$+$ $\times$

$-$ $\div$

INVERSE TRIGONOMETRIC FUNCTION

CLAIM: $\cos\Big|_{[0,\pi]} : [0,\pi] \to [-1,1]$

CODOMAIN

RESTRICT DOMAIN

IT IS STRICTLY DECREASING AND BIJECTIVE

ITS INVERSE IS

$\arccos : [-1,1] \to [0,\pi]$ ARCO SINE FUNCTION

$\sin\Big|_{[-\frac{\pi}{2},\frac{\pi}{2}]} : [-\frac{\pi}{2},\frac{\pi}{2}] \to [-1,1]$

STRICTLY INCREASING AND BIJECTIVE

ITS INVERSE IS

$\arcsin : [-1,1] \to [-\frac{\pi}{2},\frac{\pi}{2}]$ ARCSINE FUNCTION

$\tan\Big|_{(-\frac{\pi}{2},\frac{\pi}{2})} : (-\frac{\pi}{2},\frac{\pi}{2}) \to \mathbb{R}$

STRICTLY INCREASING AND BIJECTIVE

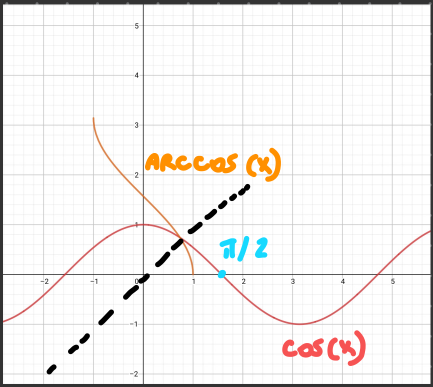
ITS INVERSE IS

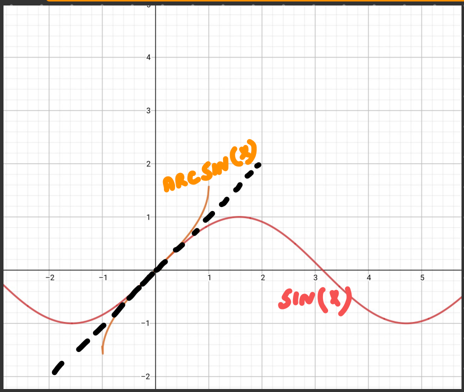
$\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ARCTANGENT FUNCTION

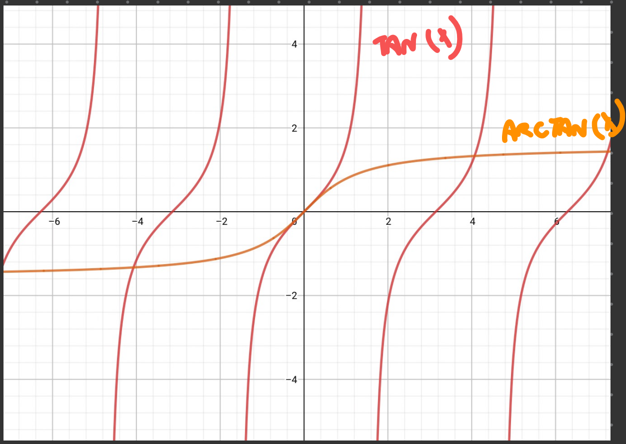
REMARK

ARCOS STRICTLY DECREASING

ARCSIN AND ARCTAN STRICTLY INCREASING AND ODD







IMPORTANT REMARK

ARCCOS, ARCSIN AND ARCTAN ARE THE INVERSE IN A LIMITED DOMAIN

IT IS LIKE

$(\sqrt{x})^2 = x \quad \forall x \geq 0$

$\sqrt{y^2} = y \quad \forall y \geq 0$

$\sqrt{(-1)^2} = 1 \neq -1$

$\cos(\arccos(x)) = x \qquad \forall x \in [-1, 1]$

$\arccos(\cos(y)) = y \qquad \forall y \in [0, \pi]$

$\arccos(\cos(2\pi)) = \arccos(1) = 0 \neq 2\pi$

$\sin(\arcsin(x)) = x \qquad \forall x \in [-1, 1]$

$\arcsin(\sin(y)) = y \qquad \forall y \in [-\frac{\pi}{2}, \pi/2]$

$\arcsin(\sin(\pi)) = \arcsin(0) = 0 \neq \pi$

$\tan(\arctan(x)) = x \qquad \forall x \in \mathbb{R}$

$\arctan(\tan(y)) = y \qquad \forall y \in (-\frac{\pi}{2}, \pi/2)$

$\arctan(\tan(\pi)) = \arctan(0) = 0 \neq \pi$

EXERCISE:

