$+$ $\times$

$-$ $\div$

BERNULLI INEQUALITY

$\forall \alpha > -1 \quad \forall_{n \in \mathbb{N}}$ WE HAVE

$$(1+\alpha)^n \geq 1 + n\alpha$$

REMARK: $(1+\alpha)^n > 1 + n\alpha$ IF $\alpha \neq 0$ AND $n \geq 2$

DEFINITION: NAPIER'S NUMBER

e IS DEFINED AS $\lim_n \left(1+\frac{1}{n}\right)^n \quad a_s = \left(1+\frac{1}{n}\right)^n$

WE SHOW THAT

a) $\{a_n\}_{n \in \mathbb{N}}$ IS STRICTLY INCREASING $a_n < a_{n+1} \; \forall n \in \mathbb{N}$

b) $2 = a_1 < a_n < 3 \quad \forall_{n \geq 2}$

NOTE:

a) + b) $\Rightarrow$ e IS WELL DEFINED SINCE

$$\exists \lim_n a_n \in \mathbb{R} \text{ AND } 2 < e \leq 3$$

ACTUALLY, e IS AN IRRATIONAL NUMBER, $e < 3$

$e = 2,718\ldots$

LET'S PROVE a) AND b)

a) $a_1 = 2 < a_2 = \frac{9}{4}$

$n \geq 3$ I WANT TO SHOW $a_{n-1} < a_n$ THAT IS

$$1 < \frac{a_n}{a_{n-1}} = \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{1}{n-1}\right)^{n-1}} = \frac{(1+1/n)(1+1/n)^{n-1}}{\left(\frac{n}{n-1}\right)^{n-1}} =$$

$$= \frac{\left(1+\frac{1}{n}\right)\left(\frac{n+1}{n}\right)^{n-1}}{\left(\frac{n}{n-1}\right)^{n-1}} = \left(1+\frac{1}{n}\right)\left(\frac{(n+1)(n-1)}{n^2}\right)^{n-1} =$$

$$= \left(1+\frac{1}{n}\right)\left(\frac{n^2-1}{n^2}\right)^{n-1} = \left(1+\frac{1}{n}\right)\left(1-\frac{1}{n^2}\right)^{n-1} \geq$$

$$\geq \left(1+\frac{1}{n}\right)\left(1+(n-1)\left(-\frac{1}{n^2}\right)\right) = 1 + \frac{1}{n^3} > 1$$

BERNULLI WITH

$a = -\frac{1}{n^2} > -1$ AND POWER $n-1$

$\Rightarrow$ **a)** IS PROVED

b) PROVE THAT $a_n < 3 \quad \forall n \in \mathbb{N}$

NEWTON $\Rightarrow$

$$a_n = \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} \cdot \frac{1}{n^k} =$$

$$= 1 + n \cdot \frac{1}{n} + \frac{1}{2!}\frac{n(n-1)}{n^2} + \frac{1}{3!}\frac{n(n-1)(n-2)}{n^3} + \dots$$

$$\dots + \frac{1n}{(n-1)!}\frac{(n-1)\dots(n-(n-2))}{n^{n-1}} + \frac{1}{n!}\frac{n(n-1)\dots(n(n-1))}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\underbrace{\left(1-\frac{1}{N}\right)}_{\leq 1} + \frac{1}{3!}\underbrace{\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)}_{\leq 1} + \cdots$$

$$\cdots \frac{1}{(N-1)!}\underbrace{\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\cdots\left(1-\frac{(N-2)}{N}\right)}_{\leq 1} + \frac{1}{N!}\underbrace{\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{(N-1)}{N}\right)}_{\leq 1}$$

$$< 1 + 1 + \frac{1}{2!} + \cdots \frac{1}{N!} \leq 2^{N-1} \leq N! \quad \forall N \in \mathbb{N}$$

$$1 + \frac{1}{2^0} + \frac{1}{2^1} + \cdots \frac{1}{2^{N-1}} = 1 + \sum_{K=0}^{N-1} \left(\frac{1}{2}\right)^K =$$

