Exercises - Calculus

Academic Year 2021-2022

Sheet 6

1. Establish whether the following limit is an indeterminate form and, if this is the case, establish which kind of indeterminate form it is.

   (a) $\lim_n \left(\frac{1}{2}\right)^n \sqrt[3]{n}$

   (b) $\lim_n (\log n)^{1/2)^n}$

   (c) $\lim_n \left(\log\left(1+\frac{1}{n}\right)\right)^n$

   (d) $\lim_n \left(\log\left(1+\frac{1}{n}\right)\right)^{1/n}$

   (e) $\lim_n\, (1/2)^n)^{\frac{1}{1+\sqrt{n}}}$

   (f) $\lim_n (e^{-n}+1)^{1/\log n}$

   (g) $\lim_n (\log n)^{2^n}$

2. Compute, if it exists,

   (a) $\lim_{n\to+\infty} \frac{2^n}{3n^3-n+5}$

   (b) $\lim_{n\to+\infty} \big(-\log_{10} n + \sqrt{n}\big)$

   (c) $\lim_{n\to+\infty} \frac{2^n - \log n}{3n^3-n+5}$

   (d) $\lim_{n\to+\infty} \frac{5^n + n^3}{n!}$

   (e) $\lim_{n\to+\infty} \frac{\sqrt{2n-1}}{n} \log n$

   (f) $\lim_{n\to+\infty} \frac{(\log n)^4}{n}$

   (g) $\lim_{n\to+\infty} \Big(\frac{n+1}{2n}\Big)^n$

   (h) $\lim_{n\to+\infty} \sqrt[n]{2+\sin n}$

   (i) $\lim_{n\to+\infty} \Big(\frac{n^2+5}{n^2}\Big)^{n^2}.$

3. Compute, if it exists, the limit $\lim_{n\to+\infty} a_n$ where

(a) $a_n = \dfrac{n}{\log(n)}$ $+\infty$

(b) $a_n = \dfrac{e^n}{e^{2n}}$ 0

(c) $a_n = \dfrac{e^n + \log(n)}{n^5 + 1}$ $+\infty$

(d) $a_n = \dfrac{e^n - \log(n)}{5^n + n^5}$ $\dfrac{e^N\left(1-\frac{\log(n)}{e^n}\right)}{5^N\left(1+\frac{5^N}{N^5}\right)}$ $\dfrac{e^N}{5^N}$ $\left(\dfrac{e}{5}\right)^N = 0^+$

(e) $a_n = \dfrac{e^n + \log(1/n)}{5^n + n^5}$

(f) $a_n = \dfrac{e^n + n!}{5^n + n^5}$

(g) $a_n = \dfrac{n - n^3 + \log^4(n) + \log(n^4)}{\sqrt{\log(n^7) + n^6}}$

(h) $a_n = \dfrac{e^{(n+1)(n-1)}}{e^{\frac{n^2+4}{n-3}}(e^{n+3})}$

(i) $a_n = \dfrac{e^{n^2-1}}{(e^{n-1})^2}$

(j) $a_n = \dfrac{n^n}{7^{n^2}}$

Hint: rewrite numerator and denominator as powers of e.

(k) $a_n = \dfrac{e^{n^2}}{n!}$

(l) $a_n = \dfrac{n^n + n!}{n! + 1}$

(m) $a_n = \dfrac{n^6 - n^n + n!}{-7^n + (7^n)^2 + 7^{n^2}}$

(n) $a_n = \dfrac{\log^3(n) - \log(n^3) + 2}{n\log(n^4) + 1}$

(o) $a_n = n^3 - \log(n) + e^n$

(p) $a_n = n^3 - \log(n) + e^n - n!$

4. Compute, if it exists, the limit $\lim_{n\to+\infty} a_n$ where

(a) $a_n = \dfrac{\sin\left(\log\left(1+\frac{1}{n}\right)\right)}{\log\left(1+\frac{1}{n}\right)}$

