$+$ $\times$

$-$ $\div$

- $\lim_{x \to +\infty} \left(\sin(x) - \frac{2x}{3x^2 - \cos x} \right)$ DOES NOT EXIST

$$\frac{2x}{3x^2 - \cos} = \overbrace{\underbrace{\frac{x}{x^2}}_{0} \left(\frac{2}{3 - \underbrace{\frac{\cos(x)}{x^2}}_{0}} \right)}^{\to 0 \cdot \frac{2}{3} = 0}$$

$\underbrace{\quad 0 \quad}_{2/3}$

$\sin(x)$ NO LIMIT

- $\lim_{x \to 0} \frac{\sqrt{1-x^2} - 1}{x^2}$

$$\frac{\sqrt{1+x^2} - 1}{x^2} = \frac{\left(\sqrt{1+x^2} - 1\right)}{x^2} \frac{\left(\sqrt{1+x^2} + 1\right)}{\left(\sqrt{1+x^2} + 1\right)}$$

$$\frac{\cancel{1} + \cancel{x^2} \cancel{- 1}}{\cancel{x^2}\left(\sqrt{1+x^2} + 1\right)} = \frac{1}{\sqrt{1+x^2} + 1} = \frac{1}{2}$$

- $\lim_{x \to +\infty} \left(1 + \frac{1}{2x^3}\right)^{x^4}$ $\qquad$ $\lim_{x \to +\infty} \left(1 + \frac{1}{2x^3}\right)^{x^2}$

$\downarrow$

$$\left(\underbrace{\left(1 + \frac{1}{2x^3}\right)^{2x^3}}_{\downarrow\, e}\right)^{x/2 \to +\infty} \to +\infty \qquad x \to +\infty$$

$$\left(\underbrace{\left(1 + \frac{1}{2x^3}\right)^{2x^3}}_{e}\right)^{\frac{1}{2x}} \qquad \frac{1}{2x} \to 0$$

$\to e^0 = 1$

$$\left(1+\frac{1}{2x^3}\right)^{x^K} = \qquad K \in \mathbb{N}$$

$$= e^{\log\left(1+\frac{1}{2x^3}\right)x^K}$$

$$\frac{\log\left(1+\frac{1}{2x^3}\right)}{\frac{1}{2x^3}} \cdot \frac{1}{2x^3} x^K \qquad x \to +\infty$$

$\to 1$

$$\frac{1}{2}x^{K-3} \to \begin{cases} +\infty & K>3 \\ \frac{1}{2} & K=3 \\ 0 & 0<K<3 \end{cases}$$

$$\frac{\log(1+x)}{x} = 1$$

$$\left(1+\frac{1}{2x^3}\right)^{x^K} \to \begin{cases} +\infty & K>3 \\ e^{1/2} = \sqrt{e} & K=3 \\ e^0 = 1 & 0<K<3 \end{cases}$$

- $\lim\limits_{x \to 0^+} x^{\sin(x)}$

$$x^{\sin(x)} = e^{\log(x)\sin(x)}$$

$$\log(x)\sin(x) = \underbrace{\log(x)\,x}_{\xrightarrow{x\to 0}\ 0}\ \underbrace{\frac{\sin(x)}{x}}_{\xrightarrow{x\to 0^+}\ 1} \to 0 \quad x \to 0^+$$

- $$\lim_{x\to 0^+} x^{\sin\left(1/\log(3x)\right)}$$

$$e^{\log(x)\sin\left(1/\log(3x)\right)}$$

$$\log(x)\sin\left(1/\log(3x)\right)$$

$$\log(x)\underbrace{\frac{\sin\left(1/\log(3x)\right)}{1/\log(3x)}}_{\xrightarrow{x\to 0^+}\ 1}\cdot\frac{1}{\log(3x)} \xrightarrow{x\to 0^+} 1$$

$$\log(3x) = \log(3) + \log(x)$$

- $$\lim_{x\to 0} \frac{e^{x^3}-1}{\sin(x\sin(x))}$$

$$\underbrace{\frac{e^{x^3}-1}{x^3}}_{1}\cdot\underbrace{\frac{x^3}{\sin(x\sin(x))}\cdot\frac{x\sin(x)}{x\sin(x)}}_{1}\cdot\frac{x}{x} = \frac{x^3}{x\cdot x} \xrightarrow{x\to 0} 0$$

$$1$$

• EXERCISE 1)

$$f(x) = \begin{cases} a e^{x} & x \leq 0 \\ \dfrac{\sin(x^2)}{2x^2+x^3} & x > 0 \end{cases}$$

FOR WHICH $a \in \mathbb{R}$ f IS CONTINOUS ON $\mathbb{R}$?