$$= 1 + \frac{1-(1/2)^N}{1-1/2} = 1 + 2 - \frac{1}{2^{N-1}} < 3$$

$$1 + x + \cdots x^{N-1} = \frac{1-x^N}{1-x}$$

$$x \neq 1$$

$$x = 1/2$$

---

e IS WELL DEFINED

DEFINITION: THE EXPONENTIAL FUNCTION IS

$\exp: \mathbb{R} \to (0, +\infty)$ GIVEN BY

$\exp(x) = e^x \quad \forall x \in \mathbb{R}$

ITS INVERSE IS THE LOGARITHMIC FUNCTION OR NATURAL LOGARITHM

$\log: (0, +\infty) \to \mathbb{R}$ GIVEN BY

$\log(x) = \log_e(x) \quad \forall x \in (0, +\infty)$

SOMETIMES $\log = \ln$

SPECIAL LIMITS

POWERS LET $\alpha > 0$

a) $\lim_n n^\alpha = +\infty$

PROOF: OK IF $\alpha = k \in \mathbb{N}$

n^α IS INCREASING, SO $\exists \lim_n n^\alpha = l \in [1, +\infty]$

$\alpha = \frac{1}{k}$, $k \in \mathbb{N}$ BY CONTRADICTION ASSUME

$\lim_n n^{1/k} = l \in \mathbb{R}$

BUT $\underbrace{\lim_n}_{+\infty} = n = \lim_n \left(n^{1/k}\right)^k = l^k$

CONTRADICTION

$\alpha > 0$ LET $k \in \mathbb{N}$ BE SUCH THAT $\frac{1}{k} < \alpha$

$+\infty \leftarrow n^{1/k} < n^\alpha$ BY COMPARISON $\lim_n n^\alpha = +\infty$

b) $\alpha > 0$ $x_n \to +\infty$

THEN $\lim_n x_n^\alpha = +\infty$

PROOF WE KNOW

- $\forall N > 0$ $\exists n_0 \in \mathbb{N}$ S.T. $\forall n \geq n_0$ WE HAVE $n^\alpha > N$
- $\forall N_1 > 0$ $\exists n_1 \in \mathbb{N}$ S.T. $\forall n \geq n_1$ WE HAVE $x_n > N_1$

FIX $N>0$ PICK $N_1 = N_0$ THEN $\exists\, n_{N_1} \in \mathbb{N}$ S.T. $\forall\, n \geq n_1$

WE HAVE $x_n > N_1 = N_0$ HENCE

$$N < N_0^\alpha < x_n^\alpha \Rightarrow \lim_n x_n^\alpha = +\infty$$

c) $\alpha > 0$ LET $x_n \to 0$ WITH $x_n \geq 0$ THEN

$$\lim_n x_n^\alpha = 0$$

PROOF: ASSUME $x_n \to 0^+$ THEN

$$x_n^\alpha = \frac{1}{\left(\frac{1}{x_n}\right)^\alpha} \to 0^+ \quad \square$$

d) $\alpha > 0$ $x_n \to x$ WITH $x > 0$ THEN

$$\lim_n x_n^\alpha = x^\alpha$$

PROOF:

$$\lim_n x_n^\alpha = x^\alpha \iff \lim_n \frac{x_n^\alpha}{x^\alpha} = \lim_n \left(\frac{x_n}{x}\right)^\alpha = 1$$

$(x>0$ SO $x^\alpha > 0)$ NOTE THAT $y_n = \frac{x_n}{x} \to 1$

ITS ENOUGH TO PROVE THE RESULT FOR $\boxed{x=1}$

LET $\alpha > 0$, $k \in \mathbb{N}$ BE SUCH THAT $k > \alpha$

$$\underset{\substack{\downarrow \\ 1}}{\left(1-\frac{1}{n}\right)^k} < \left(1-\frac{1}{n}\right)^\alpha \leq 1 \leq \left(1+\frac{1}{n}\right)^\alpha < \underset{\substack{\downarrow \\ 1}}{\left(1+\frac{1}{n}\right)^k}$$

$$\Rightarrow \lim_n \left(1-\frac{1}{n}\right)^{\alpha} = \lim_n \left(1+\frac{1}{n}\right)^{\alpha} = 1$$