(b) $a_n = n^2 \sin\left(\dfrac{1}{n}\right)$

(c) $a_n = n^2 \left(\cos\left(\frac{1}{n}\right) - 1\right)$

(d) $a_n = \dfrac{\sin\left(\frac{n+1}{n^2-1}\right)}{2n-2}$

(e) $a_n = \dfrac{\sin(e^{-n})}{e^{-2n}}$

(f) $a_n = \sin\left(\dfrac{n}{e^n+1}\right) e^n$

(g) $a_n = \dfrac{\sin^2(1/n)}{1-\cos(1/n)}$

(h) $a_n = \dfrac{1}{n}\left(\cos\left(\log\left(\dfrac{n+1}{n}\right)\right) - 1\right)$

(i) $a_n = \dfrac{1-\cos(3^{-n})}{4^{-2n}}$

5. Compute, if it exists, the limit $\lim\limits_{n\to+\infty} a_n$ where

(a) $a_n = \sqrt{n+1} - \sqrt{n-1}$

(b) $a_n = \sqrt{n^2+n} - \sqrt{n^2+1}$

(c) $a_n = \sqrt{2n^2+1} - \sqrt{2n^2n+5n-1}$

(d) $a_n = \left(1 + \dfrac{1}{3n}\right)^{2n}$

(e) $a_n = \dfrac{\sqrt{n} - n + n^2}{2n^2 - n^{3/2} + 1}$

(f) $a_n = \dfrac{2^n - 3^n}{1+3^n}$

(g) $a_n = \dfrac{2^n + n^2}{n^3 + 3^n}$

(h) $a_n = \left(\dfrac{n+3}{n+1}\right)^n$

(i) $a_n = \dfrac{n^2-1}{(-1)^n n - 3n^2}$

(j) $a_n = \sqrt{n}\sin\left(\dfrac{\sqrt{n+1}}{n}\right)$

(k) $a_n = \dfrac{1}{n}\left(\sin\left(\dfrac{1}{n}\right) - n\right)$

6. Compute, if it exists,

(a) $\lim\limits_{n\to+\infty} \left(\sqrt[3]{n^2+3(\sqrt[4]{n})^3+1}\right)$

(b) $\lim_{n\to+\infty} \left(\frac{1}{2}\right)^n 3^n$

(c) $\lim_{n\to+\infty} \left(1+\frac{1}{2n}\right)^{3n}$

(d) $\lim_{n\to+\infty} \left(1+\frac{1}{\log n}\right)^{\log(n^2)}$

(e) $\lim_{n\to+\infty} \frac{\sqrt[3]{2n^5+n^4+n^2+1}}{\sqrt[6]{n^{10}+n^9+7n^7+32}}$

(f) $\lim_{n\to+\infty} \frac{\sqrt[4]{2n^4+n^3+n^2+2}}{\sqrt[3]{7n^4+n^3+7n^2+6}}$

(g) $\lim_{n\to+\infty} \left(1+\frac{1}{n^3}\right)^{n^2}$

(h) $\lim_{n\to+\infty} \frac{\left(1+\frac{1}{n^2}\right)^{n^2}}{n}$

7. Compute the limits of Exercise 6 in Sheet 5 and the limits of Exercise 1 in this Sheet.

8. For any value of the parameter $\alpha \in \mathbb{R}$, compute

$$\lim_{n\to+\infty} \frac{\sqrt{n^4+n^3}-\sqrt{n^4-n^3}}{n^\alpha+n}.$$

9. Determine for which values of the parameter $\alpha \in \mathbb{R}$, the sequence

$$a_n = \frac{(n+\sqrt{n})^\alpha}{3n^5+2\log n}$$

is infinitesimal.

10. Prove that the sequence

$$a_n = \frac{n^2-\sqrt{n}-\log n}{2n^2-n+2\sin(3n^3+1)}$$

is definitely positive.

