Exercises - Calculus

Academic Year 2021-2022

Sheet 21

1. Find the general solution of the following linear scalar differential equations and solve the corresponding Cauchy problems. Write explicitly the maximal interval of definition of the solution to the Cauchy problem.

   (a) $(1+t^2)y' - ty = \sqrt{1+t^2}$, $y(0) = 1$
   (b) $y' - e^t y = e^t$, $y(1) = 0$
   (c) $y' + (\tan t)y = \sin t$, $-\pi/2 < t < \pi/2$, $y(0) = 0$
   (d) $y' - 2\dfrac{y}{t} = \dfrac{t^2}{t-2}$, $0 < t < 2$, $y(1) = 2$
   (e) $y' = -\tan(t)y + \cos(t)$, $t \neq \pi/2 + k\pi$, $k \in \mathbb{Z}$, $y(\pi) = -1$
   (f) $y' = -3y + \mathrm{e}^{-3t}\arctan(t)$, $y(0) = 1$
   (g) $y' = \dfrac{y}{t} + 1 - t^2$, $0 < |t| < 1$, $y(1/2) = 0$

2. Find the general solution of the following scalar differential equations and solve the corresponding Cauchy problems. About the maximal interval of definition of the solution to the Cauchy problem, if possible write it explicitly, otherwise try to establish its qualitative properties, for example if it is bounded from below or from above. For instance, for problem (d) the interval is $(0, +\infty)$, $(a, +\infty)$ for $0 < a < 2$, $(0, b)$ for $b > 2$, (a, b) for $0 < a < 2 < b$ or something else?

   (a) $y^4 y' = t^3 - t$, $y \neq 0$, $y(0) = -1$
   (b) $\dfrac{xy'}{y^3} = 1 - x^4$, $x \neq 0$, $y \neq 0$, $y(1) = 1$
   (c) $(t^2 - yt^2)y' + y^2 + ty^2 = 1 + t$, $t \neq 0$, $y \neq 1$, $y(1) = -1$
   (d) $t\dfrac{y'}{1+y^2} = 2$, $t \neq 0$, $y(2) = 0$
   (e) $\sin(y)y' = t$, $0 < y < \pi$, $y(0) = \pi/2$
   (f) $y' = \dfrac{t\sqrt[3]{y^2-1}}{y}$, $y \neq 0$, $y(0) = -2$
   (g) $y' = \dfrac{y^3+y}{2}t$, $y(0) = -1$

3. Find the general solution of the following linear scalar differential equations and solve the corresponding Cauchy problems.

   (a) $y'' + 3y' + 2y = 0$
   Initial conditions: $y(0) = 0$, $y'(0) = 1$
   (b) $y'' + 4y' + 5y = 0$
   Initial conditions: $y(1) = 1$, $y'(1) = 0$

(c) $y'' + 2y' + y = 0$
Initial conditions: $y(0) = 0$, $y'(0) = 1$

(d) $y'' - 3y' = 0$
Initial conditions: $y(0) = 1$, $y'(0) = 1$

(e) $y'' = 0$
Initial conditions: $y(0) = 0$, $y'(0) = 1$

1. Find the general solution of the following linear scalar differential equations and solve the corresponding Cauchy problems. Write explicitly the maximal interval of definition of the solution to the Cauchy problem.

   (a) $(1+t^2)y' - ty = \sqrt{1+t^2}$, $y(0) = 1$

$$y' - \frac{t}{1+t^2} y = \frac{1}{\sqrt{1+t^2}}$$

$$a(t) = \frac{-t}{1+t^2} \qquad A(t) = \int a(t)\,dt = -\int \frac{t}{1+t^2}\,dt = -\frac{1}{2}\int \frac{2t}{1+t^2}\,dt = -\frac{1}{2}\ln(1+t^2)$$

