PROOF OF CONVERGENT SEQUENCES

GIVEN THAT $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ AND $\{b_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ AND LET $\{a_n\} \to a$ AND $\{b_n\} \to b$ PROVE $a, b \in \mathbb{R}$

(IF a_n CONVERGES TO a AND b_n CONVERGES TO b)

$$\{a_n + b_n\} \to a + b$$

THE SUM OF CONVERGENT SEQUENCES CONVERGE TO THE SUM OF THEIR LIMITS

WE WANT TO PROVE THAT

$$|(a_n + b_n) - (a + b)| < \varepsilon \qquad \forall \varepsilon > 0$$

THAT IS

$$|(a_n - a) + (b_n - b)|$$

SINCE $a_n \to a$ $\quad |a_n - a| < \varepsilon \quad \forall \varepsilon > 0$

SINCE $b_n \to b$ $\quad |b_n - b| < \varepsilon \quad \forall \varepsilon > 0$

USING THE TRIANGLE INEQUALITY. BY THAT THE ABSOLUTE VALUE OF THE SUM OF TWO TERMS IS LESS OR EQUAL TO THE SUM OF THE ABSOLUTE VALUE OF THOSE TERMS

$$|x + y| \leq |x| + |y|$$

THAT IS

$$|(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|$$

LET $\varepsilon > 0$ AND SINCE $|a_n - a|$ AND $|b_n - b|$ ARE BOTH ARBITRARY SMALL

THIS IS WHAT WE WANT TO PROVE

$$|(a_n - a) + (b_n - b)| \le |a_n - a| + |b_n - b| < \varepsilon$$

[THAT IS $a_n - a < \frac{\varepsilon}{2}$ AND $b_n - b < \frac{\varepsilon}{2}$]

BY DEFINITION OF A CONVERGING SEQUENCE:

LET $\varepsilon > 0$

$\exists N_1 \in \mathbb{N}$ FOR ALL $n > N_1$

$$|a_n - a| < \frac{\varepsilon}{2}$$

$\exists N_2 \in \mathbb{N}$ FOR ALL $n > N_2$

$$|b_n - b| < \frac{\varepsilon}{2}$$

$$N = \max\{N_1, N_2\}$$

FOR ALL $n > N$ GIVEN

$$|(a_n - a) + (b_n - b)| \le |a_n - a| + |b_n - b|$$

SINCE $n > N$

$$|a_n - a| < \frac{\varepsilon}{2} \quad \text{AND} \quad |b_n - b| < \frac{\varepsilon}{2}$$

THAT IS

$$|(a_n - a) + (b_n - b) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

SO FOR AN ARBITRARY POSITIVE ε WE FOUND A NUMBER N SO THAT EVERY TERM OF THE SEQUENCE AFTER N IS WITHIN ε OF $a+b$. THUS A SEQUENCE OF TERMS COMPOSED OF THE SUM OF THE TERMS OF TWO CONVERGING SEQUENCES CONVERGES TO THE SUM OF THE LIMIT OF THOSE CONVERGING SEQUENCES