ZX-Calculus

Introduction and Applications

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‘Berufsschnuppertag’ at PSI

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1.Foreword

Berufsschnuppertag

- Two-day program organized by my school

mng rämibühl
Mathematisch-Naturwissenschaftliches Gymnasium

- Goal: Get insights into the jobs we are interested in

This presentation

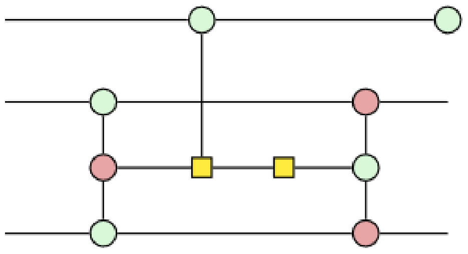
- ~45min
- Questions during the presentation are welcome + time at the end
- I’m not an expert, but I’ll try my best :^)

- This presentation, additional resources and my contact information can be found here:

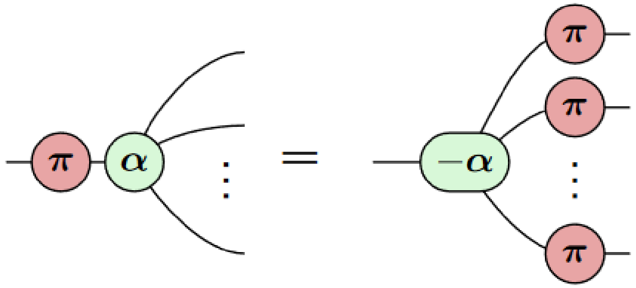
2. Introduction

ZX-Calculus in 30 seconds

- Quantum Theory/Computation?
- Easy?
- → Graphical language!
  - Diagrams*:



  - Rewrite rules:

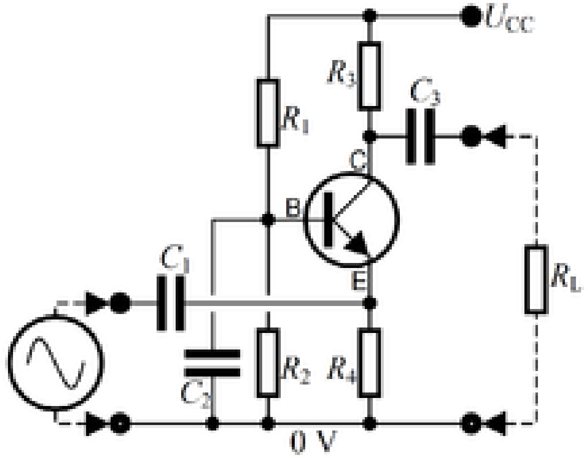


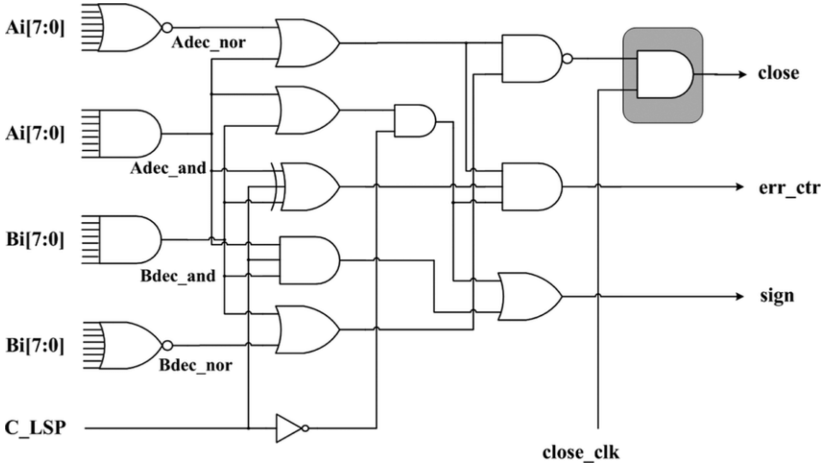
→ ZX-Calculus!

