$+$ $\times$

$-$ $\div$

DIMENSIONALITY ANALYSIS AND SCALING ARGUMENT

+ TORICELLI + CENTRIPEDAL FORCE

- CAN ALWAYS BE DONE → JUST CHECK CONSISTENCY
- USE TO CHECK RESULT OF CALCULATION AND UNIT OF MEASUREMENT
- USE TO GUESS THE RESULT → REYLEIGTH ALGORITHM

$\{M, L, T\}$ ⇒ USE TO SOLVE EVERYTHING

MASS, LENGTH, TIME

THE DIMENSIONS OF THESE...

$$\left[\frac{dA}{dB}\right] = \frac{[A]}{[B]}$$

DIMENSION OF A, DIMENSION OF B

EXERCISE Assuming $\{M, L, T\}$ (mass, length, time) as the set of fundamental dimensions, and $\{F, V, \rho\}$ as the set of relevant physical quantities, use Rayleigh's method to find an expression for the energy density ϵ (i.e., energy per unit volume). F is force, V is volume, ρ is volumetric density.

FORCE, VOLUME, VOLUMETRIC DENSITY

$$\text{ENERGY DENSITY} = F^{\alpha} V^{\beta} \rho^{\gamma} = M^{\alpha}\frac{L^{\alpha}}{T^{2\alpha}} L^{3\beta} \frac{M^{\gamma}}{L^{3\gamma}} = M^{\alpha+\gamma} L^{\alpha+3\beta-3\gamma} T^{-2\alpha}$$

(FORZA · SPOSTAMENTO)

ENERGY

$$\frac{FL}{L^3} = \frac{F}{L^2} = \frac{ML}{L^2T^2} = \frac{M}{L^2T^2}$$

$$\frac{M^1}{L^2 T^2} = M^{\alpha+\gamma} L^{\alpha+3\beta-3\gamma} T^{-2\alpha}$$

$$\begin{cases} 1 = \alpha+\gamma \\ -2 = \alpha+3\beta-3\gamma \\ -2 = -2\alpha \end{cases} \Rightarrow \{1, -2/3, 0\}$$

POWER AND EXPONENTIAL LAW

$y = cx^{\alpha}$ ← SCALE INVARIANT

- $\alpha < 1$ SUBLINEAR
- $\alpha = 1$ LINEAR
- $\alpha > 1$ SUPERLINEAR

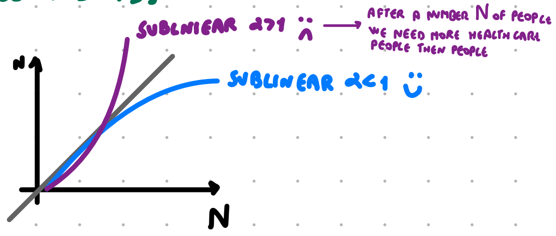
IF I CHANGE THE SCALE OF Y AND X AT THE SAME TIME I CAN GET BACK EXACTLY THE SAME PLOT

$y = ce^{\beta x}$ ← CHARACTERISTIC SCALE $x_0 = \frac{1}{\beta}$

SCALING ARGUMENT USES

- PREDICTING $C = AB$ → PREDICTING THE SCALE OF SOMETHING YOU DON'T KNOW — TO PREDICT THE SCALE OF A NEW QUANTITY GIVEN THE SCALE OF KNOWN QUANTITIES
- PREDICT EXISTENCE OF BOUNDS
- MODELLING

POPULATION OF A CITY N ≥ PERSON THAT WORK IN HEALTHCARE $n \simeq N^{\alpha > 1}$

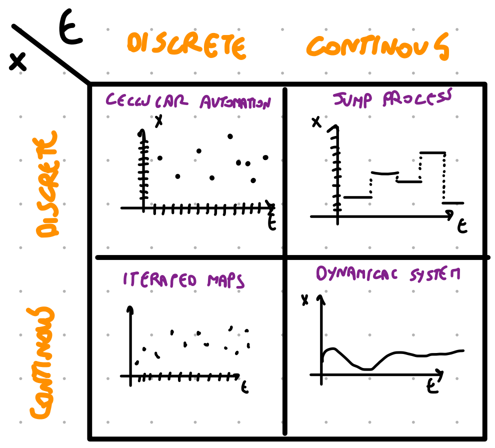


+ COW + NAIL EXAMPLE

DYNAMICAL SYSTEM

→ GOAL: TRY TO PREDICT THE FUTURE STATE OF A SYSTEM BASING ON THE PRESENT

COMPUTE MOD 37 mod 4
1. $\frac{37}{4} = 9.25$
2. SUBTRACT 9
3. MULTIPLY BY 4 ⇒ $0.25 \cdot 4 = \boxed{1}$



