Exercises - Calculus

Academic Year 2021-2022

Sheet 9

1. For each of the following functions, determine their domain of existence. Compute the limits at the extremes of the domain and establish whether they can be extended by continuity in some in some of the extreme points belonging to $\mathbb{R}$.

   (a) $f(x) = \dfrac{x^2}{x^2 - 4}$

   (b) $f(x) = \dfrac{\log(x+1)}{\sqrt{x}}$

   (c) $\arctan\left(\dfrac{1}{x}\right)$

   (d) $\arctan\left(\dfrac{1}{x^2}\right)$

   (e) $\dfrac{1 - \cos(x)}{x^{3/2}}$

2. Prove that the so-called *Dirichlet function*, defined as follows,

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

   is not continuous in x for any $x \in \mathbb{R}$. Establish the set of points where the function $g(x) = xf(x)$ is continuous.

3. Prove that the function $f(x) = x^3 - 3x^2 + 5x + 9$ admits a zero.

4. Generalize the previous exercise to show that any polynomial of degree n, with n odd, admits a zero.

5. Prove that there exists $x \in (0, +\infty)$ solving the following equation

$$2\arctan(x) + \log(1 + e^{-x^2}) - 1 = 0$$

6. Compute, where it exists, the derivative of the following functions

$$f(x) = \log(|x|) \quad \text{and} \quad f(x) = e^{|x|}$$

   For which $x \in \mathbb{R}$ the previous functions are not differentiable?

7. Compute, where it exists, the derivative of the following functions

   (a) $f(x) = x^6 - 2x^3 + 6x$ and $f(x) = \dfrac{1}{x^6 - 2x^3 + 6x}$

   (b) $f(x) = \left(x^3 - \dfrac{1}{x^3} + 3\right)^4$ and $f(x) = \left(\dfrac{1+x^2}{1+x}\right)^5$

(c) $f(x) = \sqrt{x^3+1} - x$ and $f(x) = \sqrt[3]{x^3+1} - x$

(d) $f(x) = x\log(x)$ and $f(x) = e^x(\log(-x) - 1)$

(e) $f(x) = \dfrac{1-x^2}{3+x}$ and $f(x) = \cos\left(\dfrac{1-x^2}{3+x}\right)$

(f) $f(x) = (\log(x^3))^2$ and $f(x) = \dfrac{e^{1/x}}{x}$

(g) $f(x) = x\tan(x) + \log(\cos(x))$ and $f(x) = x\arctan(x) + e^{\sin(x)}$

(h) $f(x) = x^2\arcsin(x-2)$ and $f(x) = 2\arcsin(\sqrt{x-2})$

(i) $f(x) = \sqrt{x}\sin(\sqrt{x})$. Does it exists $f'_+(0)$?

(j) $f(x) = \arctan(\cos(x))$ and $f(x) = \log(\log(x))$ and $f(x) = x^{\log(x)}$

8. Find the equation of the tangent line to the graph of f in the given points.

   (a) $f(x) = \sqrt{1+2x}$ at $(0,1)$

   (b) $f(x) = \sqrt[3]{1-4x}$ at $(0,1)$

   (c) $f(x) = e^{x^2-3}$ at $(1, f(1))$

   (d) $f(x) = \log(x^2 - 2x + 1)$ at $(2,0)$

   (e) $f(x) = \cos(\pi/x)$ at $(1,-1)$

   (f) $f(x) = \arctan(e^x)$ at $(0, \pi/4)$

9. For parameters α, $\beta \in \mathbb{R}$, let

$$f(x) = \begin{cases} \alpha x + \beta & \text{if } x \leq 1 \\ x^3 & \text{if } x > 1 \end{cases}$$

   Determine for which values of α and β we have that f is continuous on $\mathbb{R}$, and for which values α and β we have that f is differentiable on $\mathbb{R}$.

10. For parameters α, $\beta \in \mathbb{R}$, let

$$f(x) = \begin{cases} \alpha x + 2 & \text{if } x < 0 \\ \beta & \text{if } x = 0 \\ 2e^{\beta x} & \text{if } x > 0 \end{cases}$$

   Determine for which values of α and β we have that f is continuous on $\mathbb{R}$, and for which values α and β we have that f is differentiable on $\mathbb{R}$.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is differentiable in $\mathbb{R}$, that is, it is well defined $f' : \mathbb{R} \to \mathbb{R}$. Show that if f is even, then f' is odd and in particular $f'(0) = 0$. Show that if f is odd, then f' is even.