$$\underbrace{y' e^{-1/2 \ln(1+t^2)} - \frac{t}{1+t^2} y e^{-\frac{1}{2}\ln(1+t^2)}}_{\left[y e^{-\frac{1}{2}\ln(1+t^2)}\right]'} = \frac{1}{\sqrt{1+t^2}} e^{-1/2\ln(1+t^2)}$$

$$\int \sim\sim = \int \sim\sim$$

$$y e^{-1/2\ln(1+t^2)} = \int \frac{1}{\sqrt{1+t^2}} e^{-1/2\ln(1+t^2)}\,dt + c$$

$$y e^{-1/2\ln(1+t^2)} =$$

$$y' = \frac{t}{1+t^2} y + \frac{1}{\sqrt{1+t^2}}$$

$$a(t) = \frac{t}{1+t^2} \qquad A(t) = \int a(t)\,dt = \int \frac{t}{1+t^2}\,dt = \frac{1}{2}\int \frac{2t}{1+t^2}\,dt = \frac{1}{2}\ln(1+t^2)$$

$$b(t) = \frac{1}{\sqrt{1+t^2}} \qquad B(t) = \int e^{-A(t)} b(t)\,dt = \int e^{-1/2\ln(1+t^2)} \cdot \frac{1}{\sqrt{1+t^2}} = \int e^{\ln(1+t^2)^{-1/2}} \frac{1}{\sqrt{1+t^2}}\,dt = \int (1+t^2)^{-1/2} \frac{1}{\sqrt{1+t^2}}\,dt =$$

$$\int \frac{1}{1+t^2}\,dt = \arctan(t)$$

$$y(t) = e^{A(t)}\left[B(t) + c\right] = e^{1/2\ln(1+t^2)} \left[\arctan(t) + c\right] \qquad y(0) = 1$$

$$1 = \underbrace{e^{1/2\ln(1)}}_{1} \left[\arctan(0) + c\right]$$

$$\downarrow$$

$$\frac{\sin(x)}{\cos(x)} = 0; \quad \sin(x) = 0 \quad x = 0$$

$$1 = c$$

$$y(t) = e^{1/2\ln(1+t^2)}\left[\arctan(t) + 1\right]$$

(b) $y' - e^t y = e^t$, $y(1) = 0$

$y' = e^t y + e^t$

$A(t) = \int e^t \, dt = e^t$

$B(t) = \int e^{-e^t} e^t \, dt = -\int -e^{-e^t} e^t \, dt = -\int e^w dw = -e^w dw = -e^{-e^t}$

$w = -e^t$

$dw = -e^t dt$

$y(t) = e^{e^t}\left[-e^{-e^t} + c\right]$ $\quad y(1) = 0$

$0 = e^e\left[-e^{-e} + c\right]$ $\boxed{-e^e \cdot \frac{1}{e^e}} = -1$

$0 = -1 + c$; $c = 1$

(c) $y' + (\tan t)y = \sin t$, $-\pi/2 < t < \pi/2$, $y(0) = 0$

$A(t) = -\int \tan t \, dt = \int \frac{-\sin(t)}{\cos(t)} dt = \ln|\cos(t)|$

$B(t) = \int e^{-\ln(\cos(t))} \sin(t) \, dt = \int \cos(t)^{-1} \sin(t) \, dt = -\int -\cos(t)\sin(t) \, dt = \ln(-\cos(t))$

$y(t) = e^{\ln(\cos(t))}\left[-\ln(\cos(t)) + c\right]$ $\quad y(0) = 0$

$0 = e^{\ln(1)}\left[-\ln(1) + c\right] \Rightarrow c = 0$

(d) $y' - 2\dfrac{y}{t} = \dfrac{t^2}{t-2}$, $0 < t < 2$, $y(1) = 2$

$$y' = \frac{2}{t}y + \frac{t^2}{t-2}$$

$$a(t) = \frac{2}{t} \qquad b(t) = \frac{t^2}{t-2}$$

SINCE $0<t<2$

$$A(t) = \int a(t) = \int \frac{2}{t}\,dt = 2\int \frac{1}{t}\,dt = 2\ln|t| = 2\ln(t)$$