- CELLULAR AUTOMATON

CONSTRAINTS

1. NO — WOULD HAVE SOMETHING UNDETERMINED
2. NO — NO COMING OUT ARROW ARE NOT ALLOWED

ONE AND ONLY ONE ARROW GOING OUT

REVERSIBILITY: A GRAPH THAT CAN BE REVERSET AND THAT ITS REVERSE DOES NOT VIOLATE ANY RULE. IT IS A STRONG CONSTRAINT

ALLOWED ONLY TO CYCLES

T, T

EXERCISE Consider the dynamical system $x_{n+1} = f(x_n)$, with

$$f(x) = \sin(\pi x)$$

The dynamics has 2 fixed points. Linearize around the smaller fixed point $\bar{x}$. Regarding the stability of $\bar{x}$, which one of the following 4 possibilities is realized?

(a) $\bar{x}$ is stable (not a spiral)

(b) $\bar{x}$ is a stable spiral

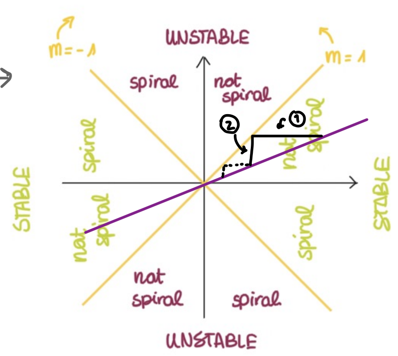
(c) $\bar{x}$ is unstable (not a spiral)

(d) $\bar{x}$ is an unstable spiral

1. FIXED POINT
   - $\bar{x} = 0$
   - $\bar{x} = \sim$
2. $S(x) = \sin(\pi x) \Rightarrow S'(x) = \pi\cos(\pi x)$
3. $S'(0) = \pi \Rightarrow$ UNSTABLE NOT SPIRAL

- CONTINOUS VALUES $x \in \mathbb{R}$ $x_{n+1} = S(x_n)$
- WE NEED TO CONSIDER THE STABILITY OF A FIXED POINT $\bar{x} = S(\bar{x})$

1. FIND FIXED POINT $\bar{x} = S(\bar{x})$
2. LINEARIZE THE SYSTEM AROUND THE FIXED POINT $S'(\bar{x}) = \sim$
3. CHECK ON COBWEB PLOT →



