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THEORY OF BIVARIATE, CONTINUOUS RANDOM VECTORS

1) Definition [RANDOM VECTOR] RANDOM VARIABLES BELONGS TO SOME INTERVAL

$$(X,Y) : \Omega \to \mathbb{R}^2$$

where both $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ are random variables.

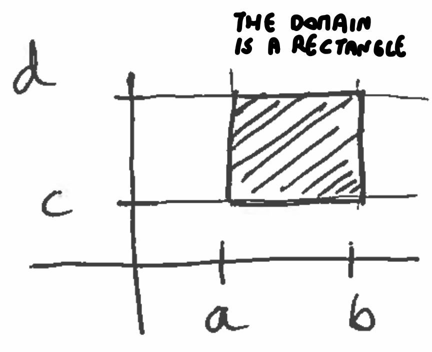
2) PROBABILITY DISTRIBUTION OF A CONTINUOUS RANDOM VECTOR

[$a<b$ FOR X, $c<d$ FOR Y fixed]

$$P[(X,Y) \in (a,b) \times (c,d)] =$$

$$= P[X \in (a,b), Y \in (c,d)]$$

$$= \boxed{\int_a^b \int_c^d f_{(X,Y)}(u,v)\,du\,dv}$$



where $f_{(X,Y)} : \mathbb{R}^2 \to \mathbb{R}$ is the (JOINT) DENSITY FUNCTION of the random vector (X,Y)

3) <u>Definition</u> [DENSITY FUNCTION]

A density function $f_{(X,Y)} : \mathbb{R}^2 \to \mathbb{R}$ is a function with the following properties:

3A) $f_{(X,Y)}(u,v) \geq 0 \quad \forall (u,v) \in \mathbb{R}^2$

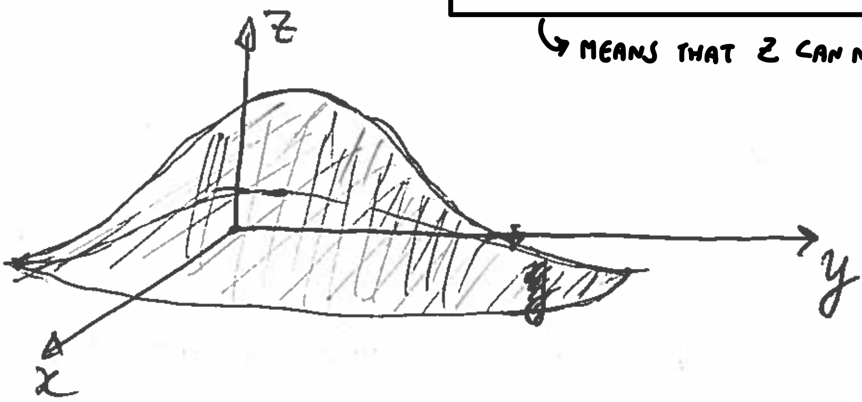
3B) $f_{(X,Y)}$ is <u>integrable</u> on every rectangle

3C) $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{(X,Y)}(u,v)\, du\, dv = 1$

<u>Remarks</u>

3A) Geometrically speaking, $f_{(X,Y)}(u,v) \geq 0$ means that the graph of $f_{(X,Y)}$ is a <u>surface</u> lying <u>upon</u> the xy-plane.

MEANS THAT z CAN NOT BE NEGATIVE



3B) Recall that the CONTINUITY of $f_{(X,Y)}$ is a sufficient condition yielding integrability.

3c) $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{(X,Y)}(u,v)\,du\,dv = 1$ means that the volume included between the xy-plane and the surface defined by the graph of $f_{(X,Y)}$ is 1. It is related to the FIRST AXIOM of probability: $\boxed{P(\Omega)=1.}$

4) DISTRIBUTION FUNCTION of (X,Y)

