Knowledge Representation and Reasoning

Exercise Session 1

Exercise 1. Truth Tables (∗)

Build the **truth tables** of the following formulas, remembering the precedence of the operators.

1. $\neg(x \wedge y) \vee z$
2. $(x \wedge y \vee \neg x \wedge \neg w) \wedge z$
3. $(x \vee y) \wedge x$
4. $\neg(p \wedge (\neg p \vee q \vee s)) \vee q \vee s$

Exercise 2. Boolean Functions (∗)

For each of the following truth tables, build a formula expressing the same Boolean function.

x	y	z	φ_1
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

x	z	φ_2
0	0	1
0	1	0
1	0	0
1	1	1

x	y	z	w	φ_3
0	0	0	0	0
0	0	0	1	1
0	0	1	0	0
0	0	1	1	0
0	1	0	0	1
0	1	0	1	0
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	1
1	0	1	0	0
1	0	1	1	0
1	1	0	0	1
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

If the tables represent Boolean functions, what is their arity?

Exercise 3. Types of Formulas $(*)$

Use truth tables (and, if needed the equivalences seen during the lecture) to verify whether the following formulas are tautologies, contradictions, or non-tautological satisfiable formulas.

1. $x \to \neg x$
2. $(x \to y) \land (\neg y \to \neg x)$
3. $(x \to y) \to (\neg y \to \neg x)$
4. $x \land y \land (\neg x \lor \neg y) \land z$

Exercise 4. NNF $(**)$

A formula is in **negation normal form** (NNF). For example, the second and third formulas in Exercise 1 are in NNF, while the other two formulas are not.

1. Using the DeMorgan equivalences, transform the formulas of Exercise 1 to NNF
2. Do you notice something from the constructions in Exercise 2?

Exercise 5. Sheffer Functions $(* * *)$

We have seen that $\neg, \land, \lor$ form a **functionally complete** set of logical connectives in the sense that every Boolean function can be described using those connectives. In fact, we $\neg, \land$ and $\neg, \lor$ are functionally complete by themselves.

A logical connective is called a **Sheffer function** or **sole sufficient operator** if it is functionally complete by itself.

1. show that the NAND connective (denoted as $\uparrow$) is a Sheffer function
2. are there other Sheffer functions?
3. could a unary connective be a Sheffer function?

Exercise 6. Knowledge Bases (∗∗)

For the following truth table, construct a knowledge base (expressed as a set of propositional rules) which represents the same knowledge.

x	y	z	K
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Exercise 7. Expressivity (∗∗)

Can any propositional formula be expressed as a knowledge base?

If the answer is **yes**, argue why; if it is **no**, give at least two examples of formulas which cannot be expressed.

Exercise 8. Reasoning (*)

Consider the following knowledge base K:

$$\begin{aligned}
x &\leftarrow \\
y &\leftarrow x,\ z,\ w \\
x &\leftarrow v \\
w &\leftarrow y,\ z \\
z &\leftarrow v,\ x \\
z &\leftarrow y,\ w \\
z &\leftarrow u,\ x \\
u &\leftarrow \\
p &\leftarrow \\
t &\leftarrow w,\ u \\
r &\leftarrow s,\ t
\end{aligned}$$

1. Compute the redux $\widehat{K}$
2. Find all the facts that are entailed by K
3. Decide whether the following clauses are consequences of K
   a) $v \leftarrow u$
   b) $t \leftarrow y$
   c) $q \leftarrow q$
   d) $r \leftarrow w$

Exercise 9. Revision (**)

In the knowledge base from Exercise 8, substitute the fact $x \leftarrow$ with $u \leftarrow$. Call this new knowledge base K'.

1. Do your answers from Exercise 8 change?
2. Which fact(s) should you remove to ensure that z is **not** a consequence of K'?
3. If facts cannot be removed, which rules would you remove to ensure that z is not entailed?

Exercise 10. Tautologies (∗ ∗ ∗)

The soundness of the fact extraction algorithm seen during the lecture depends on the following property:

> if $\varphi \to \psi$ and $\psi \to \xi$ are both tautologies, then $\varphi \to \xi$ is also a tautology.

Show that this property holds always in propositional logic.

Exercise 1. Truth Tables (*)

Build the **truth tables** of the following formulas, remembering the precedence of the operators.

