$+ \quad \times$

$- \quad \div$

EXAMPLE: A SET WITH N ELEMENTS, $N \in \mathbb{N} \cup \{0\}$

HOW MANY SUBSETS OF A ARE THERE?

$N=0$ $\quad A=\emptyset \quad \boxed{1}$ — EMPTY SET

2^0

$$\sum_{K=0}^{N} \binom{N}{K} = 2^N = (1+1)^N = \sum_{K=0}^{N} \binom{N}{K} 1^{N-K} 1^K$$

1 1

COMPLEX NUMBER

EQUATION $q^2=2$ HAVE NO SOLUTION IN $\mathbb{Q}$ ($\nexists \sqrt{2}$ IN $\mathbb{Q}$)

BUT $\sqrt{2}$ DOES EXIST IN $\mathbb{R}$ IT IS THE POSITIVE SOLUTION TO $x^2=2$, $x \in \mathbb{R}$

WHICH HAS 2 SOLUTIONS $+\sqrt{2}$, $-\sqrt{2}$

EQUATION $x^2+1=0$ HAS NO SOLUTION IN $\mathbb{R}$ ($\nexists \sqrt{-1}$ IN $\mathbb{R}$)

DEFINITION

LET i BE THE IMAGINARY UNIT AND LET

$$\mathbb{C} := \{ z = x + yi, \ x, y \in \mathbb{R} \}$$ SET OF COMPLEX NUMBERS

DEFINITION

LET $z = x + yi \in \mathbb{C}$, $x, y \in \mathbb{R}$ WE CALL

$x = Re(z)$ REAL PART OF z

$y = Im(z)$ IMAGINARY PART OF z

WE SAY $z = x+yi \in \mathbb{C}$ IS REAL IF $y = Im(z) = 0$,

THAT IS, $z = x + 0i$ WHICH IS IDENTIFIED WITH $x \in \mathbb{R}$

IN THIS SENSE $\boxed{\mathbb{R} \subseteq \mathbb{C}}$

WE SAY $z = x+yi \in \mathbb{C}$ IS PUREACY IMMAGINARY IF

$x = Re(z) = 0$ AND $y \neq 0$, THAT IS $z = 0 + yi$, $y \neq 0$

REMARK

THE REAL AND IMAGINARY PART OF A COMPLEX NUMBER ARE BOTH REAL NUMBER

WE CAN DEFINE TWO OPERATIONS IN $\mathbb{C}$: SUM $+$ AND PRODUCT $\cdot$

$$z_1 = x_1 + y_1 i \in \mathbb{C} \qquad x_1 y_1 \in \mathbb{R}$$

$$z_2 = x_2 + y_2 i \in \mathbb{C} \qquad x_2 y_2 \in \mathbb{R}$$

THEN

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2) i$$

$$z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2) i$$

REMARK

IF $z_1 = x_1 + 0i = x_1 \in \mathbb{R}$ AND

$z_2 = x_2 + 0i = x_2 \in \mathbb{R}$, THESE OPERATIONS IN $\mathbb{C}$ ARE CONSISTENT WITH THOSE IN $\mathbb{R}$, THAT IS,

$$z_1 + z_2 = x_1 + x_2 + 0i = x_1 + x_2$$

$$z_1 \cdot z_2 = x_1 \cdot x_2 + 0i = x_1 \cdot x_2$$

MOROVER

$$\alpha \in \mathbb{R},\ z = x + yi \in \mathbb{C} \text{ THEN}$$

$$\alpha \cdot z = (\alpha + 0i) \cdot z = \alpha x + \alpha y i$$

THEOREM WITH THESE OPERATIONS

$\mathbb{C}$ IS A FIELD, THAT IS, PROPERTIES $\boxed{1-5)}$ OF OPERATIONS ON $\mathbb{R}$ HOLDS

FIRST LECTURE

PROOF

- $\exists$ NEUTRAL ELEMENT FOR THE SUM

  $0 = 0 + 0i$

  $$0 + z = (0 + 0i) + (x + yi) = (0 + x) + (0 + y)i = z$$

- $\exists$ NEUNTRAL ELEMENT FOR THE PRODUCT

  $1 = 1 + 0i$

  $1 \cdot z = (1 + 0i)(x + yi) = z$

- $\forall z = x + yi$ $\exists$ OPPOSITE $-z = -x + (-yi) = -x - yi$

  SUCH THAT $z + (-z) = 0$

- $\forall z = x + yi \neq 0 = 0 + 0i$ THAT IS, x, y ARE NOT BOTH 0

$\exists$ INVERSE $z^{-1} = \frac{1}{z} = \frac{x}{x^2 + y^2} + \frac{(-yi)}{x^2 + y^2} = \frac{x - yi}{x^2 + y^2}$