JOINT PROBABILITY of $X \leq s$ AND $Y \leq t$

FIXED NUMBERS

JOINT DISTRIBUTION FUNCTION

$$F_{(X,Y)}(s,t) = P[X \leq s,\ Y \leq t] \qquad (s,t) \in \mathbb{R}^2$$

INTEGRATE

$$= \int_{-\infty}^{s}\int_{-\infty}^{t} f_{(X,Y)}(u,v)\,du\,dv$$

5) MARGINAL DISTRIBUTIONS

Given the random vector $(X,Y): \Omega \to \mathbb{R}^2$ with joint density function $f_{(X,Y)}: \mathbb{R}^2 \to \mathbb{R}$ we define:

OBTAIN SOMETHING THAT DEPENDS ON X, SO YOU DERIVATE WRT Y

FIRST MARGINAL DENSITY: $\boxed{f_X(x) = \int_{-\infty}^{+\infty} f_{(X,Y)}(x,y)\,dy}$

($x \in \mathbb{R}$ fixed)

SECOND MARGINAL DENSITY: $\boxed{f_Y(y) = \int_{-\infty}^{+\infty} f_{(X,Y)}(x,y)\,dx}$

OBTAIN SOMETHING THAT DEPENDS ON Y, SO YOU DERIVATE WRT TO X

($y \in \mathbb{R}$ fixed)

Probabilistic meaning of the marginals.

Start by considering

$$P[X \le s] = P[X \le s, Y < +\infty] =$$

DOES NOT ADD ANY CONSTRAINTS
$P(A) = P(A \cap \Omega)$
IT IS AGAIN A

$s \in \mathbb{R}$, fixed

$$= \lim_{t \to +\infty} P[X \le s, Y \le t].$$

JOINT DISTRIBUTION FUNCTION

By resorting to the notion of (joint) distribution function, this fact can be written as:

$$F_X(s) = P[X \le s] = \lim_{t \to +\infty} F_{(X,Y)}(s,t)$$

$$= \lim_{t \to +\infty} \int_{-\infty}^{s} \int_{-\infty}^{t} f_{(X,Y)}(x,y)\,dx\,dy$$

$$= \underset{(x)}{\int_{-\infty}^{s}} \underset{(y)}{\int_{-\infty}^{+\infty}} f_{(X,Y)}(x,y)\,dx\,dy$$

We deduce that $F_X(s) = \int_{-\infty}^{s} f_X(x)\,dx$

with $\boxed{f_X(x) = \int_{-\infty}^{+\infty} f_{(X,Y)}(x,y)\,dy}$.

FIRST MARGINAL FORMULA OF BEFORE

A similar argument works for the second marginal. $f_Y(y) = \int_{-\infty}^{+\infty} f_{(X,Y)}(x,y)\,dx$

SAME AS $P(A \cap B) = P(A)P(B)$ BUT DO NOT USE $X =$ "SOMETHING" BECAUSE FOR CONTINUOUS $= 0$

6) INDEPENDENCE OF RANDOM VARIABLES.

Remark. Independence is not a property of the two random variables X, Y, thought of as two disjoint entities. Rather, Independence is a property of the random vector (X, Y). Then, if you only know the marginal distributions / densities, you cannot decide ~~[illegible]~~ whether X, Y are independent or not.

Definition [For general random vectors]
Given the random vector (X,Y) with joint distribution function $F_{(X,Y)}$, we say that its component X and Y are independent if

$$\boxed{F_{(X,Y)}(s,t) = F_X(s) \cdot F_Y(t), \quad \forall (s,t) \in \mathbb{R}^2}$$

Theorem. If (X,Y) is continuous random vector with density $f_{(X,Y)}$, then

X and Y are independent iff

$$\boxed{f_{(X,Y)}(u,v) = f_X(u) \cdot f_Y(v) \quad \forall (u,v) \in \mathbb{R}^2}$$

Proposition. Let (X,Y) be a random vector with independent components X and Y. If $\varphi: \mathbb{R} \to \mathbb{R}$ and $\psi: \mathbb{R} \to \mathbb{R}$ are two functions then the transformed random vector $(\varphi(X), \psi(Y))$ has independent components, that is $\varphi(X)$ and $\psi(Y)$ are independent

