Exercises - Calculus

Academic Year 2021-2022

Sheet 14

1. Let us consider $(x, y, z) \in \mathbb{R}^3$ with $(x, y, z) \neq 0$. Let (ρ, θ, ψ) be its spherical coordinates. Prove that
$$\psi = \arccos(z/\rho).$$

2. Write the following points in $\mathbb{R}^3$ in cylindrical and in spherical coordinates.
$P = (1/2, \sqrt{3}/2, 1/\sqrt{3})$; $P = (-1, 0, \sqrt{3})$; $P = (\sqrt{2}, -\sqrt{2}, -2)$

3. Let us consider the following functions f. Establish whether they are linear or affine.
   (a) $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 2x - 3$ A
   (b) $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = -x + x^2$ NO
   (c) $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = -x$ L
   (d) $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x, y) = 3x + 2y + 5$ A
   (e) $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x, y) = 3x + 2y + 5xy$ NO
   (f) $f : \mathbb{R}^3 \to \mathbb{R}$ such that $f(x, y, z) = 3x^2 + 2y^2 - z$ NO
   (g) $f : \mathbb{R}^3 \to \mathbb{R}$ such that $f(x, y, z) = 2x - y + 6z$ L
   (h) $f : \mathbb{R}^3 \to \mathbb{R}$ such that $f(x, y, z) = x - 3z + 8$ A
   (i) $f : \mathbb{R}^3 \to \mathbb{R}$ such that $f(x, y, z) = 2(x - 5) - (y + 1) + 6z$ A
   (j) $f : \mathbb{R}^3 \to \mathbb{R}^2$ such that $f(x, y, z) = (3x, y - 2z)$ L
   (k) $f : \mathbb{R}^3 \to \mathbb{R}^2$ such that $f(x, y, z) = (2x + 5, yz)$ NO
   (l) $f : \mathbb{R}^4 \to \mathbb{R}^2$ such that $f(x_1, x_2, x_3, x_4) = (3x_1 + 2x_2 - 5, 4x_3 - x_4 + 2)$ A

4. Let $f : \mathbb{R}^N \to \mathbb{R}^M$ with components $f_i : \mathbb{R}^N \to \mathbb{R}$, $i = 1, \ldots, M$. Prove that f is linear or affine if and only if all components f_i are linear or affine, respectively.

5. Prove that $\min_{\theta \in [0, 2\pi)}(|\cos(\theta)| + |\sin(\theta)|)$ exists and it is positive. Prove that there exists constants $0 < C_0 < C_1$ such that
$$C_0\sqrt{x^2 + y^2} \leq |x| + |y| \leq C_1\sqrt{x^2 + y^2} \quad \text{for any } (x, y) \in \mathbb{R}^2.$$

6. Study the continuity of the following functions
   (a) $f(x, y) = \begin{cases} \dfrac{x^3 - 3xy^2}{x^2 + y^2} & \text{se } (x, y) \neq (0, 0) \\ 0 & \text{se } (x, y) = (0, 0) \end{cases}$ 1) $x = \rho\cos\theta$, $y = \rho\sin\theta$ $\Rightarrow x^2 + y^2 = \rho^2$
   (b) $f(x, y) = \begin{cases} \dfrac{x^2 - 3y^2}{x^2 + y^2} & \text{se } (x, y) \neq (0, 0) \\ 0 & \text{se } (x, y) = (0, 0) \end{cases}$ 2) $y = mx$

(c) $f(x,y) = \begin{cases} \dfrac{x^2}{\sqrt{x^2+y^2}} & \text{se } (x,y) \neq (0,0) \\ 0 & \text{se } (x,y) = (0,0) \end{cases}$

(d) $f(x,y) = \begin{cases} \dfrac{\log(1+(x-y)^3)}{x^2+y^2} & \text{se } (x,y) \neq (0,0) \\ 0 & \text{se } (x,y) = (0,0) \end{cases}$

(e) $f(x,y) = \begin{cases} \dfrac{\log(1+|x-y|)}{\sqrt{x^2+y^2}} & \text{se } (x,y) \neq (0,0) \\ 0 & \text{se } (x,y) = (0,0) \end{cases}$

