$A \in \mathbb{R}^{2\times 2}$ $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $a_{11}a_{22} - a_{12}a_{21} \neq 0 \Rightarrow A^{-1} = \dfrac{1}{a_{11}a_{22}-a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

THEN THE MATRIX A IS INVERTIBLE

$$A^{-1}A = \frac{1}{a_{11}a_{22}-a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \frac{1}{a_{11}a_{22}-a_{12}a_{21}} \begin{bmatrix} \overset{\nearrow 1}{a_{22}a_{11}-a_{12}a_{21}} & \overset{\nearrow 0}{a_{22}a_{12}-a_{12}a_{22}} \\ \underset{\searrow 0}{-a_{21}a_{11}+a_{11}a_{21}} & \underset{\searrow 1}{-a_{21}a_{12}+a_{11}a_{22}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

DETERMINANT OF A: $\det A = a_{11}a_{22} - a_{12}a_{21} \longrightarrow$ TELL IF A SQUARE MATRIX IS INVERTIBLE

DEF $A \in \mathbb{R}^{N\times N}$ <u>DETERMINANT OF A</u>

- CHOOSE ANY ROW $i \in \{1, 2, \dots N\}$

THEN

$$\det A = (-1)^{i+1} a_{i1} \det A_{i1} + (-1)^{i+2} a_{i2} \det A_{i2} + (-1)^{i+3} a_{i3} \det A_{i3} + \dots + (-1)^{i+N} a_{iN} \det A_{iN}$$

WHERE $A_{ij} \in \mathbb{R}^{(N-1)\times(N-1)}$ OBTAINED BY CROSS ELIMINATING THE i^{TH} ROW AND THE j^{TH} COLUMN FROM A

AS AN ALTERNATIVE CHOOSE ANY COLUMN $j \in \{1, 2, \dots N\}$

$$\det A = (-1)^{j+1} a_{1j} \det A_{1j} + (-1)^{j+2} a_{2j} \det A_{2j} + (-1)^{j+3} a_{3j} \det A_{3j} + \dots + (-1)^{j+N} a_{Nj} \det A_{Nj}$$

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix} \quad i = 1$$

MAKE A CROSS ON THE COLUM AND ROW OF a_{i1} AND SO ON

$$\det A = (-1)^2\, 2 \begin{bmatrix} -3 & 1 \\ 7 & 5 \end{bmatrix} + (-1)^3 (-2) \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix} + (-1)^4 \cdot 4 \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} = 2(-15-7) + 2(5-3) + 4(7+9) = 24$$

SAME NUMBER

I CHOOSE A COLUMN

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix} \quad j = 2$$

$$\det A = (-1)^3 \, -2 \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix} (-1)^4 \, -3 \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} + (-1)^5\, 7 \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix} = 2(5-3) - 3(10-12) - 7(2-4) = 24$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -1 & -2 \\ 5 & 1 & 1 \end{bmatrix} \quad i = 1$$

$$\det A = +1(-1+2) - 2(4+10) + 3(4+5) = 0$$

THEN THE MATRIX IS NOT INVERTIBLE

BETTER TO CHOOSE THIS COLUM BECAUSE THERE ARE 2 ZEROS

$$A = \begin{bmatrix} -2 & 1 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 4 & 5 & 6 \\ -2 & 7 & 6 & 10 \end{bmatrix}$$

I CAN SKIP IT (BECAUSE 2 ZEROS)

$$\det A = (-1)^2(-2)\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 6 & 10 \end{bmatrix} (-1)^3\, 0 \begin{bmatrix} \dots \end{bmatrix} + (-1)^4\, 0 \begin{bmatrix} \dots \end{bmatrix} + (-1)^5 (-2) \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

DET(A) SHOULD BE 0 → TEST

$i \in \{1, 2, \dots N\}$

$$\det A = \sum_{j=1}^{N} (-1)^{i+j} a_{ij} \det A_{ij}$$

$$\det A = \sum_{i=1}^{N} (-1)^{i+j} a_{ij} \det A_{ij}$$

THEOREM

$A \in \mathbb{R}^{M \times N}$. THEN A IS INVERTIBLE IF AND ONLY IF $\det A \neq 0$

OBSERVATION IF $A \in \mathbb{R}^{N \times N}$ IS UPPER OR LOWER TRIANGULAR THEN THE DETERMINANT OF A IS THE PRODUCT OF THE VALUES ON THE DIAGONAL

UPPER TRIANGULAR
$$\begin{bmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix}$$

TO MAKE THE DET A CHOSE THIS COLUMN WHEN CROSS

+COMP

THEOREM $A \in \mathbb{R}^{N \times N}$, $B \in \mathbb{R}^{N \times N}$

- $\det(A^T) = \det A$
- $\det(AB) = \det(A) \det(B)$

ELEMENTARY ROW OPERATION

1. IF WE EXCHANGE TWO ROWS OF A THEN THE DETERMINANT IS MULTIPLIED BY (-1)
2. IF WE MULTIPLY A ROW BY ANY α (ALSO $\alpha = 0$) THEN $\det A$ IS MULTIPLY BY α
3. IF WE SUBTRACT A MULTIPLE OF A ROW TO ANOTHER ROW THE DETERMINANT DOES NOT CHANGE

$A \to EF$ $\qquad A \in \mathbb{R}^{N \times N}$ $\quad A \xrightarrow{\text{ROW REDUCTION PROCESS}} EF \text{ of } A$

