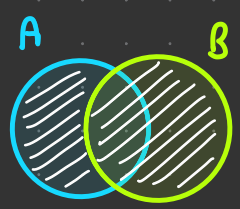
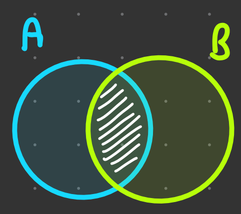
REMEMBER

$P(A^c) = 1 - P(A)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$



$P(A \cap B) \overset{?}{=} P(A) \cdot P(B)$ [TO CHECK INDEPENDENCE]



$P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

LAW OF TOTAL PROBABILITY: $P(A \cap B) = P(A|B) \cdot P(B) = P(B|A)\,P(A)$

$P(A^c \cap B) = P(B) - P(A \cap B)$

$\boxed{P(A^c|B)} = \dfrac{P(A^c \cap B)}{P(B)} = \dfrac{P(B) - P(A \cap B)}{P(B)} = 1 - \dfrac{P(A \cap B)}{P(B)} = \boxed{1 - P(A|B)}$

$\binom{N}{K}$ $\boxed{N \geq K}$ $\binom{N}{K} = \dfrac{N!}{K!(N-K)!}$

$P(B|A) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A|B)\,P(B)}{P(A|B)P(B) + P(A|B^c)\,P(B^c)}$

REMEMBER

$VAR(3X) = 3\,VAR(X)$

$COV(X,Y) = COV(Y,X)$

$COV(X,X) = VAR(X)$

$COV(X,Y) = E[XY] - E[X]E[Y]$

IF INDEPENDENT: $E[XY] = E[X]E[Y]$

$\rho = \dfrac{COV(X,Y)}{\sqrt{\sigma_X^2 \sigma_Y^2}} = \dfrac{COV(X,Y)}{\sigma_X \sigma_Y}$

$VAR(XY) = E[(XY)^2] - (E[XY])^2$

- VARIANCE IS ALWAYS
- COVARIANCE CAN BE POSITIVE OR NEGATIVE
  - $\ominus$ OPPOSITE RELATION
  - $\oplus$ DIRECTLY PROPORTIONAL
  - $\odot$ NO CORRELATION $\longrightarrow$ [X AND Y INDEP. $\Rightarrow COV(X,Y) = 0$ BUT $COV(X,Y) = 0 \not\Rightarrow$ X AND Y INDEP.

IF INDIPENDENT

- $E[3X - 2Y + 7] = 3E[X] - 2E[Y] + 7$
- $VAR(3X - 2Y + 7) = VAR(3X) + VAR(-2Y) = 9\,VAR(X) + 4\,VAR(Y)$ THE CONSTANT DISAPPEAR
- $VAR(XY) = E[(XY)^2] - (E[XY])^2 = E[X^2]E[Y^2] - (E[X])^2(E[Y])^2$

$\downarrow$

$VAR(X) = E[X^2] - (E[X])^2$

USE THE INVERSE FUNCTION

DISCRETE

- BERNULLI: $X \sim BERN(P)$ $\begin{cases} P & \text{SUCCESS} \\ 1-P & \text{FAILURE} \end{cases}$
- BINOMIAL: $X \sim BIN(N,P)$ $P(S_N = K) = \binom{N}{K} P^K (1-P)^{N-K}$
- HYPERGEOMETRIC: $X \sim HYPERGEOM(N, q, \underline{N})$ $P(X=K) = \dfrac{\binom{N}{K}\binom{N-n}{q-K}}{\binom{N}{q}}$
- POISSON: $X \sim POISSON(\lambda)$ $P(X=K) = e^{-\lambda} \dfrac{\lambda^K}{K!}$
- GEOMETRIC: $X \sim GEOM(P)$ $P(X=K) = (1-P)^{K-1}(P)$

- UNIFORM $X \sim UNIF(a,b)$ $P = \dfrac{1}{N} = \dfrac{1}{b-a}$

CONTINOUS

- EXPONENTIAL $X \sim EXP(\lambda)$ PROB. DENSITY FUNCTION $\lambda e^{-\lambda t}$
- NORMAL $X \sim NORM(N, \sigma^2)$ PROB. DENSITY FUNCTION $\dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$

X RANDOM VARIABLE: FIND $\boxed{\text{MEAN}}$

X DISCRETE

ES:

x	1	2	3	4	5
$p_X(x)$	1/7	1/14	3/14	2/7	2/7

$$E[x] = 1\cdot\frac{1}{7} + 2\cdot\frac{1}{14} + 3\cdot\frac{3}{14} + 4\cdot\frac{2}{7} + 5\cdot\frac{2}{7}$$

X CONTINOUS

$$f(x) = \begin{cases} 3e^{-3x}, & x > 0 \\ 0, & \text{else.} \end{cases}$$

$$E[x] = \int_{-\infty}^{+\infty} x\, f(x)\, dx$$

⚠ THE INTEGRAL "DOMAIN" CAN BE USUALLY REDUCED

X RANDOM VARIABLE: FIND $\boxed{E[\text{ SOMETHING}(X)]}$

X DISCRETE

ES FIND $E[|x-2|]$

x	1	2	3	4	5
$p_X(x)$	1/7	1/14	3/14	2/7	2/7

$$E[|x-2|] = |1-2|\cdot\frac{1}{7} + |2-2|\cdot\frac{1}{14} + |3-2|\cdot\frac{3}{14} + |4-2|\cdot\frac{2}{7} + |5-2|\cdot\frac{2}{7}$$

X CONTINOUS

ES: FIND $E[e^{2x}]$

$$f(x) = \begin{cases} 3e^{-3x}, & x > 0 \\ 0, & \text{else.} \end{cases}$$

$$E[e^{2x}] = \int_{-\infty}^{+\infty} e^{2x} f(x)\, dx = \int_0^{+\infty} e^{2x} e^{-3x}\, dx$$

X RANDOM VARIABLE: FIND $\boxed{\text{VARIANCE}}$

X DISCRETE

x	1	2	3	4	5
$p_X(x)$	1/7	1/14	3/14	2/7	2/7

$$VAR(x) = E[x^2] - (E[x])^2$$

$$E[x^2] = 1^2\cdot\frac{1}{7} + 2^2\cdot\frac{1}{14} + 3^2\cdot\frac{3}{14} + 4^2\cdot\frac{2}{7} + 5^2\cdot\frac{2}{7}$$

$$E[x] = 1\cdot\frac{1}{7} + 2\cdot\frac{1}{14} + 3\cdot\frac{3}{14} + 4\cdot\frac{2}{7} + 5\cdot\frac{2}{7}$$

X CONTINOUS

$$f(x) = \begin{cases} 3e^{-3x}, & x > 0 \\ 0, & \text{else.} \end{cases}$$

$$VAR(x) = E[x^2] - (E[x])^2$$

$$E[x^2] = \int_{-\infty}^{+\infty} x^2 f(x)\, dx = \int_0^{+\infty} x^2\, 3e^{-3x}\, dx$$

$$E[x] = \int_{-\infty}^{+\infty} x^1 f(x)\, dx = \int_0^{+\infty} x^1\, 3e^{-3x}\, dx$$

⚠ REMEMBER TO SQUARE THE RESULT OF $E[x]$ TO HAVE $(E[x])^2$ BEFORE DOING THE SUBTRACTION

X NORMAL RANDOM VARIABLE: FIND $\boxed{P(X > \sim),\ P(\sim < X < \sim),\ P(X < \sim)}$ KNOWING μ AND σ^2

ES: $\mu = -2$; $\sigma^2 = 7$

$$\frac{3.5-\mu}{\sqrt{\sigma^2} = \sigma}$$

- $P(x > 3.5) = P\left(S_N > \frac{3.5-(-2)}{\sqrt{7}}\right) = P(S_N > \underline{2.08}) = 1 - \Phi(2.08)$
- $P(-2.1 < x < -1.9) = P\left(\underbrace{\frac{-2.1-(-2)}{\sqrt{7}}}_{a} < S_N < \underbrace{\frac{-1.9-(-2)}{\sqrt{7}}}_{b}\right) = \Phi(b) - \Phi(a)$
- $P(x < 3.5) = P\left(S_N < \underbrace{\frac{3.5-(-2)}{\sqrt{7}}}_{a}\right) = \Phi(a)$

⚠ • THE NUMBER MUST BE BETWEEN -3 AND $+3$

- $\Phi(-1.7) = 1 - \Phi(1.7)$
- FOR "$X > \sim$" USE $1 - \Phi(\sim)$, WHILE FOR "$X < \sim$" USE $\Phi(\sim)$

$\boxed{\text{EXERCISE WITH A POPULATION SIZE, A \% AND } \varepsilon}$

BEFORE STARTING YOU NEED TO FIND

- ε: A SMALL NUMBER, IS THE CONFIDATION INTERVAL
- %: THE % OF HOW MUCH SURE YOU WANT TO BE
- N: THE SIZE OF THE SAMPLE

YOU HAVE TO FIND ONE OF THESE

Exercise 4.6. A pollster would like to estimate the fraction p of people in a population who intend to vote for a particular candidate. How large must a random sample be in order to be at least 95% certain that the fraction $\widehat{p}$ of positive answers in the sample is within 0.02 of the true p?