• f IS CONTINUOUS ON $\mathbb{R} \setminus \{0\}$ $\forall a \in \mathbb{R}$

• IN 0?

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) \, ?$$

$$\downarrow$$

$$f(0) = a$$

$$\lim_{x \to 0^-} a e^{x} = a$$

$$\lim_{x \to 0^+} \frac{\sin(x^2)}{2x^2+x^3}$$

$x \to 0$ $\quad 2x^2+x^3 \quad x^2(2+\overset{\overset{0}{\uparrow}}{x})$ $\quad x \to 0$

$$\frac{\sin(x^2)}{x^2(2+x)} \qquad \frac{\sin(x^2)}{x^2} \to 1 \qquad (2+x) \xrightarrow{x \to 0} 2$$

$$\boxed{a = \frac{1}{2}}$$

IF $a \neq \frac{1}{2}$ $f(x)$ IS NOT CONTINOUS

2) $f(x) = \begin{cases} \dfrac{e^{x+1}-1}{x+1} & x > -1 \\ a x^2 - 5 & x \leq -1 \end{cases}$ $\quad a \in \mathbb{R}$

$\exists\, a \in \mathbb{R}$ SUCH THAT f IS CONTINUOUS IN $\mathbb{R}$?

f IS CONTINUOUS IN x $\forall\, x \neq -1$

IN -1? $\quad \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to -1^+} f(x) = f(-1)$?

$f(-1) = \lim\limits_{x \to -1^-} f(x) = a - 5$

$\lim\limits_{x \to -1^+} f(x) = \lim\limits_{x \to -1^+} \dfrac{e^{x+1}-1}{x+1} \qquad \lim\limits_{y \to 0^+} \dfrac{e^y - 1}{y} = 1$

$x \to -1^+ \quad y = x + 1 \to 0^+$

$a - 5 = 1 \Rightarrow a = 6$

THEOREM ON CONTINUOUS FUNCTION

WEIERSTRASS THEOREM

LET $C \subseteq \mathbb{R}^N$ CLOSED AND BOUNDED

[FOR EXAMPLE, $C = [a,b] \subseteq \mathbb{R}$ $a, b \in \mathbb{R}$ $a < b$]

LET $f : C \to \mathbb{R}$ CONTINUOUS THE

$$\exists \max_{x \in C} f(x) \quad \text{AND} \quad \exists \min_{x \in C} f(x)$$

THAT IS $\exists\, x_{MAX} \in C$ AND $\exists\, x_{MIN} \in C$ SUCH THAT

$$f(x_{MAX}) = \max_{x \in C} f(x) = \max f(C)$$

$$f(x_{MIN}) = \min_{x \in C} f(x)$$

NOTATION

x_{MIN} IS CALLED A MINIMIZER

x_{MAX} IS CALLED A MAXIMISER

PROOF

JUST THE MINIMUM CASE

WE PROVE $\exists \{x^N\}_{N \in \mathbb{N}} \subseteq C$ SUCH THAT

$\lim_N f(x^N) = \inf_C f$ (MINIMIZING SEQUENCE)

TWO CASES

i) $\inf_C f = -\infty \Rightarrow \forall N \in \mathbb{N} \ \exists x^N \in C$ SUCH THAT $f(x^N) < -N$

$\Rightarrow \lim_N f(x^N) = -\infty$

ii) $\inf_C f = l \in \mathbb{R} \Rightarrow \forall N \in \mathbb{N} \ \exists x^N \in C$

SUCH THAT $l \leq f(x^N) < l + \frac{1}{N}$

$\Rightarrow \lim_N f(x^N) = l$

C BOUNDED $\Rightarrow \{x^N\}_{N \in \mathbb{N}}$ BOUNDED

BOLZANO-WEIERSTRASS THEOREM

(TRUE IN $\mathbb{R}^N$ AS WELL)

$\exists \{x^{N_K}\}_{K \in \mathbb{N}}$ CONVERGING, THAT IS $\exists \bar{x} \in \mathbb{R}^N$ SUCH THAT

$\lim_K x^{N_K} = \bar{x} \in \mathbb{R}^N$

C CLOSED $\Rightarrow \bar{x} \in C$

$f(x^n) \to \inf_C f \Rightarrow f(x^{n_k}) \xrightarrow[k\to\infty]{} \inf_C f$

- $x^{n_k} \to \bar{x} \overset{\text{CONTINUITY}}{\Rightarrow} f(x^{n_k}) \to f(\bar{x})$

