Exercises - Calculus

Academic Year 2021-2022

Sheet 18

1. For any of the following functions $f : C \subset \mathbb{R}^N \to \mathbb{R}$, determine, if it exists,

$$\min_C f$$

where

(a) $f(x,y) = e^{xy} + xy$ and $C = \{(x,y) \in \mathbb{R}^2 :\ x^2 + 2y^2 \leq 1\}$

(b) $f(x,y) = y^2 - \sqrt{6}x^2$ and $C = \{(x,y) \in \mathbb{R}^2 :\ x \in [-1,1],\ x^2 - 1 \leq y \leq 1 - x^4\}$ or $C = \{(x,y) \in \mathbb{R}^2 :\ x \in [-1,0],\ x^2 - 1 \leq y \leq 1 - x^4\}$

(c) $f(x,y,z) = 2y^3 - 3y^2 + x^2 - z^2$ and $C = \{(x,y,z) \in \mathbb{R}^3 :\ x^2 + y^2 + z^2 \leq 9\}$

2. Solve the following constrained minimum problems. Determine, if it exists,

$$\min_C f$$

where

(a) $f(x,y) = |y| - |x|$ and $C = \{(x,y) \in \mathbb{R}^2 :\ x^2 + y^2 - 2y = 0\}$

(b) $f(x,y) = -x^2 - x - y^2$ and $C = \{(x,y) \in \mathbb{R}^2 :\ y + 2yx = 1,\ x \in [-2,-1] \cup [0,1]\}$

(c) $f(x,y,z) = x - 2y - 2z^2$ and $C = \{(x,y,z) \in \mathbb{R}^3 :\ x^2 + 2y^2 + z^2 = 1\}$

3. Determine, if it exists,

$$\min_C f$$

where

$$C = \{(x,y,z) \in \mathbb{R}^3 :\ 4(x^2 + y^2) \leq z^2,\ 0 \leq z \leq 1\}$$

and $f(x,y,z) = x(z - 1/2)^2$.

4. Determine, if it exists,

$$\min_C f$$

where

$$C = \{(x,y,z) \in \mathbb{R}^3 :\ x^2 + y^4 + z^4 \leq 1\}$$

and $f(x,y,z) = -x^2 + y - z$.

5. Determine, if it exists,

$$\min_C f$$

where

$$C = \{(x,y,z) \in \mathbb{R}^3 :\ 0 \leq z \leq 1 - x^2 - y^2\}$$

and $f(x,y,z) = (x^4 + y^4)(z - 1)$.

6. Determine, if it exists,
$$\min_C f$$
where
$$C = \{(x, y, z) \in \mathbb{R}^3 : \ 4 \leq x^2 + y^2 + z^2 \leq 9\}$$
and $f(x, y, z) = x^3 - y^2 + z^2$.

7. Determine, if it exists,
$$\min_C f$$
where
$$C = \{(x, y) \in \mathbb{R}^2 : \ x^2 \leq y \leq \sqrt{x}\}$$
and $f(x, y) = -x\sqrt{1 - xy}$.

8. Determine, if it exists,
$$\min_C f$$
where
$$C = \{(x, y, z) \in \mathbb{R}^3 : \ x^2 + y^2 + z^2 \leq 1, \ z \geq y\}$$
and $f(x, y, z) = (z - y)^2 x$.

MINIMUM PROBLEMS

STRATEGY

EXISTENCE OF THE MINIMUM

WEIERSTRASS: f CONTINOUS, C CLOSED AND BOUNDED $\Rightarrow \exists \min_C f ; \exists \max_C f$

INTERIOR CANDIDATES:

- $(x,y) \in \mathring{C}$ S.T. f IS NOT DIFFERENTIABLE IN (x,y)
- $(x,y) \in$ S.T. f IS DIFFERENTIABLE AND $\nabla f(x,y)=0$

BOUNDARY CANDIDATES

① $(x,y) \in \partial C$ S.T. f IS NOT DIFFERENTIABLE IN (x,y)