(f) $f(x,y) = \begin{cases} \dfrac{(\sin(x+y))^2}{|x|+|y|} & \text{se } (x,y) \neq (0,0) \\ 0 & \text{se } (x,y) = (0,0) \end{cases}$

(g) $f(x,y) = \begin{cases} (x-1)\dfrac{\sin\left(2(x^2+y^2)\right)}{x^2+y^2} & \text{se } (x,y) \neq (0,0) \\ -2 & \text{se } (x,y) = (0,0) \end{cases}$

(h) $f(x,y) = \begin{cases} \dfrac{(\arctan(xy^2))^2}{(x^2+y^2)^2} & \text{se } (x,y) \neq (0,0) \\ 0 & \text{se } (x,y) = (0,0) \end{cases}$

(i) $f(x,y) = \begin{cases} \dfrac{\sqrt{|x|}-\sqrt{|y|}}{\sqrt{|x|}+\sqrt{|y|}} & \text{se } (x,y) \neq (0,0) \\ 0 & \text{se } (x,y) = (0,0) \end{cases}$

(j) $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{|x|^3+|y|^3}} & \text{se } (x,y) \neq (0,0) \\ 0 & \text{se } (x,y) = (0,0) \end{cases}$

(k) $f(x,y) = \begin{cases} \dfrac{x\sqrt{|y|}}{\sqrt{|x|^3+|y|^3}} & \text{se } (x,y) \neq (0,0) \\ 0 & \text{se } (x,y) = (0,0) \end{cases}$

7. Determine the domain of existence A of the following functions and compute, where they exist, the partial derivatives.

(a) $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ where $f(x,y) = \cos(x-y) + e^{xy^2}$

(b) $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ where $f(x,y) = \arctan(2xy) + x^3/y$

(c) $f : A \subset \mathbb{R}^3 \to \mathbb{R}$ where $f(x,y,z) = \log(z^2+y^2) - 3x\cos(z)$

(d) $f : A \subset \mathbb{R}^3 \to \mathbb{R}^{\nvDash}$ where $f(x,y,z) = (2y^3\sin(yz), x^4-4)$

(e) $f : A \subset \mathbb{R}^4 \to \mathbb{R}$ where $f(x_1,x_2,x_3,x_4) = \sin(x_1x_3) - e^{x_3-x_4^2} + x_2^2x_4$

(f) $f : A \subset \mathbb{R}^4 \to \mathbb{R}$ where $f(x_1,x_2,x_3,x_4) = \mathrm{e}^{x_1/x_4} - \log(x_3^2)$

(g) $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ where $f(x,y) = 2y + |x-y|$

(h) $f : A \subset \mathbb{R}^3 \to \mathbb{R}$ where $f(x,y,z) = \arcsin(xy) - z^2$

(i) $f : A \subset \mathbb{R}^3 \to \mathbb{R}^2$ where $f(x,y,z) = \left(\tan\left(\frac{\pi}{2}(x+z+y)\right), \cos(z^3+y)\right)$

6. Study the continuity of the following functions

$$\text{(a)}\quad f(x,y)=\begin{cases}\dfrac{x^3-3xy^2}{x^2+y^2} & \text{se } (x,y)\neq(0,0)\\ 0 & \text{se } (x,y)=(0,0)\end{cases}$$

a) $f(x,y)=\begin{cases}\dfrac{x^3-3xy^2}{x^2+y^2} & \text{IF } (x,y)\neq(0,0)\\ 0 & \text{IF } (x,y)=(0,0)\end{cases}$ $\qquad x=\rho\cos\theta$, $y=\rho\sin\theta$

$$\lim_{(x,y)\to(0,0)}\frac{\rho^3\cos^3\theta-3\rho^3\cos\theta\sin^2\theta}{\underbrace{\rho^2\cos^2\theta+\rho^2\sin^2\theta}_{1}}=\lim_{(x,y)\to(0,0)}\frac{\cancel{\rho^3}(\cos^3\theta-3\cos\theta\sin^2\theta)}{\cancel{\rho^2}}=\lim_{(x,y)\to(0,0)}\rho(\cos^3\theta-\cos\theta\sin^2\theta)\to 0 \quad \rho\to 0$$

