Written exam of Calculus - Part 1 - Sample 1

IT IS FORBIDDEN TO USE CALCULATORS, BOOKS, NOTES, AND SO ON

TIME ALLOWED: 90 MINUTES

NAME:.................................... **ID NUMBER:**....................................

SURNAME:..

PART A Write **only** the answer.

1.1 (3 POINTS)

Solve the following inequality

$$\frac{(2+\cos(x))(x-1)^3}{x-2} \geq 0.$$

$2+\cos(x) \geq 0; \ \forall x \in \mathbb{R}$

$(x-1)^3 \geq 0 \quad x \geq 1$

$x-2>0; \ x>2$

		1		2	
N_1	+		+		+
N_2	−		+		+
D_1	−		−		+
	+	•	−	∄	+

ANSWER: $(-\infty, 1] \cup (2, +\infty)$

1.2 (3 POINTS)

Compute the derivative of $f(x) = \cos\left(x^2 + e^{2x}\right)$.

ANSWER: $f'(x) =$ $-\operatorname{sen}(x^2+e^{2x}) \cdot 2x \cdot 2e^{2x}$

1.3 (3 POINTS)

Establish the character of the series $\displaystyle\sum_{n=2}^{\infty} \frac{n^2-1}{n^4 \log(n)}$.

SCOMMETTO DI GIORNO SCOMMETTO DI NOTTE
SULLE FORMICHE CON LE
ZAMPE STORTE

ANSWER:

PART B Write a **complete** solution.

1.4 (8 POINTS)

Compute, if it exists,

$$\lim_{x\to 0} \frac{(\sin(x))^2 - 2(1-\cos(x))}{\log(1+3x^4)}.$$

Other possible examples: $\lim_{x\to 0} \frac{\sin(x) - x}{\log(1-2x^3)}$ or $\lim_{x\to 1} \frac{e^{(x-1)^2} - 1 - (x-1)^2}{(\sin(2(x-1)^2))^2}$.

Other possible examples: $\lim_{n\to +\infty} \frac{e^{3n} + n^3 e^{-2n}}{(e^n)^2 + 5n^7}$ or $\lim_{n\to +\infty} \frac{(1/n)\log(n) + (1/\sqrt{n})}{1/(\sqrt{n}+1)}$ or $\lim_{n\to +\infty} \frac{\sqrt{n^2+1} - \sqrt{n}}{2n+5}$.

SOLUTION:

$$\lim_{x\to 0} \frac{(\sin(x))^2 - 2 + 2\cos(x)}{\log(1+3x^4)}$$

$\left[\frac{0}{0}\right]$ APPLY LAGRANGE

$$\lim_{x\to 0} \frac{2\sin(x)\cos(x) - 2\sin(x)}{\frac{12x^3}{1+3x^4}}$$

$$\lim_{x\to 0} \left(2\sin(x)\cos(x) - 2\sin(x)\right) \cdot \frac{(1+3x^4)}{12x^3} = \lim_{x\to 0} \frac{(\sin(x)\cos(x) - \sin(x))(1+3x^4)}{6x^3}$$

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

$$= \lim_{x\to 0} \frac{\sin(x)(\cos(x)-1)(1+3x^4)}{6x^3} = \lim_{x\to 0} \frac{(\cos(x)-1)(1+3x^4)}{6x^2} = \lim_{x\to 0} \frac{\cos(x) + 3x^4\cos(x) - 1 - 3x^4}{6x^2} \quad \left[\frac{0}{0}\right]$$

APPLY HOPITAL

$$\lim_{x\to 0} \frac{-\sin(x) + 12x^3\cos(x) - 3x^4\sin(x) - 12x^3}{12x} \quad \left[\frac{0}{0}\right] \text{ APPLY HOPITAL}$$

$$\lim_{x\to 0} \frac{-\cos(x) + 36x^2\cos(x) - 12x^3\sin(x) - 12x^3\sin(x) - 3x^4\cos(x) - 36x^2}{12} = -\frac{1}{12}$$

1.5 (8 POINTS)

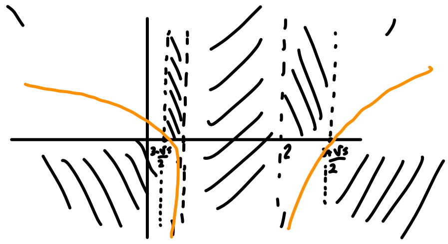
Study the following function

$$f(x) = \log(x^2 - 3x + 2)$$

and draw its graph.

