$+ \quad \times$

$- \quad \div$

DIFFERENTIAL CALCULUS

DEFINITION

$f : C \subseteq \mathbb{R}^N \to \mathbb{R}$. LET $x^0 \in C$

- WE SAY THAT x^0 IS AN ABSOLUTE MINIMIZER OR ABSOLUTE MINIMUM POINT FOR f ON C IF

$$f(x^0) \leq f(x) \quad \forall x \in C$$

THAT IS $f(x^0) = \min_{x \in C} f(x) = \min f(C)$ MINIMUM OR ABSOLUTE MINIMUM OF f ON C

WE SAY THAT x^0 IS AN ABSOLUTE MAXIMISER OR ABSOLUTE MAXIMUM POINT FOR f ON C IF

$$f(x^0) \geq f(x) \quad \forall x \in C$$

THAT IS $f(x^0) = \max_{x \in C} f(x) = \max f(C)$ MAXIMUM OR ABSOLUTE MAXIMUM OF f ON C

REMARK

C CLOSED AND BOUNDED, f CONTINUOUS ON C

$\overset{\text{WEIERSTRASS}}{\Longrightarrow}$ $\exists \min_C f$, $\exists \max_C f$ SO

THERE EXIST AT LEAST ONE ABSOLUTE MINIMIZER

(x_{min}) AND ONE ABSOLUTE MAXIMIZER (x_{max})

DEFINITION: $f: C \subseteq \mathbb{R}^n \to \mathbb{R}$. LET $\boxed{x^0 \in \mathring{C}}$ ← INTERIOR OF C

THAT IS $\exists\, \delta_0 > 0$ SUCH THAT $B_{\delta_0}(x^0) \subseteq C$

- WE SAY THAT x^0 IS A LOCAL OR RELATIVE MINIMIZER (OR MINIMUM POINT) FOR f IF

  MAXIMISER, MAXIMUM POINT

$$\exists\, \delta > 0 \ (0 < \delta \leq \delta_0) \text{ SUCH THAT } f(x_0) \underset{\geq}{\leq} f(x) \ \forall x \in B_\delta(x^0)$$

- WE SAY THAT x^0 IS A STRICT LOCAL MINIMUM POINT FOR f IF $\exists\, \delta > 0 \ (0 < \delta \leq \delta_0)$ SUCH THAT

  MAXIMUM

$$f(x^0) \underset{>}{<} f(x) \quad \forall x \in B_\delta(x^0) \setminus \{x^0\}$$

NOTATION

LOCAL MINIMUM/MAXIMUM POINTS = LOCAL EXTREMUM POINTS

GLOBAL OR ABSOLUTE MINIMUM/MAXIMUM POINTS = ABSOLUTE EXTREMUM POINTS

PROPOSITION $f: C \subseteq \mathbb{R}^N \to \mathbb{R}$. LET $x^0 \in C$ BE AN ABSOLUTE MINIMIZER FOR f ON C.

IF $x^0 \in \mathring{C}$, THEN x^0 IS A LOCAL MINIMIZER FOR f

PROOF TAKE $\delta = \delta_0$

FERMAT THEOREM

$$f : C \subseteq \mathbb{R} \to \mathbb{R} \; ; \; \boxed{x_0 \in \mathring{C}}$$

IF x_0 IS A LOCAL EXTREMUM POINT FOR f AND

$\exists f'(x_0)$, THEN $f'(x_0) = 0$

REMARK

THIS IS A NECESSARY CONDITION BUT IT IS NOT SUFFICENT

$f(x) = x^3 \quad x_0 = 0 \quad f'(0) = 0$ BUT 0 IS

NOT A LOCAL EXTRIMUM POINT!

PROOF ASSUME x_0 IS A LOCAL MINIMUM POINT, THAT

IS $\exists \delta > 0$ SUCH THAT $(x_0 - \delta, x_0 + \delta) \subseteq C$ AND

$f(x_0) \leq f(x) \quad \forall x \in (x_0 - \delta, x_0 + \delta)$

LET x BE SUCH THAT $x_0 < x < x_0 + \delta$

$$\frac{f(x) - f(x_0)}{\underbrace{x - x_0}_{>0}} \geq 0 \quad (\geq 0)$$

LET x BE SUCH THAT $x_0 - \delta < x < x_0$

$$\frac{f(x) - f(x_0)}{\underbrace{x - x_0}_{<0}} \leq 0 \quad (\geq 0)$$

$\exists\ f'(x_0) \Rightarrow \exists f'_+(x_0)\ ,\ \exists f'_-(x_0)$ AND

$$f'(x) = f'_+(x_0) = f'_-(x_0)$$

BUT

$$f'_+(x_0) = \lim_{x \to x_0^+} \underbrace{\frac{f(x) - f(x_0)}{x - x_0}}_{\geq 0} \geq 0$$

