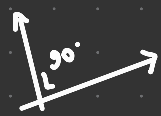
24/MAY/22

SCALAR PRODUCT $\underline{v}, \underline{w} \in \mathbb{R}^N$ $\quad (\underline{v}, \underline{w}) = \underline{v} \cdot \underline{w} = v_1 w_1 + v_2 w_2 + v_3 v_3 + \dots + v_N w_N$

- NORM: $v \in \mathbb{R}^N$ $\quad \|v\| = \sqrt{(\underline{v}, \underline{v})} = \sqrt{\underline{v_1}^2 + \underline{v_2}^2 + \dots + \underline{v_N}^2}$

- ORTOGONAL VECTOR: $\underline{v}, \underline{w} \in \mathbb{R}^N$ ARE ORTOGONAL IF $(\underline{v}, \underline{w}) = 0$



DEFINITION A SET OF VECTORS $\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_N}\} \in \mathbb{R}^N$ COSTITUTE AN HORTOGONAL SET IF THEY ARE ALL HORTOGONAL

(THAT IS $(\underline{v_i} \cdot \underline{v_j}) = 0$ FOR ANY $i \neq j$)

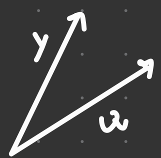
IF IN ADDITION $\|v_1\| = \|v_2\| = \dots = \|\underline{v_N}\| = 1$ THEN THE CONSITUTE AN ORTOGONAL SET

- AN ORTOGONAL BASIS IS A BASIS THAT IS ALSO AN ORTOGONAL SET
- AN ORTOGONAL BASIS IS A BASIS IF IS ALSO AN ORTOGONAL SET

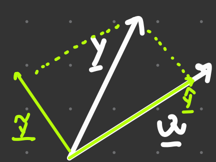
EXAMPLE: $\mathbb{R}^3$ $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ BASIS $\qquad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ORTHONORMAL BASIS FOR $\mathbb{R}^3$

$\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ LIN. IND. BASIS ORTOHOGAL

ORTOGONAL PROJECTION $\underline{y} \in \mathbb{R}^N$, $\underline{w} \in \mathbb{R}^N$, $\underline{w} \neq \underline{0}$



DECOMPSE y IN 2 PARTS



$\underline{y} = \hat{\underline{y}} + \tilde{\underline{y}}$

ORTOGONAL PROJECTION OF $\underline{y}$ ONTO $\underline{w}$

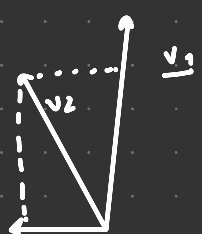
- $\hat{\underline{y}} = \alpha \underline{w} \quad \alpha \in \mathbb{R}^N$
- $(\tilde{y}, \underline{w}) = 0 \implies 0 = (\tilde{y}, \underline{w}) = (\underline{y} - \hat{\underline{y}}, \underline{w}) = (\underline{y} - \alpha \underline{w}, \underline{w}) = (\underline{y}, \underline{w}) - \alpha \overbrace{(w, w)}^{\|v\|^2} = (\underline{y}, \underline{w}) - \alpha \|\underline{w}\|^2 \implies \alpha = \frac{(\underline{y}, \underline{w})}{\|\underline{w}\|^2}$

$\hat{y} = \alpha \underline{w} = \frac{(\underline{y}, \underline{w})}{\|\underline{w}\|^2} \underline{w}$; $\tilde{\underline{y}} = \underline{y} - \hat{\underline{y}}$ $\quad \tilde{\underline{y}}$ IS ORTOGONAL TO $\underline{w}$

$y = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \quad \underline{w} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} \qquad \alpha = \frac{(\underline{y}, \underline{w})}{\|\underline{w}\|^2} = \frac{(-1 - 6 + 0)}{10} = -\frac{7}{10}$