$\to x^2$ * $\to e^y$ *

* TWO RANDOM FUNCTIONS IF INDEPENDENT BEFORE $\Rightarrow$ INDEPENDENT ALSO NOW

NOT INVOLVES Y NOT INVOLVES X

$U = \varphi(X) \qquad V = \psi(Y)$

X INDIPENDENT of $Y \Rightarrow U$ INDIPENDENT of V

7) EXPECTATION

Let (X,Y) be a continuous random vector with joint density $f_{(X,Y)}$. Let $h: \mathbb{R}^2 \to \mathbb{R}$ be a function. Then, we put

$$\boxed{\mathbb{E}[h(X,Y)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(x,y) f_{(X,Y)}(x,y)\,dx\,dy}$$

7) COVARIANCE

$$\text{Cov}(X,Y) = \mathbb{E}[X \cdot Y] - \mathbb{E}[X] \cdot \mathbb{E}[Y]$$

where :

$$\mathbb{E}[X \cdot Y] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy \, f_{(X,Y)}(x,y)\,dx\,dy$$

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x f_X(x)\,dx = m_X$$

MARGINAL of X

$$\mathbb{E}[Y] = \int_{-\infty}^{+\infty} y f_Y(y)\,dy = m_Y$$

MARGINAL of Y

Proposition [Properties* of the COVARIANCE]

A) $\boxed{\text{Cov}(X,Y) = \mathbb{E}[(X-m_X)\cdot(Y-m_Y)]}$

B) $\text{Cov}(X,X) = \text{Var}(X)$

C) Covariance is BILINEAR, i.e.

- $\text{Cov}(a_1X_1 + a_2X_2, Y) = a_1\text{Cov}(X_1,Y) + a_2\text{Cov}(X_2,Y)$, $a_1, a_2 \in \mathbb{R}$

- $\text{Cov}(X, b_1Y_1 + b_2Y_2) = b_1\text{Cov}(X,Y_1) + b_2\text{Cov}(X,Y_2)$, $b_1, b_2 \in \mathbb{R}$

D) $Cov(X,Y) = Cov(Y,X)$

E) Cauchy-Schwarz inequality:

$$|Cov(X,Y)| \leq \sqrt{Var(X) \cdot Var(Y)}$$

F) Geometrically speaking, if we restrict the Covariance to the space (linear space)

$$L^2_0 = \{ \underset{\text{R.V.}}{X : \Omega \to \mathbb{R}} \mid \mathbb{E}[X^2] < +\infty, \ \mathbb{E}[X] = 0 \}$$

then,

$$\boxed{Cov : L^2_0 \times L^2_0 \longrightarrow \mathbb{R}}$$ is a scalar product

8) TRANSFORMATION OF RANDOM VECTORS.

OPERATION TO DO ON RANDOM VECTOR

Given the random vector X, Y there are two kinds of transformation:

FIRST KIND

$$Z = g(X,Y), \quad g : \mathbb{R}^2 \longrightarrow \mathbb{R}$$ 1

Z is a $\boxed{\text{random variable}}$

E.g. $Z = X + Y$, or $Z = X \cdot Y$

SECOND KIND

$(U,V) = h(X,Y)$, $h : \mathbb{R}^2 \to \mathbb{R}^2$.

(U,V) is a random vector.

E.g. $(U,V) = (X+Y,\ X-Y)$ or

$(U,V) = (X+3Y,\ XY)$

Problem. Given that (X,Y) is a continuous random vector, find f_Z and $f_{(U,V)}$.

Problem 1 : f_Z

Split into 3 steps, starting from

$F_Z(s) = P[Z \leq s]$, $s \in \mathbb{R}$.

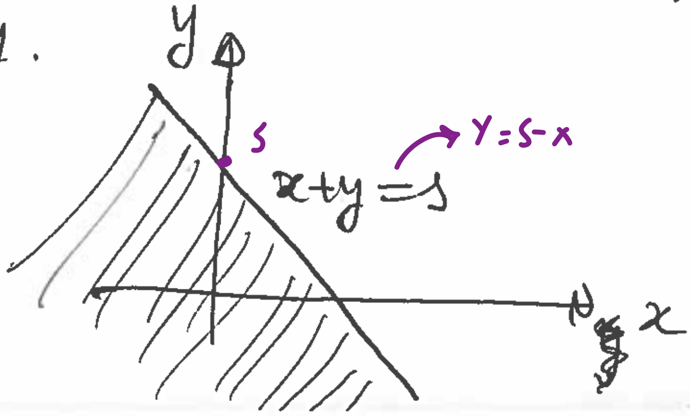
Step 1. $\{ g(x,y) \leq s \} = \{ (x,y) \in \mathbb{R}^2 \mid g(x,y) \leq s \}$

It is a problem of ANALYTIC GEOMETRY,

E.g. $g(x,y) = x+y$.

