Exercises - Calculus
Academic Year 2021-2022

Sheet 16

1. Compute the Hessian matrix of the function f in any point of its domain where f is given by
   (a) $f(x,y) = e^x y + xy^2$
   (b) $f(x,y) = \tan(x^2+y^2) - \dfrac{1}{xy}$
   (c) $f(x,y,z) = xz^3 - \log(x+y^2z^2)$
2. Let $f(x,y) = 2 - x + 2y - xy + x^3 + 2x^2y - xy^2$. Compute the Taylor polynomial of degree 2 of the function f in the point $(0,0)$ and in the point $(-1,0)$ (in the canonical representation). Then compute
$$\frac{\partial^2 f}{\partial x \partial y}(0,0) \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial x}(-1,0).$$
3. Let $f(x,y,z) = 3 - 2y + xz + y^2 + 2x^2z - yz^2 + z^3$. Compute the Taylor polynomial of degree 2 of the function f in the point $(0,0,0)$ and in the point $(1,0,-1)$ (in the canonical representation). Then compute
$$\frac{\partial^2 f}{\partial z \partial x}(0,0,0) \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial z}(1,0,-1).$$
4. Let $f(x,y) = \sin(x-y^2)$. Compute the Taylor polynomial of degree 2 of the function f in the point $(0,0)$ and in the point $(0,1)$.
5. Let $f(x,y) = \arctan(xy) + xy^3$. Compute the Taylor polynomial of degree 2 of the function f in the point $(0,0)$ and in the point $(1,1)$.
6. Let $f(x,y,z) = ze^{2x+yz}$. Compute the Taylor polynomial of degree 2 of the function f in the point $(0,0,0)$ and in the point $(1,0,1)$.
7. Let $f(x,y,z) = z^2\cos(x) + ye^{xz}$. Compute the Taylor polynomial of degree 2 of the function f in the point $(0,0,0)$ and in the point $(0,-1,1)$.
8. Let $f : \mathbb{R} \to \mathbb{R}$ be such that for $x \in \mathbb{R}$
$$f(x) = \begin{cases} x+1 & \text{if } x \leq -1 \\ 0 & \text{if } -1 < x < 1 \\ x-1 & \text{if } x \geq 1. \end{cases}$$
   Let
$$A = f^{-1}((-\infty,0)) = \{x \in \mathbb{R} : \ f(x) \in (-\infty,0)\} = \{x \in \mathbb{R} : \ f(x) < 0\},$$
$$B = f^{-1}(\{0\}) = \{x \in \mathbb{R} : \ f(x) \in \{0\}\} = \{x \in \mathbb{R} : \ f(x) = 0\},$$

$$C = f^{-1}((-\infty, 0]) = \{x \in \mathbb{R} : \ f(x) \in (-\infty, 0]\} = \{x \in \mathbb{R} : \ f(x) \leq 0\}.$$

Determine if A is open and if B and C are closed. Determine the closure, the interior and the boundary of A, B and C.

C is the closure of A? A is the interior of C? The interior di B is empty? B is the boundary of A, B or C?

9. Let $f : \mathbb{R} \to \mathbb{R}$ be such that for $x \in \mathbb{R}$

$$f(x) = \begin{cases} x+2 & \text{if } x \leq -1 \\ 0 & \text{if } -1 < x < 1 \\ x-2 & \text{if } x \geq 1. \end{cases}$$

Let

$$A = f^{-1}((-\infty, 0)) = \{x \in \mathbb{R} : \ f(x) \in (-\infty, 0)\} = \{x \in \mathbb{R} : \ f(x) < 0\},$$

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Determine if A is open and if B and C are closed. Determine the closure, the interior and the boundary of A, B and C.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be such that for $x \in \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Let $A = f^{-1}((-1/2, 1/2))$ and $C = f^{-1}([1/2, 3/2])$. Determine if A is open and if C is closed.

11. Let $C = \{(x, y) \in \mathbb{R}^2 : \ x^2 + y^2 < 1\} \subset \mathbb{R}^2$.

Determine if C is open or closed, if it is bounded. Determine and describe the interior, the closure and the boundary of C.

12. Let $C = \{(x, y, z) \in \mathbb{R}^3 : \ x^2 + y^2 + z^2 < 1\} \subset \mathbb{R}^3$.

Determine if C is open or closed, if it is bounded. Determine and describe the interior, the closure and the boundary of C.

13. Let $C = \{(x, y) \in \mathbb{R}^2 : \ x^2 + 3y^4 \leq 1\} \subset \mathbb{R}^2$.

Determine if C is open or closed, if it is bounded. Determine and describe the interior, the closure and the boundary of C.

14. Let $C = \{(x, y, z) \in \mathbb{R}^3 : \ x^2 + 3y^4 \leq 1\} \subset \mathbb{R}^3$.

Determine if C is open or closed, if it is bounded. Determine and describe the interior, the closure and the boundary of C.