$\hat{\underline{y}} = \alpha \underline{w} = \left(-\frac{7}{10}\right) \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 7/10 \\ -21/10 \\ 0 \end{bmatrix}$

$\tilde{\underline{y}} = \underline{y} - \hat{\underline{y}} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} - \begin{bmatrix} 7/10 \\ -21/10 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/10 \\ 1/10 \\ 3 \end{bmatrix} \qquad \tilde{\underline{y}} = \underline{y} - \frac{(\underline{y}, \underline{w})}{\|\underline{w}\|^2} \underline{w}$

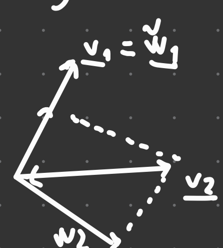


$\{\underline{v_1}, \underline{v_2}\} \in \mathbb{R}^N$ BASIS FOR $H = \text{SPAN}\{\underline{v_1}, \underline{v_2}\}$

ORTONOGAL BASIS FOR H?? $\{\underline{w_1}, \underline{w_2}\}$

$\tilde{\underline{w}}_1 = \underline{v_1}$

$\tilde{\underline{w}}_2 = \underline{v_2} - \frac{(\underline{v_2}, \underline{v_1})}{\|v_1\|^2} \underline{v_1}$



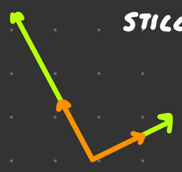
EXAMPLE

$\underline{v_1} = \begin{bmatrix}1\\0\\2\\1\end{bmatrix} \quad \underline{v_2} = \begin{bmatrix}0\\-1\\1\\4\end{bmatrix} \quad H = \text{SPAN}\{\underline{v_1}, \underline{v_2}\}$ LIN. IND.

ORTOGONAL BASIS

$$\underline{\tilde{w}_1} = \underline{v_1} = \begin{bmatrix}1\\0\\2\\1\end{bmatrix} \qquad \underline{\tilde{w}_2} = \begin{bmatrix}0\\-1\\1\\4\end{bmatrix} - \boxed{\overset{1}{\frac{(0+0+2+4)}{1+0+2^2+1}}}\begin{bmatrix}1\\0\\2\\1\end{bmatrix} = \begin{bmatrix}0\\-1\\1\\4\end{bmatrix} - \begin{bmatrix}1\\0\\2\\1\end{bmatrix} = \begin{bmatrix}-1\\-1\\-1\\3\end{bmatrix}$$

$$\left\{\begin{bmatrix}1\\0\\2\\1\end{bmatrix}\begin{bmatrix}-1\\-1\\-1\\3\end{bmatrix}\right\}$$ ORTOGONAL BASIS



STILL ORTOGONAL EVEN IF IS LENGTH 1 $\quad \underline{\tilde{w}} \in \mathbb{R}^2 \quad \underline{w} = \frac{\underline{\tilde{w}}}{\|\underline{\tilde{w}}\|}$ THEN $\|\underline{w}\| = 1$

$$\|\underline{w}\| = \left\|\frac{\underline{\tilde{w}}}{\|\underline{\tilde{w}}\|}\right\| = \frac{\|\underline{\tilde{w}}\|}{\|\underline{\tilde{w}}\|} = 1$$

$$\underline{w_1} = \frac{\underline{\tilde{w}_1}}{\|\underline{\tilde{w}_1}\|} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\0\\2\\1\end{bmatrix}$$

$$\underline{w_2} = \frac{\underline{\tilde{w}_2}}{\|\underline{\tilde{w}_2}\|} = \frac{1}{\sqrt{12}}\begin{bmatrix}-1\\-1\\-1\\3\end{bmatrix}$$

$$\{\underline{w_1}, \underline{w_2}\} = \left\{\begin{bmatrix}1/\sqrt{6}\\0\\2/\sqrt{6}\\1/\sqrt{6}\end{bmatrix}, \begin{bmatrix}-1/\sqrt{12}\\-1/\sqrt{12}\\-1/\sqrt{12}\\3/\sqrt{12}\end{bmatrix}\right\}$$ ORTOGONAL BASIS FOR H

