Exercises - Calculus

Academic Year 2021-2022

Sheet 20

1. Compute the volume of the truncated straight cone whose base has radius 3 cm and whose height is 10 cm.

2. Compute the volume of the truncated straight cone obtained by rotating around its major side the right triangle whose sides have lengths 3 cm and 4 cm, respectively.

3. Let us consider the triangle with vertices $A = (0, 2)$, $B = (5, 0)$ and $C = (0, 6)$. Compute the volume of the solid of rotation obtained by rotating the triangle around its side AC.

4. Compute the volume of the straight pyramid whose base is a rectangle with sides 3 and 5 cm and whose height is 10 cm.

5. Let $E = \{(x, y) \in \mathbb{R}^2 : \ -x < y < 2 - x, \ 0 < x < 2\}$. Compute the area of E and
$$\iint_E xy\mathrm{e}^y \mathrm{d}x\mathrm{d}y \quad \text{and} \quad \iint_E xy\mathrm{e}^x \mathrm{d}x\mathrm{d}y$$

6. Let $E = \{(x, y) \in \mathbb{R}^2 : \ 0 < y < 1, \ -y < x < y^2 + 1\}$. Compute the area of E and
$$\iint_E \log(1 + y)\mathrm{d}x\mathrm{d}y$$

7. Let $E = \{(x, y) \in \mathbb{R}^2 : \ 0 < y < 1, \ y^2 < x < 2\}$. Write E as a normal region with respect to the y-axis. Compute the area of E and
$$\iint_E \sqrt{x}y\mathrm{d}x\mathrm{d}y$$

8. Let $E = B_3(0) \backslash \overline{B_2(0)} \subset \mathbb{R}^2$. Compute
$$\iint_E e^x |y| \mathrm{d}x\mathrm{d}y$$

9. Let $A = \{(y, z) \in \mathbb{R}^2 : \ 0 \leq z \leq 1, \ z^4 \leq y \leq 2 + e^z\}$. Let V be the solid obtained by rotating in the clockwise sense A around the z-axis of an angle $\pi/4$. Compute the volume of A. Let instead V_1 be the solid obtained by rotating in the clockwise sense A around the y-axis of an angle 2π. Compute the volume of V_1.

10. Let $y = \varphi(z) = |z|e^{|z|}$, $-1 \leq z \leq 1$. Let $A = \{(y, z) \in \mathbb{R}^2 : \ -1 \leq z \leq 1, \ 0 \leq y \leq \varphi(z)\}$. Compute the barycentre di A and
$$\iint_A y^2 \mathrm{d}y\mathrm{d}z \quad \text{and} \quad \iint_A z^2 \mathrm{d}y\mathrm{d}z.$$
Finally, compute the volume of the solid obtained by rotating A around the z-axis of an angle $\pi/3$ and of that obtained by rotating A around the y-axis of an angle π.

11. Let $A = \{(x,y) \in \mathbb{R}^2 : \ \sqrt{x} + \sqrt{y} \leq 1, \ x \geq 0, \ y \geq 0\}$. Compute the barycentre of A and
$$\iint_A y^2 \mathrm{d}x\mathrm{d}y.$$
Let V be the solid obtained by rotating A of an angle π around the x-axis. Compute the volume of V.

12. Let C be the open annulus in $\mathbb{R}^3$, centred at the origin, with inner radius 2 and outer radius 3, that is, $C = B_3(0) \backslash \overline{B_2(0)} \subset \mathbb{R}^3$. Compute
$$\iiint_C y^2 xz \mathrm{d}x\mathrm{d}y\mathrm{d}z \quad \text{and} \quad \iiint_C y^2 |x||z| \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

13. Let V be the unit ball in $\mathbb{R}^3$ intersected with the cone $C = \{(x,y,z) \in \mathbb{R}^3 : \ z \geq 0, (x^2+y^2) \leq z^2\}$. Compute the volume of V and its barycentre. Then compute
$$\iiint_V x^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z \quad \text{and} \quad \iiint_V z \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

