Exercises - Calculus

Academic Year 2021-2022

Sheet 4

$\frac{N!}{(N-K)!}$

1. How many are the natural numbers with three digits that are all different? $10 \cdot 9 \cdot 8$
2. How many are the natural numbers with three digits that are all different and whose first digit (the first on the left) is an even number? $4 \cdot 9 \cdot 8$
3. How many are the natural numbers with three digits whose first digit is an odd number and the last digit is a positive number that can be divided by 3? $5 \cdot 9 \cdot 3$ 90 XOR 135
4. How many 5 letters words can be written using only the letters A, B, C and D? N^K $N=4$ $K=K$ 4^5
5. How many 4 letters words, with all different letters, can be written using only the letters A, B, C, D and E? $5 \cdot 4 \cdot 3 \cdot 2$ $\frac{5}{(5-4)!} = 5!$ $\frac{N!}{(N-K)!}$
6. How many are the subset with five elements of the set $\{1, 2, 3, 4, 5, 6, 7\}$?
7. How many are the subsets with 3 elements of the set $\{3, 7, \pi, 12, 6e, 81\}$?
8. Let $A = \{a, b, c, d, e\}$. How many are the subsets of A with 2 or 3 elements?
9. Let $A = \{a, b, c, d, e\}$. How many are the elements of $A \times A$?
10. Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{7, 8, 9\}$. How many are the elements of $(A \times A) \times B$?
11. Let $A = \{1, 2, 3, 4\}$. How many are the pairs $(a, b) \in A \times A$ such that $a \neq b$?
12. Let $A = \{1, 2, 3, 4\}$. How many are the pairs $(a, b) \in A \times A$ such that $a \leq b$?
13. Generalize the previous two exercises to the case $A = \{1, \ldots, n\}$, for some $n \in \mathbb{N}$. Show that the number of pairs $(a, b) \in A \times A$ such that $a \neq b$ is $n^2 - n$. Show that the number of pairs $(a, b) \in A \times A$ such that $a \leq b$ is $\dfrac{n^2 + n}{2}$.
14. Find the number of anagrams of the word PIPPO.
15. How may are the anagrams of the word PATACCA?
16. Compute the real part of the complex number $z = \dfrac{\pi + 3i}{i}$.
17. Find the imaginary part of the complex number

$$z = \frac{\sqrt{2}}{1 + i}.$$

18. Let z be a complex number satisfying

$$iz = \frac{1}{i}.$$

Determine z and its imaginary part.

19. Determine all complex numbers satisfying

$$(1+i)z = i.$$

Write the solutions in Cartesian form.

20. Let z be a complex number satisfying

$$\frac{z}{1-i} = \frac{1}{2i}.$$

Determine the conjugate of z. Write the solution in Cartesian form.

21. Let z be a complex number satisfying

$$(2+i)z = 2-i$$

Determine the imaginary part of z.

22. Let z be a complex number satisfying

$$\overline{z} = 2-i$$

Determine the imaginary part of z.

23. Let z be a complex number satisfying

$$\frac{(2+i)}{z} = -i$$

Determine the imaginary part of z.

24. Let z be a complex number satisfying

$$(1+i)z = 2+i$$

Determine the modulus of z.

25. Let us consider the following complex numbers

$$a = 2+i, \qquad b = 1-2i, \qquad c = \sqrt{3}.$$

Compute

$$\frac{|a|^2\,|b|^2}{5} - |c|.$$

26. Let us consider the following complex numbers

$$a = 1+2i, \qquad b = 3-i.$$

Compute $a\bar{b}$. Write the solution in Cartesian form.

27. Let us consider the following complex numbers

$$a = 1 + 3i, \qquad b = 1 + i.$$

Compute $a\bar{b} + |a + b|$. Write the solution in Cartesian form.

28. Let us consider the following sets

$$\begin{aligned} A &:= \{1 + i,\, 3 - i,\, -5 + 2i\} \\ B &:= \{i,\, i^2,\, i^3\} \\ C &:= \{z \in \mathbb{C} : \text{ real part of } z \text{ is greater than } 0\} \end{aligned}$$

Determine the set

$$(A \cup B) \cap C$$

and draw it in the Gauss plane.

29. Let $z = 3 + \sqrt{3}i$. Write z in trigonometric form.

30. Let z be the complex number written in trigonometric form as

$$z = 2(\cos(5\pi/6) + \sin(5\pi/6)i).$$

Write z in Cartesian form.

31. Let $z = 1 + i$. Compute z^5, writing it in trigonometric form and in Cartesian form.

32. Determine all complex numbers satisfying the following equation

$$(z + 1)^2 = -1.$$

Write the solutions in Cartesian form and draw them in the Gauss plane.

33. Find all complex numbers satisfying the following equation

$$z^2 = -2i.$$

Write the solutions in Cartesian form and draw them in the Gauss plane.

34. Compute the (complex) square, cubic and fourth roots of

$$-2\sqrt{3} + 2i, \quad 2 - 2i, \quad 2i, \quad -3.$$

Draw the solutions in the Gauss plane.