UNIQUENESS OF THE LIMIT $\Rightarrow f(\bar{x}) = \inf_C f$

$\Rightarrow \bar{x} = x_{MIN}$ AND $\inf_C f = \min_C f \in \mathbb{R}$

EXISTENCE OF ZEROS THEOREM

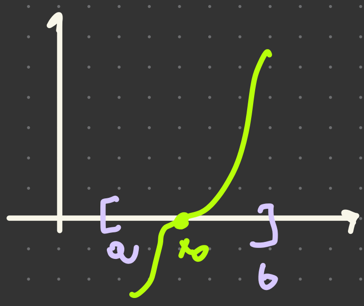
LET $a, b \in \mathbb{R}$ WITH $a < b$ LET $f : [a, b] \to \mathbb{R}$ CONTINUOUS

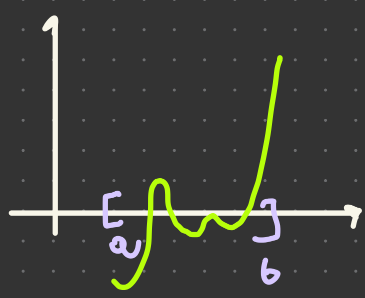
IF $f(a) \cdot f(b) \leq 0$, THEN $\exists x_0 \in [a, b]$ SUCH THAT

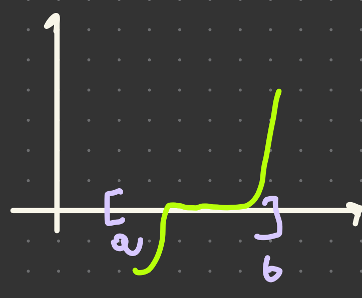
$f(x_0) = 0$

x_0 IS CALLED A ZERO FOR THE FUNCTION f

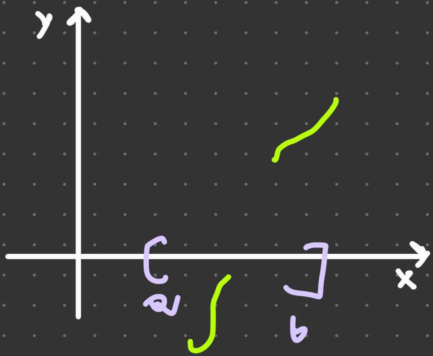
REMARKS $\exists$ AT LEAST ONE ZERO, THERE CAN BE MANY



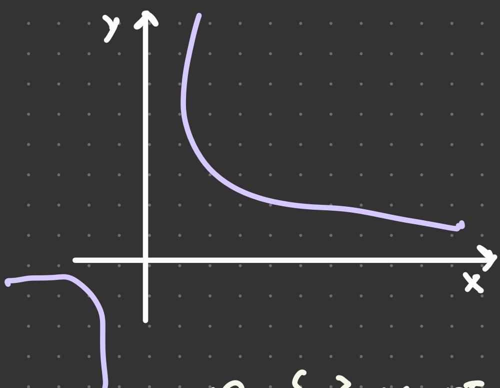




- CONTINUITY IS ESSENTIAL



$[a,b]$ IS AN INTERVAL. THIS IS ESSENTIAL!



$f(-1) = 1 < 0$

$f(1) = 1 > 0$

BUT f HAS **NO** ZERO

$\mathbb{R} - \{0\}$ IS NOT AN INTERVAL

f IS CONTINUOUS ON $\mathbb{R} - \{0\}$

$f(a) \cdot f(b) = 0 \Rightarrow$ EITHER $f(a) = 0$ OR $f(b) = 0$

INTERESTING CASE iJ

$f(a)\cdot f(b)<0 \Rightarrow \exists x_0 \in (a,b)$ SUCH THAT

$f(x_0)=0$

CONNECTION THEOREM

$I \subseteq \mathbb{R}$ INTERVAL

$f: I \subseteq \mathbb{R} \to \mathbb{R}$ CONTINUOUS

THEN $f(I)$ IS AN INTERVAL

IN PARTICULAR

$$\left(\inf_I f, \sup_I f\right) \subseteq f(I)$$

IS CONTAINED IN THE IMAGE

PROOF

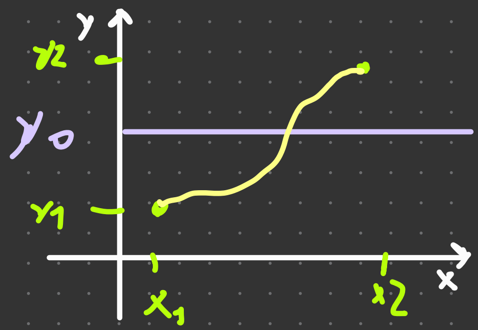
$y_1, y_2 \in f(I)$. ASSUME THAT $y_1 < y_2$

