Exercises - Calculus
Academic Year 2021-2022

Sheet 1

1. Prove by induction that
   (a) $\forall\, n \in \mathbb{N}$ we have $n < 10^n$
   (b) $\forall\, n \in \mathbb{N}$ we have $2^{n-1} \leq n!$ (and $2^{n-1} < n!\ \forall, n \geq 3$)

2. Prove by induction that for any $n \in \mathbb{N}$ we have
   (a) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
   (b) $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$
   (c) $\displaystyle\sum_{k=1}^{n} (3k(k-1)) = n^3 - n$

3. Let us consider the following subsets of $\mathbb{R}$

   $A = \{n \in \mathbb{N} :\ n \text{ is even}\};\quad B = \{n \in \mathbb{N} :\ n < 12\};\quad C = \{n \in \mathbb{N} :\ n \leq 12\}.$

   Determine the supremum and the infimum and establish if they are the maximum and the minimum, respectively, for the following sets: A, B, C, $A \cap B$, $A \cap C$, $A \cup C$, $A \backslash C$.

4. For the following subsets of $\mathbb{R}$, determine the supremum and the infimum and establish if they are the maximum and the minimum, respectively.
   - $A = [-3, 5) \cup \{7, 8\}$
   - $B = (0, \sqrt{2}]$
   - $C = (0, \sqrt{2}] \cap \mathbb{Q}$

5. For the following subsets of $\mathbb{R}$, determine the supremum and the infimum and establish if they are the maximum and the minimum, respectively.
   - $A = \{x \in \mathbb{R} :\ (1-x)(x^2-3) > 0 \text{ and } (x+2)(x-5) \leq 0\}$
   - $B = \{x \in \mathbb{R} :\ |x-2| > 1 \text{ and } |x+1| - 2 < 0\}$
   - $C = \{x \in \mathbb{R} :\ |x+1| + 1 > 0 \text{ and } x^3 - 27 < 0\}$
   - $D = \{x \in \mathbb{R} :\ x^3 + 3 > 0 \text{ and } x^2 \leq 1/4\}$

6. For the following subsets of $\mathbb{R}$, determine the supremum and the infimum and establish if they are the maximum and the minimum, respectively.
   - $A = \mathbb{N} \cap \{x \in \mathbb{R} :\ x^2 + 8x \geq 6x\}$
   - $B = \left\{x \in \mathbb{R} :\ x = \dfrac{(-1)^n}{n},\ n \in \mathbb{N}\right\}$

- $C = \{x \in \mathbb{R} : \sqrt{x+5} > 3 - x\}$
- $D = \left\{x \in \mathbb{R} : \dfrac{x+1}{\sqrt{4-x^2}} \leq 1\right\}$

7. Let
$$A = \left\{n^2 + \frac{1}{n^2} : n \in \mathbb{N}\right\}.$$
Determine the supremum and the infimum of A and establish if they are the maximum and the minimum of A, respectively.

8. Let
$$A = \left\{\frac{2n^2}{n+1} : n \in \mathbb{N}\right\}.$$
Determine the supremum and the infimum of A and establish if they are the maximum and the minimum of A, respectively.

9. Let
$$A = \left\{\frac{n^2+2n}{n+1} : n \in \mathbb{N}\right\}.$$
Determine the supremum and the infimum of A and establish if they are the maximum and the minimum of A, respectively.

10. Let
$$A = \left\{-1 - \frac{n}{n^2+1} : n \in \mathbb{N}\right\}.$$
Determine the supremum and the infimum of A and establish if they are the maximum and the minimum of A, respectively.

11. Let
$$A = \left\{x = |t^2 - 4t| : t \in (-1,5)\right\}.$$
Determine the supremum and the infimum of A and establish if they are the maximum and the minimum of A, respectively.

12. Let
$$A = \left\{x = \frac{|t|}{1+|t|} : t \in \mathbb{R}\right\}.$$
Determine the supremum and the infimum of A and establish if they are the maximum and the minimum of A, respectively.

13. Let α, $\beta \in \mathbb{R}\backslash\mathbb{Q}$ and a, $b \in \mathbb{Q}\backslash\{0\}$. Can you establish whether the following numbers are rational numbers or not?
$$\alpha+\beta; \quad a+b; \quad a+\alpha; \quad \alpha\cdot\beta; \quad \alpha\cdot a; \quad -\alpha; \quad \frac{1}{\alpha}$$

14. Prove that there does not exist any rational number c such that $c^2 = n_0$ for any n_0 which is a prime natural number (in other words, show that $\sqrt{n_0} \notin \mathbb{Q}$ for any prime natural number n_0).

15. Let m, $n \in \mathbb{N}$. Prove that if $\sqrt{m} \notin \mathbb{Q}$, then $\sqrt{n} + \sqrt{m} \notin \mathbb{Q}$ as well.