$$B(t) = \int e^{-A(t)} b(t)\,dt = \int e^{-2\ln(t)} \frac{t^2}{t-2}\,dt = \int t^{-2}\cdot\frac{t^2}{t-2}\,dt = \int \frac{1}{t-2}\,dt = \ln|t-2|$$

$$y(t) = e^{A(t)}\left[B(t) + c\right] = e^{2\ln(t)}\left[\ln|t-2| + c\right] \quad y(1)=2 \Rightarrow 2 = e^0\left[\ln|1-1|+c\right] \Rightarrow \boxed{2 = c}$$

$$\Downarrow$$

$$y(t) = e^{2\ln(t)}\left[\ln|t-2| + c\right]$$

$$y(t) = t^2\left[\ln|t-2| + 2\right]$$

(e) $y' = -\tan(t)y + \cos(t)$, $t \neq \pi/2 + k\pi$, $k \in \mathbb{Z}$, $y(\pi) = -1$

$$a(t) = -\tan(t) \qquad A(t) = \int a(t)\,dt = -\int \tan(t)\,dt = \int \frac{-\sin(t)}{\cos(t)}\,dt = \ln|\cos(t)|$$

$$b(t) = \cos(t) \qquad B(t) = \int e^{-A(t)} b(t)\,dt = \int e^{-\ln|\cos(t)|}\cos(t)\,dt = \int \cancel{\cos(t)}^{-1}\cdot\cancel{\cos(t)}\,dt = t$$

$y(\pi) = -1$

$$y(t) = e^{A(t)}\left[B(t) + c\right] \Rightarrow y(t) = e^{\ln(\cos t)}\left[t + c\right] \Longrightarrow -1 = \cos(t)\left[t + c\right]$$

$$-1 = -1\left[\pi + c\right];\ c = \pi - 1$$

$$y(t) = \ln(\cos(t))\left[t + \pi - 1\right]$$

(f) $y' = -3y + \mathrm{e}^{-3t}\arctan(t)$, $y(0) = 1$

$$a(t) = -3 \qquad A(t) = -3t$$

$$b(t) = e^{-3t}\arctan(t) \qquad B(t) = \int \cancel{e^{3t}}\,\cancel{e^{-3t}}\arctan(t)\,dt = t\arctan(t) - \int \frac{t}{1+t^2}\,dt = t\arctan(t) - \frac{1}{2}\int \frac{2t}{1+t^2}\,dt =$$

$f\ \arctan(t) \quad f'\ \dfrac{1}{1+t^2}$

$g\ t \quad g'\ 1$

$$t\arctan(t) - \frac{1}{2}\ln|1+t^2|$$

$$y(t) = e^{A(t)}\left[B(t) + c\right] = e^{-3t}\left[e^{t\arctan(t) - \frac{1}{2}\ln(1+t^2)} + c\right] = e^{-3t}\left[e^{t\arctan(t)}\cdot\frac{1}{\sqrt{1+t^2}} + c\right]$$

$$1 = 1\left[e^0\ \frac{1}{1} + c\right] \Rightarrow \boxed{c = 0}$$

$$\arctan(t) = 0$$

$$\frac{\sin(t)}{\cos(t)} = 0\ ;\ \sin(t) = 0 \qquad t = 0$$

$$\boxed{y(t) = e^{-3t}\left[e^{t\arctan(t)}\cdot\frac{1}{\sqrt{1+t^2}}\right]}$$

$$y' = \frac{y}{t} + 1 - t^2$$

$$A(t) = \int \frac{1}{t}\, dt = \ln(t)$$

$$B(t) = \int e^{-\ln(t)} (1-t^2)\,dt = \int \frac{1}{t}(1-t^2)\,dt = \ln(t) - \frac{t^2}{2}$$

$$y(t) = e^{\ln(t)}\left[\ln(t) - \frac{t^2}{2} + c\right] \qquad y\left(\frac{1}{2}\right) = 0$$