WE WANT TO SHOW THAT THAT $\forall y$ SUCH THAT $y_1 < y_0 < y_2$ THERE EXIST $x \in I$ SUCH THAT $f(x_0) = y_0$

I KNOW $\exists x_1 \in I$ SUCH THAT $f(x_1) = y_1$

$\exists x_2 \in I$ SUCH THAT $f(x_2) = y_2$

FOR SIMPLICITY, ASSUME $x_1 < x_2$



$f(x_0) = y_0 \iff g(x) = f(x) - y_0 = 0$

AND g IS CONTINUOUS AND $g(x_1) = f(x_1) - y_0 = y_1 - y_0 < 0$

$g(x_2) = f(x_2) - y_0 = y_2 - y_0 > 0$

APPLY ZERO THEOREM TO g ON $[x_1, x_2] \subseteq I$

DARBOUX THEOREM

LET $a, b \in \mathbb{R}$, $a<b$

LET $f:[a,b] \to \mathbb{R}$ CONTINUOUS. THEN

$$f([a,b]) = [m, M]$$

WHERE $m = \min_{[a,b]} f$ AND $M = \max_{[a,b]} f$

PROOF
- BY CONN. THEOREM $f([a,b])$ INTERVAL
- BY WEIERSTRASS $\exists m = \min_{[a,b]} f \in \mathbb{R}$ AND $\exists M = \max_{[a,b]} f$

CLEARLY $m, M \in f([a,b])$ SO $[m, M] \subseteq f([a,b])$

BUT $\forall x \in [a,b]$ $m \leq f(x) \leq M$

$\Rightarrow f([a,b]) \subseteq [m, M]$

CONTINUITY AND MONOTONICITY

THEOREM I interval $f: I \to \mathbb{R}$ monotone

f continuous $\iff$ $f(I)$ is an interval

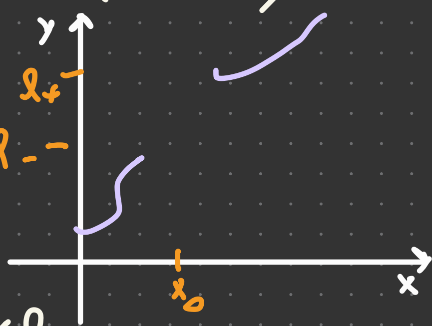
REMARK "$\Rightarrow$" always true, even without the monotonicity assumption

The interesting implication is "$\Leftarrow$" that is

monotone + $f(I)$ interval $\Rightarrow$ f cont

IDEA By contradiction f monotone ↗

not continuous in $x_0 \in \mathring{I}$



$\exists \lim_{x \to x_0} f(x) = l_\pm$

and $l_- \leq f(x_0) \leq l_+$

so f not cont. in $x_0 \Rightarrow l_- < l_+$

$y \notin f(I)$ $\Rightarrow$ $f(I)$ is not an interval

THEOREM

I interval $f: I \to \mathbb{R}$ continuous

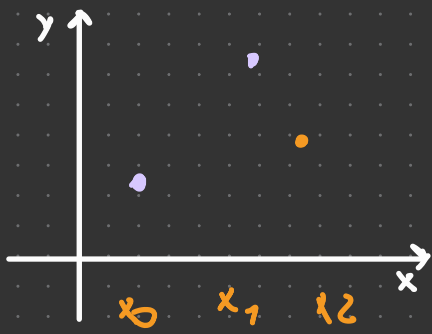
f injective $\Leftrightarrow$ f is strictly monotone

REMARK "$\Leftarrow$" always true, even if domain an interval and continuity not needed

Interesting application is "$\Rightarrow$" that is

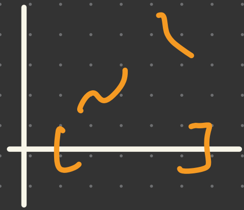
I interval, f continuous and injective $\Rightarrow$ f strictly monotone