1. Prove by induction that

(a) $\forall n \in \mathbb{N}$ we have $n < 10^n$

(b) $\forall n \in \mathbb{N}$ we have $2^{n-1} \leq n!$ (and $2^{n-1} < n!\ \forall, n \geq 3$)

X a) $\forall N \in \mathbb{N}$ WE HAVE $N < 10^N$

I) BASE CASE: PROVE FOR 1

$N = 1 \quad 1 < 10$

II) INDUCTION: ASSUME TRUE FOR K

$S(N): N < 10^N$

$S(N+1): N+1 < 10^{N+1}$

$N+1 < 10^N + 10$

$N+1 < 10(10^{N-1} + 1)$

$N+1 < 10 \cdot 10^N$

$N+1 < 10^{N+1}$

(IF $a < b$, $A + c < b + c$ FOR ANY c)

(b) $\forall n \in \mathbb{N}$ we have $2^{n-1} \leq n!$ (and $2^{n-1} < n!\ \forall, n \geq 3$)

X b) $2^{N-1} \leq N!$

$N = 1;\ 2^0 \leq 1! \quad 1 \leq 1$

$N+1:\ 2^{N+1-1} \leq (N+1)!$

$2^N \leq N!(N+1)$

2. Prove by induction that for any $n \in \mathbb{N}$ we have

(a) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(b) $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

(c) $\displaystyle\sum_{k=1}^{n} (3k(k-1)) = n^3 - n$

b $\displaystyle\sum_{i=1}^{N} i^3 = \left(\frac{N(N+1)}{2}\right)^2 = \left(\frac{(N+1)(N+2)}{2}\right)^2$

$$\left(\frac{N(N+1)}{2}\right)^2 = \left(\frac{N^2+3N+2}{2}\right)^2$$

$$\left(\frac{N(N+1)}{2}\right)^2 = \frac{N^4+9N^2+4+4N^2+6N^3+12N}{2}$$

3. Let us consider the following subsets of $\mathbb{R}$

$$A = \{n \in \mathbb{N} : n \text{ is even}\}; \quad B = \{n \in \mathbb{N} : n < 12\}; \quad C = \{n \in \mathbb{N} : n \leq 12\}.$$

Determine the supremum and the infimum and establish if they are the maximum and the minimum, respectively, for the following sets: A, B, C, $A \cap B$, $A \cap C$, $A \cup C$, $A \backslash C$.

A: $\sup(\{n \in \mathbb{N} : n \text{ is even}\}) = \emptyset$

$\inf(\{n \in \mathbb{N} : n \text{ is even}\}) = 0$

B: $\sup(\{n \in \mathbb{N} : n < 12\}) = 12$

$\inf(\{n \in \mathbb{N} : n < 12\}) = 0$

C: $\sup(\{n \in \mathbb{N} : n \leq 12\}) = 12$

$\inf(\{n \in \mathbb{N} : n \leq 12\}) = 0$

$A \cap B(\{n \in \mathbb{N} : n \text{ is even}, n < 12\})$

$\max A \cap B: 12$

$\min A \cap B: 0$

$A \cap C(\{n \in \mathbb{N} : n \text{ is even}, n \leq 12\})$

$\max A \cap C: 12$

$\min A \cap C: 0$

$A \cup C(\{n \in \mathbb{N} : n \text{ is even or } n \leq 12\})$

$\max A \cup C: \emptyset$

$\min A \cup C: 0$

$A \setminus C(\{n \in \mathbb{N} : n \text{ is even not } n \leq 12\})$

$\max A \setminus C: \emptyset$

$\min A \setminus C: 12$

4. For the following subsets of $\mathbb{R}$, determine the supremum and the infimum and establish if they are the maximum and the minimum, respectively.

- $A = [-3, 5) \cup \{7, 8\}$
- $B = (0, \sqrt{2}]$
- $C = (0, \sqrt{2}] \cap \mathbb{Q}$

SUP A: 5 MAX A: 8

INF A: 7 MIN A: -3

SUP B: $\sqrt{2}$ MAX B: $\sqrt{2}$

INF B: 0 MIN B: $\emptyset$

SUP C: $\sqrt{2}$ MAX C: $\emptyset$

INF C: 0 MIN C: $\emptyset$

5. For the following subsets of $\mathbb{R}$, determine the supremum and the infimum and establish if they are the maximum and the minimum, respectively.

