PRODUCT RULE

PROVE THAT $\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f(x)\cdot g'(x) + f'(x)\cdot g(x)$

LET $F(x) = f(x)\cdot g(x)$ THEN $F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$

THAN CAN BE REWRITED AS

$$F'(x) = \lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h}$$

SUBTRACT AND ADD $f(x+h)\,g(x)$ IN THE NUMERATOR

$$F'(x) = \lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x+h)\,g(x) + f(x+h)\,g(x) - f(x)\,g(x)}{h}$$

$$F'(x) = \lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x+h)\,g(x)}{h} + \lim_{h\to 0} \frac{f(x+h)\,g(x) - f(x)\,g(x)}{h}$$

$$F'(x) = \lim_{h\to 0} f(x+h)\,\frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x)\,\frac{f(x+h) - f(x)}{h}$$

SUBSTITUTE $h=0$

$$F'(x) = f(x)\cdot \underbrace{\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}}_{g'(x)} + g(x)\,\underbrace{\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}_{f'(x)}$$

THAT IS

$$F'(x) = f(x)\cdot g'(x) + g(x)\cdot f'(x)$$