$+$ $\times$

$-$ $\div$

DEFINITION

$S \subseteq \mathbb{R}^n$ $x \in \mathbb{R}^n$ WE SAY

- x IS ADHERENT TO S IF $\forall r > 0$ WE HAVE $B_r(x) \cap S \neq \emptyset$

  $\Leftrightarrow \exists \{x^n\}_{n \in \mathbb{N}} \subseteq S$ SUCH THAT $x^n \longrightarrow x$

- x IS AN ACCUMULATION POINT FOR S IF $\forall r > 0$ WE HAVE $\left(\left(B_r(x) \setminus \{x\} \right) \cap S \neq \emptyset \right)$

  $\Leftrightarrow \exists \{x^n\}_{n \in \mathbb{N}} \subseteq S \setminus \{x\}$ SUCH THAT $x^n \to x$

- x IS AN ISOLATED POINT OF S IF $\exists r > 0$ SUCH THAT $B_r(x) \cap S = \{x\}$

WE CALL

$\bar{S} = \{ x \in \mathbb{R}^n : x \text{ IS ADHERENT TO } S \}$ CLOSURE OF S

NOTE

- x ACCUMULATION POINT OF S

  $\Rightarrow x$ IS ADHERENT TO S

- x IS ISOLATED POINT, THEN IT IS NOT

AN ACCUMULATION POINT

- IF X IS AN ACCUMULATION POINT THAN IT IS NOT AN ISOLATED POINT

- IN $\mathbb{R}^N$, x INTERIOR, THEN X IS AN ACCUMULATION POINT

- $S \subseteq \bar{S}$ (CHOOSE $x^N = x$ $\forall N \in \mathbb{N}$)

- IF $x \in S$ IS NOT AN ACCUMULATION POINT THAN IT IS AN ISOLATED POINT

- $\bar{S} = S \cup \partial S = \mathring{S} \cup \partial S$

- $\partial S = \bar{S} \cap \overline{S^c}$

IN FACT

$x \in \partial S \Leftrightarrow \exists \{x^N\}_{N \in \mathbb{N}} \subseteq S$ SUCH THAT $x^N \to x$

AND $\exists \{y^N\}_{N \in \mathbb{N}} \subseteq S^c$ SUCH THAT $y^N \to x$

DEFINITION

- $A \subseteq \mathbb{R}^N$ IS OPEN IF $A = \mathring{A}$, THAT IS,

$\forall x \in A$ $\exists r > 0$ SUCH THAT $B_r(x) \subseteq A$

- $C \subseteq \mathbb{R}^N$ IS CLOSED IF C^c IS OPEN, THAT IS,

$C = \overline{C}$

REMARK

- A IS OPEN $\iff A \cap \partial A = \emptyset$
- C IS CLOSED $\iff \partial C \subseteq C$

THEOREM

$\forall x \in \mathbb{R}^N, \forall r > 0$ WE HAVE $B_r(x)$ IS OPEN

PROPOSITION C CLOSED $\iff$

$\forall \{x^n\}_{n \in \mathbb{N}} \subseteq C$ SUCH THAT $x^n \to x$,

WE HAVE $x \in C$

PROPERTIES OF OPEN AND CLOSE SETS

PROPOSITION (MONOTONICITY) $S, S_1 \subseteq \mathbb{R}^N$

- $S \subseteq S_1 \Rightarrow \mathring{S} \subseteq \mathring{S}_1, \overline{S} \subseteq \overline{S}_1$
- $(\mathring{S})^{\circ} = \mathring{S}$, THAT IS, $\mathring{S}$ IS OPEN
- $(\overline{S}) = \overline{S}$, THAT IS $\overline{S}$ IS CLOSED