Another possible example: $f(x) = e^{x^2-3x+2}$.

SOLUTION:



$f(x) = \log(x^2-3x+2)$

D: $x^2-3x+2>0$ $\quad$ $(-\infty,1)\cup(2,+\infty)$

$\frac{3\pm1}{2} \langle^{2}_{1}$

SIGN:

$\log(x^2-3x+2)>0; \quad x^2-3x+2>1; \quad x^2-3x+1>0 \quad \frac{3\pm\sqrt5}{2}$

$\frac{3-\sqrt5}{2} \simeq 0.5 \qquad \frac{3+\sqrt5}{2} \simeq 2.5$

VERTICAL ASYNDOTES $x=2 \vee x=1$

LIMIT

$\lim_{x\to+\infty} f(x) = +\infty$

$\lim_{x\to-\infty} f(x) = +\infty$

$\lim_{x\to1^-} f(x) = -\infty$

$\lim_{x\to2^+} f(x) = -\infty$

ASYNDOTES: $y = mx+q \quad m,q \in \mathbb{R}$

$m = \lim_{x\to+\infty} \frac{f(x)}{x} = 0$

$q = \lim_{x\to+\infty} f(x) - mx = \lim_{x\to+\infty} f(x) = +\infty$ NO ASYNDOTES OBLIQUES

DERIVATE

$f'(x) = \frac{2x-3}{x^2-3x+2}$ $\quad$ STUDY OF SIGN OF $f'(x)$

$2x-3>0; \quad x>\frac{3}{2}$

$x^2-3x+2>0; \quad \frac{3\pm1}{2}$

	1		3/2		2	
N_1	$-$	$-$		$+$		$+$
D_1	$+$	$-$		$-$		$+$
	$-$	$+$		$-$		$+$
	↘ MIN	↗	MAX	↘	MIN	↗

$f(x)$ DOES NOT EXIST IN $[1,2]$

1.6 (7 POINTS)

State and prove the Theorem of limit of a monotone sequence.

STATEMENT: IF A SEQUENCE IS BOUNDED AND MONOTONE THAN IT IS CONVERGENT

PROOF: CONSIDER THE SET $A = \{a_n \mid n \in \mathbb{N}\}$ IT IS AN NON EPTY AND BOUNDED ABOVE SET

SO IT MUST HAVE A SUPREMUM $L = \sup\{A\}$

BY DEFINITION OF SUPPEMUM

$\exists n_0 \in \mathbb{N} \quad L - \varepsilon < a_{n_0} \leq L \quad \forall \varepsilon > 0$

ASSUME $n \geq n_0$ SO

SINCE IT IS INCREASIN

$a_{n_0} \leq a_n$

$L - \varepsilon < a_{n_0} < a_n \leq L$

SO $L - \varepsilon < a_{n_0} < a_n \leq L < L + \varepsilon$

SO

$L - \varepsilon < a_n < L + \varepsilon$

Other possible examples: $\lim\limits_{x\to 0} \dfrac{\sin(x)-x}{\log(1-2x^3)}$

$$\lim_{x\to 0} \frac{\sin(x)-x}{\log(1-2x^3)} \quad \left[\frac{0}{0}\right] \quad \text{APPLY HOPITAL}$$

$$\lim_{x\to 0} \frac{\cos(x)-1}{\dfrac{-6x^2}{1-2x^3}} = \lim_{x\to 0} \frac{(\cos(x)-1)(1-2x^3)}{-6x^2}$$

$$\lim_{x\to 0} \frac{\cos(x)-1-2x^3\cos(x)+2x^3}{-6x^2} \quad \left[\frac{0}{0}\right] \quad \text{APPLY HOPITAL}$$

$$\lim_{x\to 0} \frac{-\operatorname{sen}(x)-6x^2\cos(x)+2x^3\sin(x)+6x^2}{-6x} \quad \left[\frac{0}{0}\right] \quad \text{APPLY HOPITAL}$$

$$\lim_{x\to 0} \frac{-\cos(x)-12x\cos(x)+6x^2\operatorname{sen}(x)-9x^2\sin(x)+2x^3\cos(x)+12x}{-12} = \boxed{\frac{1}{12}}$$