15. Let $C = \{(x, y) \in \mathbb{R}^2 : \ x^2 + y^2 \leq 1, \ y \geq 0\} \subset \mathbb{R}^2$.

Determine if C is open or closed, if it is bounded. Determine and describe the interior, the closure and the boundary of C.

16. Let $C = \{(x, y, z) \in \mathbb{R}^3 : \; x^2 + y^2 \leq 1, \; 0 \leq z \leq 2\} \subset \mathbb{R}^3$.

    Determine if C is open or closed, if it is bounded. Determine and describe the interior, the closure and the boundary of C.

17. Let $C = \{(x, y, z) \in \mathbb{R}^3 : \; x^2 + y^2 + z^2 \leq 1, \; z \geq 0\} \subset \mathbb{R}^3$.

    Determine if C is open or closed, if it is bounded. Determine and describe the interior, the closure and the boundary of C.

18. Let $C = \{(x, y) \in \mathbb{R}^2 : \; x^2 + y^2 \leq 1, \; (x - 3/2)^2 + y^2 \leq 1\} \subset \mathbb{R}^2$.

    Determine if C is open or closed, if it is bounded. Determine and describe the interior, the closure and the boundary of C.

19. Let $C = \{(x, y) \in \mathbb{R}^2 : \; x^2 - 1 \leq y \leq \cos(2x/\pi)\} \subset \mathbb{R}^2$.

    Determine if C is open or closed, if it is bounded. Determine and describe the interior, the closure and the boundary of C.

20. Let $C = \{(x, y) \in \mathbb{R}^2 : \; x^2 - 1 \leq y \leq 1 - x^4, \; x > 0\} \subset \mathbb{R}^2$.

    Determine if C is open or closed, if it is bounded. Determine and describe the interior, the closure and the boundary of C.

21. Let $C = \{(x, y, z) \in \mathbb{R}^3 : \; z = x - y\} \subset \mathbb{R}^3$.

    Determine if C is open or closed, if it is bounded. Determine and describe the interior, the closure and the boundary of C.

1. Compute the Hessian matrix of the function f in any point of its domain where f is given by

   (a) $f(x,y) = e^x y + xy^2$

   (b) $f(x,y) = \tan(x^2+y^2) - \dfrac{1}{xy}$

   (c) $f(x,y,z) = xz^3 - \log(x + y^2z^2)$

HESSIAN

$$H_f(x,y) = \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{bmatrix}$$

$\tan(g) = \sec^2(g)$

$\sec(g) = \tan(g)\sec(g)$

a) $f(x,y) = e^x y + xy^2$

$f'_x(x,y) = e^x y + y^2$ $f'_y(x,y) = e^x + 2xy$

$f''_{xx}(x,y) = e^x y$ $f''_{xy} = e^x + 2y$ $f''_{yy}(x,y) = 2x$ $f''_{yx}(x,y) = e^x + 2y$

$$H_f(x,y) = \begin{bmatrix} e^x y & e^x + 2y \\ e^x + 2y & 2x \end{bmatrix}$$

$(xy)^{-1} = -1\,x\,(xy)^{-2} = \dfrac{-x}{x^2y^2} = -\dfrac{1}{xy^2}$

b) $f(x,y) = \tan(x^2+y^2) - \dfrac{1}{xy}$ $(xy)^{-1}$

$f'_x = 2x\sec^2(x^2+y^2) + \dfrac{y}{(xy)^2} = 2x\sec^2(x^2+y^2) + \dfrac{1}{x^2y}$

$f'_y = 2y\sec^2(x^2+y^2) + \dfrac{1}{xy^2}$

$f''_{xx}(x,y) = 2\sec^2(x^2+y^2) + 2x\dfrac{dx}{dx}\sec^2(x^2+y^2)$

$2x\tan(s)\dfrac{d}{dx}($

c) $f(x,y,z) = xz^3 - \cos(x+y^2z^2)$

$f'_x(x,y,z) = z^3 - \dfrac{1}{x+y^2z^2}$ $f'_y(x,y,z) = -\dfrac{2yz^2}{x+y^2z^2}$ $f'_z(x,y,z) = 3xz^2 - \dfrac{2zy^2}{x+y^2z^2}$

$f''_{xx}(x,y,z) = \dfrac{1}{(x+y^2z^2)^2}$

$f''_{yy} = -\dfrac{2z^2(x+y^2z^2) - \overbrace{2yz^2(2yz^2)}^{4y^2z^4}}{(x+y^2z^2)^2}$

$f''_{zz}(x,y,z) = 6xz - \dfrac{2y^2(x+y^2z^2) - 2zy^2(2y^2z)}{(x+y^2z^2)^2}$

$f''_{xy}(x,y,z) = \dfrac{2yz^2}{(x+y^2z^2)^2}$ $f''_{yx} = \dfrac{2yz^2}{(x+y^2z^2)^2}$ $f''_{zx} = 3z^2 + \dfrac{2zy^2}{(x+y^2z^2)^2}$

