+ ×

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$$\cosh^2(x) - \sinh^2(x) = 1$$

$$\frac{d}{dx}\cosh(x) = \sinh(x) \qquad \frac{d}{dx}\sinh(x) = \cosh(x)$$

$$\lim_{x\to+\infty}\cosh(x) = \lim_{x\to+\infty}\sinh(x) = +\infty$$

$$\lim_{x\to+\infty}\frac{\cosh(x)}{\sinh(x)} = 1 \qquad \frac{\infty}{\infty}$$

L'HOPITAL NOT USEFUL

$$\lim_{x\to+\infty}\frac{\cosh'(x)}{\sinh'(x)} = \lim_{x\to+\infty}\frac{\sinh(x)}{\cosh(x)} \qquad \frac{\infty}{\infty}$$

$$\lim_{x\to+\infty}\frac{\sinh'(x)}{\cosh'(x)} = \lim_{x\to+\infty}\frac{\cosh(x)}{\sinh(x)} \qquad \frac{\infty}{\infty}$$

EXAMPLE

$$\lim_{x\to 0}\frac{e^x - 1 - x}{x^2} \qquad \frac{0}{0}$$

$$f(x) = e^x - 1 - x \ ; \ g(x) = x^2$$

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0}\frac{e^x - 1}{2x} = \frac{1}{2}$$

- $$\lim_{x\to 0} \frac{e^x - 1 - x - \frac{x^2}{2} - \frac{x^3}{6}}{(1-\cos(x))^2} \qquad \frac{0}{0}$$

$$f(x) = e^x - 1 - x - \frac{x^2}{2} - \frac{x^3}{6} \qquad g(x) = (1-\cos(x))^2$$

$$f'(x) = e^x - 1 - x - \frac{x^2}{2} \qquad g'(x) = 2\sin(x)(1-\cos(x))$$

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} \qquad \frac{0}{0}$$

$$f''(x) = e^x - 1 - x$$

$$g''(x) = 2\cos(x)(1-\cos(x)) + 2\sin^2(x)$$

$$\lim_{x\to 0} \frac{f''(x)}{g''(x)} \qquad \frac{0}{0}$$

$$f'''(x) = e^x - 1 \qquad g'''(x) = \ldots\ldots$$

$$\xrightarrow[x\to 0]{} 4$$

$$h(x) = x^4$$

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f(x)}{h(x)} \cdot \frac{h(x)}{(1-\cos(x))^2} = \frac{1}{24} \cdot 4 = \frac{1}{6}$$

$$h'(x) = 4x^3 \qquad h''(x) = 12x^2 \qquad h''' = 24x$$

$$\lim_{x\to 0} \frac{f'(x)}{h'(x)} = \frac{1}{24}\ \frac{0}{0}\ ; \quad \lim_{x\to 0} \frac{f''(x)}{h''(x)} = 1/24\ \frac{0}{0}$$

$$\lim_{x\to 0} \frac{f''(x)}{h''(x)} = \frac{1}{24}\ \frac{0}{0}\ ; \quad \lim_{x\to 0} \frac{f'''(x)}{h'''(x)} = \lim_{x\to 0} \frac{e^x - 1}{24x} = \frac{1}{24}$$

$$\lim_{N} \frac{-\cos\left(\frac{1}{N}\right) + e^{-1/2N^2}}{\sin\left(3/N^3\right)} = ?$$

$$\lim_{x\to 0} \frac{-\cos(x) + e^{-1/2x^2}}{\sin(3x^3)} = \lim_{x\to 0} \frac{3x^3}{\sin(3x^3)}\ \frac{f(x)}{3x^3}$$

$$\lim_{x\to 0} \frac{f(x)}{x^3} = ? \qquad \frac{0}{0}$$

$$\lim_{x\to 0} \frac{f'(x)}{3x^2} = \lim_{x\to 0} \frac{\sin(x) - xe^{-1/2x^2}}{3x^2}$$

$$\lim_{x\to 0} \frac{f''(x)}{6x} = \lim_{x\to 0} \frac{\cos(x) - e^{-1/2x^2} + x^2e^{-1/2x^2}}{6x}$$

