Knowledge Representation and Reasoning

Exercise Session 3

Exercise 1. Subsumption (*)

Use the **homomorphism method** to verify whether the following subsumption relations hold:

1. $A \sqcap C \sqcap \exists r.B \sqsubseteq C \sqcap \exists r.\top$
2. $\exists r.(B \sqcap \exists s.\top) \sqsubseteq \exists s.\exists s.\top$
3. $\exists r.(B \sqcap \exists s.\top) \sqsubseteq \exists r.\exists s.B$
4. $\exists r.(B \sqcap \exists s.A \sqcap \exists s.A)) \sqsubseteq \exists s.\exists s.A$
5. $B \sqcap \exists r.A \sqcap \exists r.B \sqcap \exists r.C \sqsubseteq \exists r.(A \sqcap C) \sqcap \exists r.(B \sqcap C)$

Exercise 2. Couter-Models (*)

For the following pairs of concepts C, D, find an interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} \not\subseteq D^{\mathcal{I}}$

1. $C = \exists r.\top, \ \ D = \exists r.A$
2. $C = A \sqcap C \sqcap \exists r.(A \sqcap B), \ \ D = \exists r.(A \sqcap C)$
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5. $C = A \sqcap C \sqcap \exists r.(A \sqcap B), \ \ D = \exists r.(A \sqcap \exists s.\bot)$

Exercise 3. TBox Models (*)

Construct a model of the following TBox which has **at least three elements**

$$\{\exists r.(A \sqcap B) \sqsubseteq A \sqcap \exists r.C, \quad C \sqcap \exists s.\top \sqsubseteq B \sqcap \exists s.B, \quad B \sqcap D \sqsubseteq \bot, \quad \exists s.D \sqsubseteq A \sqcap C\}$$

Exercise 4. Disjoint Unions (***)

Recall the notion of **disjoint union** of interpretations from the previous exercise session. Is the disjoint union of two models of an $\mathcal{EL}_\bot$ TBox $\mathcal{T}$ also a model of $\mathcal{T}$? **Justify**

Exercise 5. Model Size (***)

Construct an $\mathcal{EL}_\perp$ TBox that has only models with **at most** 5 elements, or argue why it cannot exist.

Exercise 6. Normalisation (*)

Transform the TBox from Exercise 3 to normal form.

Exercise 7. Reasoning (*)

Let $\mathcal{T}$ be the TBox from Exercise 3.

1. Apply the completion algorithm to check whether the following consequences hold:
   - $\exists r.\exists s.D \sqsubseteq A \sqcap \exists r.\exists s.B$
   - $D \sqcap \exists s.D \sqsubseteq B \sqcap \exists r.\top$
   - $B \sqcap \exists r.\top \sqsubseteq D \sqcap \exists s.D$
2. Construct eventual **countermodels**

Exercise 8. Completeness (***)

In the lecture we showed how to build an interpretation $\mathcal{I}$ from the data structure resulting from the completion algorithm, and **claimed it was a model**. As a step towards proving this claim, show that if the TBox contains the GCI $X \sqsubseteq Y$, then the interpretation $\mathcal{I}$ satisfies this GCI.

Exercise 9. Inverse Roles (***)

Using **inverse roles** build a TBox that expresses the knowledge that *humans can only have human children.*

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4. $\exists r.(B \sqcap \exists s.A \sqcap \exists s.A)) \sqsubseteq \exists s.\exists s.A$
5. $B \sqcap \exists r.A \sqcap \exists r.B \sqcap \exists r.C \sqsubseteq \exists r.(A \sqcap C) \sqcap \exists r.(B \sqcap C)$

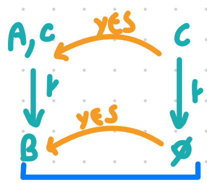
HOMOMORPHISM METHOD: CHECK WHETHER TWO CONCEPTS ARE IN A SUBSUMPTION RELATION

REMEMBER!! THE HOMOMORPHISM METHOD IS ONLY ABOUT CONCEPTS WITHOUT A T-BOX. IT IS JUST THE CONCEPTS

BUILD A HOMOMORPHISM BETWEEN TWO TREES (THEY REPRESENTS THE CONCEPTS THAT WE ARE INTERESTED WITH)

