$+$ $\times$

$-$ $\div$

$$\frac{d}{dt}\partial_{\dot{x}} L(x,\dot{x}) = \partial_x L(x,\dot{x})$$

$L = K - U$ (K: KINETIC, U: POTENTIAL)

$\{q_i\}_{i=1,\dots,N}$

DERIVATIVE WRT TIME - DEPENDS ON ALL COORDINATES

$$\frac{d}{dt}\partial_{\dot{q}_J} L(\{q_i\},\{\dot{q}_i\})$$

DIFFERENTIATE FOR ANY VARIABLE

$$\partial_{q_J} L(\{q_i\},\{\dot{q}_i\}) \quad J = 1,\dots,N$$

A POINT IN 3D

$$\begin{cases} q_1 = x \\ q_2 = y \\ q_3 = z \end{cases}$$

DIFFERENTIATE TO ALL OF THEM

$$\frac{d}{dt}\partial_{\dot{y}} L(x,y,z,\dot{x},\dot{y},\dot{z}) = \partial_y L(\dots) \qquad L = \frac{1}{2}m\dot{x}^2$$

1D POINT PARTICLE W POTENTIAL

(SINGLE POINT IN SPACE (1 DEGREE OF FREEDOM))

$$L = \underbrace{\frac{1}{2}m\dot{x}^2}_{\text{KINETIC}} - \underbrace{U(x)}_{\text{POTENTIAL}}$$

$\partial_{\dot{x}} L = m\dot{x} = p$ (MOMENTUM)

DERIVATIVE WRT X

$$\partial_x L = -\partial_x U(x) = F(x) \qquad \boxed{\dot{p} = F(x)}$$

IF I HAVE POTENTIAL ENERGY EVERYTHING WORKS CORRECTLY

LAGRANGIAN > NEWTONIAN 2 CASES

1. CHANGE OF COORDINATES → CHANGE OF VARIABLE
   COMPLICATE W/ NEWTON EQUATIONS

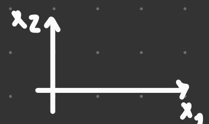
$$\{x_i\}_{i=1,\dots,N} \sim \{y_i\}_{i=1,\dots,N}$$

THE NUMBER OF COORDINATES AND DEGREE OF FREEDOM DOES NOT CHANGE

2. WHEN I HAVE CONSTRAINTS

$$y_J(\{x_i\})$$

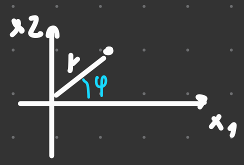
$\{x_1, x_2\}$ HORIZONTAL AND VERTICAL VARIABLES IN 2D



↓ MOVE TO POLAR COORDINATES

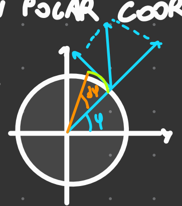
$\{r, \varphi\}$

$$\begin{cases} x_1 = r\cos\varphi \\ x_2 = r\sin\varphi \end{cases}$$



FREE PARTICLE IN POLAR COORDINATES

$$L = K = \frac{1}{2}mv^2$$



HOW MUCH φ CHANGE IN A SMALL AMOUNT OF TIME

$d\varphi$ HOW MUCH φ CHANGED IN Δt

$\dot{\varphi}r$ = LENGTH OF ARC

$$v^2 = \frac{dr^2 + r^2 d\varphi^2}{dt^2}$$

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2)$$

$$\begin{cases} \partial_r L = mr\dot{\varphi}^2 \\ \partial_{\dot{r}} L = m\dot{r} \end{cases}$$

$\partial_{\dot{q}_J} L = p_J$

GENERALIZED MOMENTUM

$$m\ddot{r} = mr\dot{\varphi}^2$$

$$\{x_i\}_{i=1,\dots,N} \qquad f(x_1, x_2, \dots, x_N) = 0$$

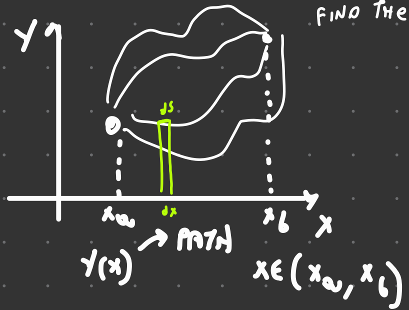
$$x_1, x_2 \quad x_1^2 + x_2^2 - R^2 = 0$$

$$r^2 - R^2 = 0$$

$$L = K = \frac{1}{2}mv^2$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) = \frac{1}{2}mR^2\cdot\dot{\varphi}^2$$