DEMOSTRATE THAT $l=0$ IS THE LIMIT

$$|f(\rho\cos\theta,\rho\sin\theta)-0|=|\rho(\cos^3-3\cos\theta\sin^2\theta)|=\rho|\cos^3\theta-\cos\theta\sin^2\theta|\le\rho\equiv g(\rho)$$

SINCE $g(\rho)\to 0$ AS $\rho\to 0$ THEN $\lim_{(x,y)\to(0,0)} f(x,y)=0$ $\quad f$ IS CONTINUOUS

b) $f(x,y)=\begin{cases}\dfrac{x^2-3y^2}{x^2+y^2} & \text{IF } (x,y)\neq(0,0)\\ 0 & \text{IF } (x,y)=(0,0)\end{cases}$

SWITCH TO POLAR COORDINATES

$x=\rho\cos\theta$

$y=\rho\sin\theta$

$$\lim_{(x,y)\to(0,0)}\frac{x^2-3y^2}{x^2+y^2}=\lim_{(x,y)\to(0,0)}\frac{\rho^2\cos^2\theta-3\rho^2\sin^2\theta}{\underbrace{\rho^2\cos^2\theta+\rho^2\sin^2\theta}_{\rho^2}}=\lim_{(x,y)\to(0,0)}\frac{\cancel{\rho^2}(\cos^2\theta-3\sin^2\theta)}{\cancel{\rho^2}}=\underbrace{\cos^2\theta-3\sin^2\theta}$$

DEPENDS ON θ, SO THE LIMIT DOES NOT EXISTS

$\Downarrow$

THE FUNCTION IS NOT CONTINOUS

c) $f(x,y)=\begin{cases}\dfrac{x^2}{\sqrt{x^2+y^2}} & \text{IF } (x,y)\neq(0,0)\\ 0 & \text{IF } (x,y)=(0,0)\end{cases}$

SWITCH TO POLAR COORDINATES
$\begin{cases}x=\rho\cos\theta\\ y=\rho\sin\theta\end{cases}$

$$\lim_{(x,y)\to(0,0)}\frac{\rho^{\cancel{2}}\cos^2\theta}{\cancel{\rho}}=\lim_{(x,y)\to(0,0)}\rho\cos^2\theta=0$$

THE LIMIT DOES NOT DEPENDS ON θ

$$|\rho\cos^2\theta-0|\le\rho=g(\rho)\to 0 \text{ AS } \rho\to 0$$

$l=0$ IS THE LIMIT

f IS CONTINUOS

d)

$$f(x,y) = \begin{cases} \dfrac{\log(1+(x-y)^3)}{x^2+y^2} & \text{IF } (x,y) \neq (0,0) \\ 0 & \text{IF } (x,y) = (0,0) \end{cases}$$

REMEMBER

$$\lim_{\epsilon \to 0} \frac{\log(1+\epsilon)}{\epsilon} = 1$$

a) $\epsilon = 0$ $\quad \epsilon = (x-y)^3 = 0 \ ; \ x = y$

$$\lim_{(x,y)\to(0,0)} \frac{\log(1)}{2x^2} = 0 \quad \text{IF } (x \neq 0)$$

b) $\epsilon \neq 0$ $\quad x \neq y$

$$\lim_{(x,y)\to(0,0)} \underbrace{\frac{\log(1+(x-y)^3)}{(x-y)^3}}_{\to 1} \cdot \frac{(x-y)^3}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{(\rho\cos\theta - \rho\sin\theta)^3}{\rho^2} = \lim_{(x,y)\to(0,0)} \frac{\rho^{\cancel{3}}(\cos\theta - \sin\theta)^3}{\cancel{\rho^2}} \xrightarrow[\rho \to 0]{} 0$$

TEST

$$|\rho(\cos\theta - \sin\theta)^3 - 0| = \rho|(\cos\theta - \sin\theta)^3| \leq \rho = g(\rho) \longrightarrow 0 \text{ AS } \rho \to 0$$

e)

$$f(x,y) = \begin{cases} \dfrac{\log(1+|x-y|)}{\sqrt{x^2+y^2}} & \text{IF } (x,y) \neq (0,0) \\ 0 & \text{IF } (x,y) = (0,0) \end{cases}$$

