Knowledge Representation and Reasoning

Exercise Session 1

Exercise 1. Truth Tables (∗)

Build the **truth tables** of the following formulas, remembering the precedence of the operators.

1. $\neg(x \wedge y) \vee z$
2. $(x \wedge y \vee \neg x \wedge \neg w) \wedge z$
3. $(x \vee y) \wedge x$
4. $\neg(p \wedge (\neg p \vee q \vee s)) \vee q \vee s$

Exercise 2. Boolean Functions (∗)

For each of the following truth tables, build a formula expressing the same Boolean function.

x	y	z	φ_1
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

x	z	φ_2
0	0	1
0	1	0
1	0	0
1	1	1

x	y	z	w	φ_3
0	0	0	0	0
0	0	0	1	1
0	0	1	0	0
0	0	1	1	0
0	1	0	0	1
0	1	0	1	0
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	1
1	0	1	0	0
1	0	1	1	0
1	1	0	0	1
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

If the tables represent Boolean functions, what is their arity?

Exercise 3. Types of Formulas (∗)

Use truth tables (and, if needed the equivalences seen during the lecture) to verify whether the following formulas are tautologies, contradictions, or non-tautological satisfiable formulas.

1. $x \rightarrow \neg x$
2. $(x \rightarrow y) \wedge (\neg y \rightarrow \neg x)$
3. $(x \rightarrow y) \rightarrow (\neg y \rightarrow \neg x)$
4. $x \wedge y \wedge (\neg x \vee \neg y) \wedge z$

Exercise 4. NNF (∗∗)

A formula is in **negation normal form** (NNF). For example, the second and third formulas in Exercise 1 are in NNF, while the other two formulas are not.

1. Using the DeMorgan equivalences, transform the formulas of Exercise 1 to NNF
2. Do you notice something from the constructions in Exercise 2?

Exercise 5. Sheffer Functions (∗ ∗ ∗)

We have seen that $\neg, \wedge, \vee$ form a **functionally complete** set of logical connectives in the sense that every Boolean function can be described using those connectives. In fact, we $\neg, \wedge$ and $\neg, \vee$ are functionally complete by themselves.

A logical connective is called a **Sheffer function** or **sole sufficient operator** if it is functionally complete by itself.

1. show that the NAND connective (denoted as $\uparrow$) is a Sheffer function
2. are there other Sheffer functions?
3. could a unary connective be a Sheffer function?

Exercise 6. Knowledge Bases (∗∗)

For the following truth table, construct a knowledge base (expressed as a set of propositional rules) which represents the same knowledge.

x	y	z	K
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Exercise 7. Expressivity (∗∗)

Can any propositional formula be expressed as a knowledge base?

If the answer is **yes**, argue why; if it is **no**, give at least two examples of formulas which cannot be expressed.

Exercise 8. Reasoning (∗)

Consider the following knowledge base K:

$$\begin{aligned}
x &\leftarrow \\
y &\leftarrow x,\ z,\ w \\
x &\leftarrow v \\
w &\leftarrow y,\ z \\
z &\leftarrow v,\ x \\
z &\leftarrow y,\ w \\
z &\leftarrow u,\ x \\
u &\leftarrow \\
p &\leftarrow \\
t &\leftarrow w,\ u \\
r &\leftarrow s,\ t
\end{aligned}$$

1. Compute the redux $\widehat{K}$
2. Find all the facts that are entailed by K
3. Decide whether the following clauses are consequences of K
   a) $v \leftarrow u$
   b) $t \leftarrow y$
   c) $q \leftarrow q$
   d) $r \leftarrow w$

Exercise 9. Revision (∗∗)

In the knowledge base from Exercise 8, substitute the fact $x \leftarrow$ with $u \leftarrow$. Call this new knowledge base K'.

1. Do your answers from Exercise 8 change?
2. Which fact(s) should you remove to ensure that z is **not** a consequence of K'?
3. If facts cannot be removed, which rules would you remove to ensure that z is not entailed?

Exercise 10. Tautologies $(* * *)$

The soundness of the fact extraction algorithm seen during the lecture depends on the following property:

> if $\varphi \to \psi$ and $\psi \to \xi$ are both tautologies, then $\varphi \to \xi$ is also a tautology.

Show that this property holds always in propositional logic.

Exercise 1. Truth Tables (*)

Build the **truth tables** of the following formulas, remembering the precedence of the operators.

1. $\neg(x \wedge y) \vee z$
2. $(x \wedge y \vee \neg x \wedge \neg w) \wedge z$
3. $(x \vee y) \wedge x$
4. $\neg(p \wedge (\neg p \vee q \vee s)) \vee q \vee s$

1) $\neg(x \wedge y) \vee z$

x	y	z	$x \wedge y$	$\neg(x \wedge y)$	$\neg(x \wedge y) \vee z$
0	0	0	0	1	1
0	0	1	0	1	1
0	1	0	0	1	1
1	0	0	0	1	1
0	1	1	0	1	1
1	0	1	0	1	1
1	1	0	1	0	0
1	1	1	1	0	1

3) $(x \wedge y \vee \neg x \wedge \neg w) \wedge z$

x	y	w	z	$x \wedge y$	$\neg x \wedge \neg w$	$x \wedge y \vee \neg x \wedge \neg w$	$\wedge z$
0	0	0	0	0	1	1	0
0	0	0	1	0	1	1	1
0	0	1	0	0	0	0	0
0	1	0	0	0	1	1	0
1	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	1	1	1
1	0	0	1	0	0	0	0
0	1	1	0	0	0	0	0
1	0	1	0	0	0	0	0
1	1	0	0	1	0	1	0
0	1	1	1	0	0	0	0
1	0	1	1	0	0	0	0
1	1	0	1	1	0	1	1
1	1	1	0	1	0	1	0
1	1	1	1	1	0	1	1