ACTUALLY

$\mathring{S}$ IS THE LARGEST OPEN SET CONTAINED IN S

$\bar{S}$ THE SMALLEST CLOSED SET CONTAINING S

PROPOSITION

1) LET $\{A_i\}_{i\in I}$ BE AN ARBITRARY FAMILY OF OPEN SUBSET OF $\mathbb{R}^N$, THAT IS, A_i OPEN $\forall i \in I$

THEN $A = \bigcup_{i\in I} A_i$ IS OPEN (ANY UNION OF OPEN SETS IS OPEN)

PROOF $x\in A \Rightarrow \exists\, i\in I$ SUCH THAT $x \in A_i$

$\overset{A_i \text{ OPEN}}{\Rightarrow}$ $\exists\, r>0$ SUCH THAT $B_r(x) \subseteq A_i \subseteq \bigcup_{i\in I} A_i = A$

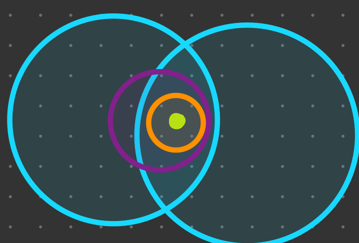
2) $A_1,\dots A_N$ A **FINITE** NUMBER OF OPEN SETS OF $\mathbb{R}^N$

$A_i = \bigcap_{i=1}^{N} A_i$ IS OPEN (THE INTERSECTION OF A FINITE NUMBER OF OPEN SETS IS OPEN)

PROOF $A_i \cap A_2$ IS OPEN?

TAKE $x \in A_1 \cap A_2$

$\exists\, r_1 > 0$ SUCH THAT



$B_{r_1}(x) \subseteq A_1$ (A_1 OPEN)

$\exists r_2 > 0$ SUCH THAT $B_{r_2}(x) \subseteq A_2$ (A_2 OPEN)

LET $r = \min(r_1, r_2)$ THEN

$B_r(x) \subseteq B_{r_1}(x) \cap B_{r_2}(x) \subseteq A_1 \cup A_2$

3) LET $\{C_i\}_{i \in I}$ BE AN ARBITRARY FAMILY OF CLOSED SETS, THAT IS, C_i CLOSED $\forall i \in I$

THEN $C = \bigcap_{i \in I} C_i$ IS CLOSED (ANY INTERSECTION OF CLOSED SETS IS CLOSED)

4) LET $C_1, .. C_N$ BE A FINITE NUMBER OF CLOSED SETS. THEN

$C_U = \bigcup_{i=1}^{N} C_i$ IS CLOSED (THE UNION OF A FINITE NUMBER OF CLOSED SETS IS CLOSED)

PROOF 3)-4)

$C = \bigcap_{i \in I} C_i = \left(\bigcup_{i \in I} C_i^c \right)^c$; $C_U = \bigcup_{i=1}^{N} C_i = \left(\bigcap_{i=1}^{N} C_i^c \right)^c$

EXAMPLE

- $\mathbb{R}^N, \emptyset$

$\mathring{\mathbb{R}}^N = \mathbb{R}^N$ SO $\mathbb{R}^N$ IS OPEN, $\overline{\mathbb{R}^N} = \mathbb{R}^N$ IS CLOSED

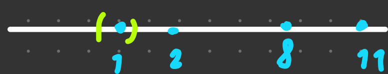
$\emptyset^{\circ} = \emptyset$ SO $\emptyset$ IS OPEN $\bar{\emptyset} = \emptyset$ SO $\emptyset$ IS CLOSED

WHAT IS $\partial \mathbb{R}^N = \partial \emptyset = \emptyset$

ANY POINT OF $\mathbb{R}^N$ IS AN ACCUMULATION POINT FOR $\mathbb{R}^N$, NO ISOLATED POINTS

- $S_1 = \{1, 2, 8, 11\} \subseteq \mathbb{R}$

$\mathring{S}_1 = \emptyset$



$\overline{S_1} = S_1 = \partial S$ CLOSED

NO ACCUMULATION POINT, ALL POINTS OF S_1 ARE ISOLATED!