SUCH THAT $z \cdot z^{-1} = 1 \cdot 0i = 1$

REMARK

NEUTRAL ELEMENTS FOR SUM AND PRODUCT IN $\mathbb{C}$ COINCIDES WITH THOSE IN $\mathbb{R}$

MOROVER, $\forall a \in \mathbb{R}$

$$-(a + 0i) = -a + 0i = -a$$

$$a \neq 0 \quad (a + 0i)^{-1} = \frac{1}{a} + 0i = \frac{1}{a}$$

IMPORTAN REMARK THE IMAGINARY UNIT

$i = 0 + 1 \cdot i$ THEN

$$i^2 = (0 + 1i)(0 + 1i) = (0 \cdot 0 - 1 \cdot 1) + (0 \cdot 1 + 1 \cdot 0)i = -1 \cdot 0i = -1$$

SO IN $\mathbb{C}$ THE EQUATION $z^2 + 1 = 0$ HAS TWO SOLUTION i AND $-i$

REMARK $z_1 = x_1 + y_1 i \in \mathbb{C} \quad x_1 y_1 \in \mathbb{R}$

$z_2 = x_2 + y_2 i \in \mathbb{C} \quad x_2 y_2 \in \mathbb{R}$

$$z_1 \cdot z_2 = (x_1 + y_1 i) \cdot (x_2 + y_2 i) = x_1 x_2 + x_1 y_2 i + y_1 i x_2 + y_1 i y_2 i =$$

$= x_1x_2 + x_1y_2 i + y_1x_2 i + y_1y_2 \underbrace{i}_{-1} =$

$= (x_1x_2 - y_1y_2) + (x_1y_2 + y_1x_2) i$

EXAMPLE

$z = 2+3i$

$Re(z) = 2$

$Im(z) = 3$ IT IS NOT $3i$

$(2+3i) + (4-2i) = (2+4) + (3+(-2))i = 6+i$

$(2+3i)\cdot(4-2i) = 2\cdot 4 + 2(-2)i + (3i)4 + (3i)(-2i) =$

$= 8 - 4i + 12i - 6\underbrace{i^2}_{-1} = 8+6+8i = 14+8i$

POLYNOMIALS (REAL AND COMPLEX)

DEF LET $n \in \mathbb{N} \cup \{0\}$ LET $a_0, a_1 \dots, a_n \in \mathbb{R}$

DEFINE

$P_n(x) = a_n x^2 + a_{n-1} x^{n-1} + \dots a_1 x + a_0 \quad \forall x \in \mathbb{R}$

(REAL) POLYNOMIAL OF DEGREE AT MOST N

- IF $a_n \neq 0$ DEGREE IS N
- IF $a_0 = \dots a_n = 0$ $\quad P_n(x) = 0$ THIS HAS DEGREE $-\infty$ (BY CONVENTION)

P_N POLYNOMIAL OF DEGREE $N \geq 1$ WE SAY THAT

x_1 IS A ROOT OF P_N IF $P_N(x_1) = 0$

IN THIS CASE $P_N(x) = (x - x_1)\, P_{N-1}(x)$

WHERE $P_{N-1}(x)$ IS A POLYNOMIAL OF DEGREE $N-1$

$\Rightarrow$ P_N HAS AT MOST N ROOTS

REMARK

- POLYNOMIAL OF DEGREE AT MOST 0: CONSTANTS
- POLYNOMIAL OF DEGREE AT MOST 1: LINEAR FUNCTIONS

EX:

$P(x) = 3$

$P(x) = 2x + 1$

$P(x) = \frac{4}{3}x^2 - 3x + 1$

$P(x) = 3x^4 - x^2 + \pi$ $\quad a_4 = 3 \quad a_3 = 0 \quad a_2 = -1 \quad a_0 = \pi$

$$P(x) = 0 \cdot x^5 - 3x^3 + 1 = \underset{a_5}{0 \cdot x^5} \underset{a_4}{+ 0x^4} \underset{a_3}{- 3x^3} \underset{a_2}{+ 0x^2} \underset{a_1}{+ 0x} \underset{a_0}{+ 1}$$

$$P_N(x) = \sum_{k=0}^{N} a_k x^k \quad x \in \mathbb{R} \qquad (x^0 = 1)$$

DEFINITION

A POLYNOMIAL OF DEGREE 2 IS **IRRIDUCIBLE** IF IT HAS **NO ROOTS**

FOR EXAMPLE

x^2+1 IS IRRIDUCIBLE

DEFINITION

LET $N \in \mathbb{N} \cup \{0\}$ LET $b_0, \dots b_N \in \mathbb{C}$

DEFINE

$$Q_N(z) = b_N z^N + b_{N-1} z^{N-1} + \dots + b_1 z + b_0$$

$$= \sum_{k=0}^{N} b_k z^k \quad z \in \mathbb{C}$$

($z^0 = 1$ BY CONVENTION)

