# Knowledge Representation and Reasoning

Exercise Session 3

**Exercise 1. Subsumption** (*)

Use the **homomorphism method** to verify whether the following subsumption relations hold:

1. $A \sqcap C \sqcap \exists r.B \sqsubseteq C \sqcap \exists r.\top$
2. $\exists r.(B \sqcap \exists s.\top) \sqsubseteq \exists s.\exists s.\top$
3. $\exists r.(B \sqcap \exists s.\top) \sqsubseteq \exists r.\exists s.B$
4. $\exists r.(B \sqcap \exists s.A \sqcap \exists s.A)) \sqsubseteq \exists s.\exists s.A$
5. $B \sqcap \exists r.A \sqcap \exists r.B \sqcap \exists r.C \sqsubseteq \exists r.(A \sqcap C) \sqcap \exists r.(B \sqcap C)$

**Exercise 2. Couter-Models** (*)

For the following pairs of concepts $C$, $D$, find an interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} \not\subseteq D^{\mathcal{I}}$

1. $C = \exists r.\top, \;\; D = \exists r.A$
2. $C = A \sqcap C \sqcap \exists r.(A \sqcap B), \;\; D = \exists r.(A \sqcap C)$
3. $C = \exists r.A \sqcap \exists r.B, \;\; D = \exists r.(A \sqcap B)$
4. $C = \exists r.(A \sqcap B), \;\; D = \exists r.A \sqcap \exists s.B$
5. $C = A \sqcap C \sqcap \exists r.(A \sqcap B), \;\; D = \exists r.(A \sqcap \exists s.\bot)$

**Exercise 3. TBox Models** (*)

Construct a model of the following TBox which has **at least three elements**

$$\{\exists r.(A \sqcap B) \sqsubseteq A \sqcap \exists r.C, \quad C \sqcap \exists s.\top \sqsubseteq B \sqcap \exists s.B, \quad B \sqcap D \sqsubseteq \bot, \quad \exists s.D \sqsubseteq A \sqcap C\}$$

**Exercise 4. Disjoint Unions** (***)

Recall the notion of **disjoint union** of interpretations from the previous exercise session. Is the disjoint union of two models of an $\mathcal{EL}_\bot$ TBox $\mathcal{T}$ also a model of $\mathcal{T}$? **Justify**

### Exercise 5. Model Size (∗∗∗)

Construct an $\mathcal{EL}_\bot$ TBox that has only models with **at most** 5 elements, or argue why it cannot exist.

### Exercise 6. Normalisation (∗)

Transform the TBox from Exercise 3 to normal form.

### Exercise 7. Reasoning (∗)

Let $\mathcal{T}$ be the TBox from Exercise 3.

1. Apply the completion algorithm to check whether the following consequences hold:
   - $\exists r.\exists s.D \sqsubseteq A \sqcap \exists r.\exists s.B$
   - $D \sqcap \exists s.D \sqsubseteq B \sqcap \exists r.\top$
   - $B \sqcap \exists r.\top \sqsubseteq D \sqcap \exists s.D$
2. Construct eventual **countermodels**

### Exercise 8. Completeness (∗∗∗)

In the lecture we showed how to build an interpretation $\mathcal{I}$ from the data structure resulting from the completion algorithm, and **claimed it was a model**. As a step towards proving this claim, show that if the TBox contains the GCI $X \sqsubseteq Y$, then the interpretation $\mathcal{I}$ satisfies this GCI.

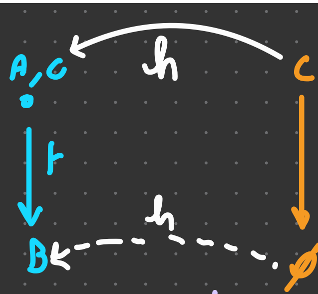
### Exercise 9. Inverse Roles (∗∗∗)

Using **inverse roles** build a TBox that expresses the knowledge that *humans can only have human children.*

Use the **homomorphism method** to verify whether the following subsumption relations hold:

1. $A \sqcap C \sqcap \exists r.B \sqsubseteq C \sqcap \exists r.\top$ yes
2. $\exists r.(B \sqcap \exists s.\top) \sqsubseteq \exists s.\exists s.\top$
3. $\exists r.(B \sqcap \exists s.\top) \sqsubseteq \exists r.\exists s.B$
4. $\exists r.(B \sqcap \exists s.A \sqcap \exists s.A)) \sqsubseteq \exists s.\exists s.A$
5. $B \sqcap \exists r.A \sqcap \exists r.B \sqcap \exists r.C \sqsubseteq \exists r.(A \sqcap C) \sqcap \exists r.(B \sqcap C)$

1)

