E_2A=\begin{bmatrix}1&0&0\\0&3&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&4\\2&5\\3&6\end{bmatrix}=\begin{bmatrix}1&4\\6&15\\3&6\end{bmatrix}$$

$$E_3A=\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&4\\2&5\\3&6\end{bmatrix}=\begin{bmatrix}1&4\\0&-3\\3&6\end{bmatrix}$$

OBSERVATION: • (IMPORTANT!) APPLYING AN ELEMENTARY ROW OPERATION TO A MATRIX $A\in\mathbb{R}^{m\times n}$ IS EQUIVALENT TO CALCULATE THE PRODUCT EA, WHERE $E\in\mathbb{R}^{m\times m}$ IS THE ELEMENTARY MATRIX ASSOCIATED TO THAT SAME ROW OPERATION (THAT IS, THE MATRIX OBTAINED BY APPLYING THE SAME ROW OPERATION TO I_m)

- ALL ELEMENTARY ROW OPERATION ARE REVERSIBLE THERFORE ALL MATRIXES ARE INVERTIBLE

CAN GO TO THE ORIGINAL

THE INVERSE MATRIX IS THE ELEMENTARY MATRIX ASSOCIATED TO THE REVERSE ROW OPERATION

$$E_2=\begin{pmatrix}1&0&0\\0&3&0\\0&0&1\end{pmatrix}$$

• DIVIDE THE SECOND ROW BY 3 = MULTIPLY THE 2ND ROW BY 1/3 $\rightarrow \begin{pmatrix}1&0&0\\0&1/3&0\\0&0&1\end{pmatrix}=E$

$$EE_2=\begin{pmatrix}1&0&0\\0&1/3&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&3&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$E_1^{-1}=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$$

$$E_3^{-1}=\begin{pmatrix}1&0&0\\2&1&0\\0&0&1\end{pmatrix}$$

$A\in\mathbb{R}^{N\times N}$ ECHELON FORM $A\in\mathbb{R}^{4\times 4}$

IF THE MATRIX IS SQUARED AND YOU MUST HAVE A PIVOUT POSITIO IN EVERY ROW THERE IS ONLY ONE SOLUTION

$$\begin{pmatrix}\boxed{*}&*&*&*\\0&\boxed{*}&*&*\\0&0&\boxed{*}&*\\0&0&0&\boxed{*}\end{pmatrix}$$

THAT IS THE SAME TO HAVE ONE PIVOUT POSITION IN EVERY COLUMN

THEOREM $A\in\mathbb{R}^{N\times N}$ (SQUARE) THE FOLLOWING STATEMENTS ARE EQUIVALENT

- A HAS A PIVOT POSITION IN EVERY ROW
- A HAS A PIVOT POSITION IN EVERY COLUMN
- THE PIVOT POSITION ARE ALL ON THE DIAGONAL
- (THERE ARE EXACTLY N PIVOT POSITION)

THEOREM $A \in \mathbb{R}^{N \times N}$ IS INVERTIBLE IF AND ONLY IF IS ROW EQUIVALENT TO THE IDENTITY MATRIX I_N.

THE INVERSE MATRIX A^{-1} CAN BE COMPUTED BY APPLYING TO I_N THE SAME ROW OPERATIONS THAT WHERE APPLIED IN ORDER TO TRANSFORM A INTO I_N

LETS EXPLAIN WHY THIS IS TRUE ("PROOF")

WHY IF A IS INVERTIBLE THEN IT MUST BE ROW EQUIVALENT TO I_N?

22/03

TODAY A IS INVERTIBLE $\Rightarrow$ THE SYSTEM $A\underline{x}=\underline{b}$ ALWAYS HAVE A SOLUTION $\forall \underline{b} \in \mathbb{R}^N$ $\Rightarrow$ THE MATRIX A HAS A PIVOT POSITION IN EVERY ROW

TODAY $\Rightarrow$ THE MATRIX A HAS ALL PIVOT POSITION ON THE DIAGONAL $\Rightarrow$ WITH ROW OPERATION

$$\begin{bmatrix} \boxed{*} & * & * & * \\ 0 & \boxed{*} & * & * \\ 0 & 0 & \boxed{*} & * \\ 0 & 0 & 0 & \boxed{*} \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

WHY A MATRIX THAT IS ROW EQUIVALENT TO THE IDENTITY MUST BE INVERTIBLE?

$$A \xrightarrow{\text{ELEMENTARY ROW OPERATION}} I_M$$

SAME AS DOING ELEMENTARY ROW OPERATION

$$\boxed{\underbrace{E_p \dots E_3 E_2 E_1}_{\text{ELEMENTARY MATRIX}} A}$$

$$(E_p \dots E_3 E_2 E_1) A = I_N$$

A IS INVERTIBLE AND $A^{-1} = E_p \dots E_3 E_2 E_1 = E_p \dots E_3 E_2 E_1 I_N$

EXAMPLE

PIVOUT POSITION ON THE DIAGONAL $\Rightarrow$ A IS INVERTIBLE

COMPUTE THE INVERSE

IDENTITY MATRIX

$$\boxed{A} = \begin{bmatrix} \boxed{1} & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix} \begin{matrix} 3 \\ -2 \end{matrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & \boxed{1} & -2 \\ 0 & -3 & 8 \end{bmatrix} 3 = \begin{bmatrix} \boxed{1} & 0 & -2 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & \boxed{2} \end{bmatrix} \begin{matrix} 1 \\ 1 \end{matrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \cdot \tfrac{1}{2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} 3 \\ -2 \end{matrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} 3 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 7 & 3 & 1 \end{bmatrix} \begin{matrix} 1 \\ 1 \end{matrix} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 0 \\ 7 & 3 & 1 \end{bmatrix} \cdot \tfrac{1}{2} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 0 \\ 7/2 & 3/2 & 1/2 \end{bmatrix} = \boxed{A^{-1}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[I_3 \mid A^{-1} \right]$$

A $\qquad$ I_N

"DO THE SAME CALCULATION ON THE 3X6 MATRIX"

$$[A \mid I] \xrightarrow{RRA} [I \mid A^{-1}]$$

ROW REDUCTION ALGORITHM

THEOREM

$A \in \mathbb{R}^{n \times n}$ (SQUARE) THE FOLLOWING STATEMENTS ARE EQUIVALENT

- A IS INVERTIBLE
- A IS ROW EQUIVALENT TO I_n
- A HAS n PIVOT POSITIONS (ALL ON THE DIAGONAL)
- THE SYSTEM $A\underline{x} = \underline{0}$ HAS ONLY THE TRIVIAL SOLUTION
- THE EQUATION $A\underline{x} = \underline{b}$ HAS (AT LEAST) ONE SOLUTION FOR EACH $\underline{b} \in \mathbb{R}^n$
- THE COLUMNS OF A SPAN ALL $\mathbb{R}^n$
- THERE IS $C \in \mathbb{R}^{n \times n}$ SUCH THAT $CA = I_n$
- THERE IS $C \in \mathbb{R}^{n \times n}$ SUCH THAT $AC = I_n$
- A^T IS AN INVERTIBLE MATRIX