$\to 0$

$$\lim_{x\to 0} \frac{-\sin(x) + x e^{-1/2 x^2}}{6} = 0$$

$$\lim_{x\to 0^+} (e^x - 1)\tan\left(\frac{\pi}{2} - x\right) \quad 0\cdot\infty$$ NO HOPITAL

$$\lim_{x\to 0^+} \frac{e^x - 1}{\frac{1}{\tan\left(\frac{\pi}{2} - x\right)}} \quad \frac{0}{0}$$ OK HOPITAL

$$\lim_{x\to 0^+} \frac{e^x - 1}{\frac{\cos\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right)}}$$ 1

$$\lim_{x\to 0^+} \frac{e^x - 1}{\cos\left(\frac{\pi}{2} - x\right)}$$ HOPITAL

$$\lim_{x\to 0^+} \frac{e^x}{-\left(-\sin\left(\frac{\pi}{2} - x\right)\right)} = 1$$

TAYLOR FORMULA OR TAYLOR EXPANSION

$f : (a,b) \to \mathbb{R}$ $\quad -\infty \le a < b \le +\infty$, $x_0 \in (a,b)$

RECALL THAT

f IS DIFF. IN x_0, WITH DERIVATIVE $f'(x_0) \Leftrightarrow$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + R_1(x)$$

WHERE $R_1(x_0) = 0$ AND $\lim_{x\to x_0} \frac{R_1(x)}{x - x_0} = 0$

NOTE THAT

$P_1(x) = f(x_0) + f'(x_0)(x - x_0)$ IS A POLYNOMIAL OF DEGREE AT MOST 1 CENTER AT x_0

$\to A_0 + A_1(x - x_0)$

THEOREM

TAYLOR FORMULA WITH PEANO AND LAGRANGE REMINDERS

$f:(a,b) \to \mathbb{R}$ $\quad -\infty \leq a < b \leq +\infty \quad x_0 \in (a,b)$

- LET $N \geq 1$ SUPPOSE f IS DIFFERENTIABLE N TIMES IN x_0. THEN $\forall x \in (a,b)$ WE HAVE

$$\boxed{f(x) = \sum_{K=0}^{N} \frac{f^{(K)}(x_0)}{K!}(x - x_0)^K + R_N(x)}$$

TAYLOR FORMULA OF ORDER N FOR f IN x_0

WHERE

$$P_N(x) = \sum_{K=0}^{N} \frac{f^{(K)}(x_0)}{K!}(x - x_0)^K$$ TAYLOR POLYNOMIAL OF ORDER N FOR f IN x_0

WHICH IS A POLYNOMIAL OF DEGREE AT MOST N CENTER IN x_0, THAT IS

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots a_N(x - x_0)^N$$

AND THE REMAINDER R_N SATISFIES $R_N(x_0) = 0$ AND

$$\boxed{\lim_{x \to x_0} \frac{R_N(x)}{(x-x_0)^N} = 0 \quad \text{REMAINDER IN THE PEANO FORM}}$$

- IF f IS DIFFERENTIABLE $(N+1)$-TIMES ON (a,b), THEN $\forall x \in (a,b) \setminus \{x_0\}$ $\exists y$ STRICTLY BETWEEN x_0 AND x (THAT IS, $x < y < x_0$ OR $x_0 < y < x$)

SUCH THAT

$$R_N(x) = \frac{f^{(N+1)}(y)}{(N+1)!}(x-x_0)^{N+1} \quad \text{REMAINDER IN THE LAGRANGE FORM}$$

REMARK

y DEPENDS ON x AND ON N!

REMARK RECALL THAT $f^{(0)} = f$; $b^0 = 1$ $\forall b \in \mathbb{R}$

$\boxed{N=1}$ $$P_1(x) = \frac{f^{(0)}(x_0)}{0!}(x-x_0)^0 + \frac{f^{(1)}(x_0)}{1!}(x-x_0)^1 =$$

$$f(x_0) + f'(x_0)(x-x_0)$$

$\boxed{N=2}$ $$P_2(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f^2(x_0)}{2!}(x-x_0)^2$$

$$= f(x_0) + f(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2$$

$\boxed{N=3}$ $P_3(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \frac{f'''(x_0)}{6}(x-x_0)^3$

