Knowledge Representation and Reasoning

Exercise Session 1

Exercise 1. Truth Tables (∗)

Build the **truth tables** of the following formulas, remembering the precedence of the operators.

1. $\neg(x \wedge y) \vee z$
2. $(x \wedge y \vee \neg x \wedge \neg w) \wedge z$
3. $(x \vee y) \wedge x$
4. $\neg(p \wedge (\neg p \vee q \vee s)) \vee q \vee s$

Exercise 2. Boolean Functions (∗)

For each of the following truth tables, build a formula expressing the same Boolean function.

x	y	z	φ_1
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

x	z	φ_2
0	0	1
0	1	0
1	0	0
1	1	1

x	y	z	w	φ_3
0	0	0	0	0
0	0	0	1	1
0	0	1	0	0
0	0	1	1	0
0	1	0	0	1
0	1	0	1	0
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	1
1	0	1	0	0
1	0	1	1	0
1	1	0	0	1
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

If the tables represent Boolean functions, what is their arity?

Exercise 3. Types of Formulas $(*)$

Use truth tables (and, if needed the equivalences seen during the lecture) to verify whether the following formulas are tautologies, contradictions, or non-tautological satisfiable formulas.

1. $x \to \neg x$
2. $(x \to y) \wedge (\neg y \to \neg x)$
3. $(x \to y) \to (\neg y \to \neg x)$
4. $x \wedge y \wedge (\neg x \vee \neg y) \wedge z$

Exercise 4. NNF $(**)$

A formula is in **negation normal form** (NNF). For example, the second and third formulas in Exercise 1 are in NNF, while the other two formulas are not.

1. Using the DeMorgan equivalences, transform the formulas of Exercise 1 to NNF
2. Do you notice something from the constructions in Exercise 2?

Exercise 5. Sheffer Functions $(* * *)$

We have seen that $\neg, \wedge, \vee$ form a **functionally complete** set of logical connectives in the sense that every Boolean function can be described using those connectives. In fact, we $\neg, \wedge$ and $\neg, \vee$ are functionally complete by themselves.

A logical connective is called a **Sheffer function** or **sole sufficient operator** if it is functionally complete by itself.

1. show that the NAND connective (denoted as $\uparrow$) is a Sheffer function
2. are there other Sheffer functions?
3. could a unary connective be a Sheffer function?

Exercise 6. Knowledge Bases (∗∗)

For the following truth table, construct a knowledge base (expressed as a set of propositional rules) which represents the same knowledge.

x	y	z	K
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Exercise 7. Expressivity (∗∗)

Can any propositional formula be expressed as a knowledge base?

If the answer is **yes**, argue why; if it is **no**, give at least two examples of formulas which cannot be expressed.

Exercise 8. Reasoning (∗)

Consider the following knowledge base K:

$$\begin{aligned}
x &\leftarrow \\
y &\leftarrow x,\ z,\ w \\
x &\leftarrow v \\
w &\leftarrow y,\ z \\
z &\leftarrow v,\ x \\
z &\leftarrow y,\ w \\
z &\leftarrow u,\ x \\
u &\leftarrow \\
p &\leftarrow \\
t &\leftarrow w,\ u \\
r &\leftarrow s,\ t
\end{aligned}$$

1. Compute the redux $\widehat{K}$
2. Find all the facts that are entailed by K
3. Decide whether the following clauses are consequences of K
   a) $v \leftarrow u$
   b) $t \leftarrow y$
   c) $q \leftarrow q$
   d) $r \leftarrow w$

Exercise 9. Revision (∗∗)

In the knowledge base from Exercise 8, substitute the fact $x \leftarrow$ with $u \leftarrow$. Call this new knowledge base K'.

1. Do your answers from Exercise 8 change?
2. Which fact(s) should you remove to ensure that z is **not** a consequence of K'?
3. If facts cannot be removed, which rules would you remove to ensure that z is not entailed?

Exercise 10. Tautologies $(***)$

The soundness of the fact extraction algorithm seen during the lecture depends on the following property:

> if $\varphi \to \psi$ and $\psi \to \xi$ are both tautologies, then $\varphi \to \xi$ is also a tautology.

Show that this property holds always in propositional logic.