PERMANENCE OF SIGN

$$f'_-(x_0) = \lim_{x \to x_0^-} \underbrace{\frac{f(x) - f(x_0)}{x - x_0}}_{\leq 0} \leq 0$$

$\Rightarrow f'(x_0) \geq 0$ AND $f'(x_0) \leq 0 \Rightarrow f'(x_0) = 0$

DEFINITION $f: C \subseteq \mathbb{R} \to \mathbb{R}$ LET $x_0 \in \overset{\circ}{C}$

WE SAY THAT x_0 IS A CRITICAL POINT FOR f IF

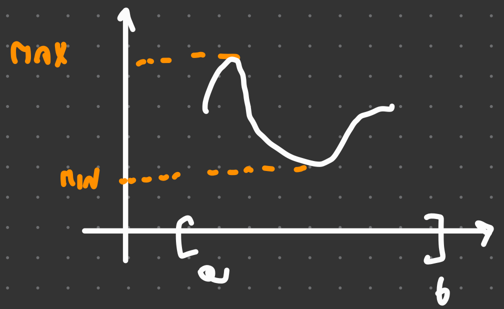
$$\nexists f'(x_0) \text{ OR } \underbrace{\exists f'(x_0) \text{ AND } f'(x_0) = 0}_{\text{STATIONARY POINT}}$$

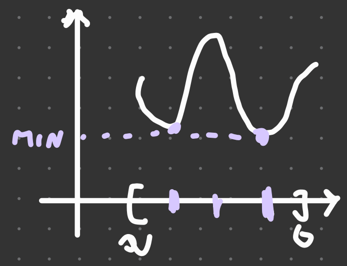
PROPOSITION

$f: C \subseteq \mathbb{R} \to \mathbb{R}$. LET $x_0 \in \overset{\circ}{C}$

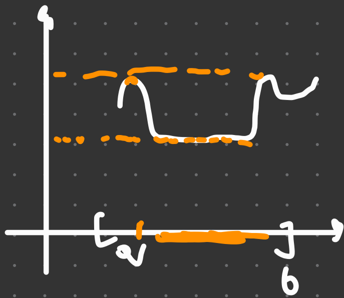
- x_0 LOCAL EXTREMUM POINT $\Rightarrow$ x_0 CRITICAL POINT
- x_0 GLOBAL EXTREMUM POINT $\Rightarrow$ x_0 LOCAL EXTREMUM POINT
  $\Rightarrow$ x_0 CRITICAL POINT

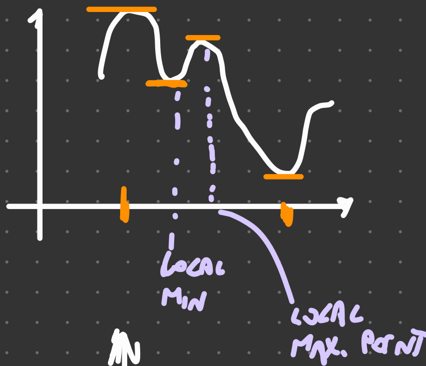
EXAMPLE $C = [a, b]$ $a, b \in \mathbb{R}$ $a < b$

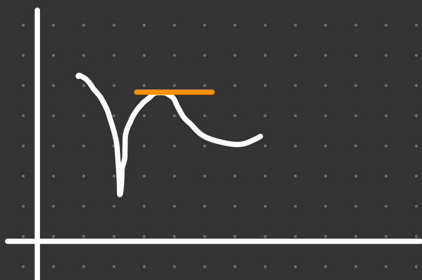


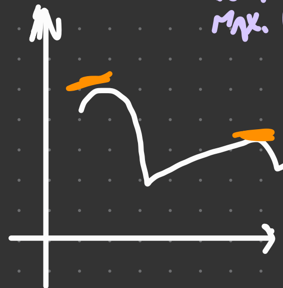


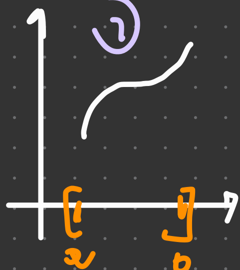
— TANGENT LINE

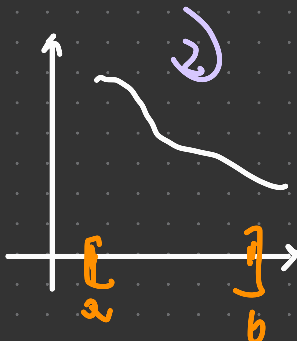


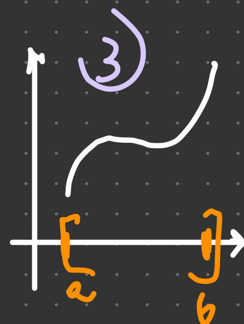












a/ GLOBAL MINIMUM POINT $\Rightarrow$

EITHER $\nexists f'_{+}(a)$ 3 OR $\exists f'_{+}(a)$ AND $f'_{+}(a) \geq 0$ 1

b GLOBAL MAX. POINT $\Rightarrow$

EITHER $\nexists f'_{-}(b)$ 3 OR $\exists f'_{-}(b)$ AND $f'_{-}(b) \geq 0$ 1

MAXIMUM AND MINIMUM PROBLEMS

PROBLEM $f: C \subseteq \mathbb{R}^n \to \mathbb{R}$ DETERMINE, IF IT EXISTS, $\min_C f$