(NO ROW EXCHANGES)

$A = LU$

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ * & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

PRODUCT OF THE VALUES ON THE DIAGONAL

$$\det(A) = \det(LU) = \det(L)\det(U) = \det(U)$$

LOWER TRIANGULAR MATRIX

IS THE PRODUCT OF THE DIAGONAL $1 \cdot 1 \cdot 1 \dots 1 = 1$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -1 & -2 \\ 5 & 1 & 1 \end{bmatrix} \begin{matrix} 4 \\ 5 \end{matrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -9 & -14 \\ 0 & -9 & -14 \end{bmatrix} 1 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -9 & -14 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \det(A) = 1 \cdot (-9) \cdot 0 = 0$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix}$$

OBSERVATION

If A is transformed in a echelon form U by using "Row multiplication and subtraction" and "Row switching" then

$\det(A) =$ "Product of the diagonal terms of U times $(-1)^{x}$" where x is the number of row switched

RIVEDERE

$$A = \begin{bmatrix} 0 & 3 & -1 \\ 1 & 2 & -7 \\ 2 & -1 & 3 \end{bmatrix}$$

In this case you must switch or can not "kill"; the $\det(A_2)$ is $-1\det(A)$

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix} \qquad \det(A) = 24$$

CORRECT

SWITCH

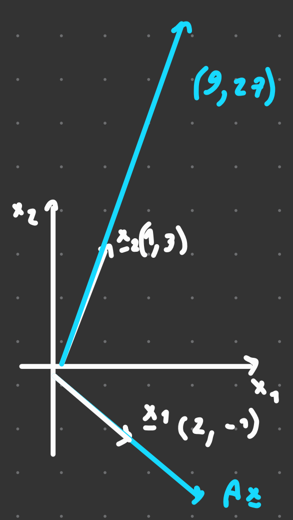
$$A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -2 & 4 \\ 3 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 4 & 2 \\ 0 & 16 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & -6 \end{bmatrix} \longrightarrow \det A = 1 \cdot 4 \cdot (-6) = -24 \longrightarrow \det(A) = -24 \cdot (-1)^1 = 24$$

use one switch

$$A = \begin{bmatrix} 2 & -1 & 2 \\ -6 & 0 & -2 \\ 8 & -1 & 5 \end{bmatrix} \qquad \det(A) = -6$$

$$A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix} \qquad T(\underline{x}) = A\underline{x} \qquad x \in \mathbb{R}^2$$

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\underline{x} \longmapsto A\underline{x}$$



$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$A\underline{x}_1 = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$A\underline{x}_2 = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 27 \end{bmatrix} = 9 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$A \begin{bmatrix} 2 \\ 6 \end{bmatrix} = A\,2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 2A \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 2 \cdot 9 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 9 \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$$

$$A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix} \qquad \underline{w} = c_1 \overset{A}{\begin{bmatrix} 2 \\ -1 \end{bmatrix}} + c_2 \overset{A}{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}$$

DEFINITION $A \in \mathbb{R}^{n \times n}$ let $\lambda \in \mathbb{R}$ and $\underline{v} \in \mathbb{R}^m$ such that $(\underline{v} \neq 0)$ $\quad A\underline{v} = \lambda\underline{v}$

Then $\underline{v}$ is called an <u>eigenvector</u> for A and λ is the corresponding <u>eigenvalue</u>

$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ IS AN EIGENVECTOR

EIGEN VALUES

$A\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \end{bmatrix} \neq \lambda \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ NOT AN EIGENVECTOR

$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ IS -4 AN EIGENVALUE FOR A?

YES IF I CAN FIND $\underline{x} \in \mathbb{R}^2$ SUCH THAT $A\underline{x} = -4\,\underline{v}$

$A\underline{v} = -4\,\overbrace{I_2\underline{v}}^{=\underline{v}}$ REMIND $\underline{v} \neq \underline{0}$

$A\underline{v} + 4I\underline{v} = \underline{0}$

$\underline{v}(A+4I) = \underline{0}$; $\underline{v} \in \mathbb{R}^2$, $\underline{v} \neq \underline{0}$

$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix}$

$\begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix}\underline{x} = \underline{0}?$ $\begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix} \xrightarrow{\text{ef}} \begin{bmatrix} 5 & 6 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} -6 \\ 5 \end{bmatrix} = -4\begin{bmatrix} -6 \\ 5 \end{bmatrix}$

$A \in \mathbb{R}^{N \times N}$ $\lambda \in \mathbb{R}$ IS λ AN EIGENVALUE OF A?

YES <u>IFF</u> $\exists\, \underline{x} \in \mathbb{R}^N$ SUCH THAT $A\underline{x} = \lambda\underline{x}$
$x \neq 0$

IFF $Ax - \lambda I\underline{x} = \underline{0}$ HAS A SOLUTION $\underline{x} \in \mathbb{R}^N$, $\underline{x} \neq 0$ IFF $(A - \lambda I)\underline{x} = 0$ HAS A SOLUTION $\underline{x} \in \mathbb{R}^N$, $\underline{x} \neq \underline{0}$

THERFORE WHAT I NEED TO CHECK IS IF THE HOMOGENEOUS EQUATION $(A - \lambda I)\underline{x} = \underline{0}$ HAS NON-TRIVIAL SOLUTION ($\underline{x} \in \mathbb{R}^N$ SOLUTION, $\underline{x} \neq \underline{0}$)