REMEMBER!! IF WE WANT TO CHECK THAT THE CONCEPT OF THE LEFT IS SUBSUMED BY THE CONCEPT ON THE RIGTH WE HAVE TO CHECK HOMOMORFISM FROM THE RIGTH TO THE LEFT (REVERSE DIRECTION)

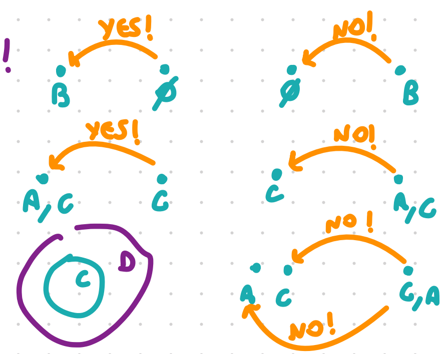
1.



TOP IS LABEL WITH THE EMPTY SET

THE EMPTY SET IS A SUBSET OF B

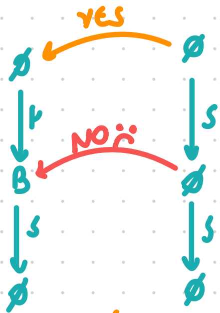
REMEMBER!!



$C \sqsubseteq D$

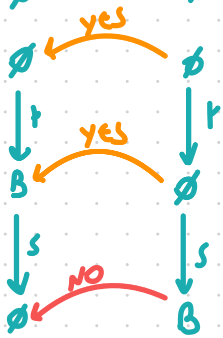
YES THE SUBSUMPTION HOLDS

2.



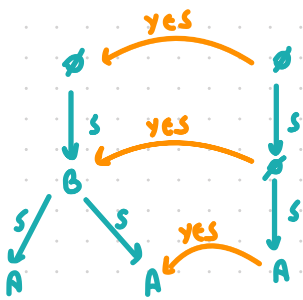
NO, THE SUBSUMPTION DOES NOT HOLDS

3.



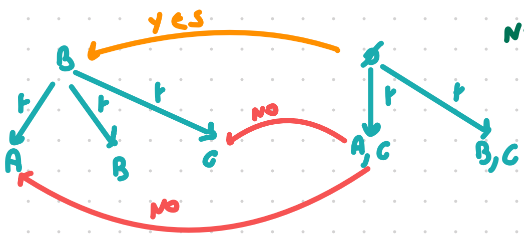
NO, THE SUBSUMPTION DOES NOT HOLDS

4.



YES, THE SUBSUMPTION HOLDS

5.



NO, THE SUBSUMPTION DOES NOT HOLDS

Exercise 2. Couter-Models (*)

For the following pairs of concepts C, D, find an interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} \not\subseteq D^{\mathcal{I}}$

1. $C = \exists r.\top, \;\; D = \exists r.A$
2. $C = A \sqcap C \sqcap \exists r.(A \sqcap B), \;\; D = \exists r.(A \sqcap C)$
3. $C = \exists r.A \sqcap \exists r.B, \;\; D = \exists r.(A \sqcap B)$
4. $C = \exists r.(A \sqcap B), \;\; D = \exists r.A \sqcap \exists s.B$
5. $C = A \sqcap C \sqcap \exists r.(A \sqcap B), \;\; D = \exists r.(A \sqcap \exists s.\bot)$

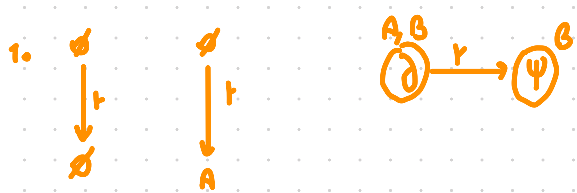
WE WANT TO FIND AN INTERPRETATION THAT SHOW THAT THE FIRST CONCEPT IS NOT A SUBCONCEPT OF THE SECOND ONE

FIND A CONCEPT THAT BELONGS TO C BUT NOT TO D $\exists \partial \in C^I / D^I$

THAT IS $\partial \in C^I$ AND $\partial \notin D^I$

YOU NEED TO FIND AN OBJECT THAT HAVE THE PROPRIETY OF C BUT NOT THE PROPERTY OF D

1.