IF THE MINIMUM EXISTS, DETERMINE ALL (OR AT LEAST ONE) ABSOLUTE MINIMUM POINTS

HOW TO SOLVE THIS PROBLEM

1) DETERMINE EXISTENCE OF THE MINIMUM

(USUALLY VIA WEIERSTRASS THEOREM)

2) ONLY IF STEP 1) IS OK, FIND ALL POSSIBLE CANDIDATES FOR ABSOLUTE MINIMIZER

(NECESSARY CONDITION)

2a) CANDIDATES AT THE BOUNDARY

BOUNDARY POINTS $x \in \partial C$

2b) INTERIOR CANDIDATES

$x \in \mathring{C}$ WHICH ARE CRITICAL POINTS

3) COMPARE THE VALUE OF f FOR THE CANDIDATES

THE LOWEST VALUE IS THE MINIMUM AND ALL CANDIDATES POINTS REACHING ARE THE GLOBAL MINIMIZER

EXAMPLE

$$C=[a,b] \quad a,b \in \mathbb{R} \quad a<b$$

2a) $\partial C = \{a,b\}$ ENDPOINTS. CONSIDER BOTH AS CANDIDATES $\boxed{a, b}$

2b) CANDIDATE ARE $x \in (a,b)$ SUCH THAT $\nexists f'(x)$ OR $\exists f'(x)$ AND $\boxed{f'(x)=0}$

EXERCISES

DETERMINE, IF IT EXISTS, $\min_C f$ WHERE

$$f(x) = x^3 - 6x^2 + 9x - 1 \qquad C = [0,2]$$

1) $\exists \min_C f$? $\left.\begin{array}{l} C \text{ CLOSED, BOUNDED} \\ f \text{ CONT. ON } C \end{array}\right\} \Rightarrow$

$\exists \min_C f$ AND $\exists \max_C f$

2) 2a) CANDIDATES AT THE BOUNDARY:
ENDPOINTS 0, 2

2b) INTERIOR CANDIDATES

- $\boxed{x \in (0,2)}$ SUCH THAT $\nexists f'(x)$ NONE (f IS DIFF. ON $\mathbb{R}$)

- $\boxed{x \in (0,2)}$ SUCH THAT $\exists f'(x)$ AND $f'(x) = 0$

$$f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3) \quad \forall x \in \mathbb{R}$$

SOLVE THE EQUATION

$$f'(x) = 0 \text{ THAT IS } 3(x-1)(x-3) = 0$$

$f'(x) = 0 \iff x = 1$ or $x = 3$

↳ out of $(0,2)$

CANDIDATE $\boxed{1}$

3) CANDIDATES $0, 1, 2$

$f(1) = 3$ $f(0) = -1$ $f(2) = 1$

MAX MIN

-1 MIN

0 ABSOLUTE MINIMUM POINT

3 MAXIMUM 1 ABSOLUTE MAX. POINT

— x ————

$$f(x) = \frac{2 - 2x}{x^2+3} \qquad C = [-3, 0]$$

f CONT $[-3, 0] \Rightarrow \exists \min_C f, \ \exists \max_C f$

2) 2a) $-3, 0$ CANDIDATES AT THE BOUNDARY

2b) INTERIOR CANDIDATES

• $x \in (-3,0)$ SUCH THAT $\nexists f'(x)$ NONE

(f IS DIFF. ON $\mathbb{R}$)