EXERCISE PROVE THAT

$$P_N^{(j)}(x_0) = f^{(j)}(x_0) \quad \forall \ 0 \leq j \leq N$$

REMARK LET $q_N(x) = b_0 + b_1 x + \dots b_N x^N$ $N \geq 1$

IF $\lim_{x \to 0} \frac{q_N(x)}{x^N} = 0$; THEN $q_N = 0$, THAT IS,

$b_0 = b_1 = \dots = b_N = 0$

UNIQNESS OF THE TAYLOR EXPANSION

IF f IS DIFFERENTIABLE N TIME IN x_0 AND

$f(x) = \tilde{P}_N(x) + \tilde{R}_N(x)$ WITH $\tilde{P}_N$ POLY. OF DEGREE AT MOST N CENTERED AT x_0

AND $\tilde{R}_0$ SUCH THAT $\tilde{R}_N(x_0) = 0$

AND $\lim_{x \to x_0} \frac{\tilde{R}_N(x)}{(x-x_0)^N} = 0$

THEN $\tilde{P}_N = P_N$, THAT IS TAYLOR POLYNOMIAL

IF $\tilde{P}_N(x) = a_0 + a_1(x-x_0) + \dots + a_N(x-x_0)^N$, THEN

$$a_j = \frac{f^{(j)}(x_0)}{j!} \qquad 0 \le j \le N$$

SO $f^{(j)}(x_0) = j!\, a_j$ $\qquad 0 \le j \le N$

APPLICATION

$f:(a,b) \to \mathbb{R}$ $\quad -\infty \le a < b \le +\infty$ $\quad x_0 \in (a,b)$

SUPPOSE f IS DIFF. 2 TIME IN x_0 AND $\boxed{f'(x_0) = 0}$

(x_0 IS A CRITICAL POINT! OR A STATIONARY POINT)

IF $f''(x_0) \neq 0$ THEN x_0 IS A LOCAL EXTREMUM POINT, IN PARTICULAR

- $f''(x_0) > 0 \Rightarrow x_0$ STRICT LOCAL MINIMUM POINT
- $f''(x_0) < 0 \Rightarrow x_0$ STRICT LOCAL MAXIMUM POINT

REMARK $f''(x_0) = 0$ NO INFORMATION

$f(x) = x^3$ $\quad f'(0) = f''(0)$ $\quad 0$ NOT INFORMATION

$f(x) = x^4$ $\quad f'(0) = f''(0) = 0$ $\quad 0$ IS A STRICT LOCAL (GLOBAL) MINIMUM POINT

PROOF CASE $f''(x_0) > 0$

TAYLOR $\Rightarrow$

$$f(x) = f(x_0) + \cancel{f'(x_0)(x - x_0)}^{0} + \frac{f''(x_0)}{2}(x - x_0)^2 + R_2(x) \quad \forall x \neq x_0$$

$$\frac{f(x) - f(x_0)}{(x - x_0)^2} = \underbrace{\frac{f''(x_0)}{2}\frac{\cancel{(x - x_0)^2}}{\cancel{(x - x_0)^2}} + \frac{R_2(x)}{(x - x_0)^2}}_{} \qquad / (x - x_0)^2$$

$$\Big\downarrow x \to x_0$$

$$\frac{f''(x_0)}{2} > 0$$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{(x - x_0)^2} > 0$$

PERMANCE OF SIGN

$\exists\, \delta > 0$ SUCH THAT $\forall x \in (x_0 - \delta, x_0 + \delta) \setminus (x_0)$ WE HAVE

$$\frac{f(x) - f(x_0)}{(x - x_0)^2} > 0 \quad \text{THAT IS} \quad f(x) > f(x_0)$$

EXERCISE WRITE THE TAYLOR POLYNOMIAL OF ORDER 2 IN $X_0 = 1$ FOR THE FUNCTION

$$f(x) = \arctan(x)$$

- WRITE THE TAYLOR FORMULA 2 IN $X_0 = 1$ FOR $f(x) = \arctan(x)$ WITH PEANO AND LAGRANGE REMAINDER

- $P_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$ POLY
- $f(x) = P_2(x) + R_2(x)$ FORMULA

WHERE

$$\lim_{x \to x_0} \frac{R_2(x)}{(x - x_0)^2} = 0$$ PEANO

OR FOR ANY $x \neq x_0$ $\exists y$ STRICTLY BETWEEN x AND x_0 SUCH THAT $R_2(x) = \frac{f^{(3)}(y)}{3!}(x - x_0)^3$ LAGRANGE

