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# FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS (SHORT)

- LET $f:[a,b]\to\mathbb{R}$ RIEMANN INTEGRABLE
- WE CAN DEFINE $\forall x\in[a,b]$ A NEW FUNCTION $F(x)=\int_a^x f(t)dt$ NAMED INTEGRAL FUNCTION
- IF $f$ IS CONTINOUS THEN $F$ IS A PRIMITIVE OF $f$ THAT IS $\forall x\in[a,b]$ $\exists F'(x)=f(x)$

## PROOF

LET $h\neq 0$ S.T. $x+h\in(a,b)$. CONSIDER $\dfrac{F(x+h)-F(x)}{h}=\dfrac{\int_a^{x+h}f(t)dt-\int_a^x f(t)dt}{h}$

$\boxed{\text{* CASE 1: } h>0}$

$$\int_a^{x+h}f(t)dt=\int_a^x f(t)dt+\int_x^{x+h}f(t)dt$$

MEAN THEOREM + $f$ IS CONTINOUS

$$\frac{F(x+h)-F(x)}{h}=\frac{1}{h}\int_x^{x+h}f(t)dt=\fint_x^{x+h}f(t)dt=f(x_1) \text{ FOR SOME } x_1=x_1(h) \text{ WITH } x<x_1<x+h$$

$$h\to 0^+\implies x_1(h)\to x\implies f(x_1(h))\to f(x)\Rightarrow \exists F'_+(x)=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=f(x)$$

$\boxed{\text{* CASE 2: } h<0}$

$h=-|h|$

$$\int_a^x f(t)dt=\int_a^{x-|h|}f(t)dt+\int_{x-|h|}^x f(t)dt \qquad x-|h|=x+h$$

$$\frac{F(x+h)-F(x)}{h}=\frac{\int_a^{x+h}f(t)dt-\int_a^x f(t)dt}{h}=\frac{-\int_{x-|h|}^x f(t)dt}{h}=\frac{1}{|h|}\int_{x-|h|}^x f(t)dt=f(x_2) \text{ FOR SOME } x_2=x_2(h) \text{ WITH } x-|h|\leq x_2\leq x$$

$$h\to 0^-\implies x_2(h)\to x\implies f(x_2(h))\to f(x)\implies \exists F'_-(x)=\lim_{h\to 0^-}\frac{F(x+h)-F(x)}{h}=f(x)$$

SO $\exists F'(x)=f(x)$

# FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

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PROOF TEO 22/03/14 00:00:00

## THEOREM

- LET $f:[a,b]\to\mathbb{R}$ RIEMANN INTEGRABLE

WE CAN DEFINE $\forall x\in[a,b]$ WE DEFINE A NEW FUNCTION $F(x)=\int_a^x f(t)\,dt$ NAMED INTEGRAL FUNCTION

IF $f$ IS CONTINOUS THEN $F$ IS A PRIMITIVE OF $f$, THAT IS $\boxed{\forall x\in[a,b]\ \exists F'(x)=f(x)}$

### CROLLARY FOUNDAMENTAL FORMULA OF INTEGRAL CALCULUS

- LET $f:[a,b]\to\mathbb{R}$ CONTINOUS
- LET $G$ BE ANY PRIMITIVE OF $f$, THEN $\int_a^b f(t)\,dt = G(b)-G(a)$

## PROOF OF THE FUNDAMENTAL THEOREM

WE NEED TO PROVE THAT $\forall x\in[a,b]\ \exists F'(x)=f(x)$

FOR SIMPLICITY, $x\in(a,b)$

LET $h\neq 0$ SUCH THAT $x+h\in(a,b)$. WE CONSIDER $\dfrac{F(x+h)-F(x)}{h}=\dfrac{\int_a^{x+h} f(t)\,dt-\int_a^x f(t)\,dt}{h}$

SI SEMPLIFICA

- CASE 1: $h>0$

$$\int_a^{x+h} f(t)\,dt=\int_a^x f(t)\,dt+\int_x^{x+h} f(t)\,dt$$

$$\frac{F(x+h)-F(x)}{h}=\frac{1}{h}\int_x^{x+h} f(t)\,dt = f \int_x^{x+h} f(t)\,dt = f(x_1) \text{ FOR SOME } x_1=x_1(h) \text{ WITH } x<x_1<x+h$$

MEAN THEOREM + $f$ IS CONTINOUS

$$h\to 0^+ \Rightarrow x_1(h)\to x \underset{\substack{f \text{ CONTINOUS}\\ \text{AT } x}}{\Longrightarrow} f(x_1(h))\to f(x) \Rightarrow \exists F'_+(x)=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=f(x)$$

- CASE 2: $h<0$

$h=-|h|$

$$\int_a^x f(t)\,dt=\int_a^{x-|h|} f(t)\,dt+\int_{x-|h|}^x f(t)\,dt \qquad x-|h|=x+h$$

$$\frac{F(x+h)-F(x)}{h}=\frac{\int_a^{x+h} f(t)\,dt-\int_a^x f(t)\,dt}{h}=-\frac{\int_{x-|h|}^x f(t)\,dt}{h}=\frac{1}{|h|}\int_{x-|h|}^x f(t)\,dt = f(x_2) \text{ FOR SOME } x_2=x_2(h)$$

WITH $x-|h|\le x_2\le x$

MEAN THEOREM + $f$ IS CONTINOUS

$$h\to 0^- \Rightarrow x_2(h)\to x \Longrightarrow f(x_2(h))\to f(x) \Rightarrow \exists F'_-(x)=\lim_{h\to 0^-}\frac{F(x+h)-F(x)}{h}=f(x)$$

$f$ CONTINOUS AT $x$

SO $\exists F'(x)=f(x)$