11. Compute, if it exists, $\lim_n a_n$ where

(a) $a_n = \frac{\sqrt[3]{n}-2}{2\sqrt[3]{n}-1}$

(b) $a_n = \frac{\log^5(n)-\log^3(n)}{2\log^5(n)-1}$

(c) $a_n = \sqrt{n}\log\left(\frac{n+1}{n^2-\log n}\right)$

(d) $a_n = \sqrt[5]{n} + (-1)^n n$

(e) $a_n = \dfrac{n^2 + (-1)^n + 3n^5}{\sin n + 7n^5}$

(f) $a_n = \sqrt[n]{4^{2n} + 3^{2n}}$

(g) $a_n = \sqrt{n} \log\left(\dfrac{n^2+1}{n - \log n}\right)$

(h) $a_n = \left(\dfrac{n+1}{n^3}\right)^{\frac{1}{\sqrt{n}}}$

1. Establish whether the following limit is an indeterminate form and, if this is the case, establish which kind of indeterminate form it is.

(a) $\lim_n \left(\frac{1}{2}\right)^n \sqrt[3]{n}$ $\left[\frac{\infty}{\infty}\right]$

(b) $\lim_n (\log n)^{1/2)^n}$

(c) $\lim_n \left(\log\left(1+\frac{1}{n}\right)\right)^n$ $(\log 1)^\infty = [0^\infty]$

(d) $\lim_n \left(\log\left(1+\frac{1}{n}\right)\right)^{1/n}$ $(\log 1)^0 = [0^0]$

(e) $\lim_n (1/2)^n)^{\frac{1}{1+\sqrt{n}}}$ $[1^\infty]$

(f) $\lim_n (e^{-n}+1)^{1/\log n}$ $(0+1)^{1/\infty} = 1^0 = 1$

(g) $\lim_n (\log n)^{2^n}$ $(\log \infty)^\infty = \infty$

b) $\lim_N (\log N)^{1/2^N} \to \frac{1^\infty}{\infty}$

$\lim_N \sqrt[2^N]{\log N}$

c) $\lim_N$ $\in$

2. Compute, if it exists,

(a) $\displaystyle\lim_{n\to+\infty} \frac{2^n}{3n^3 - n + 5}$ ∞

(b) $\displaystyle\lim_{n\to+\infty} \left(-\log_{10} n + \sqrt{n}\right)$ $\log \frac{1}{N} + \sqrt{N}$ $\boxed{-\infty}$

(c) $\displaystyle\lim_{n\to+\infty} \frac{2^n - \log n}{3n^3 - n + 5}$ $\boxed{\infty}$

(d) $\displaystyle\lim_{n\to+\infty} \frac{5^n + n^3}{n!}$ $\boxed{0}$??

(e) $\displaystyle\lim_{n\to+\infty} \frac{\sqrt{2n-1}}{n} \log n$

(f) $\displaystyle\lim_{n\to+\infty} \frac{(\log n)^4}{n}$

(g) $\displaystyle\lim_{n\to+\infty} \left(\frac{n+1}{2n}\right)^n$

(h) $\displaystyle\lim_{n\to+\infty} \sqrt[n]{2 + \sin n}$

(i) $\displaystyle\lim_{n\to+\infty} \left(\frac{n^2+5}{n^2}\right)^{n^2}.$

e) $\lim_{N} \frac{\sqrt{2N-1}}{N} \log N$

$$\frac{2N-1}{N\sqrt{2N-1}} \log N = \frac{\cancel{N}(2-1/N)}{\cancel{N}\sqrt{2N-1}} \log N$$