IDEA OF THE PROOF By contradiction



$I \supseteq [x_0, x_2$

REMARK CONTINUITY IS ESENTIAL



INTERVAL IS ESSENTIAL $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$

$f(x) = \frac{1}{x}$ IS INJECTIVE, CONTINUOUS BUT IS NOT STRICTLY MONOTONE

$\mathbb{R} \setminus \{0\}$ IS NOT STRICTLY MONOTONE

$\mathbb{R} - \{0\}$ IS NOT AN INTERVAL

THEOREM

$I \subseteq \mathbb{R}$ INTERVAL

$f: I \subseteq \mathbb{R} \to \mathbb{R}$ CONTINUOUS IS INJECTIVE

THEN IT IS INVERTIBLE, THAT IS

$\exists f^{-1}: f(I) \to I$ INVERSE FUNCTION

THEN f^{-1} IS CONTINUOUS

PROOF

$f(I) = J$ IS AN INTERVAL. (CONNECTED)

∘ f STRICTLY MONOTONE $\Rightarrow f^{-1}$

INCOMPLETE

CONSEQUENCES

- WE KNOW $0^n = 0$ AND $\lim_{x \to +\infty} x^n = +\infty$ $\forall n \geq 2$??

SO $f_n(x) = x^n$, WE HAVE

$$f_n([0,+\infty)) = [0,+\infty]$$

$\Rightarrow \exists f_n^{-1} : [0,+\infty) \to [0,+\infty]$ CONTINUOUS

$$f_n^{-1}(x) = \sqrt[n]{x} \quad \forall x \geq 0$$

WE KNOW $e^x > 0$ $\forall x \in \mathbb{R}$ AND

$\lim_{x \to -\infty} e^x = 0$ AND $\lim_{x \to +\infty} e^x = +\infty$

$\Rightarrow \exp(\mathbb{R}) = (0,+\infty)$ SINCE EXP IS CONTININUOUS

$\Rightarrow \exists \exp^{-1} = \log : (0,+\infty) \to \mathbb{R}$ CONTINUOUS

ANALOGOUSLY, WE CAN SHOW THAT

$$\cos([0,\pi]) = [-1,1], \quad \sin([-\pi/2, \pi/2]) = [-1,1]$$

AND $\tan\left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right) = \mathbb{R}$

$\Rightarrow$ $\exists$ AND ARE CONTINUOUS THE RESPECTIVE INVERSE FUNCTION, THAT IS,

$\arccos : [-1,1] \to [0,\pi]$ CONT

$\arcsin : [-1,1] \to [-\pi/2, \pi/2]$ CONT

$\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ CONT

EXAMPLE

$$\lim_{x \to 0} \frac{\arcsin(x)}{x} = 1$$

$x \to 0$, WITH $x \neq 0$ THEN $y = \arcsin(x) \to 0$ WITH $y \neq 0$

TAKE $y = f(x) = \arcsin(x)$; $g(y) = \frac{y}{\sin(y)} \xrightarrow{y \to 0} 1$

THEN $\lim_{x \to 0} g(f(x)) = \lim_{y \to 0} g(y) = 1$

$$g(f(x)) = \frac{\arcsin(x)}{\sin(\arcsin(x))} = \frac{\arcsin(x)}{x}$$

ANALOGOUSLY, WE CAN SHOW $\lim_{x \to 0} \frac{\arctan(x)}{x} = 1$

SINCE $\lim_{y \to 0} \frac{\tan(y)}{y} = \lim_{y \to 0} \frac{y}{\tan(y)} = 1$

DERIVATES

VELOCITY: LET US CONSIDER A POINT MOVING ALONG THE X-AXIS. ITS POSITION AT TIME $t \in \mathbb{R}$ IS $s(t) \in \mathbb{R}$

LET $t \in \mathbb{R}$ AND $h > 0$. AT TIME $(t+h)$ THE POINT IS AT $s(t+h)$. THE MEAN VELOCITY IN THE INTERVAL $[t, t+h]$ IS

$$\frac{s(t+h) - s(t)}{h}$$

IT IS THE VELOCITY THE POINT SHOULD HAVE IF IT MOVED WITH CONSTANT SPEED (UNIFORM LINEAR MOTION) TO GO FROM $s(t)$ TO $s(t+h)$ IN TIME h

IF $h < 0$

$$\frac{s(t+h)-s(t)}{h} = \frac{s(t)-s(t+h)}{-h} = \frac{s(t)-s(t-|h|)}{|h|}$$

MEAN VELOCITY IN THE INTERVAL $[t-|h|, t]$

LET $t \in \mathbb{R}$, $h \neq 0$, DEFINE

$$\frac{s(t+h)-s(t)}{h}$$ DIFFERENT QUOTIENT

THE ISTANTANEOUS VELOCITY AT TIME $t \in \mathbb{R}$ IS
IF IT EXIST AND IT IS FINITE

$$\lim_{h \to 0} \frac{s(t+h)-s(t)}{h}$$

IN THIS CASE WE DEFINE

$$v(t) = \lim_{h \to 0} \frac{s(t+h)-s(t)}{h} \in \mathbb{R}$$ VELOCITY AT TIME $t \in \mathbb{R}$

