$\sin 2\theta = 2\sin\theta\cos\theta$
$1 = \cos^2\theta + \sin^2\theta$
$\cos^2\theta - \sin^2\theta = \cos(2\theta)$

Exercises - Calculus
Academic Year 2021-2022

Sheet 19

1. Compute

   (a) $\displaystyle\iint_Q xy\log(xy)\mathrm{d}x\mathrm{d}y$ where $Q = [1,2]\times[2,3]$

   (b) $\displaystyle\iint_Q \frac{xy}{x+y}\mathrm{d}x\mathrm{d}y$ where $Q = [1,2]\times[2,3]$

   (c) $\displaystyle\iint_Q xye^y\mathrm{d}x\mathrm{d}y$ where $Q = [0,2]\times[0,1]$

   (d) $\displaystyle\iint_Q \frac{y}{1+x+y}\mathrm{d}x\mathrm{d}y$ where $Q = [0,1]\times[0,2]$

   (e) $\displaystyle\iint_Q (x+y)\log(1+x)\mathrm{d}x\mathrm{d}y$ where $Q = [0,1]\times[0,1]$

   (f) $\displaystyle\iint_Q x\sqrt{1-y^2}\mathrm{d}x\mathrm{d}y$ where $Q = [1,2]\times[0,1/2]$

   (g) $\displaystyle\iint_Q (x+y)e^{2xy+y^2}\mathrm{d}x\mathrm{d}y$ where $Q = [1,2]\times[0,1]$

   (h) $\displaystyle\iint_Q \sin(x+y)\mathrm{d}x\mathrm{d}y$ where $Q = [0,\pi]\times[0,\pi]$

   (i) $\displaystyle\iint_Q \frac{y}{4x^2+y^2}\mathrm{d}x\mathrm{d}y$ where $Q = [1,2]\times[2,3]$

2. Let a, $b > 0$. Compute the area of the ellipse

$$E = \left\{(x,y)\in\mathbb{R}^2 : \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\right\}$$

3. Let $E\subset\mathbb{R}^2$ be the fourth of disc of radius 1 contained in the first quadrant. Compute

$$\iint_E \log(1+x^2)y^3\mathrm{d}x\mathrm{d}y \quad\text{and}\quad \iint_E x\arcsin(y/\sqrt{3})\mathrm{d}x\mathrm{d}y$$

4. Let $E\subset\mathbb{R}^2$ be the region contained in the first quadrant bounded by the x-axis and by the curve $y=-x^2+2x$. Write E as a normal region with respect to the y-axis and as a normal region with respect to the x-axis. Compute the area of E and

$$\iint_E xy\mathrm{d}x\mathrm{d}y \quad\text{and}\quad \iint_E x\sqrt{1-y}\mathrm{d}x\mathrm{d}y \quad\text{and}\quad \iint_E (x-1)e^{(x-1)^2/(y+1)}\mathrm{d}x\mathrm{d}y$$

5. Let

$$f(x,y) = \begin{cases} \frac{1+x^2}{1+2y} & \text{se } (x,y) \in [-1,1]\times[0,2],\ 0 \le y \le 1-x^2 \\ 1 & \text{se } (x,y) \in [-1,1]\times[0,2],\ y > 1-x^2 \end{cases}$$

Establish whether f is integrable on $[-1,1]\times[0,2]$ and in that case compute

$$\iint_{[-1,1]\times[0,2]} f(x,y)\,\mathrm{d}x\mathrm{d}y$$

6. Compute the area of

$$E = \left\{ (x,y) \in \mathbb{R}^2 : \ \frac{x^2}{4} + \frac{y^2}{9} \le 1 \text{ and } y \ge 1 \right\}.$$

7. Compute

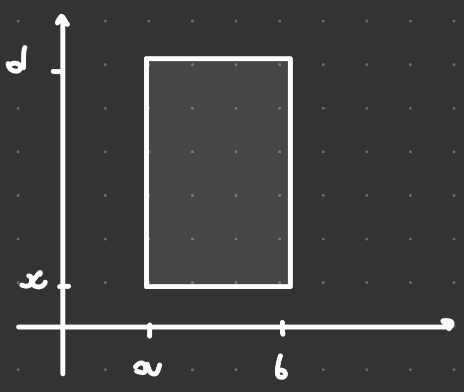
$$\iiint_Q \frac{1+2xy}{1+z^2}\mathrm{d}x\mathrm{d}y\mathrm{d}z \text{ where } Q = [-1,1]\times[0,1]\times[2,3]$$

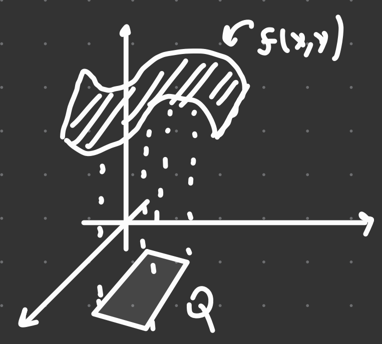
$$\iiint_Q \frac{1}{1+x+y+z}\mathrm{d}x\mathrm{d}y\mathrm{d}z \text{ where } Q = [-1,1]\times[0,1]\times[2,3]$$