FIX $\varepsilon > 0$ $\exists\, n_0 \in \mathbb{N}$ S.T. $\forall n \geq n_0$

$$1-\varepsilon < \left(1-\frac{1}{n}\right)^{\alpha} \leq \left(1+\frac{1}{n}\right)^{\alpha} < 1+\varepsilon$$

IN PARTICULAR $x_n \to 1 \Rightarrow$ DEFINITELY $1-\frac{1}{n_0} < x_n < 1+\frac{1}{n_0}$, SO

$$1-\varepsilon < \left(1-\frac{1}{n_0}\right)^{\alpha} \leq x_n^{\alpha} \leq \left(1+\frac{1}{n_0}\right) < 1+\varepsilon$$

EXPONENTIALS:

1) $\omega \in \mathbb{R}$ $\lim_n \omega^n = \begin{cases} +\infty & \text{IF } \omega > 1 \\ 1 & \text{IF } \omega = 0 \\ 0 & \text{IF } -1 < \omega < 1 \end{cases}$

DOES NOT EXIST IF $\omega \leq -1$

PROOF $\boxed{\omega > 1}$ $\omega = 1+\alpha,\ \alpha > 0$

BERNOULLI $\Rightarrow$ $\omega^n = (1+\alpha)^n \geq 1 + n\alpha \longrightarrow +\infty$

OK BY COMPARISON

$\boxed{\omega = 1}$ OBVIOUS!

$\boxed{-1 < \omega < 1}$ $|\omega^n| = |\omega|^n = \frac{1}{\left(\frac{1}{|\omega|}\right)^n} \to 0^+ \Rightarrow$

$\omega^n \to 0$

$\boxed{\omega = -1}$ $(-1)^n$ IS IRREGULAR

$\boxed{\omega < -1}$ $\omega^n = (-1)^n \underbrace{|\omega|^n}_{+\infty}$ IRREGULAR

$a^{2N} \to +\infty \qquad a^{2N-1} \to -\infty$

2) $\boxed{a>0}$ $x_N \to +\infty$ THEN

$$\lim_N a^{x_N} = \begin{cases} +\infty & a>1 \\ 1 & a=1 \\ 0 & 0<a<1 \end{cases}$$

$$\lim_N a^{-x_N} = \begin{cases} 0 & a>1 \\ 1 & a=1 \\ +\infty & 0<a<1 \end{cases}$$

EXAMPLE: $\lim_N \left(\frac{1}{3}\right)^{\frac{2N^2-1}{N+1}} = 0$; $x_N = \frac{2N^2-1}{N+1} \to +\infty$

3) $\boxed{a>0}$ $x_N \to 0$ THEN $\lim_N a^{x_N} = 1$

PROOF: WE SHOW $\forall a>0 \ \lim_N a^{1/N} = \lim_N \sqrt[N]{a} = 1$

$\left(\Rightarrow a^{-1/N} = \frac{1}{a^{1/N}} \to 1 \right)$

$\boxed{a=1}$ OK

$\boxed{a>1}$ BERNULLI

$$a = (1 + \underbrace{\sqrt[N]{a} - 1}_{=\,a>0})^N \overset{\text{BERNULLI}}{\geq} 1 + N\left(\sqrt[N]{a} - 1\right)$$

$\Rightarrow 0 \leq \sqrt[N]{a} - 1 \leq \frac{a-1}{N} \to 0$

$\Rightarrow \sqrt[N]{a} - 1 \to 0 \Rightarrow \sqrt[N]{a} \to 1$

$\boxed{0<a<1}$ $\sqrt[n]{a} = \left(\sqrt[n]{1/a}\right)^{-1} \to 1$

4) $a>0$ $x_n \to x \in \mathbb{R}$ THEN $\lim_n a^{x_n} = a^x$

PROOF $|a^{x_n} - a^x| = a^x |a^{x_n - x} - 1|$

BUT $x_n - x \to 0$ SO $a^{x_n - x} \to 1$ SO

$|a^{x_n - x} - 1| \to 0 \Rightarrow |a^{x_n} - a^x| \to 0$

LOGARITHMS

PROPOSITION

$$x_n \to \pm\infty \Rightarrow \lim_n \left(1 + \frac{1}{x_n}\right)^{x_n} = e = \lim \left(1 + \frac{1}{n}\right)^n$$