$\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_m}\} \in \mathbb{R}^N$ BASIS FOR $H = \text{SPAN}\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_m}\}$

ORTOGONAL BASIS $\{\underline{\tilde{w}_1}, \underline{\tilde{w}_2}, \dots, \underline{\tilde{w}_m}\}$

$$\underline{\tilde{w}_1} = \underline{v_1}$$

$$\underline{\tilde{w}_2} = \underline{v_2} - \frac{(\underline{v_1}, \underline{v_2})}{\|\underline{v_1}\|^2}\underline{v_1}$$

$$\underline{\tilde{w}_3} = \underline{v_3} - \frac{(\underline{v_1}, \underline{v_3})}{\|\underline{v_1}\|^2}\underline{v_1} - \frac{(v_2, v_3)}{\|\underline{v_2}\|^2}v_2$$

$$\underline{\tilde{w}_4} = \underline{v_4} - \frac{(v_1, \underline{v_4})}{\|v_1\|^2}\underline{v_1} - \frac{(\underline{v_2}, \underline{v_4})}{\|\underline{v_2}\|^2} - \frac{(\underline{v_3}, \underline{v_4})}{\|v_3\|^2}v_3$$

$\underline{w_m} = \quad i = 1, 2, 3, \dots, m \quad \underline{\tilde{w}_i} = \underline{v_i} - \sum_{j=1}^{i-1} \frac{(\underline{w_j}, \underline{v_i})}{\|\underline{w_j}\|^2}\underline{w_j}$

ORTHONORMAL BASIS $\{\underline{w_1}, \underline{w_2}, \dots, \underline{w_m}\}$

$$\underline{w_1} = \underline{\tilde{w}_1}/\|\underline{w_1}\|$$

$$\underline{w_2} = \underline{\tilde{w}_2}/\|\underline{\tilde{w}_2}\|$$

$$\vdots$$

$$\underline{w_m} = \underline{\tilde{w}_m}/\|\underline{\tilde{w}_m}\|$$

EXAMPLE

$$\underline{v_1} = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \quad \underline{v_2} = \begin{bmatrix}2\\0\\1\\0\end{bmatrix} \quad \underline{v_3} = \begin{bmatrix}2\\-2\\1\\1\end{bmatrix}$$

- ARE THEY A BASIS FOR $H = \text{SPAN}\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$?
- IF THE ANSWER IS YES, BUILD AN ORTOGONAL AND AN ORTOGONAL BASIS FOR H

WE CHECK IF $\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ ARE LIN. INDIPENDENT?

$$\begin{bmatrix}1&2&2\\1&0&-2\\1&1&1\\0&0&1\end{bmatrix} \xrightarrow{1,\ 1} \begin{bmatrix}1&2&2\\0&-2&-4\\0&-1&-1\\0&0&1\end{bmatrix} \xrightarrow{1/2} \begin{bmatrix}1&2&2\\0&-2&-4\\0&0&1\\0&0&1\end{bmatrix} \xrightarrow{1} \begin{bmatrix}1&2&2\\0&-2&-4\\0&0&1\\0&0&0\end{bmatrix}$$

NO FREE VARIABLE

$\Downarrow$

ONLY ONE SOLUTION ($\underline{0}$)

$\Downarrow$

$\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ LINEARLY INDIPENDENT

$\{\underline{\tilde{w}_1}, \underline{\tilde{w}_2}, \underline{\tilde{w}_3}\}$ $\quad \underline{\tilde{w}_1} = \underline{v_1} = \begin{bmatrix}1\\1\\1\\0\end{bmatrix}$

$$\underline{\tilde{w}_2} = \underline{v_2} - \frac{(\underline{v_1}, \underline{v_3})}{\|v_1\|^2}\underline{v_1} = \begin{bmatrix}2\\0\\1\\0\end{bmatrix} - \frac{3}{3}\begin{bmatrix}1\\1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\-1\\0\\0\end{bmatrix}$$