$$= \frac{2-1/N}{\sqrt{2N-1}} \log N ;$$

3. Compute, if it exists, the limit $\lim_{n\to+\infty} a_n$ where

(a) $a_n = \dfrac{n}{\log(n)}$

$\lim_{N} \dfrac{N}{\log(N)}$

(b) $a_n = \dfrac{e^n}{e^{2n}}$

(c) $a_n = \dfrac{e^n + \log(n)}{n^5 + 1}$

$$\text{(e)}\quad a_n = \frac{e^n + \log(1/n)}{5^n + n^5}$$

$$\frac{e^N\left(1 + \underbrace{\frac{\log(1/N)}{e^N}}_{\to 0}\right)}{5^N\left(1 + \underbrace{\frac{N^5}{5^N}}_{\to 0}\right)} = \frac{e^N}{5^N} = \left(\frac{e}{5}\right)^N = 0$$

$$\text{(f)}\quad a_n = \frac{e^n + n!}{5^n + n^5}$$

TRASCURABILE

$$\lim_{N} N! = +\infty$$

$$\text{(g)}\quad a_n = \frac{n - n^3 + \log^4(n) + \log(n^4)}{\sqrt{\cancel{\log(n^7)} + n^6}}$$

$$\frac{N - N^3}{N^3} = \frac{N}{N^3} - 1 = 0 - 1 = -1$$

$$\text{(h)}\quad a_n = \frac{e^{(n+1)(n-1)}}{e^{\frac{n^2+4}{n-3}}\left(e^{n+3}\right)}$$

$$e^{(N+1)(N-1)} \cdot e^{-\left(\frac{N^2+4}{N-3}\right)} \cdot e^{-(N+3)}$$

$$e^{N^2-1+\frac{-N^2-4}{N-3}-N-3}$$

$$e^{\frac{(N-3)(N^2-1)-N^2-4-N(N-3)-3(N-3)}{N-3}}$$

$$e^{\frac{N^3-N-3N^2+3-N^2-4-N^2+\cancel{3N}-\cancel{3N}+9}{N-3}}$$

$$e^{\frac{N^3-5N^2-N+8}{N-3}}$$

$$e^{\infty} = \infty$$

$$\text{(i)}\quad a_n = \frac{e^{n^2-1}}{(e^{n-1})^2}$$

$$\lim_{N} \frac{e^{N^2-1}}{e^{2N-2}}$$

$$\lim_{N} e^{N^2-1} \cdot e^{-2N+2}$$

$$\lim_{N} e^{N^2-1-2N+2}$$

$$\lim_{N} e^{N^2-2N-1} = +\infty$$

(j) $a_n = \frac{n^n}{7^{n^2}}$

Hint: rewrite numerator and denominator as powers of e.

$$\lim_{N} \frac{N^N}{7^{N^2}} = +\infty$$

(k) $a_n = \frac{e^{n^2}}{n!}$

0

(l) $a_n = \frac{n^n + n!}{n! + 1}$

$+\infty$

(m) $a_n = \frac{n^6 - n^n + n!}{-7^n + (7^n)^2 + 7^{n^2}}$

$$\frac{N^6 - N^N + N!}{7^{N^2}\left(\frac{-7^N}{7^{N^2}} + \frac{(7^N)^2}{7^{N^2}} + 1\right)}$$

$$\frac{N^6 - N^N + N!}{7^{2N}\left(\frac{-7^N}{7^{2N}} + \frac{49^{2N}}{7^{2N}} + 1\right)}$$

$$\frac{N^N\left(\frac{N^6}{N^N} - 1 + \frac{N!}{N^N}\right)}{7^{2N}\left(-(7 \cdot 7^{-2N}) + 7^{4N} \cdot 7^{-2N} + 1\right)}$$

$$\frac{N^N}{7^{2N}\left(-7^{1-2N} + 7^{2N} + 1\right)} = -\infty$$

(n) $a_n = \dfrac{\log^3(n) - \log(n^3) + 2}{n\log(n^4) + 1}$

$$\frac{\log(n)\left(\log^2(n) - 3\log n + \frac{2}{\log(n)}\right)}{n\left(\log(n^4)+1\right)}$$