$\{ (x,y) \in \mathbb{R}^2 \mid x+y \leq s \}$

$=$ HALF PLANE



Step 2 Putting $B(s) = \{(x,y)\in\mathbb{R}^2 \mid g(x,y)\le s\} \subseteq \mathbb{R}^2$

We consider the problem of evaluating

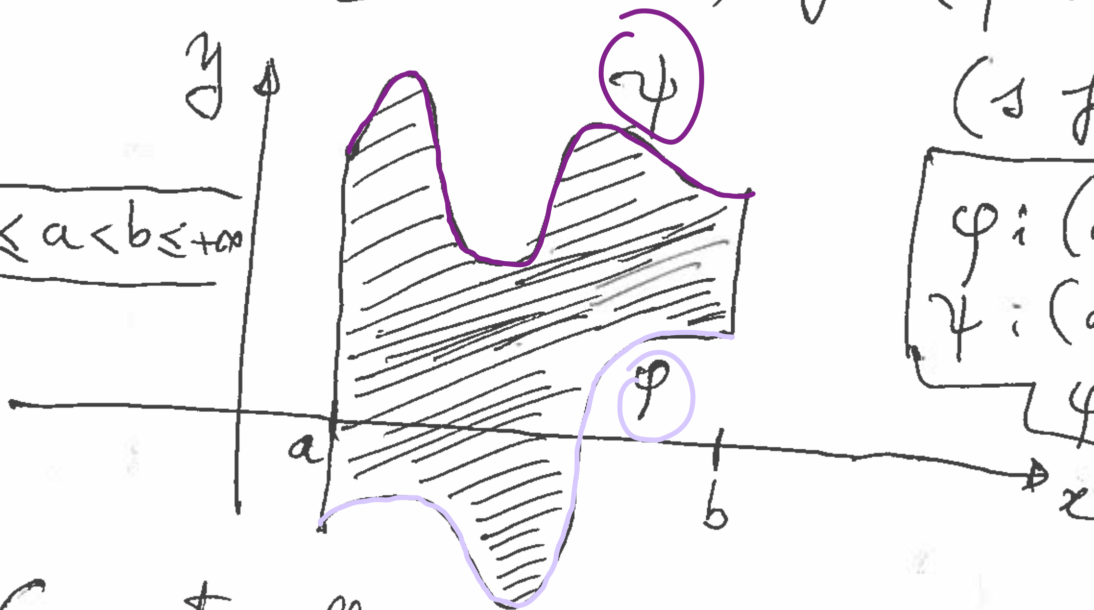
$P[(X,Y)\in B(s)]$. [s fixed]

Assume that we can write

$$B(s) = \{ x\in(a,b),\ y\in(\varphi(x),\psi(x)) \}$$

(s fixed)

$-\infty \le a < b \le +\infty$



$$\boxed{\begin{array}{l} \varphi : (a,b) \to \overline{\mathbb{R}} \\ \psi : (a,b) \to \overline{\mathbb{R}} \\ \varphi(x) < \psi(x) \end{array}}$$

Geometrically, φ, ψ are two CURVES with φ lying below of ψ.

$$\boxed{P[(X,Y)\in B(s)] = \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} f_{(X,Y)}(x,y)\,dy \right) dx}$$

Eg. For the half plane; $a=-\infty$, $b=+\infty$

$\varphi(x) \equiv -\infty$, $\psi(x) = s - x$

$$\text{DENSITY} \underset{\text{DIFFERENTIATE}}{\overset{\text{INTEGRATE}}{\rightleftarrows}} \text{DISTRIBUTION}$$

