$+$ $\times$

$-$ $\div$

COMPLEX ROOTS

DEFINITION

LET $n \in \mathbb{N}$ $n \geq 2$ LET $w \in \mathbb{C}$
WE SAY THAT $z \in \mathbb{C}$ IS A COMPLEX n^{TH}-ROOTH OF w IF IT SOLVES

$$z^n = w$$

EX:

$$z^5 = 2+i$$

COMPLEX ROOT

THE NOTATION $\sqrt[n]{w}^{\,\mathbb{C}}$ DENOTES ALL POSSIBLES n-ROOT OF w

REMARK

$$z^n = 0 \iff z = 0 \quad \forall n \geq 1$$

SO FROM NOW ON WE DEFINE $w \in \mathbb{C}$, $n \neq 0$

IMPORTANT REMARK

IN $\mathbb{R}$, $\sqrt{4} = 2$ AND $\nexists \sqrt{-4}$

IN $\mathbb{C}$, $\sqrt[\mathbb{C}]{4} = \pm 2 \quad (z^2 = 4)$

$$\sqrt[\mathbb{C}]{-4} = \pm 2i \quad (z^2 = -4)$$

$w \mapsto \sqrt[n]{w}^{\,\mathbb{C}}$ IS NOT A FUNCTION!

IN GENERAL, FOR ANY $a \in \mathbb{R}$ $a > 0$

$$\sqrt[\mathbb{C}]{a} = \pm\sqrt{a} \quad \text{AND} \quad \sqrt[\mathbb{C}]{-a} = \pm\sqrt{a}\, i$$

REMARK

$$z^N = w \iff z^N - w = 0$$

$\Rightarrow \exists$ AT <u>MOST</u> N DIFFERENT SOLUTIONS!

$$(b_N = 1,\ b_{N-1} = \dots b_1 = 0\ \ b_0 = w)$$

THEOREM

$N \geq 2$ $w \in \mathbb{C}$ $w \neq 0$ THERE EXIST N DIFFERENT N-TH ROOTS OF w, $z_0, \dots z_{N-1}$

IF $w = \rho(\cos\theta + i\sin\theta)$, $\rho > 0$ $\theta \in \mathbb{R}$ THEN

$$z_k = \rho^{1/N}\left(\cos\left(\frac{\theta}{N} + \frac{2k\pi}{N}\right) + i\sin\left(\frac{\theta}{N} + \frac{2k\pi}{N}\right)\right) \quad k = 0, 1, \dots N-1$$

IN THE GAUSS PLANE THE POINTS $z_0, \dots z_{N-1}$ FORM A REGULAR POLYGON WITH N SIDES INSCRIBED IN THE CIRCLE CENTRED AT 0 WITH RADIUS $\rho^{1/N}$

NOTATION $w = 1$ $z^N = 1$

ITS SOLUTIONS ARE CALLED N-TH ROOTS OF UNITY

REMARK

$N=2 \quad z_1 = -z_0$

PROOF $\quad w = \rho(\cos\theta + i\sin\theta) \quad \rho > 0$

$z = \gamma(\cos\alpha + i\sin\alpha) \quad \gamma > 0$

$z^N = w \iff$

$z^N = \gamma^N(\cos(N\alpha) + i\sin(N\alpha)) = \rho(\cos\theta + i\sin\theta)$

$\iff$

$$\boxed{\gamma^N = \rho \text{ AND } N\alpha = \theta + 2k\pi, \text{ FOR SOME } k \in \mathbb{Z}}$$

$$z_k = \rho^{1/N}\left(\cos\left(\frac{\theta}{N} + \frac{2k\pi}{N}\right) + i\sin\left(\frac{\theta}{N} + \frac{2k\pi}{N}\right)\right), \quad k \in \mathbb{Z}$$

$z_N = z_0$ AND $z_0, \dots z_{N-1}$ ARE DIFFERENT AND THERE CAN NOT BE ANY MORE SOLUTIONS!