$I = \{\Delta^I, \cdot^I\}$

$\Delta^I = \{\partial, \psi\}$

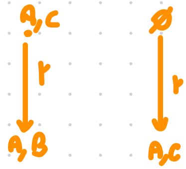
$r^I = \{(\partial, \psi)\}$

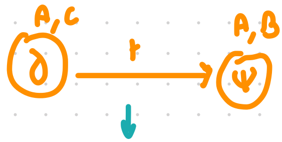
$A^I = \{\partial\}$

$B^I = \{\partial, \psi\}$

OTHERWISE THOSE ARE ALSO CORRECT

2.





THIS OBJECT BELONGS TO THE CONCEPT ON THE LEFT BUT NO TO THE CONCEPT ON THE RIGTH

$I = \{\Delta^I, \cdot^I\}$

$\Delta^I = \{\partial, \psi\}$

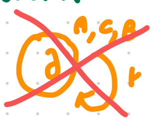
$r^I = \{(\partial, \psi)\}$

$A^I = \{\partial, \psi\}$

$B^I = \{\psi\}$

$C^I = \{\partial\}$

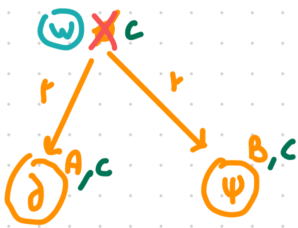
IF WE TRY TO HAVE JUST ONE ELEMENT IN THE DOMAIN IT WILL NOT WORK



NOT A COUNTERMODEL

3.

3. $C = \exists r.A \sqcap \exists r.B$, $D = \exists r.(A \sqcap B)$



$I = \{\Delta^I, \cdot^I\}$

$\Delta^I = \{\partial, \psi, w\}$

$r^I = \{(w, \partial), (w, \psi)\}$

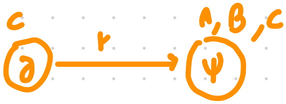
$A^I = \{\partial,\}$

$B^I = \{\psi\}$

$C^I = \{\partial, \psi, w\}$

IS ∅ PART OF THE DOMAIN
SHOUL WE ADD IT?

4. 4. $C = \exists r.(A \sqcap B)$, $D = \exists r.A \sqcap \exists s.B$



$I = \{\Delta^I, \cdot^I\}$ $\quad A^I = \{\psi\}$

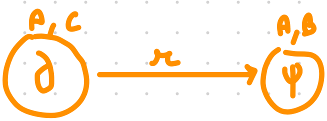
$\Delta^I = \{\partial, \psi\}$ $\quad B^I = \{\psi\}$

$r^I \{(\partial, \psi)\}$ $\quad C^I = \{\partial, \psi\}$

5. 5. $C = A \sqcap C \sqcap \exists r.(A \sqcap B)$, $D = \exists r.(A \sqcap \exists s.\perp)$

REMEMBER!! IN $AL_\perp$ ANY CONCEPT WITH $\perp$ IS JUST $\perp$

D CONTAINS $\perp$ SO IT IS EQUAL TO $\perp$



THIS IS A MODEL

THIS IS ALSO A MODEL

Exercise 3. TBox Models (*)

Construct a model of the following TBox which has **at least three elements**

$$\{\exists r.(A \sqcap B) \sqsubseteq A \sqcap \exists r.C, \quad C \sqcap \exists s.\top \sqsubseteq B \sqcap \exists s.B, \quad B \sqcap D \sqsubseteq \bot, \quad \exists s.D \sqsubseteq A \sqcap C\}$$

FIND AN INTERPRETATION THAT SATISFY ALL THE CONSTRAINTS
AND THAT HAS THREE ELEMENTS

BREAKS THE RULE:

$\exists r.(A \sqcap B) \sqsubseteq A \sqcap \exists r.C$ MEANS THAT IF I HAVE $\exists r.(A \sqcap B)$ THEN I MUST ALSO HAVE $A \sqcap \exists r.C$. BUT I CAN CREATE 2 SUCCESSOR AND THREE ELEMENTS THAT ARE NOT IN THE TBOX

$$E \xrightarrow{q} F \xrightarrow{p} G$$

Exercise 4. Disjoint Unions (***)