$X_0 = 1$ $\quad f(x_0) = f(1) = \arctan(1) = \pi/4$

$f'(x) = \frac{1}{1 + x^2}$ $\quad f'(x_0) = f'(1) = \frac{1}{2}$

$f''(x) = \frac{-2x}{(1 + x^2)^2}$ $\quad f''(x_0) = f''(1) = -\frac{1}{2}$

$$f'''(x) = \frac{2(3x^2-1)}{(1+x^2)^3}$$ (ONLY FOR LAGRANGE)

- $P_2(x) = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{2}\frac{(x-1)^2}{2}$

- $f(x) = P_2(x) + R_2(x)$ WITH $\lim_{x \to 1} \frac{R_2(x)}{(x-1)^2} = 0$

- $f(x) = P_2(x) + R_2(x)$ SUCH THAT $\forall x \neq 1 \ \exists y$ STRICTLY BETWEEN x AND 1 SUCH THAT

$$R_2(x) = \frac{f'''(y)}{3!}(x-1)^3 = \frac{1}{6}\frac{2(3y^2-1)}{(1+y^2)^3}(x-1)^3$$

TAYLOR EXPANSION OF SOME ELEMENTARY FUNCTION IN $x_0=0$

- LET $n \in \mathbb{N}$ FOR ANY $x \neq 0$ $e^x = P_n(x) + R_n(x)$

WHERE

$$P_n(x) = \sum_{j=0}^{n} \frac{x^j}{j!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}; \quad \lim_{x \to 0} \frac{R_n(x)}{x^n} = 0$$

MOREOVER $\exists y_n$, STRICTLY BETWEEN 0 AND x SUCH THAT

$$R_n(x) = \frac{e^{y_n}}{(n+1)!} x^{n+1}$$

SO

$$e^{x} = \sum_{j=0}^{N} \frac{x^{j}}{j!} + \frac{e^{y_n}}{(N+1)} x^{N+1}$$

$$\sum_{j=0}^{N} \frac{x^{j}}{j!} \xrightarrow{N\to+\infty} \sum_{j=0}^{+\infty} \frac{x^{j}}{j!}$$

$$\frac{e^{y_n}}{(N+1)} x^{N+1} \xrightarrow{N\to+\infty} 0$$

$$\Rightarrow \quad e^{x} = \sum_{j=0}^{+\infty} \frac{x^{j}}{j!} \quad \forall x \in \mathbb{R}$$

$$e = \sum_{j=0}^{+\infty} \frac{1}{j!}$$

- LET $n \in \mathbb{N} \cup \{0\}$ FOR ANY $x \neq 0$

$\cos(x) = P_{2n+1}(x) + P_{2n+1}(x)$ WHERE

$$P_{2n+1}(x) = \sum_{j=0}^{N} (-1)^{j} \frac{x^{2j}}{(2j)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6!} + \dots (-1)^{n} \frac{x^{2n}}{(2n)!}$$

$$\lim_{x\to 0} \frac{R_{2n+1}(x)}{x^{2n+1}} = 0$$

MOREOVER $\exists y$ STRICTLY BETWEEN 0 AND X SUCH THAT

$$\cos(x) = \sum_{j=0} (-1)^{j} \frac{x^{2j}}{(2j)!} + (-1)^{n+1} \frac{\cos(y_n)\, x^{2n+2}}{(2n+2)!}$$

REMARK

$\forall n \in \mathbb{N}$ $P_{2n} = P_{2n+1}$

$$\Rightarrow \cos(x) = \sum_{j=0}^{+\infty} (-1)^j \frac{x^{2j}}{(2j)!} \quad \forall x \in \mathbb{R}$$

LET $n \in \mathbb{N} \cup \{0\}$ FOR ANY $x \neq 0$, $\sin(x) = P_{2n+2}(x) + R_{2n+2}(x)$

WHERE

$$P_{2n+2}(x) = \sum_{j=0}^{n} (-1)^j \frac{x^{2j+1}}{(2j+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\lim_{x \to 0} \frac{R_{2n+2}(x)}{x^{2n+2}} = 0$$

MOREOVER $\exists y_n$ STRICTLY BETWEEN 0 AND x SUCH THAT

$$\sin(x) = \sum_{j=0}^{n} (-1)^j \frac{x^{2j+1}}{(2j+1)!} + (-1)^{n+1} \frac{\cos(y_n)}{(2n+3)!} x^{2n+3}$$

