# Exercises - Calculus

Academic Year 2021-2022

Sheet 17

1. For any of the following functions $f$, determine all stationary point and establish, if possible, if they are local minimum point, local maximum points or saddle points.
   (a) $f(x, y) = x^3 - 3x - y^2$
   (b) $f(x, y) = e^{x^2y - y^2 - y}$
   (c) $f(x, y) = x^4/4 - x^3/3 - x^2 + y^4 - y^2$
   (d) $f(x, y) = x^4 + 3x^2 + 2y^2 + xy - 4$
   (e) $f(x, y, z) = x^2 - 2x + y^3 - y + z^6 + z^4 - z^2$
   (f) $f(x, y, z) = \log(1 + x^2 - x + y^2 + z^2)$
2. Determine the tangent line to the curve $(1 + x)y\cos(y) + x^2 + e^x = 1$ in the point $(0, 0)$ passing through $(0, 0)$.
3. Determine the tangent line to the curve $x\arctan(x)y - (\pi/4)e^{y-1} = 0$ in the point $(1, 1)$ passing through $(1, 1)$.
4. Determine the tangent plane to the surface $x^2 + y^2 - z^2 = 9$ in the point $(0, 5, 4)$ passing through $(0, 5, 4)$.
5. Determine the tangent plane to the surface $y^3 - xe^{zy} + z^2y^2 + e^x + z = 5$ in the point $(0, 0, 4)$ passing through $(0, 0, 4)$.
6. Determine the tangent plane to the level set at level 2 of the function $F(x, y, z) = x^3 - xy^2 + e^{zx} + \cos(y - 1)$ in the point $(0, 1, 3)$ passing through $(0, 1, 3)$ and in the point $(1, 1, 0)$ passing through $(1, 1, 0)$.
7. Let $f : A \subset \mathbb{R}^3 \to \mathbb{R}$ be differentiable in $A$, $A$ open. Let $F : A \subset \mathbb{R}^3 \to \mathbb{R}$ and let $C_1 = \{x \in A : F(x) = 0\}$. Suppose that $F$ is given by $F(x, y, z) = x^2 - y^2 - z$ and note that $P_0 = (2, 1, 3) \in C_1$. Assuming that $f(P_0) = \min_{C_1} f$ and that $\dfrac{\partial f}{\partial x}(P_0) = 2$, compute $\nabla f(P_0)$.

X TROVARE PIANO TANG AD UNA SUPERFICE

$$\frac{df}{dx}(P_0)(x - x_0) + \frac{df}{dy}(P_0)(y - y_0) + \frac{df}{dz}(P_0)(z - z_0) = 0$$

$x^2 + 3y + z = 0 \Rightarrow$ EQU. DI UN PIANO

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(b) $f(x, y) = e^{x^2 y - y^2 - y}$
(c) $f(x, y) = x^4/4 - x^3/3 - x^2 + y^4 - y^2$
(d) $f(x, y) = x^4 + 3x^2 + 2y^2 + xy - 4$
(e) $f(x, y, z) = x^2 - 2x + y^3 - y + z^6 + z^4 - z^2$
(f) $f(x, y, z) = \log(1 + x^2 - x + y^2 + z^2)$

1. DERIVATE PARZIALI

$$f(x,y) = x^3 - 3x - y^2$$

$$\frac{df}{dx}(x,y) = 3x^2 - 3 \qquad \frac{df}{dy}(x,y) = -2y$$

2. GRADIENTE $\nabla f(x,y)$

$$\nabla f(x,y) = \left(\frac{df}{dx}(x,y), \frac{df}{dy}(x,y)\right) = (3x^2 - 3, -2y)$$

3. CERCARE I PUNTI STAZIONARI DI $f$ CHE ANNULLANO IL GRADIENTE DI $f$

$$\begin{cases} \frac{df}{dx}(x,y) = 0 \\ \frac{df}{dy}(x,y) = 0 \end{cases} \qquad \begin{cases} 3x^2 - 3 = 0 \,; \quad x = \pm 1 \\ -2y = 0 \,; \quad y = 0 \end{cases}$$

TROVO DUE PUNTI

$A = (1, 0)$

$B = (-1, 0)$

4. CONTROLLARE CHE $A$ e $B \in \mathrm{DOM}(f)$

5. CALCOLARE LA MATRICE HESSIANA $Hf(x,y)$

5.1. SVOLGERE LE DERIVATE SECONDE

$$\frac{df}{dx}(x,y) = 3x^2 - 3 \qquad \frac{df}{dy}(x,y) = -2y$$

$$\frac{d^2f}{dxdx}(x,y) = 6x \qquad \frac{d^2f}{dydy}(x,y) = -2$$

$$\frac{d^2f}{dxdy}(x,y) = 0 \qquad \frac{d^2f}{dydx}(x,y) = 0$$

MATRICE HESSIANA

$$Hf(x,y) = \text{"}\begin{bmatrix} xx & xy \\ yx & yy \end{bmatrix}\text{"} = \begin{bmatrix} 6x & 0 \\ 0 & -2 \end{bmatrix}$$