$$0 = e^{\ln(1/2)}\left[\ln\left(\frac{1}{2}\right) - \frac{1}{8} + c\right]$$

$$0 = \frac{1}{2}\ln\left(\frac{1}{2}\right) - \frac{1}{16} + \frac{1}{2}c\ ;$$

$$\frac{1}{8} - \ln\left(\frac{1}{2}\right) = c$$

$$y(t) = \overbrace{e^{\ln(t)}}^{t}\left[\ln(t) - \frac{t^2}{2} + \frac{1}{8} - \ln\left(\frac{1}{2}\right)\right]$$

SCALAR: $y^{n}(t) = f(t, y, y', \dots y^{n-1})$

LINEAR: $f(t, y, y', \dots, y^{n-1}) = b(t) + a_0(t)y + \dots a_{n-1}(t)\, y^{n-1}$

$$\begin{cases} y'(t) = a(t)y + b(t) \\ A(t) \in \int a(t)\, dt \quad \text{(NO NEED FOR } c\text{)} \\ B(t) \in \int e^{-A(t)}\, b(t)\, dt \end{cases}$$

$y(t) = e^{A(t)}[B(t) + c]$

$\exists$ of SOLUTION

$f \in C^0(\Omega)$ $\boxed{\text{CONTINUA}}$ $\Rightarrow \exists$ SOLUTION TO THE CP

$f \in C^1(\Omega) \Rightarrow \exists!$ MAXIMAL SOLUTION

ONLY ONE

1. Find the general solution of the following linear scalar differential equations and solve the corresponding Cauchy problems. Write explicitly the maximal interval of definition of the solution to the Cauchy problem.

   (a) $(1+t^2)y' - ty = \sqrt{1+t^2}$, $y(0) = 1$

$(1+t^2)y' - ty = \sqrt{1+t^2}$

$y' = \underbrace{\frac{t}{1+t^2}}_{a(t)} y + \underbrace{\frac{1}{\sqrt{1+t^2}}}_{b(t)}$

$A(t) = \int \frac{t}{1+t^2}\, dt = \frac{1}{2}\int \frac{2t}{1+t^2}\, dt = \frac{1}{2}\ln(1+t^2) + \text{"}c\text{"}$

$B(t) = \int e^{-A(t)}\, dt = \int e^{-\frac{1}{2}\ln(1+t^2)} \frac{1}{\sqrt{1+t^2}}\, dt = \int (1+t^2)^{-1/2} \cdot \frac{1}{\sqrt{1+t^2}}\, dt = \int \frac{1}{1+t^2}\, dt = \arctan(t) + c$

SOLUZIONE GENERALE

$y = e^{1/2\ln(1+t^2)}[\arctan(t) + c] = \boxed{\sqrt{1+t^2}\,[\arctan(t) + c]}$

$$\begin{cases} y(t) = \sqrt{1+t^2}\,[\arctan(t) + c] \\ y(0) = 1 \end{cases} \Rightarrow 1 = \arctan(0) + c;\ \boxed{c = 1}$$

$y(t) = \sqrt{1+t^2}\,[\arctan(t) + 1]$

$$\begin{cases} y' = e^t y + e^t \\ y(1) = 0 \end{cases}$$

$A(t) = \int e^t\, dt = e^t + c$

$B(t) = \int e^{-e^t} e^t\, dt = -e^{-e^t}$

$$\begin{cases} y(t) = e^{e^t}(-e^{-e^t}) = -1 + c\, e^{e^t} \\ y(1) = 0 \end{cases}$$

$0 = -1 + c\, e^e;\ c = 1\, e^{-e}$

$y(t) = -1\, e^{-e^{t-1}}$

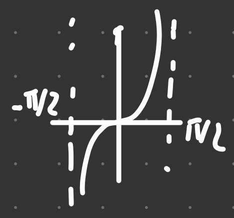
$I = \mathbb{R}$

"DOMAIN"