ALL THESE PROPERTIES ARE TRUE FOR ANY SETS

$S_1 \neq \emptyset$

$S_1 \subseteq \mathbb{R}^N$ WITH A FINITE NUMBER OF ELEMENTS

• $S_2 = (1,3] \cup \{4\}$

$\mathring{S}_2 = (1,3)$ (S_2 NOT OPEN)

$\partial S_2 = \{1,3,4\}$ $\bar{S} = [1,3] \cup \{4\}$

ACCUMULATION POINTS OF S : $[1,3]$

{ ISOLATED POINTS OF $S_2 = \{4\}$

NOTE: $x \in \bar{S} \setminus S \Rightarrow x$ IS AN ACCUMULATION POINT

• $S_3 = [5,+\infty)$

$\mathring{S}_3 = (5,+\infty)$ $\partial S_3 = \{5\}$ $\bar{S}_3 = S_3$ CLOSED

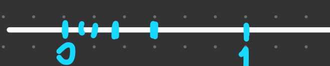
ALL POINTS OF $\bar{S}_3$ ARE ACCUMULATION POINTS NO ISOLATED POINTS

EXERCISE

$[7,9)$? $(-\infty,-1)$? $\mathbb{Z}$

$\mathring{S}_4 = \{ \frac{1}{n} : n \in \mathbb{N} \}$

$\mathring{S}_4 = \emptyset$ $\bar{S}$



$\bar{S} = \{0\} \cup S_4 \quad \partial S = \bar{S}$

0 IS THE ONLY ACCUMULATION POINT

ALL POINTS OF S_4 ARE ISOLATED

DEFINITION

$A \subseteq \mathbb{R} \quad x_0 \in \mathbb{R}$

- x_0 IS AN ACCUMULATION POINT FROM THE RIGHT / LEFT FOR A

IF $\exists \{x_n\}_{n \in \mathbb{N}} \subseteq A$ SUCH THAT $x_n \longrightarrow x$ AND $x_n > x$ / $x_n < x$

REMARK I IS A NOT DEGENERATE INTERVAL, THEN ANY $x \in \bar{I}$ IS AN ACCUMULATION POINT

(NO ISOLATED POINTS), ANY $x \in \mathring{I}$ IS AN ACCUMULATION POINT FROM RIGHT AND LEFT, IF X IS THE LEFT ENDPOINT IS AN ACC. POINT FROM THE RIGHT IF X IS THE RIGHT END POINT, THEN IS AN ACC. POINT FROM THE LEFT

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EXAMPLE $I = (1,3]$

$\bar{I} = [1,3] \quad \mathring{I} = (1,3)$

- $\mathbb{R} \setminus \{0\}$ 0 IS AN ACCUMULATION POINT FROM RIGHT AND LEFT $\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$ AND $\left\{-\frac{1}{n}\right\}_{n\in\mathbb{N}}$

PROPOSITION

$A \subseteq \mathbb{R}$

IF A IS UNBOUNDED FROM ABOVE, BELOW THEN

$\exists \{x_n\}_{n\in\mathbb{N}} \subseteq A$ SUCH THAT $x_n \to +\infty$ $-\infty$

IDEA WE KNOW $\sup A = +\infty$

LET $a_1 \in A$ BE SUCH THAT $a_1 > 1$

LET $a_2 \in A$ BE SUCH THAT $a_2 = \max\{a_1, 2\}$

LET $a_n \in A$ BE SUCH THAT $a_n > \max\{a_{n-1}, n\}$ $\forall n \geq 2$

$\{a_n\}_{n\in\mathbb{N}} \subseteq A$ AND $a_n \geq n \longrightarrow +\infty$

EXAMPLE

- $A_n = \left(-\frac{1}{n}, 1\right)$ $\forall n$ $\bigcap_{n\in\mathbb{N}} A_n = [0,1)$ NOT OPEN NOT CLOSE