$\varepsilon = 0.02$

$\% = 95$

$N = ?$ ← FIND THIS

PROCEDURE

⚠ DELETE THE DENOMINATOR $\sqrt{p(1-p)}$ AND ADD 2 IN FRONT

$$P(|\hat{p}-p| < \varepsilon) = P(-\varepsilon < \hat{p}-p < \varepsilon) = P\left(\frac{-\varepsilon\sqrt{N}}{\sqrt{p(1-p)}} < S_N < \frac{\varepsilon\sqrt{N}}{\sqrt{p(1-p)}}\right) = 2\Phi(2\varepsilon\sqrt{N}) - 1 \geq 0.95$$

← IS THE %

!!! DON'T FORGET IT !!!

NOW REWRITE S.T. Φ IS ALONE:

FIND ON THE Φ THE VALUE CORRESPONDING, IF THERE IS NONE CHOOSE THE ONE BEFORE IT

$$\Phi(2\varepsilon\sqrt{N}) \geq 0.975 \Rightarrow 2\varepsilon\sqrt{N} = 1.96 \Rightarrow N = \left(\frac{1.96}{2\varepsilon}\right)^2 \Rightarrow N = 2401$$

DISCRETE

CHECK IF IF $P(x,y)$ IS A GENUINE JOINT MASS FUNCTION

X \ Y	0	1	2	3	
1	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{1}{15}$	5/15
2	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{10}$	5/10
3	$\frac{1}{30}$	$\frac{1}{30}$	0	$\frac{1}{10}$	5/30
	$\frac{6}{30}$	$\frac{6}{30}$	$\frac{5}{15}$	$\frac{8}{30}$	

→ 1?

YOU NEED TO CHECK THAT BOTH MARGINALS IF SUMMED ARE BOTH EQUAL TO 1

MARGINAL MASS FUNCTION

X \ Y	0	1	2	3
1	0.10	0.15	0	0.05
2	0.20	0.05	0.05	0.20
3	0.05	0	0.10	0.05

→

X \ Y	0	1	2	3	
1	0.10	0.15	0	0.05	0.30
2	0.20	0.05	0.05	0.20	0.50
3	0.05	0	0.10	0.05	0.20
	0.35	0.20	0.15	0.30	1

CALCULATE THE PROBABILITY

X \ Y	0	1	2	3
1	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{1}{15}$
2	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{10}$
3	$\frac{1}{30}$	$\frac{1}{30}$	0	$\frac{1}{10}$

ES: $P(x+y^2 \le 2)$

JUST FIND THE VALUE OF X AND Y THAT SATISFY THE REQUIREMENT (1,0) (2,0) (1,1) AND SUM THEIR RESPECTIVE PROBABILITY.

$P(x+y^2 \le 2) = \frac{1}{15} + \frac{1}{10} + \frac{1}{15}$

CALCULATE THE EXPECTATION, VARIANCE AND COVARIANCE

X \ Y	0	1	2	3
1	0.10	0.15	0	0.05
2	0.20	0.05	0.05	0.20
3	0.05	0	0.10	0.05

ES: $\boxed{E[xe^y]}$ PRODUCT OF THE PROBABILITY WITH THEIR RESPECTIVE X AND Y FROM THE "EXPECTATION FORMULA". THEN SUM ALL $0.1 \cdot 1e^0 + 0.15 \cdot 1 \cdot e^1 + \dots$

⚠ FIND THE MARGINAL BEFORE

X \ Y	0	1	2	3	
1	0.10	0.15	0	0.05	0.3
2	0.20	0.05	0.05	0.20	0.5
3	0.05	0	0.10	0.05	0.2
	0.35	0.20	0.15	0.30	

$\boxed{E[x]} = \underset{\text{MARGINAL}}{0.3} \cdot 1 + \underset{\text{MARGINAL}}{0.5} \cdot 2 + \underset{\text{MARGINAL}}{0.2} \cdot 3 = 1.9$

$\boxed{VAR(x)} = E[x^2] - (E[x])^2 = 4.1 - (1.9)^2 = 0.49$ ← DON'T FORGET TO SQUARE

$E[x^2] = \underset{\text{MARGINAL}}{0.3} \cdot 1^2 + \underset{\text{MARGINAL}}{0.5} \cdot 2^2 + \underset{\text{MARGINAL}}{0.2} \cdot 3^2 = 4.1$

$\boxed{Cov(x,y)} = E[xy] - E[x]E[y]$ ← SEE BEFORE

X \ Y	0	1	2	3
1	0.10	0.15	0	0.05
2	0.20	0.05	0.05	0.20
3	0.05	0	0.10	0.05

VALUE OF X · VALUE OF Y · THEIR PROB.

$E[xy] = 1 \cdot 0 \cdot 0.10 + 1 \cdot 1 \cdot 0.15 + 1 \cdot 2 \cdot 0 + \dots$

MOMENT GENERATING FUNCTION

USEFUL TO FIND:
- $E[x]$
- $E[x^2]$
- $VAR(x)$

⚠ FIND THE MARGINAL BEFORE

X \ Y	0	1	2	3	
1	0.10	0.15	0	0.05	0.3
2	0.20	0.05	0.05	0.20	0.5
3	0.05	0	0.10	0.05	0.2
	0.35	0.20	0.15	0.30	

$E[x^n] = \underset{\text{MARGINAL}}{0.3} \cdot 1^n + \underset{\text{MARGINAL}}{0.5} \cdot 2^n + \underset{\text{MARGINAL}}{0.2} \cdot 3^n$

$E[y^n] = \underset{\text{MARGINAL}}{0.35} \cdot 0^n + \underset{\text{MARGINAL}}{0.20} \cdot 1^n + \underset{\text{MARGINAL}}{0.15} \cdot 2^n + \underset{\text{MARGINAL}}{0.30} \cdot 3^n$

⚠ TO FIND $E[x]$ AND $E[x^2]$ JUST SUBSTITUTE n WITH 1 OR 2 (SAME FOR $E[y]$ AND $E[y^2]$)

CONTINOUS

CHECK IF f IS A GENUINE JOINT DENSITY FUNCTION

$$f(x,y) = \begin{cases} \frac{12}{7}(xy+y^2), & 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

DOUBLE INTEGRAL WRT X AND Y. CHECK IF THE RESULT IS ACTUALLY EQUAL TO 1

$$\int_0^1 \int_0^1 f(x)\, dy\, dx \stackrel{?}{=} 1$$

MARGINAL DENSITY FUNCTION

FIRST MARGINAL (X) — SECOND MARGINAL (Y)

$$f(x,y) = \begin{cases} \frac{12}{7}(xy+y^2), & 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

⚠ DEPENDS ON X, SO DERIVATE WRT TO Y

$$f_x = \int_0^1 f(x)\, dy$$

⚠ DEPENDS ON Y, SO DERIVATE WRT TO X

$$f_y = \int_0^1 f(x)\, dx$$

CALCULATE THE PROBABILITY

⚠ USE $f(x,y)$ FOR CASES LIKE $P(x<y)$ USE FIRST MARGINAL INSTEAD FOR $P(x<3)$

$$f(x,y) = \begin{cases} \frac{12}{7}(xy+y^2), & 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

$P(x<y)$

$$\iint_{x<y} f(x)\, dx\, dy = \int_0^1 \left[\int_0^y \frac{12}{7}(xy+y^2)\, dx \right] dy$$

ALWAYS OPPOSITE

(x) 0 —— y —— 1, $0<x<y$

⚠
- THE BOUNDARIES OF THE INTEGRAL "OUTSIDE" MUST BE WITH NO VARIABLES AND SO IN ITS ENTIRE DOMAIN
- USE THE "LINE TRICK" TO WRITE CORRECTLY THE BOUNDARIES OF THE "INTEGRAL INSIDE"

CALCULATE THE EXPECTATION, VARIANCE AND COVARIANCE

$$f(x,y) = \begin{cases} \frac{12}{7}(xy+y^2), & 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

DO THE DOUBLE INTEGRAL OF $f(x,y)$ AND USE ADD IN FRONT THE "EXPECTATION FORMULA"

ES: $\boxed{E[x^2y]}$ $\iint x^2 y\, f(x,y)\, dx\, dy = \frac{12}{7} \int_0^1 \int_0^1 (x^2 y)\, xy + y^2\, dx\, dy$

⚠ USE THE MARGINAL, DO NOT USE $f(x,y)$ ⚠

$\boxed{E[x]} = \int_{-\infty}^{+\infty} x f_x(x,y)\, dx = \frac{12}{7} \int_0^1 x \text{ MARGINAL}_x\, dx = m_x$