14. Let V be the unit ball in $\mathbb{R}^3$ intersected with the cone $C = \{(x,y,z) \in \mathbb{R}^3 : z \geq 0, (x^2+y^2) \leq 3z^2\}$. Compute the volume of V and its barycentre. Then compute
$$\iiint_V x^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z, \quad \iiint_V y^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z, \quad \iiint_V z^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z$$
and
$$\iiint_V \log(z) z^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z \quad \text{and} \quad \iiint_V (x^2 e^z + y^2) \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

15. Let V be the portion of the ball
$$V = \{(x,y,z) \in \mathbb{R}^3 : \ \|(x,y,z)\| \leq 2, \ z \geq 1\}.$$
Write V in cartesian coordinates, a normal region with respect to the z-axis, in cylindrical coordinates and in spherical coordinates Compute the volume of V and its barycentre. Then compute
$$\iiint_V (x^2+y^2) \mathrm{d}x\mathrm{d}y\mathrm{d}z$$
and
$$\iiint_V \sqrt{x^2+y^2+z^2} \mathrm{d}x\mathrm{d}y\mathrm{d}z \quad \text{and} \quad \iiint_V \log(1+x^2+y^2) \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

5. Let $E = \{(x, y) \in \mathbb{R}^2 : \ -x < y < 2 - x, \ 0 < x < 2\}$. Compute the area of E and

$$\iint_E xy\mathrm{e}^y \mathrm{d}x\mathrm{d}y \quad \text{and} \quad \iint_E xy\mathrm{e}^x \mathrm{d}x\mathrm{d}y$$

$$\int_{-x}^{2}\int_{0}^{2} xye^y\,dx\,dy = \int_{-x}^{2} ye^y\left[\frac{x^2}{2}\right]_{x=0}^{x=2} dy = 2\int_{-x}^{2} ye^y\,dy = 2\left[ye^y\right]_{-x}^{2} - \int_{-x}^{2} e^y\,dy = 4e^2 + 2xe^{-x} - e^2 + e^{-x}$$

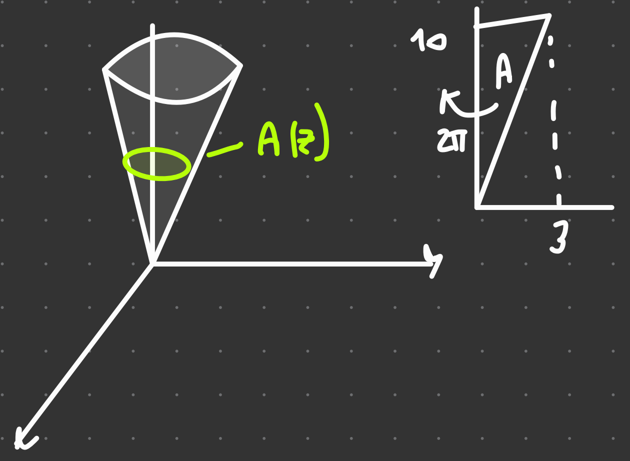
BY PARTS

$f \quad y \qquad f' \quad 1$

$g \quad e^y \qquad g' \quad e^y$

1. Compute the volume of the truncated straight cone whose base has radius 3 cm and whose height is 10 cm.

$$|V| = \iiint_V 1 \, dx\,dy\,dz$$



$$V = \{(x,y,z) \in \mathbb{R}^3 \quad 0 \le z \le 10\}$$

$$|V| = \int_0^{10} dz \iint_{A'} dx\,dy$$

$$A(z) = \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} \le \frac{3z}{10}$$

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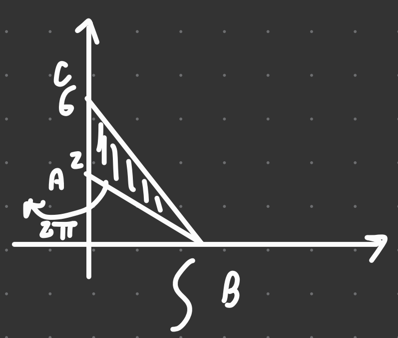
3. Let us consider the triangle with vertices $A = (0,2)$, $B = (5,0)$ and $C = (0,6)$. Compute the volume of the solid of rotation obtained by rotating the triangle around its side AC.