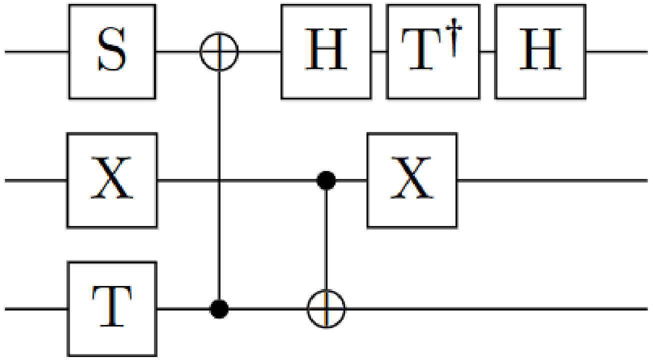
*Diagram shown is part of ZX**H**-Calculus

Reasoning using Graphical Languages

- Electronic / Logic / Quantum Circuits

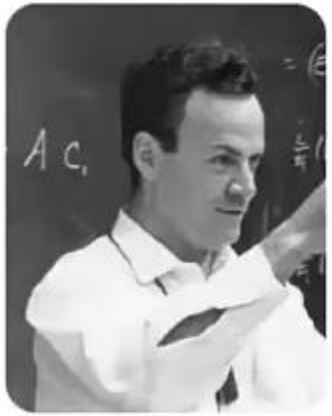


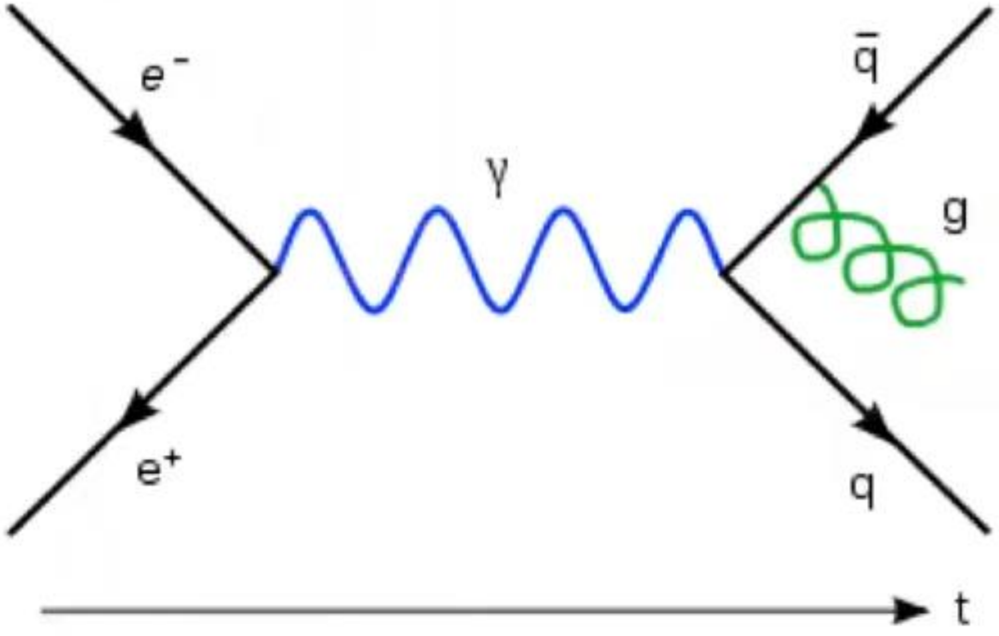




