Exercises - Calculus

Academic Year 2021-2022

Sheet 15

1. Compute, if it exists, the directional derivative $\dfrac{\partial f}{\partial v}(0)$ for the following functions f and directions v

   (a) $f(x,y) = x^2 \sin(x-y)$ and $v = (\sqrt{2}/2, -\sqrt{2}/2)$ and $v = (-1/2, \sqrt{3}/2)$

   (b) $f(x,y) = 2y + \mathrm{e}^{x^2 y}$ and $v = (\sqrt{2}/2, \sqrt{2}/2)$ and $v = (-\sqrt{2}/2, -\sqrt{2}/2)$

   (c) $f(x,y,z) = z^2 y - \cos(z+x)$ and $v = (1/2, -\sqrt{3}/4, 3/4)$

   (d) $f(x,y) = \begin{cases} \dfrac{xy\sin(x-y)}{x^2+y^2} & \text{if } (x,y) \neq 0 \\ 0 & \text{if } (x,y) = 0 \end{cases}$
   and $v = (\sqrt{2}/2, \sqrt{2}/2)$, $v = (\sqrt{2}/2, -\sqrt{2}/2)$ and $v = (1/2, \sqrt{3}/2)$

2. Let $f, g : A \subset \mathbb{R}^N \to \mathbb{R}$, with A open, two real valued functions. Suppose that f and g are differentiable in $x^0 \in A$.

   (a) Let $\lambda,\ \mu \in \mathbb{R}$. Prove that $\lambda f + \mu g$ is differentiable in x^0 and compute its gradient.

   (b) Prove that fg is differentiable in x^0 and compute its gradient.
   Note: proving that fg is differentiable is not easy. Write $f(x) = f(x^0) + \langle \nabla f(x^0), x - x^0 \rangle + R_1(x)$ and $g(x) = g(x^0) + \langle \nabla g(x^0), x - x^0 \rangle + R_2(x)$. Multiply these two and check if you can write the product as an affine function plus a suitable remainder.

3. Let $f, g : A \subset \mathbb{R}^N \to \mathbb{R}^M$, A open, two vector valued functions. Suppose that f and g are differentiable in $x^0 \in A$. Let $\langle \cdot, \cdot \rangle$ be the usual scalar product on $\mathbb{R}^M$. By using the previous exercise, show that $h : A \subset \mathbb{R}^N \to \mathbb{R}$ defined by

$$h(x) = \langle f(x), g(x) \rangle = \sum_{i=1}^{M} f_i(x) g_i(x) \quad \text{for any } x \in A$$

   is differentiable in x^0. Then show that

$$\nabla h(x^0) = \sum_{i=1}^{M} \big(f_i(x^0) \nabla g_i(x^0) + g_i(x^0) \nabla f_i(x^0) \big) .$$

4. Study the continuity and the differentiablility of the following functions.

   (a) $f(x,y) = \begin{cases} \dfrac{xy^2}{\sqrt[4]{x^2+y^2}} & \text{IF } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

   (b) $f(x,y) = \begin{cases} \dfrac{(x-y)^4}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

(c) $f(x,y) = \begin{cases} \dfrac{\arctan^2(x-y)}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

(d) $f(x,y) = \begin{cases} \dfrac{e^{x^2 y} - 1}{(x^2+y^2)^{1/4}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

(e) $f(x,y) = \begin{cases} \dfrac{\sin(x^3-y^3)}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

(f) $f(x,y) = \begin{cases} \dfrac{\log(1+xy)}{|x|+|y|} & \text{if } (x,y) \in B_1((0,0)),\ (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

5. Determine the domain of existence A of the following functions and determine in which points of A they are differentiable. In those points compute the gradient and the Taylor polynomial of order 1.

   (a) $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ where $f(x,y) = 2x^2 \log(xy) + 3$

   (b) $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ where $f(x,y) = \cos(\arctan(x-y))$

   (c) $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ where $f(x,y) = x^4 + 3y\log(1+x)$

   (d) $f : A \subset \mathbb{R}^3 \to \mathbb{R}$ where $f(x,y,z) = \dfrac{z(e^{x+y})}{x+y^2}$

6. Determine the domain of existence A of the following functions and determine in which points of A they are differentiable. In those points compute the Jacobian matrix and the Taylor polynomial of order 1.