$f''_{xz}(x,y,z) = 3z^2 + \dfrac{2zy^2}{(x+y^2z^2)^2}$ $f''_{yz} = \dfrac{-4yz(x+y^2z^2) + 2yz^2(2zy^2)}{(x+y^2z^2)^2}$ $f''_{zy} = -\dfrac{4zy(x+y^2z^2) - 2zy^2(2yz^2)}{(x+y^2z^2)^2}$

$$\begin{bmatrix} f''_{xx} & f''_{xy} & f''_{xz} \\ f''_{yx} & f''_{yy} & f''_{yz} \\ f''_{zx} & f''_{zy} & f'_{zz} \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{2yz^2}{(x+y^2z^2)^2} & \dfrac{2yz^2}{(x+y^2z^2)^2} & 3z^2 + \dfrac{2zy^2}{(x+y^2z^2)^2} \\ \dfrac{2yz^2}{(x+y^2z^2)^2} & \dfrac{4y^2z^4 - 2z^2(x+y^2z^2)}{(x+y^2z^2)^2} & \dfrac{4y^3z^3 - 4yz(x+y^2z^2)}{(x+y^2z^2)^2} \\ 3z^2 + \dfrac{2zy^2}{(x+y^2z^2)^2} & \dfrac{4y^3z^3 - 4zy(x+y^2z^2)}{(x+y^2z^2)^2} & 6xz + \dfrac{4z^2y^4 - 2y^2(x+y^2z^2)}{(x+y^2z^2)^2} \end{bmatrix}$$

TAYLOR IN CANONICAL FORM

$$f(x,y) = f(x_0,y_0) + \frac{df}{dx}(x_0,y_0)(x-x_0) + \frac{df}{dy}(x_0y_0)(y-y_0) + \frac{1}{2}\frac{d^2f}{dx}(x_0,y_0)(x-x_0)^2 + \frac{1}{\cancel{2}}\cdot\cancel{2}\frac{df}{dxdy}(x_0,y_0)(x-x_0)(y-y_0) + \frac{1}{2}\frac{d^2f}{dy}(x_0y_0)(x-x_0)^2 + R_2(x,y)$$

2. Let $f(x,y) = 2 - x + 2y - xy + x^3 + \overbrace{2x^2y - xy^2}^{\text{DEG } 3}$. Compute the Taylor polynomial of degree 2 of the function f in the point $(0,0)$ and in the point $(-1,0)$ (in the canonical representation). Then compute

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial x}(-1,0).$$

2

(0,0)

$f'_x = -1 - y + 3x^2 + 4xy - y^2$ $\qquad$ $f'_y = 2 - x + 2x^2 - 2xy$

$f''_{xx} = 6x + 4y$ $\qquad$ $f''_{yy} = -2x$

$f''_{xy} = -1 + 4x - 2y$ $\qquad$ $f''_{yx} = -1 + 4x - 2y$

NO

$f(0,0) = 2$

$\frac{df}{dx}(0,0) = -1$ $\qquad$ $\frac{df}{dy}(0,0) = 2$

$\frac{df}{dy}(0,0) = 2$

$\frac{df}{dxdy}(0,0) = \frac{df}{dydx}(0,0) = -1$

$\frac{df}{dx^2}(0,0) = \frac{df}{dy^2}(0,0) = 0$

P_2 $\quad 2 - 1(x) + 2y + \frac{1}{2}(0)x\ \frac{1}{2}\cdot 2(-1)xy + \frac{1}{2}0 = 2 - x + 2y - xy$

$$T = f(0,0)\overbrace{(-1 - y + 3x^2 + 4xy - y^2)}^{x}(x-x_0) + \overbrace{(2 - x + 2x^2 - 2xy)}^{y}(y-y_0) + \frac{1}{2}\overbrace{(6x+4y)}^{xx}(x-x_0) + \overbrace{(-1 + 4x - 2y)}^{xy}(x-x_0)(y-y_0) + \overbrace{(-2x)}^{yy}(y-y_0)$$

TAYLOR IN CANONICAL FORM

$$f(x,y) = f(x_0,y_0) + \frac{df}{dx}(x_0,y_0)(x-x_0) + \frac{df}{dy}(x_0,y_0)(y-y_0) + \frac{1}{2}\frac{d^2f}{dx}(x_0,y_0)(x-x_0)^2 + \cancel{\frac{1}{2}}\cdot\cancel{2}\frac{df}{dxdy}(x_0,y_0)(x-x_0)(y-y_0) + \frac{1}{2}\frac{d^2f}{dy}(x_0,y_0)(x-x_0)^2 + R_2(x,y)$$

2. Let $f(x,y) = 2 - x + 2y - xy + x^3 + 2x^2y - xy^2$. Compute the Taylor polynomial of degree 2 of the function f in the point $(0,0)$ and in the point $(-1,0)$ (in the canonical representation). Then compute