Reasoning using Graphical Languages

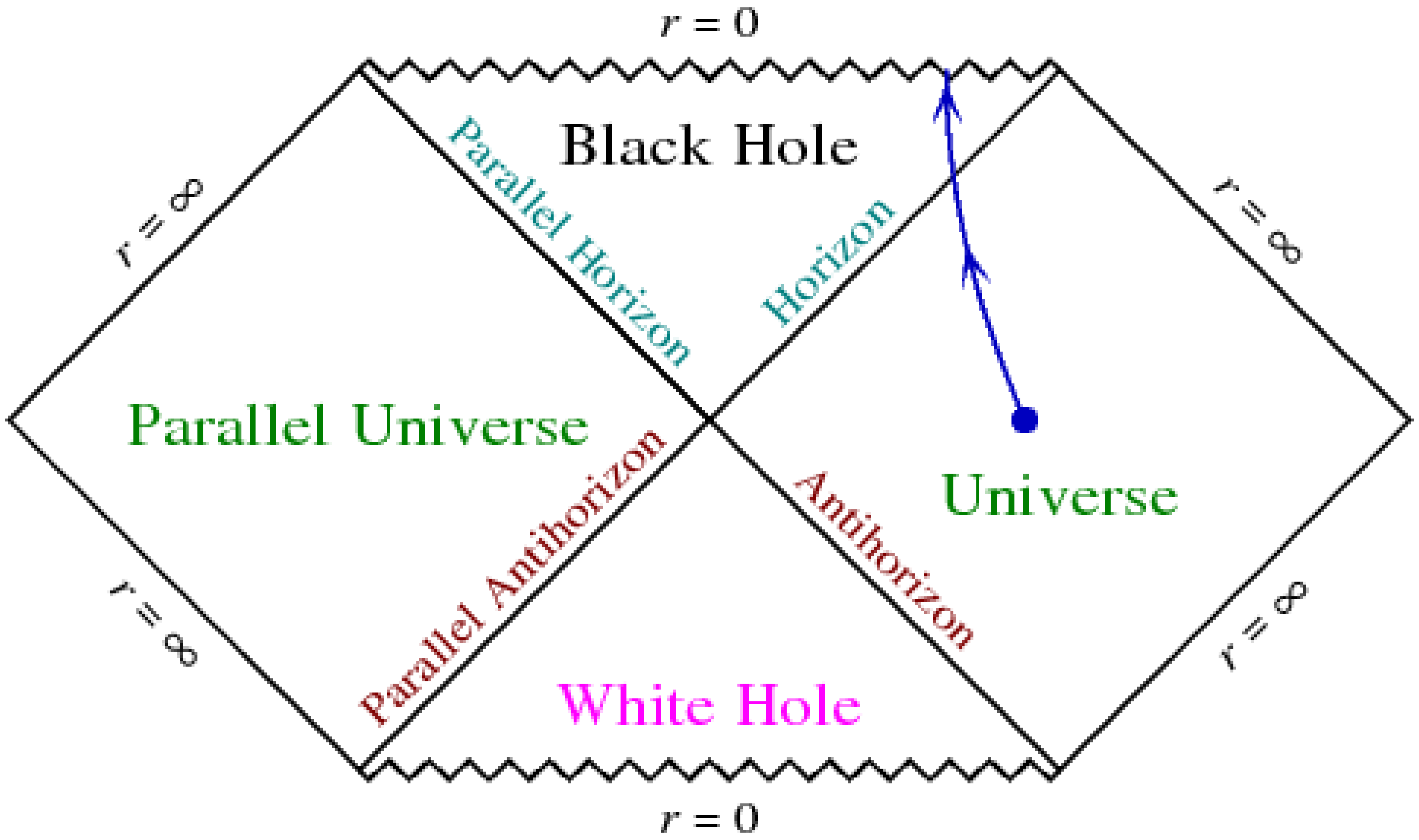
- Feynman Diagrams





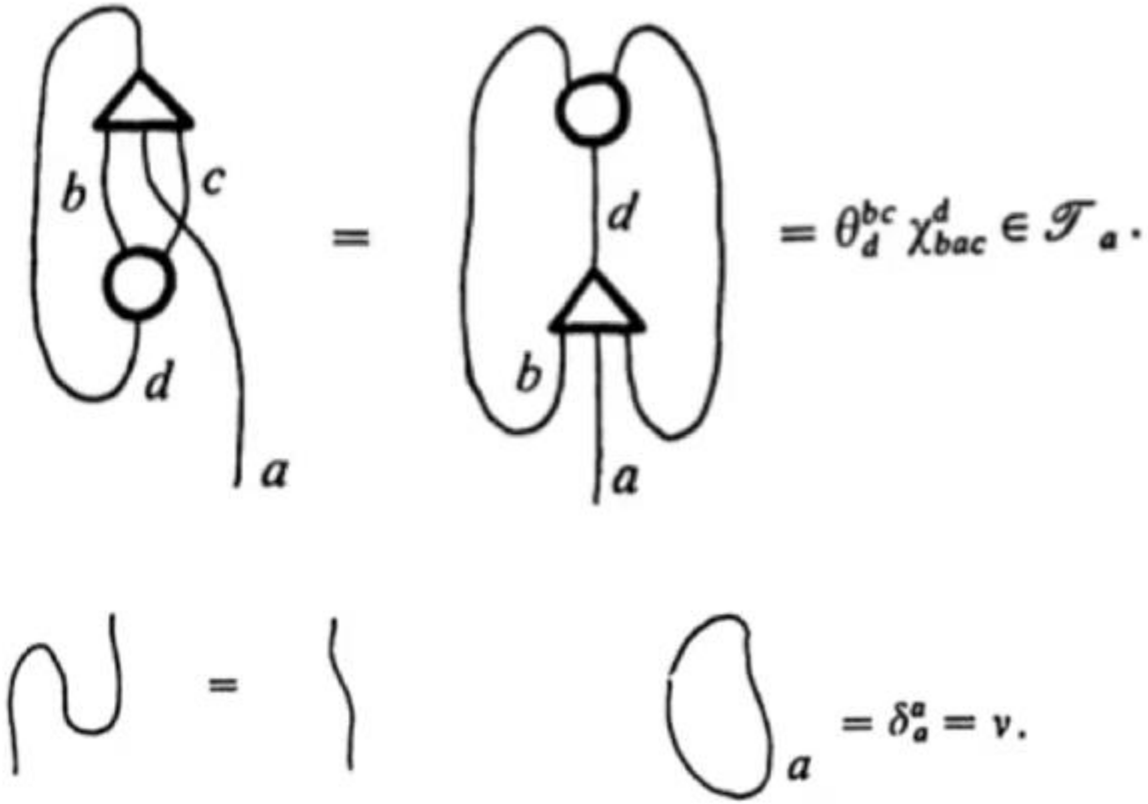
Reasoning using Graphical Languages

- Penrose Diagrams



Reasoning using Graphical Languages

