$+ \quad \times$

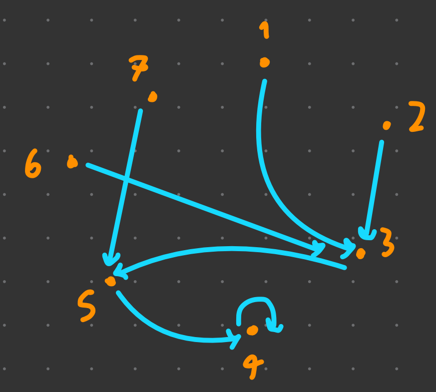
$- \quad \div$

EXERCISE Consider an iterated map $x_{n+1} = f(x_n)$, with states $\{1, 2, 3, 4, 5, 6, 7\}$. The function $f(x)$ computes its values by the following algorithm:

1. Take the string of characters that expresses the number x in English (e.g., $5 \rightsquigarrow$ "five').
2. Count the number of letters in the string; this is the value returned by the function (e.g., "five" $\rightsquigarrow 4$).

Find the fixed points, cycles, transient and recurrent states of this dynamics. How many connected components are there?

$X_{N+1} = f(X_N)$ STATES $\{1, 2, 3, 4, 5, 6, 7\}$



N=1 X_{N+1} = O-N-E = 3

N=2 X_{N+1} = T-W-O = 3

N=3 X_{N+1} = T-H-R-E-E = 5

N=4 X_{N+1} = F-O-U-R = 4

N=5 X_{N+1} = F-I-V-E = 4

N=6 X_{N+1} = S-I-X = 3

N=7 X_{N+1} = S-E-V-E-N = 5

FIXED POINTS: 1 [4]

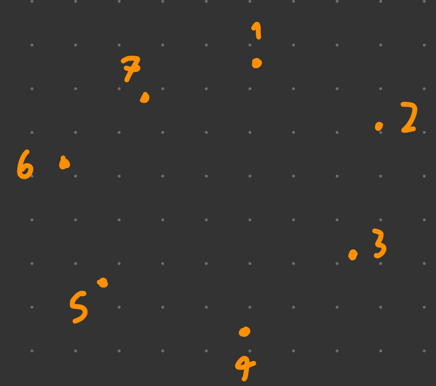
CYCLES: 1 [4]

TRANSIENT: 6 [1, 2, 3, 5, 6, 7]

RECURRENT: 1 [4]

CONNECTED COMPONENT: 1

EXERCISE Consider the same function f as in the previous exercise, but now the set of possible states is $\{0, 1, 2, 3, 4\}$. Does f represent the update rule of a dynamical system on this set of states?



N=1 X_{N+1} = O-N-E = 3

N=2 X_{N+1} = T-W-O = 3

N=3 X_{N+1} = T-H-R-E-E = 5

N=4 X_{N+1} = F-O-U-R = 4

f DOES NOT REPRESENT A RULE FOR A DYNAMICAL SYSTEM

EXERCISE Consider the iterated map $x_{n+1} = f(x_n)$, in the state space $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$, with

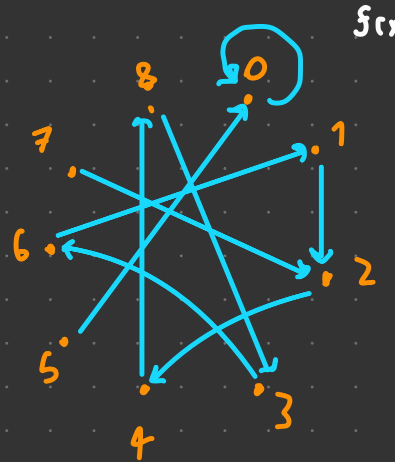
$$f(x) = \begin{cases} 2x & \text{if } x < 5 \\ x - 5 & \text{if } x \geq 5 \end{cases}$$

Find the fixed points, cycles, transient and recurrent states. How many connected components are there? Consider the function

$$Q(x) = 5x \mod 5$$

Is this a conserved quantity for the dynamics? Is it a *non-trivial* conserved quantity?

$$f(x) = \begin{cases} 2X & \text{IF } X < 5 \\ X - 5 & \text{IF } X \geq 5 \end{cases} \qquad \{0, \dots 8\}$$



FIXED POINTS: 1 [0]

CYCLES: 2 [0] [1, 2, 4, 8, 3, 6]

RECURRENT: 7 [0, 1, 2, 4, 8, 3, 6]

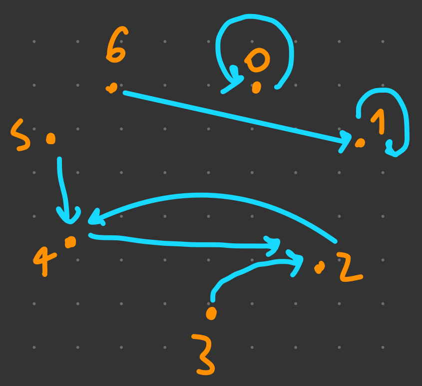
TRANSIENT: 2 [5, 7]

CONNECTED COMPONENTS: 2 [5, 0] [1, 2, 3, 4, 6, 7, 8]

EXERCISE Consider the iterated map $x_{n+1} = f(x_n)$, with

$$f(x) = x^2 \mod 7,$$

where $x \in \{0, 1, 2, 3, 4, 5, 6\}$. How many connected components are there? Can you write down a non-trivial conserved charge?