ANALOGOUSLY, $p(t)$ QUANTITY CHANGING WITH TIME
$t \in \mathbb{R}$, $h > 0$

$$\frac{P(t+h)-P(t)}{h}$$

MEAN RATE OF CHANGE PER TIME UNIT OF P IN THE INTERVAL $[t, t+h]$

>0 P IS INCREASING FROM t TO $t+h$

<0 P IS DECREASING FROM t TO $t+h$

IF IT EXIST AND IT IS FINITE

$$\lim_{h\to 0} \frac{P(t+h)-P(t)}{h} \in \mathbb{R}$$

THIS IS THE RATE OF CHANGE OF P AT TIME $t \in \mathbb{R}$

DEFINITION

LET $I=(a,b)$ WITH

$-\infty \leq a < b \leq +\infty$ BE AN OPEN INTERVAL

LET $f:(a,b) \to \mathbb{R}$ AND LET $x_0 \in (a,b)$

WE SAY THAT f IS DIFFERENTIABLE IN x_0 IF IT EXIST AND IT IS FINITE THE LIMIT

$$\exists \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = l \in \mathbb{R}$$

THE VALUE OF THE LIMIT l IS CALLED THE DERIVATIVE OF f IN x_0 AND IS DENOTED BY

$$f'(x_0) \quad \text{OR} \quad \frac{df}{dx}(x_0) \quad \text{OR} \quad \left.\frac{df(x)}{dx}\right|_{x=x_0}$$

REMARK CHANGE OF VARIABLES

$x = x_0 + h$, $h = x - x_0$, WE HAVE

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$

SO f IS DIFFERENTIABLE IN x_0 IF AND ONLY IF

$\exists\, f'(x_0) \in \mathbb{R}$ SUCH THAT

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$$

EXAMPLE

- LET $c \in \mathbb{R}$; $f(x) = c$ $\forall x \in \mathbb{R}$ THEN

$\forall x_0 \in \mathbb{R}$ $\exists\, f'(x_0) = 0$

$x_0 \in \mathbb{R}$, $h \neq 0$

$$\frac{f(x_0+h)-f(x_0)}{h} = \frac{c-c}{h} = \frac{0}{h} = 0 \to 0$$

$f(x) = x \quad \forall x \in \mathbb{R}$ THEN

$\forall x_0 \in \mathbb{R} \quad \exists f'(x_0) = 1$

$$\frac{f(x_0+h) - f(x_0)}{h} = \frac{(x_0+h) - x_0}{h} = 1 \xrightarrow{h \to 0} 1$$

- $f(x) = x^2 \quad \forall x \in \mathbb{R}$ THEN

$\forall x_0 \in \mathbb{R} \quad \exists f'(x_0) = 2x_0$

$$\frac{f(x_0+h) - f(x_0)}{h} = \frac{(x_0+h)^2 - x_0^2}{h} = \frac{x_0^2 + 2hx_0 + h^2 - x_0^2}{h} =$$

$$2x_0 + h \xrightarrow{h \to 0} 2x_0$$

- $f(x) = |x| \quad \forall x \in \mathbb{R}$ f IS NOT DIFFERENTIABLE IN $x_0 = 0$

$$\lim_{x \to 0^\pm} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^\pm} \frac{|x| - |0|}{x} = \lim_{x \to 0^\pm} \frac{|x|}{x} = \pm 1$$

$\Rightarrow \not\exists$ LIM

- $f(x) = x^{1/3} \quad \forall x \in \mathbb{R}$ f IS NOT DIFFERENTIABLE IN $x_0 = 0$

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x} = \lim_{x\to 0} \frac{x^{1/3}-0^{1/3}}{x} = \lim_{x\to 0} x^{-2/3} = +\infty$$

$\Rightarrow \exists$ LIM BUT IT IS NOT FINITE

- $f(x) = x^{2/3}$ $\forall x \in \mathbb{R}$ f IS NOT DIFFERENTIABLE IN $x_0 = 0$

$$\lim_{x\to 0^{\pm}} \frac{f(x)-f(0)}{x} = \lim_{x\to 0^{\pm}} x^{-1/3} = \pm\infty$$

$\Rightarrow \nexists$ LIM

THEOREM $f:(a,b)\to\mathbb{R}$ $-\infty \leq a < b \leq +\infty$

$x_0 \in (a,b)$

IF f IS DIFFERENTIABLE IN x_0 THEN f IS CONTINUOUS IN x_0

REMARK VICEVERSA DOES NOT HOLD!