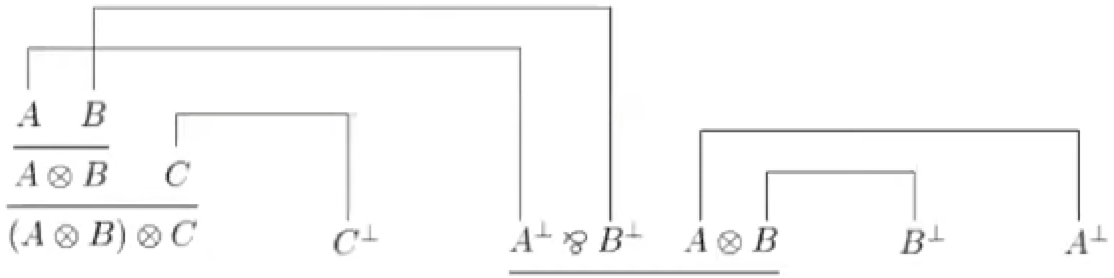
- “Applications of Negative Dimensional Tensors” by Penrose



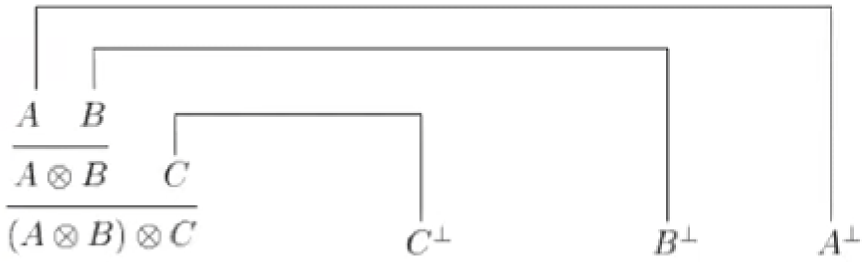
Reasoning using Graphical Languages

- Proof-Nets in Linear Logic (Girard, Danos-Regnier, Tortoro de Falco, …)