<u>Step 3</u> Once we have found

$$F_Z(s) = P[Z \leq s] = P[(X,Y) \in B(s)]$$

we conclude by resorting to

$$\boxed{f_Z(s) = F_Z'(s)} \quad \forall s \in \mathbb{R}$$

E.g. $g(x,y) = x+y \implies Z = X+Y$,

$$\boxed{f_Z(s) = \int_{-\infty}^{+\infty} f_{(X,Y)}(x, s-x)\,dx}$$

In fact, from step 2, we get

$$F_Z(s) = \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{s-x} f_{(X,Y)}(x,y)\,dy \right) dx$$

Now consider only the inner integral

$$\int_{-\infty}^{s-x} f_{(X,Y)}(x,y)\,dy = \int_{-\infty}^{s} f_{(X,Y)}(x, z-x)\,dz$$

by the transformation (change of variable) $y = z - x$.

Thus, by interchanging the order of integration (Fubini)

$$F_Z(s) = \int_{-\infty}^{s} \left(\int_{-\infty}^{+\infty} f_{(X,Y)}(x, z-x)\,dz \right) dx,$$

we have <u>conclude</u>

~~concluded~~

Second KIND of transformation.

This is more difficult, and we confine ourselves to considering $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that

i) h is one-to-one (injective and surjective)

ii) h is $C^1(\mathbb{R}^2;\mathbb{R}^2)$ [differentiable with continuous derivatives]

iii) h^{-1} is $C^1(\mathbb{R}^2;\mathbb{R}^2)$

$$f_{(U,V)}(u,v) = f_{(X,Y)}\left(h^{-1}(u,v)\right) \cdot \left| \det\left(\mathrm{Jac}[h^{-1}]_{(u,v)}\right) \right|$$

(i) $h^{-1}(u,v)$ is obtained by solving $h(x,y) = (u,v)$ and finding x e y in terms of u, v

(ii) $\mathrm{Jac}[h^{-1}](u,v) = \begin{pmatrix} \frac{\partial H_1}{\partial u} & \frac{\partial H_1}{\partial v} \\ \frac{\partial H_2}{\partial u} & \frac{\partial H_2}{\partial v} \end{pmatrix}$

$h^{-1}(u,v) = H(u,v)$

$= \left(H_1(u,v),\ H_2(u,v)\right)$

[JACOBIAN MATRIX]

(iii)

$$\left| \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| = |ad - bc|$$

9) CONDITIONAL DENSITIES

$$f_{Y|X}(y|x) = \frac{f_{(X,Y)}(x,y)}{f_X(x)} \quad [\text{if } f_X(x)>0]$$

Y GIVEN X; JOINT DENSITY; MARGINAL

$$f_{X|Y}(x|y) = \frac{f_{(X,Y)}(x,y)}{f_Y(y)} \quad [\text{if } f_Y(y)>0]$$

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10) CONDITIONAL EXPECTATIONS

SAME AS UNIVARIATE CASE

EXPECTATION OF Y GIVEN X = x

$$\mathbb{E}[Y|X=x] = \int_{-\infty}^{+\infty} y \, f_{Y|X}(y|x)\,dy$$

SEE BEFORE $\frac{f_{(x,y)}(x,y)}{f_x(x)}$

x fixed. More generally

$$\mathbb{E}[g(Y)|X=x] = \int_{-\infty}^{+\infty} g(y)\, f_{Y|X}(y|x)\,dy$$

SEE BEFORE

$$\mathbb{E}[X|Y=y] = \int_{-\infty}^{+\infty} x\, f_{X|Y}(x|y)\,dx$$

SEE BEFORE

y fixed

More generally

$$\mathbb{E}[g(X)|Y=y] = \int_{-\infty}^{+\infty} g(x)\, f_{X|Y}(x|y)\,dx$$

SEE BEFORE

EX 2 SESSION JULY 15, 2022

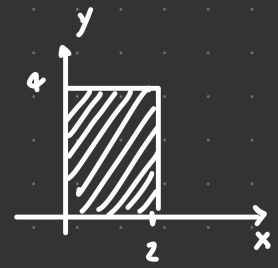
$$f_{(X,Y)}(x,y) = \begin{cases} \frac{y}{16} & \text{IF } (x,y) \in (0,2)\times(0,4) \\ 0 & \text{OTHERWISE} \end{cases}$$

