$+ \quad \times$

$- \quad \div$

EXERCISE Consider an iterated map $x_{n+1} = f(x_n)$, with states $\{1, 2, 3, 4, 5, 6, 7\}$. The function $f(x)$ computes its values by the following algorithm:

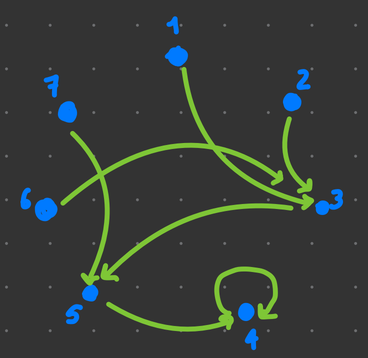
1. Take the string of characters that expresses the number x in English (e.g., $5 \rightsquigarrow$ "five').
2. Count the number of letters in the string; this is the value returned by the function (e.g., "five" $\rightsquigarrow 4$).

Find the fixed points, cycles, transient and recurrent states of this dynamics. How many connected components are there?

TAKE THE LETTERS OF THE NUMBER

$X_{N+1} = f(X_N)$ STATES $\{1,2,3,4,5,6,7\}$

$X_N = 1$	$X_{N+1} = 3$	O-N-E
$X_N = 2$	$X_{N+1} = 3$	T-W-O
$X_N = 3$	$X_{N+1} = 5$	T-H-R-E-E
$X_N = 4$	$X_{N+1} = 4$	F-O-U-R
$X_N = 5$	$X_{N+1} = 4$	F-I-V-E
$X_N = 6$	$X_{N+1} = 3$	S-I-X
$X_N = 7$	$X_{N+1} = 5$	S-E-V-E-N



FIXED POINTS:* STATE 4

CYCLE: THE ONE WITH ONLY 4

TRANSIENT STATE:** 1, 2, 3, 5, 6, 7

RECURRENT STATE: STATE 4

CONNECTED COMPONENTS:*** THE GRAPH ITSELF

* CYCLE WITH PERIOD ONE

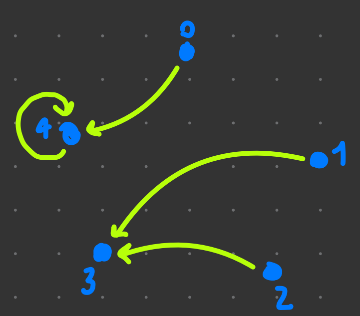
** NOT VISITED INFINITELY MANY TIMES

*** SUBSET OF THE GRAPH

EXERCISE Consider the same function f as in the previous exercise, but now the set of possible states is $\{0, 1, 2, 3, 4\}$. Does f represent the update rule of a dynamical system on this set of states?

$X_{N+1} = f(X_N)$ STATE $\{0,1,2,3,4\}$

$N = 0$	$X_{N+1} = 4$	Z-E-R-O
$N = 1$	$X_{N+1} = 3$	O-N-E
$N = 2$	$X_{N+1} = 3$	T-W-O
$N = 3$	$X_{N+1} = 5$	T-H-R-E-E
$N = 4$	$X_{N+1} = 4$	F-O-U-R



f DOES NOT REPRESENT THE UPDATE RULE OF A DYNAMICAL SYSTEM ON THIS SET OF STATES BECAUSE $f(3) = 5$ AND THE STATE 5 IS NOT REPRESENTED

EXERCISE Consider the iterated map $x_{n+1} = f(x_n)$, in the state space $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$, with

$$f(x) = \begin{cases} 2x & \text{if } x < 5 \\ x - 5 & \text{if } x \geq 5 \end{cases}$$

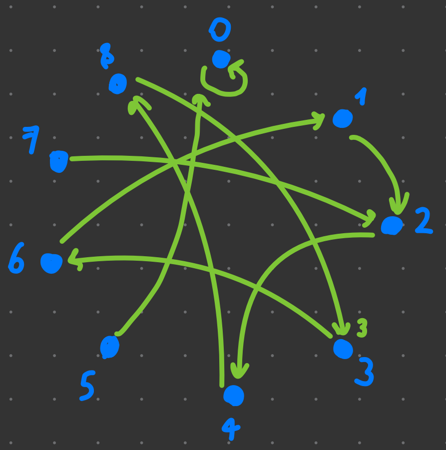
Find the fixed points, cycles, transient and recurrent states. How many connected components are there? Consider the function

$$Q(x) = 5x \mod 5$$

Is this a conserved quantity for the dynamics? Is it a *non-trivial* conserved quantity?