For example, the proof net



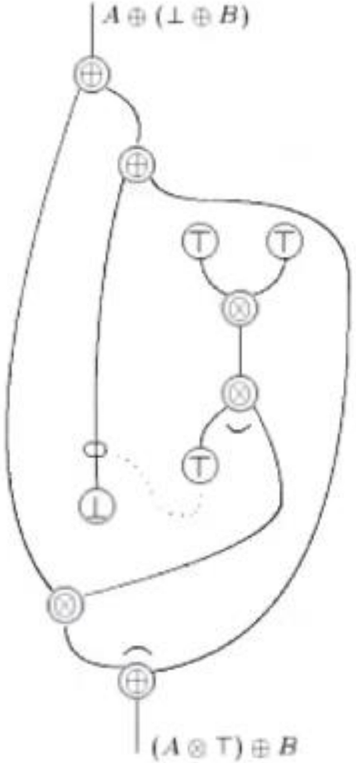
reduces (in three steps) to

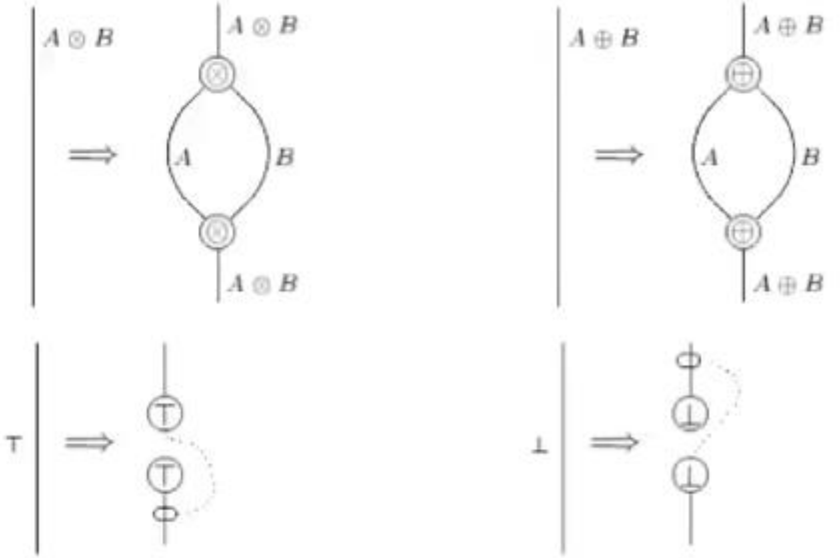


[2]

Reasoning using Graphical Languages

- Proof-Nets in Linear Logic (Blute-Cocket-Seely-Trimble)