⚠ FOR THE EXPECTATION YOU DON'T CARE ABOUT ADDITIONAL CONSTRAINT. X IS DEFINED IN THE ENTIRE DOMAIN

REMEMBER TO SQUARE IT

$\boxed{VAR(x)} = E[x^2] - (E[x])^2$

REMEMBER

$E[x^2] = \int_{-\infty}^{+\infty} x^2 f(x,y)\, dx = \frac{12}{7} \int_0^1 x^2 (xy+y^2)\, dx$

SEE BEFORE

$\boxed{Cov(xy)} = E[xy] - E[x]E[y]$

⚠ $Cov(x,x) = VAR(x)$
$Cov(x,y) = Cov(y,x)$

$E[xy] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xy\, f(x,y)\, dx\, dy = \frac{12}{7} \int_0^1 \int_0^1 (xy)(xy+y^2)\, dx\, dy$

MOMENT GENERATING FUNCTION

USEFUL TO FIND:
- $E[x]$
- $E[x^2]$
- $VAR(x)$

⚠ FIND THE MARGINAL BEFORE

$$f(x,y) = \begin{cases} \frac{12}{7}(xy+y^2), & 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

$E[x^n] = \int_{-\infty}^{+\infty} x^n f_x\, dx = \int_0^1 x^n \text{ MARGINAL}_x\, dx$ (MARGINAL)

⚠ TO FIND $E[x]$ AND $E[x^2]$ JUST SUBSTITUTE n WITH 1 OR 2 (SAME FOR $E[y]$ AND $E[y^2]$)

CUMULATIVE DISTRIBUTION FUNCTION

x	1	2	3	4	5
$p_X(x)$	1/7	1/14	3/14	2/7	2/7

$$F(s)=P(X\le s)=\begin{cases}0 & s<1\\ 1/7 & 1\le s<2\\ 3/14 & 2\le s<3\\ 6/14 & 3\le s<4\\ 10/14 & 4\le s<5\\ 1 & s\ge 5\end{cases}$$

TWO VARIABLES → (JOINT) CUMULATIVE DISTRIBUTION FUNCTION

$$f(x)=\begin{cases}3e^{-3x}, & x>0\\ 0, & \text{else.}\end{cases}$$

$P(X\le s)=\int_{-\infty}^{s} 3e^{-3x}\,dx = 1-e^{-3s}$ ⚠ USE THE LETTER s

THEN SUBSTITUTE WITH 0

IF THERE ARE 2 VAR: $P(X\le s, Y\le t)=\int_{-\infty}^{s}\int_{-\infty}^{t} f(x,y)\,dy\,dx$

CHECK THE INDEPENDENCE OF RANDOM VARIABLE X AND Y

⚠ BEFORE YOU NEED TO CALCULATE THE MARGINAL MASS FUNCTION

X \ Y	0	1	2	3	
1	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{1}{15}$	5/15
2	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{10}$	5/10
3	$\frac{1}{30}$	$\frac{1}{30}$	0	$\frac{1}{10}$	5/30
	$\frac{6}{30}$	$\frac{6}{30}$	$\frac{5}{15}$	$\frac{8}{30}$	

YOU NEED TO CHECK THAT THE PRODUCT OF ALL THE MARGINALS IS EQUAL TO THE RISPECTIVE INTERSECTION ↓

- IF YOU HAVE A ZERO ⇒ NOT INDEPENDENT = DEPENDENT
- IF IS TRUE ∀ MARGINALS ⇒ INDEPENDENT

CHECK THE INDEPENDENCE OF RANDOM VARIABLE X AND Y

$$f(x,y)=\begin{cases}\frac{12}{7}(xy+y^2), & 0\le x\le 1 \text{ and } 0\le y\le 1\\ 0, & \text{otherwise.}\end{cases}$$

⚠ BEFORE YOU NEED TO CALCULATE THE MARGINAL MASS FUNCTION

- YOU NEED TO CHECK IF THE PRODUCT OF THE TWO MARGINAL IS EQUAL TO $f(x,y)$
- $f(x,y) \overset{?}{=} \left(\int_0^1 f(x,y)\,dx\right)\left(\int_0^1 f(x,y)\,dy\right)$
- IF THE EQUATION IS FALSE ⇒ NOT INDEPENDENT = DEPENDENT
- IF THE EQUATION IS TRUE ⇒ INDEPENDENT

PROBABILITY MASS FUNCTION

Exercise 3.10. Let X have probability mass function

$P(X=-1)=\frac{1}{2}$, $P(X=0)=\frac{1}{3}$, and $P(X=1)=\frac{1}{6}$.

Calculate $E[|X|]$ using the approaches in (a) and (b) below.

(a) First find the probability mass function of the random variable $Y=|X|$ and using that compute $E[|X|]$.

GIVEN $Y=|X|$ FIND THE VALUES THAT Y CAN HAVE
- $Y=1$ ⇒ FROM $X=1$ AND $X=-1$
- $Y=0$ ⇒ FROM $X=0$

IF $Y=1$: $P(Y=1)=P(X=1)+P(X=-1)=\frac{1}{2}+\frac{1}{6}=\frac{2}{3}$

IF $Y=0$: $P(Y=0)=P(X=0)=\frac{1}{3}$

PROBABILITY DENSITY FUNCTION

⚠ REMEMBER THE FORMULA: $f_Y = f_X\left(h^{-1}(y)\right)\cdot\boxed{\frac{d}{dy}h^{-1}(y)}$

Exercise 5.7. Suppose $X\sim \text{Exp}(\lambda)$ and $Y=\ln X$. Find the probability density function of Y.

$X\sim EXP(\lambda)$ $Y=LN(X)$

h^{-1} IS THE INVERSE OF y: SINCE $y=\ln(x)$ ⇒ $h^{-1}=x=e^y$

$f_Y=\underbrace{f_X(e^y)}\cdot\left[\frac{d}{dy}e^y\right]=\lambda e^{-\lambda \underbrace{e^y}_{t}}e^y=\boxed{\lambda e^{y-\lambda e^y}}$

THIS IS THE FUNCTION OF X THAT IS $\lambda e^{-\lambda t}$ BECAUSE IS EXPONENTIAL WITH $t=h^{-1}=e^y$

CONDITIONAL

DISCRETE

CONDITIONAL MASS FUNCTION OF X GIVEN Y

X \ Y	0	1	2	3	
1	0.10	0.15	0	0.05	0.3
2	0.20	0.05	0.05	0.20	0.5
3	0.05	0	0.10	0.05	0.2
	0.35	0.20	0.15	0.30	

⚠ FIND THE MARGINAL BEFORE

$P(X=1|Y=0)=\frac{P(X=1,Y=0)}{P(Y=0)}=\frac{0.10}{0.35}$ ... .

$P(X=2|Y=0)=\frac{P(X=2,Y=0)}{P(Y=0)}=\frac{0.20}{0.35}$ ... ...

$P(X=3|Y=3)=\frac{P(X=3,Y=3)}{P(Y=3)}=\frac{0.05}{0.30}$ ... ...

CONDITIONAL MASS FUNCTION

GIVEN A JOINT PROBABILITY MASS FUNCTION FIND $f_{X|Y}(x|y)$ 1

Exercise 10.1. The joint probability mass function of the random variables (X,Y) is given by the following table:

X \ Y	0	1	2
1	0	$\frac{1}{9}$	0
2	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{1}{9}$
3	0	$\frac{1}{9}$	$\frac{1}{9}$

(a) Find the conditional probability mass function of X given $Y=y$.

FIND MARGINAL

X \ Y	0	1	2	
1	0	$\frac{1}{9}$	0	1/9
2	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{1}{9}$	6/9
3	0	$\frac{1}{9}$	$\frac{1}{9}$	2/9
	$\frac{3}{9}$	$\frac{4}{9}$	$\frac{2}{9}$	

$P(X=1|Y=0)=\frac{P(X=1,Y=0)}{P(Y=0)}=\frac{0}{3/9}$ $P(X=2|Y=0)=\frac{1/3}{3/9}=1$ $P(X=3|Y=0)=0$

$P(X=1|Y=1)=\frac{P(X=1,Y=1)}{P(Y=1)}=\frac{1/9}{4/9}=1/4$ $P(X=2|Y=1)=\frac{2/9}{4/9}=1/2$ $P(X=3|Y=1)=\frac{1/9}{4/9}=1/4$

$P(X=1|Y=2)=\frac{P(X=1,Y=2)}{P(Y=2)}=\frac{0}{2/9}=0$ $P(X=2|Y=2)=\frac{1/9}{4/9}=1/4$ $P(X=3|Y=2)=\frac{1/9}{2/9}=1/2$

CONTINOUS

CONDITIONAL DENSITY FUNCTION OF X GIVEN Y

$$f(x,y)=\begin{cases}\frac{12}{7}(xy+y^2), & 0\le x\le 1 \text{ and } 0\le y\le 1\\ 0, & \text{otherwise.}\end{cases}$$

$f(x|y)=\frac{f(x,y)}{f_Y(y)}=\frac{\frac{12}{7}(xy+y^2)}{\int_0^1\frac{12}{7}(xy+y^2)\,dx}$

MARGINAL OF Y (SO INTEGRATE WRT X)

CONDITIONAL DENSITY FUNCTION

GIVEN A JOINT DENSITY FUNCTION FIND $f_{X|Y}(x|y)$ 1

Exercise 10.6. Let the joint density function of (X,Y) be

$$f(x,y)=\frac{x+y}{4},\quad 0<x<y<2.$$

(a) Find $f_{X|Y}(x|y)$.