35. Solve in $\mathbb{C}$ the following equations of second degree

$$z^2 - \sqrt{3}iz - \frac{\sqrt{3}}{4}i = 0 \quad \text{and} \quad z^2 + \sqrt{2}z - \frac{1}{2}i = 0.$$

Draw the solutions in the Gauss plane.

6. How many are the subset with five elements of the set $\{1, 2, 3, 4, 5, 6, 7\}$?

$$\binom{N}{K} = \frac{7!}{(7-5)!\,5!} = 21$$

7. How many are the subsets with 3 elements of the set $\{3, 7, \pi, 12, 6e, 81\}$?

$$\binom{N}{K} = \frac{N!}{(N-K)!\,K!} = \frac{6!}{(6-3)!\,3!} \qquad \frac{6 \cdot 5 \cdot 4 \cdot \cancel{3!}}{3!\,\cancel{3!}}$$

$N = 6$

$K = 3$

$$= \frac{\cancel{6} \cdot 5 \cdot 4}{\cancel{3} \cdot \cancel{2}} = 20$$

8. Let $A = \{a, b, c, d, e\}$. How many are the subsets of A with 2 or 3 elements?

$\binom{N}{K}$

$N = 5$

$K = 2$

$$\frac{5!}{3!\,2!} = \frac{5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!}\,2!} = 5 \cdot 2 = 10$$

$N = 5$

$K = 3$

$$\frac{5!}{2!\,3!} = 10$$

$$10 + 10 = 20$$

9. Let $A = \{a, b, c, d, e\}$. How many are the elements of $A \times A$?

$A \times A = N \cdot N = 5 \cdot 5 = 25$

10. Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{7, 8, 9\}$. How many are the elements of $(A \times A) \times B$?

$(6 \cdot 6) \cdot 3 = 108$

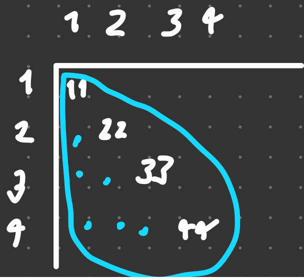
11. Let $A = \{1, 2, 3, 4\}$. How many are the pairs $(a, b) \in A \times A$ such that $a \neq b$?

$(a, b) \in A \times A \quad a \neq b$

	1	2	3	4
1	11			
2		22		
3			33	
4				44

$4^2 - 4 = 12$

12. Let $A = \{1, 2, 3, 4\}$. How many are the pairs $(a, b) \in A \times A$ such that $a \leq b$?



$(a,b) \in A \times A$ $\quad a \leq b$

$6+4=10$

13. Generalize the previous two exercises to the case $A = \{1, \ldots, n\}$, for some $n \in \mathbb{N}$. Show that the number of pairs $(a, b) \in A \times A$ such that $a \neq b$ is $n^2 - n$. Show that the number of pairs $(a, b) \in A \times A$ such that $a \leq b$ is $\dfrac{n^2 + n}{2}$.

14. Find the number of anagrams of the word PIPPO.

15. How may are the anagrams of the word PATACCA?

14) PIPPO

3P

$N = 5$

$$\frac{5!}{(5-2)!} = \frac{5!}{3!} =$$

$$\frac{5 \cdot 4 \cdot 3\cancel{!}}{\cancel{3}!} = \boxed{20}$$

15) PATACCA

$$\frac{7!}{3!\,2!} = \frac{7 \cdot 6 \cdot 5 \cdot \overset{2}{\cancel{4}} \cdot \cancel{3!}}{\cancel{3!} \cdot \cancel{2!}} = \boxed{420}$$

16. Compute the real part of the complex number $z = \dfrac{\pi + 3i}{i}$.

$i^2 = -1$

$$Z = \frac{\pi + 3i}{i} = \frac{\pi + 3i}{i} \cdot \frac{i}{i} = \frac{\pi i + 3i^2}{i^2} = \frac{i\pi - 3}{-1}$$

$$= -i\pi + 3$$

$$Re(z) = 3$$

17. Find the imaginary part of the complex number

$$z = \frac{\sqrt{2}}{1+i}.$$

$$z = \frac{\sqrt{2}}{1+i} = \frac{\sqrt{2}}{1+i} \cdot \frac{1-i}{1-i}$$

$$\frac{\sqrt{2} - i\sqrt{2}}{1 - i^2} = \frac{\sqrt{2} - i\sqrt{2}}{2}$$

$$\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}$$

$$Im(z) = -\frac{\sqrt{2}}{2}$$

18. Let z be a complex number satisfying

$$iz = \frac{1}{i}.$$

Determine z and its imaginary part.

$iz = \frac{1}{i}$

$i^2 z = 1$

$-z = 1$

$z = -1$

$\mathrm{Im}(z) = 0$

19. Determine all complex numbers satisfying

$$(1+i)z = i.$$

Write the solutions in Cartesian form.