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial x}(-1,0).$$

2

DEGREE 2

$P_2(x,y) = 2 - x + 2y - xy \qquad R_2(x,y) = x^3 + 2x^2y - xy^2$

	$P_2(0,0)$	$Q_2(-1,0)$
$\frac{df}{dx}(x,y) = -1-y$	-1	-1
$\frac{d^2f}{dxdx}(x,y) = 0$	0	0
$\frac{d^2f}{dxdy}(x,y) = -1$	-1	-1
$\frac{df}{dy}(x,y) = 2-x$	2	3
$\frac{d^2f}{dydy}(x,y) = 0$	0	0
$\frac{d^2f}{dydx}(x,y) = -1$	-1	-1

$f(0,0)$ JUST IN $P_2(0,0)$ OR ALSO IN $R_2(0,0)$??

$P_2(x,y) = 2 + (-1-y)x + (2-x)y + \frac{1}{2}\cdot(0)\cdot x^2 + \frac{1}{2}(0)\cdot y^2 - 1(xy) = 2 - x - xy + 2y - xy - xy = 2 - x + 2y - 3xy$

$P_2(0,0) = 2 + (-1)x + (2)y + 0 + 0 - xy = 2 - x + 2y - xy$

$f(-1,0) = 2 + 1 - 1 = 0$

OR 3 IF I USE Q_2

$Q_2(-1,0) = 0 - 1(x+1) + 3y + \frac{1}{2}(0)(x+1)^2 + (-1)(x+1)y + \frac{1}{2}(0)(y-0)^2$

$= -x - 1 + 3y - xy - y = -x + 2y - xy - 1$

$P_2(-1,0) =$

3. Let $f(x, y, z) = 3 - 2y + xz + y^2 + 2x^2z - yz^2 + z^3$. Compute the Taylor polynomial of degree 2 of the function f in the point $(0, 0, 0)$ and in the point $(1, 0, -1)$ (in the canonical representation). Then compute

$$\frac{\partial^2 f}{\partial z \partial x}(0, 0, 0) \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial z}(1, 0, -1).$$

$P_2(x,y,z) = 3-2y+xz+y^2 \qquad R_2(x,y,z) = 2x^2z - yz^2 + z^3$

IN $(1,0,-1)$ CHANGE x AND z VARIABLES

$\tilde{x} = x-1 \qquad \tilde{z} = z+1$

$f(x,y,z) = f((x-1)+1, y, (z+1)-1) = f(\tilde{x}+1, y, \tilde{z}-1) =$

$= 3-2y+(\tilde{x}+1)(\tilde{z}-1)+y^2+2(\tilde{x}+1)^2(\tilde{z}-1)-y(\tilde{z}-1)^2+(\tilde{z}-1)^3 =$

$= 3-2y+\tilde{x}\tilde{z}-\tilde{x}+\tilde{z}\cancel{-1}+y^2+(2\tilde{z}-2)(\tilde{x}^2+1+2\tilde{x}) - y\tilde{z}^2 - y + 2y\tilde{z} + \tilde{z}^3 \cancel{+1} + 3\tilde{z}^2 + 3\tilde{z}^1$

4. Let $f(x,y) = \sin(x - y^2)$. Compute the Taylor polynomial of degree 2 of the function f in the point $(0,0)$ and in the point $(0,1)$.

$f(0,0) = \sin(0) = 0$ $\quad$ $f(0,1) = \sin(-1) = 0$

NO

NO

$\frac{df}{dx}(x,y) = \cos(x - y^2)$ $\quad$ $\frac{df}{dx}(0,0) = 1$ $\quad$ $\frac{df}{dx}(0,1) = -1$

$\frac{d^2f}{dxdx}(x,y) = -\sin(x-y^2)$ $\quad$ $\frac{df}{dxdx}(0,0) = 0$ $\quad$ $\frac{d^2f}{dxdx}(0,1) = 0$

$\frac{d^2f}{dxdy}(x,y) = 2y\sin(x-y^2)$ $\quad$ $\frac{df}{dxdy}(0,0) = 0$ $\quad$ $\frac{d^2f}{dxdy}(0,1) = 0$

$\frac{df}{dy}(x,y) = -2y\cos(x-y^2)$ $\quad$ $\frac{df}{dy}(0,0) = 0$ $\quad$ $\frac{df}{dy}(0,1) = 2$

$\frac{d^2f}{dydy}(x,y) = -2\cos(x-y^2) - 4y^2\sin(x-y^2)$ $\quad$ $\frac{d^2f}{dydy}(0,0) = -2$ $\quad$ $\frac{d^2f}{dydy}(0,1) = 2$