1a)

$D: \mathbb{R} - \{\pm 2\}$

$$f(x) = \frac{x^2}{x^2-4}$$

$$\frac{x^2}{x^2(1-4/x^2)} = \frac{1}{1-4/x^2}$$

$$\lim_{x \to +\infty} = 1^+$$

$$\lim_{x \to -\infty} = 1^+$$

$$\lim_{x \to 2^+} = \frac{1}{0^+} = +\infty$$

$$\lim_{x \to 2^-} = +\infty = \frac{1}{0^-} = -\infty$$

NO

$$\lim_{x \to -2^+} = -\infty$$

$$\lim_{x \to -2^-} = +\infty$$

1b) $f(x) = \dfrac{\log(x+1)}{\sqrt{x}}$

$\left.\begin{array}{l} x+1>0;\ x>-1 \\ x>0;\ \ x>0 \end{array}\right]$ $\forall x \in \mathbb{R} \mid x>0$

$\lim_{x \to +\infty} = 0^+$

1

$\lim_{x \to 0^+} \dfrac{\log(1)}{\sqrt{0}} = \dfrac{\log(x+1)}{\sqrt{x}} \dfrac{\sqrt{x}}{\sqrt{x}} = \dfrac{\log(x+1)\sqrt{x}}{x} = 0$

$\left[\dfrac{0}{0}\right]$

1c) $\arctan\left(\frac{1}{x}\right)$ $x \neq 0$ $\cos(x) \neq 0$

$x \neq \frac{\pi}{2} \pm k\pi$

$\arctan\left(\frac{1}{x}\right)$ $\forall x \in \mathbb{R} \mid x \neq 0,\ x \neq \frac{\pi}{2} \pm k\pi$

$\lim_{x \to \frac{\pi}{2}^+}$

1D) $\arctan\left(\frac{1}{x^2}\right)$

$x \neq 0 \quad \cos(x^2) \neq 0$

$x^2 \neq \frac{\pi}{2} \pm k\pi$

1E) $\frac{1-\cos(x)}{x^{3/2}}$ $\quad x^{-1/2} \cdot x^{-1/2}$

$x \in \mathbb{R} \mid x \neq 0$

$\frac{1-\cos(x)}{x} \frac{1}{\sqrt{x}}$

$\lim_{x \to 0^+} \underbrace{\frac{1-\cos(x)}{x}}_{0} \frac{1}{\sqrt{x}} =$

$\lim_{x \to 0^-} \underbrace{\frac{1-\cos(x)}{x}}_{0} \frac{1}{\sqrt{x}}$

$\lim_{x \to +\infty}$

$\lim_{x \to -\infty}$

3. Prove that the function $f(x) = x^3 - 3x^2 + 5x + 9$ admits a zero.

$$0 = x^3 - 3x^2 + 5x + 9$$

	1	-3	5	9
-1		-1	4	-9
	1	-4	9	0

$$(x^2 - 4x + 9)(x + 1) = 0$$

$$\downarrow \qquad\qquad \downarrow$$

$$\emptyset \qquad\qquad x = -1$$

4. Generalize the previous exercise to show that any polynomial of degree n, with n odd, admits a zero.

5. Prove that there exists $x \in (0, +\infty)$ solving the following equation

$$2 \arctan(x) + \log(1 + e^{-x^2}) - 1 = 0$$

6. Compute, where it exists, the derivative of the following functions

$$f(x) = \log(|x|) \quad \text{and} \quad f(x) = e^{|x|}$$

$$\frac{d}{dx} \operatorname{Log}(|x|) = \frac{1}{|x|} \begin{cases} x>0; & \frac{1}{x} \\ x<0; & -\frac{1}{x} \end{cases}$$

$$\frac{d\, e^{|x|}}{dx} = e^{|x|} \begin{cases} x>0; & e^{x} \\ x<0; & -e^{-x} \end{cases}$$