SWITCH TO POLAR COORDINATES

$$\lim_{(x,y)\to(0,0)} \underbrace{\frac{\log(1+|\rho\cos\theta - \rho\sin\theta|)}{\rho|\cos\theta - \sin\theta|}}_{\to 1} \cdot \frac{\cancel{\rho}|\cos\theta - \sin\theta|}{\cancel{\rho}} = |\cos\theta - \sin\theta|$$

DEPENDS ON θ, THE LIMIT DOES NOT EXISTS, f IS NOT CONTINUOS

f)

(f) $f(x,y) = \begin{cases} \dfrac{(\sin(x+y))^2}{|x|+|y|} & \text{se } (x,y) \neq (0,0) \\ 0 & \text{se } (x,y) = (0,0) \end{cases}$

$$\lim_{(x,y)\to(0,0)} \frac{\sin^2(x+y)}{|x|+|y|}$$

USE THE SLOPE TRICK $y = mx$

$$\lim_{x\to 0} f(x, mx) = \lim_{x \to 0} \frac{\sin^2(x+mx)}{|x|+|mx|}$$

- IF $x = -mx$ $\quad (m=-1)$ $\quad \lim_{x\to 0} (x, mx) = 0$

- IF $x \neq -mx$

$$\lim_{x\to 0} \underbrace{\frac{\sin^2(x+mx)}{(x+mx)^2}}_{\to 1} \cdot \frac{(x+mx)^2}{|x|+|mx|} = \lim_{x\to 0} \frac{(x+mx)^2}{|x|\,(1+|m|)} = \lim_{x\to 0} \frac{x^2+m^2x^2+2mx^2}{|x|(1+|m|)} = \lim_{x\to 0} \frac{\cancel{|x|}|x|(1+m^2+m)}{\cancel{|x|}(1+|m|)} = 0$$

INDIPENDENT FROM m

STILL NEED TO PROVE THAT $l = 0$ IS THE LIMIT

• for $x = -y$ $f(x,-x) = 0$

• for $x \neq -y$

$$|f(x,y) - 0| = \left|\underbrace{\frac{\sin^2(x+y)}{(x+y)^2}}_{\to 1} \cdot \frac{(x+y)^2}{|x|+|y|}\right| \leq \left|\frac{(x+y)^2}{|x|+|y|}\right| \leq \left|\frac{(x+y)^2}{|x|+|y|}\right| \leq \left|\frac{(|x|+|y|)^2}{|x|+|y|}\right| = |x|+|y| = g_2(x,y)$$

ALSO $g_2(x,y) \to 0$ AS $(x,y) \to (0,0)$

THE LIMIT EXISTS, IT IS $l = 0 = f(0,0)$

f IS CONTINOUS

g) (g) $f(x,y) = \begin{cases} (x-1)\dfrac{\sin(2(x^2+y^2))}{x^2+y^2} & \text{se } (x,y) \neq (0,0) \\ -2 & \text{se } (x,y) = (0,0) \end{cases}$

$$\begin{cases} (x-1)\dfrac{\sin(2x^2+2y^2)}{x^2+y^2} & \text{IF } (x,y) \neq (0,0) \\ -2 & \text{IF } (x,y) = (0,0) \end{cases}$$

POLAR COORDINATES

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

$$\lim_{\rho\to 0} (\rho\cos\theta - 1)\underbrace{\frac{\sin(2\rho^2)}{2\rho^2}}_{\to 1} 2 = \underbrace{2\rho\cos\theta}_{\to 0} - 2 = -2$$

CHECK IF $l = -2$ IS THE LIMIT

$$|f(\rho\cos\theta, \rho\sin\theta) - (-2)| = \left|(\rho\cos\theta - 1)\frac{\sin(2\rho^2)\cdot 2}{2\rho^2} + 2\right| = \left|2\rho\cos\theta\underbrace{\frac{\sin(2\rho^2)}{2\rho^2}}_{\to 1} - \frac{2\sin(2\rho^2)}{2\rho^2} + 2\right| = \left|2\rho\cos\theta - \frac{2\sin(2\rho^2)}{2\rho^2} + 2\right|$$

$$\leq |2\rho\cos\theta| + \left|\frac{-2\sin(2\rho^2)}{2\rho^2} + 2\right| \leq 2\rho\underbrace{|\cos\theta|}_{\leq 1} + \left|2 - 2\underbrace{\frac{\sin(2\rho^2)}{2\rho^2}}_{\to 1}\right| \leq 2\rho + |2-2| = g(\rho) \to 0 \text{ AS } \rho \to 0$$