$\frac{d^2f}{dydx}(x,y) = 2y\sin(x-y^2)$ $\quad$ $\frac{d^2f}{dydx}(0,0) = 0$ $\quad$ $\frac{d^2f}{dydx}(0,1) = 0$

$$T_2(0,0) = 0 + x + 0y + \frac{1}{2}(0)(x)^2 + 0(xy) + \frac{1}{2}(-2)y^2 = x - y^2$$

$$T_2(0,1) = 0 - 1x + 2(y-1) + \frac{1}{2}(0)(x)^2 + 0(x)(y-1) + \frac{1}{2}\cdot 2(y-1)^2 =$$

$$= -x + 2y - 2 + y^2 + 1 - 2y = y^2 - x - 1$$

8. Let $f : \mathbb{R} \to \mathbb{R}$ be such that for $x \in \mathbb{R}$

$$f(x) = \begin{cases} x+1 & \text{if } x \leq -1 \\ 0 & \text{if } -1 < x < 1 \\ x-1 & \text{if } x \geq 1. \end{cases}$$

Let

$$A = f^{-1}((-\infty, 0)) = \{x \in \mathbb{R} : \ f(x) \in (-\infty, 0)\} = \{x \in \mathbb{R} : \ f(x) < 0\},$$

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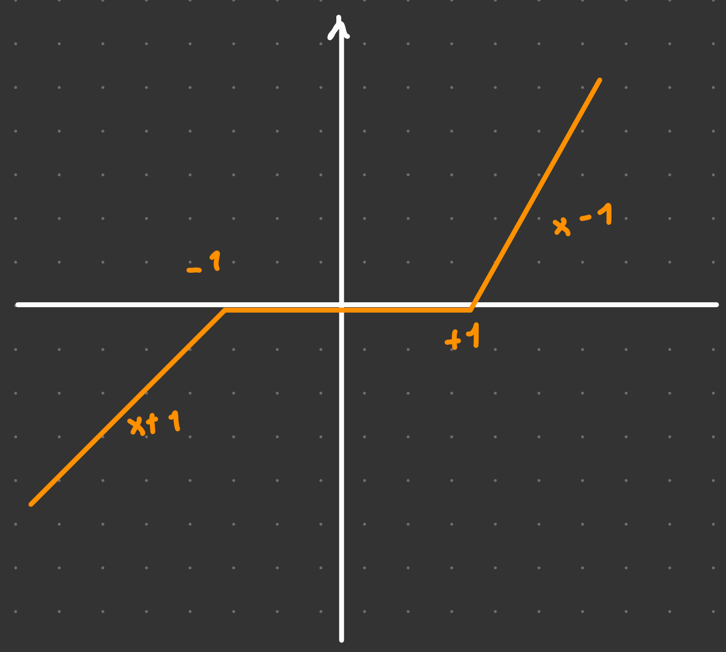
Determine if A is open and if B and C are closed. Determine the closure, the interior and the boundary of A, B and C.

C is the closure of A? A is the interior of C? The interior di B is empty? B is the boundary of A, B or C?



A IS OPEN IFF $\forall x \in A \ \exists r > 0$ S.T. $B_r(x) \subseteq A$

A IS CLOSED IFF A^c IS OPEN

IF $f \in C^0(\mathbb{R}) \Rightarrow$

$(-\infty, 0)$ IS OPEN $\Rightarrow$ A IS OPEN $\Rightarrow \mathring{A} = A$

$\{0\}$ IS CLOSED $\Rightarrow$ B IS CLOSED

$(-\infty, 0]$ IS CLOSED $\Rightarrow$ C IS CLOSED

INTERIOR $\mathring{A} = \{x \in A \mid \exists r > 0$ S.T. $B_r(x) \subseteq A\}$

CLOSURE: $\bar{A} = \{x \in \mathbb{R}^N \mid \forall \varepsilon > 0 \ B_\varepsilon(x) \cap A \neq \emptyset\}$

BOUNDARY:

$\partial A = \bar{A} \setminus \mathring{A} = \{x \in \mathbb{R}^N \mid \forall r > 0, B_r(x) \cap A \neq \emptyset$ AND $B_r(x) \cap A^c \neq \emptyset\}$

DOMAIN

$A = (-\infty, -1)$ $\quad \mathring{A} = A = (-\infty, -1)$ BECAUSE A IS OPEN $\quad \bar{A} = (-\infty, -1]$ $\quad \partial A = \bar{A} \setminus \mathring{A} = \{-1\}$

$B = [-1, 1]$ $\quad \mathring{B} = (-1, 1)$ $\quad \bar{B} = B = [-1, 1]$ $\quad \partial B = \bar{B} \setminus \mathring{B} = \{-1, 1\}$

$C = (-\infty, 1)$ $\quad \mathring{C} = (-\infty, +1)$ $\quad \bar{C} = C = (-\infty, +1]$ $\quad \partial C = \bar{C} \setminus \mathring{C} = \{1\}$