2a) CHECK THAT $f_{(X,Y)}$ IS DENSITY AND FIND f_X, f_Y

ALREADY DONE

ALREADY DONE

$$f_X(x) = \begin{cases} 1/2 & \text{IF } x \in (0,2) \\ 0 & \text{OTHERWISE} \end{cases}$$



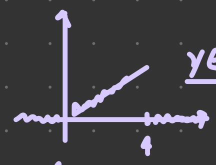
Y FIXED

$$f_Y(y) = \int_0^2 \frac{y}{16}\,dx = \frac{y}{16}[x]_0^2 = \frac{y}{8}$$

$$f_Y(y) = \begin{cases} y/8 & \text{IF } y \in (0,4) \\ 0 & \text{OTHERWISE} \end{cases}$$

EXTRA: CHECK THE COMPUTATION

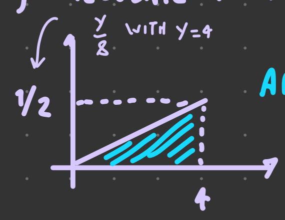
• IS IT TRUE THAT $f_Y(y) \geq 0$ (AKA NON NEGATIVE)



YES

• a) $\int_0^4 \frac{y}{8}\,dy = \frac{1}{8}\left[\frac{y^2}{2}\right]_0^4 = 1$ ü

• b) CALCULATE AREA OF RECTANGLE



$$\text{AREA} = \frac{4 \cdot \frac{1}{2}}{2} = 1$$

2b) FIND E(Y) AND VAR(Y)

$$\bullet E[Y] = \int_{-\infty}^{+\infty} y f_Y(y)\,dy = \int_0^4 y \cdot \frac{y}{8}\,dy = \frac{1}{8}\int_0^4 y^2\,dy = \frac{1}{8}\left[\frac{y^3}{3}\right]_0^4 = \frac{1}{\cancel{8}} \cdot \frac{\cancel{64}}{3} = \frac{8}{3}$$

ALREADY KNOW $= \frac{8}{3}$

$$\bullet VAR(Y) = E[Y^2] - (E[Y])^2 = 8 - \left(\frac{8}{3}\right)^2 = \frac{72-64}{9} = \frac{8}{9}$$

$$\int_{-\infty}^{+\infty} y^2 f_Y(y)\,dy = \int_0^4 y^2 \cdot \frac{y}{8}\,dy = \frac{1}{8}\left[\frac{y^4}{4}\right]_0^4 = 8$$

$$VAR(Y) = 8 - \left(\frac{8}{3}\right)^2$$

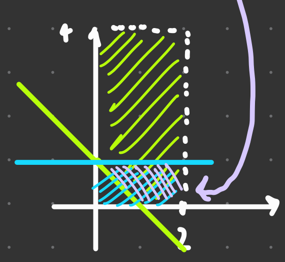
3c) $P(X+Y>1 \mid Y<1) = \frac{P(X+Y>1,\, Y<1)}{P(Y<1)} = \frac{\frac{5}{96}}{1/16} = \frac{5}{96}\cdot 16 = \frac{5}{6}$

$P[Y<1]$ REMEMBER THAT $f_Y(y) = \begin{cases} y/8 & y \in (0,4) \\ 0 & \text{OTHERWISE} \end{cases}$

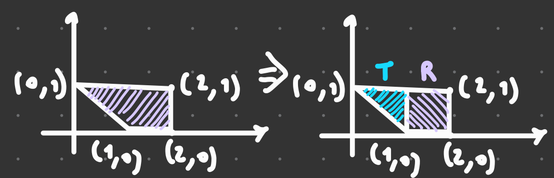
$$= \int_0^1 f_Y(y)\,dy = \int_0^1 y/8\,dy = \frac{1}{8}\left[\frac{y^2}{2}\right]_0^1 = \frac{1}{16}$$

