Exercises - Calculus

Academic Year 2021-2022

Sheet 8

1. Compute, if it exists, the following limit

   (a) $\lim_{x\to 3} \frac{1-x^2}{x-1}$

   (b) $\lim_{x\to -\infty} \frac{1-x^2}{x-1}$

   (c) $\lim_{x\to 1^+} \frac{1-x^2}{x-1}$

   (d) $\lim_{x\to 1^-} \frac{1-x^2}{x-1}$

   (e) $\lim_{x\to 1^+} \frac{1+x}{1-x}$

   (f) $\lim_{x\to 1} \frac{x^2+x+1}{x^2-1}$

   (g) $\lim_{x\to +\infty} \frac{x^2+x+1}{x^2-1}$

   (h) $\lim_{x\to 1} \frac{x^5-1}{x^4-1}$

   (i) $\lim_{x\to +\infty} \frac{x^5-1}{x^4-1}$

   (j) $\lim_{x\to -\infty} \frac{x^5-1}{x^4-1}$

2. Establish, with respect to the parameters a, b, $c \in \mathbb{R}$ if the following functions are continuous in the given points x_0.

   (a) $f(x) = \begin{cases} ax+b & \text{if } x \geq 1 \\ 2bx+a & \text{if } x < 1 \end{cases}$ in $x_0 = 0$ and in $x_0 = 1$

   (b) $f(x) = \begin{cases} ax+b & \text{if } x \geq 1 \\ bx+a & \text{if } x < 1 \end{cases}$ in $x_0 = 0$ and in $x_0 = 1$

   (c) $f(x) = \begin{cases} ax+b & \text{if } x \geq 2 \\ bx+a & \text{if } x < 2 \end{cases}$ in $x_0 = 2$ and in $x_0 = 1$

   (d) $f(x) = \begin{cases} \frac{a}{x}+b & \text{if } x \geq 2 \\ x^2+bx & \text{if } x < 2 \end{cases}$ in $x_0 = 2$, in $x_0 = 1$ and in $x_0 = 0$

   (e) $f(x) = \begin{cases} \frac{ax^2+bx+c}{x^2} & \text{if } x > 0 \\ x^2+5 & \text{if } x \leq 0 \end{cases}$ in $x_0 = 0$ and in $x_0 = 1$

3. Establish whether the following identities are correct or not. When they are not correct, if possible, change the right hand side to make them correct.

(a) $\lim_{x\to 2} \dfrac{x^2+x-6}{x-2} = 5$

(b) $\lim_{x\to 4} \log_2(x) = 2$

(c) $\lim_{x\to 0} \dfrac{9^x-1}{3^x-1} = 3$

(d) $\lim_{x\to 0^+} \log(x^3) = -\infty$

(e) $\lim_{x\to 0^-} \dfrac{1}{2e^x-2} = -\infty$

(f) $\lim_{x\to e^-} \dfrac{-3}{\log(x)-1} = +\infty$

(g) $\lim_{x\to -\infty} e^{\frac{x-2}{x+5}} = e$

4. Establish whether the following limits exist or not

$$\lim_{x\to +\infty} \frac{\sin(x)}{x^2} \quad \text{and} \quad \lim_{x\to -\infty} x\cos(x)$$

5. Compute, if it exists, the following limit

(a) $\lim_{x\to 1} \left(\dfrac{1}{\log(x)} - \dfrac{1}{\log(x^2)}\right)$

(b) $\lim_{x\to -2^+} \dfrac{|4-x^2|}{x+2}$ and $\lim_{x\to -2^-} \dfrac{|4-x^2|}{x+2}$

(c) $\lim_{x\to 0^+} (1+x^3)^{1/x^3}$

(d) $\lim_{x\to +\infty} \left(\dfrac{4x+5}{6x+1}\right)^{\frac{1-x^2}{3x+2}}$ and $\lim_{x\to -\infty} \left(\dfrac{4x+5}{6x+1}\right)^{\frac{1-x}{3x+2}}$

(e) $\lim_{x\to -\infty} \left(\dfrac{2x+3}{2x}\right)^{1-x}$

(f) $\lim_{x\to -\infty} e^{\frac{x^3-2}{x^3+5x^2}}$

(g) $\lim_{x\to +\infty} \dfrac{\log(\sqrt{x}+1)}{x}$ and $\lim_{x\to 0^+} \dfrac{\log(\sqrt{x}+1)}{x}$