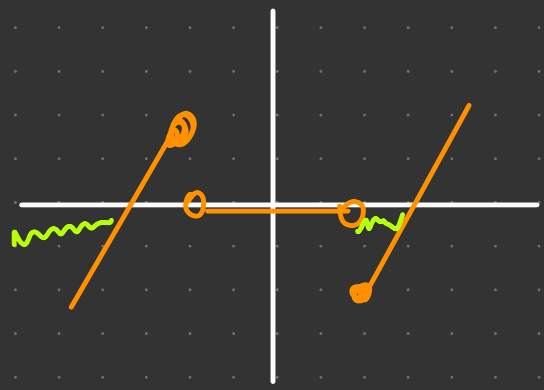
9. Let $f : \mathbb{R} \to \mathbb{R}$ be such that for $x \in \mathbb{R}$

$$f(x) = \begin{cases} x+2 & \text{if } x \leq -1 \\ 0 & \text{if } -1 < x < 1 \\ x-2 & \text{if } x \geq 1. \end{cases}$$

Let

$$A = f^{-1}((-\infty, 0)) = \{x \in \mathbb{R} : \ f(x) \in (-\infty, 0)\} = \{x \in \mathbb{R} : \ f(x) < 0\},$$
$$B = f^{-1}(\{0\}) = \{x \in \mathbb{R} : \ f(x) \in \{0\}\} = \{x \in \mathbb{R} : \ f(x) = 0\},$$
$$C = f^{-1}((-\infty, 0]) = \{x \in \mathbb{R} : \ f(x) \in (-\infty, 0]\} = \{x \in \mathbb{R} : \ f(x) \leq 0\}.$$

Determine if A is open and if B and C are closed. Determine the closure, the interior and the boundary of A, B and C.



$A = (-\infty, -1] \cup (1, 2)$

10. Let $f : \mathbb{R} \to \mathbb{R}$ be such that for $x \in \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

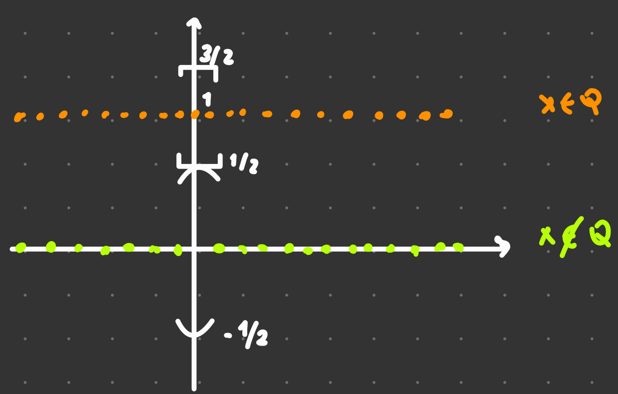
Let $A = f^{-1}((-1/2, 1/2))$ and $C = f^{-1}([1/2, 3/2])$. Determine if A is open and if C is closed.

$f: \mathbb{R} \to \mathbb{R}$ S.T. FOR $x \in \mathbb{R}$: $f(x) = \begin{cases} 1 & \text{IF } x \in \mathbb{Q} \\ 0 & \text{IF } x \notin \mathbb{Q} \end{cases}$

$\mathbb{Q}$ RATIONAL NUMBER SET

$A = f^{-1}((-\frac{1}{2}, \frac{1}{2}))$ $\quad C = f^{-1}([1/2, 3/2])$

f CAN BE ONLY 0 OR 1



$A = \{x \text{ S.T. } -1/2 < f(x) < 1/2\} = \{x \text{ S.T. } f(x) = 0\} = \{x \text{ S.T. } x \notin \mathbb{Q}\}$

11. Let $C = \{(x, y) \in \mathbb{R}^2 : \; x^2 + y^2 < 1\} \subset \mathbb{R}^2$.

Determine if C is open or closed, if it is bounded. Determine and describe the interior, the closure and the boundary of C.

C CLOSED?

$f_1(x,y) = x^2+y^2<1$ CONTINOUS $\quad f_1^{-1}(x,y) = ((-\infty, 1]) = \{(x,y) = x^2+y^2<1\}$ CLOSE

C BOUNDED?

$|x|<1 \quad |y|<1$

??? ???

$\partial C = \{x^2+y^2<1\} \cup \{x^2+y^2=1\} \cup \{x^2+y^2 \leq 1\}$

12. Let $C = \{(x, y, z) \in \mathbb{R}^3 : \; x^2 + y^2 + z^2 < 1\} \subset \mathbb{R}^3$.

Determine if C is open or closed, if it is bounded. Determine and describe the interior, the closure and the boundary of C.

C CLOSE?

$f_1(x,y,z) = x^2+y^2+z^2<1$ CONTINOUS $\quad f_1^{-1}(x,y) = ((-\infty, 1]) = \{(x,y) = x^2+y^2+z^2<1\}$ CLOSE

C BOUNDED?