$f(x) = x^2 \bmod 7$

FIXED POINTS: 2 [0] [1]

CYCLES: 3 [0] [1] [4, 2]

RECURRENT: 4 [0] [1] [4] [2]

TRANSIENT: 3 [6] [5] [3]

CONNECTED COMPONENTS: 3 [0] [1] [5, 4, 3, 2]

$$Q(x) = \begin{cases} 0 & \text{IF } x = 0 \\ 1 & \text{IF } x > 0 \end{cases}$$

EXERCISE Consider the iterated map $x_{n+1} = f(x_n)$ in the (infinite) set $\mathbb{N} = \{0, 1, 2, \dots\}$ with

$$f(x) = x + k$$

where the parameter $k \geq 2$ is an integer. Describe the dynamics (cycles, etc.) Can you construct a non-trivial conserved quantity?

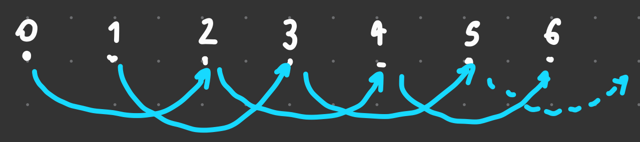
HINT: If the exercise seems too difficult, try considering the special case $k = 2$ first; then see if and how the picture changes when $k > 2$.

$x_{n+1} = f(x_n) \quad N = \{0, 1, 2, \dots\}$

$\downarrow$

$f(x) = x + k$

$k = 2$



FIXED POINTS: $\emptyset$

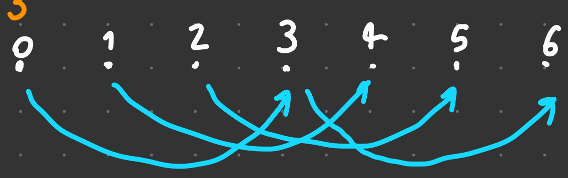
CYCLES: $\emptyset$

RECURRENT: $\emptyset$

TRANSIENT: $\mathbb{N}$

CONNECTED COMPONENTS: 2

$k = 3$



FIXED POINTS: $\emptyset$

CYCLES: $\emptyset$

RECURRENT: $\emptyset$

TRANSIENT: $\mathbb{N}$

CONNECTED COMPONENTS: 3

$\rightarrow$

FIXED POINTS: $\emptyset$

CYCLES: $\emptyset$

RECURRENT: $\emptyset$

TRANSIENT: $\mathbb{N}$

CONNECTED COMPONENTS: k

EXERCISE [difficult] Consider the iterated map $x_{n+1} = f(x_n)$ in the (infinite) set $\mathbb{N} = \{0, 1, 2, \ldots\}$ with

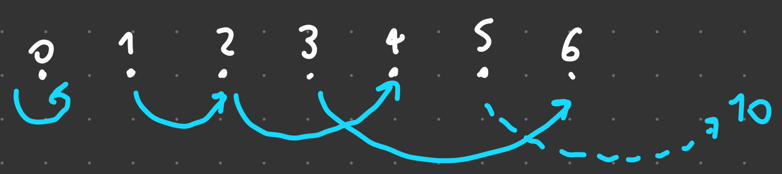
$$f(x) = kx,$$

where the parameter $k \geq 2$ is an integer. Describe the dynamics (cycles, etc.)

$x_{n+1} = f(x_n) \qquad N = \{0, 1, 2, \ldots\}$

$\hookrightarrow f(x) = kx$

$\boxed{K = 2}$



FIXED POINTS: 1 [0]

CYCLES: 1 [0]

RECURRENT: 1 [0]

TRANSIENT: $\mathbb{N} - \{0\}$

CONNECTED COMPONENTS: $\mathbb{N}/2 + 1 = \infty$

K = 3

EXERCISE Consider the dynamical system $x_{n+1} = f(x_n)$, with

$$f(x) = \sin(\pi x)$$

The dynamics has 2 fixed points. Linearize around the smaller fixed point $\bar{x}$. Regarding the stability of $\bar{x}$, which one of the following 4 possibilities is realized?

