Exercises - Calculus
Academic Year 2021-2022

Sheet 5

1. Using only the definition of limits, establish whether the next statements are true or false.
   (a) if $a_n = n^2$, then $\lim_{n\to+\infty} a_n = +\infty$
   (b) if $a_n = n$, then $\lim_{n\to+\infty} a_n = 0$
   (c) if $a_n = \dfrac{1}{n+2}$, then $\lim_{n\to+\infty} a_n = 0$
   (d) if $a_n = |1-n|$, then $\lim_{n\to+\infty} a_n = +\infty$
   (e) if $a_n = 1-n^2$, then $\lim_{n\to+\infty} a_n = +\infty$
   (f) if $a_n = 2 + \dfrac{n+1}{n^2+2n+1}$, then $\lim_{n\to+\infty} a_n = 2$
   (g) if $a_n = \dfrac{n+2}{n+3}$, then $\lim_{n\to+\infty} a_n = 2$
   (h) if $a_n = \dfrac{5-n}{n+1}$, then $\lim_{n\to+\infty} a_n = -1$
   (i) if $a_n = \log(n)$, then $\lim_{n\to+\infty} a_n = 0$
2. Let $a_n = \sqrt{n}$ for any $n \in \mathbb{N}$. Prove that
$$\lim_n a_n = \lim_n \sqrt{n} = +\infty$$
   HINT: prove that a_n is increasing and unbounded from above.
3. Let $a_n = \sqrt{1+\frac{1}{n}}$ for any $n \in \mathbb{N}$. Prove that
$$\lim_n a_n = \lim_n \sqrt{1+\frac{1}{n}} = 1$$
   HINT: prove that a_n is decreasing and show that $\inf_{n\in\mathbb{N}} a_n = 1$.
4. Assume to know that $\lim_{n\to+\infty} e^n = +\infty$ and $\lim_{n\to+\infty} \log(n) = +\infty$, a fact that will be proved in the next lectures. Compute
$$\lim_{n\to+\infty} e^{-n} \quad \text{and} \quad \lim_{n\to+\infty} \log(1/n)$$
5. Compute, if it exists, $\lim_n a_n$ where
   (a) $a_n = 1+n$
   (b) $a_n = n+n^2$

(c) $a_n = n + n^3 - 1$

(d) $a_n = 2(n + 1)$

(e) $a_n = \frac{1}{n} + 2 + n$

(f) $a_n = \frac{1}{n} \frac{1}{n+3}$

(g) $a_n = \log(n) + ne^n$

(h) $a_n = n^2 + n + e^{-n}$

(i) $a_n = \frac{3n+1}{n}$

(j) $a_n = \frac{e^{-n}+1}{\log(1/n)}$

6. Establish whether the following limit is an indeterminate form and, if this is the case, establish which kind of indeterminate form it is.

(a) $\lim_n \frac{n+3}{n+4}$

(b) $\lim_n e^n - n^2$

(c) $\lim_n \frac{n^2}{\log(n)}$

(d) $\lim_n e^n - (5 - n^2$

(e) $\lim_n e^n \log(n)$

(f) $\lim_n e^{-n} \log(n)$

(g) $\lim_n \left(\frac{1}{e}\right)^n \sqrt{n}$

(h) $\lim_n \frac{n+3}{5}$

7. Establish whether the following sequences are monotone, bounded and, if possible, compute their limit as $n \to +\infty$.

(a) $a_n = n + (-1)^n$

(b) $a_n = 3n + (-1)^n$

(c) $a_n = n^3 - 1$

(d) $a_n = 3 - e^{-n}$

(e) $a_n = \sin(2n\pi) + n(-1)^n$

(f) $a_n = \frac{n+1}{n^2+1}$

(g) $a_n = \frac{\sqrt{n}}{\sqrt[3]{n}}$

8. If possible, extract two different subsequences with two different limits from the sequence $\{a_n\}_{n\in\mathbb{N}}$ where

(a) $a_n = n + (-1)^n$

(b) $a_n = 3n + (-1)^n$
(c) $a_n = |(-e)^n|$
(d) $a_n = \tan(n\pi)$
(e) $a_n = \cos(n\pi)$
(f) $a_n = \sin(n\pi/2)$
(g) $a_n = n^2 + (-1)^n n$
(h) $a_n = (-1)^n n^2 + n$
(i) $a_n = n + (-e)^n n$