EXAMPLE

$w = 1 + \sqrt{3}\,i \qquad w = 2\left(\cos\frac{\pi}{3} + i\sin\left(\frac{\pi}{3}\right)\right)$

2^{nd}-ROOTS

$$\sqrt{2}\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right) = \frac{\sqrt{3}}{\sqrt{2}} + \frac{i}{\sqrt{2}} = z_0$$

$$\sqrt{2}\left(\cos\left(\frac{\pi}{6} + \pi\right) + i\sin\left(\frac{\pi}{6} + \pi\right)\right) =$$

$$\sqrt{2}\left(\cos\left(\frac{7}{6}\pi\right) + i\sin\left(\frac{7}{6}\pi\right)\right) = -\left(\frac{\sqrt{3}}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = z_1$$

CUBIC ROOTS

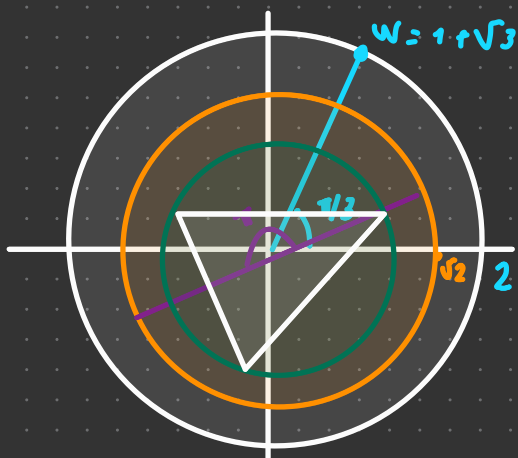
$$\sqrt[3]{2}\left(\cos\left(\frac{\pi}{9}\right)+i\sin\left(\frac{\pi}{9}\right)\right)=z_0$$

$$\sqrt[3]{2}\left(\cos\left(\frac{\pi}{9}+\frac{2\pi}{3}\right)+i\left(\frac{\pi}{9}+\frac{2\pi}{3}\right)\right)=$$

$$\sqrt[3]{2}\left(\cos\left(\frac{7}{9}\pi\right)+i\sin\left(\frac{\pi}{9}+\frac{4\pi}{3}\right)\right)=$$

$$\sqrt[3]{2}\left(\cos \frac{\pi}{9}+\frac{4}{3}\pi\right)+i\sin\left(\frac{\pi}{9}+\frac{4\pi}{3}\right)\bigg)=$$

$$\sqrt[3]{2}\left(\cos\left(\frac{13}{9}\pi\right)+i\sin\left(\frac{13}{9}\pi\right)=z_2\right.$$