• $x \in (-3,0)$ SUCH THAT $\exists f'(x)$ AND $f'(x)=0$

$$f'(x) = \frac{2(x+1)(x-3)}{(x^2+3)^2} \quad \forall x \in \mathbb{R}$$

$f'(x) = 0 \iff x = -1$ OR $x = 3$

INTERIOR CANDIDATE $\boxed{-1}$

3) CANDIDATES $-3, 0, -1$

$f(-3) = \frac{2}{3}$ $\quad f(-1) = 1$ $\quad f(0) = \frac{2}{3}$

$\frac{2}{3}$ MINIMUM; $-3, 0$ ABSOC. MINIMIZER

1 MAX $\quad -1$ ABSOC. MAXIMIZER

3) EXAMPLE FIND THE CYLINDER OF MAXIMUM VOLUME WHOSE TOTAL SURFACE AREA IS $20\ cm^2$

$h, r > 0$ VOLUME $= \pi r^2 h$

AREA LATERAL SURFACE $= 2\pi r h$

AREA BASE $= \pi r^2$

$S =$ TOTAL SURFACE AREA $= 2\pi r h + 2\pi r^2$

$20 = S = 2\pi r^2 + 2\pi r h \quad r, h > 0$

$\max V = \max(\pi r^2 h) \quad r, h > 0$

$2\pi r^2 + 2\pi r h = 20 \Rightarrow h = \dfrac{20 - 2\pi r^2}{2\pi r} > 0$

$0 < r < \sqrt{\frac{10}{\pi}}$

$V(r) = \pi r^2 h = r(10 - \pi r^2) > 0 \quad 0 < r < \sqrt{\frac{10}{\pi}}$

$r = 0, \ r = \sqrt{\frac{10}{\pi}} \quad V(r) = 0$

$\max V(t) = \max V(t) = 0$

$\max V(t) = \max V(t) \quad C = \left(0, \sqrt{\frac{10}{\pi}}\right)$

$t \in \left(0, \sqrt{\frac{10}{\pi}}\right) \quad t \in \left(0, \sqrt{\frac{10}{\pi}}\right] \quad f(x) = x(10 - \pi x^2)$

DIFFERENTING CALCULUS THEOREM

ROLLE THEOREM $a, b \in \mathbb{R} \quad a < b$

$f : [a, b] \to \mathbb{R}$ SUCH THAT

- f CONTINOUS ON $[a, b]$
- f DIFFERENTIABLE ON (a, b)
- $f(a) = f(b)$

THEN $\exists x_0 \in (a, b)$ SUCH THAT $f'(x_0) = 0$

PROOF

WEIERSTRASS $\Rightarrow \exists x_{MIN} \in [a, b]$ ABS. MIN. POINT AND $\exists x_{max} \in [a, b]$ ABS. MAX. POINT

IF $X_{MIN} \in (a,b)$ THEN X_{MIN} IS A LOCAL MIN. POINT

AND BY HYPOTHES $\exists f'(X_{MIN}) \overset{FERMAT}{\Rightarrow} f'(X_{MIN}) = 0$

TAKE $x_0 = X_{MIN}$ AND THE PROOF IS FINISHED

- IF $X_{MAX} \in (a,b)$ SAME REASONING $\Rightarrow f'(X_{MAX}) = 0$

TAKE $X_0 = X_{MAX}$ AND THEN PROOF IS FINISHED

- IF $X_{MIN} \notin (a,b)$ AND $X_{MAX} \notin (a,b) \Rightarrow$

$X_{MIN}, X_{MAX} \in (a,b)$

BUT $f(a) = f(b) \Rightarrow f(X_{MIN}) = f(X_{MAX}) = c$

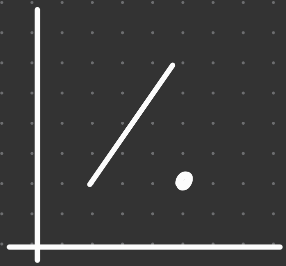
$\Rightarrow c \leq f(x) \leq c \quad \forall x \in [a,b] \Rightarrow$

f IS CONSTANT ON $[a,b]$!

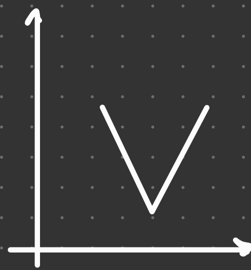
$\Rightarrow f'(x) = 0 \quad \forall x \in (a,b)$

REMARK

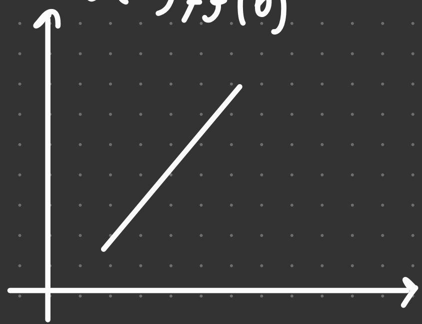
- f NOT CONT

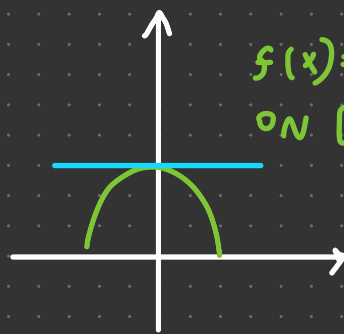


- f NOT DIFF IN (a,b)



- $f(a) \neq f(b)$





$f(x) = \sqrt{1-x^2}$
ON $[-1,1]$

LAGRANGE THEOREM

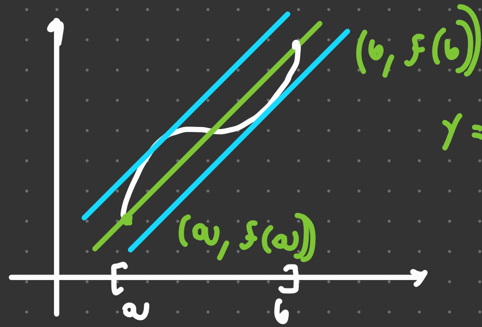
$a, b \in \mathbb{R}$ $a<b$; $f:[a,b] \to \mathbb{R}$ SUCH THAT

