PERCHE'

$(-2)^0$ NOT DEFINE

Exercises - Calculus

Academic Year 2021-2022

Sheet 2

Note: for these exercises, it is enough to know that the function $\exp : \mathbb{R} \to (0, +\infty)$, such that $\exp(x) = e^x$ for any $x \in \mathbb{R}$, is strictly increasing and surjective and that its inverse is the function $\log : (0, +\infty) \to \mathbb{R}$. Also recall that $\exp(0) = e^0 = 1$.

1. Let $f(x) = 2x+1$. Determine the following images and preimages through f:

   $f((-2, 3]);\quad f(\{3\});\quad f((-\infty, 0]);\quad f^{-1}([3, +\infty));\quad f^{-1}(0);\quad f^{-1}((-1, 1))$

2. Let $f(x) = x^2$. Determine the following images and preimages through f:

   $f((-2, 1]);\quad f([-2, -1));\quad f((1, 2]);\quad f^{-1}([-2, -1]);\quad f^{-1}([-2, 1]);\quad f^{-1}((1, 4]);$

3. Let $f(x) = 1/x$, $x \neq 0$. Determine the following images and preimages through f:

   $f([-2, -1));\quad f((1, 3]);\quad f^{-1}[-2, -1]);\quad f^{-1}([1, 5));\quad f^{-1}(\{0\});\quad f^{-1}((-3, 15)).$

   Moreover, find the preimage through f of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$.

4. Lef $f : A \to B$ be a function and let X_1 and X_2 be two subsets of A. Prove that

   $$f(X_1 \cup X_2) = f(X_1) \cup f(X_2) \quad \text{and} \quad f(X_1 \cap X_2) \subseteq f(X_1) \cap f(X_2)$$

   Show, using one of the previous exercises, that this last inclusion can be strict.

5. Lef $f : A \to B$ be a function and let Y_1 and Y_2 be two subsets of B. Prove that

   $$f^{-1}(Y_1 \cup Y_2) = f^{-1}(Y_1) \cup f^{-1}(Y_2) \quad \text{and} \quad f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2)$$

6. Let A be the preimage of $\{1\}$ through the function $f(x) = e^x$, that is,

   $$A := \{x_0 \in \mathbb{R} \text{ such that } f(x_0) = 1\}.$$

   Let $B := [0, \pi)$. Determine the sets $A \cap B$ and $B \backslash A$.

7. What is the preimage of $\{1\}$ through the function $f(x) = e^{x^2-1}$ (that is, what is the set of $x_0 \in \mathbb{R}$ such that $f(x_0) = 1$)?

8. Determine the domain of definition of the function $f(x) = \sqrt{x^4}$.

9. Determine the domain of definition of the function $f(x) = \sqrt{x^4 - 1}$.

10. Determine the domain of definition of the function $f(x) = \log(x^4)$.

11. Determine the domain of definition of the function $f(x) = \log(x^4 - 1)$.

12. Determine the domain of definition of

$$f(x) = \sqrt{\frac{x-1}{x^2-3x+1}}$$

13. Determine the domain of definition of

$$f(x) = \log\left(\frac{x-1}{x^2-3x+1}\right)$$

14. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = -3x + 2$ is bijective and compute its inverse.

15. Let $f(x) = \log(-2x + 1)$. What is the domain of definition of f? Is f injective on its domain of definition? If the answer is yes, compute its inverse function along with its domain of definition.

16. Is the function $f(x) = e^{3(x+\pi)} - 1$ injective? If the answer is yes, compute its inverse function along with its domain of definition.

17. Is the function $f(x) = \exp(3x^3)$ injective? Go to *www.wolframalpha.com* and type the following question: `Is exp(3 x^3) injective?` Compare the solution given by wolframalpha with yours.

18. Is the function $f(x) = \log(x^2 + 1)$ injective?

19. Let $f : (0, +\infty) \to \mathbb{R}$ be the function such that $f(x) = \log(x^3)$ for any $x > 0$. Show that f is bijective and compute the inverse function of f.