$\log(x^2 - 2) < 1$

$x^2 - 2 > 0; \qquad x < -\sqrt{2} \vee x > \sqrt{2}$

$x^2 - 2 < e$

$2 < x^2 < 2+\varepsilon$ STRICTLY INCREASING

SOLVE

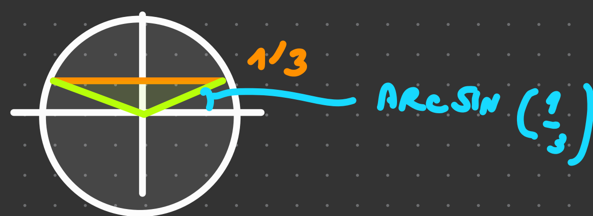
$\sin(x) = \frac{1}{2}$

$$\left\{x = \frac{\pi}{6} + 2k\pi,\ k \in \mathbb{Z}\right\} \cup \left\{x = \frac{5}{6}\pi + 2k\pi,\ k \in \mathbb{Z}\right\}$$

WHAT IS

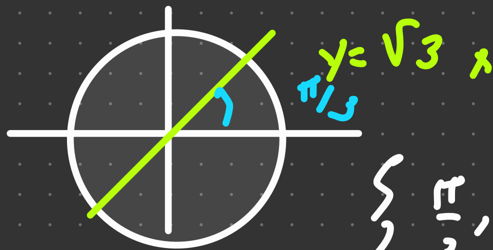
$\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$ IS THE ONLY SOLUTION IN BELONGING TO $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ THAT IS $\frac{\pi}{6}$

$\sin(x) = \frac{1}{3}$



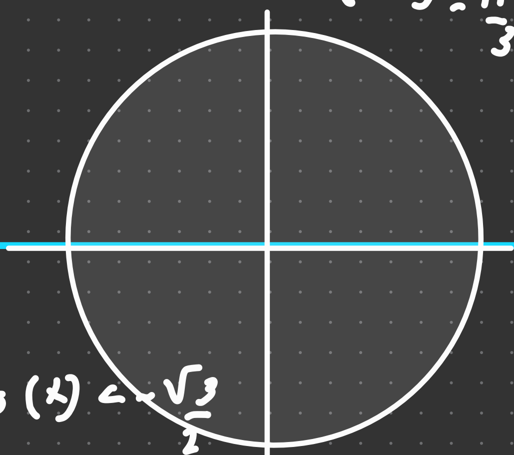
$$\left\{\arcsin\left(\frac{1}{3}\right) + 2k\pi,\ k \in \mathbb{Z}\right\} \cup \left\{\pi - \arcsin\left(\frac{1}{3}\right) + 2k\pi,\ k \in \mathbb{Z}\right\}$$

$\tan(x) = \sqrt{3}$



$\left\{ \frac{\pi}{3}, k\pi, k \in \mathbb{Z} \right\}$

$\arctan(\sqrt{3}) = \frac{\pi}{3}$ IS THE ONLY SOLUTION TO $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

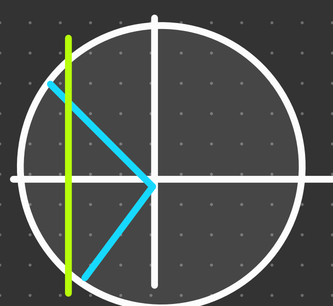


SOLVE:

$\cos(x) < -\frac{\sqrt{3}}{2}$

$\cos\left(\left[-\pi, \frac{\sqrt{3}}{2}\right]\right) = \cos^{-1}\left(\left[-1, -\frac{\sqrt{3}}{2}\right]\right)$

$\left(\frac{5}{6}\pi, \frac{7}{6}\pi\right)$



COMBINATORICS

LET $N \in \mathbb{N}$ WE DEFINE

$$N! = 1 \cdot 2 \cdot \ldots (N-1) \cdot N$$ N FACTORIAL

AND SET $0! = 1$

EXAMPLE

$1! = 1 \quad 2! = 2 \quad 3! = 6 \quad 4! = 24$

$$N! = (N-1)\, N! \quad \forall N \in \mathbb{N}$$

LET $N \in \mathbb{N}$ AND LET $A = \{a_1, \ldots, a_N\}$ BE A SET WITH N ELEMENTS (ALPHABET). LET $K \in \mathbb{N}$. AN ORDERED K-TUPLE OF ELEMENTS OF A IS A WORD OF K LETTERS