② $(x,y) \in \partial C$ S.T. F IS NOT C^1 IN A NEIGHBOURHOOD $f(x,y)$ OR $\nabla f(x,y)=(0,0)$

③ POINTS THAT NOT SATISFY ① OR ② AND SOLVE

$$\begin{cases} F(x,y)=0 \\ \nabla f(x,y) = \lambda \nabla F(x,y) \end{cases}$$

1. For any of the following functions $f : C \subset \mathbb{R}^N \to \mathbb{R}$, determine, if it exists,

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where

(a) $f(x,y) = e^{xy} + xy$ and $C = \{(x,y) \in \mathbb{R}^2 : \; x^2 + 2y^2 \leq 1\}$

$f(x,y) = e^{xy} + xy$ CONTINOUS

$C = \{(x,y) \in \mathbb{R}^2 : x^2 + 2y^2 \leq 1\}$ CLOSED

$C = \{(x,y) \in \mathbb{R}^2 . x^2 + 2y^2 \leq 1\}$ BOUNDED (SUM OF SQUARE NUMBERS BOUNDED FROM ABOVE)

INTERIOR

$\mathring{C} = \{(x,y) \in \mathbb{R}^2 : x^2 + 2y^2 < 1\}$

BOUNDARY $\partial C = \{(x,y) \in \mathbb{R}^2 : x^2 + 2y^2 = 1\} = \{F(x,y) = 0\}$

CANDIDATES

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{cases} \quad \begin{cases} y e^{x} + y = 0 \\ x e^{x} + x = 0 \end{cases} \quad \begin{matrix} y=0 \\ x=0 \end{matrix} \quad (0,0) \in C \Rightarrow \text{LOCAL EXTREMUM} \quad f(0,0) = 1$$

① NO POINTS $f \in C^\infty(\mathbb{R}^2)$

② $F(x,y) = x^2 + 2y^2 - 1 \in C^\infty(\mathbb{R}^2)$

$\nabla F(x,y) = 0 \Rightarrow \begin{cases} 2x = 0 \\ 4y = 0 \end{cases} \quad (0,0) \notin \partial C$ NO POINTS

LAGRANGE MULTIPLIERS

$$\begin{cases} ye^{xy}+y = 2\lambda x \\ xe^{xy}+x = 4\lambda y \\ x^2+2y^2=1 \end{cases} \qquad \begin{cases} y(e^{xy}+1) = 2\lambda x \\ x(e^{xy}+1) = 4\lambda y \\ x^2+2y^2=1 \end{cases} \qquad \begin{cases} \dfrac{e^{xy}+1}{2\lambda} = \dfrac{x}{y} \\ \dfrac{e^{xy}+1}{2\lambda} = \dfrac{2y}{x} \\ x^2+2y^2=1 \end{cases} \qquad \begin{cases} \dfrac{x}{y} = \dfrac{2y}{x}; \quad x^2 = 2y^2 \\ x^2+2y^2=1 \end{cases} \qquad \begin{cases} \\ 2x^2=1; \quad x = \pm\sqrt{\frac{1}{2}} \end{cases}$$

$$\Downarrow$$

$$x^2 = 2y^2; \quad \frac{1}{2} = 2y^2;$$

$$\frac{1}{4} = y^2; \quad y = \pm\frac{1}{2}$$

$$\left(-\sqrt{\tfrac{1}{2}}, \tfrac{1}{2}\right)$$
$$\left(-\sqrt{\tfrac{1}{2}}, -\tfrac{1}{2}\right)$$
$$\left(\sqrt{\tfrac{1}{2}}, -\tfrac{1}{2}\right)$$
$$\left(\sqrt{\tfrac{1}{2}}, \tfrac{1}{2}\right)$$