DEPENDS ONLY ON ρ

$l = -2$ IS THE LIMIT! AND $l = -2 = f(0,0)$

$\boxed{f \text{ IS CONTINUOS}}$

h) (h) $f(x,y) = \begin{cases} \dfrac{(\arctan(xy^2))^2}{(x^2+y^2)^2} & \text{se } (x,y) \neq (0,0) \\ 0 & \text{se } (x,y) = (0,0) \end{cases}$

$$\lim_{(x,y)\to(0,0)} \frac{\arctan^2(xy^2)}{(x^2+y^2)^2} = \lim_{(x,y)\to(0,0)} \underbrace{\frac{\arctan^2(xy^2)}{(xy^2)^2}}_{\to 1} \frac{(xy^2)^2}{(x^2+y^2)^2} = \lim_{(x,y)\to(0,0)} \frac{(\rho^3\cos\theta\sin^2\theta)^2}{\rho^4} =$$

$$\lim_{(x,y)\to(0,0)} \frac{\rho^{\cancel{6}\,2}\cos^2\theta\sin^4\theta}{\cancel{\rho^4}} \longrightarrow 0 \text{ AS } \rho \to 0 \quad l = 0 \text{ CANDIDATE}$$

$$\left|\underbrace{\frac{\arctan^2(\rho^3\cos\theta\sin^2\theta)}{(\rho^3\cos\theta\sin^2\theta)^2}}_{\leq 1} \cdot \rho^2 \underbrace{\cos^2\theta\sin^4\theta}_{\leq 1}\right| \leq \rho^2 = g(\rho) = 0 \text{ AS } \rho \to 0$$

$l = 0 = f(0,0)$ IS THE LIMIT

f IS CONTINUOS

i)

(i) $f(x,y) = \begin{cases} \dfrac{\sqrt{|x|} - \sqrt{|y|}}{\sqrt{|x|} + \sqrt{|y|}} & \text{se } (x,y) \neq (0,0) \\ 0 & \text{se } (x,y) = (0,0) \end{cases}$

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{|x|}-\sqrt{|y|}}{\sqrt{|x|}+\sqrt{|y|}} \cdot \frac{\sqrt{|x|}-\sqrt{|y|}}{(\sqrt{|x|}-\sqrt{|y|})} = \lim_{(x,y)\to(0,0)} \frac{(\sqrt{|x|}-\sqrt{|y|})^2}{|x|-|y|}$$

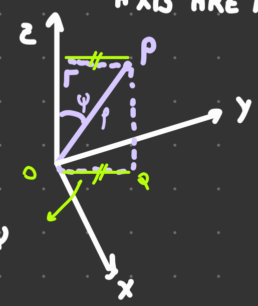
SLOPE TRICK

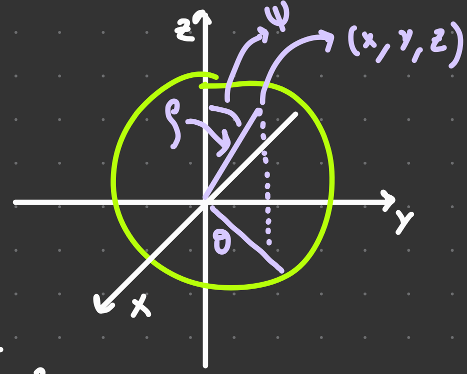
$$\lim_{(x,mx)\to(0,0)} \frac{\sqrt{|x|}-\sqrt{|mx|}}{\sqrt{|x|}+\sqrt{|mx|}} = \lim_{(x,mx)} \frac{\cancel{\sqrt{|x|}}\,(1-\sqrt{m})}{\cancel{\sqrt{|x|}}\,(1+\sqrt{m})} \to \frac{1-\sqrt{m}}{1+\sqrt{m}}$$