6 HESSIANO IN $A$ e $B$

$$Hf(1,0) = \begin{bmatrix} 6 & 0 \\ 0 & -2 \end{bmatrix} \qquad Hf(-1,0) = \begin{bmatrix} -6 & 0 \\ 0 & -2 \end{bmatrix}$$

7 CALCOLARE IL DETERMINANTE $\det(H_f(x,y))$

$$\det(H_f(x,y)) = "(xx \cdot yy) - (xy \cdot yx)"$$

$$\det(H_f(1,0)) = -12$$

$$\det(H_f(-1,0)) = 12$$

8 ELABORARE

| PUNTO | DETERMINA | "PRIMO ELEMENTO" |
| --- | --- | --- |
| $A=(1,0)$ | $-$ | $+$ |
| $B(-1,0)$ | $+$ | $-$ |

A PUNTO DI SELLA

B MASSIMO RELATIVO

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$$f(x,y) = e^{x^2y-y^2-y} \qquad \mathrm{DOM}(f(x,y)) \in \mathbb{R}$$

$$\frac{df}{dx}(x,y) = 2xye^{x^2y-y^2-y} \qquad \frac{df}{dy}(x,y) = (x^2-2y-1)e^{x^2y-y^2-y}$$

$$\begin{cases} \frac{df}{dx}(x,y)=0 \\ \frac{df}{dy}(x,y)=0 \end{cases} \qquad \begin{cases} 2xy\, e^{x^2y-y^2-y}=0 \\ (x^2-2y-1)\, e^{x^2y-y^2-y}=0 \end{cases}$$

$(0,-1/2)$

$(-1,0)$ $(1,0)$

$$\boxed{\begin{aligned} A &= (0,-1/2) \\ B &= (-1,0) \\ C &= (1,0) \end{aligned}}$$

$$\frac{df}{dxdx}(x,y) = 2y\, e^{x^2y-y^2-y} + 2xy(2xy)\, e^{x^2y-y^2-y} = e^{x^2y-y^2-y}(2y+4x^2y^2)$$

$$\frac{df}{dydy}(x,y) = -2e^{x^2y-y^2-y} + (x^2-2y-1)(x^2-2y-1)e^{x^2y-y^2-y} = e^{x^2y-y^2-y}(-2+(x^2-2y-1)^2)$$

$$\frac{df}{dxdy}(x,y) = 2x(e^{x^2y-y^2-y}) + 2xy\,(2x^2-2y-1)\, e^{x^2y-y^2-y} = e^{x^2y-y^2-y}(2x+2xy(2x^2-2y-1))$$

$$\frac{df}{dydx}(x,y) = 2x(e^{x^2y-y^2-y}) + (x^2-2y-1)(2xy)e^{x^2y-y^2-y} = e^{x^2y-y^2-y}(2x+(x^2-2y-1)\,2xy)$$

$$H_f(x,y) = (4x^2y^2+2y)\, e^{x^2y-y^2-y} \qquad 2x+2x^3y-4xy^2-2xy$$

A

$$H_f(0,-1/2) = \begin{bmatrix} -e^{-3/4} & 0 \\ 0 & -2e^{-3/4} \end{bmatrix}$$

$$\det(H_f(0,-1/2)) = 2e^{-3/2}$$

MASSIMO RELATIVO

B

$$H_f(-1,0) = \begin{bmatrix} 0 & -2 \\ -2 & -2 \end{bmatrix}$$

$$\det(H_f(-1,0)) = -4$$

SELLA

C

$$H_f(1,0) = \begin{bmatrix} 0 & 2 \\ 2 & -2 \end{bmatrix}$$

$$\det(H_f(1,0)) = -4$$

SELLA

TEOREMA DEL DINI $\qquad P_0 = (x_0, y_0)$

$$f(x_0, y_0) = 0 \qquad \frac{df}{dy}(x_0, y_0) \neq 0$$

$$\Downarrow$$

$$y(x_0) = y_0 \; y'(x_0) = - \frac{\frac{df}{dx}(x_0, y_0)}{\frac{df}{dy}(x_0, y_0)} = m$$

TEO "GENERALE"

$x^2 + y^2 = 1$ TAN IN $P_0\left(1/2, \sqrt{3}/2\right)$

$$y = \sqrt{1 - x^2}$$

$x = 1/2 \qquad y'(x) = f'(x) = \frac{-2x}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}} \Rightarrow f'\left(\tfrac{1}{2}\right) = -1/\sqrt{3} = m$

$$\left(y - \sqrt{3}/2\right) = -1/\sqrt{3}\left(x - 1/2\right)$$

2. Determine the tangent line to the curve $(1 + x)y\cos(y) + x^2 + e^x = 1$ in the point $(0, 0)$ passing through $(0, 0)$.