$$= \lim_n \left(1 - \frac{1}{n}\right)^{-n}$$

IDEA ONLY $x_n \to +\infty$

$\forall \{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$ SUCH THAT $a_n \to +\infty$, THEN

$$\lim_n \left(1 + \frac{1}{a_n}\right)^{a_n} = e$$

$b_n = \left(1 + \frac{1}{n}\right)^n$ $\left(1 + \frac{1}{a_n}\right)^{a_n} = b_{a_n}$

$x_n \to +\infty$ THEN $a_n = [x_n]$ INTEGER PART OF x_n

$x_n \geq 1$ $x_n = 5,75\ldots$ $[x_n] = 5$

$x_n \to +\infty \Rightarrow a_n = [x_n] \to +\infty$

a) IF $x_n \to +\infty$ THEN $\log(x_n) \to +\infty$

PROOF: BY MONOTONICITY $\exists \lim_n \log n = l \in [1, +\infty]$

IF $l \in \mathbb{R}$, THEN

$$e^{l} = \lim_n e^{\log(n)} = \lim_n n = +\infty \quad \underline{\text{CONTRADICTION}}$$

b) $x_n \to 0^+$ THEN $\log(x_n) \to -\infty$

PROOF: $\frac{1}{x_n} \to +\infty$ SO $\lim_n \log\left(\frac{1}{x_n}\right) = +\infty$

BUT $\log\left(\frac{1}{x_n}\right) = -\log(x_n) \to +\infty \Rightarrow \log(x_n) \to -\infty$

c) IF $x_n \to x > 0$, $x_n > 0$ THEN $\log(x_n) \to \log(x)$

PROOF: $|\log(x_n) - \log(x)| \to 0$?

$$|\log(x_n) - \log(x)| = \left|\log\left(\frac{x_n}{x}\right)\right| \to 0?$$

$y^n = \frac{x_n}{x} \to 1$ ENOUGH TO CONSIDER $x = 1$

$$\lim_n \log\left(1 - \frac{1}{n}\right) = l_1 \leq 0 \leq l_2 = \lim_n \log\left(1 + \frac{1}{n}\right)$$

(MONOTONICITY)

$$e^{l_1} = \lim_n e^{\log(1-\frac{1}{n})} = \lim_n \left(1-\frac{1}{n}\right) = 1$$

$\Rightarrow l_1 = 0$ SAME WITH l_2

CONCLUSION

$\{x_n\}_{n\in\mathbb{N}}$ $\{y_n\}_{n\in\mathbb{N}}$ WITH $x_n > 0$ $\forall n\in\mathbb{N}$

HOW TO COMPUTE, IF IT EXISTS $\lim_n x_n^{y_n}$?

CONSIDER

$$x_n^{y_n} = \left(e^{\log x_n}\right)^{y_n} = e^{(\log x_n)\, y_n}$$

IF THE LIMIT OF $z_n = (\log x_n)\, y_n$ EXISTS, APPLY THE PREVIOUS RULES FOR THE EXPONENTIAL

- $z_n \to +\infty \Rightarrow e^{z_n} = x_n^{y_n} \to +\infty$
- $z_n \to -\infty \Rightarrow e^{z_n} = x_n^{y_n} \to 0$
- $z_n \to z \in \mathbb{R} \Rightarrow e^{z_n} = x_n^{y_n} \to e^{z}$

INDETERMINATE FORMS $x_n^{y_n}$ $x_n > 0$

$$\boxed{1^{\infty} \quad 0^0 \quad (+\infty)^0}$$

$\boxed{1^{\infty}}$ $x_n \to 1$ $y_n \to \pm\infty$

$z_n = \log(x_n) \cdot y_n$ $\log(x_n) \to 0$ $y_n \to \pm\infty$

IND. FORM $0 \cdot \infty$ FOR $\log(x_n) \cdot y_n$

EXAMPLES

$\lim \left(1+\frac{1}{n}\right)^n = e$; $1^{+\infty}$

$\lim \left(1+\frac{1}{n}\right)^{n^2} = +\infty$ $1^{+\infty}$

$$\left(1+\frac{1}{n}\right)^{n^2} = \left(1+\frac{1}{n}\right)^{n \cdot n} = \left(\left(1+\frac{1}{n}\right)^n\right)^n$$