IT DEPENDS ON m $\Rightarrow \nexists$ LIMIT

(ρ, θ, φ) $\varphi = \arccos \frac{z}{\rho}$

AXIS ARE ROTATED





$\overline{OQ} = \rho \sin\varphi$

$\rho = \sqrt{x^2+y^2+z^2}$; $z = \pm\sqrt{\rho^2 - x^2 - y^2}$

$\overline{OQ} = \rho \sin\varphi$

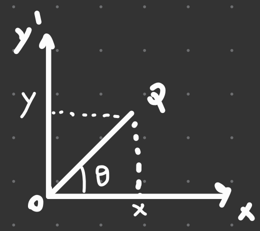
$z = \pm\sqrt{\rho^2 \underbrace{- x^2 - y^2}_{-(OQ)^2}}$

$\Downarrow$

$z = \pm\sqrt{\rho^2 - \rho^2 \sin^2\varphi}$

$= \pm\sqrt{\rho^2 \underbrace{(1-\sin^2\varphi)}_{\cos^2\varphi}} = \pm\rho|\cos\varphi| = z$

COS CAN BE + OR -



$OQ = \sqrt{x^2+y^2}$

$\varphi \in [0, \pi]$

$z>0 \iff \cos\varphi > 0 \iff \varphi \in [0, \pi/2]$

$z<0 \iff \cos\varphi < 0 \iff \varphi \in [\frac{\pi}{2}, \pi]$

$z = \begin{cases} +\rho(+\cos\varphi) & z>0 \\ -\rho(-\cos\varphi) & z<0 \end{cases}$

$\rho(\cos\varphi)$

$z = \rho\cos\varphi \Rightarrow \varphi = \arccos\frac{z}{\rho}$

CHANGE OF COORDINATES

POLAR, IN 2D

$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$

CYLINDRICAL, IN 3D

$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}$

SPHERICAL COORDINATES

$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$

1 $P = (\overset{x}{\sqrt{2}}, \overset{y}{-\sqrt{2}}, \overset{z}{-2})$

CYLINDRICAL COORD.

$\begin{cases} \rho = \sqrt{x^2+y^2} = \sqrt{2+2} = 2 \\ \theta = \arctan\frac{y}{x} + 2\pi = \frac{7}{4}\pi \end{cases}$

SPHERICAL COORDINATES

$\begin{cases} \rho = \sqrt{x^2+y^2+z^2} = \sqrt{8} = 2\sqrt{2} \\ \theta = \frac{7}{4}\pi \\ \varphi = \arccos\frac{z}{\rho} = \arccos\left(\frac{-\cancel{2}}{2\sqrt{2}}\right) = \frac{3}{4}\pi \end{cases}$

3) LINEAR $L(\underline{x}) = \underline{\underline{A}}\,\underline{x}$

$L(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 L(\underline{x}_1) + \lambda_2 L(\underline{x}_2)$

AFFINE FUNCTION

$L(\underline{x}) = \underline{\underline{A}}\,\underline{x} + \underline{y}^0$

$f(x) = x^N + (\dots)$ $\quad N \neq 0,1 \Rightarrow \begin{cases} \text{NOT LINEAR} \\ \text{NOT AFFINE} \end{cases}$ $\quad f(x) \cancel{=} x^2$

$f(x,y)\,\cancel{=}\,\frac{x}{y}$

$f(x,y\dots)$ $\quad xy$

$f(f_1, f_2, \dots)$ f IS LINEAR OR (AFFINE) $\Leftrightarrow$ f_i ARE LINEAR (OR AFFINE) $\forall i$

7) $f(x,y,z) = (\underbrace{3x}_{f_1}, \underbrace{y-2z}_{f_2})$ $\qquad f: \mathbb{R}^3 \to \mathbb{R}^2$, $\quad (x,y,z) \mapsto (f_1, f_2)$

f IS L(A) $\Leftrightarrow$ f_1, f_2 ARE L(A)

$f_1 = 3x = \underbrace{[3,0,0]}_{\underline{\underline{A}}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\underline{x}} = 3x + 0 + 0 = \underline{\underline{A}}\,\underline{x} \Rightarrow$ LINEAR

$f_2 = y - 2z = \underbrace{\begin{bmatrix} 0 & 1 & -2 \end{bmatrix}}_{\tilde{\underline{\underline{A}}}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \tilde{\underline{\underline{A}}}\,\underline{x} \Rightarrow$ LINEAR