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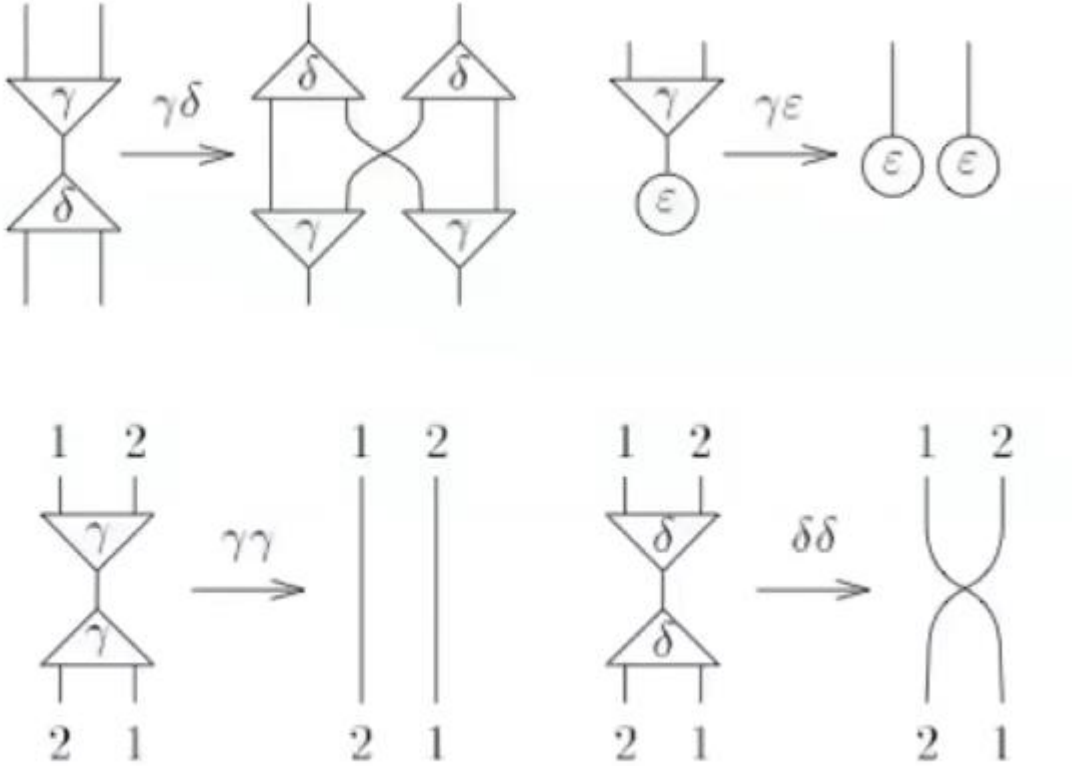
Reasoning using Graphical Languages

- Interaction Combinators



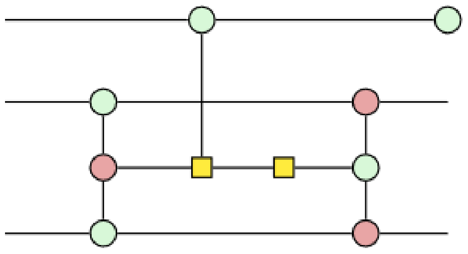
[2]

Reasoning using Graphical Languages

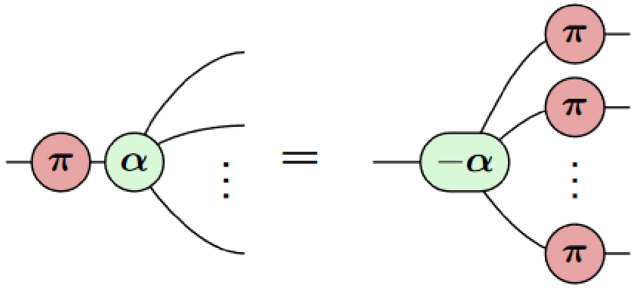
- ZX-Calculus was developed by Bob Coecke and Ross Duncan in 2007

ZX-Calculus in 30 seconds

- Quantum Theory/Computation?
- Easy?
- → Graphical language!
  - Diagrams*:



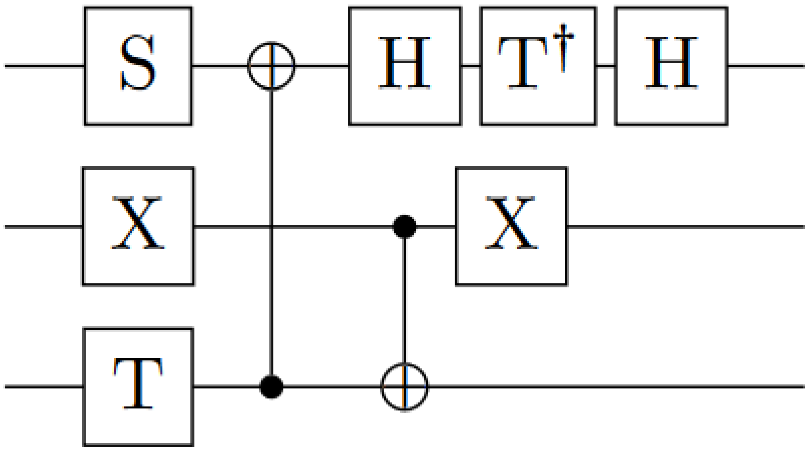
  - Rewrite rules:



→ ZX-Calculus!

