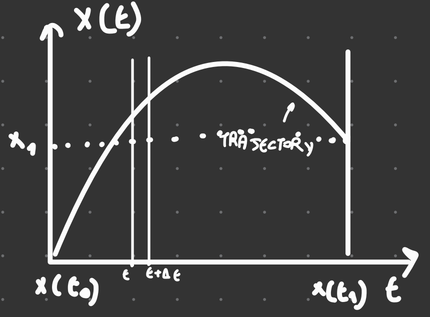
$+$ $\times$

$-$ $\div$




$g \downarrow$

FUNCTION OF WHOLE TRAJECTORY
FUNCTION OF A FUNCTION

$S[x(t)] =$

SMALL CHANGE

$= S[x(\cdot) + \delta x(\cdot)]$

↳ STATIONARITY OF $S$

$\frac{d}{dx} f(x) = 0$

SMALL CHANGE

$f(x + dx) = f(x)$

$f(x) + dx\, f'(x)$ → TAYLOR EXPANSION (STOP AT FIRST ORDER)

POSITION AT TIME $t$

VELOCITY AT TIME $t$

$$S = \int_{t_0}^{t_1} L(x(t)), \dot{x}(t))\, dt$$

ACTION

LAGRANGIAN

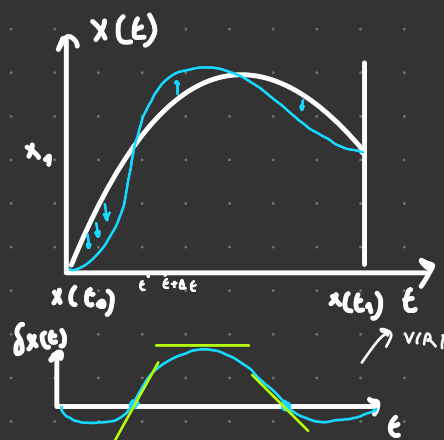
IF YOU USE $t$ INSTEAD OF $\cdot$ IT WOULD DEPENDS ON TIME.

THE VALUE $S$ DEPENDS ON THE SHAPE NOT ON TIME
CAN'T ASK VALUE OF $S$ IN $t=3$

$$S[x(\cdot) + \delta x(\cdot)] - S[x(\cdot)] = 0$$

$$\int_{t_0}^{t_1} L\left(x(t) + \delta x(t), \dot{x}(t) + \delta\dot{x}(t)\right) dt - \int_{t_0}^{t_1} L(x(t), \dot{x}(t))\, dt = 0$$

DERIVATIVES WRT TIME




$\ddot{x}(t) = -g$

THE DERIVATIVE CAN BE $=0$ OR ARBITRARY LOW/HIGH

## DO TAYLOR EXPANSION:

$$\int_{t_0}^{t_1} \Big\{ L(x(t), \dot{x}(t)) + \underline{\partial_x L(x(t), \dot{x}(t))\, \delta x(t)} +$$

↳ ON FIRST VARIABLE $x$

$$+ \underline{\partial_{\dot{x}} L(x(t), \dot{x}(t))\, \delta_{\dot{x}}(t)} - L(x(t), \dot{x}(t)) \Big\}\, dt = 0$$

↳ ON SECOND VARIABLE $\dot{x}$

↳ PUT INSIDE SAME INTEGRAL

BOUNDARY CONDITION

$x(t_0) = x_0 \quad x(t_1) = x_1$

$\Downarrow \quad \Downarrow$

$\delta x(t_0) = x_0 = \delta x(t_1) = x_1$

$$\int_{t_0}^{t_1} \Big\{ d_x L(x(t), \dot{x}(t)) + d_x L(x(t), \dot{x}(t))\, \delta x(t) + \partial_{\dot{x}} L(x(t), \dot{x}(t))\, \delta_{\dot{x}}(t) \Big\}\, dt = 0$$

## DO INTEGRATION BY PARTS

$$\int_{t_0}^{t_1} f(t) \frac{d}{dt} g(t) = f g \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \Big\{ \Big[\frac{d}{dt} f(t)\Big] g(t) \Big\}\, dt$$

BY PARTS

$$\int_{t_0}^{t_1} \partial_{\dot{x}} L\; \dot{\delta}_x\, dt = \underbrace{\partial_{\dot{x}} L\; \delta x \Big|_{t_0}^{t_1}}_{} - \int_{t_0}^{t_1} \Big\{ \frac{d}{dt} \partial_{\dot{x}} L \cdot \delta x \Big\}\, dt$$

↳ BOUNDARY TERM

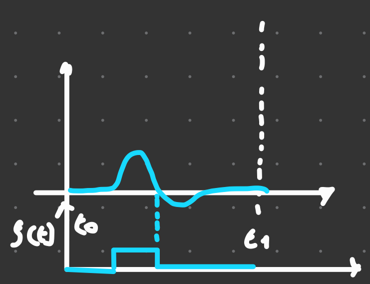
$\partial x$ MUST BE $0$ AT BOTH BOUNDARY

$\Downarrow$

THE BOUNDARY TERM IS EQUAL TO $0$

$$\int_{t_0}^{t_1} \delta x(t) \Big\{ \delta_x L(x, \dot{x}) - \frac{d}{dt} \delta_{\dot{x}} L(x, \dot{x}) \Big\}\, dt = 0$$

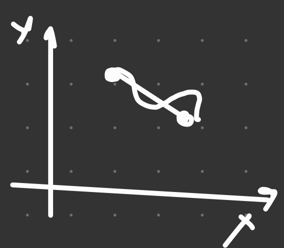
$$\int_{t_0}^{t_1} f(t)\, g(t)\, dt$$

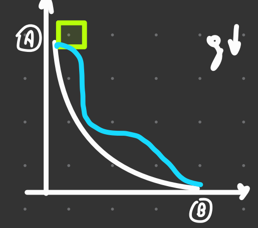



$$\Longrightarrow \boxed{\frac{d}{dt} \partial_{\dot{x}} L(x, \dot{x}) = \partial_x L(x, \dot{x})}$$

ORDINARY DIFFERENTIAL EQUATION

↳ EULER-LAGRANGE EQ




$g \downarrow$

FIND FASTEST PATH FROM A TO B

TOTAL KINETIC
TOTAL POTENTIAL

$L = K - U$

$L\left(x(t), \dot{x}(t)\right)$

NO FORCES ACTING ON THE PARTICLE $\Rightarrow U = 0$

**FREE PARTICLE IN 1 DIMENSION**

COMPLETELY SPECIFIED BY ONE VARIABLE

$L = K + \cancelto{0}{U} = \frac{1}{2} m \underbrace{\dot{x}^2(t)}_{v^2}$

$\frac{1}{2} m 2\dot{x}$

$\partial_{\dot{x}} L = m\dot{x} = p$ THE MOMENTUM

$\partial_x L = 0$

NO $x$

$\frac{d}{dt} m\dot{x} = 0 \Rightarrow \ddot{x} = 0$