FUBINI (REDUCTION) THEOREM

$N=2$ $\quad Q=[a,b]\times[c,d]$

$f(x,y)\in \mathbb{R}(Q)$





a) If $\forall y\in[c,d]\ \exists G(y)=\int_a^b f(x,y)\,dx$

$G(y)\in\mathbb{R}([c,d])$

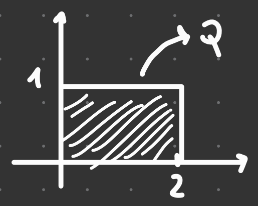
A) $\iint_Q f(x,y)\,dxdy=\int_c^d G(y)\,dy=\int_c^d\left(\int_a^b f(x,y)\,dx\right)dy=\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx$

STRATEGY

① IDENTIFY Q (DOMAIN OF INTEGRATION) AND <u>DRAW Q</u>

② APPLY FUBINI

1c) $Q=[0,2]\times[0,1]$



$$\iint_Q xye^y\,dxdy=\int_0^1\left(\int_0^2 xye^y\,dx\right)dy=\int_0^1\frac{ye^y}{2}\left[x^2\right]_{x=0}^{x=2}dy=\int_0^1 2ye^y\,dy=2\left(\left[ye^y\right]_0^1-\int_0^1 1\cdot e^y\,dy\right)=2\left(e-\left[e^y\right]_0^1\right)=$$

$$2(e-(e-1))=\boxed{2}$$

BY PART

$f\ y \qquad f'\ 1$

$g\ e^y \qquad g'\ e^y$

7.2 $\displaystyle\iiint_Q \frac{1}{1+x+y+z}\,dxdydz$ where $Q=[-1,1]\times[0,1]\times[2,3]$ (x, y, z)

SEMPRE POSITIVI MAX INT X E Y VO 1 0

$$\iiint_Q\frac{1}{1+x+y+z}\,dxdydz=\int_{-1}^1\left(\int_0^1\left(\int_2^3\frac{1}{1+x+y+z}\,dz\right)dy\right)dx=\int_{-1}^1\int_0^1\left[\ln|1+x+y+z|\right]_{z=2}^{z=3}dydx=\int_{-1}^1\int_0^1\ln|4+x+y|-\ln|3+x+y|\,dydx$$

$\int \log(x)\,dx = x\log(x)-x$

$$=\int_{-1}^1\left[(4+x+y)\log(4+x+y)-(4+x+y)-\left((3+x+y)\log(3+x+y)-(3+x+y)\right)\right]_{y=0}^{y=1}dx=$$

$$\int_{-1}^1\Big(\big((5+x)\log(5+x)-(5+x)\big)-\big((4+x)\log(4+x)-(4+x)\big)-\Big(\big((4+x)\log(4+x)-(4+x)\big)-\big((3+x)\log(3+x)-(3+x)\big)\Big)\Big)dx$$

$$\int_{-1}^1(5+x)\log(5+x)\,dx-\int_{-1}^1(5+x)\,dx-\int_{-1}^1(4+x)\log(4+x)\,dx+\int_{-1}^1(4+x)\,dx-\int_{-1}^1(4+x)\log(4+x)\,dx+\int_{-1}^1(4+x)\,dx$$

$$+\int_{-1}^1(3+x)\log(3+x)\,dx-\int_{-1}^1(3+x)\,dx$$

$$=\left[\frac{(5+x)^2}{2}\log(5+x)\right]_{-1}^1-\int_{-1}^1\frac{(5+x)^2}{2}\frac{1}{5+x}\,dx+\int_{-1}^1\underbrace{\left(\cancel{-5-x+4+x+4+x-3-x}\right)}_{=0}dx\ -$$

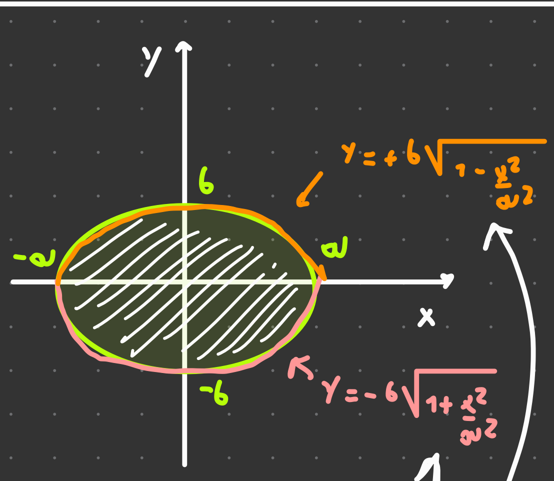
$$\left[\frac{(4+x)^2}{2}\cos(4+x)\right]_{-1}^{1} - 2\int_{-1}^{1}\frac{(4+x)^2}{2}\,\frac{1}{4+x}\,dx + \Big[$$

$$\left[\frac{(3+x)^2}{2}\cos(3+x)\right]_{-1}^{1} - \int_{-1}^{1}\frac{(3+x)^2}{2}\,\frac{1}{3+x}\,dx$$

$$= \frac{6^2}{2}\cos(6) - \frac{4^2}{2}\cos(4) - \frac{1}{2}\left(\frac{(5+x)^2}{2}\right]_{-1}^{1}$$

