Written exam of Calculus - Part 1 - Sample 2

IT IS FORBIDDEN TO USE CALCULATORS, BOOKS, NOTES, AND SO ON

TIME ALLOWED: 90 MINUTES

NAME:.....................................**ID NUMBER:**.....................................

SURNAME:..

PART A Write **only** the answer.

1.1 (3 POINTS)

Let z be the complex number satisfying

$$\frac{z}{1-i} = \frac{1}{2i}.$$

Compute the conjugate of z. Write the solution in Cartesian form.

ANSWER: $\overline{z} =$ $-\frac{1}{2} - \frac{1}{2}i$

$z = \frac{1-i}{2i}$; $z = \frac{1}{2i} - \frac{i}{2i}$; $z = \frac{1}{2i} - \frac{1}{2}$

$1 = i^4$

$z = \frac{1 i^3}{2} - \frac{1}{2} i^4$

$i^2 = -1$

$z = \frac{i^3}{2} - \frac{1}{2}$; $z = -\frac{1}{2} - \frac{1 i}{2}$

1.2 (3 POINTS)

Let $f(x) = \sin\big(\log(x^2-3)\big)$. Write the equation of the tangent line to the graph of f at the point $(2, 0)$.

ANSWER:

$y = m(x-2)$

$f'(x) = \frac{2x}{x^2-3} \cos(\log(x^2-3))$

$y = 4x - 8$

$f'(2) = \frac{4}{1} = 4$

1.3 (3 POINTS)

Let $A = \{x \in \mathbb{R} : 0 < x^2 \leq 4\}$. Determine if A is open. Determine all the accumulation points of A.

ANSWERS: $(-2, 0) \cup (0, 2)$

$(0, 4]$

PART B Write a **complete** solution.

1.4 (8 POINTS)

Let, for the parameter $a \in \mathbb{R}$,

$$f(x) = \begin{cases} a + a^2 x & \text{se } x \leq 0 \\ \dfrac{\sqrt{x}\sin(\sqrt{x})\log\left((1+3x)^2\right)}{x^2 + x^4} & \text{se } x > 0 \end{cases}$$

Determine, if they exist, the values of $a \in \mathbb{R}$ such that the function f is continuous on the whole $\mathbb{R}$.

Another possible example: Let $f : \mathbb{R} \to \mathbb{R}$ be such that for any $x \in \mathbb{R}$

$$f(x) = \begin{cases} x^2(\log(|x|) - 1) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Determine in which points $x \in \mathbb{R}$ the function is continuous. Determine in which points $x \in \mathbb{R}$ the function is derivable and in these points compute the derivative.

SOLUTION:

$$f(0) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) \qquad f(0) = a$$

$$\lim_{x \to 0^-} a + a^2 x$$

$$\frac{\sin(\sqrt{x})}{\sqrt{x}} = 1$$

$$\lim_{x \to 0^+} \frac{\sqrt{x}\sin(\sqrt{x})\log(1 + 9x^2 + 6x)}{x^2 + x^4} \cdot \frac{\sqrt{x}}{\sqrt{x}}$$

$$\lim_{x \to 0^+} \frac{x\log(1 + 9x^2 + 6x)}{x^2 + x^4} = \lim_{x \to 0^+} \frac{\log(1 + 9x^2 + 6x)}{x + x^3} \quad \left[\frac{0}{0}\right] \text{ APPLY HOPITAL} = \lim_{x \to 0^+} \frac{\frac{18x + 6}{1 + 9x^2 + 6x}}{1 + 3x^2}$$

$$\lim_{x \to 0^+} \frac{18x + 6}{1 + 9x^2 + 6x} \cdot \frac{1}{1 + 3x^2} = \lim_{x \to 0^+} \frac{18x + 6}{1 + 9x^2 + 6x + 3x^2 + 27x^4 + 18x^3} = \boxed{6} \qquad a = 6$$

$$\lim_{x \to 0^-} a + a^2 x = \lim_{x \to 0^-} 6 + 36x = \boxed{6}$$

SO IF $a = 6$ THE FUNCTION IS CONTINUOUS

1.5 (8 POINTS)

Determine, if it exists,

$$\min_{x\in[1/8,8]} f(x)$$

where $f(x) = -x^{2/3}(x-1)^2$. Determine also all minimizers.

Another possible example: $\min_{x\in[-2,1/2]} f(x)$ where $f(x) = \dfrac{2x^2-x+1}{1-x}$.

SOLUTION:

$$\min_{x\in[\frac{1}{8},8]} -\sqrt[3]{x^2}\,(x^2+1-2x)$$

$$f\left(\tfrac{1}{8}\right) = -\frac{1}{\sqrt[3]{64}}\left(\frac{1}{64}+1-\frac{1}{4}\right) = -\frac{1}{\sqrt[3]{64}}\cdot\left(\frac{1+64-16}{64}\right) = -\frac{1}{\sqrt[3]{64}}\left(\frac{49}{64}\right) = -\frac{1}{4}\cdot\frac{49}{64}$$