⚠ REMEMBER THAT $f_{X|Y}(x|y)=\frac{f(x,y)}{f_Y(x,y)}$

SECOND MARGINAL

1) FIRST FIND THE SECOND MARGINAL

SECOND BECAUSE IS GIVEN Y IN THE TEXT

REMEMBER WHERE X IS DEFINED

$f_Y(x,y)=\int_0^y\frac{x+y}{4}\,dx=\frac{1}{4}\left[\frac{x^2}{2}+xy\right]_0^y=\frac{1}{4}\cdot\frac{3}{2}y^2=\frac{3}{8}y^2$

2) FIND $f_{X|Y}(x|y)$

$f_{X|Y}(x|y)=\frac{\frac{x+y}{4}}{\frac{3}{8}y^2}=\frac{x+y}{4}\cdot\frac{8}{3y^2}=\boxed{\frac{2}{3}\frac{x+y}{y^2}}$

CONDITIONAL PROBABILITIES

GIVEN A JOINT MASS FUNCTION FIND $P(X<\sim \mid Y=\sim)$ ②

USELESS?

FAKE-ES: $P(X<2 \mid Y=y) = P(X=1|y=0) + P(X=1|y=1) + P(X=1|y=2)$

$$P(X=1|y=0) = \frac{P(X=1, Y=0)}{P(Y=0)} = \frac{0}{3/9}$$

$$P(X=1|y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{1/9}{4/9} = 1/4$$

$$P(X=1|y=2) = \frac{P(X=1, Y=2)}{P(Y=2)} = \frac{0}{2/9} = 0$$

$P(X<2|Y=y) = 0 + \frac{1}{4} + 0 = 1/4$

CONDITIONAL PROBABILITIES

GIVEN A JOINT DENSITY FUNCTION FIND $P(X<\sim \mid Y=\sim)$ ②

Exercise 10.6. Let the joint density function of (X, Y) be

$$f(x, y) = \frac{x+y}{4}, \quad 0 < x < y < 2.$$

(a) Find $f_{X|Y}(x|y)$.
(b) Compute the conditional probabilities $P(X < 1/2 \mid Y = 1)$ and $P(X < 3/2 \mid Y = 1)$.

I KNOW
- $f(x,y) = \frac{x+y}{4}$
- $0<x<y<2$

$\boxed{\text{FIND } P(X<\frac{1}{2} \mid y=1)}$

FORMULA: $P(X\in A \mid Y=y) = \int_A f_{X|Y}(x|y)\,dx$

FROM BEFORE
- $f_{X|Y}(x|y) = \frac{2}{3}\frac{x+y}{y^2}$

ALWAYS CHECK X!!!
IT MUST BE INSIDE ITS DOMAIN ⇒ IF $P(X<3/2|Y=1)$ BUT $0<x<y<2$, AND $Y=1$ SO $\boxed{0<x<1}$
IF $P(X>1/2|Y=1)$ BUT $0<x<y<2$ AND $y=1$ I GET $\boxed{\frac{1}{2}<x<1}$

⇒ TAKE $f_{X|Y}(x|y)$

- $P(X<1/2|Y=1) = \int_0^{1/2} \frac{2}{3}\frac{x+y}{y^2}\,dx = \frac{2}{3}\int_0^{1/2} x+1\,dx = \frac{2}{3}\left[\frac{x^2}{2}+x\right]_0^{1/2} = \frac{2}{3}\cdot\frac{5}{8} = \boxed{\frac{5}{12}}$

YOU KNOW $Y=1$, SO SUBSTITUTE Y WITH 1

CONDITIONAL EXPECTATION

GIVEN A JOINT DENSITY FUNCTION COMPUTE $E[X|Y=y]$ ③

Exercise 10.1. The joint probability mass function of the random variables (X, Y) is given by the following table:

X \ Y	0	1	2
1	0	$\frac{1}{9}$	0
2	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{1}{9}$
3	0	$\frac{1}{9}$	$\frac{1}{9}$

(a) Find the conditional probability mass function of X given $Y = y$.
(b) Find the conditional expectation $E[X|Y = y]$ for each of $y = 0, 1, 2$.

FIND MARGINAL

X \ Y	0	1	2	
1	0	$\frac{1}{9}$	0	1/9
2	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{1}{9}$	6/9
3	0	$\frac{1}{9}$	$\frac{1}{9}$	2/9
	$\frac{3}{9}$	$\frac{4}{9}$	$\frac{2}{9}$	

$E[X|Y=y]$ FOR EACH $Y=0,1,2$

YOU NEED TO FIND BEFORE THE CONDITIONAL MASS FUNCTION ①

- $E[X|Y=0] = 1\cdot P(X=1|y=0) + 2P(X=2|y=0) + 3P(X=3|y=0) = 1\cdot 0 + 2\cdot 1 + 3\cdot 0 = \boxed{2}$
- $E[X|Y=1] = 1\cdot P(X=1|y=1) + 2P(X=2|y=1) + 3P(X=3|y=1) = 1\cdot\frac{1}{4} + 2\cdot\frac{1}{2} + 3\cdot\frac{1}{4} = \boxed{2}$
- $E[X|Y=2] = 1\cdot P(X=1|y=2) + 2P(X=2|y=2) + 3P(X=3|y=2) = 1\cdot 0 + 2\cdot\frac{1}{2} + 3\cdot\frac{1}{2} = \boxed{5/2}$

CONDITIONAL EXPECTATION

GIVEN A JOINT DENSITY FUNCTION COMPUTE $E[X^2|Y=y]$ ③

Exercise 10.6. Let the joint density function of (X, Y) be

$$f(x, y) = \frac{x+y}{4}, \quad 0 < x < y < 2.$$

(a) Find $f_{X|Y}(x|y)$.
(b) Compute the conditional probabilities $P(X < 1/2 \mid Y = 1)$ and $P(X < 3/2 \mid Y = 1)$.
(c) Compute the conditional expectation $E[X^2|Y = y]$. Check the averaging identity

$$\int_{-\infty}^{\infty} E[X^2|Y = y]\, f_Y(y)\, dy = E[X^2].$$

$f_{X|Y}(x|y) = \frac{2}{3}\frac{x+y}{y^2}$ ①

REMEMBER $0<x<y<2$

CHOOSE WHERE X IS DEFINED

$$E[X^2|Y=y] = \int_0^y x^2 f_{X|Y}(x|y)\,dx = \int_0^y x^2\frac{2}{3}\frac{x+y}{y^2}\,dx = \frac{2}{3}\left[\frac{x^4}{4y^2} + \frac{x^3}{3y}\right]_0^y = \frac{2}{3}\left[\frac{y^2}{4} + \frac{y^2}{3}\right] =$$

USE THE SAME DEGREE

CHECK THE AVERAGIN IDENTITY

CHECK WHERE Y IS DEFINED $0<x<y<2$. DO NOT CONSIDER X SO $\boxed{0<y<2}$

$$\boxed{= \frac{7}{18}y^2} = \frac{7}{18}E[Y^2] = \frac{7}{18}\int_0^2 y^2\,\frac{3}{8}y^2\,dy = \boxed{14/15}$$

CONDITIONAL EXPECTATION (THE END)

CALCULATE $E[Y^2]$

REMEMBER TO MULTIPLY

USE THE SECOND MARGINAL

X, Y INDIPENDENT RANDOM VARIABLES. $Z = X+Y$ FIND $P(Z=3)$

ES: $X \sim \text{Poisson}(2)$, $Y \sim \text{Geom}(2/3)$

$P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}$ $\qquad$ $P(Y=k) = (1-p)^{k-1}\cdot p$

FIND ALL THE COMBINATIONS THAT SATISFY THE Z FUNCTION

- $X=0,\ Y=3 \longrightarrow P(X=0)P(Y=3) = e^{-2}\cdot\left(\frac{1}{3}\right)^2\cdot\frac{2}{3} = \frac{2}{27}e^{-2}$
- $X=1,\ Y=2 \longrightarrow P(X=1)P(Y=2) = e^{-2}\cdot 2\cdot\frac{1}{3}\cdot\frac{2}{3} = \frac{4}{9}e^{-2}$
- $X=2,\ Y=1 \longrightarrow P(X=2)P(Y=1) = e^{-2}\cdot\frac{4}{2}\cdot\frac{2}{3} = \frac{4}{3}e^{-2}$
- $X=3,\ Y=0 \longrightarrow P(X=3)P(Y=0) = e^{-2}\frac{8}{6}\cdot\left(\frac{1}{3}\right)^{-1}\cdot\frac{2}{3} = \emptyset$