   (a) $f : A \subset \mathbb{R}^3 \to \mathbb{R}^2$ where $f(x,y,z) = \big(x + y^3, \sin(x+z)\big)$

   (b) $f : A \subset \mathbb{R}^2 \to \mathbb{R}^3$ where $f(x,y) = \left(x^2/y, 2\cos(x+y), \dfrac{\arctan(x^3)}{x-y}\right)$

   (c) $f : A \subset \mathbb{R} \to \mathbb{R}^5$ where $f(t) = (t+1, 3\sin(t), e^{2t}, 1-t, \log(t+1))$

   (d) $f : A \subset \mathbb{R}^3 \to \mathbb{R}^2$ where $f(x,y,z) = (z^2xy - z/y, \log(1+x+y^2))$

   (e) $f : A \subset \mathbb{R}^4 \to \mathbb{R}$ where $f(x_1,x_2,x_3,x_4) = x_1x_2 - \sin(x_3^2x_4)$

   (f) $f : A \subset \mathbb{R}^2 \to \mathbb{R}^2$ where $f(x,y) = \left(x^3y, \dfrac{x+y}{x+4y}\right)$

7. Write the equation of the tangent plane to the graph of f in the point $P_0 = (x_0, y_0, z_0)$, passing through the point P_0, where f and P_0 are given by

   (a) $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ where $f(x,y) = x^4 + 3y\log(1+x)$ and $P_0 = (0,1,0)$

   (b) $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ where $f(x,y) = xy - \dfrac{x+y}{x+4y}$ and $P_0 = (1,2,5/3)$

8. Write the equation of the tangent space to the graph of f in the point $P_0 = (x^0, f(x^0))$, passing through the point P_0, where f and P_0 are given by

(a) $f : A \subset \mathbb{R}^3 \to \mathbb{R}$ where $f(x_1, x_2, x_3) = \sin(x_2 - x_3) + x_1 \cos(x_2)$ and $P_0 = (1, \pi, \pi, -1)$

(b) $f : A \subset \mathbb{R}^4 \to \mathbb{R}^2$ where

$$f(x_1, x_2, x_3, x_4) = \left(\frac{x_1^2 + x_3}{x_4^3}, x_2 - \frac{x_1}{x_3} \right)$$

and $P_0 = (4, 3, 2, 1, 18, 1)$

(c) $f : A \subset \mathbb{R}^2 \to \mathbb{R}^3$ where

$$f(x_1, x_2) = (\sin(x_1 x_2), x_2 - x_1 x_2^3, x_2 / x_1^2)$$

and $P_0 = (1, \pi, 0, \pi - \pi^3, \pi)$

9. Compute the Jacobian matrix of the composed function $g \circ f$ where the functions g and f are given by

(a) $f : \mathbb{R}^2 \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}^2$ where

$$f(x, y) = (2xy, x^2 + y, \sin(y)) \quad \text{and} \quad g(x, y, z) = (e^{x+y}, z^2 x)$$

Hint: consider

$$f(x, y) = (u(x, y), v(x, y), w(x, y)) = (2xy, x^2 + y, \sin(y))$$

and g given by $g(u, v, w) = (e^{u+v}, w^2 u)$

(b) $f : \mathbb{R}^2 \to \mathbb{R}^4$ and $g : \mathbb{R}^4 \to \mathbb{R}^3$ where

$$f(x, y) = (x + y, x - y, xy, 2) \quad \text{and}$$
$$g(x_1, x_2, x_3, x_4) = (\cos(x_1^2 x_3), x_4^5 x_2, \sin(x_2 x_3))$$

(c) $f : \mathbb{R} \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}$ where

$$f(t) = (t^3, t^2, t) \quad \text{and} \quad g(x, y, z) = x^2 + y^4 + \cos(xyz)$$

(d) $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}^3$ where

$$f(x, y, z) = x^2 + y^4 + \cos(xyz) \quad \text{and} \quad g(t) = (t^3, t^2, t)$$

10. Let $f : \mathbb{R} \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}$ where

$$f(t) = (\cos(t), \sin(t), \cos^2(t)) \quad \text{and} \quad g(x, y, z) = xy + z^2.$$

Compute

$$\frac{d}{dt}(g \circ f)(t)$$

11. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}^4$ where $f(x, y) = (x^2 y, xy, 2y^2)$ and

$$g(x, y, z) = (g_1(x, y, z), g_2(x, y, z), g_3(x, y, z), g_4(x, y, z)) =$$
$$(3x^2 y, z^3 + \sin(xy), z^4, x^8 + y^9).$$

Compute

$$\frac{\partial}{\partial y}(g_2 \circ f)(x, y)$$

12. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}^2$ where $f(x, y, z) = (x^2, z^2 y, y^2 x)$ and

$$g(x, y, z) = (g_1(x, y, z), g_2(x, y, z)) = (e^{xz} y, y^2).$$

Compute

$$\frac{\partial}{\partial z}(g_1 \circ f)(x, y, z)$$

13. Let $f : A \subset \mathbb{R}^N \to \mathbb{R}$ where A is open and connected. Suppose that f is differentiable in any point of A and that, for a constant $a = (a_1, \dots, a_N) \in \mathbb{R}^N$, we have $\nabla f(x) = a$ for any $x \in A$. Find all possible functions with this property.