$$x_{n+1} = \begin{cases} 2x & \text{IF } x < 5 \\ x-5 & \text{IF } x \geq 5 \end{cases}$$

STATE $\{0,1,2,3,4,5,6,7,8\}$



$N=0$ $x_{n+1}=2x$; $x_{n+1}=0$

$N=1$ $x_{n+1}=2x$; $x_{n+1}=2$

$N=2$ $x_{n+1}=2x$; $x_{n+1}=4$

$N=3$ $x_{n+1}=2x$; $x_{n+1}=6$

$N=4$ $x_{n+1}=2x$; $x_{n+1}=8$

$N=5$ $x_{n+1}=x-5$; $x_{n+1}=0$

$N=6$ $x_{n+1}=x-5$; $x_{n+1}=1$

$N=7$ $x_{n+1}=x-5$ $x_{n+1}=2$

$N=8$ $x_{n+1}=x-5$ $x_{n+1}=3$

FIXED POINTS: $\{0\}$

CYCLES: $\{0\}$

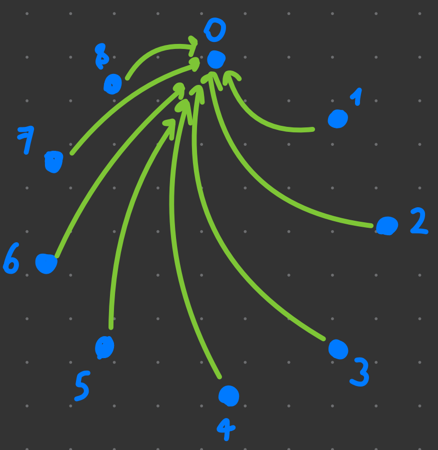
TRANSIENT STATE: $\{5,7\}$

RECURRENT STATE: $\{1,2,3,4,6,8\}$

CONNECTED COMPONENTS: $\{5,0\}\{7,2,4,8,3,6,1,2\}$

$Q(x) = 5x \mod 5$ STATE $\{0,1,2,3,4,5,6,7,8\}$

$x_{n+1} = 5x \mod 5$



$N=1$ $x_{n+1} = 5 \mod 5 = 0$

$N=2$ $x_{n+1} = 10 \mod 5 = 0$

$N=3$ $x_{n+1} = 15 \mod 5 = 0$

$N=4$ $= 0$

$N=5$ $= 0$

$N=6$ $= 0$

$N=7$ $= 0$

$N=8$ $= 0$

YES: $Q(x)$ IS A CONSERVED QUANTITY BECAUSE $Q_x = 0$ $\forall x$. IT IS ALSO A TRIVIAL CONSERVED QUANTITY BECAUSE SINCE IT TAKES THE SAME VALUE ON BOTH CONNECTED COMPONENTS

EXERCISE Consider the iterated map $x_{n+1} = f(x_n)$, with

$$f(x) = x^2 \mod 7,$$

where $x \in \{0, 1, 2, 3, 4, 5, 6\}$. How many connected components are there? Can you write down a non-trivial conserved charge?

$X_{N+1} = X_N^2 \text{ MOD } 7 \qquad X \in \{0,1,2,3,4,5,6\}$

$N=0 \quad X_{N+1} = 0 \text{ MOD } 7 = 0$

$N=1 \quad X_{N+1} = 1 \text{ MOD } 7 = 1$

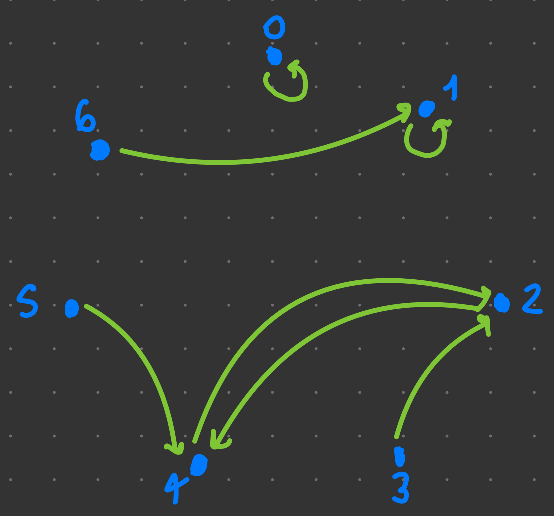
$N=2 \quad X_{N+1} = 4 \text{ MOD } 7 = 4$