- $A = \{x \in \mathbb{R} : (1 - x)(x^2 - 3) > 0 \text{ and } (x + 2)(x - 5) \leq 0\}$
- $B = \{x \in \mathbb{R} : |x - 2| > 1 \text{ and } |x + 1| - 2 < 0\}$
- $C = \{x \in \mathbb{R} : |x + 1| + 1 > 0 \text{ and } x^3 - 27 < 0\}$
- $D = \{x \in \mathbb{R} : x^3 + 3 > 0 \text{ and } x^2 \leq 1/4\}$

A: $(1-x) > 0$

$x^2 - 3 > 0$

$-\sqrt{3}$ 1 $\sqrt{3}$

$+$ $+$ $-$ $-$

$+$ $-$ $-$ $+$

$+$ $-$ $+$ $-$

$(-\infty, -\sqrt{3}) \cup (1, \sqrt{3})$

$x + 2 \geq 0$

$x - 5 \geq 0$

-2 5

$-$ $+$ $+$

$-$ $-$ $+$

$+$ $-$ $+$

$[-2, 5]$

SUP A: 1

INF A: -2

MAX A: 5

MIN A: -2

B: $|x-2|>1$ $\quad$ $|x+1|-2<0$

$-1>x-2>1$ $\quad$ $|x+1|<2$

$1>x>3$ $\quad$ $-2<x+1<2$

$(-\infty,1)\cup(3,+\infty)$ $\quad$ $-3<x<1$

$(-3,1)$

sup B: 1

inf B: 3

max B: $\emptyset$

min B: $\emptyset$

C: $|x+1|>-1$ $\quad$ $x^3<27$

$1>x+1>-1$ $\quad$ $x<3$

$-1<x+1<1$

$-2<x<0$

$(-2,0)$ $\quad$ $(-\infty,3)$

sup C: 3

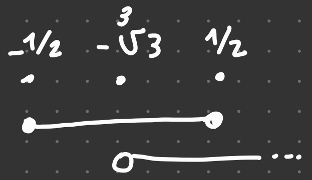
inf C: -2

max C: 3

min C: $\emptyset$

D: $x^3>-3$ $\quad$ $x^2\le\frac{1}{4}$

$x>-\sqrt[3]{3}$ $\quad$ $-\frac{1}{2}\le x\le\frac{1}{2}$



sup D: $\emptyset$

inf D: $-1/2$

max D: $\emptyset$

min D: $-1/2$

6. For the following subsets of $\mathbb{R}$, determine the supremum and the infimum and establish if they are the maximum and the minimum, respectively.

- $A = \mathbb{N} \cap \{x \in \mathbb{R} : \ x^2 + 8x \geq 6x\}$
- $B = \left\{x \in \mathbb{R} : \ x = \dfrac{(-1)^n}{n}, \ n \in \mathbb{N}\right\}$

A:

$x^2 + 2x \geq 0$

$x(x+2) \geq 0$

$-2 \quad 0$

$- \quad - \quad +$

$N \cap \quad - \quad + \quad +$

$+ \quad - \quad +$

$(0, +\infty)$

SUP A: $\emptyset$

INF A: 0

MAX A: $\emptyset$

MIN A: 0

B:

$x = \dfrac{(-1)^N}{N} \quad N \in \mathbb{N}$

$x = 1^N \cdot N^{-1}$

$x = \dfrac{1}{N}$

SUP A: 1

INF A: 0

MAX A: 1

MIN A: $\emptyset$

C: $\sqrt{x+5} > 3-x$

$x \geq 5$

$3-x \geq 0$; $x \leq 3$ AND $x+5 > 9+x^2-6x$; $x^2-7x+4<0$ $\qquad \frac{7 \pm \sqrt{33}}{2}$

$3-x<0$: $x>3$ AND $x \geq 5$

$\frac{7-\sqrt{33}}{2} \qquad 5 \qquad \frac{7+\sqrt{33}}{2}$

$- \quad + $

SUP C: $\emptyset$

INF C: 5

MAX C: $\emptyset$

MIN C: 5

D: $\frac{x+1}{\sqrt{4-x^2}} \leq 1$ $\qquad -2<x<2$

$x+1 \leq \sqrt{4-x^2}$

$x+1 \geq 0$; $x \geq -1$ AND $x^2+2x+1 \leq 4-x^2$ $\quad 2x^2+2x-3 \leq 0$ $\qquad \frac{-2 \pm \sqrt{28}}{4}$

$x+1<0$ $x<-1$ AND $-2<x$ $\qquad \frac{-2 \pm 2\sqrt{7}}{4} = \frac{-1 \pm \sqrt{7}}{2}$

SUP D: $\frac{-1+\sqrt{7}}{2}$

INF D: $\frac{-1-\sqrt{7}}{2}$

MAX D: 2

MIN D: -2

7. Let

$$A = \left\{ n^2 + \frac{1}{n^2} : n \in \mathbb{N} \right\}.$$

Determine the supremum and the infimum of A and establish if they are the maximum and the minimum of A, respectively.

$$\frac{n^4+1}{n^2}$$

sup A: $\emptyset$

inf A: 1

max A: $\emptyset$

min A: 1