- HOW MANY ARE THE K-LETTERS WORDS WITH AN ALPHABET OF N LETTERS?

$$\underbrace{N \cdot N \ldots N} = N^K \quad \left(\text{NUMBER OF ELEMENTS OF } A^K\right)$$

K TIMES

QUESTION A WITH N ELEMENTS

B WITH M ELEMENTS

HOW MANY ELEMENTS ARE IN $A \times B = N \cdot M$

LET $N \in \mathbb{N}$ AND $1 \leq K \leq N$

HOW MANY ARE THE K LETTERS WORDS WITH DIFFERENT LETTERS?

(DISPOSITION OF K ELEMENTS FROM N GIVEN ELEMENTS)

$$N \cdot (N-1) \cdot (N-K+1) = \boxed{\frac{N!}{(N-K)!}}$$

LET $N \in \mathbb{N}$

HOW MANY ARE THE DIFFERENT WAYS IN WHICH YOU CAN ORDER THE ELEMENTS OF A?

TAKE $K=N$ IN THE PREVIOUS EXAMPLE $\boxed{N!}$

THIS IS THE NUMBER OF PERMUTATIONS OF THE

ELEMENT OF A

$\pi : \{1, \dots N\} \to \{1, \dots N\}$ BIJECTIVE

$(\omega_1, \dots, \omega_N) \mapsto (\omega_{\pi(1)}, \omega_{\pi(2)}, \dots \omega_{\pi(N)})$

$\boxed{N=3}$ $(\omega_1, \omega_2, \omega_3) \to (\omega_3, \omega_1, \omega_2)$

$\pi(1) = 3 \quad \pi(2) = 1 \quad \pi(3) = 2$

$\boxed{N=4}$ $(\omega_1, \omega_2, \omega_3, \omega_4) \mapsto (\omega_1, \omega_4, \omega_2, \omega_3)$

$\pi(1) = 1 \quad \pi(2) = 4 \quad \pi(3) = 2 \quad \pi(4) = 3$

EXERCISE: HOW MANY ARE THE ANAGRAMS OF THE WORD FILIPPI?

$$\frac{7!}{3!\,2!}$$

FOR 3 i

FOR 2 P

EXERCISE

$N \in \mathbb{N}$ HOW MANY ARE THE SUBSETS OF A WITH 1 ELEMENT? AND $(N-1)$ ELEMENTS?

$\{a_i\}_{i \in (1, \dots N)}$ IN BOTH CASES $\boxed{N}$

LET $N \in \mathbb{N}$ AND $1 \leq K \leq N$ HOW MANY ARE THE SUBSETS OF A WITH K ELEMENTS?

(COMBINATION OF K ELEMENTS FROM N GIVEN ELEMENTS)

ANY WORD OF K LETTERS WHICH ARE ALL DIFFERENT PROVIDES A SUBSET OF K ELEMENTS ALL ITS ANAGRAMS GIVE THE SAME SUBSETS

$$\boxed{\frac{N!}{(N-K)!\,K!} = \binom{N}{K}}$$

DEFINITION $\forall N \in \mathbb{N} \cup \{0\}$ $\forall K \in \mathbb{N} \cup \{0\}$ $0 \leq K \leq N$

DEFINE $\binom{N}{K} = \dfrac{N!}{(N-K)!\,K!}$ BINOMIAL COEFFICIENT

NOTE THAN $\binom{N}{0} = \frac{N!}{(N-0)!\,0!} = 1 \quad \forall N \geq 0$

$\binom{N}{N} = \frac{N!}{(N-N)!\,N!} = 1 \quad \forall N \geq 0$

$\binom{N}{1} = \frac{N!}{(N-1)!\,1!} = N \qquad \binom{N}{N-1} = \frac{N!}{(N-(N-1))!\,(N-1)!} = N$

LET A BE A SET WITH N ELEMENTS

$(N \in \mathbb{N} \cup \{0\})$

$0 \leq K \leq N$, K INTEGER

THE NUMBER OF SUBSET OF A WITH K ELEMENTS IS $\binom{N}{K}$

$N = 0$ $\quad A = \emptyset$ THE SUBSET WITH 0 ELEMENT IS $\emptyset$

$N \geq 1$ $\quad K = 0$ $\quad \emptyset$ IS THE ONLY SUBSET WITH 0 ELEM.