1. For any of the following functions $f : C \subset \mathbb{R}^N \to \mathbb{R}$, determine, if it exists,

$$\min_C f$$

where

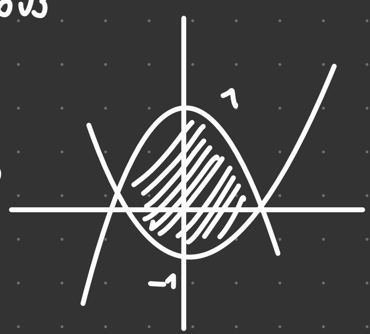
(a) $f(x,y) = e^{xy} + xy$ and $C = \{(x,y) \in \mathbb{R}^2 : \ x^2 + 2y^2 \leq 1\}$

(b) $f(x,y) = y^2 - \sqrt{6}x^2$ and $C = \{(x,y) \in \mathbb{R}^2 : \ x \in [-1,1], \ x^2 - 1 \leq y \leq 1 - x^4\}$ or $C = \{(x,y) \in \mathbb{R}^2 : \ x \in [-1,0], \ x^2 - 1 \leq y \leq 1 - x^4\}$

$f(x,y) = y^2 - \sqrt{6}x^2$ AND $C = \{(x,y) \in \mathbb{R}^2 : x \in [-1,1], x^2-1 \leq y \leq 1-x^4\}$

$f(x,y) = y^2 - 6x^2$ CONTINOUS

$-1 \leq y \leq 1$ CLOSE BOUNDED



INTERIOR: $\mathring{C}\{(x,y) \in \mathbb{R}^2 : x \in (-1,1), \ x^2-1 < y < 1-x^4\}$

BOUNDERIE $\partial C: \{(x,y) \in \mathbb{R}^2 : x \in (-1,1), \ x^2-1 = y\} \cup \{ x \in (-1,1) \quad y = 1-x^4\} \cup f_2(x,y) = 0$

(c) $f(x,y,z) = 2y^3 - 3y^2 + x^2 - z^2$ and $C = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 9\}$

$f(x,y,z) = 2y^3 - 3y^2 + x^2 - z^2$ CONTINOUS

C CLOSE AND BOUNDED

INTERIOR CANDIDATES: $\mathring{C} = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 < 1\}$

BOUNDED CANDIDATES: $\partial C = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$

CANDIDATES:

$$\begin{cases} \frac{\partial f}{\partial x}(x,y,z) = 0 \\ \frac{\partial f}{\partial y}(x,y,z) = 0 \\ \frac{\partial f}{\partial z}(x,y,z) = 0 \end{cases} \quad \begin{cases} 2x = 0 \\ 6y^2 - 6y = 0 \\ -2z = 0 \end{cases}$$

$(0,0,0) \in C$ LOCAL EXTREMUM $f(0,0,0) = 0$

$(0,1,0) \in C$ LOCAL EXTREMUM $f(0,1,0) = -1$

$$\begin{cases} 2x = 2\lambda x \\ 6y(y-1) = 2\lambda y \\ -2z = 2\lambda z \\ x^2+y^2+z^2-9=0 \end{cases} \quad \begin{cases} x = \lambda x \\ 3y(y-1) = \lambda y \\ z = -\lambda z \\ x^2+y^2+z^2-9=0 \end{cases}$$

$x \neq 0,\ y \neq 0$
$$\begin{cases} \lambda = 1 \\ 3(y-1) = \lambda;\ y = \frac{4}{3} \\ z = 0 \\ x^2+y^2+z^2-9=0 \end{cases} \quad \begin{cases} \lambda = 1 \\ y = 4/3 \\ z = 0 \\ x^2 = 9 - \frac{16}{9};\ x = \pm\frac{\sqrt{65}}{3} \end{cases}$$

$\left(\frac{\sqrt{65}}{3}, \frac{4}{3}, 0\right)$

$\left(-\frac{\sqrt{65}}{3}, \frac{4}{3}, 0\right)$

$$\begin{cases} x = 0 \\ y = 0 \\ z = -\lambda z \\ z^2 = 9;\ z = \pm 3 \end{cases}$$