1° FROM LINE TO $\boxed{m=1}$ HORIZONTALLY

2° ⊥ GO BACK TO LINE

3° REPEAT 1°

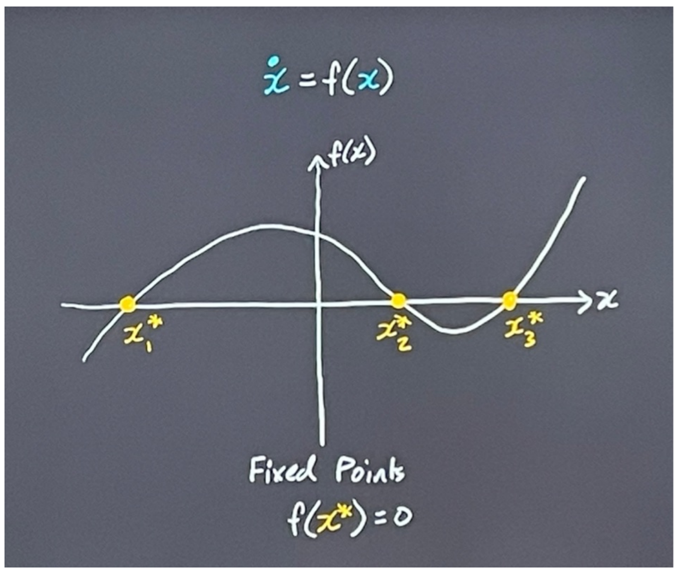
CONTINOUS DYNAMICAL SYSTEM

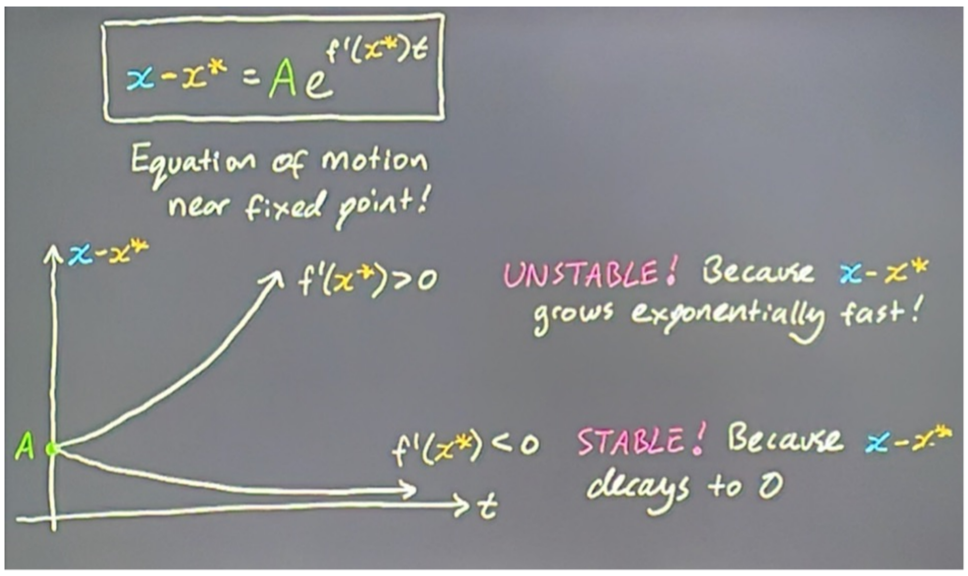
x, t CONTINOUS
- $t \in [t_0, +\infty) \in \mathbb{R}$
- $x \in \mathbb{R}^D$ → DEGREE OF FREEDOM

$$S(x(t)) = \dot{x}(t) = \frac{d}{dt}x(t) \;???$$

LOGISTIC MAP ???

$$x_{n+1} = rx_n(1-x_n)$$





5 Consider the dynamical system (with continuous time and continuous state) specified by the following ordinary differential equation:

$$\dot{x}(t) = \cos\left(\frac{\pi}{2}x(t)\right)$$

Which one of the following statements is correct?

1. The point $x = 0$ is a fixed point and it is stable
2. The point $x = 0$ is a fixed point and it is unstable
3. The point $x = 1$ is a fixed point and it is stable
4. The point $x = 1$ is a fixed point and it is unstable

- FIND FIXED POINT: $f(x) = 0$

$$0 = \cos\left(\frac{\pi}{2}x(t)\right) \Rightarrow x = 1, 3, 5, 7, \dots$$

- DERIVATE $f'(x) = \sim$

$$f'(x) = -\frac{\pi}{2}\sin\left(\frac{\pi}{2}x(t)\right)$$

- SUBSTITUTE THE FIXED POINT

$$f'(1) = -\frac{\pi}{2}\sin(\pi/2) = -\pi/2$$

- CHECK RESULT
  - IF $f'(\bar{x}) > 0$ UNSTABLE
  - IF $f'(\bar{x}) < 0$ STABLE

SINCE $-\pi/2 < 0$ THEN $x = 1$ IS A STABLE FIXED POINT

PRINCIPLE OF STATIONARY ACTION

$$S = \underbrace{\int_{T_1}^{T_2} L(x, \dot{x}, t)\, dt}$$

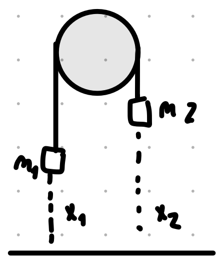
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

$\boxed{\text{LAGRANGIAN}}$ $\frac{d}{dt}\partial_{\dot{x}} L(x, \dot{x}) = \partial_x L(x, \dot{x})$ $L = K - U$

KINETIC

POTENTIAL

ATWOOD MACHINE



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

$C_1 = x_1 + x_2$; $x_2 = C_1 - x_1 \Rightarrow \dot{x}_2 = -\dot{x}_1$ $\dot{x}_2^2 = \dot{x}_1^2$