Recall the notion of **disjoint union** of interpretations from the previous exercise session. Is the disjoint union of two models of an $\mathcal{EL}_\perp$ TBox $\mathcal{T}$ also a model of $\mathcal{T}$? **Justify**

THE INTERPRETATION OF THE DOMAIN IS THE UNION OF THE INTERPRETATION OF THE TWO DOMAINS

$$\mathcal{I} \oplus \mathcal{J} = (\Delta^{\mathcal{I}} \cup \Delta^{\mathcal{J}}, \cdot^{\mathcal{IJ}})$$

Exercise 5. Model Size (***)

Construct an $\mathcal{EL}_\perp$ TBox that has only models with **at most** 5 elements, or argue why it cannot exist.

IT CANNOT EXISTS

- A MODEL MUST HAVE AT LEAST ONE ELEMENT (CAN NOT BE EMPTY)

Exercise 6. Normalisation (*)

Transform the TBox from Exercise 3 to normal form.

$$\{\overset{1.}{\exists r.(A \sqcap B) \sqsubseteq A \sqcap \exists r.C}, \quad \overset{2.}{C \sqcap \exists s.\top \sqsubseteq B \sqcap \exists s.B}, \quad \overset{3.}{B \sqcap D \sqsubseteq \bot}, \quad \overset{4.}{\exists s.D \sqsubseteq A \sqcap C}\}$$

1. $\exists r.(A \sqcap B) \sqsubseteq X_0$

$X_0 \sqsubseteq A \sqcap \exists r.C$

$\exists r.X_1 \sqsubseteq X_0$ $\quad$ $X_0 \sqsubseteq A$

$A \sqcap B \sqsubseteq X_1$ $\quad$ $X_0 \sqsubseteq \exists r.C$

2. $C \sqcap \exists s.\top \sqsubseteq X_2$

$X_2 \sqsubseteq B \sqcap \exists s.B$

$C \sqcap X_3 \sqsubseteq X_2$ $\quad$ $X_2 \sqsubseteq B$

$\exists s.\top \sqsubseteq X_3$ $\quad$ $X_2 \sqsubseteq \exists s.B$

3. IS ALREADY IN NORMAL FORM $B \sqcap D \sqsubseteq \bot$

4. $\exists s.D \sqsubseteq A$

$\exists s.D \sqsubseteq C$

Exercise 7. Reasoning (∗)

Let $\mathcal{T}$ be the TBox from Exercise 3.

1. Apply the completion algorithm to check whether the following consequences hold:
   - $\exists r.\exists s.D \sqsubseteq A \sqcap \exists r.\exists s.B$
   - $D \sqcap \exists s.D \sqsubseteq B \sqcap \exists r.\top$
   - $B \sqcap \exists r.\top \sqsubseteq D \sqcap \exists s.D$
2. Construct eventual **countermodels**

• ADD THE CONSEQUENCES THAT WE WANT TO CHECK TO THE T-BOX

$\exists r.\exists s.D \sqsubseteq Y_1$

$Y_1 \sqsubseteq A \sqcap \exists r.\exists s.B$

$\exists r.X_4 \sqsubseteq Y_1$ $\qquad$ $Y_1 \sqsubseteq A \sqcap X_5$

$\exists s.D \sqsubseteq X_4$ $\qquad$ $Y_1 \sqsubseteq A$ $\qquad$ $\exists r.\exists s.B \sqsubseteq X_5$

$\exists r.X_6 \sqsubseteq X_5$

$\exists s.B \sqsubseteq X_6$

REWRITE THE T-BOX OF THE EX. BEFORE THAT WERE IN NORMAL FORM

$X_0 \sqsubseteq A$ $\quad$ $X_0 \sqsubseteq \exists r.C$ $\quad$ $\exists s.D \sqsubseteq A$ $\quad$ $\exists s.D \sqsubseteq C$ $\quad$ $B \sqcap D \sqsubseteq \bot$ $\quad$ $\exists r.X_1 \sqsubseteq X_0$