REMARK

$\forall n \in \mathbb{N} \cup \{0\}$ $P_{2n+1} = P_{2n+2}$

$$\Rightarrow \sin(x) = \sum_{j=0}^{+\infty} (-1)^j \frac{x^{2j+1}}{(2j+1)!} \quad \forall x \in \mathbb{R}$$

- LET $N \in \mathbb{N}$ FOR ANY $x \neq 0$, $x > -1$,

$$\log(1+x) = P_N(x) + R_N(x) \text{ WHERE}$$

$$P_N(x) = \sum_{j=1}^{N} (-1)^{j+1} \frac{x^j}{j} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{N+1} \frac{x^N}{N}$$

$$\lim_{x \to 0} \frac{R_N(x)}{x^N} = 0$$

MOREOVER $\exists\, y_N$ STRICTLY BETWEEN 0 AND x SUCH THAT

$$\log(1+x) = \sum_{j=1}^{N} (-1)^{j+1} \frac{x^j}{\cancel{j!}\, j} + \frac{(-1)^N}{N+1} \frac{1}{(1+y_N)^{N+1}} x^{N+1}$$

REMARK THIS EXPANSION IS VALID $\forall x > -1$, $x \neq 0$ BUT IT IS SIGNIFICANT ONLY FOR $x \in (-1, 1]$, $x \neq 0$

IN FACT,

$$\log(1+x) = \sum_{j=1}^{+\infty} (-1)^{j+1} \frac{x^j}{j} \quad \forall x \in (-1, 1]$$

$$\Rightarrow \sum_{j=1}^{+\infty} \frac{(-1)^{j+1}}{j} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \log(2)$$

APPLICATION OF TAYLOR FORMULA

- **DETERMINATION OF ORDER OF INFINITESIMAL** (PEANO ENOUGH)
  - LIMITS (SIMILAR TO L'HOPITAL)
  - CONGRUENCE OF SERIES
  - INEQUALITIES (LAGRANGE)
  - APPROXIMATE COMPUTATIONAL OF FUNCTIONS (LAGRANGE)

EXAMPLES

- $\lim_{x \to 0} \frac{\log(1+x)-x}{\sin(x)-x}$

$$\log(1+x) = x - \frac{x^2}{2} + R_2(x) \qquad \lim_{x \to 0} \frac{R_2(x)}{x^2} = 0$$

$$\sin(x) = x - \frac{x^3}{6} + \tilde{R}_4(x) \qquad \lim_{x \to 0} \frac{\tilde{R}_4(x)}{x^4} = 0$$

$$\frac{\log(1+x)-x}{\sin(x)-x} = \frac{-x^2/2 + R_2(x)}{-\frac{x^3}{6} + R_4(x)} =$$

$$= \frac{x^2}{x^3} \frac{\left[-\frac{1}{2} + \frac{R_2(x)}{x^2}\right]}{\left[-\frac{1}{6} + \frac{R_4(x)}{x^3}\right]} \to \pm\infty \text{ as } x \to 0^{\pm}$$

$\frac{R_2(x)}{x^2} \to 0$, $\frac{R_4(x)}{x^3} \to 0$, 3

$$\bullet \lim_{x \to 0} \frac{\log(1+x) - x + x^2/2}{\sin(x) - x} =$$

$$\log(1+x) - x + \frac{x^2}{2} = \frac{x^3}{3} + R_3(x) + R_3(x) \qquad \lim_{x \to 0} \frac{R_3(x)}{x^3} = 0$$

$$\frac{f(x)}{g(x)} = \frac{x^3/3 + R_3(x)}{-\frac{x^3}{6} + \tilde{R}_4(x)} = \frac{x^3}{x^3} \frac{\left[\frac{1}{3} + R_3(x)/x^3\right]}{\left[-\frac{1}{6} + \frac{\tilde{R}_4(x)}{x^3}\right]} \to -2$$

$$\bullet \lim_{x \to 0} \frac{x \sin(x) - x^2}{\log(1+x^4)}$$

$$f(x) = x\sin(x) - x^2 = x\left(x - \frac{x^3}{6} + \tilde{R}_4(x)\right) - x^2$$