$U_1 = m_1 g x_1$ $K_1 = \frac{1}{2}m_1 v_1^2$

$U_2 = m_2 g x_2 = m_2 g(C_1 - x_1)$ $K_1 = \frac{1}{2}m_2 v_2^2$

$$L = K - U = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - m_1 g x_1 - m_2 g(C_1 - x_1) \quad L = \frac{1}{2}\dot{x}_1^2(m_1 + m_2) - m_1 g x_1 - m_2 g C_1 + m_2 g x_1$$

$\dot{x}_1^2$

$$L = \frac{1}{2}\dot{x}_1^2(m_1 + m_2) - g(m_1 - m_2)x_1 - m_2 g C_1$$

$$\frac{d}{dt}\left((m_1 + m_2)\dot{x}_1\right) = -g(m_1 - m_2) \quad (m_1 + m_2)\ddot{x} = -g(m_1 - m_2) \quad \ddot{x} = -g\frac{(m_1 - m_2)}{(m_1 + m_2)}$$

$\boxed{\text{THERMODYNAMICS}}$

A THERMODYNAMIC IS A MACROSCOPIC SYSTEM WITH A LARGE DEGREE OF FREEDOM (DOF)

SPECIFIC HEAT

$$\Delta Q = mC\Delta T$$

- TEMPERATURE $\longrightarrow$ FLUCTUATION
- ENTROPY $\longrightarrow$ ~~DISORDER~~ $\longrightarrow$ WHAT IS THE MOST LIKELY OUTCOME?

MEASURE HOW THE SYSTEM FLUCTUATES, HOW FAR FROM THE AVERAGE VALUE IT GETS

- MEASURE OF THE DISORDER OF THE SYSTEM
- ORGANIZE SYSTEM $\Rightarrow$ LOW ENTROPY

GLASS OF WATER ↑

GLASS OF ICE ↓

WE CAN CHOOSE WATER AND WE DECIDE THAT IT BOILS AT A PRECISE "VALUE" (100) AND IT FREEZES AT 0 (CALIBRATION)

$$\text{WATER}\begin{cases}\text{BOILS } 100°C\\ \text{FREEZE } 0°C\end{cases} \qquad T = \alpha\tilde{T} + b \begin{cases}100 = \alpha\tilde{T}_B + b\\ 0 = \alpha\tilde{T}_F + b\end{cases}$$

BOILING AND FREEZING TEMPERATURE INVENTED BY ME

CONSERVATION OF ENERGY $\Delta Q = 0$

$\tilde{T} = PV$ $T \propto PV$ $0°C = 273.15\,K$

- A SYSTEM IS IN THERMAL EQUILIBRIUM WHEN IT'S TEMPERATURE IS CONSTANT IN TIME $\longrightarrow$ CAN DEFINE T
- TWO SYSTEMS ARE IN THERMAL EQUILIBRIUM IF THEY HAVE SAME T

- JOULE'S EXPERIMENT: PROVE THAT HEAT IS A ENERGY BY CONVERTING MECHANICAL ENERGY INTO HEAT

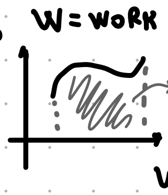
BOLTZMANN CONSTANT $= 1.38 \cdot 10^{-23}\,\frac{J}{K} = \left[\frac{m^2\,kg}{s^2\,K}\right]$

$$\boxed{K_B = \frac{PV}{TN}}$$

MACROSCOPIC THERMODYNAMIC W = WORK DONE BY THE SYSTEM

- ISOTHERMAL $W_{A\to B} = \int_{V_A}^{V_B} P(V)\,dV$

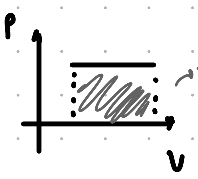
CONSTANT TEMPERATURE



NO INFORMATION (∞ THE CLASSICAL INTEGRAL)

- ISOBARIC $W_{A\to B} = (V_B - V_A) \cdot P$

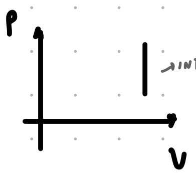
CONSTANT PRESSURE



WORK IS (BASE x ALTEZZA)

- ISOCHORIC $W_{A\to B} = \int_{V_0}^{V_0} P(V)\,dV = 0$

CONSTANT VOLUME



INTEGRAL IS 0 $W = 0$

1° LAW OF THERMODYNAMIC

$$\Delta Q - \Delta W = dE$$

WORK DONE BY THE SYSTEM (NOT AN EXACT DIFFERENTIAL)