Ex

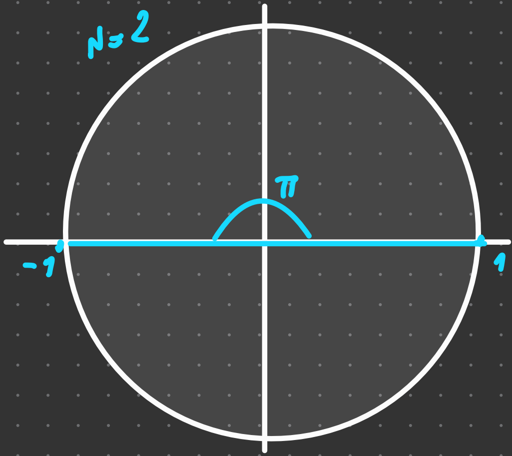
COMPUTE AND DRAW THE 4-TH ROOT OF W

EXAMPLE

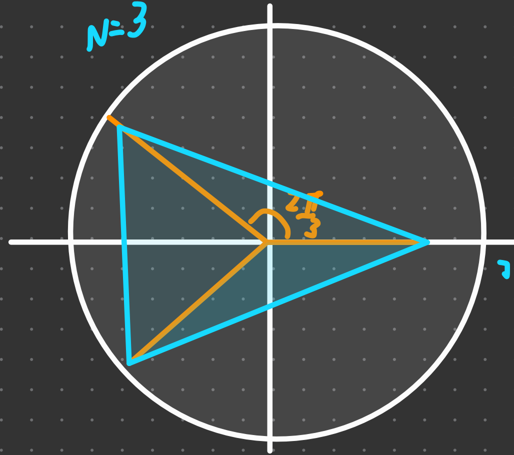
1 IS ALWAYS A N-TH ROOT OF THE UNITY

$1 = 1(\cos\theta + i\sin\theta)$

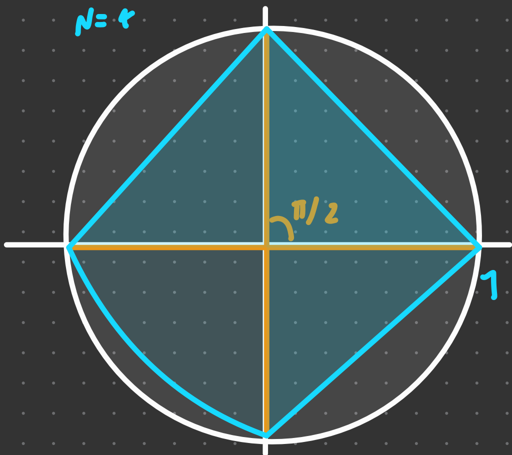
N=2



N=3

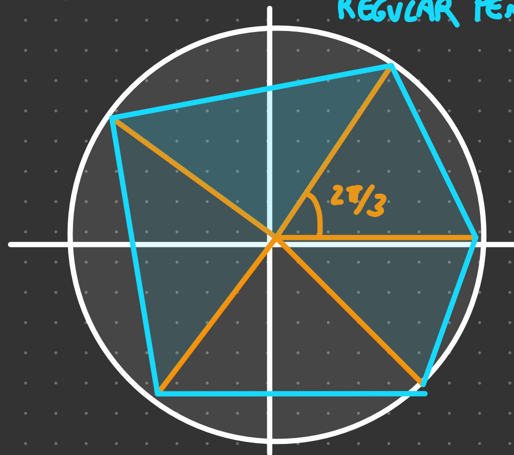


N=4



N=5

REGULAR PENTAGON



FINAL REMARK

$a, b, c \in \mathbb{C}$ WITH $a \neq 0$. SOLVE $az^2 + bz + c = 0 \quad z \in \mathbb{C}$

z IS A SOLUTION $\Leftrightarrow$ $z = \dfrac{-b + \sqrt[c]{b^2 - 4ac}}{2a}$

1 SOLUTION IF b^2-4ac $\left(z=-\frac{b}{2a}\right)$

2 DIFFERENT SOLUTION IF $b^2-4ac\neq 0$

EXAMPLE $a,b,c\in\mathbb{R}$ $a\neq 0$. IN $\mathbb{C}$ THE EQUATION

$az^2+bz+c=0$ HAS THE FOLLOWING SOLUTIONS

$\Delta=b^2-4ac>0$ $\quad z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\Delta=0$ $\quad z=-\frac{b}{2a}$

$\Delta<0$ $\quad z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{4ac-b^2}\,i}{2a}$

EXAMPLE

- $2z^2-3z+5=0$

  $\Delta=9-40=-31$

  $z_1=\frac{3\pm\sqrt{31}\,i}{4}$

- $iz^2-3z+i=0$

  $\Delta=9-4i^2=13$

  $z_\pm=\frac{3\pm\sqrt{13}}{2i}=\frac{-i}{2}\left(3\pm\sqrt{13}\right)$

SEQUENCES

LET X BE AN ARBITRARY SET $(X \neq \emptyset)$

WE CALL **SEQUENCE** OF X A FUNCTION

$$f : \mathbb{N} \to X$$
$$\mathbb{N} \ni n \to f(x) = x_n \in X$$

A SEQUENCE IS DENOTED $\{x_n\}_{n\in\mathbb{N}} \subseteq X$

NOTE DIFFERENT FROM $\{x_n : n \in \mathbb{N}\}$ ORDERING IN A SEQUENCE MATTERS!

EXAMPLE $X = \mathbb{R}$

$$f : \mathbb{N} \to \mathbb{R} \quad f(n) = x_n = n \quad \forall n \in \mathbb{N}$$

$$1, 2, 3 \dots, n, \dots \quad \{n\}_{n\in\mathbb{N}}$$

$$f : \mathbb{N} \to \mathbb{R} \quad f(n) = x_n = -n^2 \quad \forall n \in \mathbb{N}$$

$$-1, -4, -9, \dots, -n^2, \dots \quad \{-n^2\}_{n\in\mathbb{N}}$$

$$f : \mathbb{N} \to \mathbb{R} \quad f(n) = x_n = \frac{1}{n} \quad \forall n \in \mathbb{N}$$

$$1, \frac{1}{2}, \frac{1}{3}, \dots \frac{1}{n}, \dots \quad \left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$$

$$f : \mathbb{N} \to \mathbb{R} \quad f(n) = x_1 = 3 \quad \forall n \in \mathbb{N}$$

$$3, 3, 3 \dots 3, \dots \quad \{3\}_{n\in\mathbb{N}}$$

- $X = \mathcal{P}(\mathbb{N})$

$f : \mathbb{N} \to \mathcal{P}(\mathbb{N}) \quad f(n) = x_n = \{n\} \quad \forall n \in \mathbb{N}$

$\{1\}, \{2\}, \{3\}, \dots \{n\}, \dots \quad \{\{n\}\}_{n \in \mathbb{N}}$

$f : \mathbb{N} \to P(\mathbb{N}) \quad f(n) = x_n = \{1, \dots n\} \quad \forall n \in \mathbb{N}$