$N \geq 0$ $\quad K = N$ $\quad A$ IS THE ONLY SUBSET WITH N ELEM

COMPUTATION OF THE BINOMIAL COEFFICIENT

(PASCAL OR TARTAGLIA TRIANGLE)

$\forall k, n \in \mathbb{N}$ WITH $1 \le k \le n-1$, WE HAVE

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\binom{0}{0} \quad 1$$

$$\binom{1}{0} = 1 \qquad 1 = \binom{1}{1}$$

$$\binom{2}{0} = 1 \qquad \binom{2}{1} = 1+1 = 2 \qquad 1 = \binom{2}{2}$$

$$\binom{3}{0} = 1 \quad \binom{3}{1} = 3 \quad \binom{3}{2} = 3 \quad 1 = \binom{3}{3}$$

$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

$$\binom{4}{2}$$

1 — 0

1 1 — 1

1 2 1 — 2

1 3 3 1 — 3

1 4 6 4 1 — 4

1 5 10 5 1 — 5

$$\binom{N}{K} = \binom{N-1}{K-1} + \binom{N-1}{K}$$

PROVE

NEWTON'S BINOMIAL FORMULA

LET $n \in \mathbb{N}$ $\quad a, b \in \mathbb{R}$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} \, b^k$$

HERE $a^0 = 1$

$\forall_a \in \mathbb{R}$

$a^1 = 1$

$$a^n + n a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{n-2} a^2 b^{n-2} + n a b^{n-1} + b^n$$

$$\binom{n}{0} = a^{n-0} \, b^0 = 1 \, a^n \cdot 1 = a^n$$

$$\binom{n}{1} a^{n-1} b^1 = n a^{n-1} b$$

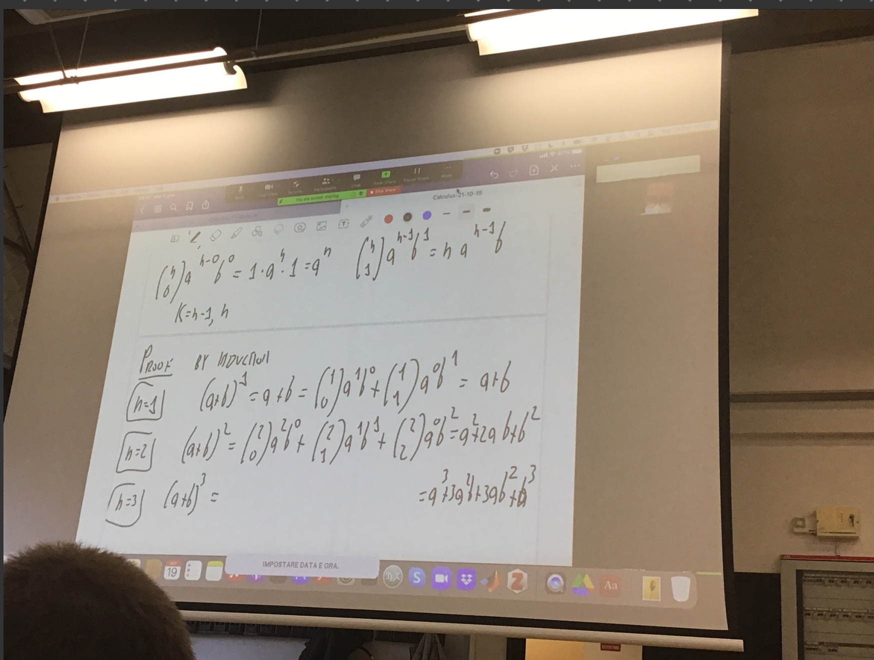
PROOF BY INDUCTION

N-1

$$(a+b)^1 = a+b = \binom{1}{0}a^1b^0 + \binom{1}{1}a^0b^1 = a+b$$

N= .

$$(a+b)^2 = \binom{2}{0}a^2b^0 + \binom{2}{1}a^1b^1 + \binom{2}{2}a^0b^2 + = a^2 + 2ab + b^2$$



BASE OF INDUCTION IS OK

INDUCTION STEP

$$(a+b)^n \stackrel{??}{=} \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$(a+b)^{n+1} = \sum_{j=0}^{n+1} \binom{n}{j} a^{(n+1)-j} b^j \ ??$$

INDUCTION HYPOTHESIS

$$(a+b)^{n+1} = (a+b)(a+b)^n = (a+b) \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} =$$

COMPLETARE