7. Compute, where it exists, the derivative of the following functions

(a) $f(x) = x^6 - 2x^3 + 6x$ and $f(x) = \dfrac{1}{x^6 - 2x^3 + 6x}$

(b) $f(x) = \left(x^3 - \dfrac{1}{x^3} + 3\right)^4$ and $f(x) = \left(\dfrac{1+x^2}{1+x}\right)^5$

a) $f'(x)\ 6x^5 - 6x^2 + 6$ a) $\dfrac{-(-6x^5 - 6x^2 + 6)}{(x^6 - 2x^3 + 6x)^2}$

b) $4\left(x^3 - \frac{1}{x^3} + 3\right)^3$ $5\left(\dfrac{1+x^2}{1-x}\right)^4$

c) $\dfrac{d}{dx}\sqrt{x^3+1} - x = \dfrac{1}{2\sqrt{x^3+1}} \cdot 3x^2 - 1$

c2) $\dfrac{d}{dx}\sqrt[3]{x^3+1} - x = \dfrac{1}{\cancel{3}\sqrt[3]{(x^3+1)^2}} \cdot \cancel{3}x^2 - 1$

D) $\frac{d}{dx} x\log(x) = \log(x) + \cancel{x}\cdot\frac{1}{\cancel{x}\,\cancel{\log_e(e)}} = \log(x)+1$

D2) $\frac{d}{dx} e^x\left(\log(-x)-1\right)$

$e^x\left(\log(-x)-1\right) + e^x\frac{1}{x\cdot\cancel{\log_e e}}$

$\log(x) = \frac{f'(x)}{f(x)} = \frac{-1}{-x}$

$e^x\left(\log(-x)-1+\frac{1}{x}\right)$

E) $\frac{d}{dx}\frac{1-x^2}{3+x} = \frac{-2x(3+x)-(1-x^2)}{(3+x)^2} = \frac{-2x^2-6x-1+x^2}{(3+x)^2} =$

$= \frac{-x^2-6x-1}{(3+x)^2}$

e2) $\frac{d}{dx}\cos\left(\frac{1-x^2}{3+x}\right)$ $\frac{-2x(3+x)-(1-x^2)}{(3+x)^2} = \frac{-x^2-6x-1}{(3+x)^2}$

$\frac{+x^2+6x+1}{(3+x)^2}\sin\left(\frac{1-x^2}{3+x}\right)$

F) $\frac{d}{dx}\left(\log(x^3)\right)^2 =$

$\log(x^3)\cdot\log(x^3)$

$$\frac{3x^{\cancel{2}}}{x^{\cancel{3}}} \quad \frac{3}{x}\cdot\cos(x^3) + \cos(x^3)\cdot\frac{3}{x} =$$

$$\frac{6}{x}\cos(x^3)$$

52) $$\frac{d}{dx}\frac{e^{1/x}}{x} = \frac{-\frac{e^{1/x}}{\cancel{x^2}}\cdot\cancel{x} - e^{1/x}}{x^2} = \frac{\frac{-e^{1/x}}{x} - e^{1/x}}{x^2}$$

$$\frac{d}{dx}e^{1/x} = e^{1/x}\cdot\frac{d}{dx}1/x = e^{1/x}\cdot -1\cdot x^{-2} = -\frac{e^{1/x}}{x^2}$$

$$= \frac{-e^{1/x}}{x^3} - \frac{e^{1/x}}{x^2}$$

8) $$\frac{d}{dx} x\tan(x) + \cos(\cos(x)) =$$

$- \tan(x)$

$$\tan(x) + \frac{x}{\cos^2(x)} + \frac{-\sin(x)}{\cos(x)} = \frac{x}{\cos^2(x)}$$

82) $$\frac{d}{dx} x\arctan(x) + e^{\sin(x)}$$

$- 1 - \cos(x)$

$$\tan^{-1}(x) = \frac{\cos(x)}{\sin(x)} = \frac{-\sin(x)\cancel{\sin(x)} - \cos(x)\cancel{\sin(x)}}{\sin^{\cancel{2}}(x)} =$$

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H1) $\frac{d}{dx} x^2 \arcsin(x-2)$

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H2) $\frac{d}{dx} 2\arcsin(\sqrt{x}-2)$