$\left(\left(1+\frac{1}{n}\right)^n\right)^n \to +\infty$ $(e-1)^n \to +\infty$

$\downarrow$

e

DEFINITELY $\left(1+\frac{1}{n}\right)^n > e-1$ SO

DEFINITELY $\left(\left(1+\frac{1}{n}\right)^n\right)^n > (e-1)^n \to +\infty$

$\boxed{0^0}$ $x_n \to 0^+$ $y_n \to 0$

$\log(x_n) \to -\infty$ SO

IND. FORM $(-\infty)\cdot 0$ FOR $\log(x_n)\,y_n$

$(+\infty)^0$ $x_n \to +\infty$ $y_n \to 0$

$\log(x_n) \to +\infty$ SO

IND. FORM $(+\infty)\cdot 0$ FOR $\log(x_n)\cdot y_n$

$\frac{0}{0}$ $\frac{\infty}{\infty}$

THEOREM LET $\{x_n\}_{n\in\mathbb{N}}$ BE A SEQUENCE OF REAL NUMBERS SUCH THAT $x_n \to 0$ AND $x_n \neq 0$ $\forall n \in \mathbb{N}$. THEN

$$\boxed{\lim_n \frac{e^{x_n}-1}{x_n} = 1} \quad \text{AND} \quad \boxed{\lim_n \frac{\log(1+x_n)}{x_n} = 1}$$

PROOF WE BEGIN WITH THE SECOND LIMIT

LET $y_n \to \pm\infty$

$$\lim_n \log\Big(\underbrace{\Big(1+\frac{1}{y_n}\Big)^{y_n}}_{e}\Big) = \log(e) = 1$$

IN PARTICULAR, IF z_n IS SUCH THAT $|z_n| \to +\infty$, THEN

$$\lim_n \log\left(\left(1+\frac{1}{z_n}\right)^{z_n}\right) = 1$$

$$\log\left(\left(1+\frac{1}{z_n}\right)^{z_n}\right) = z_n \log\left(1+\frac{1}{z_n}\right) = \frac{\log\left(1+\frac{1}{z_n}\right)}{1/z_n}$$

$x_n \to 0$, $x_n \neq 0$ THEN $z_n = \frac{1}{x_n}$ SATISFIES $|z_n| \to +\infty$