- $B_n = \left(-\frac{1}{n}, 1+\frac{1}{n}\right)$ $\forall n$ $\bigcap_{n\in\mathbb{N}} B_n = [0,]$ NOT OPEN CLOSED

- $C_n = \left[\frac{1}{n}, 2\right]$ $\forall n$ $\bigcup_{n \in \mathbb{N}} C_n = (0, 2]$ NOT CLOSED
  NOT CLOSED

- $E_n = \left[\frac{1}{2}, 2 - \frac{1}{n}\right]$ $\forall n$ $\bigcup_{n \in \mathbb{N}} E_n = (0, 2)$ NOT CLOSED
  OPEN

WHAT IS $\overline{B_r(x^0)}$? $x^0 \in \mathbb{R}^n$, $r > 0$

$$B_r(x^0) = \{ y \in \mathbb{R}^n : \|y - x^0\| \le r \} =$$

$$= \{ \text{ACC. POINT OF } B_r(x^0) \}$$

$$\partial B_r(x^0) = \{ y \in \mathbb{R}^n : \|y - x^0\| = r \}$$

$B_r(x^0)$ HAS NO ISOLATED POINT

(RATIONAL NUMBERS)

$\mathring{Q} = ?$ $\quad \mathring{Q} = \emptyset$

IF $x \in \mathring{Q}$, THEN $\exists t > 0$ SUCH THAT $(x-t, x+t) \subseteq Q$

IMPOSSIBLE BY DENSITY OF IRRATIONAL POINTS IN $\mathbb{R}$

$\overset{\circ}{\widehat{\mathbb{R} \setminus Q}} = \emptyset$

$\overline{Q} = \overline{\mathbb{R} - Q} = \mathbb{R}$ $\qquad \left(x, x + \frac{1}{n}\right)$

$\{$ACCUMULATION POINT $Q\} = \{$ACC. POINT OF $\mathbb{R} \setminus Q\} = \mathbb{R}$

Q AND $\mathbb{R} \setminus Q$ HAVE NO ISOLATED POINTS

$\partial Q = \partial(\mathbb{R} \setminus Q) = \mathbb{R}$

- $S_5 = (-1, 1) \cap Q$

$\mathring{S}_5 = \emptyset \quad \partial S_5 = [-1, 1] = \overline{S} = \{$ACC. POINTS OF $S_5\}$

S_5 HAS NO ISOLATED POINTS

LIMITS OF FUNCTIONS AND CONTINUITY

DEFINITION LET $f: A \subseteq \mathbb{R}^N \longrightarrow \mathbb{R}^M$, $N, M \geq 1$

$x = (x_1, \dots x_n) \in A$

$\mathbb{R}^M \ni f(x) = f(x_1, \dots x_N) = (f_1(x), \dots f_N(x))$

$f_i : A \subseteq \mathbb{R}^N \longrightarrow \mathbb{R} \quad i = 1, \dots, M$ COMPONENTS OF f

LET x_0 BE AN ACCUMULATION POINT FOR A

LET $y_0 \in \mathbb{R}^M$ WE SAY THAT f HAS LIMIT EQUAL TO $y_0 \in \mathbb{R}^N$

OR THAT f GOES OR CONVERGES TO y^0 AS $x \in A$ GOES

TO x^0, AND WE WRITE

$$\lim_{A \ni x \to x^0} f(x) = y^0$$

IF $\forall \varepsilon > 0$ SUCH THAT $\forall x \in A$ WITH $0 < d(x, x^0) < \delta$

WE HAVE $d(f(x), y^0) < \varepsilon$

REMARK

- $0 < d(x, x^0) < \delta \iff x \in B_\delta(x^0) \setminus (x^0) \subseteq \mathbb{R}^N$
- $d(f(x), y^0) < \varepsilon \iff f(x) \in B_\varepsilon(y^0) \subseteq \mathbb{R}^N$