9. Compute, if it exists,

   (a) $\lim_n \frac{n+3}{n+4}$
   (b) $\lim_n \frac{n^2+2n+1}{n+2}$
   (c) $\lim_n \frac{(-1)^n}{n}$
   (d) $\lim_n \sin(n\pi)$
   (e) $\lim_n \cos(2n\pi)$
   (f) $\lim_n \frac{n^3+n+1}{n+5}$
   (g) $\lim_n \frac{(n^3+1)(n+2)}{5n^5+3n^4+5n}$
   (h) $\lim_n \frac{\sqrt{n}}{n}$
   (i) $\lim_n \left(1 + \cos(n)e^{-n}\right)$
   (j) $\lim_n n^2 \left(\log(n) + (-1)^n\right)$
   (k) $\lim_n \left(2 + \frac{\cos(n)+2}{n}\right)$
   (l) $\lim_n \frac{e^{-n}}{(-1)^n - n}$
   (m) $\lim_n (-1)^n \frac{e^n+1}{e^{2n}}$

10. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $a_n < 0$ for any $n \in \mathbb{N}$. Prove that $\lim_n a_n = 0$ implies that $\sup_{n\in\mathbb{N}} a_n = 0$. Prove, by a counterexample, that $\sup_{n\in\mathbb{N}} a_n = 0$ does not imply that $\lim_n a_n = 0$.

11. Construct a sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ such that $a_n < 0$ for any $n \in \mathbb{N}$, $\lim_n a_n = 0$ and a_n is not increasing.

12. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers and $a \in \mathbb{R}$. Prove that $\lim_n a_n = a$ implies that $lim_n|a_n| = |a|$. Prove, by a counterexample, that $lim_n|a_n| = |a|$ does not imply $\lim_n a_n = a$.

13. Prove that the sequence $\{a_n\}_{n\in\mathbb{N}}$, given by $a_n = \dfrac{n!}{2^n}$ for any $n \in \mathbb{N}$, is increasing.

14. Compute, if it exists,

    (a) $\lim_n \left(n - \sqrt{n}\right)$

    (b) $\lim_n \left(\sqrt{n} - \sqrt[4]{n}\right)$

    (c) $\lim_n \left(\dfrac{\cos^2(n)}{n}\right)$

    (d) $\lim_n \left(\dfrac{n}{n+1} + \cos(n\pi/2)\right)$

    (e) $\lim_{n\to+\infty} \dfrac{\sin(n^2)}{n^2+4}$

    (f) $\lim_{n\to+\infty} \dfrac{n^3 - 3n + \sqrt{n}}{(n+1)^3}$

    (g) $\lim_{n\to+\infty} (-n\sqrt{n} + 3n)$

    (h) $\lim_n \left(\sqrt{n^4+4} - n^2\right)$

    (i) $\lim_n \left(\sqrt{n(n+1)} - n\right)$

    (j) $\lim_{n\to+\infty} \left(3n^{3/2} + (-1)^n\right)$

    (k) $\lim_{n\to+\infty} \dfrac{\sqrt{n^2+2n}}{n+1}\left(\sqrt{n^4+n^2+1} - n^2\right)$

    (l) $\lim_{n\to+\infty} \sqrt{n+2+\sin(n^3)}$

    (m) $\lim_{n\to+\infty} \dfrac{1}{n}\sqrt{n+2+\sin(n^3)}$

1. Using only the definition of limits, establish whether the next statements are true or false.

   (a) if $a_n = n^2$, then $\lim\limits_{n \to +\infty} a_n = +\infty$ T

   (b) if $a_n = n$, then $\lim\limits_{n \to +\infty} a_n = 0$

   (c) if $a_n = \dfrac{1}{n+2}$, then $\lim\limits_{n \to +\infty} a_n = 0$

   (d) if $a_n = |1-n|$, then $\lim\limits_{n \to +\infty} a_n = +\infty$

   (e) if $a_n = 1 - n^2$, then $\lim\limits_{n \to +\infty} a_n = +\infty$

   (f) if $a_n = 2 + \dfrac{n+1}{n^2+2n+1}$, then $\lim\limits_{n \to +\infty} a_n = 2$

   (g) if $a_n = \dfrac{n+2}{n+3}$, then $\lim\limits_{n \to +\infty} a_n = 2$

   (h) if $a_n = \dfrac{5-n}{n+1}$, then $\lim\limits_{n \to +\infty} a_n = -1$

   (i) if $a_n = \log(n)$, then $\lim\limits_{n \to +\infty} a_n = 0$

a) $\lim\limits_{N} N^2 = \lim\limits_{N} N \cdot N = \infty \cdot \infty = \infty$ T

b) F $\lim\limits_{N} N = \infty$ $\quad \infty \neq 0$

c) $\lim\limits_{N} \dfrac{1}{N+2} = \lim\limits_{N} = 0$

D) $\lim\limits_{N} |1 - N| =$

2. Let $a_n = \sqrt{n}$ for any $n \in \mathbb{N}$. Prove that

$$\lim_n a_n = \lim_n \sqrt{n} = +\infty$$

HINT: prove that a_n is increasing and unbounded from above.

