LEAST SQUARE PROBLEMS

27/MAY/22

$A \in \mathbb{R}^{m \times n}$ $\quad \underline{b} \in \mathbb{R}^m$ $\quad A\underline{x} = \underline{b}$ $\quad \underline{x} \in \mathbb{R}^n$

$\boxed{m > n}$ $\quad (m \gg n)$

$$\begin{array}{c c} m & \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{n} \end{array} \begin{bmatrix} \vdots \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$b \in Col(A)$

$\dim Col(A) \le n$

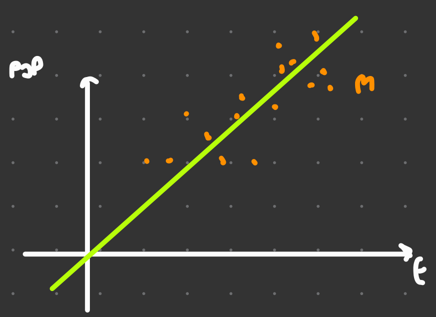
$\underline{b} \in \mathbb{R}^m \longrightarrow \dim = m$

$pop(t) = \alpha t + c \quad \alpha, c \in \mathbb{R}$

$pop(t_1), pop(t_2), \dots, pop(t_n)$] KNOWN

ONLY TWO PARAMETERS
$$\begin{cases} \alpha t_1 + c = pop(t_1) \\ \alpha t_2 + c = pop(t_2) \\ \vdots \\ \alpha t_n + c = pop(t_n) \end{cases}$$

$$\begin{bmatrix} t_1 & 1 \\ t_2 & 1 \\ t_3 & 1 \\ \vdots & \vdots \\ t_n & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ c \end{bmatrix} = \begin{bmatrix} pop(t_1) \\ pop(t_2) \\ pop(t_3) \\ \vdots \\ pop(t_n) \end{bmatrix}$$



$$A\underline{x} = \underline{b} \Rightarrow A\underline{x} - \underline{b} = \underline{0} \Rightarrow \underbrace{\|A\underline{x} - \underline{b}\|}_{} = \underline{0}$$

I WANT TO HAVE AN APPROXIMATION THAT MAKES $\|A\underline{x} - \underline{b}\|$ THE SMALL AS POSSIBLE

$A \in \mathbb{R}^{m \times n}$, $m > n$, $\underline{b} \in \mathbb{R}^m$. A LEAST SQUARE SOLUTION FOR THE PROBLEM $A\underline{x} = \underline{b}$ IS AN $\hat{\underline{x}} \in \mathbb{R}^n$ SUCH THAT $\|A\hat{\underline{x}} - \underline{b}\| = \min_{\underline{y} \in \mathbb{R}^n} \|A\underline{y} - \underline{b}\|$

OBSERVATION IF $\hat{\underline{x}}$ IS A SOLUTION OF $A\hat{\underline{x}} = \underline{b}$ THEN IT IS ALSO A LEAST SQUARE SOLUTION

$A\underline{x} = \underline{b}$

THEOREM $A \in \mathbb{R}^{m \times n}$, $m > n$ $\underline{b} \in \mathbb{R}^m$ THE SET OF LEAST SQUARE SOLUTION OF $A\underline{x} = \underline{b}$ CORRESPOND TO THE (NON EMPTY SET) OF ALL SOLUTIONS OF THE NORMAL EQUATION $A^T A \underline{x} = A^T \underline{b}$

YOU ALWAYS HAVE A LEAST SQUARE SOLUTION

EX: $A=\begin{bmatrix}1&2\\2&0\\-1&1\end{bmatrix}$ $\underline{b}=\begin{bmatrix}3\\2\\1\end{bmatrix}$

① DOES $A\underline{x}=\underline{b}$ HAVE A SOLUTION?

② FIND THE LEAST SQUARE SOLUTION OF $A\underline{x}=\underline{b}$

$$\left[\begin{array}{cc|c}1&2&3\\2&0&2\\-1&1&1\end{array}\right]\begin{matrix}2\\-1\end{matrix}=\left[\begin{array}{cc|c}1&2&3\\0&-4&-4\\0&3&4\end{array}\right]\left(\tfrac{2}{4}\right)=\left[\begin{array}{cc|c}1&2&3\\0&-1&-1\\0&3&4\end{array}\right]-3=\left[\begin{array}{cc|c}1&2&3\\0&-1&-1\\0&0&1\end{array}\right]$$