*Diagram shown is part of ZX**H**-Calculus

“Why not quantum circuits?”



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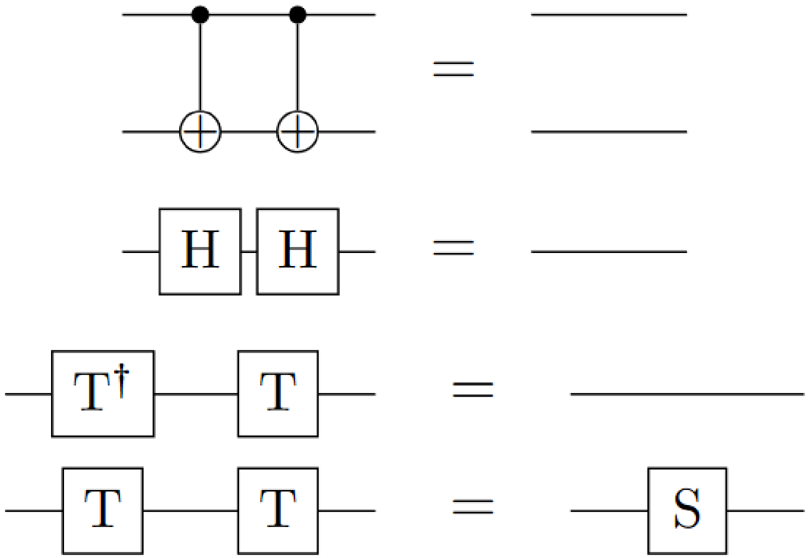
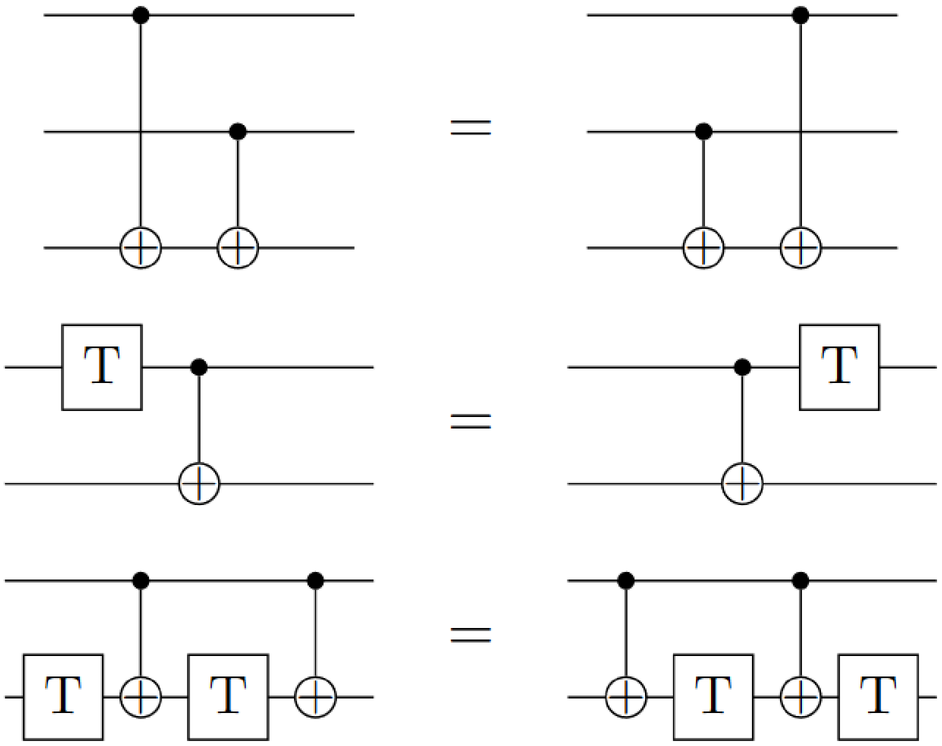
“Why not quantum circuits?”