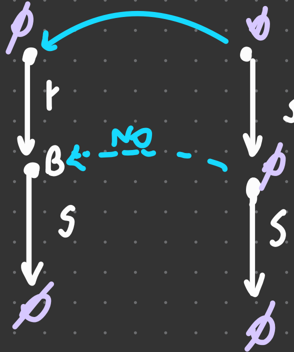



YES

LEBELO OF AN ELEMENT HAS TO BE CONTAINED IN THE IMAGE

$\xleftarrow{h}$

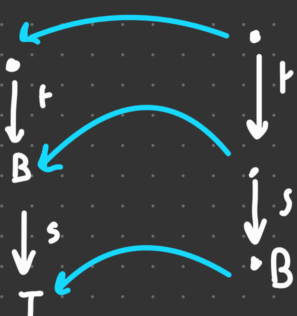
2)




IF IT WAS A r WAS A SUBSET

NO

3)

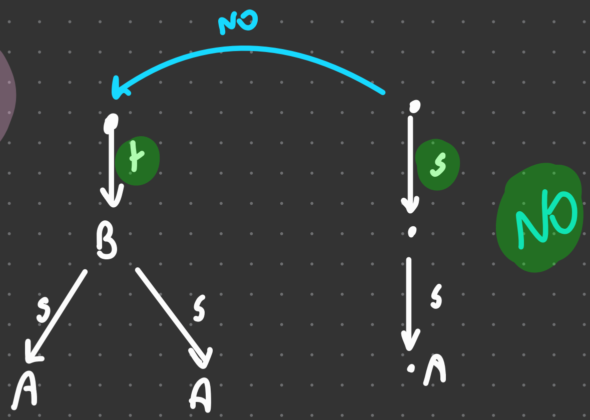



YES

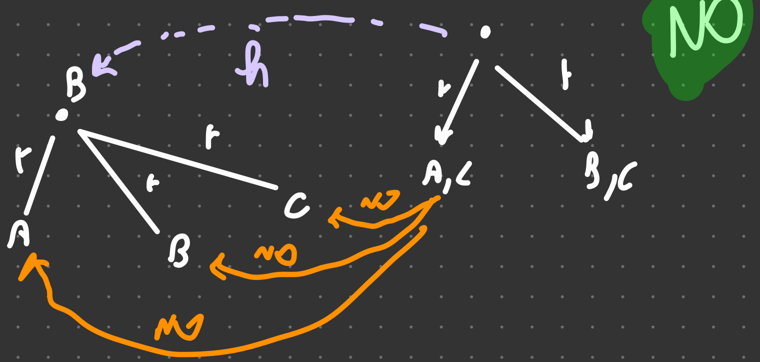
Use the **homomorphism method** to verify whether the following subsumption relations hold:

1. $A \sqcap C \sqcap \exists r.B \sqsubseteq C \sqcap \exists r.\top$
2. $\exists r.(B \sqcap \exists s.\top) \sqsubseteq \exists s.\exists s.\top$
3. $\exists r.(B \sqcap \exists s.\top) \sqsubseteq \exists r.\exists s.B$
4. $\exists r.(B \sqcap \exists s.A \sqcap \exists s.A) \sqsubseteq \exists s.\exists s.A$
5. $B \sqcap \exists r.A \sqcap \exists r.B \sqcap \exists r.C \sqsubseteq \exists r.(A \sqcap C) \sqcap \exists r.(B \sqcap C)$

4)




5)




A,C CAN'T BE MAP
ANY WHERE

**Exercise 2. Couter-Models** (*)

For the following pairs of concepts $C$, $D$, find an interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} \not\subseteq D^{\mathcal{I}}$

1. $C = \exists r.\top,\ \ D = \exists r.A$
2. $C = A \sqcap C \sqcap \exists r.(A \sqcap B),\ \ D = \exists r.(A \sqcap C)$
3. $C = \exists r.A \sqcap \exists r.B,\ \ D = \exists r.(A \sqcap B)$
4. $C = \exists r.(A \sqcap B),\ \ D = \exists r.A \sqcap \exists s.B$
5. $C = A \sqcap C \sqcap \exists r.(A \sqcap B),\ \ D = \exists r.(A \sqcap \exists s.\bot)$

FIND AN OBJECT THAT BELONGS TO C BUT NOT TO D

$\exists \delta \in C^{\mathcal{I}} \setminus D^{\mathcal{I}}$

$\delta \in C^{\mathcal{I}}$ AND $\delta \notin D^{\mathcal{I}}$

1)

$\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$

$\Delta^{\mathcal{I}} = \{\delta, y\}$

$r^{\mathcal{I}} = \{(\delta, y)\}$

$A^{\mathcal{I}} = \{\delta\}$

$B^{\mathcal{I}} = \{\delta, y\}$

2)