① NO SOLUTION

② $A^TA=\begin{bmatrix}1&2&-1\\2&0&1\end{bmatrix}\begin{bmatrix}1&2\\2&0\\-1&1\end{bmatrix}=\begin{bmatrix}6&1\\1&5\end{bmatrix}$

A^TA SYMMETRIC SQUARE MATRIX

$A^T\underline{b}=\begin{bmatrix}1&2&-1\\2&0&1\end{bmatrix}\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}6\\7\end{bmatrix}$

$A^TA\hat{\underline{x}}=A^T\underline{b}$

$\begin{bmatrix}6&1\\1&5\end{bmatrix}\begin{bmatrix}\hat{\underline{x}}\end{bmatrix}=\begin{bmatrix}6\\7\end{bmatrix}$

$\left[\begin{array}{cc|c}6&1&6\\1&5&7\end{array}\right]=\left[\begin{array}{cc|c}1&5&7\\6&1&6\end{array}\right]-6=\left[\begin{array}{cc|c}1&5&7\\0&-29&-36\end{array}\right]$

$-29x_2=-36;\ x_2=\frac{36}{29}$

$x_1+5x_2=7;\ x_1=7-\frac{180}{29}=\frac{23}{29}$

$\hat{\underline{x}}=\begin{bmatrix}23/29\\36/29\end{bmatrix}$

$A^TA\in\mathbb{R}^{n\times n}$ $M=M^T$

$(A^TA)^T=A^TA^{T^T}=A^TA$

LEAST SQUARE SOL. OF $A\underline{x}=\underline{b}$ $A^TA\hat{\underline{x}}=A^T\underline{b}$

$$\|A\hat{x}-\underline{b}\|=\min_{x\in\mathbb{R}^n}\|Ax-b|$$

$$\|Ax-b\|^2=(Ax-b,Ax-b)=(Ax-b)^T(Ax-b)=(x^TA^T-b^T)(Ax-b)=x^TA^TAx-x^TA^Tb-b^TAx+b^Tb=x^TAA^Tx-2x^TA^Tb+b^Tb\longrightarrow \cancel{2}AA^Tx-\cancel{2}A^Tb=\underline{0}$$

THEOREM $A\in\mathbb{R}^{m\times n}$, $m\geq n$, $\underline{b}\in\mathbb{R}^m$. THE FOLLOWING STATEMENTS ARE EQUIVALENT

① THE LEAST SQUARE PROBLEM HAS A UNIQUE SOLUTION $(A^TA\hat{x}=A^Tb)$

② A^TA IS INVERTIBLE

③ THE COLUMNS OF A ARE LIN. INDIPENDENT

EXAMPLE

$A=\begin{bmatrix}1&0&2\\2&2&3\\-1&2&1\\4&4&0\end{bmatrix}$ $\underline{b}=\begin{bmatrix}1\\7\\3\\8\end{bmatrix}$

① $A\underline{x}=\underline{b}$ HAS A SOLUTION?

② THE SOLUTION OF THE LEAST SQUARES PROBLEM IS UNIQUE?

③ BUILD THE NORMAL EQUATION SYSTEM

① $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 2 & 2 & 3 & 7 \\ -1 & 2 & 1 & 3 \\ 4 & 4 & 0 & 8 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 2 & -1 & 5 \\ 0 & 2 & 3 & 4 \\ 0 & 4 & -8 & 4 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 2 & -1 & 5 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & -6 & -6 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 2 & -1 & 5 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & -1 & -1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 2 & -1 & 5 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 4 & -1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 2 & -1 & 5 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -5 \end{array}\right]$

NO FREE VARIABLE

↓

NO MULTIPLE SOLUTION

①: NO

② YES

③ $A^T A \hat{x} = A^T \underline{b}$

$$A^T A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 2 & 2 & 4 \\ 2 & 3 & 1 & 0 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 2 & 2 & 3 & 7 \\ -1 & 2 & 1 & 3 \\ 4 & 4 & 0 & 8 \end{array}\right] = \begin{bmatrix} 22 & 18 & 7 \\ \boxed{18} & 24 & 8 \\ \boxed{7} & \boxed{8} & 14 \end{bmatrix}$$

NO NEED TO COMPUTE
(SYMMETRY)

$$A^T \underline{b} = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 2 & 2 & 4 \\ 2 & 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \\ 3 \\ 8 \end{bmatrix} = \begin{bmatrix} 44 \\ 52 \\ 26 \end{bmatrix}$$