3. Let $a_n = \sqrt{1 + \frac{1}{n}}$ for any $n \in \mathbb{N}$. Prove that

$$\lim_n a_n = \lim_n \sqrt{1 + \frac{1}{n}} = 1$$

HINT: prove that a_n is decreasing and show that $\inf_{n \in \mathbb{N}} a_n = 1$.

4. Assume to know that $\lim_{n \to +\infty} e^n = +\infty$ and $\lim_{n \to +\infty} \log(n) = +\infty$, a fact that will be proved in the next lectures. Compute

$$\lim_{n \to +\infty} e^{-n} \quad \text{and} \quad \lim_{n \to +\infty} \log(1/n)$$

$$\lim_{N \to +\infty} e^{-N} = \lim_N = \frac{1}{e^N} = 0$$

$$\lim_{N \to +\infty} \log\left(\frac{1}{N}\right) = -\infty$$

5. Compute, if it exists, $\lim_{n} a_n$ where

(a) $a_n = 1 + n$ ∞

(b) $a_n = n + n^2$ ∞

(c) $a_n = n + n^3 - 1$ ∞

(d) $a_n = 2(n+1)$ ∞

(e) $a_n = \frac{1}{n} + 2 + n$ ∞

(f) $a_n = \frac{1}{n}\frac{1}{n+3}$ 0

(g) $a_n = \log(n) + ne^n$ ∞

(h) $a_n = n^2 + n + e^{-n}$ ∞

(i) $a_n = \frac{3n+1}{n}$ $\frac{\cancel{n}(3+1/n)}{\cancel{n}} = 3$

(j) $a_n = \frac{e^{-n}+1}{\log(1/n)}$

$= \frac{1/\infty + 1}{\infty} = \frac{1}{-\infty} = 0^-$

6. Establish whether the following limit is an indeterminate form and, if this is the case, establish which kind of indeterminate form it is.

(a) $\lim_n \frac{n+3}{n+4}$

(b) $\lim_n e^n - n^2$

(c) $\lim_n \frac{n^2}{\log(n)}$

(d) $\lim_n e^n - (5 - n^2)$

(e) $\lim_n e^n \log(n)$

(f) $\lim_n e^{-n} \log(n)$

(g) $\lim_n \left(\frac{1}{e}\right)^n \sqrt{n}$

(h) $\lim_n \frac{n+3}{5}$

a) $\left[\frac{\infty}{\infty}\right]$ $\lim_N \frac{\cancel{N}(1+3/N)}{\cancel{N}(1+4/N)} = 1$

b) $[\infty - \infty]$ $\lim_N e^N - N^2 = \infty$

c) $\left[\frac{\infty}{\infty}\right]$ $\lim_N \frac{N^2}{\log(N)} = \infty$

d) $\lim_N e^N - (5 - N^2)$ ∞

e) $\lim_N e^N \log(N) = \infty$

f) $\lim_N e^{-\infty} \log(N)$ $[0 \cdot \infty]$

8) $\lim_{N} \left(\frac{1}{e}\right)^N \sqrt{N}$ $[0 \cdot \infty]$

h) ∞

7) • $\lim_{N} N + (-1)^N$ NON BOUNDED

NON MONOTONA $+\infty$

• $3N + (-1)^N$ MONOTONA

NON BOUNDED

$+\infty$

• $\lim_{N} N^3 - 1$ MONOTONA

$+\infty$

NON BOUNDED

• $\lim_{N} 3 - e^{-N}$ 3

BOUNDED

MONOTONE

• $\lim_{N} \sin(2N\pi) + N(-1)^N$

(e) an = sin(2nπ) + n(−1)n

$\sin(2N\pi) \in [-1, 1]$

NOT BOUNDED

NO MONOTONE

• $\lim \dfrac{N+1}{N^2+1}$ $\quad \dfrac{\cancel{N}(1+1/N)}{\cancel{N}(N+1)}$ $\quad \dfrac{1}{\infty} = 0$

8) $\dfrac{\sqrt{N}}{\sqrt[3]{N}}$

MONOTONA
INCREASING

NOT BOUNDED

$N^{1/2} \cdot N^{-1/3} = N^{-1/6} =$

8A)