DELETE THE ENTIRE LINE, MAKES NO SENSE

$$P(Z=3) = \frac{2}{27}e^{-2} + \frac{4}{9}e^{-2} + \frac{4}{3}e^{-2} = \boxed{\frac{50}{27}e^{-2}}$$

TRANSFORMATION OF A RANDOM VECTOR

GIVEN X, Y RANDOM VECTOR

FIRST KIND OF TRANSFORMATION

$Z = g(X,Y)$ $\quad g: \mathbb{R}^2 \to \mathbb{R}$ ①

RANDOM VARIABLE

SECOND KIND OF TRANSFORMATION

$(U,V) = h(X,Y)$ $\quad h: \mathbb{R}^2 \to \mathbb{R}^2$ ②

RANDOM VECTOR

ES: GIVEN THAT (X,Y) IS A CONTINOUS RANDOM VECTOR FIND f_Z

ES: GIVEN THAT (X,Y) IS A CONTINOUS RANDOM VECTOR FIND $f(U,V)$

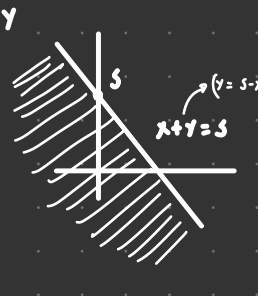
$F_Z(s) = P[Z \le s]$ $\quad s\in\mathbb{R}$

1) $\{g(X,Y) \le s\} = \{(x,y)\in\mathbb{R}^2 \mid g(x,y) \le s\}$

PROBLEM OF ANALYTIC GEOMETRY

ES: I KNOW THAT $g(x,y) = x+y$

$\{(x,y)\in\mathbb{R}^2 \mid x+y \le s\}$



TRANSFORMATION $R^1 \to R^1$

- **CONTINOUS** (SAME ES AS BEFORE)

ES: $X \sim N(1,2)$ FIND PROB. DENSITY FUNCTION OF $Y = X$

$$f_Y = f_X\left(h^{-1}(y)\right)\left[\frac{d}{dy} h^{-1}(y)\right]$$

1) CHECK THAT:
   - x^3 IS 1-TO-1 (FROM GRAPH)
   - x^3 IS DIFF.BLE (FROM GRAPH)
   - h^{-1} IS DIFF.BLE

2) $Y = x^3 \Rightarrow \boxed{h^{-1}} = x = \boxed{\sqrt[3]{y}}$

$\Rightarrow$ APPLAY FORMULA

$$f_Y = f_X(\sqrt[3]{y}) \cdot \left[\frac{d}{dy}\sqrt[3]{y}\right] \Rightarrow f_Y = f_X(\sqrt[3]{y}) \cdot \frac{1}{3}\frac{1}{\sqrt[3]{y^2}} \xRightarrow{\text{SUBSTITUTE}} f_Y = \underbrace{\frac{1}{\sqrt{2\pi}} e^{-x^2/2}}_{\text{BECAUSE } X \sim N(1,2)} \cdot \frac{1}{3}\frac{1}{\sqrt[3]{y^2}} \Rightarrow$$

$$\Rightarrow \boxed{f_Y = \frac{1}{\sqrt{2\pi}} e^{-y^{2/3}/2} \cdot \frac{1}{3}\frac{1}{\sqrt[3]{y^2}}}$$

- **DISCRETE** I KNOW: $Y = h(x)$ $P_X(1) = 0.2$ $P_X(2) = P_X(3) = 0.4 \Rightarrow \{1,2,3\} \to \{0, \sqrt{\pi}\}$ ($Y_1, Y_2 \to 0$; $Y_3 \to \sqrt{\pi}$)

YOU HAVE EITHER 0 OR $\sqrt{\pi}$ AS CODOMAIN/RANGE

- $P(0) =$ "2 CASES" $= 0.2 + 0.4 = \boxed{0.6}$
- $P(\sqrt{\pi}) =$ "1 CASE" $= \boxed{0.4}$

TRANSFORMATION $R^2 \to R^2$

- 2 VECTOR
  - $S = X + Y$
  - $T = Y$

$$h(x,y) = (x+y,\ y) = \begin{pmatrix} x+y \\ y \end{pmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{M \text{ MATRIX}} \begin{pmatrix} x \\ y \end{pmatrix}$$

1) CHECK $\det(M) \neq 0$ ⟶ $\det(M) = (1\cdot 1) - (0 \cdot 1) = \boxed{1}$ ü
   - IF $\det(M) \neq 0$ M IS INV.BLE
   - IF $\det(M) = 0$ M IS <u>NOT</u> INV.BLE

2) FIND M^{-1} (INVERSE OF M)

EXPRESS X AND Y WRT U AND V

$$h^{-1}(x,y) = \begin{cases} U = X + V \\ V = Y \end{cases} \Rightarrow h^{-1}(u,v) = \begin{cases} x = \boxed{1U - 1V}^{*} \\ y = \boxed{1V}^{*} \end{cases} \Rightarrow \underbrace{\frac{1}{\det(M)}}_{=1 \text{ BEFORE}} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = M^{-1} \Rightarrow h^{-1}(u,v) = \overbrace{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}^{M^{-1}} \begin{pmatrix} U \\ V \end{pmatrix}$$

SAME — INVERT SIGN

3) FIND $\det(M^{-1}) = (1 \cdot 1) - (-1 \cdot 0) = 1$

⟶ FROM BEFORE WE KNOW: $f_{X,Y}(x,y) = \frac{y}{16}$ WITH $0 < x < 2$ AND $0 < y < 4$

4) SOLVE

$$f_{S,T}(U,V) = f_{X,Y}\left(\boxed{U-V}^{*}, \boxed{V}^{*}\right) \cdot \underbrace{\det(M^{-1})}_{=1} \Rightarrow f_{X,Y}(U-V, V) = \frac{y}{16} \cdot 1 \xRightarrow{\text{SUBSTITUTE}} \frac{V}{16}$$

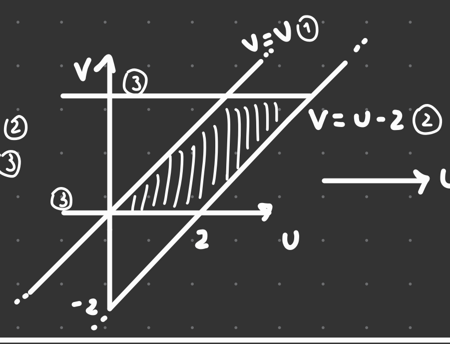
OLD DOMAIN		NEW DOMAIN
$0 < x < 2$	⟶	$0 < U - V < 2$
$0 < y < 4$	⟶	$0 < V < 4$

CHECK INDEPENDENCE

- INDEP. ⇐ DOMAIN IS RECTANGLE
- NOT INDEP. ⇐ OTHER SHAPE

NEW DOMAIN
$0 < U - V < 2$
$0 < V < 4$

$$\Rightarrow \begin{cases} V < U \ \text{①} \\ V > U - 2 \ \text{②} \\ 0 < V < 4 \ \text{③} \end{cases}$$

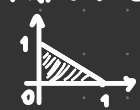


⟶ U AND Y ARE <u>NOT</u> INDEPENDENT

TRANSFORMATION $R^2 \to R^1$ (FROM SAMPLE EXAM 3E)

FIND <u>DENSITY OF X+Y</u> $\Rightarrow Z = X + Y$

I KNOW $0 < Y < 1 - X$



REWRITE THE DOMAIN: $0 < Y < 1 - X \Rightarrow 0 < Z - X < 1 - X$

$$f_{(X,Y)}(z) = \int_0 f_{(X,Y)}(x, z - x)\, dx$$

($z - x$ → Y (BECAUSE $Z = X + Y$))

$\min(x) + \min(y) \le z \le \max(x) + \max(y)$ (SINCE $Z = X + Y$)
$0 \qquad 0 \qquad 1 \qquad 1$
$0 \le z \le 2$

DOMAIN
① $0 < z - x < 1 - x$
② $0 \le z \le 2$

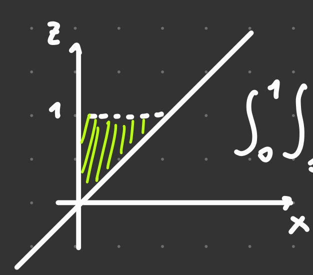
ES: $z = 1.5$ ☹

RESTRICT THE DOMAIN

$z \in (0,1)$ (MORE THEN 1 ⇒ ① IS FALSE) ⟶ $\displaystyle\int_0^{z} \frac{1}{x+y}\, dx = \int_0^{z} \frac{1}{z}\, dx = \frac{1}{z}[z - 1] = \boxed{1}$ ü

⚠ DON'T WRITE 1

$z \notin (0,1) \Rightarrow z \in (1,2)$ ⟶ $\boxed{0}$ OUTSIDE THE DOMAIN



$$\int_0^1 \int_0^2 \frac{1}{z}\, dx\, dz = \int_0^1 1\, dz = \boxed{1}$$

$$f_{(X+Y)}(z) = \begin{cases} 1 & \text{IF } z \in (0,1) \\ 0 & \text{OTHERWISE} \end{cases}$$

Exercise 3.5. Suppose that the discrete random variable X has cumulative distribution function given by

$$F(x) = \begin{cases} 0, & \text{if } x < 1 \\ 1/3, & \text{if } 1 \le x < \frac{4}{3} \\ 1/2, & \text{if } \frac{4}{3} \le x < \frac{3}{2} \\ 3/4, & \text{if } \frac{3}{2} \le x < \frac{9}{5} \\ 1, & \text{if } x \ge \frac{9}{5}. \end{cases}$$

Find the possible values and the probability mass function X.