0+

(o) $a_n = n^3 - \log(n) + e^n$ $e^n = \infty$

(p) $a_n = n^3 - \log(n) + e^n - n!$ $= -\infty$

4. Compute, if it exists, the limit $\lim_{n\to+\infty} a_n$ where

(a) $a_n = \dfrac{\sin\left(\log\left(1+\frac{1}{n}\right)\right)}{\log\left(1+\frac{1}{n}\right)}$

(b) $a_n = n^2 \sin\left(\dfrac{1}{n}\right)$

$$\frac{\sin x}{x} = 0$$

$$\frac{\sin\left(\frac{1}{N}\right)}{\frac{1}{N}} \cdot \frac{1}{N} \cdot N^2$$

$$\downarrow$$

$$0 \cdot N$$

IMP

(c) $a_n = n^2\left(\cos\left(\frac{1}{n}\right) - 1\right)$

$$N^2 \cos\left(\frac{1}{N}\right) - N^2 \to N^2 - N^2 = 0$$

$\cos 0 = 1$

(d) $a_n = \dfrac{\sin\left(\frac{n+1}{n^2-1}\right)}{2n-2}$

$$T = \frac{1}{N-1}$$

$$\frac{\sin\left(\frac{\cancel{N+1}}{\cancel{(N+1)}(N-1)}\right)}{2(N-1)} \to \frac{\sin(T)}{2T^{-1}}$$

$$\frac{\sin(T)}{T} \cdot \frac{T}{2T^{-1}}$$

(e) $a_n = \dfrac{\sin\left(e^{-n}\right)}{e^{-2n}}$

(f) $a_n = \sin\left(\dfrac{n}{e^n+1}\right)e^n$

(g) $a_n = \dfrac{\sin^2(1/n)}{1-\cos(1/n)}$

(h) $a_n = \dfrac{1}{n}\left(\cos\left(\log\left(\dfrac{n+1}{n}\right)\right)-1\right)$

(i) $a_n = \dfrac{1-\cos(3^{-n})}{4^{-2n}}$

$$\text{LIM} \quad \frac{\text{SIN } e^{-N}}{\left(e^{-N}\right)^2} = 0$$

11. Compute, if it exists, $\lim_n a_n$ where

(a) $a_n = \dfrac{\sqrt[3]{n}-2}{2\sqrt[3]{n}-1}$

(b) $a_n = \dfrac{\log^5(n)-\log^3(n)}{2\log^5(n)-1}$

(c) $a_n = \sqrt{n}\log\left(\dfrac{n+1}{n^2-\log n}\right)$

(d) $a_n = \sqrt[5]{n}+(-1)^n n$

(e) $a_n = \dfrac{n^2+(-1)^n+3n^5}{\sin n+7n^5}$

(f) $a_n = \sqrt[n]{4^{2n}+3^{2n}}$

(g) $a_n = \sqrt{n}\log\left(\dfrac{n^2+1}{n-\log n}\right)$

(h) $a_n = \left(\dfrac{n+1}{n^3}\right)^{\frac{1}{\sqrt{n}}}$

a) $\lim_n \dfrac{\sqrt[3]{n}-2}{2\sqrt[3]{n}-1}$

$$\lim_N \frac{\cancel{\sqrt[3]{N}}(1-2/\sqrt[3]{N})}{\cancel{\sqrt[3]{N}}\left(2-\frac{1}{\sqrt[3]{N}}\right)} = \frac{1-\frac{2}{\sqrt[3]{N}}}{2-\frac{1}{\sqrt[3]{N}}} = \boxed{\frac{1}{2}}$$

b) $\lim_N \dfrac{\log^5(N)-\log^3(N)}{2\log^5(N)-1}$

$$\frac{\cancel{\log^5(N)}\left(1-\frac{\log^3(N)}{\log^5(N)}\right)}{\cancel{\log^5(N)}\left(2-\frac{1}{\log^5(N)}\right)} = \frac{1-\frac{1}{\log^2(N)}}{2-\frac{1}{\log^5(N)}} = \boxed{\frac{1}{2}}$$

c) $\lim_{n} \sqrt{n} \log\left(\frac{n+1}{n^2 - \log n}\right)$

$\lim_{n} \sqrt{n} \log\left(\frac{\cancel{n^2}\left(\frac{1}{n} + 1/n^2\right)}{\cancel{n^2}\left(1 - \frac{\log n}{n^2}\right)}\right)$