(h) $\lim_{x\to +\infty} \left(\dfrac{x+2}{x+1}\right)^x$

(i) $\lim_{x\to 1} \dfrac{e^x-e}{3x-3}$ and $\lim_{x\to 1} \dfrac{e^x-e}{\sqrt{2-x}-1}$

(j) $\lim_{x\to +\infty} x^{1/\sqrt{x}}$

(k) $\lim_{x\to 0} \dfrac{\sin(5x)}{\sin(3x)}$

(l) $\lim_{x\to 0} \dfrac{1-\cos(x)}{\sin^2(x)}$ and $\lim_{x\to 0} \dfrac{e^{2-2\cos(x)}-1}{\sin^2(x)}$

(m) $\lim_{x\to 0} \dfrac{2^{3x}-1}{x}$

(n) $\lim_{x\to 0} \dfrac{\log(x+x^2)}{\log(x)}$

(o) $\lim_{x\to -\infty} (2-x)\sin(1/x)$

(p) $\lim_{x\to 4} \left(\dfrac{1}{2}\right)^{\frac{1}{|x-4|}}$ and $\lim_{x\to \frac{\pi}{6}^+} \left(\dfrac{1}{3}\right)^{\frac{1}{1-2\sin(x)}}$

6. Establish, with respect to the parameters $a \in \mathbb{R}$, if the following functions are continuous on their domain of definition

(a) $f(x) = \begin{cases} \frac{\log(1+2x)}{x} & \text{if } -\frac{1}{2} < x < 0 \\ a^2 - 2 & \text{if } x = 0 \\ a\frac{\sin(x)}{x} & \text{if } x > 0 \end{cases}$

(b) $f(x) = \begin{cases} \frac{\log(1+3x)}{x} & \text{if } -\frac{1}{2} < x < 0 \\ a^2 - 2 & \text{if } x = 0 \\ a\frac{\sin(x)}{x} & \text{if } x > 0 \end{cases}$

(c) $f(x) = \begin{cases} \frac{1-\cos(2x)}{x^2} & \text{if } x < 0 \\ \sqrt{x^2 + |a|} & \text{if } x \geq 0 \end{cases}$

1) $\frac{-8}{2} = -4$

2) $\dfrac{x\left(\frac{1}{x}-x\right)}{x\left(1-\frac{1}{x}\right)} = +\infty$

c) $\dfrac{1-x^2}{x-1} \cdot \dfrac{x+1}{(x+1)}$

$\dfrac{-\cancel{(1+x^2)}(x+1)}{\cancel{x^2-1}} = -x-1$

$-1^+-1 = 2^+$

1. Compute, if it exists, the following limi

(a) $\lim_{x \to 3} \dfrac{1-x^2}{x-1}$

(b) $\lim_{x \to -\infty} \dfrac{1-x^2}{x-1}$

(c) $\lim_{x \to 1^+} \dfrac{1-x^2}{x-1}$

(d) $\lim_{x \to 1^-} \dfrac{1-x^2}{x-1}$

(e) $\lim_{x \to 1^+} \dfrac{1+x}{1-x}$

(f) $\lim_{x \to 1} \dfrac{x^2+x+1}{x^2-1}$

(g) $\lim_{x \to +\infty} \dfrac{x^2+x+1}{x^2-1}$

(h) $\lim_{x \to 1} \dfrac{x^5-1}{x^4-1}$

(i) $\lim_{x \to +\infty} \dfrac{x^5-1}{x^4-1}$

(j) $\lim_{x \to -\infty} \dfrac{x^5-1}{x^4-1}$

$x \to 1^-$

D) $\lim_{x \to 1^-} \dfrac{1-x^2}{x-1} \cdot \dfrac{(x+1)}{(x+1)} = \dfrac{\cancel{(1-x)}(1+x)}{-\cancel{(1-x)}} = -1-x = 0^+$

(e) $\lim_{x\to 1^+} \frac{1+x}{1-x}$

(f) $\lim_{x\to 1} \frac{x^2+x+1}{x^2-1}$

(g) $\lim_{x\to +\infty} \frac{x^2+x+1}{x^2-1}$

(h) $\lim_{x\to 1} \frac{x^5-1}{x^4-1}$

(i) $\lim_{x\to +\infty} \frac{x^5-1}{x^4-1}$

(j) $\lim_{x\to -\infty} \frac{x^5-1}{x^4-1}$

e) $\lim_{x\to 1^+} \frac{1+x}{1-x} = \frac{1+x}{1-x} \cdot \frac{(1+x)}{(1+x)}$

$$\frac{1+x^2+2x}{1-x^2} = \frac{\cancel{x^2}(1/x^2+1+2/x)}{\cancel{x^2}(1/x^2-1)} = +\infty$$