$\Rightarrow$

3.5. The possible values of a discrete random variable are exactly the values where the c.d.f. jumps. In this case these are the values $1, 4/3, 3/2$ and $9/5$. The corresponding probabilities are equal to the size of corresponding jumps:

$$p_X(1) = \tfrac{1}{3} - 0 = \tfrac{1}{3},$$
$$p_X(4/3) = \tfrac{1}{2} - \tfrac{1}{3} = \tfrac{1}{6},$$
$$p_X(3/2) = \tfrac{3}{4} - \tfrac{1}{2} = \tfrac{1}{4},$$
$$p_X(9/5) = 1 - \tfrac{3}{4} = \tfrac{1}{4}.$$

Exercise 3.7. Suppose that the continuous random variable X has cumulative distribution function given by

$$F(x) = \begin{cases} 0, & \text{if } x < \sqrt{2} \\ x^2 - 2, & \text{if } \sqrt{2} \le x < \sqrt{3} \\ 1, & \text{if } \sqrt{3} \le x. \end{cases}$$

(a) Find the smallest interval $[a, b]$ such that of $P(a \le X \le b) = 1$.
(b) Find $P(X = 1.6)$.
(c) Find $P(1 \le X \le \frac{3}{2})$.
(d) Find the probability density function of X.

$\Rightarrow$

3.7. (a) If $P(a \le X \le b) = 1$ then $F(y) = 0$ for $y < a$ and $F(y) = 1$ for $y \ge b$. From the definition of F we see that $a = \sqrt{2}$ and $b = \sqrt{3}$ gives the smallest such interval.
(b) Since X is continuous, $P(X = 1.6) = 0$. We can also see this directly from F:

$$P(X = 1.6) = F(1.6) - \lim_{x \to 1.6-} F(x) = F(1.6) - F(1.6-).$$

Since $F(x)$ is continuous at $x = 1.6$ (actually, it is continuous everywhere), we have $F(1.6-) = F(1.6)$ and this gives $P(X = 1.6) = 0$ again.
(c) Because X is continuous, we have $P(1 \le X \le 3/2) = P(1 < X \le 3/2)$. We also have

$$P(1 \le X \le 3/2) = P(1 < X \le 3/2) = P(X \le 3/2) - P(X \le 1)$$
$$= F(3/2) - F(1) = ((\tfrac{3}{2})^2 - 2) - 0 = \tfrac{9}{4} - 2 = \tfrac{1}{4}.$$

We used $1 < \sqrt{2} \le 3/2 \le \sqrt{3}$ when we evaluated $F(3/2) - F(1)$.
(d) Since F is continuous, and it is differentiable apart from finitely many points ($\sqrt{2}$ and $\sqrt{3}$), we can just differentiate it to get the probability density function:

$$f(x) = F'(x) = \begin{cases} 2x & \text{if } \sqrt{2} < x < \sqrt{3} \\ 0 & \text{otherwise.} \end{cases}$$

We chose 0 for the value of f at $\sqrt{2}$ and $\sqrt{3}$, but the actual values are not important.

Problema 3 (10 punti). Su uno spazio di probabilità $(\Omega, \mathcal{F}, \mathbb{P})$ si consideri il vettore aleatorio (X, Y) avente come codominio l'insieme $\{1, 2\} \times \{-1, 3\}$. Per tale vettore sono note le leggi di probabilità marginali, come in tabella:

	$Y=-1$	$Y=3$	
$X=1$			0.3
$X=2$			0.7
	0.6	0.4	1

Si affrontino i seguenti quesiti.

3a) Determinare $\mathbb{E}(X)$, $\mathbb{E}(Y)$, $\mathrm{Var}(X)$, $\mathrm{Var}(Y)$.

3b) Determinare le distribuzioni congiunte di X e Y che, compatibilmente con le marginali assegnate, rendano massima, minima o nulla, rispettivamente, $\mathrm{Cov}(X, Y)$.

3c) Dimostrare che la distribuzione congiunta che rende massima (minima, rispettivamente) $\mathrm{Cov}(X, Y)$ rende anche massima (minima, rispettivamente) l'espressione $\mathbb{E}(|X - Y|)$.

3d) Stabilire se la distribuzione congiunta che rende nulla $\mathrm{Cov}(X, Y)$ rende anche X e Y stocasticamente indipendenti.

$\boxed{3a)}$

- $E[X] = 0.3 \cdot 1 + 0.7 \cdot 2 = \boxed{1.7}$
- $E[X^2] = 0.3 \cdot 1^2 + 0.7 \cdot 2^2 = 3.1$
- $E[Y] = 0.6 \cdot (-1) + 0.4 \cdot 3 = \boxed{0.6}$
- $E[Y^2] = 0.6(-1)^2 + 0.4(3^2) = 4.2$
- $VAR(X) = E[X^2] - (E[X])^2 = 3.1 - (1.7)^2 = \boxed{0.21}$
- $VAR(Y) = E[Y^2] - (E[Y])^2 = 4.2 - (0.6)^2 = \boxed{3.84}$

$\boxed{3b)}$

TO DO / DONE IN 3A

$COV(X,Y) = E[XY] - E[X]E[Y]$

	$Y=-1$	$Y=3$	
$X=1$			0.3
$X=2$			0.7
	0.6	0.4	

FILL WITH α $\Rightarrow$

	$Y=-1$	$Y=3$	
$X=1$	α	$0.3-\alpha$	0.3
$X=2$	$0.6-\alpha$	$0.1+\alpha$	0.7
	0.6	0.4	

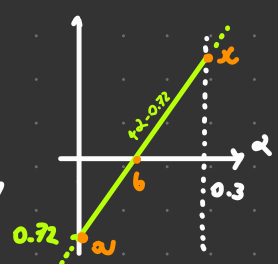
$0.4-(0.3-\alpha)$

$E[XY] = 1 \cdot (-1) \cdot (\alpha) + 1 \cdot 3 \cdot (0.3-\alpha) + 2 \cdot (-1) \cdot (0.6-\alpha) + 2 \cdot 3 \cdot (0.1+\alpha) =$

$= 4\alpha + 0.3$

$COV(X,Y) = E[XY] - E[X]E[Y] = 4\alpha + 0.3 - (1.7)(0.6) = \boxed{4\alpha - 0.72}$

GRAPH IT $\rightarrow$



- MIN: FOR $\alpha = 0 \Rightarrow COV(X,Y) = -0.72$
- NULL: FOR $0 = 4\alpha - 0.72 \Rightarrow \alpha = 0.18$ $COV(X,Y) = 0$
- MAX: FOR $\alpha = 0.3 \Rightarrow COV(X,Y) = 0.48$

???

$\boxed{3c)}$ WE KNOW:

- MAX $COV(X,Y) = 0.48$
- MIN $COV(X,Y) = -0.72$

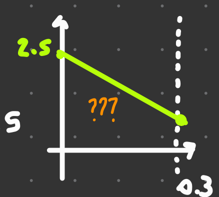
	$Y=-1$	$Y=3$	
$X=1$	α	$0.3-\alpha$	0.3
$X=2$	$0.6-\alpha$	$0.1+\alpha$	0.7
	0.6	0.4	

$\boxed{E[|X-Y|]}$

$Z = |X-Y|$

- $Z = 2$ FOR $(X=1, Y=-1)$, $(X=1, Y=3)$
- $Z = 3$ FOR $(X=2, Y=-1)$
- $Z = 1$ FOR $(X=2, Y=3)$

$\rightarrow E[Z] = 2 \cdot \alpha + 2 \cdot (0.3-\alpha) + 3 \cdot (0.6-\alpha) + 1 \cdot (0.1+\alpha) = -2\alpha + 2.5$



$\boxed{3d)}$ WE KNOW NULL: FOR $0 = 4\alpha - 0.72 \Rightarrow \alpha = 0.18$ $COV(X,Y) = 0$

	$Y=-1$	$Y=3$	
$X=1$	α	$0.3-\alpha$	0.3
$X=2$	$0.6-\alpha$	$0.1+\alpha$	0.7
	0.6	0.4	

SUBSTITUTE $\Rightarrow$

	$Y=-1$	$Y=3$	
$X=1$	0.18	0.12	0.3
$X=2$	0.42	0.28	0.7
	0.6	0.4	

CHECK INDEPENDENCE

- $0.6 \cdot 0.3 \stackrel{?}{=} 0.18$ ✓
- $0.6 \cdot 0.7 \stackrel{?}{=} 0.42$ ✓
- $0.4 \cdot 0.3 \stackrel{?}{=} 0.12$ ✓
- $0.4 \cdot 0.7 \stackrel{?}{=} 0.28$ ✓

X AND Y ARE INDEPENDENT

⚠

- X AND Y INDEP. $\Rightarrow COV(X,Y) = 0$
- $COV(X,Y) \not\Rightarrow$ X AND Y INDEP.