1. $\neg(x \wedge y) \vee z$
2. $(x \wedge y \vee \neg x \wedge \neg w) \wedge z$
3. $(x \vee y) \wedge x$
4. $\neg(p \wedge (\neg p \vee q \vee s)) \vee q \vee s$

1. $\neg x \vee \neg y \vee z$

JUST BUILD THE TRUTH TABLE

x	y	z	$\neg x \vee \neg y \vee \neg z$
0	0	0	1
0	0	1	1
0	1	0	1
1	0	0	1
0	1	1	1
1	0	1	1
1	1	0	1
1	1	1	0

Exercise 2. Boolean Functions (*)

For each of the following truth tables, build a formula expressing the same Boolean function.

x	y	z	φ_1
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

x	z	φ_2
0	0	1
0	1	0
1	0	0
1	1	1

x	y	z	w	φ_3
0	0	0	0	0
0	0	0	1	1
0	0	1	0	0
0	0	1	1	0
0	1	0	0	1
0	1	0	1	0
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	1
1	0	1	0	0
1	0	1	1	0
1	1	0	0	1
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

If the tables represent Boolean functions, what is their arity?

x	y	z	φ_1
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

$$(\neg x \wedge \neg y \wedge z) \vee (\neg x \wedge y \wedge z) \vee (x \wedge \neg y \wedge \neg z) \vee (x \wedge y \wedge z)$$

FOR THIS EXERCISE CHECK WHAT VALUATIONS MAKES THE FORMULA φ_1 TRUE (1) AND MAKE THE DISJUNCTION OF ALL THE POSSIBILITIES

THE ARITY IS HOW MANY VARIABLE THERE ARE SO THE ARITY OF φ_1 IS 3

IF φ TAKES N ARGUMENTS WE SAY THAT IT HAS ARITY N

Exercise 3. Types of Formulas (*)

Use truth tables (and, if needed the equivalences seen during the lecture) to verify whether the following formulas are tautologies, contradictions, or non-tautological satisfiable formulas.

1. $x \to \neg x$
2. $(x \to y) \land (\neg y \to \neg x)$
3. $(x \to y) \to (\neg y \to \neg x)$
4. $x \land y \land (\neg x \lor \neg y) \land z$

1. $x \to \neg x$

   $\neg x \lor \neg x$

2. $(\neg x \lor y) \land (y \lor \neg x)$ NON TAUTOLOGICAL SATISFIABLE

3. $(\neg x \lor y) \to (y \lor \neg x)$

   $(x \land \neg y) \lor (y \lor \neg x)$ TAUTOLOGY

4. CONTRADICTION

Exercise 4. NNF (**)

A formula is in **negation normal form** (NNF). For example, the second and third formulas in Exercise 1 are in NNF, while the other two formulas are not.

1. Using the DeMorgan equivalences, transform the formulas of Exercise 1 to NNF

2. Do you notice something from the constructions in Exercise 2?

1. $\neg(x \wedge y) \vee z$
2. $(x \wedge y \vee \neg x \wedge \neg w) \wedge z$
3. $(x \vee y) \wedge x$
4. $\neg(p \wedge (\neg p \vee q \vee s)) \vee q \vee s$

1. $\neg x \vee \neg y \vee z$

4. $\neg p \vee (p \wedge \neg q \wedge \neg s) \vee q \vee s$

Exercise 5. Sheffer Functions (* * *)

We have seen that $\neg, \wedge, \vee$ form a **functionally complete** set of logical connectives in the sense that every Boolean function can be described using those connectives. In fact, we $\neg, \wedge$ and $\neg, \vee$ are functionally complete by themselves.

A logical connective is called a **Sheffer function** or **sole sufficient operator** if it is functionally complete by itself.

1. show that the NAND connective (denoted as $\uparrow$) is a Sheffer function
2. are there other Sheffer functions?
3. could a unary connective be a Sheffer function?

NAND $\uparrow$

x	y	$x \uparrow y$
0	0	1
0	1	1
1	0	1
1	1	0

$(\neg x \wedge \neg y) \vee (\neg x \vee y) \vee (x \vee \neg y)$

$\neg x = x \uparrow x$

AND: $x \wedge y = \neg(x \uparrow y) \quad (x \uparrow y) \uparrow (x \uparrow y)$

NOR $x \downarrow y = \neg(x \vee y) = \neg x \wedge \neg y$

Exercise 6. Knowledge Bases (**)

For the following truth table, construct a knowledge base (expressed as a set of propositional rules) which represents the same knowledge.

