**DEFINITION** A SET OF VECTORS $\{\underline{v_1}, \underline{v_2}, \dots \underline{v_p}\} \in \mathbb{R}^N$ IS CALLED LINEARLY INDIPENDENT (ONLY ONE SOLUTION)

IF THE ONLY SOLUTION $\{x_1, \dots x_p\}$ OF THE PROBLEM $x_1 \underline{v_1} + x_2 \underline{v_2} + \dots x_p \underline{v_p} = 0$

IS THE $\{x_1, x_2, \dots x_p\} = \{0, 0, 0 \dots 0\}$ SOLUTION

ON CONTRARY, A SET OF VECTORS $\{\underline{v_1}, \underline{v_2}, \dots \underline{v_p}\} \in \mathbb{R}^N$ IS CALLED LINEARLY DEPENDENT ($\exists$ SOLUTION $\neq \underline{0}$)

IF IT EXIST $\{c_1, c_2, \dots c_p\} \in \mathbb{R}$ SUCH THAT

$$c_1 \underline{v_1} + c_2 \underline{v_2} + \dots + c_p \underline{v_p} = 0 \quad \text{WITH } \{c_1, c_2 \dots c_p\} \neq \{0, 0 \dots 0\}$$

$\{\underline{v_1}, \underline{v_2}, \dots \underline{v_p}\} \in \mathbb{R}^N$

$$c_1 \underline{v_1} + c_2 \underline{v_2} + \dots + c_p \underline{v_p} = 0 \qquad c_1, c_2, \dots c_p \in \mathbb{R}$$

$\begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix} \in \mathbb{R}^3$ ARE LINEARLY DEPENDENT OR INDIPENDENT?

$$c_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + c_2 \begin{bmatrix}0\\1\\0\end{bmatrix} + c_3 \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

IF $c_1 = c_2 = c_3 = 0 \longrightarrow$ LINEARLY INDIPENDENT

BUT AT LEAST ONE $\neq 0 \longrightarrow$ LINEARLY DEPENDENT

$\begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}1\\1\\0\end{bmatrix}$

$$c_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + c_2 \begin{bmatrix}0\\1\\0\end{bmatrix} + c_3 \begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \xrightarrow[c_1, c_2, c_3]{\text{CHOOSE}} 1 \begin{bmatrix}1\\0\\0\end{bmatrix} + 1 \begin{bmatrix}0\\1\\0\end{bmatrix} - 1 \begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

IS LINEARLY DEPENDENT BECAUSE I FIND ONE SOLUTION THAT GIVES $[0, 0, 0]$ THAT IS NOT $c_1 = c_2 = c_3 = 0$

$$\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$c_1 \underline{v_1} + c_2 \underline{v_2} + \dots + c_p \underline{v_p} = 0$ LIN. DEPENDENT ASSUME $c_2 \neq 0$ $\quad c_2 \underline{v_2} = -c_1 \underline{v_1} - c_3 \underline{v_3} - \dots - c_p \underline{v_p}$

$$\Downarrow$$

$$\underline{v_2} = -\left(\frac{c_1}{c_2}\right)\underline{v_1} - \left(\frac{c_3}{c_2}\right)\underline{v_3} - \dots - \left(\frac{c_p}{c_2}\right)\underline{v_p}$$

**THEORY** A SET OF VECTORS $\{\underline{v_1}, \underline{v_2}, \dots \underline{v_p}\} \in \mathbb{R}^N$ ARE LINEARLY DEPENDENT IF AND ONLY IF (AT LEAST) ONE OF THEM CAN BE WRITTEN AS A LINEAR COMBINATION OF THE OTHERS

SPECIAL CASE: ONLY ONE VECTOR

$$\{ \underline{v_1} \} \in \mathbb{R}^N \qquad c_1 \underline{v_1} = \underline{0} \qquad c_1 \in \mathbb{R} \quad (c_1 \neq 0)$$