- f CONTINUOUS ON $[a,b]$
- f DIFFERENTIABLE ON (a,b)

THEN THERE EXISTS $x_0 \in (a,b)$ SUCH THAT

$$f'(x_0) = \frac{f(b)-f(a)}{b-a}$$

IDEA



$$y = h(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

PROOF

$$h(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

LET $F(x) = f(x) - h(x) \quad \forall x \in [a,b]$

- h CONST ON $\mathbb{R}$ ($\Rightarrow$ ALSO IN $[a,b]$)
- h CONST ON $\mathbb{R}$ ($\Rightarrow$ ALSO IN (a,b))
- $h'(x) = \dfrac{f(b)-f(a)}{b-a} \quad \forall x \in \mathbb{R}$
- $h(a) = f(a)$; $h(b) = f(b)$

SO F' SATISFIES THE HYPOTHESIS OF ROLLE, THAT IS,

- F CONST. ON $[a,b]$
- F DIFF. IN (a,b)

- $F(a) = f(a) - h(a) = 0 \quad F(b) = f(b) - h(b) = 0$

$\Rightarrow \exists x_0 \in (a,b)$ SUCH THAT $F'(x_0) = 0$

BUT $F'(x_0) = f'(x_0) - h'(x_0) = f'(x_0) - \dfrac{f(b)-f(a)}{b-a} = 0$

$$\Rightarrow f'(x_0) = \frac{f(b)-f(a)}{b-a}$$

CAUCY THEOREM

$a, b \in \mathbb{R}$ $a<b$: $f, g : [a,b] \to \mathbb{R}$

- f, g CONT ON $[a,b]$
- f, g DIFF ON (a,b)
- $g'(x) \neq 0$ $\forall x \in (a,b)$ $\left(\overset{\text{ROLLE}}{\Rightarrow} g(b)-g(a) \neq 0 \right)$

THEN THERE EXISTS $x_0 \in (a,b)$ SUCH THAT

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

NOTE:
- LAGRANGE IS CAUCHY WITH $g(x) = x$
- ROLLE IS LAGRANGE WITH $f(a) = f(b)$

CHARATERIZATION OF MONOTONE AND CONSTANT FUNCTIONS

THEOREM $I = (a,b)$ $-\infty \leq a < b \leq +\infty$ OPEN INTERVAL

$f: (a,b) \to \mathbb{R}$ DIFFER. ON (a,b). THEN

- f INCREASING ON $(a,b) \Leftrightarrow f'(x) \geq 0$ $\forall x \in (a,b)$.

• f DECREASING ON (a,b) $\Leftrightarrow$ $f'(x) \leq 0$ $\forall x \in (a,b)$

PROOF: JUST FOR THE INCREASING CASE

"$\Rightarrow$" EASY. LET $x_0 \in (a,b)$

IF f IS INCREASING, THEN $x_0 < x$ IMPLIES $f(x) \geq f(x_0)$ SO

$$\frac{f(x)-f(x_0)}{x-x_0} \geq 0 \Rightarrow \lim_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} \geq 0$$

PERM. OF SIGN

BUT

$$f'(x_0) = f'_+(x_0) = \lim_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} \geq 0$$

"$\Leftarrow$" SUPPOSE $f'(x) \geq 0$ $\forall x \in (a,b)$

LET $x_1, x_2 \in (a,b)$ SUCH THAT $x_1 < x_2$. WE WANT TO SHOW $f(x_1) \leq f(x_2)$, THAT IS, $f(x_2) - f(x_1) \geq 0$

APPLY LAGRANGE ON $[x_1, x_2] \subseteq (a,b) \Rightarrow$

$\exists x_0 \in (x_1, x_2)$ SUCH THAT

$$0 \leq f'(x_0) = \frac{f(x_2)-f(x_1)}{x_2-x_1} \Rightarrow f(x_2) - f(x_1) \geq 0$$

HYP

REMARK $f'(x) > 0 \ \forall x \Rightarrow f$ STRICTLY INCR.
$<$ DECREASE

VICEVERSA DOES NOT HOLD

$f(x) = x^3$ STRICTLY INCR. BUT $f'(0) = 0$, SO IT IS NOT TRUE THAT $f'(x) > 0 \ \forall x$

THEOREM

$I = (a, b) \quad -\infty \leq a < b \leq +\infty$ OPEN INTERVAL

$f : (a, b) \to \mathbb{R}$

f IS CONSTANT ON (a, b) $\Leftrightarrow$ f IS DIFFER. ON (a, b) AND $f'(x) = \ \forall x \in (a, b)$