5) $D = \bot$

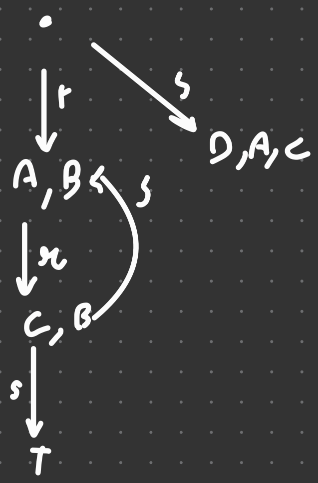
ANY CONCEPT THAT HAVE $\bot$ IS $\bot$

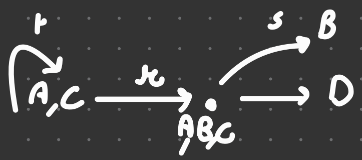
NO OBJECT CAN BELONG TO D

**Exercise 3.** **TBox Models** (*)

Construct a model of the following TBox which has **at least three elements**

$$\{\exists r.(A \sqcap B) \sqsubseteq A \sqcap \exists r.C, \quad C \sqcap \exists s.\top \sqsubseteq B \sqcap \exists s.B, \quad B \sqcap D \sqsubseteq \bot, \quad \exists s.D \sqsubseteq A \sqcap C\}$$



**Exercise 3. TBox Models** (*)

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$$\{\exists r.(A \sqcap B) \sqsubseteq A \sqcap \exists r.C, \quad C \sqcap \exists s.\top \sqsubseteq B \sqcap \exists s.B, \quad B \sqcap D \sqsubseteq \bot, \quad \exists s.D \sqsubseteq A \sqcap C\}$$

$\exists r\,(A \sqcap B) \subseteq x_0$

$x_0 \subseteq A \sqcap \exists r.C$

$\begin{cases} \exists r.x_1 \subseteq x_0 \\ A \sqcap B \subseteq x_1 \end{cases}$

$\begin{cases} x_0 \subseteq A \\ x_0 \subseteq \exists r.C \end{cases}$

$C \sqcap \exists s.\top \subseteq x_2$

$x_2 \subseteq B \sqcap \exists s.B$

$\begin{cases} C \sqcap x_3 \subseteq x_2 \\ \exists s.\top \subseteq x_3 \end{cases}$

$\begin{cases} x_2 \subseteq B \\ x_2 \subseteq \exists s.B \end{cases}$

$B \sqcap D \subseteq \bot$

$\exists s.D \subseteq A \sqcap C$

$\begin{cases} \exists s.D \subseteq A \\ \exists s.D \subseteq C \end{cases}$

**Exercise 4. Disjoint Unions** (***)

Recall the notion of **disjoint union** of interpretations from the previous exercise session. Is the disjoint union of two models of an $\mathcal{EL}_\perp$ TBox $\mathcal{T}$ also a model of $\mathcal{T}$? **Justify**

$\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ $\Delta^{\mathcal{I}} \cap \Delta^{\mathcal{J}} = \emptyset$

$\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$ $\mathcal{I} + \mathcal{J} = (\Delta^{\mathcal{I}} \cup \Delta^{\mathcal{J}}, \cdot^{\mathcal{I}\mathcal{J}})$

**Exercise 5. Model Size** (***)

Construct an $\mathcal{EL}_\perp$ TBox that has only models with **at most** 5 elements, or argue why it cannot exist.

NO IT CANNOT EXIST

CAN HAVE INFINITE ELEMENT

IF YOU ADD ANOTHER ELEMENT THAT IS OUTSIDE THE CONSTRAIN OF THE MODEL THE MODEL STILL EXISTS AND HAVE ONE ELEMENT MORE

CAN'T FORCE TWO ELEMENTS TO BE CONNECTED

**Exercise 6. Normalisation** (*)

Transform the TBox from Exercise 3 to normal form.

**Exercise 7. Reasoning** (∗)

Let $\mathcal{T}$ be the TBox from Exercise 3.

1. Apply the completion algorithm to check whether the following consequences hold:
   - $\exists r.\exists s.D \sqsubseteq A \sqcap \exists r.\exists s.B$
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   - $B \sqcap \exists r.\top \sqsubseteq D \sqcap \exists s.D$

2. Construct eventual **countermodels**

$\exists r.X_1 \subseteq X_0$

$A \sqcap B \subseteq X_1$

$X_0 \subseteq A$

$B \sqcap D \subseteq \bot$

$X_0 \subseteq \exists r.C$

$C \sqcap X_3 \subseteq X_2$

$\exists s.\top \subseteq X_3$

$\exists s.D \subseteq A$

$\exists s.D \subseteq C$

$X_2 \subseteq B$

$X_2 \subseteq \exists s.B$

| | |
|---|---|
| $A \subseteq A$ | $A \subseteq \top$ |
| $B \subseteq B$ | $B \subseteq \top$ |
| $C \subseteq C$ | $C \subseteq \top$ |
| $D \subseteq D$ | $D \subseteq \top$ |
| $X_1 \subseteq X_1$ | $X_1 \subseteq \top$ |
| $X_2 \subseteq X_2$ | $X_2 \subseteq \top$ |
| $X_3 \subseteq X_3$ | $X_3 \subseteq \top$ |

## Exercise 8. Completeness

(***)

In the lecture we showed how to build an interpretation $\mathcal{I}$ from the data structure resulting from the completion algorithm, and **claimed it was a model**. As a step towards proving this claim, show that if the TBox contains the GCI $X \sqsubseteq Y$, then the interpretation $\mathcal{I}$ satisfies this GCI.