$z = \rho(\cos\theta + i\sin\theta)$

RHO

$(1+i)z = i$

$(1+i)(1-i)z = i(1-i)$

$(1-i^2)z = i - i^2$

$2z = i + 1$

$z = \frac{i+1}{2} \Rightarrow z = \frac{i}{2} + \frac{1}{2}$

RHO → $\rho = \sqrt{a^2 + b^2}$

$a = \rho(\cos\theta)$ $\theta = \arctan\left(\frac{1/2}{1/2}\right) = \frac{\pi}{4} + 2k\pi$

$b = \rho(\sin\theta)$

$\rho = \sqrt{x^2+y^2} = \sqrt{\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$z = \frac{\sqrt{2}}{2}\left(\cos\left(\frac{\pi}{4} + 2k\pi\right) + i\sin\left(\frac{\pi}{4} + 2k\pi\right)\right)$

20. Let z be a complex number satisfying

$$\frac{z}{1-i} = \frac{1}{2i}.$$

Determine the conjugate of z. Write the solution in Cartesian form.

21. Let z be a complex number satisfying

$$(2+i)z = 2 - i$$

Determine the imaginary part of z.

$$(2+i)(2-i)z = (2-i)^2$$

$$4z - i^2 z = 4 - i^2 - 4i$$

$$4z + z = 5 - 4i$$

$$5z = 5 - 4i$$

$$z = 1 - \frac{4}{5}i$$

$$\text{Im}(z) = -\frac{4}{5}$$

22. Let z be a complex number satisfying

$$\overline{z} = 2 - i$$

Determine the imaginary part of z.

23. Let z be a complex number satisfying

$$\frac{(2+i)}{z} = -i$$

Determine the imaginary part of z.

$$\frac{(2+i)}{z} = -i$$

$$2+i = -iz$$

$$2i + i^2 = -i^2 z$$

$$2i - 1 = z$$

24. Let z be a complex number satisfying

$$(1+i)z = 2+i$$

Determine the modulus of z.

$$(1+i)z = 2+i \qquad |z| = \sqrt{(Re\, z)^2 + (Im\, z)^2}$$

$$z = \frac{2+1}{1+i}$$

$$z = \frac{(2+i)(1-i)}{1-i^2} \qquad z = \frac{2-2i+1+i}{2} = \frac{3-i}{2}$$

$$Re\, z = \frac{3}{2} \quad Im\, z = -\frac{1}{2} \quad |z| = \sqrt{\frac{9}{4} + \frac{1}{4}} = \sqrt{\frac{10}{4}} = \frac{\sqrt{10}}{2}$$

26. Let us consider the following complex numbers

$$a = 1 + 2i, \qquad b = 3 - i.$$

Compute $a\bar{b}$. Write the solution in Cartesian form.

$$A\bar{B} = (1+2i)(3+i) = 3 + i + 6i + 2i^2 = 1 + 7i$$

28. Let us consider the following sets

$$A := \{1+i,\ 3-i,\ -5+2i\}$$
$$B := \{i,\ i^2,\ i^3\}$$
$$C := \{z \in \mathbb{C} :\ \text{real part of } z \text{ is greater than } 0\}$$

Determine the set

$$(A \cup B) \cap C$$

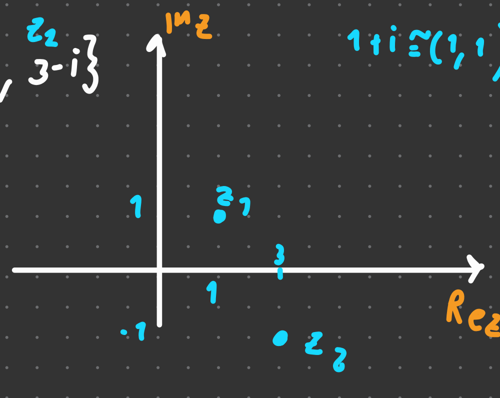
and draw it in the Gauss plane.

$$B = \{i, -1, -i\}$$

$$A \cup B = \{1+i,\ 3-i,\ -5+2i,\ i,\ -1,\ -i\}$$

$$(A \cup B) \cap C = \{\overset{z_1}{1+i},\ \overset{z_2}{3-i}\}$$

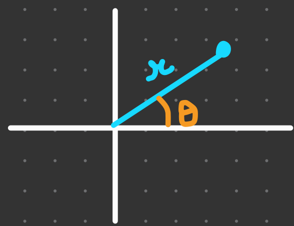
$1+i \cong (1,1)$



TRIGONOMETRIC FORM

$z = x + iy$ $\qquad$ $z = r(\cos(\theta) + i\sin(\theta))$

$\theta \in [0, 2\pi]$



29) $z = 3 + \sqrt{3}i$

$|z| = \sqrt{3^2 + \sqrt{3}^2} = 2\sqrt{3}$

$z = 2\sqrt{\ }$

31. Let $z = 1 + i$. Compute z^5, writing it in trigonometric form and i
Cartesian form.

$$z^5 = (1+i)^5 = (1+i)^2(1+i)^2 + (1+i)$$

$$(1+i^2+2i)(1+i^2+2i)(1+i)$$

$$(4i^2)(1+i) = -4 - 4i$$

DE MOIVRE $z = r(\cos(\theta) + i\sin(\theta))$

$$z^n = r^2(\cos(n\theta) + i\sin(n\theta))$$