3) $(x \vee y) \wedge x$

x	y	$x \vee y$	$(x \vee y) \wedge x$
0	0	0	0
0	1	1	0
1	0	1	1
1	1	1	1

4)

Exercise 2. Boolean Functions (*)

For each of the following truth tables, build a formula expressing the same Boolean function.

x	y	z	φ_1
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

x	z	φ_2
0	0	1
0	1	0
1	0	0
1	1	1

x	y	z	w	φ_3
0	0	0	0	0
0	0	0	1	1
0	0	1	0	0
0	0	1	1	0
0	1	0	0	1
0	1	0	1	0
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	1
1	0	1	0	0
1	0	1	1	0
1	1	0	0	1
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

If the tables represent Boolean functions, what is their arity?

1) $(\neg x \wedge \neg y \wedge z) \vee$
$(\neg x \wedge y \wedge \neg z) \vee$
$(\neg x \wedge y \wedge z) \vee$
$(x \wedge \neg y \wedge \neg z) \vee$
$(x \wedge \neg y \wedge z) \vee$
$(x \wedge y \wedge \neg z) \vee$
$(x \wedge y \wedge z$

2) $(\neg x \wedge \neg z) \wedge (x \wedge z)$

3) y $z \wedge \vee$
$\wedge$
$\wedge$
, z $\wedge$
$(1$ $\neg z \wedge w \wedge$

Exercise 3. Types of Formulas (*)

Use truth tables (and, if needed the equivalences seen during the lecture) to verify whether the following formulas are tautologies, contradictions, or non-tautological satisfiable formulas.

1. $x \to \neg x$
2. $(x \to y) \land (\neg y \to \neg x)$
3. $(x \to y) \to (\neg y \to \neg x)$
4. $x \land y \land (\neg x \lor \neg y) \land z$

1) $\neg x \lor \neg x \Rightarrow \neg x$ NON-TAUTOLOGICAL SATISFIABLE FORMULA

2) $(\neg x \lor y) \land (y \lor \neg x)$ NON-TAUTOLOGICAL SATISFIABLE FORMULA

3) $\neg(x \to y) \lor (\neg y \to \neg x)$

$(x \land \neg y) \lor y \lor \neg x$ ~~NON-TAUTOLOGICAL SATISFIABLE FORMULA~~ TAUTOLOGY

4) $x \land y \land (\neg x \lor \neg y) \land z$ CONTRADICTION

Exercise 4. NNF (**)

A formula is in **negation normal form** (NNF). For example, the second and third formulas in Exercise 1 are in NNF, while the other two formulas are not.

1. Using the DeMorgan equivalences, transform the formulas of Exercise 1 to NNF
2. Do you notice something from the constructions in Exercise 2?

1) $\neg x \lor \neg y \lor z$

4) $(\neg p \lor \neg(\neg p \lor q \lor s)) \lor q \lor s$

$(\neg p \lor p \land \neg q \land \neg s) \lor q \lor s$

5) NAND

$x \uparrow y$

x	y	
0	0	1
0	1	1
1	0	1
1	1	0

$(\neg x \wedge \neg y) \vee (x \wedge \neg y) \vee (\neg x \wedge y)$

Exercise 8. Reasoning

Consider the following knowledge base K:

$$\cancel{v \leftarrow}$$
$$y \leftarrow \cancel{x},\ z,\ w$$
$$x \leftarrow v$$
$$w \leftarrow y,\ z$$
$$z \leftarrow v,\ \cancel{x}$$
$$z \leftarrow y,\ w$$
$$z \leftarrow \cancel{y},\ \cancel{x}$$
$$\cancel{u \leftarrow}$$
$$\cancel{p \leftarrow}$$
$$t \leftarrow w,\ \cancel{u}$$
$$r \leftarrow s,\ t$$

1. Compute the redux $\widehat{K}$
2. Find all the facts that are entailed by K λ ∪P
3. Decide whether the following clauses are consequences of K
   a) $v \leftarrow u$
   b) $t \leftarrow y$
   c) $q \leftarrow q$
   d) $r \leftarrow w$

Exercise 9. Revision (**)

In the knowledge base from Exercise 8, substitute the fact $x \leftarrow$ with $u \leftarrow$. Call this new knowledge base K'.

1. Do your answers from Exercise 8 change?
2. Which fact(s) should you remove to ensure that z is **not** a consequence of K'?
3. If facts cannot be removed, which rules would you remove to ensure that z is not entailed?

$$
\begin{aligned}
&w \leftarrow \\
&y \leftarrow x,\ z,\ w \\
&x \leftarrow v \\
&w \leftarrow y,\ z \\
&z \leftarrow v,\ x \\
&z \leftarrow y,\ w \\
&z \leftarrow u,\ x \\
&\cancel{v \leftarrow}\ u \\
&\cancel{x \leftarrow} \\
&t \leftarrow w,\ u \\
&r \leftarrow s,\ t
\end{aligned}
$$

Exercise 10. Tautologies (∗ ∗ ∗)

The soundness of the fact extraction algorithm seen during the lecture depends on the following property:

if $\varphi \to \psi$ and $\psi \to \xi$ are both tautologies, then $\varphi \to \xi$ is also a tautology.

Show that this property holds always in propositional logic.