QUANTITY OF HEAT ABSORBED BY A SYSTEM (NOT AN EXACT DIFFERENTIAL)

INTERNAL ENERGY STORED IN THE SYSTEM

$$\int_A^B dE = E_B - E_A$$

FIXED POINTS: ALL THE CYCLE WITH PERIOD 1

CYCLE: LOOP OF REPEATING STATES

TRANSIENT STATE: NOT VISITED INFINITELY MANY TIMES

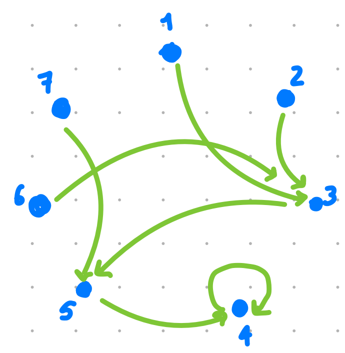
RECURRENT STATE: VISITED INFINITELY MANY TIMES

CONNECTED COMPONENTS: SUBSET OF THE GRAPH

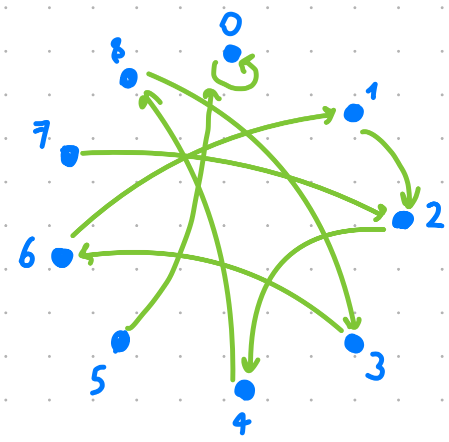
CONSERVED QUANTITY: A FUNCTION THAT LEADS TO 1 OR MORE SPECIFIC STATE

TRIVIAL CONSERVED QUANTITY: A FUNCTION THAT LEADS TO THE SAME VALUE FOR ALL CONNECTED COMPONENTS

- TRANSIENT STATE + RECURRENT STATE = ALL STATES
- IF IT "GOES OUT" OF THE STATES IT IS NOT A DYNAMICAL SYSTEM
- ALL THE STATES MUST BE IN ONE CONNECTED COMPONENT. IF TWO STATES TOUCHES EACHOTHER THEY BELONGS TO THE SAME CONNECTED COMPONENT
- THERE IS ALWAYS AT LEAST A CONNECTED COMPONENT (CAN BE THE TOTAL GRAPH)



FIXED POINTS: STATE 4
CYCLE: THE ONE WITH ONLY 4
TRANSIENT STATE: 1, 2, 3, 5, 6, 7
RECURRENT STATE: STATE 4
CONNECTED COMPONENTS: THE GRAPH ITSELF



FIXED POINTS: $\{0\}$
CYCLES: $\{0\}$ $\{8,3,6,1,2,4,8\}$
TRANSIENT STATE: $\{5,7\}$
RECURRENT STATE: $\{1,2,3,4,6,8\}$
CONNECTED COMPONENTS: $\{5,0\}$ $\{7,2,4,8,3,6,1,2\}$

- **FORCE:** $Kg\frac{m}{s^2} \Rightarrow \boxed{\frac{ML}{T^2}}$
- **ENERGY/WORK:** $J = F\Delta S = Kg\frac{m}{s^2}\cdot m \Rightarrow \boxed{\frac{ML^2}{T^2}}$
- **PRESSURE:** $P = \frac{F}{A} = \frac{Kg\frac{m}{s^2}}{m^2} = Kg\frac{1}{ms^2} \Rightarrow \boxed{\frac{M}{LT^2}}$
- **DENSITY:** $\frac{Kg}{m^3} \Rightarrow \boxed{\frac{M}{L^3}}$
- **ENERGY DENSITY:** (ENERGY PER UNIT VOLUME) $\frac{J}{m^3} \Rightarrow \frac{ML^2}{T^2L^3} = \boxed{\frac{M}{T^2L}}$
- **VOLUMETRIC DENSITY:** $\Rightarrow \boxed{\frac{M}{L^3}}$
- **POWER:** $\frac{E}{T} \Rightarrow \boxed{\frac{ML^2}{T^3}}$