$\lim_{n} \sqrt{n} \log\left(\frac{\frac{1}{n} + \frac{1}{n^2}}{1 - \frac{\log n}{n^2}}\right)$ 0

1

$\sqrt{n} \log(0) = +\infty$

GOES FASTER THAN $\sqrt{n}$

d) $\lim_{n} \sqrt[5]{n} + (-1)^n n$ $\emptyset$

e) $\emptyset$

f) $\lim_{n} \sqrt[n]{4^{2n} + 3^{2n}}$

$\lim_{n} \sqrt[n]{4^{2n} + 3^{2n}}$

$\sqrt[n]{4^{2n}\left(1 + \frac{3^{2n}}{4^{2n}}\right)}$

$$4^{2N}\sqrt[N]{1+\left(\frac{3}{4}\right)^{2N}}$$

$$16\sqrt[N]{1+0}$$

$$16\cdot 1^{\frac{1}{N}} = \boxed{16}$$

g) $$\lim_{N} \sqrt{N}\cos\left(\frac{N^2+1}{N-\cos N}\right) = +\infty$$

h) $$\lim_{N}\left(\frac{N+1}{N^3}\right)^{1/\sqrt{N}}$$

$$\lim_{N}\left(\frac{\cancel{N^3}\left(\frac{1}{N^2}+\frac{1}{N^3}\right)}{\cancel{N^3}}\right)^{1/\sqrt{N}}$$

$$\lim_{N}\left(\frac{1}{N^2}+\frac{1}{N^3}\right)^{1/\sqrt{N}}$$

$$\left(\frac{1}{N^2}\right)^{1/\sqrt{N}}+\left(\frac{1}{N^3}\right)^{1/\sqrt{N}}$$

$$N^{-2/\sqrt{N}} + N^{-3/\sqrt{N}}$$ ✱

(c) $a_n = n^2\left(\cos\left(\frac{1}{n}\right) - 1\right)$

(d) $a_n = \dfrac{\sin\left(\frac{n+1}{n^2-1}\right)}{2n-2}$

(e) $a_n = \dfrac{\sin\left(e^{-n}\right)}{e^{-2n}}$

(f) $a_n = \sin\left(\dfrac{n}{e^n+1}\right)e^n$

(g) $a_n = \dfrac{\sin^2(1/n)}{1-\cos(1/n)}$

(h) $a_n = \dfrac{1}{n}\left(\cos\left(\log\left(\dfrac{n+1}{n}\right)\right) - 1\right)$

(i) $a_n = \dfrac{1-\cos(3^{-n})}{4^{-2n}}$

c) $N^2\left(\cos\left(\frac{1}{N}\right)-1\right)$

$N^2 = \dfrac{1}{1/N^2}$

$\lim\limits_{x\to+\infty} \dfrac{\cos(x)-1}{x^2} = \dfrac{1}{2}$

$$\frac{\cos\left(\frac{1}{N}\right)-1}{\left(\frac{1}{N}\right)^2} = -\frac{1-\cos\left(\frac{1}{N}\right)}{\left(\frac{1}{N}\right)^2} = -\frac{1}{2}$$