$|x|<1 \quad |y|<1 \quad |z|<1$

$\partial C = \{x^2+y^2+z^2<1\} \cup \{x^2+y^2+z^2 \leq 1\} \cup \{x^2+y^2+z^2=1\}$

13. Let $C = \{(x,y) \in \mathbb{R}^2 : \ x^2 + 3y^4 \leq 1\} \subset \mathbb{R}^2$.

Determine if C is open or closed, if it is bounded. Determine and describe the interior, the closure and the boundary of C.

SAME?

14. Let $C = \{(x,y,z) \in \mathbb{R}^3 : \ x^2 + 3y^4 \leq 1\} \subset \mathbb{R}^3$.

Determine if C is open or closed, if it is bounded. Determine and describe the interior, the closure and the boundary of C.

C CLOSED?

$f_1(x,y,z) = x^2+3y^4 \leq 1$ CONTINOUS $\quad f_1^{-1}(x,y) = \left(\left(-\infty, \frac{1}{\sqrt[4]{3}}\right]\right) = \{(x,y): x^2+3y^4 \leq 1\}$ CLOSE

???

C BOUNDED?

$x^2+3y^4 \leq 1 \quad |x| \leq 1 \quad |y| \leq \frac{1}{\sqrt[4]{3}}$??

$\partial C = \{x^2+3y^4 \leq 1\} \cup \{x^2+3y^4 < 1\} \cup \{x^2+3y^4 = 1\}$

15. Let $C = \{(x,y) \in \mathbb{R}^2 : \ x^2 + y^2 \leq 1, \ y \geq 0\} \subset \mathbb{R}^2$.

Determine if C is open or closed, if it is bounded. Determine and describe the interior, the closure and the boundary of C.

C CLOSED?

$f_1(x,y) = x^2+y^2 \leq 1$ CONTINOUS

$f_1^{-1}(x,y) = ((-\infty, 1]) = \{(x,y): x^2+y^2 \leq 1\}$ CLOSED

$f_2(x,y) = y \geq 0$ CONTINOUS

$f_2^{-1}(x,y) = ([0,+\infty))$ CLOSED

$C = f_1^{-1}((-\infty,1]) \cap f_2^{-1}([0,+\infty))$ CLOSED (INTERSECTION OF CLOSED SETS)

C BOUNDED?

$(x,y) \in C \quad x^2+y^2 \leq 1 \quad |x| \leq 1 \quad |y| \leq 1$

$y \geq 0 \quad y \geq 0$

$\partial C = \{x^2+y^2 \leq 1; y \geq 0\} \cup \{x^2+y^2 < 1; y \geq 0\} \cup \{x^2+y^2 = 1; y \geq 0\} \cup$

$\cup \{x^2+y^2 \leq 1; y > 0\} \cup \{x^2+y^2 < 1; y > 0\} \cup \{x^2+y^2 = 1; y > 0\} \cup$

$\cup \{x^2+y^2 \leq 1; y = 0\} \cup \{x^2+y^2 < 1; y = 0\} \cup \{x^2+y^2 = 1; y = 0\}$

16. Let $C = \{(x,y,z) \in \mathbb{R}^3 : \ x^2+y^2 \le 1, \ 0 \le z \le 2\} \subset \mathbb{R}^3$.

Determine if C is open or closed, if it is bounded. Determine and describe the interior, the closure and the boundary of C.

N. 16) $C = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \le 1 \quad 0 \le z \le 2\} \subset \mathbb{R}^3$

$f_1(x,y,z) = x^2+y^2$ CONTINOUS

$f_1^{-1}(x,y,z) = ((-\infty, 1]) = \{(x,y,z) : x^2+y^2 \le 1\}$ CLOSED

$f_2(x,y,z) = z$ CONTINOUS

$f_2^{-1}(x,y,z) = ([0,2]) = \{(x,y,z) : 0 \le z \le 2\}$ CLOSED

INTERSECTION OF CLOSED SETS

$C = f_1^{-1}((-\infty, 1]) \cap f_2^{-1}([0,2])$ CLOSED

C BOUNDED?

$(x,y,z) \in C \quad x^2+y^2 \le 1 \quad |x| \le 1 \quad |y| \le 1$

$0 \le z \le 2 \quad |z| \le 2$

BOUNDED

$\overline{C}$

$\partial C = \{x^2+y^2 \le 1, \ z=0\} \cup \{x^2+y^2 \le 1, z=2\} \cup \{x^2+y^2 \le 1, \ 0 \le z \le 2\} \cup$

$\{x^2+y^2 < 1, \ z=0\} \cup \{x^2+y^2 = 1, z=0\} \cup \{x^2+y^2 < 1, \ z=2\} \cup \{x^2+y^2 = 1, z=2\} \cup$

$\{x^2+y^2 = 1, \ 0 < z < 2\}$

17. Let $C = \{(x, y, z) \in \mathbb{R}^3 : \ x^2 + y^2 + z^2 \leq 1, \ z \geq 0\} \subset \mathbb{R}^3$.

Determine if C is open or closed, if it is bounded. Determine and describe the interior, the closure and the boundary of C.

