$+$ $\times$

$-$ $\div$

GULDINO THEOREM (SHORT)

- LET A BE A PLANAR REGION S.T. $\{(x,y,z)\in\mathbb{R}^3 \quad x=0;\ y\geq 0\}$ AND AREA $A>0$
- LET V BE THE SOLID OBTAINED BY ROTATING A OF α $0<\alpha\leq 2\pi$ ON THE Z-AXIS

THEN

$$\text{VOLUME } V = \alpha \,\text{AREA} A \cdot y_b \qquad \text{WHERE } (y_b, z_b) \text{ IS THE BARYCENTRE OF } A$$

PROOF

$$V = |V| = |V \cap \Psi(\Omega)| = \iiint_V 1 \, dx\,dy\,dz = \int_{\pi/2-\alpha}^{\pi/2} d\theta \iint_A dr\,dz\, r = \alpha \iint_A r\,dr\,dz = \alpha \iint_A y\,dy\,dz = \alpha \,\text{AREA} A \frac{1}{\text{AREA} A} \iint_A y\,dy\,dz = \alpha\,\text{AREA} A\, y_b$$

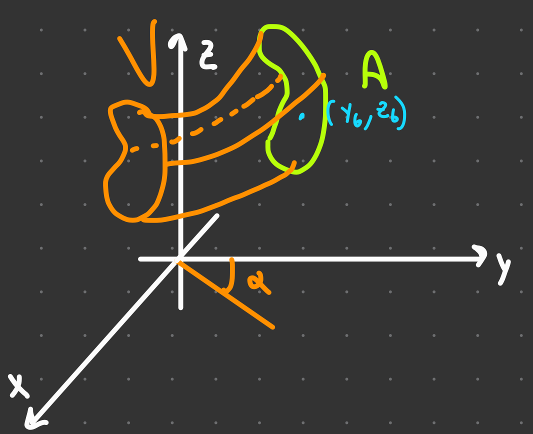
IN CHYLINDRICAL COORDINATES

RENAME r WITH y

22-05-12

GULDINO THEOREM

LET A BE A PLANAR REGION CONTAINED IN THE HALF PLANE YZ WITH $Y \geq 0$, THAT IS, IN $\{(x,y,z)\in\mathbb{R}^3 \quad x=0,\ y\geq 0\}$ LET A BE MESURABLE IN $\mathbb{R}^2$ AND AREA $A>0$. LET V BE THE SOLID OBTAINED BY ROTATING A AROUND THE Z-AXIS OF AN ANGLE α, $0<\alpha\leq 2\pi$ (2π COMPLETE ROTATION) IN THE CLOCKWISE (OR COUNTERCLOCKWISE) SENSE, THEN



$$\boxed{\text{VOLUME } V = \alpha \cdot \text{AREA } A \cdot y_b}$$

↳ WHERE (y_b, z_b) IS THE BARYCENTRE OF A

THE VOLUME IS PROPORTIONAL TO
- THE ANGLE OF ROTATION α
- THE AREA OF A
- THE Y-COORDINATE OF THE BARYCENTER → y_b IS THE MEAN DISTANCE OF THE POINTS OF A FROM THE Z-AXIS

PROOF OF THE THEOREM

USE CYLNDRICAL COORDINATES

VOLUME $$V = |V| = |V\cap\varphi(\Omega)| + \underbrace{|V-\varphi(\Omega)|}_{=0 \text{ (BOUNDED SUBSET CONTAINED IN } \mathbb{R}^3-\varphi(\Omega))} = |V\cap\varphi(\Omega)| = \iiint_V 1\,dx\,dy\,dz$$

IN CYLNDRICAL COORDINATES: V CAN BE DESCRIBED AS:

$$\left\{(r,\theta,z)\in\mathbb{R}^3 : \frac{\pi}{2}-\alpha \le\theta\le \pi/2,\ \underbrace{(r,z)\in A}\right\}$$

WHEN YOU DO THE ROTATION OF A POINT YOU DON'T CHANGE THE DISTANCE FROM THE AXIS OF ROTATION (r) AND THE HEIGTH OF THE POINT (z)

$$\iiint_V 1\,dx\,dy\,dz = \underbrace{\int_{\frac{\pi}{2}-\alpha}^{\pi/2} d\theta}\iint_A dr\,dz\, r = \alpha\iint_A r\,dr\,dz = \underbrace{\alpha \iint_A y\,dy\,dz} = \alpha \text{ AREA } A \cdot \underbrace{\frac{1}{\text{AREA } A}\iint_A y\,dy\,dz}_{=y_b \text{ (BY DEFINITION)}} = \alpha \text{ AREA } A\, y_b$$