$|x|$ IS CONTINUOUS IN $x_0 = 0$ BUT IT IS NOT DIFFERENTIABLE IN $x_0 = 0$

PROOF WE NEED TO SHOW THAT $\lim_{x\to x_0} f(x) = f(x_0)$

THAT IS $\lim_{x\to x_0} (f(x)-f(x_0)) = 0$

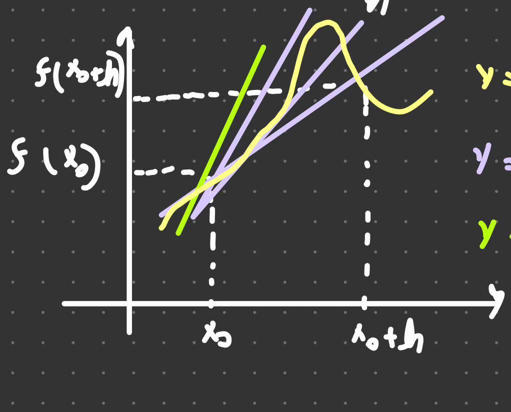
$$\underbrace{\frac{f(x)-f(x_0)}{(x-x_0)}}_{\substack{\downarrow x\to x_0 \\ f'(x_0)\in\mathbb{R}}} \cdot \underbrace{(x-x_0)}_{\substack{\downarrow x\to x_0 \\ 0}} \xrightarrow{x\to x_0} f'(x_0)\cdot 0 = 0 \qquad x \neq x_0$$

GENERIC INTERPRETATION OF DERIVATES

$f:(a,b)\to\mathbb{R}$ $\quad -\infty \leq a < b \leq +\infty \quad x_0 \in (a,b)$

$h \neq 0$ WITH $x_0+h \in (a,b)$ THE SECANT LINE (WITH RESPECT TO THE GRAPH OF f) PASSING THROUGH THE POINTS $(x_0, f(x_0))$ AND $(x_0+h, f(x_0+h))$ HAS EQUATION

$$y = f(x_0) + \frac{f(x_0+h)-f(x_0)}{h}(x-x_0)$$



$y = f(x)$

$y = f(x_0) + \frac{f(x_0+h)-f(x_0)}{h}\cdot(x-x_0)$

$y = f(x_0) + f'(x_0)(x-x_0)$

IF f IS DIFF IN x_0, IMPLIES THE SLOPE OF THE SECANT LINES AS $h \to 0$, GOES TO $f'(x_0)$. SO THE SECANT LINES GOES TO THE LINE WITH EQUATION

$$y = f(x_0) + f'(x_0)(x - x_0)$$

$$y = f(x_0) + f'(x_0)(x - x_0)$$

TANGENT LINE TO THE GRAPH OF f AT THE POINT $(x_0, f(x_0))$

EXAMPLE LET $f(x) = x^2$

COMPUTE THE TANGENT LINE TO THE GRAPH OF f AT THE POINT $(3,9)$

$$(3,9) = (3, f(3))$$

$$x_0 = 3 \quad f(x_0) = f(3) = 9$$

$$f'(x_0) = 2x_0 = 6$$

TANGENT LINE

$$y = 9 + 6(x-3) = 6x - 9$$

PROPOSITION $f(a,b) \to \mathbb{R}$, $-\infty \leq a < b \leq +\infty$, $x_0 \in (a,b)$

f IS DIFFERENTIABLE IN x_0

$$\iff$$

$\exists\, m \in \mathbb{R}$ SUCH THAT

$$f(x) = f(x_0) + m(x - x_0) + R_1(x) \qquad (*)$$

WHERE THE REMAINDER R_1 SATISFIES $R_1(x_0) = 0$

AND $\lim_{x \to x_0} \frac{R_1(x)}{x - x_0} = 0$

IN THIS CASE, $m = f'(x_0)$

REMARK THE TANGENT LINE, IF IT EXIST, IS THE UNIQUE LINE PASSING THROUGH $(x_0, f(x_0))$ SUCH THAT $(*)$ HOLDS

f DIFF. $\Rightarrow$ TAKE $m = f'(x_0)$ CLEARLY $R_1(x_0) = 0$

$$\frac{R_1(x)}{x - x_0} = \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \xrightarrow[x \to x_0]{} 0$$