BUT $\frac{1}{z_n} = x_n$ SO THE LIMIT IS PROVED

THE FIRST LIMIT

LET $x_N \to 0$ $x_N \neq 0$. CALL $W_N = e^{x_N} - 1 \to 0$ $W_N \neq 0$

SO $1 = \lim_N \frac{\log(1+W_N)}{W_N} = \lim_N \frac{x_N}{e^{x_N}-1}$

$\Rightarrow \lim_N \frac{1}{\frac{x_N}{e^{x_N}-1}} = 1$

$$\frac{1}{\frac{x_N}{e^{x_N}-1}} = \frac{e^{x_N}-1}{x_N}$$

TRIGONOMETRIC FUNCTIONS

RECALL

IF $0 < \alpha < \frac{\pi}{2}$ THEN $0 < \sin\alpha < \alpha < \tan\alpha$

IF $|\alpha| < \frac{\pi}{2}$

$0 \leq |\sin\alpha| \leq |\alpha| \leq |\tan\alpha|$

a) LET $x_N \to x \in \mathbb{R}$ THEN

$\lim_N \sin(x_N) = \sin(x)$ AND $\lim_N \cos(x_N) = \cos(x)$

PROOF FIRST $\boxed{x=0}$

$x_N \to 0$ THEN DEFINETELY $|x_N| < \frac{\pi}{2}$

$0 \leq |\sin(x_N)| \leq |x_N| \to 0$

$$\Rightarrow |\sin(x_n)| \to 0 \Rightarrow \sin(x_n) \to 0 = \sin(0)$$

$\forall \alpha$ SUCH THAT $|\alpha| < \frac{\pi}{2}$ WE HAVE

$$\cos(\alpha) > 0, \text{ HENCE } \cos(\alpha) = \sqrt{1 - \sin^2(\alpha)}$$

$$\cos(x_n) = \sqrt{1 - \sin^2(x_n)} \to 1 = \cos(0)$$

$$\sin(x_n) = \sin(\underbrace{x_n - x} + \underbrace{x})$$

$$\sin(x_n - x)\cos(x) + \sin(x)\cos(x_n - x) \to 0 \cdot \cos(x) + \sin(x) \cdot 1 = \sin(x)$$

$$\cos(x_n) = \cos(x_n - x + x) = \cos(x_n - x)\cos(x) - \sin(x_n - x)\sin(x) \to 1 \cdot \cos(x) - 0 \sin(x) = \cos(x)$$

WHAT HAPPENS IF $x_n \to \pm\infty$? IT DEPENDS ON THE SEQUENCE!

$$\nexists \lim_n \sin(n); \quad \nexists \lim_n \cos(n); \quad \nexists \lim_n \cos(n\pi) = \lim_n (-1)^n$$

BUT $\lim_n \sin(n\pi) = \lim_n 0 = 0$; $\lim_n \cos(1 + 2n\pi) = \cos(1)$

THEOREM: $\{x_n\}_{n\in\mathbb{N}}$ SEQUENCE OF REAL NUMBERS SUCH THAT

$x_n \to 0$ AND $x_n \neq 0$ $\forall n \in \mathbb{N}$ THEN

$$\boxed{\lim_n \frac{\sin(x_n)}{x_n} = 1} \quad \text{AND} \quad \boxed{\lim_n \frac{1-\cos(x_n)}{x_n^2} = \frac{1}{2}}$$

PROOF:

$x_n \to 0$, $x_n \neq 0$ THEN DEFINETLY

$0 < |x_n| < \frac{\pi}{2}$ HENCE

$$0 < |\sin(x_n)| < |x_n| < |\tan(x_n)| = \frac{|\sin(x_n)|}{|\cos(x_n)|}$$

THEN

$$1 < \frac{|x_n|}{|\sin(x_n)|} < \frac{1}{|\cos(x_n)|}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$1 \qquad 1 \qquad 1$$

$\Rightarrow \lim_n \left|\frac{x_n}{\sin(x_n)}\right| = 1$ SO, $\lim_n \left|\frac{\sin(x_n)}{x_n}\right| = 1$

SIN IS ODD SO

$$\frac{\sin(x_n)}{x_n} = \left|\frac{\sin(x_n)}{x_n}\right| \qquad 0 < |x_n| < \frac{\pi}{2}$$

$$\cos(x_n) = \cos\left(2\frac{x_n}{2}\right) = \cos^2\left(\frac{x_n}{2}\right) - \sin^2\left(\frac{x_n}{2}\right) =$$

$$1 - 2\sin^2\left(\frac{x_n}{2}\right)$$

$$1 - \cos(x_n) = 2\sin^2\left(\frac{x_n}{2}\right)$$

$$\lim_n \frac{1-\cos(x_n)}{x_n^2} = \lim_n \frac{2\sin^2\left(\frac{x_n}{2}\right)}{4\,\frac{x_n^2}{4}} =$$

$$= \lim_n \frac{1}{2}\,\frac{\sin^2\left(\frac{x_n}{2}\right)}{\left(\frac{x_n}{2}\right)^2} = \lim_n \frac{1}{2}\left(\frac{\sin\left(\frac{x_n}{2}\right)}{\frac{x_n}{2}}\right)^2 = \frac{1}{2}\cdot 1^2 = \frac{1}{2}$$

1

EXAMPLE:

$$\nexists \lim_n (n\sin(n))$$

$$\lim_n n\sin\left(\frac{1}{n}\right) = 1 \qquad \infty, 0$$

- $n\sin\left(\frac{1}{n}\right) = \dfrac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \to 1 \qquad \frac{0}{0}$