$\neg, \wedge, \vee$

1. $\neg(x \wedge y) \vee z$

x	y	z	$x \wedge y$	$\neg(x \wedge y)$	$\neg(x \wedge y) \vee z$
1	1	1	1	0	1
1	1	0	1	0	0
1	0	1	0	1	1
0	1	1	0	1	1
1	0	0	0	1	1
0	1	0	0	1	1
0	0	1	0	1	1
0	0	0	0	1	1

2^3

$\neg x \vee \neg w \vee z$ (DEMORGAN) — REMOVE PARENTESIS

IS TRUE IF AT LEAST ONE IS TRUE

2. $(x \wedge y \vee \neg x \wedge \neg w) \wedge z$

x	y	w	z	$\neg x$	$\neg w$	$x \wedge y$	$x \wedge y \vee \neg x$	$x \wedge y \vee \neg x \wedge \neg w$	$(x \wedge y \vee \neg x \wedge \neg w) \wedge z$
0	0	0	1						
0	0	1	0						
0	1	0	0						
1	0	0	0						
0	0	1	1						
0	1	0	1						
1	0	0	1						
0	1	1	0						
1	0	1	0						
1	1	0	0						
0	1	1	1						
1	0	1	1						
1	1	0	1						
1	1	1	0						
1	1	1	1						

4. $\neg(p \wedge (\neg p \vee q \vee s)) \vee q \vee s$

p q s	$\neg p$	$\neg p \vee q$	$\neg p \vee q \vee s$	$p \wedge (\neg p \vee q \vee s)$	$\neg(p \wedge (\neg p \vee q \vee s))$	$\neg(p \wedge (\neg p \vee q \vee s)) \vee q$	$\neg(p \wedge (\neg p \vee q \vee s)) \vee q \vee s$
1 1 1	0	1	1	1	0	1	1
1 1 0	0	1	1	1	0	1	1
1 0 1	0	0	1	1	0	0	1
0 1 1	1	1	1	0	1	1	1
1 0 0	0	0	0	0	1	1	1
0 1 0	1	1	1	0	1	1	1
0 0 1	1	1	1	0	1	1	1
0 0 0	1	1	1	0	1	1	1

x	y	z	φ_1
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

$(\neg x \wedge \neg y \wedge z) \vee$

$(\neg x \wedge y \wedge z) \vee$

$(\neg x \wedge z) \wedge \cancel{(\neg y \vee y)}$

$\vee$

$(x \wedge \neg y \wedge \neg z) \vee$

$(x \wedge y \wedge z)$

$x \wedge (\neg y \wedge \neg z) \vee (y \wedge z)$

ARITY 3

x	z	φ_2
0	0	1
0	1	0
1	0	0
1	1	1

$(\neg x \wedge \neg z) \vee$

OR BOTH TRUE
OR BOTH FALSE

$(x \wedge z)$

ARITY 2

x	y	z	w	φ_3
0	0	0	0	0
0	0	0	1	1
0	0	1	0	0
0	0	1	1	0
0	1	0	0	1
0	1	0	1	0
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	1
1	0	1	0	0
1	0	1	1	0
1	1	0	0	1
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

$(\neg x \wedge \neg y \wedge \neg z \wedge w) \vee$

$(\neg x \wedge y \wedge z \wedge w) \vee$

$(\neg x \wedge y \wedge z \wedge w) \vee$

$(\neg x \wedge y \wedge z \wedge w) \vee$

$(x \wedge \neg y \wedge \neg z \wedge w) \vee$

$(x \wedge y \wedge \neg z \wedge \neg w)$

ARITY-4

φ TAKES N ARGUMENT

ARITY: NUMBER OF PARAMETERS THAT THE FUNCTION HAVE TO TAKE IN CONSIDERATION

Exercise 3. Types of Formulas (*)

Use truth tables (and, if needed the equivalences seen during the lecture) to verify whether the following formulas are tautologies, contradictions, or non-tautological satisfiable formulas.

└ ALL 1 └ ALL 0 └ 0 AND 1

1. $x \to \neg x$ $a \to b$ AND $\neg a \lor b$
2. $(x \to y) \land (\neg y \to \neg x)$
3. $(x \to y) \to (\neg y \to \neg x)$
4. $x \land y \land (\neg x \lor \neg y) \land z$

$x \to \neg x$

x	$\neg x$	$x \to \neg x$	$= \neg x$
0	1	1	1
1	0	0	0

NON-TAUTOLOGICAL SATISFIABLE FORMULAS

$(x \to y) \land (\neg y \to \neg x)$

x	y	$x \to y$	$\neg y$	$\neg x$	$\neg y \to \neg x$	$(x \to y) \land (\neg y \to \neg x)$
0	0	1	1	1	1	1
0	1	1	0	1	1	1
1	0	0	1	0	0	0
1	1	1	0	0	1	1