$\dfrac{1}{\sqrt[6]{N}}$

$$\lim_{N} \left(\sqrt{N} - \sqrt[4]{N}\right)$$

$$\lim_{N} \left(\sqrt{N} - \sqrt[4]{N}\right) \frac{\left(\sqrt{N} + \sqrt[4]{N}\right)}{\left(\sqrt{N} + \sqrt[4]{N}\right)}$$

$$\lim_{N} \frac{N - \sqrt{N}}{\left(\sqrt{N} + \sqrt[4]{N}\right)} = \frac{\cancel{\sqrt{N}}\left(\sqrt{N} - 1\right)}{\cancel{\sqrt{N}}\left(1 + \frac{1}{\sqrt{N}}\right)}$$

$$= \frac{\sqrt{N} - 1}{1 - \frac{1}{\sqrt{N}}} = \frac{\infty}{1}$$

14. Compute, if it exists,

(a) $\lim_{n} \left(n - \sqrt{n}\right)$

(b) $\lim_{n} \left(\sqrt{n} - \sqrt[4]{n}\right)$

(c) $\lim_{n} \left(\frac{\cos^2(n)}{n}\right)$

(d) $\lim_{n} \left(\frac{n}{n+1} + \cos(n\pi/2)\right)$

(e) $\lim_{n\to+\infty} \frac{\sin(n^2)}{n^2+4}$

(f) $\lim_{n\to+\infty} \frac{n^3 - 3n + \sqrt{n}}{(n+1)^3}$

(g) $\lim_{n\to+\infty} \left(-n\sqrt{n} + 3n\right)$

(h) $\lim_{n} \left(\sqrt{n^4+4} - n^2\right)$

(i) $\lim_{n} \left(\sqrt{n(n+1)} - n\right)$

(j) $\lim_{n\to+\infty} \left(3n^{3/2} + (-1)^n\right)$

(k) $\lim_{n\to+\infty} \frac{\sqrt{n^2+2n}}{n+1}\left(\sqrt{n^4+n^2+1} - n^2\right)$

(l) $\lim_{n\to+\infty} \sqrt{n+2+\sin(n^3)}$

(m) $\lim_{n\to+\infty} \frac{1}{n}\sqrt{n+2+\sin(n^3)}$

c) $$\lim_{N} \frac{\cos^2(N)}{N} = 0^+$$

D) $$\lim_{N} \left(\frac{N}{N+1} + \left(\cos \frac{N\pi}{2}\right)\right)$$

$$\lim_{N} \left(\frac{\cancel{N}}{\cancel{N}\left(1 + \frac{1}{N}\right)} + \underbrace{\cos \frac{N\pi}{2}}_{\Downarrow\, 0}\right) = 1$$

$\cos^2 x + \sin^2 x = 1$

e) $\lim\limits_{n\to+\infty} \dfrac{\sin(n^2)}{n^2+4}$

$[-1, 1]$

$\lim\limits_{n\to+\infty} \dfrac{\sin(n^2)}{n^2+4} =$ IND

f) $\lim\limits_{n\to+\infty} \dfrac{n^3-3n+\sqrt{n}}{(n+1)^3}$

$\lim\limits_{n} \dfrac{n^3-3n+\sqrt{n}}{n^3+1+3n^2+3n}$

$\lim\limits_{n} \dfrac{\cancel{n^3}\left(1-3/n^2+\frac{1}{\sqrt{n^5}}\right)}{\cancel{n^3}\left(1+\frac{1}{n^3}+\frac{3}{n}+\frac{3}{n^2}\right)} = 1$

g) $\lim\limits_{n\to+\infty} \left(-n\sqrt{n}+3n\right)\dfrac{\left(-n\sqrt{n}-3n\right)}{\left(-n\sqrt{n}-3n\right)}$

$\lim\limits_{n\to+\infty} = n^3 - 9n^2$

14. Compute, if it exists,

(a) $\lim_n \left(n-\sqrt{n}\right)$

(b) $\lim_n \left(\sqrt{n}-\sqrt[4]{n}\right)$

(c) $\lim_n \left(\dfrac{\cos^2(n)}{n}\right)$

(d) $\lim_n \left(\dfrac{n}{n+1}+\cos(n\pi/2)\right)$

(e) $\lim\limits_{n\to+\infty} \dfrac{\sin(n^2)}{n^2+4}$

(f) $\lim\limits_{n\to+\infty} \dfrac{n^3-3n+\sqrt{n}}{(n+1)^3}$

(g) $\lim\limits_{n\to+\infty} (-n\sqrt{n}+3n)$

(h) $\lim_n \left(\sqrt{n^4+4}-n^2\right)$

(i) $\lim_n \left(\sqrt{n(n+1)}-n\right)$

(j) $\lim\limits_{n\to+\infty} (3n^{3/2}+(-1)^n)$

(k) $\lim\limits_{n\to+\infty} \dfrac{\sqrt{n^2+2n}}{n+1}\left(\sqrt{n^4+n^2+1}-n^2\right)$

(l) $\lim\limits_{n\to+\infty} \sqrt{n+2+\sin(n^3)}$

(m) $\lim\limits_{n\to+\infty} \dfrac{1}{n}\sqrt{n+2+\sin(n^3)}$