- $\lim_n \dfrac{\sin^2\left(\frac{1}{n}\right)}{\frac{1}{n^2}} = 1$

$$\left(\frac{\sin\left(\frac{1}{n}\right)}{1/n}\right)^2 \to 1^2 = 1$$

- $\lim_n \dfrac{\sin\left(1/n^2\right)}{1/n^2} = 1$

$$x_n = \frac{1}{n^2} \to 0 \quad x_n \neq 0$$

$$\lim_n \frac{\sin^2(1/n)}{\sin(2/n^2)} = \frac{1}{2}$$

$$\underbrace{\frac{\sin^2(1/n)}{1/n^2}}_{1} \; \frac{1/\cancel{n^2}}{2/\cancel{n^2}} \; \underbrace{\frac{2/n^2}{\sin(2/n^2)}}_{1}$$

$$\frac{\sin(1/n^k)}{1/n^k} \to 1$$

- $$\lim_n \frac{\log\left(1 + \frac{2}{n^2+1}\right)}{1/n^2} = 2$$

$$\frac{\log\left(1+\frac{2}{n^2+1}\right)}{2/(n^2+1)} \; \frac{2/(n^2+1)}{1/n^2} = \overbrace{\frac{\log\left(1+2/(n^2+1)\right)}{\frac{2}{n^2+1}}}^{1} \, 2 \, \frac{n^2}{n^2+1} \to 1 \cdot 2 \cdot 1 = 2$$

INFINITE AND INFINITESIMAL ORDER

RATIO CRITERION FOR SEQUENCES

LET $\{x_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$, WITH $x_n > 0 \quad \forall n \in \mathbb{N}$

SUPPOSE $\exists \lim_n \frac{x_{n+1}}{x_n} = \alpha \in [0, +\infty]$

THEN WE HAVE

i) IF $0 \leq \alpha < 1$, THEN $\lim_n x_n = 0$

ii) IF $1 < \ell$ (INCLUDING $\ell = +\infty$), THEN $\lim_N x_N = +\infty$

INFINITE ORDERS

WE HAVE THE FOLLOWING ORDERING (FROM THE SMALLER TO THE BIGGER) AMONG THE FOLLOWING INFINITIES

$\forall a > 0,\ 0 < b < b_1,\ \lambda > 0$

$$(\log N)^a \quad N^b \quad N^{b_1} \quad e^{\lambda N} \quad N! \quad N^N$$

THAT IS

$$\lim_N \frac{(\log N)^a}{N^b} = \lim_N \frac{N^b}{N^{b_1}} = \lim_N \frac{N^{b_1}}{e^{\lambda N}} = \lim_N \frac{e^{\lambda N}}{N!} = \lim_N \frac{N!}{N^N} = 0$$

MOREOVER $\forall x_N \to +\infty$ WE HAVE

$$\lim_N \frac{(\log(x_N))^a}{x_N^b} = \lim_N \frac{x_N^b}{x_N^{b_1}} = \lim_N \frac{x_N^{b_1}}{e^{\lambda x_N}} = \lim_N \frac{e^{\lambda x_N}}{x_N^{x_N}} = 0$$

PROOF $\lim_N \frac{x_N}{x_N^{b_1}} = 0$ COMES FROM THE LIMIT OF POWERS

$$\frac{x_N^b}{x_N^{b_1}} = \underbrace{x_N}_{\to +\infty}^{\overbrace{b-b_1}^{<0}} \to 0$$

- $\lim_N \frac{N^{b_1}}{e^{\lambda N}} = 0$

$$y_N = \frac{N^{b_1}}{e^{\lambda N}} \qquad \frac{y_{N+1}}{y_N} = \frac{\frac{(N+1)^{b_1}}{e^{\lambda(N+1)}}}{\frac{N^{b_1}}{e^{\lambda N}}} = \left(\frac{N+1}{N}\right)^{b_1} e^{-\lambda} \to 1^{b_1} \cdot e^{-\lambda}$$