COMPLEX POLYNOMIALS OF DEGREE AT MOST N

DEFINITION Q_N POLYNOMIAL OF DEGREE $N \geq 1$ $z_1 \in \mathbb{C}$

IS A ROOT OF Q_N IF $Q_N(z_1) = 0$ IN THIS CASE

$$Q_N(z) = (z - z_1)\, Q_{N-1}(z)$$

Q_{N-1} COMPLEX POLYNOMIAL OF DEGREE $N-1$

$\Rightarrow Q_N$ HAS AT MOST N ROOTS

EXAMPLE

$$Q(z) = 2z^2 - 3z + 1$$

$$Q(z) = z^2 - 1$$

$$Q(z) = (1+i)z^3 + z - 1 = \underset{b_3}{(1+i)}\, z^3 + \underset{b_2}{0}\, z^2 + \underset{b_1}{1}\, z + \underset{b_0}{(-i)}$$

FUNDAMENTAL THEOREM OF ALGEBRA

Q_N COMPLEX POLYNOMIAL OF DEGREE $N \geq 2$

THEN $\exists$ A ROOT z_1 OF Q_N, THAT IS, $\exists z_1 \in \mathbb{C}$

SUCH THAT $Q_N(z_1) = 0$

WE CONCLUDE $\exists z_1, \dots z_N$ ROOTS OF Q_N NOT NECESSARILY DIFFERENT, SUCH THAT

$$Q_N(z) = b_N (z - z_1) \dots (z - z_N)$$

OPERATION ON COMPLEX NUMBERS

LET $z = x + yi \in \mathbb{C}$ $x, y \in \mathbb{R}$. WE DEFINE

$\bar{z} = x - yi \in \mathbb{C}$ CONJUGATE OF z

$|z| = \sqrt{x^2 + y^2} \in \mathbb{R}$ MODULUS OF z

REMARK

- IF $z = x + 0i \in \mathbb{R}$ THEN $|z| = \sqrt{x^2 + 0^2} = \sqrt{x} = |x|$

- IF $z = x + yi \in \mathbb{C}$, THEN

$$|Re(z)| = |x| \leq |z| \text{ AND } |Im(z)| = |y| \leq |z|$$

PROPERTIES OF THE CONJUGATE $\forall z \in \mathbb{C}$

1) $\bar{\bar{z}} = z$

2) $\bar{z} = z \iff z \in \mathbb{R}$

$z_1 = x_1 + y_1 i \quad x_1\ y_1 \in \mathbb{R} \quad i = 1,2$

$z_2 = x_2 + y_2 i$

$z_1 = z_2 \iff x_1 = x_2$ AND $y_1 = y_2$

3) $z + \bar{z} = 2Re(z)$ AND $z - \bar{z} = 2Im(z)i$, SO

$$Re(z) = \frac{z+\bar{z}}{2} \quad Im(z) = \frac{z-\bar{z}}{2i} = \frac{-i}{2}(z-\bar{z})$$

4) $z \cdot \bar{z} = |z|^2$ HENCE

$$\forall z \neq 0 \quad z^{-1} = \frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

EXAMPLE

- $i^{-1} = \frac{1}{i} = \frac{-i}{0^2+i^2} = \frac{-1}{1} = -i$

- $(4-2i)^{-1} = \frac{4+2i}{4^2+(-2)^2} = \frac{4+2i}{20} = \frac{1}{5} + \frac{1}{10}i$

5) $\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$ AND $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$ $\forall z_1, z_2 \in \mathbb{C}$

PROPERTY OF THE MODULUS

1) $|z| \geq 0$ $\forall z \in \mathbb{C}$ AND $|z| = 0 \iff z = 0$

2) $|\lambda z| = |\lambda| |z|$ $\forall \lambda \in \mathbb{R}$ $\forall z \in \mathbb{C}$

IN PARTICULAR

$|z| = |-z|$

3) $|z_1+z_2| \leq |z_1| + |z_2|$ $\forall z_1, z_2 \in \mathbb{C}$ TRIANGLE INEQUALITY

4) $|\overline{z}| = |z|$

5) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

HENCE $\forall z \neq 0$ $|z^{-1}| = \left|\frac{\overline{z}}{|z|^2}\right| = \frac{1}{|z|^2} |\overline{z}| = \frac{|z|}{|z|^2} = \frac{1}{|z|}$

THERFORE IF $z_2 \neq 0$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

EXERCISE PROVE 1) AND 4)

PROOF: $|\lambda z| = \sqrt{(\lambda x)^2 + (\lambda y)^2} = \sqrt{\lambda^2 (x^2 + y^2)} =$

$= \sqrt{\lambda^2}\ \sqrt{x^2 + y^2} = |\lambda|\,|z|$

5) $z_1 = x_1 + y_1 i$ AND $z_2 = x_2 + y_2 i$ $\quad x_1 y_2 \in \mathbb{R}$

$|z_1 \cdot z_2|^2 = |(x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2) i|^2 =$