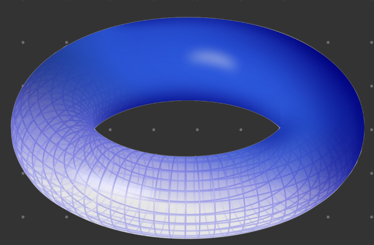
THIS FORMULA ALREADY GIVES THE VOLUME, SOMETIMES IS BETTER TO USE THIS BECAUSE YOU DON'T NEED TO CALCULATE THE AREA

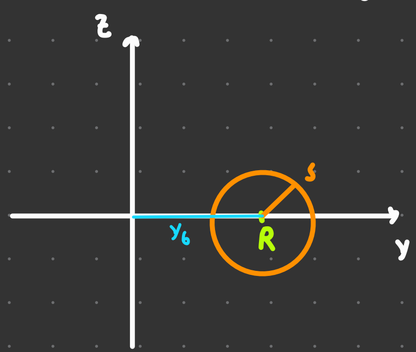
RENAME r WITH y

IF ROTATING AROUND Z USE Y, IF ROTATING AROUND Y USE Z

EXAMPLE 1: TORUS T OF RADII $0<s<R$

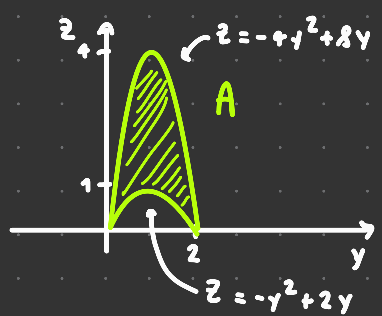
↳ TAKE A CIRCLE AND LET IT ROTATE AROUND THE Z AXIS (DONUT SHAPE)





VOLUME OF $T = \underbrace{2\pi}_{\alpha}\ \underbrace{\pi s^2}_{\text{AREA}}\ \underbrace{R}_{y_b} = 2\pi^2 s^2 R$

EXAMPLE 2: $A=\{(y,z)\in\mathbb{R}^2 \quad 0\le y\le 2 \quad -y^2+2y\le z\le -4y^2+8y\}$



COMPUTE THE VOLUME V_1, V_2 WHERE
- V_1 SOLID OBTAINED BY ROTATING A AROUND Z-AXIS OF ANGLE 2π
- V_2 SOLID OBTAINED BY ROTATING A AROUND Y-AXIS OF ANGLE π

- V_1:

$$V_1 = 2\pi \iint_A y \, dy \, dz = 2\pi \int_0^2 dy \int_{-y+2y}^{-4y^2+8y} y \, dz = 2\pi \int_0^2 y \left(-4y^2+8y - (-y^2+2y)\right) dY = 2\pi \int_0^2 -3y^3 + 6y^2 \, dy = 2\pi \left(-\frac{3}{4} y^4 + 2y^3 \Big|_{y=0}^{y=2} \right) = 2\pi(-12+16) = 8\pi$$

- V_2:

!! NOT y !!
ROTATING AROUND Y

$$V_2 = \pi \iint_A z \, dy \, dz = \pi \int_0^2 dy \int_{-y^2+2y}^{-4y^2+8y} z \, dz = \pi \int_0^2 dy \left[\frac{z^2}{2} \right]_{z=-y^2+2y}^{z=-4y^2+8y} = \frac{\pi}{2} \int_0^2 (-4y^2+8y)^2 - (-y^2+2y)^2 \, dy = \frac{\pi}{2} \int_0^2 16y^4 + 64y^2 - 64y^3 - (y^4 + 4y^2 - 4y^3) \, dy =$$

$$= \frac{\pi}{2} \int_0^2 15y^4 - 60y^3 + 60y^2 \, dy = \frac{\pi}{2} \left[3y^5 - 15y^4 + 20y^3 \right]_{y=0}^{y=2} = \frac{\pi}{2} (96 - 240 + 160) = 8\pi$$