$$f(8) = -\sqrt[3]{64}\,(49) = -4\cdot 49 = \boxed{-196}$$

$$f(x) = -x^{2/3}(x^2+1-2x)$$

$$f'(x) = -\frac{2}{3}x^{-1/3}(x^2+1-2x) - x^{2/3}(2x-2)$$

$$f'(x) = -\frac{2}{3}\frac{1}{\sqrt[3]{x}}(x^2+1-2x) - \sqrt[3]{x^2}(2x-2)$$

$$f'(x) = \frac{-2x^2-2+4x-3\sqrt[3]{x^2}\cdot\sqrt[3]{x}\,(2x-2)}{3\sqrt[3]{x}} \qquad x^{2/3}\cdot x^{1/3} =$$

$$f'(x) = \frac{-2x^2-2+4x-6x^2+6x}{3\sqrt[3]{x}}$$

$100-4(-8)(-2)$ $\qquad$ 100 / 64 / 36

$$f'(x) = \frac{-8x^2+10x-2}{3\sqrt[3]{x}}$$

SIGN

$\frac{1}{4}$ $\quad 1$ $\qquad \dfrac{-10\pm 6}{-16} < \begin{matrix}1\\ \frac{1}{4}\end{matrix}$

$3\sqrt[3]{x} > 0$; $x > 0$

	0		1/4		1	
−		−		+		−
−		+		+		+
+	∄	−	•	+	•	−
↗	•	↘	•	↗	•	↘

MIN $\left(\frac{1}{4}\right)$ $\qquad f\left(\frac{1}{4}\right) = -\sqrt[3]{x^2}\,(x^2+1-2x)$

$$-\sqrt[3]{\tfrac{1}{16}}\left(\frac{1}{16}+1-\frac{1}{2}\right)$$

$$-\frac{1}{\sqrt[3]{16}}\left(\frac{1+16-8}{16}\right)$$

$$-\frac{1}{\sqrt[3]{16}}\left(\frac{9}{16}\right)$$

1.6 (7 POINTS)

State the Theorem on the derivability of the composition of functions and write the formula of its derivative

STATEMENT AND FORMULA:

1) $F(x) = f(x) + g(x)$ PROVE $F'(x) = f'(x) + g'(x)$. GIVEN $F'(x) = \lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0}$ THEN

$F'(x) = \lim_{x \to x_0} \frac{f(x) + g(x) - f(x_0) - g(x_0)}{x - x_0}$ THAT IS $F'(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}$ SO $F'(x) = f'(x) + g'(x)$

2) $F(x) = f(x) \cdot g(x)$ PROVE $F'(x) = f'(x) g(x) + f(x) g'(x)$ SINCE $F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$ THEN

$F'(x) = \lim_{h \to 0} \frac{f(x+h) g(x+h) - f(x) g(x)}{h}$ ADD AND SUBTRACT $g(x+h) f(x)$ ON THE NOMINATOR

$F'(x) = \lim_{h \to 0} \frac{f(x+h) g(x+h) - f(x) g(x+h) + f(x) g(x+h) - f(x) g(x)}{h} = \lim_{h \to 0} \frac{g(x+h)\big(f(x+h) - f(x)\big)}{h} + \lim_{h \to 0} \frac{f(x)\big(g(x+h) - g(x)\big)}{h}$

THAT IS $F'(x) = g(x) f'(x) + f(x) g'(x)$

3) $F(x) = \frac{f(x)}{g(x)}$ PROVE THAT $F'(x) = \frac{f'(x) g(x) - f(x) g'(x)}{g(x)^2}$ $F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$

SO $F'(x) = \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h^2} = \lim_{h \to 0} \frac{f(x+h) g(x) - f(x) g(x+h)}{g(x+h) g(x) h^2} = \frac{1}{g(x)^2} \lim_{x \to x_0} \frac{f(x+h) g(x) - f(x) g(x+h)}{h^2}$

ADD AND SUBTRACT FROM THE NOMINATOR $f(x) g(x)$

$\frac{1}{g(x)^2} \lim_{h \to 0} \frac{f(x+h) g(x) - f(x) g(x) + f(x) g(x) - f(x) g(x+h)}{h^2} = \frac{1}{g(x)^2} \lim_{h \to 0} \frac{g(x)\big(f(x+h) - f(x)\big) - f(x)\big(g(x+h) - g(x)\big)}{h^2}$

THAT IS

$\frac{1}{g(x)^2} \cdot \big(g(x) f'(x) - f(x) g'(x)\big) \Rightarrow F'(x) = \frac{f'(x) g(x) - f(x) g'(x)}{g(x)^2}$

Determine, if they exist, the values of $a \in \mathbb{R}$ such that the function f is continuous on the whole $\mathbb{R}$.

Another possible example: Let $f : \mathbb{R} \to \mathbb{R}$ be such that for any $x \in \mathbb{R}$

$$f(x) = \begin{cases} x^2(\log(|x|) - 1) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Determine in which points $x \in \mathbb{R}$ the function is continuous. Determine in which points $x \in \mathbb{R}$ the function is derivable and in these points compute the derivative.

$$f(x) = \begin{cases} x^2(\log|x|) - 1 & \text{IF } x \neq 0 \\ 0 & \text{IF } x = 0 \end{cases}$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0)$$

$$\lim_{x \to 0^+} x^2(\log|x|) - 1 = \left(\lim_{x \to 0^+} x^2(\log|x|)\right) - 1 = \left(\lim_{x \to 0^+} x^2 \log(x)\right) - 1 = \lim_{x \to 0} 0$$

$$\lim_{x \to 0^-} x^2(\log|x|) - 1 = \lim_{x \to 0^+} \frac{x^3(\log|x|) - x}{x} = 0$$

$\to 1$