$(0,0,3)$

$(0,0,-3)$

$$\begin{cases} x = 0 \\ 3y(y-1) = \lambda y \\ z = 0 \\ y = \pm 3 \end{cases}$$

$(0,3,0)$

$(0,-3,0)$

$(3,0,0)$

$(-3,0,0)$

$x = 0 \quad y \neq 0 \quad z \neq 0$

$$\begin{cases} x = 0 \\ 3y - 3 = \lambda \\ 1 = -\lambda;\ \lambda = -1 \\ x^2+y^2+z^2-9=0 \end{cases} \quad \begin{cases} x = 0 \\ y = \frac{2}{3} \\ \lambda = -1 \\ z^2 = 9 - \frac{4}{9};\ z = \pm\frac{\sqrt{77}}{3} \end{cases}$$

$\left(0, \frac{2}{3}, \frac{\sqrt{77}}{3}\right)$

$\left(0, \frac{2}{3}, -\frac{\sqrt{77}}{3}\right)$

$x \neq 0 \quad y = 0 \quad z \neq 0$

$$\begin{cases} \lambda = 1 \\ y = 0 \\ 1 = -\lambda \\ \emptyset \end{cases}$$

CANDIDATES:

$f(x,y,z) = 2y^3 - 3y^2 + x^2 - z^2$

$A = (0,0,0) \quad f(A) = 0$

$B = (0,1,0) \quad f(B) = -1$

$C = \left(\frac{\sqrt{65}}{3}, \frac{4}{3}, 0\right) \quad f(C) = \frac{128}{27} - \cancel{3}^{1} \cdot \frac{16}{\cancel{9}3} + \frac{65}{9} =$

$D = \left(-\frac{\sqrt{65}}{3}, \frac{4}{3}, 0\right) \quad f(D) = f(C) =$

$27 \cdot 2$

54

$E = (0,0,3) \quad f(E) = -9$

$F = (0,0,-3) \quad f(F) = f(E) = -9$

$G = (0,3,0) \quad f(G) = 54 - 27 = 27$ MAX

$H = (0,-3,0) \quad f(H) = f(G) = 27$ MAX

$I = (3,0,0) \quad f(I) = 9$

$L = (-3,0,0) \quad f(L) = 9$

$M = \left(0, \frac{2}{3}, \frac{\sqrt{77}}{3}\right) \quad f(M) = \frac{16}{27} - \frac{4}{3} - \frac{77}{9} =$ MIN

$N = \left(0, \frac{2}{3}, -\frac{\sqrt{77}}{3}\right) \quad f(N) = f(M) =$ MIN

(a) $f(x,y) = |y| - |x|$ and $C = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 - 2y = 0\}$

$f(x,y) = |y| - |x|$ CONTINOUS

$\mathcal{E} = C = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 - 2y = 0\}$ OPEN AND CLOSE BOUNDED ???

NON-DIFF. POINT $(0,0)$ $(0,2)$

$$\begin{cases} \frac{\nabla f}{dx}(x,y) = 0 \\ \frac{\nabla f}{dy}(x,y) = 0 \end{cases} \quad \begin{cases} 2x = 0; \\ 2y - 2 = 0; \end{cases} \quad (0,1) \Rightarrow 1 - 2 = 0 \Rightarrow \text{NO CANDIDATE}$$

LAGRANGE MULTIPLIERS

$$\begin{cases} x > 0 \\ |y| = 2\lambda x \\ -\underbrace{|x|}_{-x} = 2\lambda y - 2\lambda \\ x^2 + y^2 - 2y = 0 \end{cases} \qquad \begin{cases} x < 0 \\ |y| = 2\lambda x \\ -\underbrace{|x|}_{-x} = 2\lambda y - 2\lambda \\ x^2 + y^2 - 2y = 0 \end{cases}$$

$$\begin{cases} |y| = 2\lambda x \\ x = 2\lambda(y - 1) \\ |y| = 4\lambda^2 y - 4\lambda^2 \\ x^2 + y^2 - 2y = 0 \end{cases} \qquad \begin{cases} '' \quad '' \\ '' \quad '' \\ '' \quad '' \end{cases}$$

6. Determine, if it exists,

$$\min_C f$$

where

$$C = \{(x, y, z) \in \mathbb{R}^3 : \ 4 \leq x^2 + y^2 + z^2 \leq 9\}$$

and $f(x, y, z) = x^3 - y^2 + z^2$.