Exercise 10.5. Suppose that the joint density function of X and Y is

$$f(x,y) = \frac{12}{5}x(2-x-y), \quad \text{for } 0 < x < 1 \text{ and } 0 < y < 1,$$

and zero otherwise.

(a) Find $f_{X|Y}(x|y)$.
(b) Compute $P(X > 1/2 \mid Y = 3/4)$ and $E[X \mid Y = 3/4]$.

$\Downarrow$

10.5. (a) The conditional probability density function is given by the formula:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)},$$

$\Rightarrow$

if $f_Y(y) > 0$. Since the joint density is only nonzero for $0 < y < 1$, the Y variable will have a density which is only nonzero in $0 < y < 1$. In that case we have

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(w,y)dw = \int_0^1 \frac{12}{5}w(2-w-y)dw \\ &= \frac{12}{5}(w^2 - \frac{1}{3}w^3 - \frac{1}{2}yw^2)\Big|_0^1 = \frac{12}{5}(1 - \frac{1}{3} - \frac{1}{2}y) = \frac{8}{5} - \frac{6}{5}y \end{aligned}$$

Thus, for $0 < y < 1$ we have

$$f_{X|Y}(x|y) = \frac{\frac{12}{5}x(2-x-y)}{\frac{8}{5} - \frac{6}{5}y} = \frac{6x(2-x-y)}{4-3y}.$$

(b) We have

$$\begin{aligned} P(X > \frac{1}{2}|Y = \frac{3}{4}) &= \int_{\frac{1}{2}}^1 f_{X|Y}(x|y = \frac{3}{4})dx = \int_{\frac{1}{2}}^1 \frac{6x(2-x-\frac{3}{4})}{4-\frac{9}{4}}dx \\ &= \frac{24}{7}\int_{\frac{1}{2}}^1 x(\frac{5}{4} - x)dx = \frac{24}{7}(\frac{5}{8}x^2 - \frac{1}{3}x^3)\Big|_{\frac{1}{2}}^1 = \frac{24}{7}(\frac{5}{8} - \frac{1}{3} - \frac{5}{32} + \frac{1}{24}) \\ &= \frac{24}{7}(\frac{7}{24} - \frac{11}{96}) = \frac{24}{7}\frac{17}{96} = \frac{17}{28}. \end{aligned}$$

$$\begin{aligned} E[X|Y = \frac{3}{4}] &= \int_0^1 x\frac{6x(\frac{5}{4} - x)}{\frac{7}{4}}dx = \frac{24}{7}\int_0^1 x^2(\frac{5}{4} - x)dx = \frac{24}{7}(\frac{5}{12}x^3 - \frac{1}{4}x^4)\Big|_0^1 \\ &= \frac{24}{7}\frac{1}{6} = \frac{4}{7}. \end{aligned}$$

Problem 1 (8 Points) In a certain community, 36% of the families own a dog, and 22% of the families that own a dog also own a cat. In addition, 30% of the families own a cat. Find:

$P(CAT) = 0.30$

$P(DOG) = 0.36$

$P(CAT|DOG) = 0.22$

LAW OF TOTAL PROBABILITY
$P(A \cap B) = P(A|B) \cdot P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

1a) the probability that a randomly selected family owns both a dog and a cat;

$P(CAT \cap DOG) = P(CAT|DOG)\,P(DOG) = 0.22 \cdot 0.36 = 0.0792$

1b) the conditional probability that a randomly selected family own a dog, given that it owns a cat.

SAME AS INTERSECTION "∩"

$$P(DOG|CAT) = \frac{P(DOG \text{ AND } CAT)}{P(CAT)} = \frac{P(DOG \cap CAT)}{P(CAT)} = \frac{0.0792}{0.30} = 0.264$$

Then, select at random, independently, two families out of this community. Find:

1c) the conditional probability that both the families own a cat, given that at least one of the two families owns a dog.

$$P(CAT_1 \cap CAT_2 | DOG_1 \cup DOG_2) = \frac{\boxed{P((CAT_1 \cap CAT_2) \cap (DOG_1 \cup DOG_2))}}{\boxed{P(DOG_1 \cup DOG_2)}} = \frac{0.0472}{0.5904} = 0.0698$$

SAME AS "+"

- $P((CAT_1 \cap CAT_2) \cap (DOG_1 \cup DOG_2)) = P((CAT_1 \cap CAT_2 \cap DOG_1) \cup (CAT_1 \cap CAT_2 \cap DOG_2) - (CAT_1 \cap CAT_2 \cap DOG_1 \cap DOG_2))$

INDEPENDENT, INDEPENDENT, INDEPENDENT

$= P(CAT_2)\,P(CAT_1 \cap DOG_1) + P(CAT_1)\,P(CAT_1 \cap DOG_1) - P(CAT_1 \cap DOG_1)\,P(DOG_1 \cap DOG_2)$

$= 0.30 \cdot 0.0792 + 0.30 \cdot 0.0792 - 0.0792 \cdot 0.0792 = 0.0472$

- $P(DOG_1 \cup DOG_2) = P(DOG_1) + P(DOG_2) - P(DOG_1 \cap DOG_2) = 0.36 + 0.36 - (0.36)^2 = 0.5904$

Problem 2 (10 Points). The joint probability mass funtion $p_{(X,Y)}$ of the random vector $(X,Y): \Omega \to \{1,2\} \times \{1,2\}$ is given by

$p_{(X,Y)}(1,1) = \frac{1}{8}$ $\quad p_{(X,Y)}(1,2) = \frac{1}{4}$

$p_{(X,Y)}(2,1) = \frac{1}{8}$ $\quad p_{(X,Y)}(2,2) = \frac{1}{2}$

2a) Compute the conditional mass function of X given Y.

CONDITIONAL MASS FUNCTION OF X GIVEN Y

VISUALIZE ON A MATRIX AND COMPUTE THE MARGINALS

Y \ X	1	2	
1	1/8	1/8	1/4
2	1/4	1/2	3/4
	3/8	5/8	

- $P(X=1, Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{1/8}{1/4} = \frac{1}{2}$
- $P(X=1, Y=2) = \frac{P(X=1, Y=2)}{P(Y=2)} = \frac{1/4}{3/4} = \frac{1}{3}$
- $P(X=2, Y=1) = \frac{P(X=2, Y=1)}{P(Y=1)} = \frac{1/8}{1/4} = 1/2$
- $P(X=2, Y=2) = \frac{P(X=2, Y=2)}{P(Y=2)} = \frac{1/2}{3/4} = \frac{2}{3}$

2b) ~~Xxx~~ Compute $\mathbb{E}[X]$, $\mathbb{E}[Y]$, $Var(X)$, $Var(Y)$

$E[X] = 1 \cdot \frac{3}{8} + 2 \cdot \frac{5}{8} = \frac{13}{8}$

$E[X^2] = 1^2 \cdot \frac{3}{8} + 2^2 \cdot \frac{5}{8} = \frac{23}{8}$

$VAR(X) = E[X^2] - (E[X])^2 =$

$= \frac{23}{8} - \left(\frac{13}{8}\right)^2 = 0.2343$

$E[Y] = 1 \cdot \frac{1}{4} + 2 \cdot \frac{3}{4} = \frac{7}{4}$

$E[Y^2] = 1^2 \cdot \frac{1}{4} + 2^2 \cdot \frac{3}{4} = 13/4$

$VAR(Y) = E[Y^2] - (E[Y])^2 =$

$= \frac{13}{4} - \left(\frac{7}{4}\right)^2 = 0.1875$

2c) Compute $Cov(X,Y)$

$COV(X,Y) = E[XY] - E[X]E[Y] = \frac{23}{8} - \frac{13}{8} \cdot \frac{7}{4} = \frac{1}{32}$

$1 \cdot 1 \cdot \frac{1}{8} + 2 \cdot 1 \cdot \frac{1}{8} + 1 \cdot 2 \cdot \frac{1}{4} + 2 \cdot 2 \cdot \frac{1}{2} = \frac{23}{8}$

2d) Are X, Y independent?

