1 Existence and uniqueness theorem (Theorem 2 pag 37)

THEOREM 2

Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—that is, if and only if an echelon form of the augmented matrix has *no* row of the form

$$[0 \quad \cdots \quad 0 \quad b] \qquad \text{with } b \text{ nonzero}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

2 Equivalent ways to write a linear system (Theorem 3 pag. 52)

THEOREM 3

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b} \tag{4}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \tag{5}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix} \tag{6}$$

3 Span of the columns of A (Theorem 4 pag. 53)

THEOREM 4

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
c. The columns of A span $\mathbb{R}^m$.
d. A has a pivot position in every row.

4 Decomposition of the solution of a linear system (Theorem 6 pag. 63)

THEOREM 6

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

5 Characterization of linearly dependent sets (Theorem 7 pag. 75)

THEOREM 7

Characterization of Linearly Dependent Sets

An indexed set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.

6 Cases that are surely linearly dependent (Theorem 8 pag. 76)

THEOREM 8

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

7 Invertibility implies uniqueness (Theorem 5 pag. 122)

THEOREM 5

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

8 Properties of the inverse operator (Theorem 6 pag. 123)

THEOREM 6

a. If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

b. If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}$$

c. If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,

$$(A^T)^{-1} = (A^{-1})^T$$

9 Row equivalence for invertible matrixes (Theorem 7 pag. 125)

THEOREM 7

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

10 The invertible matrix theorem (Theorem 8 pag. 130 + Theorem pag. 253)

THEOREM 8

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.
b. A is row equivalent to the $n \times n$ identity matrix.
c. A has n pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
h. The columns of A span $\mathbb{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix C such that $CA = I$.
k. There is an $n \times n$ matrix D such that $AD = I$.
l. A^T is an invertible matrix.

11 The rank theorem (Theorem 14 pag. 251)

THEOREM 14

The Rank Theorem

The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n$$

12 The basis theorem (Theorem 12 pag. 245),

THEOREM 12

The Basis Theorem

Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

13 Properties of determinants (Theorem 3 pag. 294)

THEOREM 3

Properties of Determinants

Let A and B be $n \times n$ matrices.

a. A is invertible if and only if $\det A \neq 0$.
b. $\det AB = (\det A)(\det B)$.
c. $\det A^T = \det A$.
d. If A is triangular, then $\det A$ is the product of the entries on the main diagonal of A.
e. A row replacement operation on A does not change the determinant. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same scalar factor.

14 Similar matrixes theorem (Theorem 4 pag. 295)

THEOREM 4 If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

15 The diagonalization theorem (Theorem 5 pag. 300)

THEOREM 5 **The Diagonalization Theorem**

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

16 The QR factorization (Theorem 12 pag. 375)

THEOREM 12 **The QR Factorization**

If A is an $m \times n$ matrix with linearly independent columns, then A can be factored as $A = QR$, where Q is an $m \times n$ matrix whose columns form an orthonormal basis for Col A and R is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.

17 equivalent statements for least square problems (Theorem 14 pag. 381)

THEOREM 14 Let A be an $m \times n$ matrix. The following statements are logically equivalent:

a. The equation $A\mathbf{x} = \mathbf{b}$ has a unique least-squares solution for each $\mathbf{b}$ in $\mathbb{R}^m$.
b. The columns of A are linearly independent.
c. The matrix A^TA is invertible.

When these statements are true, the least-squares solution $\hat{\mathbf{x}}$ is given by

$$\hat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} \tag{4}$$

18 QR factorization for least square problems (Theorem 15 pag. 383).

THEOREM 15 Given an $m \times n$ matrix A with linearly independent columns, let $A = QR$ be a QR factorization of A as in Theorem 12. Then, for each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique least-squares solution, given by

$$\hat{\mathbf{x}} = R^{-1}Q^T\mathbf{b} \tag{6}$$

A **linear equation** in the variables $x_1, \dots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

where b and the **coefficients** $a_1, \dots, a_n$ are real or complex numbers, usually known in advance. The subscript n may be any positive integer. In textbook examples and

solution set of a linear system- A solution of a linear system is an assignment of values to the variables x1, x2, ..., xn such that each of the equations is satisfied. The set of all possible solutions is called the solution set.