I) $\frac{d}{dx} \sqrt{x}\sin(\sqrt{x}) = \frac{1}{2\sqrt{x}} \cdot \sin(\sqrt{x}) + \cancel{\sqrt{x}} \cdot \cos(\sqrt{x}) \frac{1}{2\cancel{\sqrt{x}}}$

$= \frac{1}{2\sqrt{x}} \sin(\sqrt{x}) + \frac{1}{2}\cos(\sqrt{x})$

DOES NOT EXISTS $f'(0)$

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J1)

$\frac{f'(x)}{f(x}$

J2) $\frac{d}{dx}\cos(\cos(x)) = \frac{\frac{1}{x}}{\cos(x)} = \frac{1}{x}\cos(x)$

* J 3) $\frac{d}{dx} x^{\log(x)} = \log(x) \cdot x^{\log(x)-1} \cdot \frac{1}{x} =$

$\frac{1}{x} \cdot \log(x) \cdot x^{\log(x)-1}$

8A) $f(x) = \sqrt{1+2x}$ $(0,1)$

$$\frac{d}{dx}\sqrt{1+2x} = \frac{1}{\cancel{2}\sqrt{1+2x}} \cdot \cancel{2}$$

$f'(x) = \frac{1}{\sqrt{1+2x}}$ $f'(0) = 1$

$(y-1) = 1(x)$

$y = x+1$

8B) $f(x) = \sqrt[3]{1-4x}$ $(0,1)$

$$\frac{d}{dx}(1-4x)^{1/3} = \frac{1}{3}(1-4x)^{-2/3} = \frac{1}{3}\frac{-4}{\sqrt[3]{(1-4x)^2}}$$

$f'(0) = -\frac{4}{3} = m$

$y-1 = \frac{4}{3}x$; $y = -\frac{4}{3}x+1$

8c) $f(x) = e^{x^2-3}$

$$\frac{d}{dx} e^{x^2-3} = (2x)e^{x^2-3} \qquad (1, f(1))$$

$$(1, e^{-2})$$

$$m = f'(1) = 2 \cdot e^{-2}$$

$$y - y_0 = m(x - x_0)$$

$$y = 2e^{-2}(x-1) + e^{-2}$$

$$y = 2e^{-2}x - 2e^{-2} + e^{-2} = \boxed{2e^{-2}x - e^{-2}}$$

8D) $f(x) = \log(x^2 - 2x + 1) \qquad (2, 0)$

$$f'(x) \qquad \frac{f'(x)}{f(x)} = \frac{2x - 2}{x^2 - 2x + 1}$$

$$m = f'(2) = 2$$

$$y = 2x - 4$$

8E) $f(x) = \cos(\pi/x)$ $(1,-1)$

$$f'(x) = -\sin(\pi/x) \cdot \left(-\pi/x^2\right) = \frac{\pi}{x^2} \sin(\pi/x)$$

$\pi \cdot \sin(\pi) = 0$

$y = 0(x-1) - 1$

$y = -1$

* 8f) $\arctan(e^x)$ $(0, \pi/4)$

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* 9)

$$f(x) = \begin{cases} Ax+B & \text{if } x \leq 1 \\ x^3 & \text{if } x > 1 \end{cases}$$

IS DIFFERENTABLE

THE DERIVATIVE EXIST IN ALL THE FUNCTION DOMAIN

f CONTINUOUS

$$f(1) = \lim_{x \to 1^+} x^3 = \lim_{x \to 1^-} Ax+B$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$A+B \qquad 1$

$$\boxed{A+B=1}$$

10) *

$$f(x) = \begin{cases} Ax+2 & \text{IF } x<0 \\ B & \text{IF } x=0 \\ 2e^{Bx} & \text{IF } x>0 \end{cases}$$

f CONTINUOUS

$$f(0) = \lim_{x\to0^+} 2e^{Bx} = \lim_{x\to0^-} Ax+2$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$B \qquad 2 \qquad 2$$

$$\boxed{\begin{matrix} A=0 \\ B=2 \end{matrix}}$$

IS DIFFERENTIABLE BECAUSE IT EXIST IN ALL PART) OF THE FUNCTION DOMAIN

11) *

f EVEN $\Rightarrow f'$ ODD

$$\downarrow \qquad \downarrow$$

$$f(-x) = f(x) \qquad -f(x) = f(x)$$