$P[X+Y>1, Y<1] =$

$X+Y>1$
$\Downarrow$
• $Y>1-X$
• $Y<1$



$$\begin{cases} X+Y>1 \\ Y<1 \\ 0<X<2 \\ 0<Y<1 \end{cases} = \begin{cases} X+Y>1 \\ 0<Y<1 \\ 0<X<2 \end{cases}$$



$$P[X+Y>1, Y<1] = P[(X,Y)\in B] = \iint_B f_{(X,Y)}(x,y)\,dx\,dy = \overbrace{\iint_T f_{(X,Y)}(x,y)\,dx\,dy}^{T} + \overbrace{\iint_R f_{(X,Y)}(x,y)\,dx\,dy}^{R} = \int_0^1\left(\int_{1-x}^1 \frac{y}{16}\,dy\right)dx + \int_1^2\int_0^1 \left(\frac{y}{16}\right)dy\,dx =$$

$$= \frac{1}{16}\int_0^1 \left[\frac{y^2}{2}\right]_{1-x}^1 dx + \frac{1}{16}\int_1^2 \left[\frac{y^2}{2}\right]_0^1 dx = \frac{1}{32}\int_0^1 1-(1-x)^2\,dx + \frac{1}{32}\int_1^2 dx = \frac{1}{32}\int_0^1 2x - x^2\,dx + \frac{1}{32} = \frac{1}{32}\left[x^2 - \frac{x^3}{3}\right]_0^1 + \frac{1}{32} = \frac{1}{\cancel{32}_{16}}\cdot\frac{\cancel{2}^{1}}{3} + \frac{1}{32} = \frac{1}{48} + \frac{1}{32} = \frac{5}{96}$$

2e) $S = X+Y$; $T = Y$ (S,T NEW TRANSFORMED RANDOM VECTOR). FIND JOINT DENSITY $f_{(S,T)}$

$h(x,y) = (x+y, y)$ LINEAR TRANSFORMATION

$$\Downarrow$$

$$\begin{pmatrix} x+y \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{M}\begin{pmatrix} x \\ y \end{pmatrix}$$

h IS INVERTIBLE (ONE-TO-ONE) IFF M IS NON-SINGULAR

$DET(M) \neq 0$ $\quad DET(M) = (1\cdot 1 - 1\cdot 0) = 1 \Rightarrow M$ IS INVERTIBLE

FIND THE INVERSE

$$h(x,y) = \begin{pmatrix} u \\ v \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix} \rightarrow h^{-1}(w,v) = \begin{cases} x = w - v \\ y = v \end{cases}$$

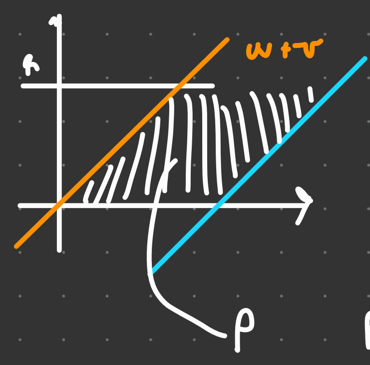
$$h^{-1}(w,v) = M^{-1}\begin{pmatrix} w \\ v \end{pmatrix} \qquad \begin{cases} x+y = w \,;\; x = u - v \\ y = v \end{cases}$$

COMPUTE THE JACOBIAN

$$h^{-1}(w,v) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} w \\ v \end{pmatrix}$$

$$JAC\left[h^{-1}\right] = M^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad \left|DET\left(JAC\left(h^{-1}\right)\right)\right| = \left|DET\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right| = 1$$

$$f_{(s,r)}(w,v) = f_{(x,y)}(w-v,v) \cdot 1 = \begin{cases} \frac{v}{16} & \text{IF } \begin{array}{l} 0 < w - v < 2 \\ 0 < w < 4 \end{array} \\ 0 & \text{OTHERWISE} \end{cases}$$



$0 < w - v \Leftrightarrow v < w$

$v - v < 2 \Leftrightarrow w < 2 + v$

$$P = \begin{cases} \frac{v}{16} & \text{IF } w,p \in P \\ 0 & \text{OTHERWISE} \end{cases}$$