20. Let $f(x) = \log(x)$ and $g(x) = 3x^2 + 2$. Compute $g \circ f$ and determine its domain of definition. Compute $f \circ g$ and determine its domain of definition.

21. Let $f(x) = \sqrt{x}$ and $g(x) = 3x^2 - 2$. Compute $g \circ f$ and determine its domain of definition. Compute $f \circ g$ and determine its domain of definition.

22. Let $f : A \to B$, $g : B \to C$ and $h : C \to D$. Show that the composition of functions is associative, that is,

$$h \circ (g \circ f) = (h \circ g) \circ f$$

which can therefore be simply denoted by $h \circ g \circ f$.

23. Let $f(x) = 1/x$, $x \neq 0$, $g = x^2 + 1$ and $h = x^2 - 1$. Find the domain of definition and compute the following composition of functions

$$g \circ f, \;\; f \circ g, \;\; h \circ f, \;\; f \circ h, \;\; g \circ f \circ h$$

24. For m, $q \in \mathbb{R}$, let $f(x) = mx + q$. Find all pairs (m, q) such that f is even. Find all pairs (m, q) such that f is odd.

25. Let $A \subseteq \mathbb{R}$ be symmetric with respect to 0. Let $f : A \to \mathbb{R}$ be odd. If $0 \in A$, what is $f(0)$?

26. Let $A \subseteq \mathbb{R}$ be symmetric with respect to 0. Let $f : A \to \mathbb{R}$ be odd. Show that if f is injective, then $B = f(A)$ is symmetric with respect to) and the inverse of f, $f^{-1} : B \to A$, is also odd.

27. Let $A \subseteq \mathbb{R}$ be symmetric with respect to 0. If $A \neq \{0\}$ and $f : A \to \mathbb{R}$ is even, can f be injective?

28. Let $A \subseteq \mathbb{R}$ be symmetric with respect to 0. Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be two functions. Let us consider the sum of the two functions, $f + g$, and the product of the two functions, $f \cdot g$. Prove that if f and g are even, then $f + g$ and $f \cdot g$ are also even. What happens if f and g are both odd? What can be said of $f + g$ and $f \cdot g$ if f is even and g is odd?

29. Let $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ be symmetric with respect to 0. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be two functions, with $f(A) \subseteq B$. Let us consider $g \circ f : A \to \mathbb{R}$. Show that if f and g are even, then $g \circ f$ is even and that if f and g are odd, then $g \circ f$ is odd. What happens if f is odd and g is even? Prove that if f is even, then $g \circ f$ is even, independently on the properties of g!

30. The function *integer part of* $x \in \mathbb{R}$ is defined as the greatest integer which is less than or equal to x. This is usually denoted $[x]$. In other words, $[x]$ is the number in $\mathbb{Z}$ is such that $[x] \leq x < [x] + 1$. Compute $[2.5]$, $[3]$, $[\pi]$, $[-0.7]$ and draw the graph of the function $\mathbb{R} \ni x \mapsto [x]$.

31. Show that the function $\mathbb{R} \ni x \mapsto x - [x]$ is periodic of period 1 and draw its graph.

Note: for these exercises, it is enough to know that the function $\exp : \mathbb{R} \to (0, +\infty)$, such that $\exp(x) = e^x$ for any $x \in \mathbb{R}$, is strictly increasing and surjective and that its inverse is the function $\log : (0, +\infty) \to \mathbb{R}$. Also recall that $\exp(0) = e^0 = 1$.

1. Let $f(x) = 2x+1$. Determine the following images and preimages through f:

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4. Lef $f : A \to B$ be a function and let X_1 and X_2 be two subsets of A. Prove that

$$f(X_1 \cup X_2) = f(X_1) \cup f(X_2) \quad \text{and} \quad f(X_1 \cap X_2) \subseteq f(X_1) \cap f(X_2)$$

Show, using one of the previous exercises, that this last inclusion can be strict.

TAKE ANY $x \in f(X_1 \cup X_2)$ THUS MEANS

$x \in X_1$ OR $x \in X_2$ SO

6. Let A be the preimage of $\{1\}$ through the function $f(x) = e^x$, that is,

$$A := \{x_0 \in \mathbb{R} \text{ such that } f(x_0) = 1\}.$$

Let $B := [0, \pi)$. Determine the sets $A \cap B$ and $B \backslash A$.