x	y	z	K
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

$$(x \land \neg y \land \neg z) \lor (x \land y \land z)$$

APPLY DISTRIBUTIVITY

THIS IS JUST X

$$\underbrace{(x \lor x)} \land (x \lor y) \land (x \lor z) \land (\neg y \lor x) \land (\neg y \lor y) \land (\neg y \lor z)$$
$$\land (\neg z \lor x) \land (\neg z \lor y) \land (\neg z \lor z)$$

TAUTOLOGY, CAN BE REMOVED

- IF WE HAVE A CLAUSE (DISJUNCTION OF LITERALS) WHERE ONE IS POSITIVE AND ONE NEGATIVE (LIKE $\neg z \lor y$) IT CAN BE EASLY RE-WRITTEN AS $(z \to y)$
- THE PROBLEM IS RELATED WITH THE CLOSE COMPOSED BY ALL POSITIVE OR ALL NEGATIVE LITERALS

<u>BUT</u> SINCE THE FIRST CLAUSE IS JUST X THEN OF COURSE X MUST BE TRUE, THAT IS THAT THAT THE NEXT TWO CLAUSES $(x \lor y) \land (x \lor z)$ [THAT WE HAD SOME TRUBLES TO "CONVERT"] NOW ARE TRUE SINCE X IS TRUE

NOW IF WE WANT TO BUILD A MODEL WE HAVE:

- $x \leftarrow$
- $\cancel{x \leftarrow y}$
- $z \leftarrow y$
- $\cancel{x \leftarrow z}$
- $y \leftarrow z$

- FURTHERMORE SINCE X IS A FACT (ALWAYS TRUE) WE CAN APPLY THE SAME IDEA USED WITH $(x \vee y) \wedge (x \vee z)$ WITH $(\neg y \vee x) \wedge (\neg z \vee x)$ ARE REMOVE TWO CONSTRAINT OF OUR MODEL. THIS WILL NOT CHANGE OUR KB

- IF WE TAKE A LOOK AT OUR KB WE CAN SEE THAT $z \leftarrow y$ AND $y \leftarrow z$ MEANS THAT Y AND Z MUST HAVE THE SAME TRUTH VALUE (BOTH TRUE OR BOTH FALSE)

x	y	z	K
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

X TRUE AND Y AND Z SAME TRUTH VALUE

WE KNOW THAT IT IS CORRECT SINCE

- X MUST BE TRUE
- Y AND Z MUST HAVE THE SAME TRUTH VALUE

AND THIS IS CORRECT IF WE CHECK OUR TRUTH TABLE IT CAN BE SPOTTED EASLY

Exercise 7. Expressivity (**)

Can any propositional formula be expressed as a knowledge base?

If the answer is **yes**, argue why; if it is **no**, give at least two examples of formulas which cannot be expressed.

FORMULA: BOOLEAN FUNCTION.

IN THE EXERCISE BEFORE WE WERE ABLE TO EXPRESS ALL THE CLAUSES AS KB. BUT IS THIS ALWAYS POSSIBLE?

THE PROBLEM IS WHEN A CLAUSE IS NOT A HORN CLAUSE

↳ CLAUSE WITH EXACTLY ONE POSITIVE LITERAL

- $x \vee \neg y$ → THIS IS AN HORN CLAUSE IT CAN BE EXPRESSED AS AS A KB IN THE FORM $x \leftarrow y$
- $x \vee y$ → IN THE EXERCISE BEFORE WE WERE LUCKY BECAUSE THERE WAS A CLAUSE $(x \vee x)$ AND SO x MUST BE TRUE, SO ALL THE CLAUSE WITH x WERE ALSO TRUE

  SADLY, IT IS NOT ALWAYS THE CASE, AND IF WE ARE NOT IN THE "LUCK SITUATION" WE CAN NOT REPRESET THIS CLAUSE AS KB

IN OUR RULE LANGUANGE WE CAN NOT HAVE TWO VARIABLES IN THE HEAD SO WE CAN NOT HAVE $(x \vee y)$ ← THIS IS BECAUSE THIS KIND OF RULE STATES THAT OR x OR y OR x AND y MUST BE TRUE, BUT YOU DON'T KNOW WHICH ONE

REMEMBER! IF YOU CAN TRANSFORM A FORMULA IN A SET OF HORN CLAUSES THAN IT CAN BE EXPRESSED IN PREDICATE LOGIC.