$$\frac{d}{dt}\partial_{\dot{x}} L(x,\dot{x}) = \partial_x L(x,\dot{x})$$



FIND THE TRAJECTORY THAT MINIMIZE LENGTH

$$\underbrace{S}_{\text{LENGTH}} = \int_A^B ds = \int_A^B \sqrt{dx^2 + dy^2}\,\frac{dx}{dx}$$

$$= \int_A^B \sqrt{1 + \frac{dy^2}{dx^2}}\,dx = \int_A^B \sqrt{1 + (\dot{y}(x))^2}\,dx = \int_A^B L(y,\dot{y})\,dx$$

$$L = \sqrt{1 + (\dot{y})^2}$$

PHYSICS 12.12

LAGRANGIAN : $L(x, \dot{x})$

Particle in 3D $\begin{cases} q_1 = x \\ q_2 = y \\ q_3 = z \end{cases}$

$L(x, y, z, \dot{x}, \dot{y}, \dot{z})$

PARTIAL ∂ = derivative when others are kept constant

$\{q_i\}_{i=1\dots n}$ $\quad \underbrace{\partial_{q_i} L(\{q_i\}, \{\dot{q}_i\}) = d/dt \, \partial_{\dot{q}_i} L(\{q_i\}, \{\dot{q}_i\})}$

example: $\frac{d}{dt}\partial_{\dot{y}} L(\dots) = \partial_{y} L(\dots)$

link with Newtonian? ↓

no internal degree of freedom!

POINT PARTICLE IN ONE DIMENSION:

In Potential: $L = K - V = \frac{1}{2} m\dot{x}^2 - U(x)$

$\partial_{\dot{x}} L = m\dot{x}$ MOMENTUM p

$\dot{p} = F(x)$

$\partial_x L = -\partial_x U(x)$ FORCE $F(x)$

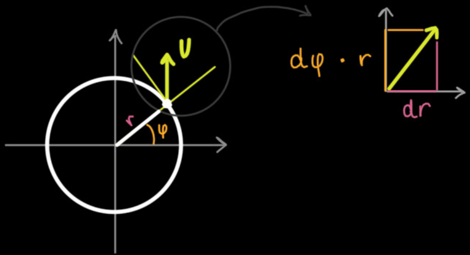
lagrangian mechanics allows us to deal with:

→ CHANGE OF COORDINATES $\{x_i\}_{i=1\dots n} \rightsquigarrow \{y_i\}_{i=1\dots n}$

→ CONSTRAINTS $\{x_i\}_{i=1\dots n}$ with $f(x_1 \dots x_n) = 0$

① EXAMPLE [change of coordinates]

$$\{x_1, x_2\} \rightsquigarrow \overset{\text{POLAR}}{\{r, \varphi\}} \quad \text{with} \quad \begin{aligned} x_1 &= r\cos\varphi \\ x_2 &= r\sin\varphi \end{aligned}$$



$$L(r, \varphi, \dot r, \dot\varphi) = K = \tfrac{1}{2} m \underbrace{(\dot r^2 + r^2 \dot\varphi^2)}_{v^2}$$

$$\partial_r L = m r \dot\varphi^2$$

$$\partial_{\dot r} L = m \dot r$$

$$m \ddot r = m r \dot\varphi^2$$

Newtanian only

warks in cartesian !

energy is energy in any frame of reference, force is not !

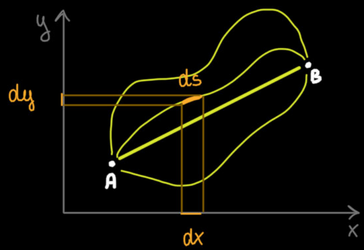
② EXAMPLE [CONSTRAINT]

X_1, X_2 "must be on surface of a sphere" $\underbrace{X_1^2 + X_2^2}_{r^2} - R^2 = 0$

$$L(\varphi, \dot\varphi) = \tfrac{1}{2} m (\dot r^2 + r^2 \dot\varphi^2) = \tfrac{1}{2} m R^2 \dot\varphi^2$$

[...]

PROVING SHORTEST PATH BETWEEN TWO POINTS :



$$S = \int_A^B ds = \int_A^B \sqrt{dx^2 + dy^2} \; dx/dx =$$

$$= \int_A^B \sqrt{1 + \frac{dy^2}{dx^2}}$$

$$\int_A^B \sqrt{1 + \left(\dot{y}(x)\right)^2} = \int_A^B L(y, \dot{y}) \, dx$$

E.L. equation $\longrightarrow$ $\dot{y}$ constant $\longrightarrow$ SEGMENT!