$(1\cdot 2)+(1\cdot(-2))+(1\cdot 1)+(0\cdot 1)=1$

$$\underline{\tilde{w}_3} = \underline{v_3} - \frac{(\underline{v_1}, \underline{v_3})}{\|\underline{v_1}\|^2}\underline{v_1} - \frac{(\underline{v_2}, \underline{v_3})}{\|\underline{v_2}\|^2}\underline{v_2} = \begin{bmatrix}2\\-2\\1\\1\end{bmatrix} - \frac{1}{3}\begin{bmatrix}1\\1\\1\\0\end{bmatrix} - \frac{5}{5}\begin{bmatrix}2\\-2\\1\\1\end{bmatrix} = \begin{bmatrix}-1/3\\-7/3\\-1/3\\1\end{bmatrix}$$

NO!

$$\underline{\tilde{w}_1} = \underline{v_1}$$

$$\underline{\tilde{w}_2} = \underline{v_2} - \frac{(\underline{\tilde{w}_1}, \underline{v_3})}{\|\underline{\tilde{w}_1}\|^2}\underline{\tilde{w}_1} = \underline{v_2} - \frac{(\underline{v_1}, \underline{v_2})}{\|\underline{v_1}\|^2}\underline{v_1}$$

NO!

$$\underline{\tilde{w}_3} = \underline{v_3} - \frac{(\underline{\tilde{w}_1}, \underline{v_3})}{\|\underline{\tilde{w}_1}\|^2}\underline{\tilde{w}_1} - \frac{(\underline{w_2}, \underline{v_3})}{\|\underline{\tilde{w}_2}\|^2}\underline{\tilde{w}_2} \neq \underline{v_3} - \frac{(\underline{\tilde{w}_1}, \underline{v_3})}{\|\underline{\tilde{w}_1}\|^2}\underline{\tilde{w}_1} - \frac{(\underline{w_2}, \underline{v_3})}{\|\underline{\tilde{v}_2}\|^2}\underline{\tilde{v}_2}$$

$$\underline{\tilde{w}_2} = \underline{v_2} - \frac{(\underline{v_1}, \underline{v_3})}{\|v_1\|^2}\underline{v_1} = \begin{bmatrix}2\\0\\1\\0\end{bmatrix} - \frac{3}{3}\begin{bmatrix}1\\1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\-1\\0\\0\end{bmatrix}$$

$(1\cdot 2)+(1\cdot(-2))+(1\cdot 1)+(0\cdot 1)=1$

$$\underline{\tilde{w}_3} = \underline{v_3} - \frac{(\underline{v_1}, \underline{v_3})}{\|\underline{v_1}\|^2}\underline{v_1} - \frac{(\underline{v_2}, \underline{v_3})}{\|v_2\|^2}\underline{v_2} = \begin{bmatrix}2\\-2\\1\\1\end{bmatrix} - \frac{1}{3}\begin{bmatrix}1\\1\\1\\0\end{bmatrix} - \frac{4}{2}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} = \begin{bmatrix}-1/3\\-1/3\\2/3\\1\end{bmatrix}$$

$\|\underline{\tilde{w}_1}\| = \sqrt{3}$

$\|\underline{\tilde{w}_2}\| = \sqrt{2}$

$\|\underline{\tilde{w}_3}\| = \frac{\sqrt{15}}{3}$

$$\underline{\tilde{w}_1} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\\0\end{bmatrix} \quad \underline{w_2} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} \quad w_3 = \frac{3}{\sqrt{15}}\begin{bmatrix}-1/3\\-1/3\\2/3\\1\end{bmatrix}$$