$1 = f(e^x)(x)$

$1 = e^x \quad x = 0$

$B := [0, \pi)$

$0 = e^x \quad \nexists x \mid e^x = 0$

$\pi = e^x \quad x = \ln \pi \simeq 1.14$

7. What is the preimage of $\{1\}$ through the function $f(x) = e^{x^2-1}$ (that is, what is the set of $x_0 \in \mathbb{R}$ such that $f(x_0) = 1$)?

8. Determine the domain of definition of the function $f(x) = \sqrt{x^4}$.

9. Determine the domain of definition of the function $f(x) = \sqrt{x^4 - 1}$.

10. Determine the domain of definition of the function $f(x) = \log(x^4)$.

$f(x) = e^{x^2-1}$

$1 = e^{x^2-1}$

$e^0 = e^{x^2-1}$

$0 = x^2-1$

$x = \pm 1$

8. Determine the domain of definition of the function $f(x) = \sqrt{x^4}$.

$\mathbb{R}$

9. Determine the domain of definition of the function $f(x) = \sqrt{x^4 - 1}$.

$x \neq \pm 1$

10. Determine the domain of definition of the function $f(x) = \log(x^4)$.

$\mathbb{R} - \{0\}$

11. Determine the domain of definition of the function $f(x) = \log(x^4 - 1)$.

$x^4 - 1 > 0$; $x^4 > 1$ $\quad x < -1 \vee x > 1$

12. Determine the domain of definition of

$$f(x) = \sqrt{\frac{x-1}{x^2-3x+1}}$$

$x - 1 \geq 0$; $x \geq 1$

$x^2 - 3x + 1 > 0$

$$\frac{3 \pm \sqrt{5}}{2}$$

$\frac{3-\sqrt{5}}{2} \quad 1 \quad \frac{3+\sqrt{5}}{2}$

$- \quad - \quad \bullet + \quad +$

$D>0 \quad + \circ - \quad - \circ +$

$- \quad \cancel{\exists} \quad + \quad \circ \quad - \quad \cancel{\exists} \quad +$

$\left(\frac{3-\sqrt{5}}{2}, 1\right] \cup \left(\frac{3+\sqrt{5}}{2}, +\infty\right)$

13. Determine the domain of definition of

$$f(x) = \log\left(\frac{x-1}{x^2-3x+1}\right)$$

NEGATIVA

$x > 1$

$$\left(\frac{3-\sqrt{5}}{2}, 1\right] \cup \left(\frac{3+\sqrt{5}}{2}, +\infty\right)$$

14. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = -3x + 2$ is bijective and compute its inverse.

PROVE IT IS WELL DEFINE

PROVE IT IS INJECTIVE

f IS INJECTIVE IF DIFFERENT ELEMENTS OF A HAVE DIFFERENT IMAGES THROUGH f, THAT IS, $\forall\, a_1, a_2 \in A$

$$a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$$

f IN SUBJECTIVE IF FOR ANY $b \in B$ THERE EXISTS AT LEAST ONE ELEMENT $a \in A$ SUCH THAT $f(a) = b$

BIJECTIVE: INJECTIVE AND SUBJECTIVE

$$f(a) = f(b)$$

$$-3A + 2 = -3B + 2$$

$$\Downarrow$$

$A = B \rightarrow f$ IS INJECTIVE

TAKE ANY $b \in \mathbb{R}$ THEN $Y = -3x + 2$

$$-3a + 2 = B$$

$$a = -\frac{B-2}{3}$$

$-3a+2$

$$f(a) \quad \cancel{-3}\left(-\frac{B-2}{\cancel{3}}\right)+2$$

$$B-2+2=b$$

SUBJECTIVE

$$\boxed{f(a)=b}$$

Since f is 1-1 AND ONTO IT IS A BIJECTION

INVERSE

$$f(x)=-3x+2$$

$$y=-3x+2$$

$$x=-3y+2$$

$$y=\frac{x-2}{-3} \leftarrow \text{INVERSE}$$

$$f^{-1}(x)=\frac{-x+2}{3}$$

15. Let $f(x) = \log(-2x+1)$. What is the domain of definition of f? Is f injective on its domain of definition? If the answer is yes, compute its inverse function along with its domain of definition.