$\{1\}, \{1,2\}, \{1,2,3\}, \dots \{1,2,\dots n\} \quad \{\{1,\dots n\}\}_{n \in \mathbb{N}}$

DEFINITION

LET $P(n)$ BE A STATEMENT DEPENDING ON $n \in \mathbb{N}$
WE SAY THAT $P(n)$ IS DEFINETLY TRUE IF
$\exists\, n_0 \in \mathbb{N}$ SUCH THAT $\forall n \in \mathbb{N}$ WITH $n \geq n_0$ WE HAVE
THAT $P(n)$ IS TRUE

REMARK

LET $P(n)$ BE DEFINETLY TRUE
($\exists\, n_0 \in \mathbb{N}$ SUCH THAT $\forall n \geq n_0$ $P(n)$ IS TRUE)

LET $q(n)$ BE DEFINETLY TRUE
($\exists\, m_0 \in \mathbb{N}$ SUCH THAT $\forall n \geq m_0$ $q(n)$ IS TRUE)

THEN $\{P(n)$ AND $q(n)\}$ IS DEFINETLY TRUE

IN FACT, LET $K_0 = \max(N_0, M_0)$

$\forall n \geq K_0$, $n \geq N_0$ SO $P(n)$ IS TRUE AND $n \geq M_0$ SO $Q(n)$ IS TRUE HENCE

$\forall n \geq K_0$ BOTH $P(n)$ AND $Q(n)$ ARE TRUE

LIMITS OF SEQUENCES

DEFINITION LET $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ BE A SEQUENCE OF REAL NUMBER

- $\{a_n\}_{n\in\mathbb{N}}$ IS **CONVERGENT** IF $\exists\, a \in \mathbb{R}$ SUCH THAT

  $\forall \varepsilon > 0$ $\exists\, n_0 \in \mathbb{N}$ SUCH THAT $\forall n \in \mathbb{N}$ WITH $n \geq n_0$ WE HAVE $|a_n - a| < \varepsilon$

IN THIS CASE WE SAY THAT THE **LIMIT** OF $(a_n)_{n\in\mathbb{N}}$ AS n GOES TO $+\infty$ IS a OR THAT a_n GOES OR **CONVERGES** TO a AS n GOES TO $+\infty$ AND WE WRITE

$$\lim_{n\to+\infty} a_n = a \quad \text{OR} \quad a_n \to a \text{ AS } n \to +\infty$$

REMARK

- $|a_n - a| < \varepsilon \iff a - \varepsilon < a_n < a + \varepsilon$
  $\iff a_n \in (a - \varepsilon, a + \varepsilon)$
  $\iff d(a_n - a) < \varepsilon$
  $\iff a_n \in B_\varepsilon(a)$

- LET $C > 0$ BE A CONSTANT. WE CAN REPLACE $< \varepsilon$ WITH $\leq \varepsilon$, $< C\varepsilon$, $\leq C\varepsilon$

PROOF

$< \varepsilon \iff \leq 2\varepsilon$

$\boxed{< \varepsilon}$ $\forall \varepsilon > 0 \ \exists N_0 \in \mathbb{N} \ \forall n \geq N_0$ WE HAVE $|a_n - a| < \varepsilon$

$\boxed{\leq 2\varepsilon}$ $\forall \varepsilon_1 > 0 \ \exists M_0 \in \mathbb{N} \ \forall n \geq M_0$ WE HAVE $|a_n - a| \leq 2\varepsilon_1$

"$\Rightarrow$" FIX $\varepsilon_1 > 0$ LET $\varepsilon = \varepsilon_1$ THEN $\exists N_0 \in \mathbb{N}$ SUCH THAT

$\forall n \geq N_0$ WE HAVE $|a_n - a| < \varepsilon \leq \varepsilon \leq 2\varepsilon = 2\varepsilon_1$

$\Rightarrow \leq 2\varepsilon$ HOLDS WITH $M_0 = N_0$

"$\Leftarrow$" FIX $\varepsilon > 0$ LET $\varepsilon_1 = \frac{\varepsilon}{3}$ THEN $\exists M_0 \in \mathbb{N}$ SUCH THAT

$\forall n \geq M_0$ WE HAVE $|a_n - a| \leq 2\varepsilon_1 = \frac{2\varepsilon}{3} < \varepsilon$