$f(x, y) = (1 + x)y\cos(y) + x^2 + e^x - 1 \qquad P_0 = (0, 0)$

TEST DINI

$f(0,0) = 0 \qquad \frac{df}{dy}(x,y) = (1+x)\cos(y) - (1+x)\sin(y) \qquad \frac{df}{dx}(x,y) = y\cos(y) + 2x + e^x$

$$\frac{df}{dy}(0,0) = 1 \neq 0$$

I CAN APPLY DINI THEOREM

$$y(x_0) = y_0 \; y'(x_0) = - \frac{\frac{df}{dx}(x_0, y_0)}{\frac{df}{dy}(x_0, y_0)} = - \frac{1}{1} = -1$$

$$y - 0 = m(x - 0) \Rightarrow \boxed{y = -x}$$

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$$\langle \nabla f(x_0, y_0), (x - x_0, y - y_0) \rangle = 0$$

$$f(x, y) = (1 + x)y\cos(y) + x^2 + e^x - 1$$

$$\frac{df}{dx}(x, y) = 1(y\cos y) + 2x + e^x \to 1$$

$$\frac{df}{dy}(x, y) = (1 + x)[\cos y - y\sin(y)] \underset{(0,0)}{\to}$$

3. Determine the tangent line to the curve $x \arctan(x)y - (\pi/4)e^{y-1} = 0$ in the point $(1,1)$ passing through $(1,1)$.

$$f(x,y) = x\arctan(x)y - \left(\frac{\pi}{4}\right)e^{y-1} \qquad P = (1,1)$$

TEST DINI

$$f(1,1) = \pi/4 - \pi/4 = 0$$

NO DINI :(

$$\frac{df}{dy}(x,y) = x\arctan(x) - \left(\frac{\pi}{4}\right)e^{y-1} \qquad \frac{df}{dy}(1,1) = \pi/4 - \pi/4 = 0$$

$$\frac{\pi}{4}e^{y-1} = x\arctan(x)y$$

$$\frac{e^{y-1}}{y} = \frac{4}{\pi}x\arctan(x)$$

$$e^{y-1} \cdot y^{-1} = \frac{4}{\pi}x\arctan(x)$$

$$e^{y-1} \cdot e^{-\ln y} = \frac{4}{\pi}x\arctan(x)$$

$$e^{y-1-\ln y} = \frac{4}{\pi}x\arctan(x)$$

$$y - \ln y = \ln\left(\frac{4}{\pi}\arctan(x)\right) + 1$$

4. Determine the tangent plane to the surface $x^2 + y^2 - z^2 = 9$ in the point $(0,5,4)$ passing through $(0,5,4)$.

STEP 1. FIND PARTIAL DERIVATES OF $f(x,y) = x^2 + y^2 - z^2 - 9$

$$\frac{df}{dx}(x,y,z) = 2x \quad \Rightarrow \frac{df}{dx}(P_0) = 0$$

$$\frac{df}{dy}(x,y,z) = 2y \quad \Rightarrow \frac{df}{dy}(P_0) = 10$$

$$\frac{df}{dz}(x,y,z) = -2z \quad \Rightarrow \frac{df}{dy}(P_0) = -8$$

STEP 2. FIND THE EQUATION OF THE PLANE

$$T = \frac{df}{dx}(P_0)(x - x_0) + \frac{df}{dy}(P_0)(y - y_0) + \frac{df}{dz}(P_0)(z - z_0)$$

$$T = 0(x-0) + 10(y-5) - 8(z-4)$$

$$T = 10y - 50 - 8z + 32$$

$$T = 10y - 8z - 18$$

5. Determine the tangent plane to the surface $y^3 - xe^{zy} + z^2y^2 + e^x + z = 5$ in the point $(0, 0, 4)$ passing through $(0, 0, 4)$.

$$f(x,y,z) = y^3 - xe^{zy} + z^2y^2 + e^x + z - 5$$

$$\frac{df}{dx}(x,y,z) = -e^{zy} + e^x \qquad \frac{df}{dx}(P_0) = 0$$

$$\frac{df}{dy}(x,y,z) = 3y^2 - zxe^{zy} + 2z^2y \qquad \frac{df}{dy}(P_0) = 0$$

$$\frac{df}{dz}(x,y,z) = -xye^{zy} + 2zy^2 + 1 \qquad \frac{df}{dz}(P_0) = 1$$

$$T = \frac{df}{dx}(P_0)(x-x_0) + \frac{df}{dy}(P_0)(y-y_0) + \frac{df}{dz}(P_0)(z-z_0)$$

$$T = z - 4$$