$f(x,y,z) = x^3 - y^2 + z^2$

$C = \{(x,y,z) \in \mathbb{R}^3 : 4 \leq x^2 + y^2 + z^2 \leq 9\}$ CLOSE AND BOUNDED

INTERNAL $\quad \mathring{C} = \{(x,y,z) \in \mathbb{R}^3 : 4 < x^2 + y^2 + z^2 < 9\}$? SPLIT?

BOUNDARIES $\quad \partial\mathring{C} = \{(x,y,z) \in \mathbb{R}^3 : 4 = x^2 + y^2 + z^2 = 9\}$

$$\begin{cases} \frac{\nabla f}{dx}(x,y,z) = 0 \\ \frac{\nabla f}{dy}(x,y,z) = 0 \\ \frac{\nabla f}{dz}(x,y,z) = 0 \end{cases} \qquad \begin{cases} 2x = 0 \\ 2y = 0 \\ 2z = 0 \end{cases}$$

$(0,0,0)$ CANDIDATE

$$\begin{cases} 3x = 2\lambda x \\ -2y = 2\lambda y \\ 2z = 2\lambda z \\ x^2 + y^2 + z^2 = 9 \end{cases} ? \qquad \begin{cases} '' \quad '' \\ '' \quad '' \\ '' \quad '' \\ x^2 + y^2 + z^2 = 4 \end{cases} ?$$

1. For any of the following functions $f : C \subset \mathbb{R}^N \to \mathbb{R}$, determine, if it exists,

$$\min_C f$$

where

(a) $f(x, y) = e^{xy} + xy$ and $C = \{(x, y) \in \mathbb{R}^2 : x^2 + 2y^2 \leq 1\}$

$f(x,y) = e^{xy} + xy$ CONTINOUS FUNCTION $\Rightarrow \exists \min_{C} f$

STEP 1. DERIVATES = 0

$$\begin{cases} \frac{df}{dx}(x,y) = ye^{xy} + y \\ \frac{df}{dy}(x,y) = xe^{xy} + x \end{cases} \Rightarrow \begin{cases} \frac{df}{dx}(x,y) = 0 \\ \frac{df}{dy}(x,y) = 0 \end{cases} = \begin{cases} ye^{xy} + y = 0 \\ xe^{xy} + x = 0 \end{cases} \Rightarrow$$

POINT $A = (0,0)$ IT IS IN C

$x^2 + 2y^2 \leq 1$; $0 \leq 1$

2. WRITE LAGRANGE FORMULA

$$L(x,y,\lambda) = f(x,y) + \lambda(C(x,y))$$

$$L(x,y,\lambda) = e^{xy} + xy + \lambda(x^2 + 2y^2 - 1)$$

3. WRITE THE NEW SYSTEM

$$\begin{cases} L'x = 0 \\ L'y = 0 \\ L'\lambda = 0 \end{cases} \qquad \begin{cases} ye^{xy} + y + 2x\lambda = 0 \text{ ; } \lambda = \frac{-ye^{xy}}{2x} - \frac{y}{2x} \\ xe^{xy} + x + 4y\lambda = 0 \text{ ; } xe^{xy} + x + \frac{2y^2}{x}e^{xy} - \frac{2y^2}{x} = 0 \\ x^2 + 2y^2 - 1 = 0 \text{ ;} \end{cases}$$