f) $\lim_{x\to 1} \frac{x^2+x+1}{x^2-1} = \frac{\cancel{x^2}(1+1/x+1/x^2)}{\cancel{x^2}(1-1/x^2)} = +\infty$

g) $\lim_{x\to +\infty} \frac{x^2+x+1}{x^2-1} = \frac{\cancel{x^2}(1+1/x+1/x^2)}{\cancel{x^2}(1-1/x^2)} = 1$

h) $\lim_{x\to 1} \frac{x^5-1}{x^4-1} = \frac{\cancel{x^5}(1-1/x^5)}{\cancel{x^5}(1/x-1/x^5)} =$

$$\frac{x^5-1}{(x^4-1)}\frac{(x^4+1)}{(x^4+1)} = \frac{x^9+x^5-x^4-1}{x^8-1} =$$

$$\lim_{x\to 1} \frac{\cancel{x^9}(1+1/x^4-\frac{1}{x^5}-1/x^9)}{\cancel{x^9}(1/x-1/x^9)} \quad \frac{1+1-1-1}{1-1}$$

$$\boxed{\frac{5}{4}}$$

(i) $\lim_{x\to+\infty} \frac{x^5-1}{x^4-1}$

(j) $\lim_{x\to-\infty} \frac{x^5-1}{x^4-1}$

i) $\lim_{x\to+\infty} \frac{x^5-1}{x^4-1}$ $\quad \frac{\cancel{x^5}\left(1-\frac{1}{x^5}\right)}{\cancel{x^4}\left(1-1/x^4\right)} = +\infty$

j) $\lim_{x\to-\infty} \frac{x^5-1}{x^4-1}$ $\quad \frac{\cancel{x^5}\left(1-1/x^5\right)}{\cancel{x^5}\left(\frac{1}{x}-1/x^3\right)}$ $\quad \frac{1}{0} = -\infty$

2. Establish, with respect to the parameters a, b, $c \in \mathbb{R}$ if the following functions are continuous in the given points x_0.

(a) $f(x) = \begin{cases} ax + b & \text{if } x \geq 1 \\ 2bx + a & \text{if } x < 1 \end{cases}$ in $x_0 = 0$ and in $x_0 = 1$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1)$$

$$\lim_{x \to 1^+} ax+b = A+B$$

$$f(1) \to A+B$$

$$\lim_{x \to 1^-} 2bx+A = 2b+A$$

$$2B+\cancel{A} = \cancel{A}+B$$

$$\boxed{B=0}$$

b)

(b) $f(x) = \begin{cases} ax + b & \text{if } x \geq 1 \\ bx + a & \text{if } x < 1 \end{cases}$ in $x_0 = 0$ and in $x_0 = 1$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

$$A+B \qquad A+B \qquad A+B$$

CONTINUA

c)

(c) $f(x) = \begin{cases} ax + b & \text{if } x \geq 2 \\ bx + a & \text{if } x < 2 \end{cases}$ in $x_0 = 2$ and in $x_0 = 1$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$$

$$2B+A \qquad 2A+B \qquad 2A+B$$

$$2B+A = 2A+B; \quad 3A=3B; \quad A=B$$

D)

(d) $f(x) = \begin{cases} \frac{a}{x} + b & \text{if } x \geq 2 \\ x^2 + bx & \text{if } x < 2 \end{cases}$ in $x_0 = 2$, in $x_0 = 1$ and in $x_0 = 0$

$$\lim_{x \to 2^-} x^2 + Bx = \lim_{x \to 2^+} \frac{A}{x} + B = f(2)$$

$$\lim_{x \to 2^-} x^2 + Bx \to 4 + 2B \qquad \lim_{x \to 2^+} \frac{A}{x} + B \to \frac{A}{2} + B \qquad f(2) \to \frac{A}{2} + B$$

$$4 + 2B = \frac{A}{2} + B$$

$$2B + 8 = A$$

(e) $f(x) = \begin{cases} \frac{ax^2+bx+c}{x^2} & \text{if } x > 0 \\ x^2 + 5 & \text{if } x \leq 0 \end{cases}$ in $x_0 = 0$ and in $x_0 = 1$

$$\lim_{x \to 0^-} x^2 + 5 = \lim_{x \to 0^+} \frac{Ax^2 + Bx + C}{x^2} = f(0)$$

$$\lim_{x \to 0^-} x^2 + 5 \to 5 \qquad f(0) \to 5$$

$$\frac{\cancel{x^2}(A + B/x + C/x^2)}{\cancel{x^2}} =$$

3. Establish whether the following identities are correct or not. When they are not correct, if possible, change the right hand side to make them correct.