MAXIMAL INTERVAL $I = \mathbb{R}$

1c)

$$\begin{cases} y' = -y\tan(t) + \sin(t) \\ y(0) = 0 \\ -\frac{\pi}{2} < t < \frac{\pi}{2} \Rightarrow \text{OPEN INTERVAL} \end{cases}$$



$$A(t) = \int -\tan(t)\,dt = \int -\frac{\sin(t)}{\cos(t)}\,dt = \ln|\cos(t)|$$

$$B(t) = \int e^{-\ln|\cos(t)|} \sin(t)\,dt = \int \frac{1}{\cos(t)} \sin(t)\,dt = -\ln(\cos(t))$$

GENERAL SOLUTION: $y(t) = e^{\ln(\cos(t))}\left[-\ln(\cos(t)) + c\right] = \cos(t)\left[-\ln(\cos(t)) + c\right]$

CP: $0 = -\ln(1) + c$; $c = 0$

$$y(t) = -\cos(t)\ln(\underbrace{\cos(t)}_{})$$

PROBLEM: CAN BE < 0

MAXIMAL INTERVAL

BUT $\underbrace{-\frac{\pi}{2} < t < \pi/2}_{\cos(t) > 0}$

MAXIMAL INTERVAL $I = (-\frac{\pi}{2}, \pi/2)$

1d)

$$\begin{cases} y' = \frac{2y}{t} + \frac{t^2}{t-2} \\ y(1) = 2 \\ 0 < t < 2 \end{cases}$$

$$A(t) = \int \frac{2}{t} = 2\ln|t|$$

$$B(t) = \int e^{-2\ln|t|} \frac{t^2}{t-2}\,dt = \int \frac{1}{\cancel{t^2}} \frac{\cancel{t^2}}{t-2}\,dt = \ln|t-2| = \ln(2-t)$$

$$y(t) = e^{2\ln|t|}\left[\ln(2-t) + c\right] = t^2\left[\ln(2-t) + c\right]$$

CP: $2 = 1\left[\ln(1) + c\right]$; $c = 2$

$$y(t) = t^2\left[\ln(2-t) + 2\right]$$

MAXIMAL INTERVAL $\boxed{0 < t < 2}$

ES 2

- SCALAR $\quad y^{(n)} = g(t, y, y', \dots y^{n-1})$
- NON-LINEAR $\quad g(t, y, y', \dots, y^{n-1}) \neq b(t) + a_0(t)y + \dots + \dots$
- VARIABILI SEPARABILI $\quad y' = f(t)\,h(y)$
- PRIMO ORDINE $(n=1)$

$y_0 \in \mathbb{R} : h(y_0) = 0 \Rightarrow y(t) = y_0$ È UNA SOLUZIONE

$$\int \frac{y'}{h(y)}\,dt = \int f(t)\,dt$$

$$\int \frac{1}{h(y)}\,dx$$

$$y' = \frac{dy}{dt}$$

$$\frac{1}{h(y)}\frac{dx}{dt} = f(t)\,\boxed{dt}$$

$$\int \frac{1}{h(y)}\,dx = \int f(t)\,dt$$

2. Find the general solution of the following scalar differential equations and solve the corresponding Cauchy problems. About the maximal interval of definition of the solution to the Cauchy problem, if possible write it explicitly, otherwise try to establish its qualitative properties, for example if it is bounded from below or from above. For instance, for problem (d) the interval is $(0, +\infty)$, $(a, +\infty)$ for $0 < a < 2$, $(0, b)$ for $b > 2$, (a, b) for $0 < a < 2 < b$ or something else?