D) $\lim\limits_{N} \dfrac{\sin\left(\frac{1}{N-1}\right)}{2N-2} = \dfrac{\sin\left(\frac{1}{N-1}\right)}{2(N-1)}\left(\dfrac{\frac{1}{N-1}}{\frac{1}{N-1}}\right)$ 1

$$\frac{1}{N-1}\cdot\frac{1}{2(N-1)} = 0$$

e) $\lim\limits_{N} \dfrac{\sin(e^{-N})}{e^{-2N}} \cdot \dfrac{\sin(e^{-N})}{e^{-N}\cdot e^{-N}}$ 1 $\quad \dfrac{1}{e^{-N}} = e^N = \infty$

b) $a_n = \sqrt{N^2+N} - \sqrt{N^2+1}\;\dfrac{\left(\sqrt{N^2+N}+\sqrt{N^2+1}\right)}{\sqrt{N^2+N}+\sqrt{N^2+1}}$

$$\frac{\cancel{N^2}+N-\cancel{N^2}+1}{\sqrt{N^2+N}+\sqrt{N^2+1}} = \frac{N+1}{\sqrt{N^2\left(1+\frac{1}{N}\right)}+\sqrt{N^2\left(1+\frac{1}{N^2}\right)}}$$

$$\frac{N+1}{N\sqrt{1+\frac{1}{N}}+N\sqrt{1+\frac{1}{N^2}}} = \frac{N+1}{2N} = \frac{\cancel{N}(1+1/N)}{2\cancel{N}} = \frac{1}{2}$$

$$\frac{(N+\sqrt{N})^2}{3N^5+2\cos N}$$

INFINITESIMAL

$\boxed{\alpha<5}$

5) $\sin\left(\frac{N}{e^N+1}\right)e^N$

$$\frac{\sin\left(\frac{N}{e^N+1}\right)e^N}{\frac{N}{e^N+1}} \quad \frac{N}{e^N+1} \qquad \frac{e^N \cdot N}{e^N+1} = \infty$$

1

$\sin^2 - \cos^2 = 1$

6) $$\frac{\sin^2(1/N)}{1-\cos(1/N)} = \frac{1+\cos^2\left(\frac{1}{N}\right)}{1-\cos\left(\frac{1}{N}\right)}$$

$$\frac{1-\cos(x)}{x^2} = \frac{1}{2}$$

$$\frac{\sin\left(\frac{1}{N}\right)\cdot\sin\left(\frac{1}{N}\right)}{\frac{1}{N^2}} \cdot \frac{1}{N^2}\;\frac{1}{1-\cos\left(\frac{1}{N}\right)} = 2$$

1

2

H) $\frac{1}{N}\left(\cos\left(\log\left(\frac{N+1}{N}\right)\right)-1\right)$

$-\frac{1}{N}\left(1-\cos\left(\log\left(\frac{N+1}{N}\right)\right)\right)$ $\frac{\log^2\left(\frac{N+1}{N}\right)}{\log^2\left(\frac{N+1}{N}\right)}$

$\lim_{x\to\infty}\frac{1-\cos x}{x^2}=\frac{1}{2}$

$-\frac{1}{2N}\cdot\log^2\left(1+\frac{1}{N}\right)=\frac{\frac{1}{N^2}}{\frac{1}{N^2}}$ $\lim_{x\to\infty}\frac{\log(1+x)}{x}=\frac{1}{2}$

$-\frac{1}{2N}\cdot 1\ \frac{1}{N^2}=-\frac{1}{2N^3}=0$

l) $a_N=\frac{1-\cos(3^{-N})}{4^{-2N}}$ $\boxed{\frac{1-\cos(x)}{x^2}}$

$\frac{1-\cos(3^{-N})}{4^{-2N}}\cdot\frac{(3^{-N})^2}{(3^{-N})^2}$

$\frac{1}{2}\frac{(3^{-N})^2}{4^{-2N}}$ $\frac{1}{2}$

$\frac{1}{2}\left(\frac{3}{4}\right)^{-2N}=\frac{1}{2}\left(\frac{4}{3}\right)^{2N}=\infty$