$= (x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + y_1 x_2)^2 =$

$= (x_1^2 + x_2^2) \cdot (y_1^2 + y_2^2)$

5) $z_1 = x_1 + y_1 i$ AND $z_2 = x_2 + y_2 i$ $\quad x_i, y_i \in \mathbb{R}$

$|z_1 \cdot z_2|^2 = |(x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2) i|^2 =$

$= (x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + y_1 x_2)^2 =$

$= x_1^2 x_2^2 + y_1^2 y_2^2 \cancel{- 2 x_1 x_2 y_1 y_2} + x_1^2 y_2^2 + y_1^2 x_2^2 \cancel{+ 2 x_1 y_2 y_1 x_2}$

$\underset{\substack{| \\ \text{CHECK}}}{=} (x_1^2 + y_1^2) \cdot (x_2^2 + y_2^2) = |z_1|^2 \cdot |z_2|^2$

3) $|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2}) =$

$= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) =$

$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2 =$

$= \overline{z_1 z_2} = \bar{z}_1 z_2$

$= |z_1|^2 + 2Re(z_1\bar{z}_2) + |z_2|^2 \leq$

$\leq |z_1|^2 + 2|Re(z_1\bar{z}_2)| + |z_2|^2 \leq$

$\leq |z_1|^2 + 2|z_1\tilde{z}_2| + |z_2|^2 =$

$= |z_1|^2 + 2|z_1| \cdot |z_2| + |z_2|^2 = (|z_1| + |z_2|)^2$

THE EUCLIDEAN SPACE $\mathbb{R}^2$, $N \geq 2$ $(\mathbb{R}^1 = \mathbb{R})$

$\vec{x} = (x_1, \dots x_N) \in \mathbb{R}^N$ $\quad x_1, \dots, x_N \in \mathbb{R}$

$x_1, \dots, x_N$ COORDINATES OF $\vec{x}$

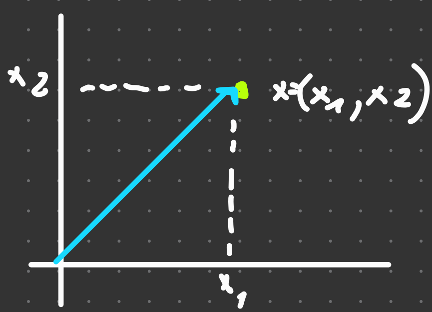
WARNING USUALLY WE JUST WRITE $x \in \mathbb{R}^2$ INSTEAD OF $\vec{x}$

IF $N = 2$ $\quad P = (x, y) \in \mathbb{R}^2 \quad x, y \in \mathbb{R}$

IF $N = 3$ $\quad P = (x, y, z) \in \mathbb{R}^3 \quad x, y, z \in \mathbb{R}$

$x = \vec{x} = (x_1, x_2, x_3) \qquad P = (x, y, z)$

$x \in \mathbb{R}^2$ REPRESENT BOTH A POINT IN $\mathbb{R}^2$ AND A VECTOR IN $\mathbb{R}^2$ ($N = 2$ PLANE, $N = 3$ SPACE)



OPERATORS ON VECTORS

- SUM $+$ $\quad x = (x_1 \dots x_N) \quad y = (y_1, \dots y_N) \in \mathbb{R}^2$

$$x + y = (x_1 + y_1 \dots x_N + y_N) \in \mathbb{R}^2$$

REMARKS ASSOCIATIVE, COMMUTATIVE

$0 = (0, 0, \dots, 0)$ NEUTRAL ELEMENT

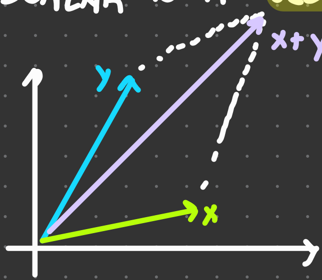
$-x = (-x_1, \dots -x_N)$ OPPOSITE

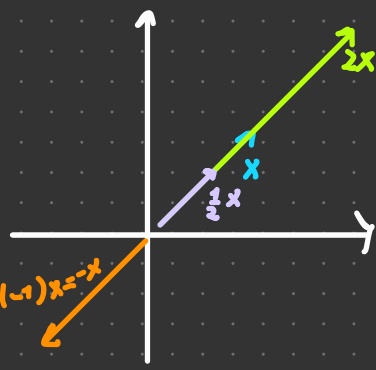
- MULTIPLICATION $\cdot$ BY A SCALAR $\quad x \in \mathbb{R}^2 \quad \lambda \in \mathbb{R}$

$$\lambda x = (\lambda x_1, \dots \lambda x_N) \in \mathbb{R}^2$$

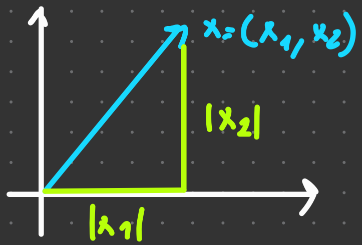
REMARKS $\quad -x = -1 \cdot x \qquad \underset{\mathbb{R}}{0} \cdot \underset{\mathbb{R}}{x} = \underset{\mathbb{R}}{0}$