2:

$$E = \left\{ (x,y) \in \mathbb{R}^2 \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \right\}$$



? AREA

$$\chi_E(x,y) = \begin{cases} 1 & (x,y) \in E \\ 0 & (x,y) \notin E \end{cases}$$

$$\iint_E 1 \, dx dy$$

BORDO DELL'ELLISSE

$x \in [-a, a]$ $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; $y^2 = b^2 - b^2\frac{x^2}{a^2}$; $y = \pm b\sqrt{1-\frac{x^2}{a^2}} \Rightarrow -b\sqrt{1-\frac{x^2}{a^2}} \le y \le b\sqrt{1-\frac{x^2}{a^2}}$

$$E = \left\{ (x,y) \in \mathbb{R}^2 \quad x \in [-a, a) \text{ AND } -b\sqrt{1-\frac{x^2}{a^2}} \le y \le b\sqrt{1-\frac{x^2}{a^2}} \right\}$$

$\frac{x}{a} = t$

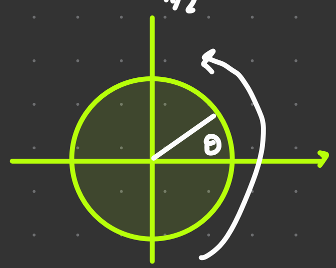
$dx = a\,dt$

$$\int_{-a}^{a}\left(\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} 1\,dy\right)dx = \int_{-a}^{a}[y]_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-x^2/a^2}}\,dx = \int_{-a}^{a} 2b\sqrt{1-\frac{x^2}{a^2}}\,dx =$$

$t = \sin\theta$

$dt = \cos\theta\, d\theta$

$$= 2ab\int_{-1}^{1}\sqrt{1-t^2}\,dt = 2ab\int_{\arcsin(-1)=-\frac{\pi}{2}}^{\arcsin(1)=\pi/2} \boxed{\underbrace{\sqrt{1-\sin^2\theta}}_{\cos^2\theta}\cos\theta}_{\cos^2\theta}\,dt = 2ab\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta$$



$\cos^2\theta - \sin^2\theta = \cos(2\theta)$

$\cos^2\theta - (1-\cos^2\theta) = \cos(2\theta)$

$2\cos^2\theta - 1 = \cos(2\theta)$

$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$

$$= 2ab\int_{-\pi/2}^{\pi/2}\frac{1+\cos(2\theta)}{2}\,d\theta = 2ab\left[\frac{\theta}{2}\right]_{-\pi/2}^{\pi/2} + \underbrace{2ab\left[\frac{\sin 2\theta}{2}\right]_{-\pi/2}^{\pi/2}}_{\to 0 \quad \sin(\pm\pi)=0} =$$

$$= 2ab\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = \pi ab$$

(a) $\displaystyle\iint_Q xy\log(xy)\mathrm{d}x\mathrm{d}y$ where $Q = [1,2]\times[2,3]$

$$\int_2^3\int_1^2 xy\log(xy)\,dx\,dy$$

BY PARTS

$f\ \log(xy)$ $\qquad f'\ \frac{\cancel{y}}{x\cancel{y}}$

$g\ \frac{x^2y}{2}$ $\qquad g'\ xy$

$$\int_2^3\left[\frac{x^2y}{2}\log(xy) - \int\frac{1}{\cancel{x}}\cdot\frac{x^{\cancel{2}}y}{2}\,dx\right]_1^2 dy = \int_2^3\left[\frac{x^2y}{2}\log(xy) - \frac{1}{2}\int xy\,dx\right]_1^2 dy = \int_3^2\left[\frac{x^2y}{2}\log(xy) - \frac{1}{4}x^2y\right]_1^2 dy =$$

$$\int_3^2 2y\log(2y) - y - \frac{1}{2}y\log(y) + \frac{1}{4}y\,dy = \int_3^2 \underbrace{2y\log(2y)} - \underbrace{\frac{1}{2}y\log(y)} - \frac{3}{4}y\,dy =$$

$f\ \log(2y)$ $\quad f'\ \frac{1}{y}$ | $f\ \log(y)$ $\quad f'\ \frac{1}{y}$

$g\ y^2$ $\quad g'\ 2y$ | $g\ -\frac{y^2}{4}$ $\quad g'\ -\frac{1}{2}y$

$$= \left[y^2\log(2y) - \int_3^2 y\,dy - \frac{y^2}{4}\log(y) + \int_3^2\frac{y}{4}\,dy\right]_2^3 = \left[y^2\log(2y) - \frac{y^2}{2} - \frac{y^2}{2}\log(y) + \frac{y^2}{8}\right]_2^3$$

$$9\log(6) - \frac{9}{2} - \frac{9}{2}\log(3) + \frac{9}{8} - 4\log(4) + 2 + 2\log(2) - \frac{1}{2}$$

$$9\log(6) - 4\log(4) - \frac{9}{2}\log(3) + 2\log(2) - \frac{15}{8}$$