C OPEN OR CLOSE?

$f_1(x,y,z) = x^2+y^2+z^2 \leq 1$ CONTINOUS

$f_1^{-1}(x,y,z) = ((-\infty, 1]) = \{(x,y,z) : x^2+y^2+z^2 \leq 1\}$ CLOSED

$f_2(x,y,z) = z \geq 0$ CONTINOUS

$f_2^{-1}(x,y,z) = ([0,+\infty)) = \{(x,y,z) : z \geq 0\}$ CLOSED

$C = f_1^{-1}((-\infty,1]) \cap f_2^{-1}([0,+\infty))$ C CLOSED (INTERSECTION OF CLOSED SETS)

C BOUNDED?

$(x,y,z) \in C \quad x^2+y^2+z^2 \leq 1 \quad |x| \leq 1 \quad |y| \leq 1 \quad |z| \leq 1$

$z \geq 0 \quad z \geq 0$

$\bar{C}$

$\partial C \quad \{x^2+y^2+z^2 \leq 1, z=0\} \cup \{x^2+y^2+z^2 \leq 1, z>0\} \cup \{x^2+y^2 \leq 1; z \geq 0\}$

$\cup \{x^2+y^2+z^2 < 1, z=0\} \cup \{x^2+y^2+z^2 < 1, z>0\} \cup \{x^2+y^2 < 1; z \geq 0\}$

$\cup \{x^2+y^2+z^2 = 1, z=0\} \cup \{x^2+y^2+z^2 = 1, z=0\} \cup \{x^2+y^2 = 1; z \geq 0\}$

18. Let $C = \{(x, y) \in \mathbb{R}^2 : \ x^2 + y^2 \leq 1, \ (x - 3/2)^2 + y^2 \leq 1\} \subset \mathbb{R}^2$.

Determine if C is open or closed, if it is bounded. Determine and describe the interior, the closure and the boundary of C.

$f_1(x,y) = x^2 + y^2 \leq 1$ CONTINOUS

$f_1^{-1}(x,y) = ((-\infty, 1]) = \{(x,y) : x^2 + y^2 \leq 1\}$ CLOSE

$f_2(x,y) = \left(x - \frac{3}{2}\right)^2 + y^2 \leq 1$ CONTINOUS

$f_2^{-1}(x,y) = ((-\infty, \frac{3}{2}]) = \{(x,y) : (x - \frac{3}{2})^2 + y^2 \leq 1\}$ CLOSE

$C = f_1^{-1}(-\infty, 1] \cap f_2^{-1}(-\infty, \frac{3}{2}]$ C CLOSED, INTERSECTION OF CLOSED SETS

C BOUNDED?

$(x,y) \in C \quad x^2 + y^2 \leq 1 \quad |x| \leq 1 \quad |y| \leq 1$

$\left(x - \frac{3}{2}\right)^2 + y^2 \leq 1 \quad |x - \frac{3}{2}| \leq 1 \quad |y| \leq 1$

$\partial C : \{x^2+y^2 \leq 1 ; (x - \frac{3}{2})^2 + y^2 \leq 1\} \cup \{x^2+y^2 < 1 ; (x - \frac{3}{2})^2 + y^2 \leq 1\} \cup \{x^2+y^2 = 1 ; (x - \frac{3}{2})^2 + y^2 \leq 1\} \cup$

$\cup \{x^2+y^2 < 1 ; (x - \frac{3}{2})^2 + y^2 \leq 1\} \cup \{x^2+y^2 < 1 ; (x - \frac{3}{2})^2 + y^2 < 1\} \cup \{x^2+y^2 \leq 1 ; (x - \frac{3}{2})^2 + y^2 \leq 1\} \cup$

$\cup \{x^2+y^2 = 1 ; x - \frac{3}{2})^2 + y^2 \leq 1\} \cup \{x^2+y^2 = 1 ; (x - \frac{3}{2})^2 + y^2 < 1\} \cup \{x^2+y^2 = 1 ; (x - \frac{3}{2})^2 + y^2 = 1\}$

19. Let $C = \{(x, y) \in \mathbb{R}^2 : x^2 - 1 \leq y \leq \cos(2x/\pi)\} \subset \mathbb{R}^2$.

Determine if C is open or closed, if it is bounded. Determine and describe the interior, the closure and the boundary of C.

20. Let $C = \{(x, y) \in \mathbb{R}^2 : x^2 - 1 \leq y \leq 1 - x^4,\ x > 0\} \subset \mathbb{R}^2$.

Determine if C is open or closed, if it is bounded. Determine and describe the interior, the closure and the boundary of C.

21. Let $C = \{(x, y, z) \in \mathbb{R}^3 : \; z = x - y\} \subset \mathbb{R}^3$.

Determine if C is open or closed, if it is bounded. Determine and describe the interior, the closure and the boundary of C.