} SO f IS LINEAR

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5) $\exists_1 > 0$ $\quad \min_{\theta \in [0, 2\pi]} (|\cos\theta| + |\sin\theta|)$

$[0, 2\pi] \Rightarrow$ WEIESTRASS EXIST MINIMUM $\Rightarrow \exists$ MIN, MAX

$x = \rho\cos\theta$, $\quad y = \rho\sin\theta$

$\exists c_0, c_1 \quad c_0\sqrt{x^2+y^2} \leq |x| + |y| \leq c_1 \underbrace{\sqrt{x^2+y^2}}_{\rho}$

$\Rightarrow$ POLAR COOR

$c_0\rho \leq \rho(|\cos\theta| + |\sin\theta|) \leq c_1\rho$

$c_0 \leq (|\cos\theta| + |\sin\theta|) \leq c_1$

$c_0 = \min$

$c_1 = \max$

7) h_7 $\quad f(x,y,z) = \arcsin(xy) - z^2$

$\sin\theta \in [-1,1]$ $\quad \sin\theta = xy \Rightarrow xy \in [-1,1]$

$-1 \le xy \le 1 \iff |xy| \le 1$

$|x|\,|y| \le 1 \quad |x| \le \frac{1}{|y|}$

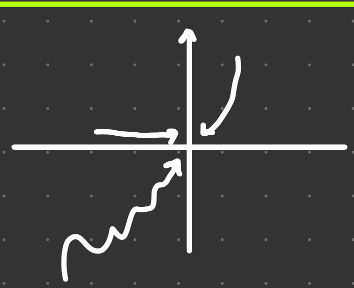
$A = \{ (x,y,z) \in \mathbb{R}^3 \text{ s.t. } |x| \le \frac{1}{|y|} \}$

$$\frac{df}{dx} = \frac{1}{\sqrt{1-(xy)^2}} \underbrace{\frac{d(xy)}{dx}}_{y} - \underbrace{\frac{d(z^2)}{dx}}_{=0} = \frac{1}{\sqrt{1-(x,y)^2}} \cdot y$$

$$\frac{df}{dy} = \frac{1}{\sqrt{1-(xy)^2}} x + 0$$

$$\frac{df}{dz} = 0 - 2z$$

LIMIT



$\boxed{\text{Ex 6}}$ 1. TEST OF THE NON-EXISTENCE OF THE LIMIT

1. "SLOPE TRICK" $\quad y = mx \Rightarrow \lim_{x\to 0} f(x, mx) = l$

IF l DEPENDS ON $m \Rightarrow \nexists$ LIMIT

IF l DOES NOT DEPEND ON $m \Rightarrow$ YOU CAN SAY NOTHING

2. "SWITCH TO POLAR COORDINATES"

$\lim_{\rho\to 0} f(x = \rho\cos\theta, y = \rho\sin\theta) = l$

IF l DEPENDS ON $\theta \Rightarrow \nexists$ LIMIT

IF l DOES NOT DEPEND ON $\theta \Rightarrow$ YOU CAN SAY NOTHING $\rightarrow l$ CANDIDATE LIMIT

FIND THE LIMIT

2. POLAR COORDINATES, FIND A FUNCTION $g(\rho)$, $g(\rho) \xrightarrow[\rho\to 0]{} 0$

$$|f(x = \rho\cos\theta, y = \rho\sin\theta) - l| \le g(\rho)$$

$f(\rho,\theta) \xrightarrow[\rho\to 0]{} l$ UNIFORMLY

FIND $g(x,y)$ S.T $|f(x,y)-\lambda| \le g(x,y)$

IF $\lim_{(x,y)\to(0,0)} g(x,y) = 0 \Rightarrow \lambda$ IS THE LIMIT

$g \quad f(x,y) = \begin{cases} (x-1)\dfrac{\sin(2(x^2+y^2))}{x^2+y^2} & (x,y)\neq(0,0) \\ -2 & (x,y)=(0,0) \end{cases}$