   (a) $y^4 y' = t^3 - t$, $y \neq 0$, $y(0) = -1$

   (b) $\dfrac{xy'}{y^3} = 1 - x^4$, $x \neq 0$, $y \neq 0$, $y(1) = 1$

   (c) $(t^2 - yt^2)y' + y^2 + ty^2 = 1 + t$, $t \neq 0$, $y \neq 1$, $y(1) = -1$

   (d) $t\dfrac{y'}{1+y^2} = 2$, $t \neq 0$, $y(2) = 0$

   (e) $\sin(y)y' = t$, $0 < y < \pi$, $y(0) = \pi/2$

   (f) $y' = \dfrac{t\sqrt[3]{y^2 - 1}}{y}$, $y \neq 0$, $y(0) = -2$

   (g) $y' = \dfrac{y^3 + y}{2}t$, $y(0) = -1$

2a)
$$\begin{cases} y^4 y' = t^3 - t \\ y \neq 0 \\ y(0) = -1 \end{cases}$$

$$y' = \underbrace{\frac{1}{y^4}}_{h(y)}\underbrace{(t^3 - t)}_{f(t)}$$

$h(y) = J \to \mathbb{R}$ $\quad J_1 = (-\infty, 0)$, $J_2 = (0, +\infty)$

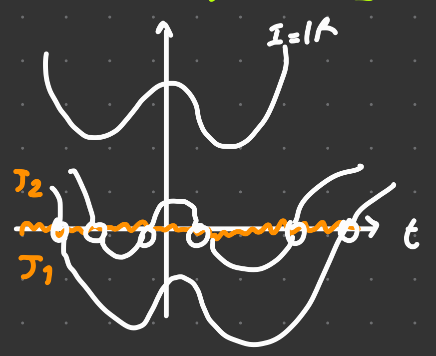
$f(t) = I \to \mathbb{R}$ $\quad I = \mathbb{R}$

J_1: $\int y^4 y'\,dt = \int t^3 - t\,dt$

$$\frac{y^5}{5} = \frac{t^4}{4} - \frac{t^2}{2} + c$$

$$y = \sqrt[5]{\frac{5}{4}t^4 - \frac{5}{2}t^2 + 5c}$$ GENERAL SOLUTION

$I : y(t) < 0 \ \forall t \in I$



$$y = \sqrt[5]{\frac{5}{4}t^4 - \frac{5}{2}t^2 + 5c}$$

$y(0) = -1$

$-1 = \sqrt[5]{5c}$; $\quad -1 = 5c$; $\quad c = -\frac{1}{5}$

$$y(t) = \sqrt[5]{\frac{5}{4}t^2 - \frac{5}{2}t^2 - 1}$$ STUDY WHEN $y(t) < 0$

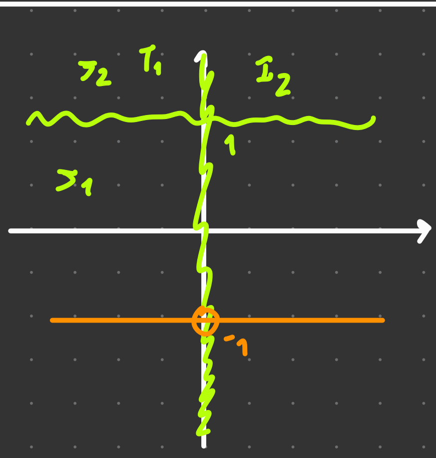
$\frac{5}{4}t^4 - \frac{5}{2}t^2 - 1 < 0 \qquad \frac{5}{4}k^2 - \frac{5}{2}k - 1 < 0$

$5k^2 - 10 - 4 < 0 \qquad k_{1,2} = \dfrac{5 \pm \sqrt{45}}{5}$

$$\frac{5 - \sqrt{45}}{5} < k < \frac{5 + \sqrt{45}}{5}$$

$\frac{S-\sqrt{4s}}{S} < t^2 < S+\sqrt{\frac{4s}{S}}$

ES 2.1



$$\begin{cases} (t^2 - yt^2)\,y' + y^2 + t y^2 = 1 + t & t \neq 0 \quad y \neq 1 \\ y(1) = -1 \end{cases}$$

$$t^2(1-y)\,y' + y^2(1+t^2) = 1+t$$

$$t^2(1-y)\,y' = (1+t)(1-y^2)$$

$$y' = \underbrace{\frac{1+t}{t^2}}_{f(t)} \, \frac{(1-y^2)}{1-y}$$

$h(y_0) = 0$

$y_0 = \pm 1$

$y(t) = -1$

$$\int \left(\frac{1}{1-y^2} - \frac{y}{1-y^2}\right) y' \, dt = \int \frac{1}{t^2 + \frac{1}{t}} \, dt$$