NOTE $\mathbb{R}^2$ WITH THIS SUM AND THIS MULTIPLICATION BY A SCALAR IS A VECTOR SPACE





DEFINITION $\forall x = (x_1, \dots x_N) \in \mathbb{R}^2 \quad x_1, \dots x_N \in \mathbb{R}$ WE DEFINE $\|x\| = \sqrt{x_1^2 + \dots x_N^2}$ EUCLIDEAN NORM OF X (LENGTH OF VECTOR X)



IT SATISFIES

1) $\|x\| \geq 0 \quad \forall x \in \mathbb{R}^2$ AND $\|x\| = 0 \iff x = 0 \in \mathbb{R}^2$

2) $\|\lambda x\| = |\lambda| \, |x| \quad \forall \lambda \in \mathbb{R} \quad \forall x \in \mathbb{R}^2$

IN PARTICULAR

$$\|-x\| = \|x\|$$

3) $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in \mathbb{R}^2$ TRIANGLE INEQUALITY

EX PROVE 1, 2

DEFINITION

$\forall x, y \in \mathbb{R}^2$ WE DEFINE $d(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + \dots + (x_1 - y_1)^2}$

EUCLIDEAN DISTANCE BETWEEN X AND Y

IT SATISFIES

i) $d(x,y) \geq 0$ AND $d(x,y) = 0 \iff x = y$

ii) $d(x,y) = d(y,x)$ SYMMETRY $\forall x, y \in \mathbb{R}^2$

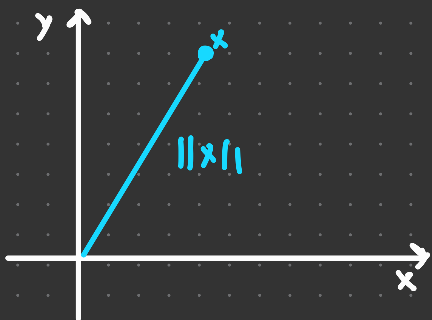
iii) $d(x,z) \leq d(x,y) + d(y,z)$ $\forall x, y, z \in \mathbb{R}^2$

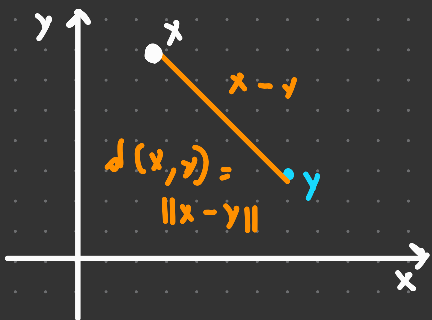
TRIANGLE INEQUALITY

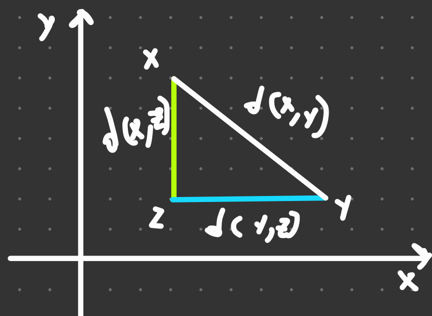
REMARK

$\|x\|$, $x \in \mathbb{R}^2$ LENGTH OF THE VECTOR x (ALSO THE DISTANCE OF THE POINT x FROM THE ORIGIN)

$d(x,y)$ IS THE LENGTH OF THE VECTOR CONNECTING x TO y







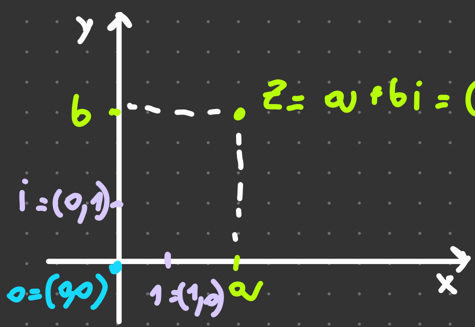
REMARK

I-II-III come from 1) 2) 3) as in $\mathbb{R}$ (with $|\cdot|$ replaced with $\|\cdot\|$)

GAUSS PLANE

We identify $\mathbb{C}$ with $\mathbb{R}^2$

$$\mathbb{C} \ni z = a + bi \longleftrightarrow (a, b) \in \mathbb{R}^2$$



NOTATION x-axis real axis

y-axis imaginary axis

REMARKS: sum of complex numbers = sum of vectors

- $\lambda \in \mathbb{R}$ $z \in \mathbb{C}$ λz multiplication by a scalar
- $x, y \in \mathbb{R}$ $P = (x, y)$ $z = x + yi$