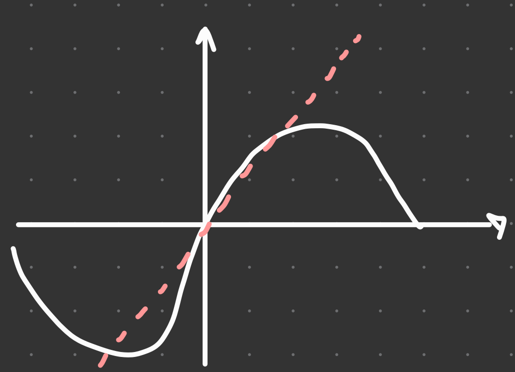
(a) $\bar{x}$ is stable (not a spiral)

(b) $\bar{x}$ is a stable spiral

(c) $\bar{x}$ is unstable (not a spiral)

(d) $\bar{x}$ is an unstable spiral

$x_{n+1} = f(x_n) \qquad f(x) = \sin(\pi x)$



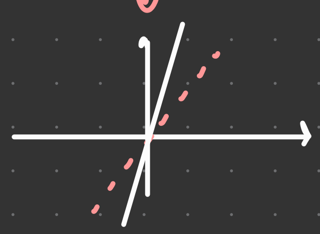
FIND FIXED POINT: (EQUAL TO ITSELF)

$\bar{x} = \sin(\pi \bar{x})$; $0 = \sin(0)$

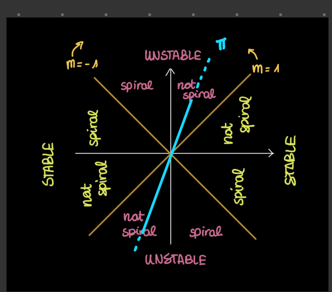
DO THE DERIVATIVE

LINEARIZE $f(x) = \sin(\pi x) \Rightarrow f'(x) = \pi\cos(\pi x)$

ZOOM



$f'(0) = \pi\cos(0) = \pi$



NOT SPIRAL ⇒ UNSTABLE

EXERCISE Consider the map

$$f(x) = \frac{1}{\alpha x + 1}$$

with state $x \geq 0$ and parameter $\alpha > 0$. Show that the dynamical system $x_{n+1} = f(x_n)$ has a single fixed point, which is always a stable spiral.

1) FIND FIXED POINT: $\bar{x} = f(\bar{x})$

$$\bar{x} = \frac{1}{\alpha \bar{x} + 1} = \alpha \bar{x}^2 + \bar{x} + 1 = 0$$

Δ: $\Delta = b^2 - 4ac = 1 - 4(\alpha)(+1) = = 1 - 4\alpha$

$$x_{1,2} = \frac{-1 \pm \sqrt{1-4\alpha}}{2\alpha}$$

$\downarrow$

$X \geq 0$ AND $\alpha > 0$ $\Rightarrow$ 1 FIXED POINT AT

$$\bar{x} = \frac{-1 + \sqrt{1-4\alpha}}{2\alpha}$$

2) DERIVATE

$$f(x) = \frac{1}{\alpha x + 1} = (\alpha x + 1)^{-1} \qquad \frac{d}{dx} x^3 = 3x^2$$

$$f'(x) = -1\alpha (\alpha x + 1)^{-2} = \boxed{\frac{-\alpha}{(\alpha x + 1)^2}}$$

3) SUBSTITUTE X

$$f'(x) = \frac{-\alpha}{\left(\cancel{\alpha} \frac{-1 + \sqrt{1-4\alpha}}{2\cancel{\alpha}} + 1\right)^2} = \frac{\alpha}{\left(\frac{\sqrt{1-4\alpha}}{2} + \frac{1}{2}\right)^2} = \frac{\alpha}{\left(\frac{1-4\alpha}{4} + \frac{1}{4} + \cancel{2}\frac{\sqrt{1-4\alpha}}{\cancel{2}} \cdot \frac{1}{2}\right)}$$

$$= \frac{\alpha}{\frac{1}{4} - \alpha + \frac{1}{4} + \frac{\sqrt{1-4\alpha}}{2}}$$

$$= \frac{4\alpha}{2 - 4\alpha + 2\sqrt{1-4\alpha}}$$

EXERCISE Consider the dynamical system obtained by iteration of the logistic map

$$f(x) = rx(1-x),$$

with state $x \in [0,1]$ and parameter $r \in [0,4]$. Obtain analytically the critical values $r_1 = 1$, $r_2 = 2$, $r_3 = 3$, that separate the following 3 regimes of the dynamical system $x_n = f(x_{n-1})$:

- $0 < r < r_1$ — x_n converges to 0
- $r_1 < r < r_2$ — x_n converges to $(r-1)/r$ monotonically
- $r_2 < r < r_3$ — x_n converges to $(r-1)/r$ with oscillations

???

• CHECK STABILITY w $\bar{x} = 0$

$$f'(x) = r(1-x) - rx = \underbrace{r - 2rx}$$

↳ CHECK ON 0 $\quad f'(0) = r \Rightarrow x$ STABLE FOR $r < r_1 = 1$