$- e^{-\alpha} < 1$ $(\alpha > 0)$

• $\lim_n \frac{e^{\alpha n}}{n!} = 0$

$$\frac{e^{\alpha n}}{n!} = \frac{e^\alpha}{1} \cdot \frac{e^\alpha}{2} \cdots \frac{e^\alpha}{n}$$

LET $n_0 \in \mathbb{N}$ BE SUCH THAT $\frac{e^\alpha}{n} \leq 1$ $\forall n \geq n_0$

$$\frac{e^{\alpha n}}{n!} = \underbrace{\frac{e^\alpha}{1} \cdots \frac{e^\alpha}{n_0}}_{C} \cdot \underbrace{\frac{e^\alpha}{n_0+1} \cdots \frac{e^\alpha}{n-1}}_{\leq 1} \cdot \frac{e^\alpha}{n} \leq \quad \forall n \geq 0$$

$$\leq C \frac{e^\alpha}{n} \to 0$$

• $\lim_n \frac{n!}{n^n} = 0$

$$\frac{n!}{n^n} = \underbrace{\frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdots \frac{n}{n}}_{\leq 1} \leq \frac{1}{n} \to 0$$

• ASSUME $\lim_n \frac{x_n^{b_1}}{e^{\alpha x_n}} = 0$ $\forall x_n \to +\infty$

WE PROVE $\lim_N \frac{(\log x_N)^a}{x_N{}^b} = 0 \quad \forall x_N \to +\infty$

$$\lim_N \left(\frac{\log x_N}{x_N{}^{b/a}}\right)^a = \lim_N \left(\frac{y_N}{(e^{y_N})^{b/a}}\right)^a = \lim_N \left(\left(\frac{y_N}{e^{(b/a)y_N}}\right)\right)^a = 0$$

$$y_N = \log(x_N) \to +\infty \quad x_N = e^{y_N}$$

0 $\quad \alpha = \frac{b}{a}$

INFINITESIMAL ORDERS

WE HAVE THE FOLLOWING ORDERING (FROM THE SMALLER TO THE BIGGER) AMONG THE FOLLOWING INFINITESIMAL

$$\forall a > 0, \quad 0 < b < b_1, \quad \alpha > 0$$

$$(\log N)^{-a} \quad \frac{1}{N^b} \quad \frac{1}{N^{b_1}} \quad e^{-\alpha N} \quad \frac{1}{N!} \quad N^{-N}$$

THAT IS

$$\lim_N \frac{1/N^b}{(\log N)^{-a}} = \lim_N \frac{1/N^{b_1}}{1/N^b} = \lim_N \frac{e^{-\alpha N}}{1/N^{b_1}} = \lim_N \frac{1/N!}{e^{-\alpha N}} =$$

$$= \lim_N \frac{N^{-N}}{1/N!} = 0$$

MOROVER $\forall y_n \to 0^+$

$$\lim_n \frac{y_n^{\,b}}{(\log 1/y_n)^{-a}} = \lim_n \frac{y_n^{\,b_1}}{y_n^{\,b}} = \lim_n \frac{e^{-\alpha/y_n}}{y_n^{\,b_1}} =$$

$$= \lim_n \frac{(1/y_n)^{-1/y_n}}{e^{-\alpha/y_n}} = 0$$

PROOF TAKE $x_n = \frac{1}{y_n}$ AND USE THE INFINITE ORDERS

REMARK $a > 0$, $b > 0$ $\alpha > 0$

$$\lim_n x_n^{\,b} e^{-\alpha x_n} = 0 \quad \text{IF } x_n \to +\infty$$

$$\lim_n y_n^{\,b} |\log(y_n)|^{a} = 0 \quad \text{IF } y_n \to 0^+$$

$$|\log(y_n)| = -\log y_n = \log \frac{1}{y_n} \quad \text{IF } 0 < y_n < 1$$

EXAMPLE

$$\lim_n \sqrt[n]{n} = 1 \quad \lim_n n^{1/n} \quad (\text{IND FORM OF } (+\infty)^0)$$

$$n^{1/n} = (e^{\log n})^{1/n} = e^{(\log n)\cdot 1/n} \to e^0 = 1$$

$$\frac{\log n}{n} \to 0$$