$$\lim_{x\to 1} \frac{e^{(x-1)^2}-1-(x-1)^2}{(\sin(2(x-1)^2))^2}.$$

$\left[\frac{0}{0}\right]$ APPLY HOPITAL

$$\lim_{x\to 1} \frac{e^{x^2+1-2x}-1-x^2-1+2x}{(\sin(2x^2+2-4x))^2}$$

$$\lim_{x\to 1} \frac{(\cancel{2}x-\cancel{2})e^{x^2+1-2x}-\cancel{2}x+\cancel{2}\,1}{\cancel{2}(4x-4)(\sin(2x^2+2-4x))}$$

$$\lim_{x\to 1} \frac{(x-1)e^{x^2+1-2x}-x+1}{4x\sin(2x^2+2-4x)-4\sin(2x^2+2-4x)} = \lim_{x\to 1} \frac{xe^{x^2+1-2x}-e^{x^2+1-2x}-x+1}{4x\sin(2x^2+2-4x)-4\sin(2x^2+2-4x)}$$

$$\lim_{x\to 1} \frac{\quad}{\quad}$$ $\left[\frac{0}{0}\right]$ HOPITAL

$$\lim_{x\to 1} \frac{e^{(x-1)^2}-1-x^2-1+2x}{\sin(2x^2+2-4x)^2} = \left[\frac{0}{0}\right]$$ APPLY HOPITAL

$$\lim_{x\to 1} \frac{(\cancel{2}x+\cancel{2}\,1)e^{(x-1)^2}-\cancel{2}x+\cancel{2}}{\cancel{2}(4x-4)\sin(2x^2+2-4x)\cos(2x^2+2-4x)}$$

$$\lim_{x\to 1} \frac{(x+1)e^{(x-1)^2}-x+1}{4x\sin(2x^2+2-4x)\cos(2x^2+2-4x)-4\sin(2x^2+2-4x)\cos(2x^2+2-4x)} = \frac{2}{0} = +\infty$$

$$\lim_{n\to+\infty} \frac{e^{3n}+n^3e^{-2n}}{(e^n)^2+5n^7}$$

$$\lim_{N\to+\infty} \frac{e^{3N}+\underbrace{N^3e^{-2N}}_{\to 0}}{(e^N)^2+5N^7}$$

$$\lim_{x\to+\infty} \frac{e^{x}-1}{x}=1$$

$$\lim_{N\to+\infty} \frac{e^{3N}}{e^{2N}+5N^7} \Rightarrow e^N = = +\infty$$

$$\lim_{N\to+\infty} \frac{\left(\frac{1}{N}\right)\cos(N)+\left(1/\sqrt{N}\right)}{\frac{1}{(\sqrt{N}+1)}}$$

$$\lim_{N\to+\infty} \left(\frac{1}{N}\right)\cos(N)+\left(\frac{1}{\sqrt{N}}\right)\cdot(\sqrt{N}+1)$$

$$\lim_{N\to+\infty} \frac{1}{\sqrt{N}}\cdot\cos(N)+1+\frac{1}{N}\cos(N)+\frac{1}{\sqrt{N}}$$

$t=\frac{1}{\sqrt{N}}$ $\quad t\to 0$

$$\lim_{t\to 0} t\cdot\cos(t^{-2})+1+t^2\cos(t^{-2})+t=1$$

$$\lim_{N\to+\infty} \frac{\sqrt{N^2+1} - \sqrt{N}}{2N+5} = \frac{1}{2}$$

$$\frac{N - \sqrt{N}}{2N+5}$$

$$\frac{\cancel{N}\left(\frac{1}{N} - \frac{1}{\sqrt{N}}\right)}{\cancel{N}\left(2 + 5/N\right)} = \frac{1}{2}$$

$$\lim_{N\to+\infty} \frac{\left(\sqrt{N^2+1} - \sqrt{N}\right)\left(\sqrt{N^2+1} + \sqrt{N}\right)}{(2N+5)\left(\sqrt{N^2+1} + \sqrt{N}\right)}$$

$$\lim_{N\to+\infty} \frac{N^2+1-N}{(2N+5)\left(\sqrt{N^2+1} + \sqrt{N}\right)}$$

$$\lim_{N\to+\infty} \frac{\cancel{N}\left(N + 1/N - 1\right)}{\cancel{N}\left(2 + \frac{5}{N}\right)\left(\sqrt{N^2+1} + \sqrt{N}\right)}$$

$$\lim_{N\to+\infty} \frac{\cancel{N}\left(1 + 1/N^2 - \frac{1}{N}\right)}{\cancel{N}\left(2 + \frac{5}{N}\right)\left(\frac{\sqrt{N^2+1} + \sqrt{N}}{N}\right)}$$