$-2x+1>0$; $x < -\frac{1}{2}$

INJECTIVE 1 TO 1

$f(a) = f(b)$

$\log(-2a+1) = \log(-2b+1)$

$-2a+1 = -2b+1$

$a = b$

$y = \log(-2x+1)$

$x = \log(-2y+1)$

$e^x = -2y+1$

$y = \frac{e^x - 1}{-2}$

$y = \frac{1-e^x}{2}$

$f^{-1}(x) = \frac{1-e^x}{2}$

16. Is the function $f(x) = e^{3(x+\pi)} - 1$ injective? If the answer is yes, compute its inverse function along with its domain of definition.

INJECTIVE 1-1

$f(a) = f(b)$

$e^{3(a+\pi)} - 1 = e^{3(b+\pi)} - 1$

$e^{3(a+\pi)} = e^{3(b+\pi)}$

$3(a+\pi) = 3(b+\pi)$

$a+\pi = b+\pi$

$a = b$ ← IS INJECTIVE

$y = e^{3(x+\pi)} - 1$

$x = e^{3(y+\pi)} - 1$

$x+1 = e^{3(y+\pi)}$

$\ln(x+1) = 3(y+\pi)$

$\ln(x+1) = 3y + 3\pi$

$\ln(x+1) - 3\pi = 3y$

$\frac{\ln(x+1)}{3} - \pi = y$ $\quad f^{-1}(x) = \frac{\ln(x+1)}{3}$

17. Is the function $f(x) = \exp(3x^3)$ injective? Go to *www.wolframalpha.com* and type the following question: `Is exp(3 x^3) injective?` Compare the solution given by wolframalpha with yours.

IS NOT INSECTIVE

1 to 1

$f(a) = f(b)$

$e^{3a^3} = e^{3b^3}$

$3a^3 = 3b^3$

$a^3 = b^3$

$a = b$

18. Is the function $f(x) = \log(x^2 + 1)$ injective?

INJECTIVE?

$x^2+1>0 \quad \forall x \in \mathbb{R}$

$f(a) = f(b)$

$\log(a^2+1) = \log(b^2+1)$

$f(a)=b$

$\log(a^2+1)=b$

$a^2+1 = b^2+1$

$a^2 = b^2$

$a \neq b$ IS NOT INJECTIVE

19. Let $f : (0, +\infty) \to \mathbb{R}$ be the function such that $f(x) = \log(x^3)$ for any $x > 0$. Show that f is bijective and compute the inverse function of f.

$f(x) = \log(x^3)$ BIJECTIVE

INJECTIVE

SURJECTIVE $f(a)=b$

$f(a) = f(b)$

$\log(a^3)=b$

$\log(a^3) = \log(b^3)$

$a^3 = e^b \,;\; a = \sqrt[3]{e^b}$

$a=b$

$\log\left(\sqrt[3]{e^b}\right)^3 = b$

INJ AND SUB

$\Downarrow$

BIJE

$\log e^b = b$

$b=b$

INVERSE

$y = \log(x^3)$

$x = \log(y^3)$

$e^x = y^3$

$\sqrt[3]{e^x} = y$

20. Let $f(x) = \log(x)$ and $g(x) = 3x^2 + 2$. Compute $g \circ f$ and determine its domain of definition. Compute $f \circ g$ and determine its domain of definition.

a) $3(\log^2(x)) + 2$ $\quad D: x > 0$

b) $\log(3x^2 + 2)$ $\quad \forall x \in \mathbb{R}$

21. Let $f(x) = \sqrt{x}$ and $g(x) = 3x^2 - 2$. Compute $g \circ f$ and determine its domain of definition. Compute $f \circ g$ and determine its domain of definition.