② $|V| = 12\pi$

③ $A = (0,2) \quad B = (5,0) \quad C = (0,6)$



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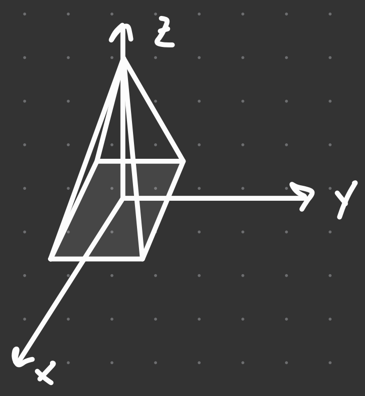
$$A = \left\{(y,z) : 0 \le y \le 5 \quad 2 - \frac{2}{5}y \le z \le 6 - \frac{6}{5}y\right\}$$

$$|V| = 2\pi \iint_A y\,dy\,dz = 2\pi \int_0^5 dy \int_{2-\frac{2}{5}y}^{6-\frac{6}{5}y} dz = \int_0^5 y\,dy \int_{2-\frac{2}{5}y}^{6-\frac{6}{5}y} 1\,dz =$$

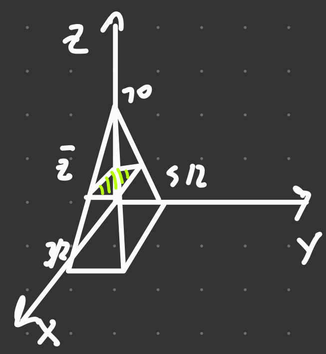
$$2\pi \int_0^5 dy\; y \Big[z\Big]_{2-\frac{2}{5}y}^{6-\frac{6}{5}y} = 2\pi \int_0^5 dy\; y\left(6 - \frac{6}{5}y - 2 + \frac{2}{5}y\right) = 2\pi \left[\frac{4y^2}{2} - \frac{4}{15}y^3\right]_0^5 = \frac{100}{3}\pi$$

$$4 \qquad -\frac{4}{5}y$$

(4) $|V| = 50\ cm$



$$|V| = \iiint_V 1\, dx\, dy\, dz$$

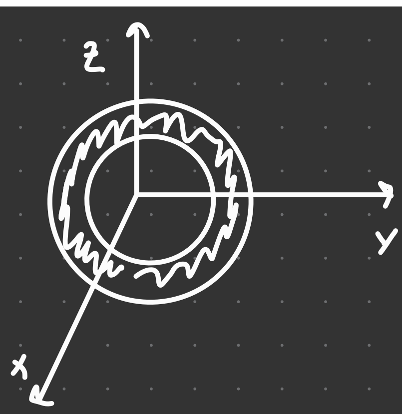


$V = \{ (x,y,z) : 0 \le z \le 10,\ 0 \le x \le \quad x\ y \le \}$

(5) $E = \{ (x,y) \in \mathbb{R}^2 :$

12. Let C be the open annulus in $\mathbb{R}^3$, centred at the origin, with inner radius 2 and outer radius 3, that is, $C = B_3(0)\backslash\overline{B_2(0)} \subset \mathbb{R}^3$. Compute

$$\iiint_C y^2 xz\,dx\,dy\,dz \quad \text{and} \quad \iiint_C y^2|x||z|\,dx\,dy\,dz$$



$$I_1 = \iiint_C y^2 x z \, dx\, dy\, dz$$

$$f(x,y,z) = -f(-x,y,z)$$

$$I_1 = \iiint_{C\cap\{x>0\}} f(x,y,z) + \iiint_{C\cap\{x<0\}} f(x,y,z) \qquad \boxed{x' = -x}$$

$$\iiint_{C\cap\{x>0\}} f(x,y,z) + \iiint_{C\cap\{x<0\}} f(-x',y,z) = 0 \qquad \boxed{f(-x',y,z) = -f(x',y,z)}$$