$$\Downarrow$$

$$\underline{v_1} = \frac{1}{c_1} \cdot \underline{0} = \underline{0}$$

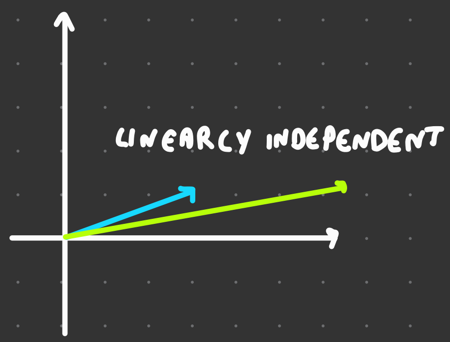
CAN'T HAVE BOTH $c_1 = c_2 = 0$

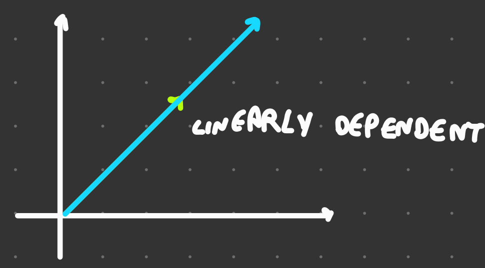
$$\{ \underline{v_1}, \underline{v_2} \} \in \mathbb{R}^N \qquad c_1 \underline{v_1} + c_2 \underline{v_2} = \underline{0} \quad \text{WITH } c_1, c_2 \in \mathbb{R} \qquad \{c_1, c_2\} \neq \{0, 0\}$$

LINEARLY DEPENDENT. ASSUME $c_1 \neq 0$

$$c_1 v_1 = -c_2 v_2$$

$$v_1 = \underbrace{\left(-\frac{c_2}{c_1}\right)}_{\in \mathbb{R}} v_2$$




$$\{ \underline{v_1}, \underline{v_2}, \dots, \underline{v_p} \} \in \mathbb{R}^N \qquad \{x_1, x_2, \dots x_p\} \in \mathbb{R} \qquad A \in \mathbb{R}^{N \times P} \qquad A = [\underline{v_1} | \underline{v_2} | \underline{v_3} | \dots | \underline{v_p}]$$

USE X INSTEAD OF X

$$x_1 \underline{v_1} + x_2 \underline{v_2} + \dots + x_p \underline{v_p} = \underline{0} \qquad \{x_1, x_2, \dots x_p\} \neq \{0, 0, \dots 0\}$$

$$x = [x_1, x_2, \dots x_p] \in \mathbb{R}^P$$

$$A\underline{x} = \underline{v_1} x_1 + \underline{v_2} x_2 + \dots \underline{v_p} x_p = 0$$

CAN I FIND $\underline{x} \in \mathbb{R}^P$ WITH $\underline{x} \neq \underline{0}$ SUCH THAT $A\underline{x} = \underline{0}$

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$\{ \underline{v_1}, \underline{v_2}, \dots \underline{v_p} \} \in \mathbb{R}^N$ ARE THEY LINEARLY DEPENDENT?

1. BUILD THE MATRIX $A = [\underline{v_1}, \underline{v_2}, \underline{v_3}, \dots \underline{v_p}] \in \mathbb{R}^{N \times P}$

   AND THEN ASK YOURSELF IF THE HOMOGENEOUS LINEAR SYSTEM $A\underline{x} = \underline{0}$

   HAS A NON-TRIVIAL SOLUTION $\underline{x} \neq \underline{0}$

2. HOW DO WE DO THIS?

   BRING $A$ INTO ECHELON FORM SO THE VECTOR ARE LINEARLY DEPENDENT IF AND ONLY IF THERE IS AT LEAST ONE FREE VARIABLE (IN OTHER WORD, IF AND ONLY IF THERE IS AT LEAST ONE COLUM OF THE MATRIX WITHOUT A PIVOT POSITION)

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$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 8 \end{bmatrix} \in \mathbb{R}^3 \text{ ARE LINEARLY DEPENDENT OR INDIPENDENT?}$$

ECHELON FORM

$$A = \begin{bmatrix} \boxed{1} & 2 & 4 \\ 0 & 3 & 3 \\ 2 & 4 & 8 \end{bmatrix} \quad A\underline{x} = \underline{0} \quad \begin{bmatrix} \boxed{1} & 2 & 4 \\ 0 & \boxed{3} & 3 \\ 0 & 0 & 0 \end{bmatrix} \qquad \underline{v_1} x_1 + \underline{v_2} x_2 + \dots + \underline{v_p} x_p$$

X3 FREE VARIABLE

LINEARLY DEPENDENT

CAN YOU EXPLICIT A LINEAR COMBINATION OF SUCH VECTOR (NON-TRIVIAL) THAT GIVES THE $\underline{0}$ VECTOR?