$\Leftarrow$

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{f(x_0) + m(x - x_0) + R_1(x) - f(x_0)}{x - x_0} =$$

$$= m + \frac{R_1(x)}{x - x_0} \to m \quad \Rightarrow f \text{ DIFF. AND } m = f'(x_0)$$

DEFINITION

$$f:(a,b)\to\mathbb{R} \quad -\infty\le a<b\le+\infty,\ x_0\in(a,b)$$

f ADMITS RIGTH LEFT DERIVATIVE IN x_0 IF IT EXISTS AND IS FINITE

$$\lim_{h\to 0^{\pm}}\frac{f(x_0+h)-f(x_0)}{h}=\lim_{x\to x_0^{\pm}}\frac{f(x)-f(x_0)}{x-x_0}=$$

$$=f'_{\pm}(x_0)\in\mathbb{R}$$

$f'_{\pm}(x_0)$ RIGHT LEFT DERIVATIVE OF f AT x_0

NOTATION $f'_{\pm}(x_0)=\left.\dfrac{df}{dx}\right|_{x=x_0^{\pm}}$

REMARK f IS DIFFERENTIABLE IN $x_0 \iff$ THERE EXISTS AND ARE EQUAL $f'_+(x_0)$ AND $f'_-(x_0)$

IN THIS CASE

$$f'(x_0)=f'_+(x_0)=f'_-(x_0)\in\mathbb{R}$$

DEFINITION $I = (a, b)$ OPEN INTERVAL $-\infty \le a < b \le +\infty$

$f : (a,b) \to \mathbb{R}$ IS DIFFERENTIABLE ON I IF

IT IS DIFFERENTIABLE IN ANY $x \in (a,b)$

- $I = [a,b]$ WITH $-\infty < a < b < +\infty$

NOT DEGENERATE $f : [a,b] \to \mathbb{R}$ IS

DIFFERENTIABLE ON I IF IT IS DIFFERENTIABLE

ON (a, b) **AND**

$$\exists f'_+(a) \text{ AND } \exists f'_-(b)$$

REMARK $f : I \to \mathbb{R}$ I IS **NOT** DEGENERATE INTERVAL

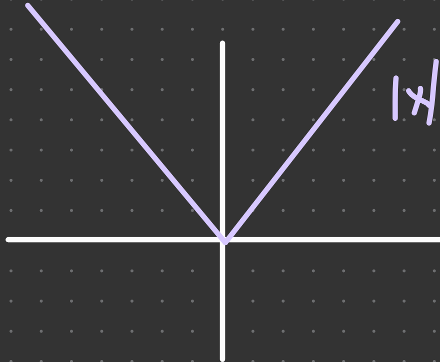
f DIFF ON I $\Rightarrow$ f CONTINUOUS ON I

EXAMPLES OF NOT DIFFERENTIABILITY

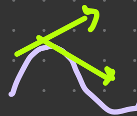
$$f(x) = |x| \quad x_0 = 0$$

$\exists f'_+(0) = 1 \in \mathbb{R}$ $\exists f'_-(0) = -1 \in \mathbb{R}$ AND

THEY ARE DIFFERENT



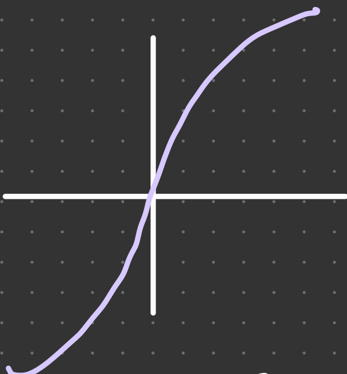
ANGLE



$\exists f'_+(x_0),\ \exists f'_-(x_0)$

BUT THEY ARE DIFF.

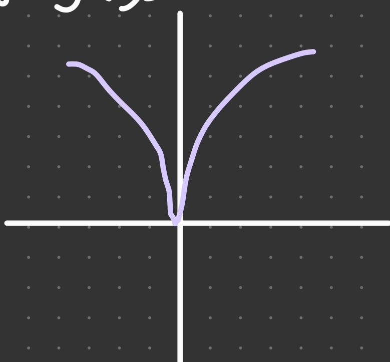
- $f(x) = x^{1/3}$



VERTICAL TANGENT

$$\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \pm\infty$$

- $f(x) = x^{2/3}$



CUSP

$$\lim_{x \to x_0^{\pm}} \frac{f(x)-f(x_0)}{x-x_0} = \pm\infty$$

$(\mp\infty)$