$\Rightarrow$ $\boxed{< \varepsilon}$ HOLDS WITH $N_0 = M_0$

EXAMPLE

$a_n = \frac{1}{n}$ THEN $\lim_{n \to +\infty} a_n = \lim_{n \to +\infty} \frac{1}{n} = 0$

FIX $\varepsilon > 0$ PICK $N_0 \in \mathbb{N}$ SUCH THAT $N_0 > \frac{1}{\varepsilon}$ THAN

$\forall n \geq N_0$ WE HAVE

$-\varepsilon < 0 < |a_n - 0| = \frac{1}{n} \leq \frac{1}{N_0} < \varepsilon$

- $a_n = \dfrac{n^2-1}{n^2+1}$ THEN $\lim_n$

NOTATION $\boxed{\lim_{n \to +\infty} = \lim_n}$

$|a_n - 1| = \left|\frac{n^2-1}{n^2+1} - 1\right| = \left|\frac{-2}{n^2+1}\right| = \frac{2}{n^2+1} \leq \frac{2}{n^2} \leq \frac{2}{n} \overset{?}{<} \varepsilon$

FOR n BIG ENOUGH?

LET $N_0 \in \mathbb{N}$ S.T. $N_0 > \frac{2}{\varepsilon}$ THEN $\forall n \geq N_0$ WE HAVE

$|a_n - 1| \leq \frac{2}{n} \leq \frac{2}{N_0} < \varepsilon$

UNIQUENESS OF THE LIMIT

LET $\{a_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ BE A SEQUENCE OF REAL NUMBERS. IF $\{a_n\}_{n \in \mathbb{N}}$ IS CONVERGENT, IT

CAN NOT CONVERGE TO TWO DIFFERENT LIMITS,

THAT IS, IF $\exists\, a, b \in \mathbb{R}$ SUCH THAT

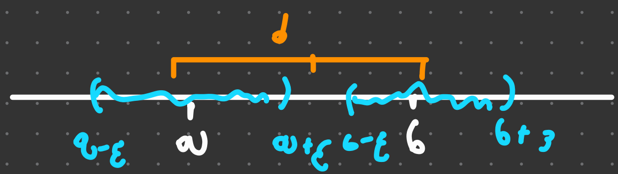
$\lim_N a_N = a$ AND $\lim_N a_N = b$, THEN $a = b$

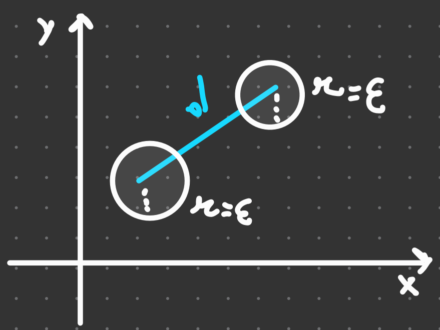
PROOF

LET $a \neq b$ THEN $d = d(a, b) = |b - a| > 0$

TAKE $0 < \varepsilon \leq \frac{d}{2}$

THEN $\underbrace{(a-\varepsilon, a+\varepsilon)}_{\parallel \; B_\varepsilon(a)} \cap \underbrace{(b-\varepsilon, b+\varepsilon)}_{\parallel \; B_\varepsilon(b)} = \emptyset$





IN FACT, BY CONTRADICTION, ASSUME $B_\varepsilon(a) \cap B_\varepsilon(b) \neq 0$

SO $\exists\, c \in B_\varepsilon(a) \cap B_\varepsilon(b)$ THAT IS

$d(c,a) < \varepsilon$ AND $d(c,b) < \varepsilon$ BUT

$0 \leq d = d(a,b) \leq d(a,c) + d(c,b) < \varepsilon + \varepsilon = 2\varepsilon \leq \frac{2d}{2} = d$

CONTRADICTION!

BY CONTRADICTION, ASSUME $a \neq b$ PICK $0 < \varepsilon \leq \frac{d}{2}$

$\lim_n a_n = a \Rightarrow \exists n_0 \in \mathbb{N}$ SUCH THAT $\forall n \geq n_0$ WE HAVE $a_n \in B_\varepsilon(a)$

$\lim_n a_n = b \Rightarrow \exists m_0 \in \mathbb{N}$ SUCH THAT $\forall n \geq m_0$ WE HAVE $a_n \in B_\varepsilon(b)$

$a_n \in B_\varepsilon(a)$ AND $a_n \in B_\varepsilon(b)$ THAT IS

$a_n \in B_\varepsilon(a) \cap B_\varepsilon(b)$ CONTRADICTION!

$B_\varepsilon(a) \cap B_\varepsilon(b) = \emptyset$ $\square$