↳ NOT ALL ARE 0

$$x_1\begin{bmatrix}1\\0\\2\end{bmatrix}+x_2\begin{bmatrix}2\\3\\4\end{bmatrix}+x_3\begin{bmatrix}4\\3\\8\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \qquad x_1, x_2, x_3 \neq \{0,0,0\}$$

2ND) $3x_2+3x_3=0$; $x_2=-x_3$

1ST) $x_1+2x_2+4x_3=0$; $x_1=-2x_2-4x_3$; $x_1=-2x_3$

$$\begin{cases}x_1=-2x_3\\x_2=-x_3\\x_3=x_3\in\mathbb{R}\end{cases} \qquad \underline{x}=\begin{bmatrix}-2\\-1\\1\end{bmatrix}x_3$$

ES: $x_3=1$ $\begin{bmatrix}-2\\-1\\1\end{bmatrix}$ IS A SOLUTION

$$-2\begin{bmatrix}1\\0\\2\end{bmatrix}-1\begin{bmatrix}2\\3\\4\end{bmatrix}+1\begin{bmatrix}4\\3\\8\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

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$$\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}0\\-1\\-1\end{bmatrix},\begin{bmatrix}2\\-1\\4\end{bmatrix}$$

$$\begin{bmatrix}\boxed{1}&0&2\\2&-1&-1\\3&-1&4\end{bmatrix}\begin{matrix}\\2\\3\end{matrix}=\begin{bmatrix}\boxed{1}&0&2\\0&\boxed{-1}&-5\\0&-1&-2\end{bmatrix}\begin{matrix}\\\\1\end{matrix}=\begin{bmatrix}\boxed{1}&0&2\\0&\boxed{-1}&-5\\0&0&\boxed{3}\end{bmatrix}$$

ECHELON FORM

NO FREE VARIABLE

LINEARLY INDIPENDENT
ONLY SOLUTION IS THE $\underline{0}$ VECTOR

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$$\begin{bmatrix}-1\\0\\2\\1\end{bmatrix}\begin{bmatrix}1\\-1\\0\\3\end{bmatrix}\begin{bmatrix}-2\\1\\2\\-2\end{bmatrix}$$

ECHELON FORM

$$A=\begin{bmatrix}\boxed{-1}&1&-2\\0&-1&1\\2&0&2\\1&3&-2\end{bmatrix}\begin{matrix}\\\\-2\\-1\end{matrix}=\begin{bmatrix}-1&1&-2\\0&\boxed{-1}&1\\0&2&-2\\0&4&-4\end{bmatrix}\begin{matrix}\\\\-2\\-4\end{matrix}=\begin{bmatrix}\boxed{-1}&1&-2\\0&\boxed{-1}&1\\0&0&0\\0&0&0\end{bmatrix}$$

LINEARLY DEPENDENT

$x_3$ FREE VARIABLE

2ND $-x_2+x_3=0$; $x_2=x_3$

1ST $-x_1+x_2-2x_3=0$; $x_1=x_2-2x_3$; $x_1=-x_3$

$$\begin{cases}x_1=-x_3\\x_2=x_3\\x_3=x_3\in\mathbb{R}\end{cases} \qquad \underline{x}=\begin{bmatrix}-1\\1\\1\end{bmatrix}x_3$$

CHOICE $x_3=1$ $\underline{x}=\begin{bmatrix}-1\\1\\1\end{bmatrix}$

$$-1\begin{bmatrix}-1\\0\\2\\1\end{bmatrix}+1\begin{bmatrix}1\\-1\\0\\3\end{bmatrix}+1\begin{bmatrix}-2\\1\\2\\-2\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

THINGS TO THINK $\{V_1, V_2, \dots V_p\} \in \mathbb{R}^N$

6 VECTORS OF 3 COMPONENTS

$\downarrow$ P $\quad$ $\downarrow$ $N=3$ $\mathbb{R}^3$

- IF $P>N$ THE MUST BE LINEARLY DEPENDENT
- IF ONE OF THE VECTOR IS EQUAL TO $\underline{0}$ THE VECTORS ARE LINEARLY DEPENDENT

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$A \in \mathbb{R}^{m\times N}$, $B \in \mathbb{R}^{m\times N}$, $s \in \mathbb{R}$

$$A+B = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1N} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2N} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mN} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1N} \\ b_{21} & b_{22} & \dots & b_{2N} \\ \vdots & & & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mN} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1N}+b_{1N} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2N}+b_{2N} \\ \vdots & & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mN}+b_{mN} \end{bmatrix}$$

$$sA = \begin{bmatrix} sa_{11} & sa_{12} & \dots & s\,a_{1N} \\ sa_{21} & sa_{22} & \dots & sa_{2N} \\ \vdots & & & \vdots \\ sa_{m1} & sa_{m2} & \dots & sa_{mN} \end{bmatrix}$$