A FORMULA LIKE $(x \vee y)$ CAN NOT BE EXPRESSED AS A SET OF HORN CLAUSES

↳ A SET OF HORN

CLAUSES IS A DISJUNCTION OF HORN CLAUSES

ALSO A FORMULA LIKE $(\neg X \vee \neg Y)$ [ALSO $(\neg X)$] WITHOUT POSITIVE LITERALS CAN'T BE TRANSFORMED IN A SET OF HORN CLAUSES.

REMEMBER! ANY FORMULA CAN BE TRANSFORMED IN CNF (CONJUNCTION NORMAL FORM). KB IS A SET OF "SPECIAL" SET OF CLAUSES BECAUSE THEY HAVE TO BE HORN CLAUSES (EXACTLY ONE POSITIVE LITERAL)

Exercise 8. Reasoning (*)

Consider the following knowledge base K:

$$
\begin{aligned}
x &\leftarrow \\
y &\leftarrow x,\ z,\ w \\
x &\leftarrow v \\
w &\leftarrow y,\ z \\
z &\leftarrow v,\ x \\
z &\leftarrow y,\ w \\
z &\leftarrow u,\ x \\
u &\leftarrow \\
p &\leftarrow \\
t &\leftarrow w,\ u \\
r &\leftarrow s,\ t
\end{aligned}
$$

1. Compute the redux $\widehat{K}$
2. Find all the facts that are entailed by K
3. Decide whether the following clauses are consequences of K
   a) $v \leftarrow u$
   b) $t \leftarrow y$
   c) $q \leftarrow q$
   d) $r \leftarrow w$

$$
\begin{aligned}
&\cancel{x \leftarrow} \\
y &\leftarrow \cancel{x},\ \cancel{z},\ w \\
x &\leftarrow v \\
w &\leftarrow y,\ \cancel{z} \\
z &\leftarrow v,\ \cancel{x} \\
z &\leftarrow y,\ w \\
&\cancel{z \leftarrow u,\ x} \\
&\cancel{u \leftarrow} \\
&\cancel{p \leftarrow} \\
t &\leftarrow w,\ \cancel{u} \\
r &\leftarrow s,\ t
\end{aligned}
$$

- REDUX: REMOVE ALL THE LITERALS THAT ARE TRUE (AKA FACTS) UNTIL NO LITERAL CAN BE REMOVED

- THE FACTS ARE WHICH LITERAL ARE TRUE, THAT IS
  - $z \leftarrow$
  - $w \leftarrow$
  - $p \leftarrow$
  - $x \leftarrow$

3. Decide whether the following clauses are consequences of K

a) $v \leftarrow u$

b) $t \leftarrow y$

c) $q \leftarrow q$

d) $r \leftarrow w$

TO CHECK IF A CLAUSE IS A CONSEQUENCE OF K
YOU NEED TO ADD THE RIGTH PART (THE BODY) IN THE KNOWLEDGE BASE
AND SEE IF THE LEFT PART (THE HEAD) BECOMES TRUE

1)

$\cancel{x \leftarrow}$

$y \leftarrow \cancel{x}, \cancel{z}, w$

$x \leftarrow v$

$w \leftarrow y, \cancel{z}$

$z \leftarrow v, \cancel{x}$

$z \leftarrow y, w$

$\cancel{z \leftarrow u, x}$

$\cancel{u \leftarrow}$

$\cancel{v \leftarrow}$

$t \leftarrow w, \cancel{u}$

$r \leftarrow s, t$

$u \leftarrow$

SINCE THE REDUX IS EQUIVALENT YOU CAN ADD THE "BODY-FACT" IN THE REDUX

ADD U DOES NOT DERIVE V SO V ← U IS NOT A CONSEGUENCE OF K

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2)

$t \leftarrow y$

$\cancel{x \leftarrow}$

$y \leftarrow \cancel{x}, \cancel{z}, \cancel{w}$

$x \leftarrow v$

$\cancel{w} \leftarrow \cancel{y}, \cancel{z}$

$z \leftarrow v, \cancel{x}$

$z \leftarrow \cancel{y}, w$

$\cancel{z \leftarrow u, x}$

$\cancel{u \leftarrow}$

$\cancel{v \leftarrow}$

$t \leftarrow w, \cancel{u}$

$r \leftarrow s, t$

$y \leftarrow$

ADDING Y AS A FACT ALLOWS t TO BE DERIVED SO t ← Y IS A CONSEGUENCE OF K

3)