1. For any of the following functions $f : C \subset \mathbb{R}^N \to \mathbb{R}$, determine, if it exists,

$$\min_C f$$

where

(b) $f(x,y) = y^2 - \sqrt{6}x^2$ and $C = \{(x,y) \in \mathbb{R}^2 : \ x \in [-1,1], \ x^2 - 1 \leq y \leq 1 - x^4\}$ or $C = \{(x,y) \in \mathbb{R}^2 : \ x \in [-1,0], \ x^2 - 1 \leq y \leq 1 - x^4\}$

$f(x,y) = y^2 - \sqrt{6x^2}$ $\qquad C = \{(x,y) \in \mathbb{R}^2 : x \in [-1, 1], \ x^2 - 1 \leq y \leq 1 - x^4$

$$\begin{cases} \frac{df}{dx}(x,y) = -2\sqrt{6}x \\ \frac{df}{dy}(x,y) = 2y \end{cases} \quad \begin{cases} \frac{df}{dx}(x,y) = 0 \\ \frac{df}{dy}(x,y) = 0 \end{cases} \quad \begin{cases} x = 0 \\ y = 0 \end{cases} \Rightarrow -1 \leq 0 \leq 1 \ \checkmark \quad \text{CANDIDATE } A = (0,0)$$

LAGRANGE MOLTIPLICATORS

$L(x,y,\lambda) = f(x,y) + \lambda$

1. For any of the following functions $f : C \subset \mathbb{R}^N \to \mathbb{R}$, determine, if it exists,

$$\min_C f$$

where

(c) $f(x, y, z) = 2y^3 - 3y^2 + x^2 - z^2$ and $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 9\}$

$f(x,y,z) = 2y^3 - 3y^2 + x^2 - z^2$ CONTINOUS $\Rightarrow \exists \min_C f$

$$\begin{cases} \frac{df}{dx}(x,y,z) = 2x \\ \frac{df}{dy}(x,y,z) = 6y^2 - 6y \\ \frac{df}{dz}(x,y,z) = -2z \end{cases} \quad \begin{cases} \frac{df}{dx}(x,y,z) = 0 \\ \frac{df}{dy}(x,y,z) = 0 \\ \frac{df}{dz}(x,y,z) = 0 \end{cases} \quad \begin{cases} 2x = 0 \\ 6y(y-1) = 0 \\ -2z = 0 \end{cases}$$

CANDIDATES:

$A = (0,0,0) \Rightarrow 0 \leq 9$

$B = (0,1,0) \Rightarrow 0 \leq 9$

LAGRANGE MULTIPLIERS

$$L(x,y,z,\lambda) = 2y^3 - 3y^2 + x^2 - z^2 + \lambda(x^2 + y^2 + z^2 - 9)$$

$$\begin{cases} L'x = 2x + 2\lambda x \\ L'y = 6y^2 - 6y + 2\lambda y \\ L'z = -2z + 2\lambda z \\ L'\lambda = x^2 + y^2 + z^2 - 9 \end{cases} \quad \begin{cases} L'x = 0 \\ L'y = 0 \\ L'z = 0 \\ x^2 + y^2 + z^2 - 9 = 0 \end{cases} \quad \begin{cases} 2x(1+\lambda) = 0 \\ y(6y - 6 + 2\lambda) = 0 \\ 2z(\lambda - 1) = 0 \\ x^2 + y^2 + z^2 - 9 = 0 \end{cases}$$

$$\begin{cases} x = -\lambda x \\ y(3y - 3 + \lambda) = 0 \\ z = \lambda z \\ x^2 + y^2 + z^2 - 9 = 0 \end{cases}$$

$x = 0$

$y = 0$; $3y - 3 + \lambda = 0$; $y = 1$

$z = 0$; $\lambda = 0$; $z = 0$; $\lambda = 0$

$(0,0,0)$; $(0,1,0)$

$y \neq 0 \Rightarrow \lambda = 0$

$y = 0$; $y = 1$

0