PROOF

"$\Rightarrow$" EASY

"$\Leftarrow$" $\left.\begin{array}{l} f' \geq 0 \quad f \text{ IS INCREASING} \\ f' \leq 0 \quad f \text{ IS DECREASING} \end{array}\right\} \Rightarrow f$ IS CONSTANT

REMARK

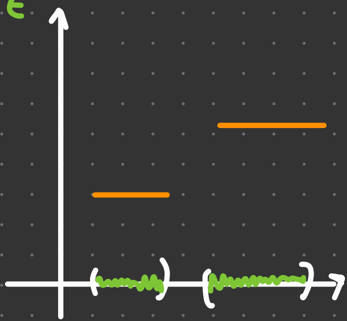
- ALL PREVIOUS RESULTS WORK FOR ANY NONDEGENERATE INTERVAL I

$f : I \to \mathbb{R}$ $\quad f$ CONT. ON I, f DIFF. ON $\mathring{I}$

• INTERVAL IS ESSENTIAL

EXAMPLE

•



• $f(x) = \frac{1}{x}$, $x \neq 0$ $\mathbb{R} \setminus \{0\}$

$f'(x) = -\frac{1}{x^2} < 0 \quad \forall x \in \mathbb{R} - \{0\}$

BUT f IS NOT DECREASING ON $\mathbb{R} - \{0\}$

APPLICATIONS

CHARACTERIZATION OF CRITICAL POINT

$f : (a,b) \to \mathbb{R}$ $\quad -\infty \leq a < b \leq +\infty$ $\quad f$ CONTINUOUS

LET $x_0 \in (a,b)$ BE A CRITICAL POINT FOR f

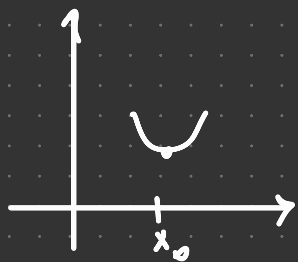
HOW TO ESTABLISH IF IT IS A LOCAL EXTREMUM POINT?

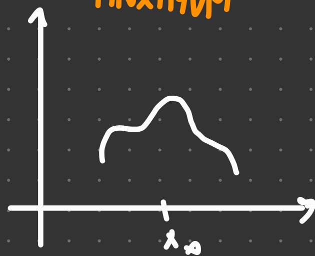
LET US ASSUME $\exists \delta_0 > 0$ SUCH THAT $(x_0 - \delta_0, x_0 + \delta_0) \subseteq (a,b)$

AND

- IF f IS (STRICTLY) DECREASING ON $(x_0-\delta_0, x_0]$
INCREASING
AND IS (STRICTLY) INCREASING ON $[x_0, x_0+\delta_0)$,
DECREASING

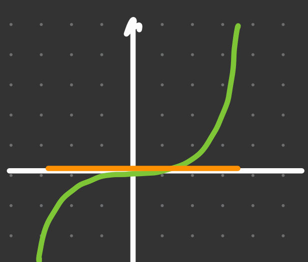
THEN x_0 IS A STRICT LOCAL MINIMUM POINT
MAXIMUM

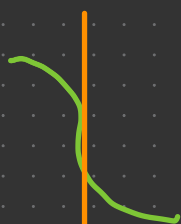


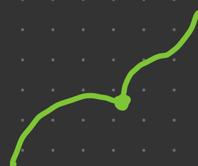


- IF f IS <u>STRICTLY</u> INCR. ON $(x_0-\delta_0, x_0]$ AND
DECR.
<u>STRICTLY</u> INCR. ON $[x_0, x_0+\delta_0)$, THEN
DECR.

x_0 IS <u>NOT</u> A LOCAL EXTREMUM POINT







APPLICATION

INVERTIBILITY OF TRIGONOMETRIC FUNCTIONS

- $\frac{d}{dx}\cos(x) = -\sin(x)$

  $-\sin(x) < 0$ ON $(0, \pi)$ $\Rightarrow$ COS IS STRICTLY DECREASING ON $[0, \pi]$

- $\frac{d}{dx}\sin(x) = \cos(x)$

  $\cos(x) > 0$ ON $(-\pi/2, \pi/2)$ $\Rightarrow$ SIN IS STRICTLY INC. ON $[-\pi/2, \pi/2]$

- $\frac{d}{dx}\tan(x) = 1 + (\tan^2(x)) > 0 \quad \forall x \in (-\pi/2, \pi/2) \Rightarrow$