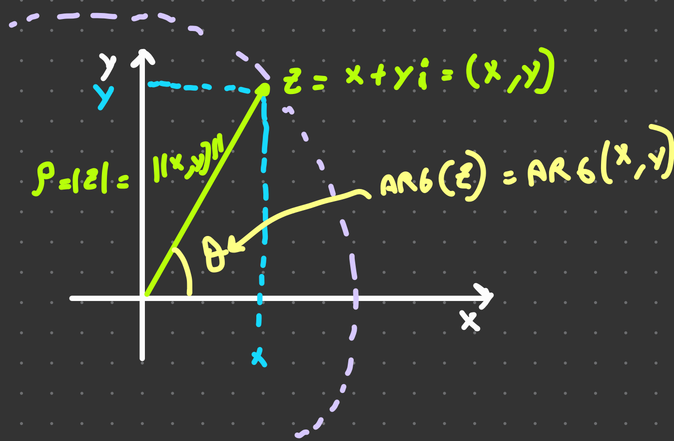
$$\|P\| = \sqrt{x^2 + y^2} = |z|$$

$\forall z \in \mathbb{C}$ $|z|$ is the distance in the Gaussian plane of z from the origin

NOTE $|z_1 + z_2| \leq |z_1| + |z_2|$ → TRIANGLE INEQUALITY FOR $\|\cdot\|$ IN $\mathbb{R}^2$!

POLAR COORDINATES IN $\mathbb{R}^2$ AND TRIGONOMETRIC REPRESENTATION OF COMPLEX NUMBER

LET $z = x + yi \in \mathbb{C}$ AND $(x, y) \in \mathbb{R}^2$ $x, y \in \mathbb{R}$



DEFINITION

$\rho = |z| = \|(x, y)\|$ IS THE DISTANCE FROM THE ORIGIN

$\theta = \mathrm{ARG}(z) \in \mathbb{R}$ (WITH $z \neq 0$) ARGUMENT. ITS THE ANGLE OF THE LINE PASSING THROUGH O AND z WITH RESPECT TO THE POSITIVE AXIS

CARTESIAN FORM OF z (CARTESIAN COORDINATE OF $P \in \mathbb{R}^2$)
$P \in \mathbb{R}^2$

LET $z \in \mathbb{C}$

x, y REAL AND θ

IMAGINARY PART OF z

x COORDINATE AND y-COORDINATE OF P $\quad P = (x, y)$

$$z = x + yi$$

TRIGONOMETRIC FORM OF z (POLAR COORDINATES OF $P \in \mathbb{R}^2$)

LET $z \in \mathbb{C}$ $\quad P \in \mathbb{R}^2$

$\rho = |z| \geq 0$ $\quad \rho = \|P\| \geq 0$

$\theta = ARG(z)$ $\quad \theta = ARG(P)$

THEN

$$z = \rho(\cos\theta + i\sin\theta) \qquad P = (\rho\cos\theta, \rho\sin\theta)$$

TRIGONOMETRIC FORM OF z

REMARK

IF $z = 0 = (0,0) = P$ THEN

$\rho = 0$ AND $ARG(z) = ARG(P)$ IS ANY $\theta \in \mathbb{R}$

IF $z \neq 0$ $\quad P \neq (0,0)$

THE ARGUMENT IS NOT UNIQUELY DEFINED!

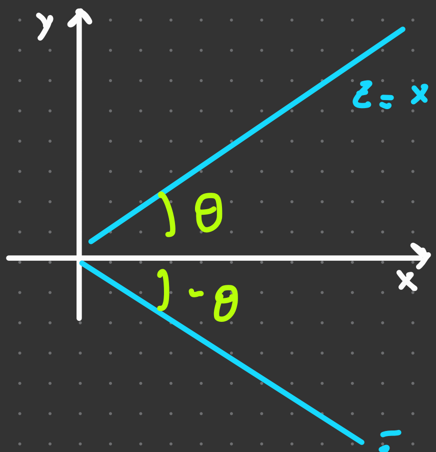
$z_1, z_2 \neq 0 \quad z_1 = \rho_1(\cos\theta_1 + i\sin\theta_1) \quad \rho_1 > 0 \quad \theta_1 \in \mathbb{R}$

$\qquad\qquad\quad z_2 = \rho_2(\cos\theta_2 + i\sin\theta_2) \quad \rho_2 > 0 \quad \theta_2 \in \mathbb{R}$

$z_1 = z_2 \Leftrightarrow \rho_1 = \rho_2$ AND $\theta_1 = \theta_2 + 2k\pi$ FOR SOME $k \in \mathbb{Z}$

IF $z \neq 0$ THE ARGUMENT IS UNIQUE IF WE CONSIDER $\theta \in [0, 2\pi]$ (PRINCIPAL ARGUMENT) OR $\theta \in [-\pi, \pi]$