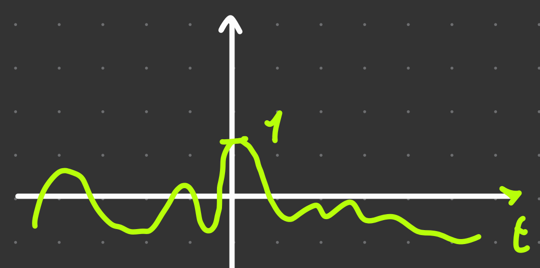
TEST: SHIFT TO POLAR COORDINATES

$$f(\rho\cos\theta, \rho\sin\theta) = (\rho\cos\theta - 1)\frac{\sin(2\rho^2)}{\rho^2} = (\rho\cos\theta-1)\,2\,\frac{\sin(2\rho^2)}{2\rho^2}$$

$$\xrightarrow[\rho\to 0]{} -2$$

$$|f(\rho\cos\theta,\rho\sin\theta) - (-2)| = \left|(\rho\cos\theta-1)\,2\,\frac{\sin(2\rho^2)}{(2\rho^2)} + 2\right| = \left|2\rho\cos\theta\,\frac{\sin(2\rho^2)}{(2\rho^2)} - 2\frac{\sin(2\rho^2)}{2\rho^2} + 2\right| \le$$

$$\le \Bigg|2\rho\cos\theta\,\underbrace{\frac{\sin(2\rho^2)}{2\rho^2}}_{\le 1}\Bigg| + \left|-2\frac{\sin(2\rho^2)}{2\rho^2} + 2\right|$$



$$\le 2\rho|\cos\theta| + \left|2 - 2\frac{\sin(2\rho^2)}{2\rho^2}\right| \le 2\rho\cdot 1 + \left|2 - \frac{2\sin(2\rho^2)}{2\rho^2}\right| = g(\rho)$$

$$\xrightarrow[\rho\to 0^+]{} 0 + |2-2| = 0$$

$$\Rightarrow \lambda = -2 \text{ IS } \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = -2$$

f IS CONTINUOUS

6a)

$$\lim_{\rho \to 0} \frac{\rho^{\cancel{3}} \cos^3\theta - 3\rho^{\cancel{3}} \cos\theta \sin^2\theta}{\cancel{\rho^2}} = \lim_{\rho \to 0} \rho \cos^3 \sin\theta - 3\rho \cos\theta \sin^2\theta \xrightarrow[\text{AS } \rho \to 0]{} 0$$

$l = 0$ CANDIDATE

$$|f(\rho\cos\theta, \rho\sin\theta) - l| = |\rho(\cos^3\theta - \cos\theta\sin^2\theta) - 0| = \rho\underbrace{|\cos^3\theta - \cos\theta\sin^2\theta|}$$

(a) $f(x,y) = \begin{cases} \dfrac{x^3 - 3xy^2}{x^2+y^2} & \text{se } (x,y) \neq (0,0) \\ 0 & \text{se } (x,y) = (0,0) \end{cases}$

STEP 1. SWITCH TO POLAR COORDINATES

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \qquad f(\rho\cos\theta, \rho\sin\theta) = \frac{\cancel{\rho^3}^{\,\rho}\cos^3\theta - 3\cancel{\rho^3}^{\,\rho}\cos\theta\sin^2\theta}{\cancel{\rho^2}} = \rho(\cos^3\theta - 3\cos\theta\sin^2\theta)$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{\rho\to 0} f(\rho\cos\theta, \rho\sin\theta) = \lim_{\rho\to 0} \rho\underbrace{(\cos^3\theta - 3\cos\theta\sin^2\theta)}_{\substack{-1-3 \quad 1+3 \\ -4<\dots<4}} = 0$$

THE LIMIT IS INDIPENDENT FROM θ

$\ell = 0$ CANDIDATE

STEP 2. PROVE THAT $\ell=0$ (THE CANDIDATE) IS THE LIMIT

$$|f(x,y) - \ell| = |f(\rho\cos\theta, \rho\sin\theta) - 0| = |\rho(\cos^3\theta - 3\cos\theta\sin^2\theta)| = \rho\underbrace{|\cos^3\theta - 3\cos\theta\sin^2\theta|}_{\leq 1} \leq \rho = g(\rho)$$

SINCE $g(\rho)\to 0$ AS $\rho\to 0$ THEN $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0) \Rightarrow f$ IS CONTINOUS