(a) $f(x,y) = \begin{cases} \dfrac{xy^2}{\sqrt[4]{x^2+y^2}} & \text{IF } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Q1) CHECK CONTINUITY

SWITCH TO POLAR COORDINATES

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{\sqrt[4]{x^2+y^2}} = \lim_{\rho\to 0} \frac{\rho^3 \cos\theta \sin^2\theta}{\rho^{1/2}} = \lim_{\rho\to 0} \rho^{5/2} \cos\theta \sin^2\theta = 0 \text{ AS } \rho \to 0$$

SINCE $\lim_{(x,y)\to(0,0)} f(x,y)$ DOES NOT DEPENDS ON θ IT IS CONTINOUS

Q2) CHECK DIFFERENTIABILITY

$$f(x,y) = \frac{xy^2}{\sqrt[4]{x^2+y^2}}$$

$$\frac{df}{dx} = \frac{y^2\sqrt[4]{x^2+y^2} - xy^2\left(2x\left(\frac{1}{4}(x^2+y^2)^{-3/4}\right)\right)}{\sqrt[2]{x^2+y^2}} = \frac{y^2}{\sqrt[4]{x^2+y^2}} - \frac{1}{2}x^2y^2\frac{1}{(x^2+y^2)^{5/4}}$$

$$\frac{df}{dy} = \frac{2yx\sqrt[4]{x^2+y^2} - xy^2\cdot\left(2y\left(\frac{1}{4}(x^2+y^2)^{-3/4}\right)\right)}{\sqrt{x^2+y^2}} = \frac{2yx}{\sqrt[4]{x^2+y^2}} - \frac{1}{2}xy^3\frac{1}{(x^2+y^2)^{5/4}}$$

$$\frac{df}{dx}(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{df}{dy}(0,0) = \lim_{x\to 0} \frac{f(0,y)-f(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

3) CHECK THAT THE PARTIAL DERIVATES ARE CONTINUOS

- $$\lim_{\rho\to 0} \frac{df}{dx} = \lim_{\rho\to 0} \frac{\rho^2\sin^2\theta}{\rho^{1/4}} - \frac{1}{2}\rho^4\cos^2\theta\sin^2\theta\cdot\frac{1}{\rho^{5/2}} = \lim_{\rho\to 0} \rho^{7/4}\sin^2\theta - \frac{1}{2}\rho^{3/2}\cos^2\theta\sin^2\theta \longrightarrow 0 \text{ AS } \rho\to 0$$

- $$\lim_{\rho\to 0} \frac{df}{dy} \lim_{\rho\to 0} \frac{2\rho^2\sin\theta\cos\theta}{\rho^{1/2}} - \frac{1}{2}\frac{\rho^4\cos\theta\sin\theta}{\rho^{5/2}} = \lim_{\rho\to 0} 2\rho^{3/2}\sin\theta\cos\theta - \frac{1}{2}\rho^{3/2}\cos\theta\sin\theta \longrightarrow 0 \text{ AS } \rho\to 0$$

f IS DIFFERENTIABLE IN $\mathbb{R}^2$

(b) $f(x,y) = \begin{cases} \dfrac{(x-y)^4}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

b1) CHECK CONTINUITY OF $f(x,y)$

$$\lim_{(x,y)\to(0,0)} \frac{(x-y)^4}{x^2+y^2}$$

SINCE DEG(N) > DEG(D) THEN THE FUNCTION IS PROBABLY CONTINOUS

b2) CHECK DIFFERENTIABILITY

$$\frac{df}{dx} = \frac{4(x-y)^3(x^2+y^2) - 2x(x-y)^4}{(x^2+y^2)^2} = \frac{4(x-y)^3}{x^2+y^2} - \frac{2x(x-y)^4}{(x^2+y^2)^2}$$

$$\frac{df}{dy} = \frac{-4(x-y)^3(x^2+y^2) - 2y(x-y)^4}{(x^2+y^2)^2} = \frac{-4(x-y)^3}{(x^2+y^2)} - \frac{2y(x-y)^4}{(x^2+y^2)^2}$$

$$\frac{df}{dx}(0,0) = \lim_{x\to 0} \frac{f(x,0)-(0,0)}{x} = \lim_{x\to 0} \frac{\frac{x^4}{x^2}}{x} = 0$$