REMARK OF CONJUGATE



$|z| = |\bar{z}| = \rho$

$\arg(\bar{z}) = -\arg(z)$

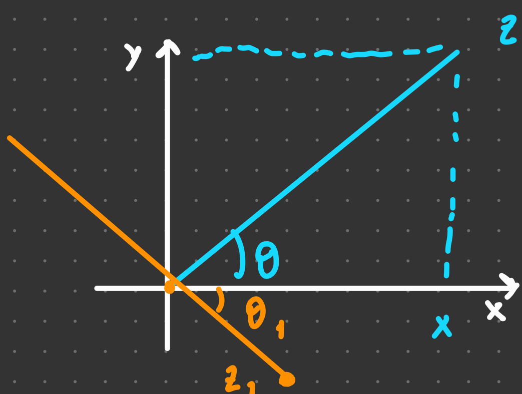
$z = \rho(\cos\theta + i\sin\theta)$ $\quad \bar{z} = \rho(\cos(-\theta) + i\sin(-\theta))$

ISSUE $z \neq 0$ $(\rho \neq 0)$

HOW TO COMPUTE THE TRIGONOMETRIC FORM FOR z?

POLAR COORDINATES (ρ, θ) OF P?

$\rho = |z| = \|P\|$ WHAT ABOUT THE ARGUMENT



$x > 0$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$

HENCE $\theta = \arctan\left(\frac{y}{x}\right)$

$x = 0 \quad y > 0 \quad \theta = \frac{\pi}{2}$

$x = 0 \quad y < 0 \quad \theta = \frac{3}{2}\pi$ OR $\theta = \frac{\pi}{2}$

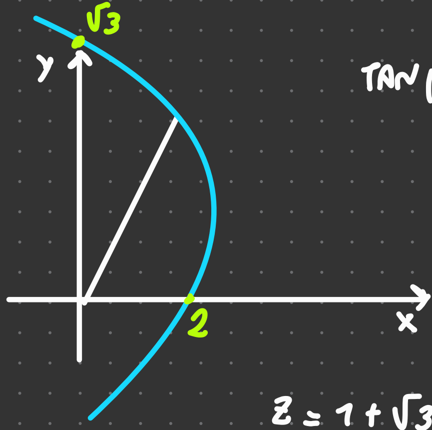
EX FIND THE FORMULA OF θ WHEN $x < 0$

$$\begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \end{cases} \qquad \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \theta = \text{ARG}((x, y)) = \arctan\left(\frac{y}{x}\right) \text{ IF } x > 0 \end{cases}$$

EXAMPLE: $z = 1 + \sqrt{3}i$ CARTESIAN FORM

TRIGONOMETRIC FORM OF z?

$|z| = \sqrt{1^2 + (\sqrt{3})^2} = 2 \qquad \theta = \arctan(\sqrt{3}) = \frac{\pi}{3}$

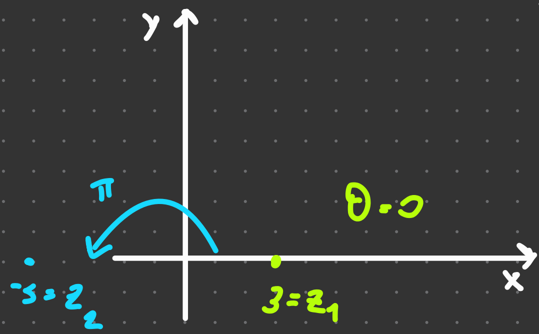


$\tan(\theta) = \sqrt{3}$

$z = 1 + \sqrt{3}i = 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$

- $z_1 = 3 \quad |z_1| = 3 \quad \theta = 0$
- $z_1 = 3 = 3 + 0i = 3(\cos(0) + i\sin(0))$
- $z_2 = -5 \quad |z_2| = 5 \quad \theta = \pi$

$z_2 = -5 = -5 + 0i = 5(\cos(\pi) + i\sin(\pi))$



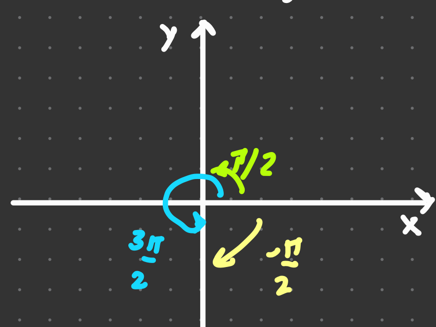
- $z_3 = i \quad |z_3| = 1 \quad \theta = \frac{\pi}{2}$

$z_4 = -2i \quad |z_4| = 2 \quad \theta = -\frac{\pi}{2}$ OR $\theta = \frac{3}{2}\pi$