(a) $\lim_{x\to 2} \frac{x^2+x-6}{x-2} = 5$

(b) $\lim_{x\to 4} \log_2(x) = 2$

(c) $\lim_{x\to 0} \frac{9^x-1}{3^x-1} = 3$

(d) $\lim_{x\to 0^+} \log(x^3) = -\infty$

(e) $\lim_{x\to 0^-} \frac{1}{2e^x-2} = -\infty$

(f) $\lim_{x\to e^-} \frac{-3}{\log(x)-1} = +\infty$

(g) $\lim_{x\to -\infty} e^{\frac{x-2}{x+5}} = e$

$\frac{(x-2)(x+3)}{x-2} = x+3 \quad 5$

b) $\lim_{x\to 4} \log_2(x) = 2$ ✓

c) $\lim_{x\to 0} \frac{9^x-1}{3^x-1} = \frac{\cancel{(3^x-1)}(3^x+1)}{\cancel{3^x-1}} = 2$

D) $\lim_{x\to 0^+} \log(x^3) = -\infty$

e) $\lim_{x\to 0^-} \frac{1}{2(e^x-1)} = -\infty$

f) $\lim_{x\to e^-} \frac{-3}{\log(x)-1} = \frac{-3}{0} = +\infty$

g) $\lim_{x\to -\infty} e^{\frac{x-2}{x+5}} = e$

$\frac{x-2}{x+5} \quad \frac{\cancel{x}(1-2/x)}{\cancel{x}(1+5/x)} = 1$

4. Establish whether the following limits exist or not

$$\lim_{x\to+\infty} \frac{\sin(x)}{x^2} \quad \text{and} \quad \lim_{x\to-\infty} x\cos(x)$$

$\to 0$ NOT DEFINE

5. Compute, if it exists, the following limit

(a) $\lim_{x\to 1}\left(\frac{1}{\log(x)} - \frac{1}{\log(x^2)}\right)$

(b) $\lim_{x\to -2^+}\frac{|4-x^2|}{x+2}$ and $\lim_{x\to -2^-}\frac{|4-x^2|}{x+2}$

(c) $\lim_{x\to 0^+}(1+x^3)^{1/x^3}$

(d) $\lim_{x\to +\infty}\left(\frac{4x+5}{6x+1}\right)^{\frac{1-x^2}{3x+2}}$ and $\lim_{x\to -\infty}\left(\frac{4x+5}{6x+1}\right)^{\frac{1-x^2}{3x+2}}$

(e) $\lim_{x\to -\infty}\left(\frac{2x+3}{2x}\right)^{1-x}$

(f) $\lim_{x\to -\infty} e^{\frac{x^3-2}{x^3+5x^2}}$

(g) $\lim_{x\to +\infty}\frac{\log(\sqrt{x}+1)}{x}$ and $\lim_{x\to 0^+}\frac{\log(\sqrt{x}+1)}{x}$

(h) $\lim_{x\to +\infty}\left(\frac{x+2}{x+1}\right)^{x}$

(i) $\lim_{x\to 1}\frac{e^x-e}{3x-3}$ and $\lim_{x\to 1}\frac{e^x-e}{\sqrt{2-x}-1}$

(j) $\lim_{x\to +\infty} x^{1/\sqrt{x}}$

(k) $\lim_{x\to 0}\frac{\sin(5x)}{\sin(3x)}$

(l) $\lim_{x\to 0}\frac{1-\cos(x)}{\sin^2(x)}$ and $\lim_{x\to 0}\frac{e^{2-2\cos(x)}-1}{\sin^2(x)}$

(m) $\lim_{x\to 0}\frac{2^{3x}-1}{x}$

a) $\lim_{x\to 1}\left(\cos^{-1}(x) - 2\cos^{-1}(x)\right)$

$-\cos^{-1}(x) = -\frac{1}{\cos(x)} = -\infty$

b) $\lim_{x\to -2^+}\frac{|4-x^2|}{x+2}$ $\quad \frac{|(2-x)(2+x)|}{2+x}$

$\downarrow$

$4-x^2>0$

$+ \quad - \quad +$

$-2 \quad\quad 2$

i) $\lim_{x\to 1}\frac{e^x-e}{3x-3}$ $\quad 3x-3=t$

$t=3x-3$

$\lim_{t\to 0}\frac{e^{t/3+1}-e}{t} = \frac{e^{t/3}\cdot e-e}{t}$

$= \frac{e(e^{t/3}-1)}{t}\cdot\frac{1/3}{\frac{1}{3}}$ $\quad \lim_{x\to 0}\frac{a^x-1}{x} = \ln a$