THE QR FACTORIZATION

$A = LU \qquad A = QR$

THEOREM $A \in \mathbb{R}^{n\times m}$ WITH LINEARLY INDEPENDENT COLUMNS THEN A CAN BE WRITTEN AS $A = QR$, WHERE $Q \in \mathbb{R}^{n\times m}$ AND ITS COLUMNS CONSITITUTES AN ORTONORMAL SET SPANNING $\text{Col}(A)$ AND WHERE $R \in \mathbb{R}^{m\times m}$ IS AN UPPER TRIANGULAR MATRIX WITH STRICTLY POSITIVES ENTRIES IN THE DIAGONAL

"PROOF"

$$A = \left[\underline{v_1} \middle| \underline{v_2} \middle| \underline{v_3} \middle| \cdots \middle| \underline{v_m}\right] \qquad \{\underline{w_1}, \underline{w_2}, \dots, \underline{w_m}\} \text{ AN ORTONOGAL BASIS FOR } \text{Col}(A) \qquad Q = \left[\underline{w_1} \middle| \underline{w_2} \middle| \cdots \middle| \underline{w_m}\right] \in \mathbb{R}^{n\times m}$$

$$A = \left[v_1 \,|\, v_2 \,|\, \dots \,|\, v_m \right] = \left[Q x_1 \,|\, Q x_2 \,|\, \dots \,|\, Q x_m \right] = QR \qquad R = \left[x_1 \,|\, x_2 \,|\, \dots \,|\, x_m \right] \qquad w_1 = v_1 / \|v_1\|$$

$Col(A)$

HOW TO COMPUTE $A = QR$ $\quad A \in \mathbb{R}^{n \times m}$

$$A = \left[v_1 \,|\, v_2 \,|\, \dots \,|\, v_m \right] \quad \{v_1, v_2, \dots, v_m\} \xrightarrow[\text{PROCESS}]{\text{ORTHONORMALIZATION}} \{w_1, w_2, \dots, w_m\}$$

$$Q = \left[w_1 \,|\, w_2 \,|\, \dots \,|\, w_m \right]$$ HOW DO WE COMPUTE R??

$$Q^T Q = \overset{Q^T}{\begin{bmatrix} \text{---}\ w_1\ \text{---} \\ \text{---}\ w_2\ \text{---} \\ \vdots \\ \text{---}\ w_m\ \text{---} \end{bmatrix}} \overset{Q}{\left[w_1 \,|\, w_2 \,|\, \dots \,|\, w_m \right]} = \begin{bmatrix} 1 & 0 & 0 \dots 0 & 0 \\ 0 & 1 & 0 \dots 0 & 0 \\ \vdots & & & \vdots \\ 0 & 0 & 0 \dots 0 & 1 \end{bmatrix} = I$$

$$A = QR \Rightarrow Q^T A = \underbrace{Q^T Q}_{I} R \Rightarrow \boxed{Q^T A = R}$$

EXAMPLE:

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = A \quad \underset{v_1}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} \underset{v_2}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \qquad \tilde{w}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \tilde{w}_2 = v_2 - \frac{(\tilde{w}_1, v_2)}{\|\tilde{w}_1\|^2} \tilde{w}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{3}{\underset{2^2+1^2}{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/5 \\ 2/5 \end{bmatrix}$$

$$w_1 = \tilde{w}_1 / \|\tilde{w}_1\| = \frac{1}{\sqrt{5}} \overset{v_1}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$w_2 = \tilde{w}_2 / \|\tilde{w}_2\| = \sqrt{5} \begin{bmatrix} -1/5 \\ 2/5 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$Q = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5/\sqrt{5} & 3/\sqrt{5} \\ 0 & 1/\sqrt{5} \end{bmatrix}$$

$$\{v_1, v_2, \dots, v_n\} \longrightarrow \{w_1, w_2, \dots, w_n\}$$ **GRAM-SCHMIDT PROCESS**