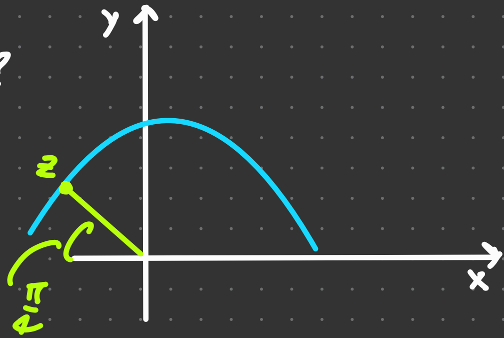
$z_3 = i = 1\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$

$z_4 = -2i = 2\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$

$= 2\left(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3}{2}\pi\right)\right)$



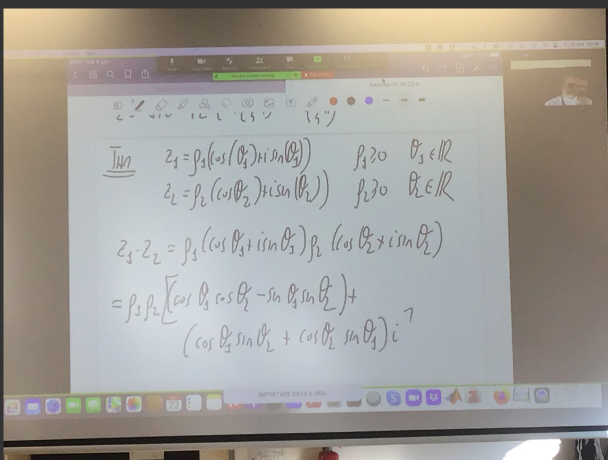
$z = -1+i \quad |z| = \sqrt{2} \quad \theta = ?$



$$z = -1+i = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$$

THR

$$z_1 = \rho_1(\cos(\theta_1) + \sin(\theta_1))$$



Thm $z_1 = \rho_1(\cos(\theta_1) + i\sin(\theta_1)) \quad \rho_1 \geq 0 \quad \theta_1 \in \mathbb{R}$

$z_2 = \rho_2(\cos(\theta_2) + i\sin(\theta_2)) \quad \rho_2 \geq 0 \quad \theta_2 \in \mathbb{R}$

$z_1 \cdot z_2 = \rho_1(\cos\theta_1 + i\sin\theta_1)\rho_2(\cos\theta_2 + i\sin\theta_2)$

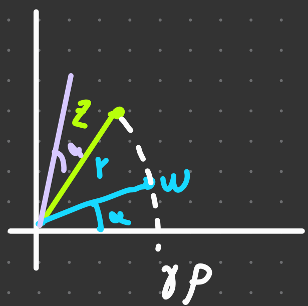
$= \rho_1\rho_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) +$
$(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)i]$

$$= \rho_1\rho_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

THR $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

AND

$\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$

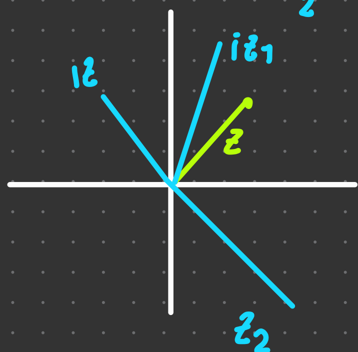


$w = r(\cos\alpha + i\sin\alpha)$

$z = \rho(\cos\theta + i\sin\theta)$

$i = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$

$i \cdot z$ ROTATION OF $\frac{\pi}{2}$ IN COUNTERCLOCKWISE SENSE



$(-i)z$ = ROTATION OF $\frac{\pi}{2}$ IN CLOCKWISE SENSE

$(-1) = 1\,(\cos\pi + i\sin\pi) = 1 \cdot (\cos(-\pi) + i\sin(-\pi))$

$(-1) \cdot z$ ROTATION OF π IN COUNTERCLOCKWISE OR CLOCKWISE SENSE

DE MOIVRE FORMULA

- $\forall n \in \mathbb{N} \quad 0^n = 0$
- $z \in \mathbb{C} \quad z \neq 0 \quad z = \rho(\cos\theta + i\sin\theta) \quad \rho > 0 \quad \theta \in \mathbb{R}$

THEN

$$z^{-1} = \frac{1}{z} = \frac{1}{\rho}\left(\cos(-\theta) + i\sin(-\theta)\right)$$

IN FACT $\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{\rho\left(\cos(-\theta) + i\sin(-\theta)\right)}{\rho^2}$

$$z^2 = \rho(\cos\theta + i\sin\theta)\,\rho(\cos\theta + i\sin\theta)$$
$$= \rho^2\left(\cos(\underset{\theta+\theta}{\underset{\parallel}{2\theta}}) + i\sin(\underset{\theta+\theta}{\underset{\parallel}{2\theta}})\right)$$

$$\boxed{z^k = \rho^k\left(\cos(k\theta) + i\sin(k\theta)\right) \quad \forall k \in \mathbb{Z}}$$

DE MOIVRE FORMULA