$$\boxed{f(x,y) = \frac{(x-y)^4}{x^2+y^2}}$$

$$\frac{df}{dy}(0,0) = \lim_{y\to 0} \frac{f(0,y)-(0,0)}{y} = \lim_{y\to 0} \frac{\frac{y^4}{y^2}}{y} = 0$$

b3) CHECK THE PARTIAL DERIVATES

METHOD (a) CHECK ALSO METHOD b)

$$\frac{(x-y)^4}{x^2+y^2}$$

1) b) $f(x,y) = 2y + e^{x^2 y}$

$\underline{v_1} = (\sqrt{2}/2, \sqrt{2}/2)$ $\quad$ $\underline{v_3} = (-\sqrt{2}/2, -\sqrt{2}/2)$

$\underline{v_2} = -\underline{v_1}$

$$\frac{df}{dv_1}(\underline{0}) = \lim_{t \to 0} \frac{f(0 + t\underline{v_1}) - f(0)}{t}$$

$(0,0)$

$$\lim_{t \to 0} \frac{f(x = \sqrt{2}/2,\ y = t\sqrt{2}/2) - e^0}{} = \lim_{t \to 0} \frac{2t\sqrt{2}/\cancel{2} + e^{\overbrace{2/4\, t^2}^{x^2} \cdot \overbrace{\frac{\sqrt{2}}{2} t}^{y}} - 1}{t}$$

$$= \lim_{t \to 0} \frac{\sqrt{2}\cancel{t}}{\cancel{t}} + \frac{e^{t^3 \frac{\sqrt{2}}{4}} - 1}{t} = \sqrt{2} + \lim_{t \to 0} \frac{(\cancel{1} + t^3 \frac{\sqrt{2}}{4} + \dots) - \cancel{1}}{t} = \boxed{\sqrt{2}} + \lim_{t \to 0} \frac{\sqrt{2}}{4} t^2 + \dots$$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

POWER of t

$\to t \to 0$

$$\frac{df}{dv_2}(\underline{0}) = -\frac{df}{dv_1}(\underline{0}) = -\sqrt{2}$$

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(2) $f, g \quad A \subset \mathbb{R}^N \to \mathbb{R}$

f, g DIFF IN $\underline{x}^0 \in A$

a) $\lambda f + wg$ IS DIFF $\nabla(\lambda f + wg)$

$$\lim_{\underline{h} \to \underline{0}} \frac{f(\underline{x}^0 + \underline{h}) - f(\underline{x}^0) - \underline{L}\,\underline{x}}{\|\underline{h}\|} = \underline{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$L = Jf = \nabla f$

"x_1" $\xi = \lambda f + wg$ $\qquad$ $\nabla \xi = \nabla(\lambda f + wg) = \lambda \nabla f + w \nabla g$

$$\lim_{\underline{h} \to \underline{0}} \frac{\xi(\underline{x}^0 + \underline{h}) - \xi(\underline{x}^0) - \nabla \xi(\underline{x}^0) \cdot \underline{x}}{\|\underline{h}\|}$$

$$= \lim_{\underline{h} \to \underline{0}} \frac{\lambda f(\underline{x}^0 + \underline{h}) + w g(\underline{x}^0 + \underline{h}) - \lambda f(\underline{x}^0) - w g(\underline{x}^0) - \lambda \nabla f(\underline{x}^0)\underline{x} - w \nabla g(\underline{x}^0) \cdot \underline{x}}{\|\underline{h}\|}$$

$$= \lambda \left[\lim_{\underline{h} \to \underline{0}} \frac{f(\underline{x}^0 + \underline{h}) - f(\underline{x}_0) - \nabla f(\underline{x}^0)}{\|\underline{h}\|}\right] + w \lim_{\underline{h} \to \underline{0}} \left[\dots\right] = \underline{0} + \underline{0}$$

$\to g$ $\qquad$ BY HYPOTESIS

f is diff in $\underline{x}^0$ $\Rightarrow$ $f(\underline{x}) = f(\underline{x}^0) + \langle \nabla f(\underline{x}^0), \underline{x} - \underline{x}^0 \rangle + R_1(\underline{x})$

g is diff in $\underline{x}^0$ $\Rightarrow$ $g(\underline{x}) = g(\underline{x}^0) + \langle \nabla g(\underline{x}^0), \underline{x} - \underline{x}^0 \rangle + R_2(\underline{x})$