$\ln|1-y^2|$

$$\frac{1}{(1+y)(1-y)} = \frac{A_1}{1+y} + \frac{A_2}{1-y} \qquad \begin{cases} A_1(1-y) + A_2(1+y) = 1 \\ A_1 + A_2 = 1 \\ A_1 y + A_2 y = 0 \end{cases}$$

$$\frac{1}{2}\left\{ \log|1+y| - \ln|1-y| + \ln|1-y^2| \right\} = -\frac{1}{t} \log|t| + c$$

3. Find the general solution of the following linear scalar differential equations and solve the corresponding Cauchy problems.

(a) $y'' + 3y' + 2y = 0$
Initial conditions: $y(0) = 0$, $y'(0) = 1$

(b) $y'' + 4y' + 5y = 0$
Initial conditions: $y(1) = 1$, $y'(1) = 0$

- LINEAR
- SCALAR
- SECOND ORDER $\quad N=2 \quad y'(t) = b(t) + a_0(t)y + a_1(t)y'$
- COEFFICIENTI CONSTANTI $\quad a_i(t) = a_i \;\; \forall t \in I$ (IS CONSTANT)
- HOMOGENEOUS $\quad b(t) = 0$

$y'' + a_1 y' + a_0 y = 0 \qquad \exists!$ MAXIMAL SOLUTION IN $\mathbb{R}$ $\quad I = \mathbb{R}$

$P(\lambda) \qquad \lambda^2 + a_1\lambda^1 + a_0 \underbrace{\lambda^0}_{1} = 0 \qquad$ • SOLVE λ

2 SOL REALI: $y(t) = a e^{\lambda_1 t} + b e^{\lambda_2 t}$

1 SOL REALE $\quad y(t) = a e^{\lambda t} + b t e^{\lambda t}$

2 SOL COMPLESSE $\quad y(t) = a e^{\alpha t}\cos(\omega t) + b e^{\alpha t}\sin(\omega t)$
$\lambda = \alpha \pm i\omega$

CP $\begin{cases} y'' + 3y' + 2x = 0 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$

$P(\lambda) = \lambda^2 + 3\lambda + 2 = 0$

$9 - 4(2) \qquad \frac{-3 \pm 1}{2} < \begin{matrix} \lambda_1 = -2 \\ \lambda_2 = -1 \end{matrix}$

$\begin{cases} y(t) = a e^{-t} + b e^{-2t} \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$

$0 = a + b$
$1 = -a - 2b \quad \Rightarrow \begin{cases} a = 1 \\ b = -1 \end{cases} \quad I = \mathbb{R}$