$y \leftarrow \cancel{x}, \cancel{z}, w$

$x \leftarrow v$

$w \leftarrow y, \cancel{z}$

$z \leftarrow v, \cancel{x}$

$z \leftarrow y, w$

$t \leftarrow w, \cancel{u}$

$r \leftarrow s, t$

$q \leftarrow$

REMEMBER! ADDING $q \leftarrow$ (THE FACT q) MAKES ALREADY BY ITSELF $q \leftarrow q$ A CONSEQUENCE OF K

4)

$\cancel{y} \leftarrow \cancel{x}, \cancel{z}, \cancel{w}$

$x \leftarrow v$

$w \leftarrow \cancel{y}, \cancel{z}$

$z \leftarrow v, \cancel{x}$

$z \leftarrow \cancel{y}, \cancel{w}$

$t \leftarrow \cancel{w}, \cancel{u}$

$r \leftarrow s, t$

$w \leftarrow$

$R \leftarrow W$

ADDING THE FACT W TO THE KNOWLEDGE K MAKES R DERIVABLE. SO $R \leftarrow W$ IS A CONSEQUENCE OF K

Exercise 9. Revision (**)

In the knowledge base from Exercise 8, substitute the fact $x \leftarrow$ with $u \leftarrow$. Call this new knowledge base K'.

1. Do your answers from Exercise 8 change?
2. Which fact(s) should you remove to ensure that z is **not** a consequence of K'?
3. If facts cannot be removed, which rules would you remove to ensure that z is not entailed?

SUBSTITUDING SOME KNOWLEDGE WITH OUR KNOWLEDGE WILL CHANGE K OF COURSE

IS ALREADY HERE!

$$
\begin{aligned}
&u \leftarrow \\
&y \leftarrow x,\ z,\ w \\
&x \leftarrow v \\
&w \leftarrow y,\ z \\
&z \leftarrow v,\ x \\
&z \leftarrow y,\ w \\
&z \leftarrow u,\ x \\
&\cancel{u \leftarrow} \\
&\cancel{p \leftarrow} \\
&t \leftarrow w,\ u \\
&r \leftarrow s,\ t
\end{aligned}
$$

THIS IS THE REDUX OF K'. WE LOSE SOME CONSEQUENCES

THE FACTS ENTAILED BY K' NOW ARE:

$u \leftarrow$

$p \leftarrow$

REMEMBER!! BY ADDING MORE FACTS WE DON'T LOSE KNOWLEDGE BUT RATHER IT INCREASE

SOMETIMES FACTS CAN NOT BE REMOVED THEY CAN NOT BE MANIPULATE BY THE KB CAN BE MANIPULATED. SO WE CAN JUST REMOVE THE RULES THAT HAVE Z IN THE HEAD. OTHERWISE WE CAN REMOVE THE RULES THAT MAKES FACTS THE BODY OF Z.

Exercise 10. Tautologies (* * *)

The soundness of the fact extraction algorithm seen during the lecture depends on the following property:

if $\varphi \to \psi$ and $\psi \to \xi$ are both tautologies, then $\varphi \to \xi$ is also a tautology.

Show that this property holds always in propositional logic.

a) WHENEVER WE HAVE φ THEN WE ALSO HAVE ψ, THAT IS:

- $V(\varphi)=1$ THEN $V(\psi)=1$

IF $\varphi \to \psi$ IS A TAUTOLOGY THEN $\neg\varphi \vee \psi$ IS ALSO A TAUTOLOGY MEANS THAT FORALL VALUATIONS $V(\neg\varphi \vee \psi)=1$

$$\Downarrow$$

(1) $\max\{V(\neg\varphi), V(\psi)\}=1$

SINCE $\psi \to \xi$ IS ALSO A TAUTOLOGY THEN $\neg\psi \vee \xi$ IS A TAUTOLOGY THAT IS $V(\neg\psi \vee \xi)=1$

$$\Downarrow$$

(2) $\max\{V(\neg\psi), V(\xi)\}=1$

WE WANT TO CHECK $\max\{V(\neg\varphi), V(\xi)\}=1$ (3)

- IF $V(\neg\varphi)=0$ THEN $V(\psi)=1$ (2) THEN $V(\neg\psi)=0$ AND SO $V(\xi)=1$ (2)

- IF $V(\neg\varphi)=1$ THEN WE ALREADY KNOW THAT (3) IS 1

SO REGARDLESS OF THE CASES (3) IS ALWAYS 1 SO $\varphi \to \xi$ IS A TAUTOLOGY