  TAN IS STRICTLY INCR. ON $(-\pi/2, \pi/2)$

ASYNPTOTIES

- $f : (a, +\infty) \to \mathbb{R}$ WITH $-\infty \leq a < +\infty$

A LINE $y = mx + q$ $m, q \in \mathbb{R}$ IS AN ASYNPTOTE FOR f AS $x \to +\infty$ IF

$$\lim_{x \to +\infty} (f(x) - (mx + q)) = 0$$

- $f: (-\infty, b) \to \mathbb{R}$ WITH $-\infty < b \leq +\infty$

A LINE $y = mx + q$ $m, q \in \mathbb{R}$ IS AN ASYNPTOTE FOR f AS $x \to -\infty$ IF

$$\lim_{x \to -\infty} \left(f(x) - (mx + q) \right) = 0$$

NOTATION

- $m = 0$ HORIZONTAL ASYMPTOTE $\left(\lim_{x \to +\infty} f(x) + q \right)$
- $m \neq 0$ OBLIQUE ASYMPTOTE
- $x_0 \in \mathbb{R}$ $f: A \to \mathbb{R}$ x_0 ACC. POINT FOR A FROM THE RIGHT IF

$\lim_{x \to x_0^+} f(x) = \pm\infty$, IT IS SAID THAT THE VERTICAL LINE

$x = x_0$ IS A VERTICAL ASYNPTOTE FOR f AS $x \to x_0^+$

HOW TO COMPUTE ASYMPTOTES?

$y = mx + q$ IS AN ASYMPTOTE FOR $x \to \pm\infty$

$$\Longleftrightarrow$$

$$\exists \lim_{x \to \pm\infty} \frac{f(x)}{x} = m \in \mathbb{R} \quad \text{AND} \quad \exists \lim_{x \to \pm\infty} \left(f(x) - mx \right) = q \in \mathbb{R}$$

- $\lim_{x \to \pm\infty} (f(x) - mx) = q \iff \lim_{x \to \pm\infty} (f(x) - (mx+q)) = 0$

SO " $\Rightarrow$ " EASY

- " $\Leftarrow$ " IF $f(x) - (mx+q) \xrightarrow{x \to \pm\infty} 0 \Rightarrow$

$$\frac{f(x) - (mx+q)}{x} \xrightarrow{x \to \pm\infty} 0$$

$$\hookrightarrow \frac{f(x)}{x} - m + q/x$$

STUDY OF FUNCTIONS STUDY THE FUNCTION f AND DRAW A QUALITATIVE GRAPH

1) $f(x) = x^3 - 6x^2 + 9x - 1$

- DOMAIN OF EXISTENCE $\boxed{\mathbb{R}}$
- SYMMETRIES (ODD, EVEN, PERIODIC...) $\boxed{\text{NONE EVIDENT}}$
- SIGN OF f AND INTERESECTION WITH X-AXIS AND Y-AXIS

Y-AXIS $f(0) = -1$ $(0, -1)$

X-AXIS $f(x) = 0$?

SIGN OF f $f(x) > 0$? $f(x) < 0$?

NOT EASY

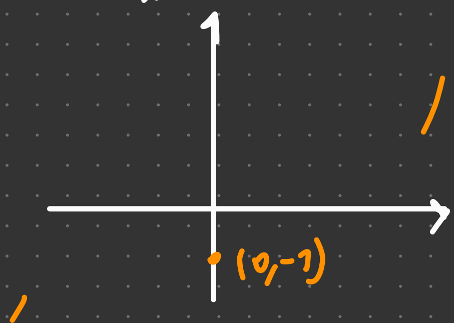
- CONTINUITY, LIMITS AND THE EXTREMES OF THE DOMAIN (INCLUDING POSSIBLE ASYMTOTICS AND EXTENSIONS BY CONTINUITY)

f CONT. ON $\mathbb{R}$

$$\lim_{x \to \pm\infty} f(x) = \pm\infty$$

ASYMPTOTES

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = +\infty \Rightarrow \text{NO ASYMTOTES FOR } x \to \pm\infty$$



- DIFFERENTIABILITY AND COMPUTATION OF DERIVATE

f DIFFER. ON $\mathbb{R}$,

$$f'(x) = 3(x-1)(x-3) \quad \forall x \in \mathbb{R}$$

- LOCAL (AND POSSIBLY ABSOLUTE) EXTREMUM POINTS, MONOTONICITY

- CANDIDATES TO EXTREMUM POINTS: CRITICAL POINTS

$\nexists f'(x)$ $\boxed{\text{NONE}}$ OR $\exists f'(x)=0$ $\boxed{x=1 \text{ AND } x=3}$

- SIGN f'

$f'>0$ ON $(-\infty, 1)$ AND ON $(3, +\infty)$

$f'(x)=0 \iff x=1$ OR $x=3$

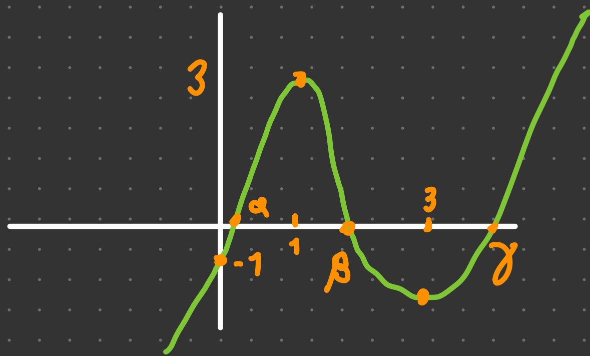
$f'<0$ ON $(1,3)$

$\Rightarrow$ f STRICTLY INCR. ON $[-\infty, 1]$
f STRICTLY DECR. ON $[1,3]$
f STRICTLY INCR ON $[3, +\infty)$