$\begin{cases} y'' + 4y' + 5y = 0 \\ y(1) = 1 \\ y'(1) = 0 \end{cases}$

$P(\lambda) = \lambda^2 + 4\lambda + 5 = 0$

$16 - 20$

$\frac{-4 - 2i}{2} = \overset{\alpha}{-2} \pm i \qquad W = 1$

$y(t) = a\, e^{-2t}\cos(t) = b e^{-2t}\sin(t)$

3 $\begin{cases} y'' = 0 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$

$p(\lambda) = \lambda^2 \qquad \lambda_{1,2} = 0$

$y(t) = a e^{0t} + b t e^{0t} = a + bt \qquad a = 0 \;;\; a = 0 \qquad y(t) = t$

$1 = 0 + b \quad b = 1$

PROBLEMI DI CAUCHY PRIMO ORDINE OMOGENEA

$y' = a(t)y + b(t)$

$A(t) = \int a(t)\,dt$

$B(t) = \int e^{-A(t)} b(t)\,dt$

$y = e^{A(t)} \left[B(t) + c \right]$

PROBLEMI DI CAUCHY 2° ORDINE OMOGENEE

2 SOL. REALI: $y(t) = a e^{\lambda_1 t} + b e^{\lambda_2 t}$

1 SOL. REALE: $y(t) = a e^{\lambda t} + b t e^{\lambda t}$

2 SOL. COMPLESSE: $y(t) = a e^{\alpha t} \cos(\omega t) + b e^{\alpha t} \sin(\omega t)$

$\lambda = \alpha + i\omega$

3. Find the general solution of the following linear scalar differential equations and solve the corresponding Cauchy problems.

(a) $y'' + 3y' + 2y = 0$
Initial conditions: $y(0) = 0$, $y'(0) = 1$

$y'' + 3y' + 2y = 0$

$\lambda^2 + 3\lambda + 2 = 0$

$9 - 4(2)$ $\quad \lambda_{1,2} = \frac{-3 \pm 1}{2}$ $\begin{cases} -2 \\ -1 \end{cases}$

$y(t) = a e^{\lambda_1 t} + b e^{\lambda_2 t}$

$y(t) = a e^{-2t} + b e^{-t}$ $\quad y(0) = 0 \Rightarrow 0 = a + b$

$y'(0) = 1 \Rightarrow 1 = -2a e^{2t} - b e^{t}$

$1 = -2a - b$

$\begin{cases} 0 = a + b \\ b = -2a - 1 \end{cases}$ $\begin{cases} 0 = a - 2a - 1; \ a = -1 \\ b = 1 \end{cases}$

$\boxed{y(t) = -e^{-2t} + e^{-t}}$

(b) $y'' + 4y' + 5y = 0$
Initial conditions: $y(1) = 1$, $y'(1) = 0$

$\lambda^2 + 4\lambda + 5 = 0$

$16 - 4(5)$ $\quad \sqrt{-4} = \sqrt{4 \cdot (-1)} = 2i$

$\lambda_{1,2} \ \frac{-4 \pm 2i}{2} =$ $\begin{cases} \lambda_1 = -2 + i \\ \lambda_2 = -2 - i \end{cases}$

$y(t) = a e^{\alpha t} \cos(\omega t) + b e^{\alpha t} \sin(\omega t)$

$\alpha = -2$

$\omega = -1$

$y(t) = a e^{-2t} \cos(-t) + b e^{-2} \sin(-t)$

$y(1) = 1 \Rightarrow 1 = a e^{-2}$

FANCULO

$$\begin{cases} (1+t^2)y' - ty = \sqrt{1+t^2} \\ y(0)=1 \end{cases}$$

$$y' = \overbrace{\frac{t}{1+t^2}}^{a(t)} y + \overbrace{\frac{1}{\sqrt{1+t^2}}}^{b(t)}$$

$$A(t) = \int a(t)\,dt = \int \frac{t}{1+t^2}\,dt = \frac{1}{2}\int \frac{2t}{1+t^2}\,dt = \frac{1}{2}\ln(1+t^2)$$

$$B(t) = \int e^{-A(t)}\, b(t)\,dt = \int e^{\ln(1+t^2)^{-1/2}} \frac{1}{\sqrt{1+t^2}}\,dt = \int \frac{1}{1+t^2}\,dt = \arctan(t)$$

$$y(t) = e^{A(t)}\left[B(t) + c\right] = e^{\ln(1+t^2)^{1/2}}\left[\arctan(t) + c\right] = \sqrt{1+t^2}\left[\arctan(t) + c\right]$$

$$1 = \arctan(0) + c \Rightarrow c = 1$$

$$\downarrow$$

$$\frac{\sin(x)}{\cos(x)} = 0; \quad \sin(x) = 0; \quad x = 0$$

$$y(t) = \sqrt{1+t^2}\left[\arctan(t) + 1\right]$$