$(\neg x \lor y) \land (y \lor \neg x)$

NON TAUTOLOGICAL SATISFIABLE FORMULA

$(x \to y) \to (\neg y \to \neg x)$

x	y	$x \to y$	$\neg y$	$\neg x$	$(\neg y \to \neg x)$	$(x \to y) \to (\neg y \to \neg x)$
0	0	1	1	1	1	1
0	1	1	0	1	1	1
1	0	0	1	0	0	0
1	1	1	0	0	1	1

TAUTOLOGY

$x \land y \land (\neg x \lor \neg y) \land z$

x	y	z	$\neg x$	$\neg y$	$(\neg x \lor \neg y)$	$x \land y \land (\neg x \lor \neg y) \land z$
0	0	0	1	1	1	0
0	0	1	1	1	1	0
0	1	0	1	0	1	0
1	0	0	0	1	1	0
0	1	1	1	0	1	0
1	0	1	0	1	1	0
1	1	0	0	0	0	0
1	1	1	0	0	0	0

CONTRADICTION

Exercise 4. NNF (**)

A formula is in **negation normal form** (NNF). For example, the second and third formulas in Exercise 1 are in NNF, while the other two formulas are not.

1. Using the DeMorgan equivalences, transform the formulas of Exercise 1 to NNF
2. Do you notice something from the constructions in Exercise 2?

???

1. $\neg(x \wedge y) \vee z$
2. $(x \wedge y \vee \neg x \wedge \neg w) \wedge z$
3. $(x \vee y) \wedge x$
4. $\neg(p \wedge (\neg p \vee q \vee s)) \vee q \vee s$

$\neg(x \wedge y) \vee z$

$(\neg x \vee \neg y) \vee z$

$\neg x \vee \neg y \vee z$

$\neg(p \wedge (\neg p \vee q \vee s)) \vee q \vee s$

$\neg((p \wedge \neg p) \vee (p \wedge q) \vee (p \wedge s)) \vee q \vee s$

$\neg(p \wedge \neg p) \wedge \neg(p \wedge q) \wedge \neg(p \wedge s) \vee q \vee s$

$(\neg p \vee \neg p) \wedge (\neg p \vee \neg q) \wedge (\neg p \vee \neg s) \vee q \vee s$

$q \vee s \vee \neg p \vee p \wedge \neg q \wedge \neg s$

X

Exercise 5. Sheffer Functions (∗∗∗)

We have seen that $\neg, \wedge, \vee$ form a **functionally complete** set of logical connectives in the sense that every Boolean function can be described using those connectives. In fact, we $\neg, \wedge$ and $\neg, \vee$ are functionally complete by themselves.

A logical connective is called a **Sheffer function** or **sole sufficient operator** if it is functionally complete by itself.

1. show that the NAND connective (denoted as $\uparrow$) is a Sheffer function
2. are there other Sheffer functions?
3. could a unary connective be a Sheffer function? NO, NEED TO BE AT LEAST BINARY

NAND

x	y	$x \uparrow y$
1	1	0
1	0	1
0	1	1
0	0	1

$$(\neg x \wedge y) \vee (x \wedge \neg y) \vee (\neg x \wedge \neg y)$$

$$\neg x \wedge y \vee x \wedge \neg y \vee \neg x \wedge \neg y$$

MORE CONNECTIVES → MORE COMPACT

$$\neg x = \boxed{x \uparrow x}$$

$$x \wedge y = \neg(x \uparrow y) = (x \uparrow y) \uparrow (x \uparrow y)$$

$$x \vee y = (x \uparrow y)$$

TEST IT

NOR ($\downarrow$) IS A SHREFFER FUNCTION

$$x \downarrow y = \neg(x \vee y) = \neg x \wedge \neg y$$

Exercise 6. Knowledge Bases (**)

For the following truth table, construct a knowledge base (expressed as a set of propositional rules) which represents the same knowledge.

x	y	z	K
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

REPRESENT TRUTH TABLE AS A FORMULA

$$(x \wedge \neg y \wedge \neg z) \vee (x \wedge y \wedge z)$$

$$x \wedge (\neg y \wedge \neg z) \vee (y \wedge z$$

$$(x \vee x) \wedge (x \vee y) \wedge (x \vee z) \wedge (\cancel{\neg y \vee y}) \wedge (\neg y \vee z) \wedge (\neg z \vee x) \wedge (\neg x \vee y) \wedge (\cancel{\neg z \vee z})$$

$x \wedge (x \vee y) \equiv x$

x MUST BE 1

THE MAX OF $x \vee y$ MUST BE 1
WE KNOW x IS 1
REGARDLESS OF y IF IT IS 0 OR 1

$x \leftarrow$

$\cancel{x \leftarrow y}$ DOESN'T AFFECT OUR KNOWLEDGE

$z \leftarrow y$

$\cancel{x \leftarrow z}$ " "

$y \leftarrow z$

BOTH TRUE OR BOTH FALSE

Exercise 7. Expressivity (**)

Can any propositional formula be expressed as a knowledge base?