TEST $(1,1)$: $\frac{3}{8} \cdot \frac{1}{4} \overset{?}{=} \frac{1}{8}$ NO $\Rightarrow$ NOT INDEPENDENT

2e) Compute $P[XY \leq 3]$, $P[X+Y>2]$ and $P[X/Y > 1]$

$P(XY \leq 3)$ IS SATIFIED IN $(1,1)\ (2,1)\ (1,2)$

$\hookrightarrow = \frac{1}{8} + \frac{1}{8} + \frac{1}{4} = \frac{1}{2}$

$P(X+Y>2)$ IS SATISFIED IN $(2,1)\ (1,2)\ (2,2)$

$\hookrightarrow \frac{1}{8} + \frac{1}{4} + \frac{1}{2} = \frac{7}{8}$

$P(X/Y > 1)$ IS SATISFIED IN $(2,1)$

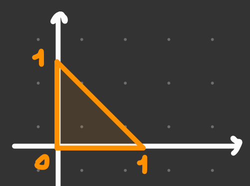
$\hookrightarrow \frac{1}{8}$

Problem 3 (12 Points). Let $D \subseteq \mathbb{R}^2$ be the triangle with vertices $(0,0)$, $(0,1)$ and $(1,0)$. Consider the continuous random vector $(X,Y): \Omega \to \mathbb{R}^2$ with joint density function

$$f_{(X,Y)}(x,y) = \begin{cases} \frac{1}{x+y} & \text{if } (x,y) \in D \\ 0 & \text{otherwise} \end{cases}$$

Find:

3a) the marginal densities f_X and f_Y



MARGINALS

- **FIRST MARGINAL (X)**

BECAUSE X IS "FIXED" AND Y CAN VARY ONLY IN 1-X → BECAUSE X AND Y INFLUENCE EACHOTHER

$$f_x = \int_0^{1-x} \frac{1}{x+y}\,dy = \Big[\ln(x+y)\Big]_0^{1-x} = \ln(\cancel{x}+1-\cancel{x}) - \ln(x+0) = -\ln(x)$$

- **SECOND MARGINAL (Y)**

BY SIMMETRY

$$f_y = \int_0^{1-y} \frac{1}{x+y} = -\ln(y)$$

3b) $E[X]$, $Var(X)$, $E[Y]$, $Var(Y)$

⚠ NOT 1-Y BECAUSE X AND Y ARE NOT INFLUENCING EACHOTHER

BY SIMMETRY

$$E[X] = \int_0^1 x\, f_x\,dx = -\int_0^1 \underbrace{x}_{g'}\underbrace{\ln(x)}_{f}\,dx = -\underbrace{\left[\ln(x)\cdot\frac{x^2}{2}\right]_0^1}_{0} + \int_0^1 \frac{1}{\cancel{x}}\cdot\frac{x^{\cancel{2}}}{2}\,dx = \left[\frac{x^2}{4}\right]_0^1 = \frac{1}{4} = E[Y]$$

f $\ln(x)$ f' $1/x$

g $x^2/2$ g' x

BY SIMMETRY

$$VAR(X) = E[X^2] - (E[X])^2 = \frac{1}{9} - \left(\frac{1}{4}\right)^2 = \frac{1}{9} - \frac{1}{16} = \frac{7}{144} = VAR(Y)$$

↓

BY SIMMETRY

$$E[X^2] = \int_0^1 x^2 f_x\,dx = -\int_0^1 \underbrace{x^2}_{g'}\underbrace{\ln(x)}_{f}\,dx = -\underbrace{\left[\frac{x^3}{3}\ln(x)\right]_0^1}_{0} + \int_0^1 \frac{x^{\cancel{3}\,2}}{3}\cdot\frac{1}{\cancel{x}}\,dx = \left[\frac{x^3}{9}\right]_0^1 = \frac{1}{9} = E[Y^2]$$

f $\ln(x)$ f' $1/x$

g $x^3/3$ g' x^2

3c) the conditional density $f_{Y|X}(y|x)$

$$f_{Y|X}(y|x) = \frac{f_{(X,Y)}(x,y)}{\underbrace{f_Y(x,y)}_{\text{MARGINAL}}} = \frac{\frac{1}{x+y}}{-\ln(y)} = \frac{-1}{(x+y)\ln(y)} \Rightarrow \begin{cases} \frac{-1}{(x+y)\ln(y)} & \text{IF } y \in (0, 1-x) \\ 0 & \text{OTHERWISE} \end{cases}$$

3d) $Cov(X,Y)$

$$COV(X,Y) = E[XY] - E[X]E[Y] = \frac{1}{18} - \frac{1}{4}\cdot\frac{1}{4} = \boxed{\frac{1}{18} - \frac{1}{16}}$$

↓

$$\iint xy\frac{1}{x+y}\,dx\,dy = \int_0^1 y\int_0^{1-y}\frac{x}{x+y}\,dx\,dy = \int_0^1 y\int_0^{1-y}\frac{x+y-y}{x+y}\,dx\,dy = \int_0^1 y\int_0^{1-y} 1 - \frac{y}{x+y}\,dx\,dy = \int_0^1 y\Big[x - y\ln(x+y)\Big]_0^{1-y}dy = \int_0^1 y\Big[1-y-0+y\log(y)\Big]dy$$

$$= \int_0^1 y - y^2 + \underbrace{y^2}_{g'}\underbrace{\log(y)}_{f}\,dy = \underbrace{\left[\frac{y^2}{2}-\frac{y^3}{3}\right]_0^1}_{\frac12-\frac13} + \underbrace{\left[\frac{y^3}{3}\log(y)\right]_0^1}_{0} - \int_0^1\frac{y^2}{3}\,dy = \frac{1}{6} - \left[\frac{y^3}{9}\right]_0^1 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$$

f $\log(y)$ f' $1/y$

g $\frac{y^3}{3}$ g' y^2

3e) the density of $X+Y$

DEFINE $Z = X+Y$ AND FIND ITS DOMAIN: $Z \in (0, 2)$

MIN(X)+MIN(Y) MAX(X)+MAX(Y)

$y = z - x$

I TRANSFORM: $f(x,y) \Rightarrow f(x, z-x)$

FROM 1 → TO 2

$$f_{x+y}(z) = \int_0^1 \frac{1}{x+(z-x)}\,dx = \frac{1}{z}\int_0^1 \mathbb{1}\{z-x\in(0,1-x)\}\,dx = \frac{1}{z}\big[x\big]_0^z = 1\,??$$

IF $z-x \in (0,1-x)$ $\mathbb{1}=1$

ELSE $\mathbb{1}=0$

Problem 4 (10 Points) Let (X,Y) be a continuous random vector having bivariate GAUSSIAN distribution with mean vector $(m_X, m_Y) = (2, 0)$ and covariance matrix $\begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 9 \end{pmatrix}$

I KNOW:

$m_X = 2$ $\quad \sigma_X^2 = 1$ $\quad$ "RHO" RELATIONAL COEFFICIENT

$m_Y = 0$ $\quad \sigma_Y^2 = 9$ $\quad \rho\sigma_X\sigma_Y = 1$

FIRST STEP: FIND ρ

I KNOW: $\sigma_X^2 = 1 \Rightarrow \sigma_X = 1$ $\quad$ SO, $\rho \cdot 1 \cdot 3 = 1$; $\boxed{\rho = 1/3}$

$\sigma_Y^2 = 9 \Rightarrow \sigma_Y = 3$

4a) $E[3X - 4Y]$

$E[3X-4Y] = 3E[X] - 4E[Y] = 3\cdot 2 - 4 \cdot 0 = 6$

4b) $Cov(2X+Y, -X+3Y)$

FIND ALL THE COMBINATIONS POSSIBLE (LIKE $(2X+Y)(-X+3Y)$)

$Cov(2X+Y, -X+3Y) = -2Cov(X,X) + 6Cov(X,Y) - Cov(Y,X) + 3Cov(Y,Y) =$

$= -2VAR(X) + 6Cov(X,Y) - Cov(X,Y) + 3VAR(Y) =$

$= -2VAR(X) + 5Cov(X,Y) + 3VAR(Y) =$

$= -2\cdot 1 + 5\cdot(1) + 3\cdot 9 = \boxed{30}$

⚠ $cov(X,Y) = \rho\sigma_X\sigma_Y = \frac{1}{3}\cdot 1 \cdot 3 = 1$

REMEMBER

$VAR(3X) = 3VAR(X)$

$COV(X,Y) = COV(Y,X)$

$COV(X,X) = VAR(X)$

$Cov(X,Y) = E[XY] - E[X]E[Y]$

IF INDEPENDENT: $E[XY] = E[X]E[Y]$

$\rho = \frac{cov(X,Y)}{\sqrt{\sigma_X^2\sigma_Y^2}} = \frac{cov(X,Y)}{\sigma_X\sigma_Y}$

- VARIANCE IS ALWAYS
- COVARIANCE CAN BE POSITIVE OR NEGATIVE
  - (−) OPPOSITE RELATION
  - (+) DIRECTLY PROPORTIONAL
  - (0) NO CORRELATION

3c) the distribution of the random vector $(U,V) = (2X+Y, -X+3Y)$

??

3d) the conditional distribution of X given Y.

??