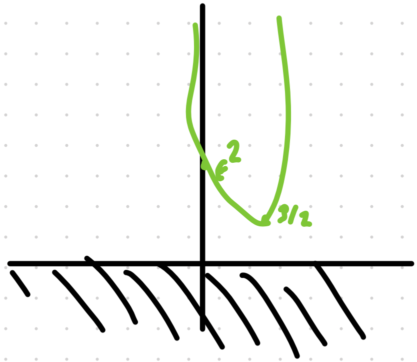
$$\lim_{N\to+\infty}$$

Another possible example: $f(x) = e^{x^2-3x+2}$.

$D: \mathbb{R}$

$f(x) > 0 \quad \forall \in \mathbb{R}$

$f(0) = e^2$



LIMITS

$\lim_{x \to +\infty} f(x) = +\infty$

ASYMPTOTES

$y = mx + q$

$m = \lim_{x \to +\infty} \frac{f(x)}{x} = +\infty$

DERIVATES

$f'(x) = (2x - 3)\, e^{x^2-3x+2}$

SIGN OF $f'(x)$

$$\lim_{x \to 0} \frac{\sin(x)^2 - x^2 + \frac{x^4}{12}}{3x^4} = \quad \left[\frac{0}{0}\right] \text{HOPITAL}$$

$$\lim_{x \to 0} \frac{2\sin(x)\cos(x) - 2x + \frac{4x^3}{12}}{12x^3}$$

$$\lim_{x \to 0} \frac{\sin(x)\cos(x) - x + \frac{2}{12}x^3}{6x^3}$$

1/6

$$\lim_{x \to 0} \frac{\frac{6\sin(x)\cos(x) - 6x + x^3}{6}}{6x^3}$$

$$\lim_{x \to 0} \frac{6\sin(x)\cos(x) - 6x + x^3}{6} \cdot \frac{1}{6x^3}$$

$$\lim_{x \to 0} \frac{6\sin(x)\cos(x) - 6x + x^3}{36x^3} \quad \left[\frac{0}{0}\right] \text{HOPITAL}$$

$$\lim_{x \to 0} \frac{6\cos(x)^2 - 6\sin(x)^2 - 6 + 3x^2}{108x^2} \quad \left[\frac{0}{0}\right] \text{HOPITAL}$$

$$\lim_{x \to 0} \frac{-12\cos(x)\sin(x) - 12\sin(x)\cos(x) + 6x}{216x}$$

$$\lim_{x \to 0} \frac{-24\sin(x)\cos(x) + 6}{216x}$$

$$\lim_{x\to 0} \frac{\sin(x)^2 - x^2 + \frac{x^4}{12}}{3x^4} = \lim_{x\to 0} \frac{\frac{12\sin(x)^2 - 12x^2 + x^4}{12}}{3x^4}$$

$$\lim_{x\to 0} \frac{12\sin(x)^2 - 12x^2 + x^4}{36x^4} \quad \left[\frac{0}{0}\right] \text{ HOPITAL}$$

$$\lim_{x\to 0} \frac{12\sin(x)\cos(x) + 12\sin(x)\cos(x) - 24x + 4x^3}{144x^3}$$

$$\lim_{x\to 0} \frac{24\sin(x)\cos(x) - 24x + 4x^3}{144x^3} \quad \left[\frac{0}{0}\right] \text{ APPLY HOPITAL}$$

$$\lim_{x\to 0} \frac{24\cos(x)^2 - 24\sin(x)^2 - 24 + 12x^2}{432x^2} \quad \left[\frac{0}{0}\right] \text{ APPLY HOPITAL}$$

$$\lim_{x\to 0} \frac{-24\cos(x)\sin(x) - 24\cos(x)\sin(x) - 24\sin(x)\cos(x) - 24\sin(x)\cos(x) + 24x}{864x}$$

$$\lim_{x\to 0} \frac{\overset{30}{N}\cos(x)\sin(x) + 24\cancel{x}}{864\cancel{x}} = \frac{1}{36}$$

NO CAZZO