If the answer is **yes**, argue why; if it is **no**, give at least two examples of formulas which cannot be expressed.

NO $x \lor y$ CAN'T BE EXPRESS AS HORN CLAUSES

KB: HAVE SPECIFIC SHAPE, NEED TO BE HORN CLAUSES, NEED 1 POSITIVE LITERAL

$\neg x$

THERE ARE THINGS THAN CAN'T BE EXPRES

Exercise 8. Reasoning

Consider the following knowledge base K:

$$
\begin{aligned}
x &\leftarrow \\
y &\leftarrow \cancel{x},\ \cancel{z},\ \cancel{w} \\
x &\leftarrow v \\
\cancel{w} &\leftarrow \cancel{y},\ \cancel{z} \\
z &\leftarrow v,\ \cancel{x} \\
z &\leftarrow \cancel{y},\ \cancel{w} \\
z &\leftarrow \cancel{u},\ \cancel{x} \\
u &\leftarrow \\
p &\leftarrow \\
\cancel{t} &\leftarrow \cancel{w},\ \cancel{u} \\
r &\leftarrow s,\ \cancel{t}
\end{aligned}
$$

Y ← X Y ←

X ← V W ←

W ← Y

Z ← V

Z ← Y, W

t ← W, V

r ← S, t

1. Compute the redux $\widehat{K}$
2. Find all the facts that are entailed by K

   x, z, v, p
3. Decide whether the following clauses are consequences of K
   - a) $v \leftarrow u$ TO CHECK WE ASSUME U (THE BODY) IS TRUE WE CONSIDER U ← AS A FACT

     NO
   - b) $t \leftarrow y$ YES
   - c) $q \leftarrow q$ ALWAYS TRUE ADD q ← AND AND q IS A FACT
   - d) $r \leftarrow w$ NO

Exercise 9. Revision (**)

In the knowledge base from Exercise 8, substitute the fact $x \leftarrow$ with $u \leftarrow$. Call this new knowledge base K'.

1. Do your answers from Exercise 8 change?
2. Which fact(s) should you remove to ensure that z is **not** a consequence of K'?
3. If facts cannot be removed, which rules would you remove to ensure that z is not entailed?

IT IS SUFFICES TO REMOVE $v \leftarrow$ OR IN K $x \leftarrow$

K'

$u \leftarrow$

$y \leftarrow x,\ z,\ w$

$x \leftarrow v$

$w \leftarrow y,\ z$

$z \leftarrow v,\ x$

$z \leftarrow y,\ w$

$z \leftarrow \cancel{u},\ x$

$\boxed{u \leftarrow}$ IS ALREADY HERE

$p \leftarrow$

$t \leftarrow w,\ \cancel{u}$

$r \leftarrow s,\ t$

Z IS NOT A FACT ANYMORE

EXPERIMENT $v \leftarrow$

$\cancel{x}\ v \leftarrow$

$y \leftarrow x,\ z,\ w$

$x \leftarrow \cancel{v}$

$w \leftarrow y,\ z$

$z \leftarrow \cancel{v},\ x$

$z \leftarrow y,\ w$

$z \leftarrow u,\ x$

$u \leftarrow$

$p \leftarrow$

$t \leftarrow w,\ \cancel{u}$

$r \leftarrow s,\ t$

- IF THERE IS NO Z IN THE HEAD
- OR REMOVE $x \leftarrow v$

Exercise 10. Tautologies (* * *)

The soundness of the fact extraction algorithm seen during the lecture depends on the following property:

if $\varphi \to \psi$ and $\psi \to \xi$ are both tautologies, then $\varphi \to \xi$ is also a tautology.

Show that this property holds always in propositional logic.

- WHENEVER φ IS 1 THAN ψ IS 1
- IF $V(\varphi)=1$ THEN $V(\psi)=1$
- $V(\neg\varphi \vee \psi) = 1$ IS A TAUTOLOGY

$$\| $$

$$\text{MAX}\{V(\neg\varphi), V(\psi)\}$$

IF $=0$ THEN $\to$ MUST BE 1

$$\text{MAX}\{V(\neg\psi), V(\varphi)\} = 1$$

GNE w $\text{MAX}\{w(\neg\varphi), w(\xi)\} = \begin{cases} \text{IF } w(\neg\varphi)=0 \quad w(\xi)=1 \\ \text{IF } w(\neg\varphi)=1 \end{cases}$

$= 1$ ← TAUTOLOGY EVERY VALUATION MARKS IT TRUE