$N=3 \quad X_{N+1} = 9 \text{ MOD } 7 = 2$

$N=4 \quad X_{N+1} = 16 \text{ MOD } 7 = 2$

$N=5 \quad X_{N+1} = 25 \text{ MOD } 7 = 4$

$N=6 \quad X_{N+1} = 36 \text{ MOD } 7 = 1$



THERE ARE 3 CONNECTED COMPONENTS

- $\{0\}$
- $\{6, 1\}$
- $\{2, 3, 4, 5\}$

NON-TRIVIAL CONSERVED CHARGE

$$Q_X = \begin{cases} 1 & X < 3 \\ 0 & X \geq 3 \end{cases}$$

EXERCISE Consider the iterated map $x_{n+1} = f(x_n)$ in the (infinite) set $\mathbb{N} = \{0, 1, 2, \ldots\}$ with

$$f(x) = x + k$$

where the parameter $k \geq 2$ is an integer. Describe the dynamics (cycles, etc.) Can you construct a non-trivial conserved quantity?

HINT: If the exercise seems too difficult, try considering the special case $k = 2$ first; then see if and how the picture changes when $k > 2$.

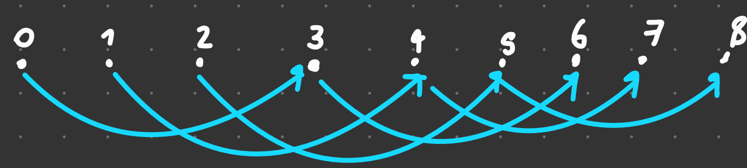
$X_{N+1} = X + K \qquad$ INFINITE SET $\mathbb{N} = \{0, 1, 2, \ldots\}$

$K \geq 2$

TEST: $K=2$

- THERE ARE TWO CONNECTED COMPONENTS, ONE FOR THE EVEN NUMBERS AND ONE FOR THE ODDS
- ALL THE STATES ARE TRANSIENT STATES
- THERE ARE NO CYCLE
- NO FIXED POINTS
- ALL TRANSIENT STATE, NO RECURRENT STATE

TEST $K>2$



0 - 3 - 6 - 9 - ...
1 - 4 - 7 - 10 - ...
2 - 5 - 8 - 11 - ...

- THERE ARE K CONNECTED COMPONENTS
- ALL THE STATE ARE TRANSIENT STATES
- THERE ARE NO CYCLE
- NO FIXED POINTS
- ALL TRANSIENT STATE, NO RECURRENT STATE

GENERAL RULES $(K \geq 2)$

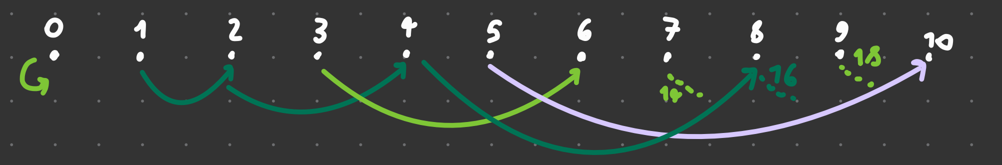
EXERCISE [difficult] Consider the iterated map $x_{n+1} = f(x_n)$ in the (infinite) set $\mathbb{N} = \{0, 1, 2, \ldots\}$ with

$$f(x) = kx,$$

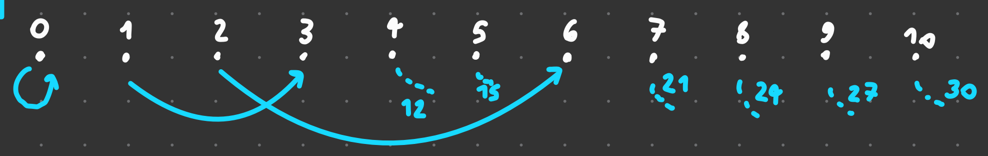
where the parameter $k \geq 2$ is an integer. Describe the dynamics (cycles, etc.)

$X_{N+1} = KX \quad K \geq 2 \quad N = \{0, 1, 2, \ldots\}$

K=2



K=3



GENERALIZE

- THERE ARE N/2 +1 CYCLES (N/2: ODD NUMBERS; +1: THE 0)
- N/2+1 CONNECTED COMPONENTS
- ALL STATES (EXCEPT THE 0) ARE TRANSIENT STATE
- 1 CYCLE (THE 0)
- 1 FIXED POINT (THE 0)