elementary row operations - Elementary row operations are used to transform a system of linear equations into a new system that has the same solutions as the original one (i.e., into an equivalent system).

pivot position - a position of a leading entry in an echelon form of the matrix

reduced echelon form of a matrix - A matrix is in reduced row echelon form (also called row canonical form) if it satisfies the following conditions:
It is in row echelon form.
The leading entry in each nonzero row is a 1 (called a leading 1).
Each column containing a leading 1 has zeros in all its other entries.

linear combinations of vector - A linear combination of two or more vectors is the vector obtained by adding two or more vectors (with different directions) which are multiplied by scalar values.

span of vectors - The span of a set of vectors is the set of all linear combinations of the vectors

matrix-vector multiplication - Matrix-vector multiplication is an operation between a matrix and a vector that produces a new vector.

homogeneous linear systems - A homogeneous system of linear equations is one in which all of the constant terms are zero.A homogeneous system of linear equations is one in which all of the constant terms are zero

linear independence - a set of vectors is said to be linearly dependent if there is a nontrivial linear combination of the vectors that equals the zero vector.

matrix operations - a matrix operation can be defined as a set of operations that can be applied in rows and columns .
few of the operations are multiplications, addition,subtraction ,etc

invertible matrixes and inverse of a matrix - The inverse of matrix is another matrix, which on multiplication with the given matrix gives the multiplicative identity.

elementary matrixes - an elementary matrix is a matrix which differs from the identity matrix by one single elementary row operation.

triangular matrixes - A triangular matrix is a special type of square matrix where all the values above or below the diagonal are zero

subspaces of R^n - A subspace of Rn is a set H of vectors in Rn such that. 1. The zero vector “0” is in H.

Col space of a matrix - In linear algebra, the column space (also called the range or image) of a matrix A is the span (set of all possible linear combinations) of its column vectors.

Nul space of a matrix - The nullspace of the matrix A, denoted N(A), is the set of all n-dimensional column vectors x such that Ax = 0

basis -a set B of vectors in a vector space V is called a basis if every element of V may be written in a unique way as a finite linear combination of elements of B.

dimension - The dimension of a vector space is the number of coordinates you need to describe a point in it.

rank - The rank of a matrix is defined as (a) the maximum number of linearly independent column vectors in the matrix or (b) the maximum number of linearly independent row vectors in the matrix.

coordinates of a vector with respect to a basis - The coefficients x1,x2,...,xn are called the coordinates of v with respect to the ordered basis v1,v2,...,vn.

determinant of a square matrix - The determinant of a square, n times n matrix A, denoted det A, is defined by an algebraic formula of the coefficients of A.

norm of a vector or matrix -

eigenvalues - Eigenvalues are a special set of scalars associated with a linear system of equations or matrices equations. Eigenvalues are also called characteristic roots.

eigenvectors and eigenspaces of a square matrix -

characteristic equation of a square matrix - $\det(A - \lambda I) = 0$ is called the characteristic equation of the matrix A.

similarity of square matrixes - Two square matrices are said to be similar if they represent the same linear operator under different bases.

diagonal matrixes - a diagonal matrix is a matrix in which the entries outside the main diagonal are all zero

transpose of a matrix - The transpose of a matrix is found by interchanging its rows into columns or columns into rows
or
In linear algebra, the transpose of a matrix is an operator which flips a matrix over its diagonal; that is, it switches the row and column indices of the matrix A by producing another matrix, often denoted by A^T

identity matrix - In linear algebra, an identity matrix is a matrix of order nxn such that each main diagonal element is equal to 1, and the remaining elements of the matrix are equal to 0.

scalar product of vectors - The scalar product of two vectors is the sum of the product of the corresponding components of the vectors.

orthogonal and orthonormal vectors - A set of vectors S is orthonormal if every vector in S has magnitude 1 and the set of vectors are mutually orthogonal.

orthogonal projection - he vector (e.g a time-series) that is closest in the space C(A), where distance is measured as the sum of squared errors.

least square problem - The linear least-squares problem occurs in statistical regression analysis; it has a closed-form solution