$$= -\frac{x^4}{6} + x\tilde{R}_4(x)$$

$$\lim_{x \to 0} \frac{\tilde{R}_4(x)}{x^4} = 0$$

$$\lim_{x \to 0} \frac{x\tilde{R}_4(x)}{x^5} = 0$$

$$\lim_{x \to 0} \frac{f(x)}{\log(1+x^4)} = \lim_{x \to 0} \frac{x^4}{\log(1+x^4)} \frac{-x^4/6 + x\tilde{R}_4(x)}{x^4} = -\frac{1}{6}$$

$$f(x) = -\frac{x^4}{6} + x\tilde{R}_4(x) \qquad \text{WITH} \lim_{x \to 0} \frac{x\tilde{R}_4(x)}{x^5} = 0$$

$$P_5(x) = -\frac{x^4}{6}$$

$f^{(5)} = 0 = a_5\, 5! = 0$ SINCE $a_5 = 0$

$f^{(4)} = a_4\, 4! = -\frac{1}{6} 4!$

$\log(1+x^4)$

$\log(1+y) = y - \frac{y^2}{2} + R_2(y)$ WITH $\frac{R_2(y)}{y^2} \to 0$ $_{y \to 0}$

$\log(1+x^4) = x^4 - \frac{x^8}{2} + R_2(x^4)$ WITH $\frac{R_2(x^4)}{(x^4)^2} = \to 0$ $_{x \to 0}$

$P_8(x) \to x^4 - \frac{x^8}{2}$

$$\lim_{x \to 0} \frac{\frac{\sin x}{x} - 1}{x}$$

$$\frac{\sin x}{x} = \frac{x - x^3/6 + \tilde{R}_4(x)}{x} = 1 - \frac{x^2}{6} + \frac{\tilde{R}_4(x)}{x}$$

$$\lim_{x \to 0} \frac{\frac{\sin(x)}{x} - 1}{x} = \lim_{x \to 0} \left(-\frac{x}{6} + \frac{\tilde{R}_4(x)}{x^2} \right) = 0$$

$f(x) = \frac{\sin(x)}{x}$, $x \neq 0$; EXTEND IT BY CONT.

SETTING $f(0) = 1$

SUCH AN EXTENSION IS DIFF. IN 0, $f'(0) = 0$

$$g(x) = \frac{\sin(x)}{x} - 1 = \frac{x^2}{6} + R_3(x) \qquad \frac{R_3(x)}{x^3} \to 0$$

$$\sum_{N=1}^{\infty} g\left(\frac{1}{N}\right) \qquad a_N = g\left(\frac{1}{N}\right); \quad \frac{a_N}{\frac{1}{N^2}} \to \frac{-1}{6}$$

$\Rightarrow$ SERIES IS CONVERGING

$$\sum_{N=1}^{\infty} g\left(\frac{1}{\sqrt{N}}\right) \qquad b_N = g\left(\frac{1}{\sqrt{N}}\right) \qquad \frac{b_N}{\frac{1}{(\sqrt{N})^2}} \to -\frac{1}{6}$$

$\Rightarrow$ SERIES IS DIVERGING TO $-\infty$

- $$\lim_{x \to 0} \frac{(1-3x^2)^{1/6} - 1}{x^2} \qquad \boxed{\frac{0}{0}}$$

$$g(x) = (1-3x^2)^{1/6} - 1 \qquad g(0) = 0$$

$$g'(x) = -x(1-3x^2)^{-5/6} \qquad g'(0) = 0$$

$$g''(x) = -(1-3x^2)^{-5/6} + 6x^2\left(-\frac{5}{6}\right)(1-3x^2)^{-11/6} \qquad g''(0) = -1$$

$$g(x) = (-1)\frac{x^2}{2} + R_2(x) \qquad \lim_{x \to 0} \frac{R_2(x)}{x^2} = 0$$

$$\frac{g(x)}{x^2} = \frac{-x^2/2 + R_2(x)}{x^2} \xrightarrow[x \to 0]{} -\frac{1}{2}$$

$$e^{1/10} = 1 + \frac{1}{10} + \frac{1}{2}\,\frac{1}{10}^2 + \dots + \frac{1}{N!}\left(\frac{1}{10}\right)^N + \frac{e^{y_N}}{(N+1)!}\left(\frac{1}{10}\right)^{N+1}$$

$0 < y_N < \frac{1}{10}$ $\quad$ $e^{y_N} \leq e^1 = e$