$g \circ f$ $\quad 3x - 2$ $\quad x \geq 0$

$f \circ g$ $\quad \sqrt{3x^2 - 2}$

$3x^2 - 2 \geq 0$

$x^2 \geq \frac{2}{3}$ $\quad x \leq -\sqrt{\frac{2}{3}} \vee x \geq \sqrt{\frac{2}{3}}$

$(\infty, -\sqrt{\frac{2}{3}}] \cup [\sqrt{\frac{2}{3}}, +\infty)$

22. Let $f : A \to B$, $g : B \to C$ and $h : C \to D$. Show that the composition of functions is associative, that is,

$$h \circ (g \circ f) = (h \circ g) \circ f$$

which can therefore be simply denoted by $h \circ g \circ f$.

A B C D

$g \circ f = C$

$h \circ (g \circ f) = D$

$H \circ g$

23. Let $f(x) = 1/x$, $x \neq 0$, $g = x^2 + 1$ and $h = x^2 - 1$. Find the domain of definition and compute the following composition of functions

$$g \circ f,\ f \circ g,\ h \circ f,\ f \circ h,\ g \circ f \circ h$$

$g \circ f = \frac{1}{x^2} + 1$ $\quad x \neq 0$

$f \circ g = \frac{1}{x^2+1}$ $\quad \forall x \in \mathbb{R}$

$h \circ f = \frac{1}{x^2} - 1$ $\quad x \neq 0$

$f \circ h = \frac{1}{x^2-1}$ $\quad x \neq \pm 1$

$g \circ f \circ h = \frac{1}{(x^2-1)^2} + 1$ $\quad x \neq \pm 1$

24. For $m, q \in \mathbb{R}$, let $f(x) = mx + q$. Find all pairs (m, q) such that f is even. Find all pairs (m, q) such that f is odd.

f IS EVEN

$f(x) = f(-x)$

$mx + q = -mx + q$

f is even $\{\forall m : m = 0\}$ $\quad$ EVEN $(0, q)$

ODD $m \neq 0$

f is ODD $\forall m, a : \neg\{m = 0\}$

25. Let $A \subseteq \mathbb{R}$ be symmetric with respect to 0. Let $f : A \to \mathbb{R}$ be odd. If $0 \in A$, what is $f(0)$?

0

26. Let $A \subseteq \mathbb{R}$ be symmetric with respect to 0. Let $f : A \to \mathbb{R}$ be odd. Show that if f is injective, then $B = f(A)$ is symmetric with respect to) and the inverse of f, $f^{-1} : B \to A$, is also odd.

27. Let $A \subseteq \mathbb{R}$ be symmetric with respect to 0. If $A \neq \{0\}$ and $f : A \to \mathbb{R}$ is even, can f be injective?

f IS INJECTIVE $\Rightarrow B$ IS SYMMETRIC WITH RESPECT $f(0)$ AND $f^{-1} : B \to A$ IS ALSO ODD

f ODD $\Leftrightarrow f(-x) = -f(x)$

A SYMMETRIC WITH RESPECT TO $x \in A \Rightarrow -x \in A$

WE HAVE TO TO PROVE: B IS SYMMETRIC $\Leftrightarrow y \in B \Rightarrow -y \in B$ B IS $f(A)$

$y \in f(A) \Rightarrow \exists x \in A : f(x) \in B \Leftarrow y \in f(x)$

$-y = -f(x) = f(-x)$ AND $-x \in A$

28. Let $A \subseteq \mathbb{R}$ be symmetric with respect to 0. Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be two functions. Let us consider the sum of the two functions, $f + g$, and the product of the two functions, $f \cdot g$. Prove that if f and g are even, then $f + g$ and $f \cdot g$ are also even. What happens if f and g are both odd? What can be said of $f + g$ and $f \cdot g$ if f is even and g is odd?

29. Let $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ be symmetric with respect to 0. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be two functions, with $f(A) \subseteq B$. Let us consider $g \circ f : A \to \mathbb{R}$. Show that if f and g are even, then $g \circ f$ is even and that if f and g are odd, then $g \circ f$ is odd. What happens if f is odd and g is even? Prove that if f is even, then $g \circ f$ is even, independently on the properties of g!

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