1 STRICT LOCAL MAX POINT

3 STRICT LOCAL MIN POINT

$f(1)=3$ $\quad f(3)=-1$



$f(x)=3$ 2 SOL

$f(x)=5$ 1 SOL

$f(x)=2$ 3 SOL

$\exists\ \alpha, \beta, \gamma \quad 0<\alpha<1 \quad 1<\beta<3 \quad \gamma>3$

SUCH THAT

$f<: \quad (-\infty, \alpha)$

$f(\alpha)=0$

$f(\beta)=0$

$f<0 \quad (\beta, \gamma)$

$f(\gamma)=0$

$f>0 \quad (\gamma, +\infty)$

2) $f(x) = \dfrac{2-2x}{x^2+3}$

- DOMAIN OF EXISTENCE $\mathbb{R}$
- SYMMETRIES NOT EVIDENT
- SIGN OF f AND INTERESECTION WITH AXIS

$f(x)=0 \iff x=1$

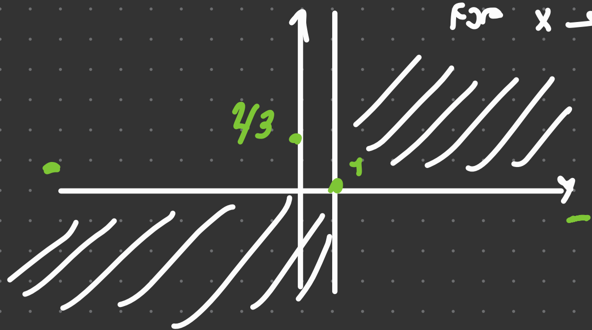
$f>0$ IF $x<1$; $f<0$ IF $x>1$

$f(0)=\frac{2}{3}$

- CONTINUITY AND LIMITS AT THE EXTREMES OF THE DOMAIN

  f CONTINUOUS ON $\mathbb{R}$

  $\lim_{x \to \pm\infty} f(x) = 0 \Rightarrow y = 0$ HORIZONTAL ASYMPTOTE FOR $x \to +\infty$ AND FOR $x \to -\infty$



- DIFFERENT AND COMP. OF DERIVATIVE

  f DIFFER ON $\mathbb{R}$ AND

  $$f'(x) = \frac{2(x+1)(x-3)}{(x^2+3)^2} \quad \forall x \in \mathbb{R}$$

- LOCAL (GLOBAL) EXTREMUM POINTS; MONOTONICITY

  - CANDIDATES, THAT IS, CRITICAL POINTS

    $\nexists f'(x)$ $\boxed{\text{NO ONE}}$ OR $\exists f'(x) = 0$ $x = -1$ AND $x = 3$

  - SIGN OF f'

    $f' > 0$ ON $(-\infty, -1)$ AND ON $(3, +\infty)$

$f'(x)=0 \iff x=-1$ OR $x=3$

$f'(x)=0 \iff x=-1$ OR $x=3$

$f'<0$ ON $(-1,3)$

$\Longrightarrow$

f STR. INCR. ON $(-\infty,-1]$

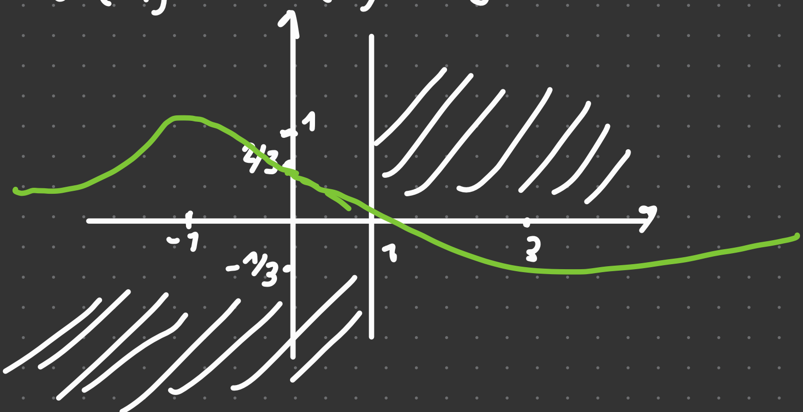
f STR. DECR. ON $[-1,3]$

f STR. INCR. ON $[3,+\infty)$

-1 STRICT LOCAL MAX POINT

3 STRICT LOCAL MIN POINT

$f(-1)=1 \quad f(3)=-1/3$



1 ABS. MAX OF f; -1 ABS MAX POINT

$-1/3$ ABS MIN OF f; 3 ABS MIN POINT