$A \sqcap B \sqsubseteq X_1$ $\quad$ $X_2 \sqsubseteq B$ $\quad$ $X_2 \sqsubseteq \exists s.B$ $\quad$ $C \sqcap X_3 \sqsubseteq X_2$ $\quad$ $\exists s.\top \sqsubseteq X_3$

THE COMPLETION ALGORITHM STARTS WITH ALL THE CONCEPT NAMES SUBSUMED BY THEMSELF AND SUBSUMED BY TOP

$P \sqsubseteq P$ $\qquad$ $P \in N_C$; P REPRESENT $A, B, C, X_0, X_1 \ldots$

$P \sqsubseteq \top$

THEN WE APPLY OUR RULES, FIRST DERIVE ALL THE SIMPLE AXIOM

$X_0 \sqsubseteq A$ $\qquad$ $X_2 \sqsubseteq B$ $\qquad$ $X_0 \sqsubseteq \exists r.C$ $\qquad$ $X_2 \sqsubseteq \exists s.B$

SINCE $X_2 \sqsubseteq \exists s.B$ AND $B \sqsubseteq T$ THEN $X_2 \sqsubseteq \exists s.T$

SINCE $X_2 \sqsubseteq \exists s.T$ AND $\exists s.T \sqsubseteq X_3$ THEN $X_2 \sqsubseteq X_3$

↑ THIS IS THE COMPLETION FOR THE T-BOX ITSELF ↑

WE NEED TO ADD THE CONSEQUENCES IN NORMAL FORM

$X_0 \sqsubseteq A$ $\quad X_2 \sqsubseteq B$ $\quad X_0 \sqsubseteq \exists t.C$ $\quad X_2 \sqsubseteq \exists s.B$ $\quad X_2 \sqsubseteq \exists s.T$ $\quad X_2 \sqsubseteq X_3$

$\exists r.X_4 \sqsubseteq Y_1$ $\quad \exists s.D \sqsubseteq X_4$

WE ONLY ADDED THE RIGTH SIDE ONE, BECAUSE WE WANT TO CHECK IF THE LEFT PART WILL POP-UP

$$\left.\begin{array}{l} Y_1 \sqsubseteq A \\ \exists r.X_6 \sqsubseteq X_5 \\ \exists s.B \sqsubseteq X_6 \end{array}\right\}$$

THIS IS WHAT WE HOPE TO FIND

SINCE $X_2 \sqsubseteq \exists s.B$ AND $\exists s.B \sqsubseteq X_6$ THEN $X_2 \sqsubseteq X_6$

...

2. Construct eventual **countermodels**

$$\exists r.\exists s.D \sqsubseteq A \sqcap \exists r.\exists s.B$$

BUILD AND CHECK IF YOU CAN REACH BY USING THE RULE BEFORE

$$\begin{array}{c} \\ \downarrow r \\ \downarrow s \\ D \end{array} \qquad \qquad \begin{array}{c} A \\ \downarrow r \\ \downarrow s \\ B \end{array}$$

Exercise 8. Completeness (***)

In the lecture we showed how to build an interpretation $\mathcal{I}$ from the data structure resulting from the completion algorithm, and **claimed it was a model**. As a step towards proving this claim, show that if the TBox contains the GCI $X \sqsubseteq Y$, then the interpretation $\mathcal{I}$ satisfies this GCI.

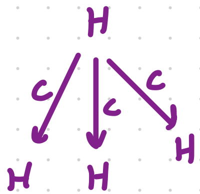
BORING

Exercise 9. Inverse Roles (***)

Using **inverse roles** build a TBox that expresses the knowledge that *humans can only have human children.*

$\exists r^- A$ $\quad$ $A \xrightarrow{r} \exists r^-.A$

WE ARE TRYING TO SAY THAT IF I HAVE A HUMAN AND THERE ARE SEVERAL CHILDREN THEY ARE ALL NECESSARLY HUMAN



$\text{HUMAN} \xrightarrow{\text{HASCHILD}} \text{CHILDREN OF HUMAN}$

$\exists \text{HASCHILD}^-.\text{HUMAN}$

$\exists \text{HASCHILD}^- \text{HUMAN} \sqsubseteq \text{HUMAN}$