$= \frac{1}{3}\, e\, \frac{(e^{t/3}-1)}{\frac{t}{3}}$ $\quad \ln e = 1$

$\frac{e}{3}$

1) $\lim_{x \to 1} \dfrac{e^x - e}{\sqrt{2-x} - 1}$

$$\frac{e^x - e}{2-x-1} \sqrt{2-x} + 1$$

$$\frac{(e^x - e)(\sqrt{2-x} + 1)}{1-x}$$

$$= \frac{e^{2-t^2} - e}{t-1}$$

$t = \sqrt{2-x}$ $\quad x = 2-t^2$

$x \to 1$

$t \to 1$

$$\frac{e^2 \cdot e^{-t^2} - e}{t-1}$$

$$\frac{\cancel{e^2}\left(e^{-t^2} - \frac{1}{e}\right)}{\cancel{e^2}\left(\frac{t}{e^2} - \frac{1}{e^2}\right)} =$$

$$\lim_{x\to 1} \frac{e^x - e}{\sqrt{2-x} - 1}$$

$t = 2-x$

$$\lim_{t\to 1} \frac{e^{2-t} - e}{\sqrt{t} - 1} = \frac{e^2 \cdot e^{-t} - e}{(\sqrt{t}-1)(\sqrt{t}+1)}(\sqrt{t}+1) = \frac{e(e^{1-t}-1)(\sqrt{t}+1)}{t-1}$$

$$= - \frac{e\boxed{(e^{1-t}-1)}}{\boxed{1-t}}(\sqrt{t}+1)$$

$v = 1-t$

$\ln e = 1$

$$\lim_{v\to 0} = \frac{e(e^v - 1)(\sqrt{1-v}+1)}{v}$$

$$\lim_{v\to 0} -e(\sqrt{1-v}+1) = -2e$$

$1-v \to 1-t \to 2-x$

È LA SOLUZIONE GIUSTA! :)

(PHOTOMATH)

$1-2+x$

$$\lim_{x\to 1} -e(\sqrt{1-1-2+x}+1)$$

$\Downarrow$

$$\lim_{x\to 1} -e(\sqrt{-2+x}+1) =$$

???

MA SE RISOSTITUISCO NON VIENE PIÙ :(

(d) $\lim_{x\to+\infty}\left(\frac{4x+5}{6x+1}\right)^{\frac{1-x^2}{3x+2}}$ and $\lim_{x\to-\infty}\left(\frac{4x+5}{6x+1}\right)^{\frac{1-x}{3x+2}}$

$$\lim_{x\to+\infty}\left(\frac{4x+5}{6x+1}\right)^{\frac{1-x^2}{3x+2}}$$

$$e^{\log\left(\frac{4x+5}{6x+1}\right)\frac{1-x^2}{3x+2}}$$

$$\log\left(\frac{4x+5}{6x+1}\right)\frac{1-x^2}{3x+2}$$

$$\downarrow$$

$$\frac{\cancel{x}(4+5/x)}{\cancel{x}(6+1/x)}\quad\frac{x^{\cancel{2}}(1/x^2-1)}{\cancel{x}(3+2/x)}$$

$$\log\left(\frac{4}{6}\right)\frac{-x}{3}$$

$$e^{+\infty}=+\infty$$

$$\lim_{x\to-\infty} e^{\log\left(\frac{4x+5}{6x+1}\right)\frac{1-x}{3x+2}}$$

$$\log\left(\frac{\cancel{x}(4+5/x)}{\cancel{x}(6+1/x)}\right)\frac{\cancel{x}(1/x-1)}{\cancel{x}(3+2/x)}$$

$$\log\left(\frac{4}{6}\right)\frac{-1}{3}$$

$$e^{-1/3\log\left(\frac{4}{6}\right)}$$

$$\sqrt[3]{\frac{1}{e}}^{\log\left(\frac{4}{6}\right)}$$

(e) $\lim_{x \to -\infty} \left(\frac{2x+3}{2x} \right)^{1-x}$

$\lim_{x \to -\infty}$ $e^{\log\left(\frac{2x+3}{2x}\right)(1-x)}$

$\log\left(\frac{\cancel{x}(2+3/x)}{2\cancel{x}}\right)(1-x)$

$\log(1)\,(1-x)$

$\downarrow$

0