PROPERTY: $\underbrace{A, B, C}_{\text{MATRIX}} \in \mathbb{R}^{m\times N}$ $\quad \underbrace{r, s}_{\text{SCALAR}} \in \mathbb{R}$

$$A+B = B+A$$

$$(A+B)+C = A+(B+C)$$

$\underline{0}$ MATRIX WITH ALL 0

$$A + \underline{0} = A$$

$$s(A+B) = sA + sB$$

$$(r+s)A = rA + sA$$

$$r(sA) = (rs)A$$

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MATRIX MULTIPLICATIONS $\quad A \in \mathbb{R}^{m\times N}$, $B \in \mathbb{R}^{N\times P}$ THEN $C = AB \in \mathbb{R}^{m\times P}$

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}} \underset{B}{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix}} = \begin{bmatrix} (a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}+a_{14}b_{41}) & (a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}+a_{14}b_{42}) \\ (a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}+a_{24}b_{41}) & (\dots) \\ (\dots) & (a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}+a_{34}b_{42}) \end{bmatrix}$$

$$\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}$$

$$\overset{A}{\begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 1 \end{bmatrix}} \quad \overset{B}{\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -2 & 6 \end{bmatrix}} \qquad A \times B = \begin{bmatrix} 6 & -4 \\ 4 & 10 \end{bmatrix}$$

$$A \in \mathbb{R}^{m \times n} \quad B \in \mathbb{R}^{n \times p} \qquad B = [\underline{b_1}, \underline{b_2}, \dots \underline{b_p}]$$

$$[AB] = [A\underline{b_1} \mid A\underline{b_2} \mid \dots \mid A\underline{b_p}]$$

**PROPERTIES** $A \in \mathbb{R}^{m \times n}$ $B, C$ GENERAL MATRIXES

$$A(BC) = (AB)C \qquad (B+C)A = BA + CA$$

$$A(B+C) = AB + AC \qquad \alpha(AB) = (\alpha A)B = A(\alpha B)$$

$$I_m A = A = A I_m$$

↳ $I_m$ IDENTITY MATRIX

$I_m \in \mathbb{R}^{m \times m} \rightarrow$ SAME NUMBER OF ROWS AND COLUMS

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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$\boxed{\text{WARNINGS!}}$

$A, B \in \mathbb{R}^{n \times n}$

$AB \neq BA$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 7 \end{bmatrix}$$

↕ DIFFERENT

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 3 & 4 \end{bmatrix}$$

$A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$

IF $AB = \underline{0} \in \mathbb{R}^{m \times p}$

DOES NOT IMPLY $A$ or $B = \underline{0}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

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$A \in \mathbb{R}^{m \times n}$ $B, C \in \mathbb{R}^{n \times p}$

$AB = AC \not\Rightarrow B = C$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 2 & -1 \\ 0 & -3 \end{pmatrix} \quad C = \begin{pmatrix} 2 & -1 \\ 4 & 7 \end{pmatrix}$$

$$AB = \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \quad BC = \begin{pmatrix} 0 & -9 \\ -12 & -21 \end{pmatrix}$$

$A \times B = A \times C$ BUT $B \neq C$

$k \in \mathbb{N}, k \geq 1$

$A \in \mathbb{R}^{n \times n}$ $\quad A^k = \underbrace{A \cdot A \cdot A \dots A}_{k \text{ TIMES}}$

$A^0 = I_n$ ONLY FOR SQUARE MATRIX

## TRANSPOSE OF A MATRIX

$A \in \mathbb{R}^{m \times n}$ $\quad A^T \in \mathbb{R}^{n \times m}$ TRANSPORT MATRIX

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

DIANAGONAL DID NOT MOVE

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{1n} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

## PROPERTIES

$A \in \mathbb{R}^{m \times n}$

$B \in \mathbb{R}^{m \times n}$

$C \in \mathbb{R}^{n \times p}$

$$(A^T)^T = A \qquad A^T + B^T = (A+B)^T$$

$$(\kappa A)^T = \kappa A^T \qquad (AC)^T = C^T A^T$$