$\eta = f g$

$$\eta = \left(f(\underline{x}^0) + \langle \nabla f(\underline{x}^0), \underline{x} - \underline{x}^0 \rangle + R_1(\underline{x})\right)\left(g(\underline{x}^0) + \langle \nabla g(\underline{x}^0), \underline{x} - \underline{x}^0 \rangle + R_2(\underline{x})\right) =$$

$$f(\underline{x}^0) + g(\underline{x}^0) + f(\underline{x}^0)\langle \nabla g, \underline{x} - \underline{x}^0 \rangle + g(\underline{x}^0)\langle \nabla f, \underline{x} - \underline{x}^0 \rangle + \underbrace{R_1(\ldots) + R_2(\ldots)}_{R_3(\underline{x})}$$

4. Study the continuity and the differentiablility of the following functions.

$$\text{(a)}\quad f(x,y)=\begin{cases}\dfrac{xy^2}{\sqrt[4]{x^2+y^2}} & \text{IF } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

SWITCH TO POLAR COORDINATES

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{\sqrt[4]{x^2+y^2}}$$

$$\lim_{(x,y)\to(0,0)} \frac{\rho^3\cos\theta\sin^2\theta}{\sqrt{\rho}} = \lim_{(x,y)\to(0,0)} \rho^2\sqrt{\rho}\cos\theta\sin\theta \xrightarrow[\text{AS } \rho\to 0]{} 0$$

$\ell = 0$ CANDIDATE

$$|f(\rho\cos\theta,\rho\sin\theta)-0| = \left|\frac{\rho^3\cos\theta\sin^2\theta}{\sqrt{\rho}}-0\right| = \rho^2\sqrt{\rho}\,|\cos\theta\sin\theta| \leq \rho = g(\rho)$$

SINCE $g(\rho)\to 0$ AS $\rho\to 0$ THEN $\lim_{(x,y)\to(0,0)} f(x,y)=0$ f IS CONTINOUS

DIFFERENTIABILITY

$$\frac{df}{dx}\,\frac{xy^2}{(x^2+y^2)^{1/4}} = \frac{y^2(x^2+y^2)^{1/4} - 2x\frac{1}{4}(x^2+y^2)^{-3/4}\cdot xy^2}{(x^2+y^2)^{1/2}} = \frac{y^2}{(x^2+y^2)^{1/4}} - \frac{1}{2}\,\underline{\qquad}$$

$$\frac{df}{dy}\,\frac{xy^2}{(x^2+y^2)^{1/4}} = \frac{2xy(x^2+y^2)^{1/4} - xy^2\left(2y\,(x^2+y^2)^{-3/4}\right)}{(x^2+y^2)^{1/2}} = \frac{2xy(x^2+y^2)^{1/4} - 2xy^3(x^2+y^2)^{-3/4}}{(x^2+y^2)^{1/2}} =$$

$$\frac{2xy}{(x^2+y^2)^{1/4}} - \frac{2xy^3}{(x^2+y^2)^{1/4}}$$

$$\text{(b)}\quad f(x,y)=\begin{cases}\dfrac{(x-y)^4}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

$$\lim_{(x,y)\to(0,0)} \frac{\rho^{\cancel{4}\,2}(\cos\theta-\sin\theta)}{\cancel{\rho^2}} \xrightarrow[\text{AS } \rho\to 0]{} 0$$

$x^2+y^2 = \|(x,y)\|^2$

$$|f(x,y)-0| = \left|\frac{((x-y)^2)^2}{x^2+y^2}\right| \leq \left|\frac{(4\|(x,y)\|^2)^2}{x^2+y^2}\right| = \left|\frac{16\|(x,y)\|^{\cancel{4}\,2}}{\cancel{\|(x,y)\|^2}}\right| \to 0$$

AS $\|(x,y)\|^2$ f CONTINOUS IN $(0,0)$

$$(x-y)^2 = x^2+y^2-2xy \leq (|x|+|y|)^2$$

DIFFERENTIABILITY

$$f(x,y) = \frac{(x-y)^4}{x^2+y^2}$$

$$\frac{df}{dx} = \frac{4(x-y)^3\cancel{(x^2+y^2)} - (x-y)^4 2x}{(\;^2+y^2)^{\cancel{2}}} = \frac{4(x-y)^3}{(x^2+y^2)} - \frac{2x(x-y)^4}{(x^2+y^2)^2}$$

$$\frac{df}{dy} = \frac{-4(x-y)^4(x^2+y^2) - (x-y)^4 2y}{(x^2+y^2)^2} = \frac{-4(x-y)^4}{(x^2+y^2)} - \frac{2y(x-y)^4}{(x^2+y^2)^2}$$

N. 4)

$$\begin{cases} \dfrac{\arctan^2(x-y)}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$