[1]

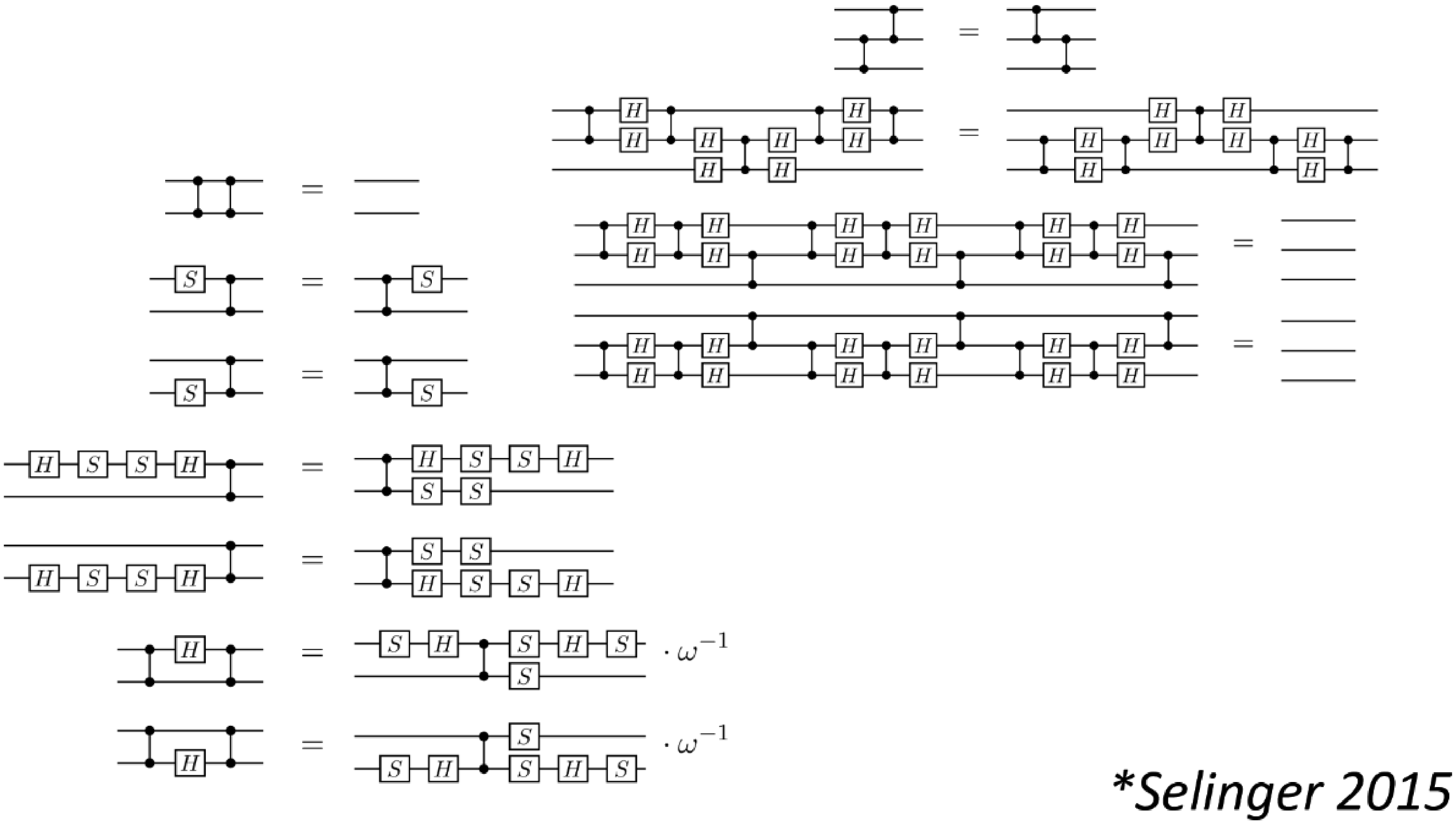
“Why not quantum circuits?”



but…

[1]

“Why not quantum circuits?”



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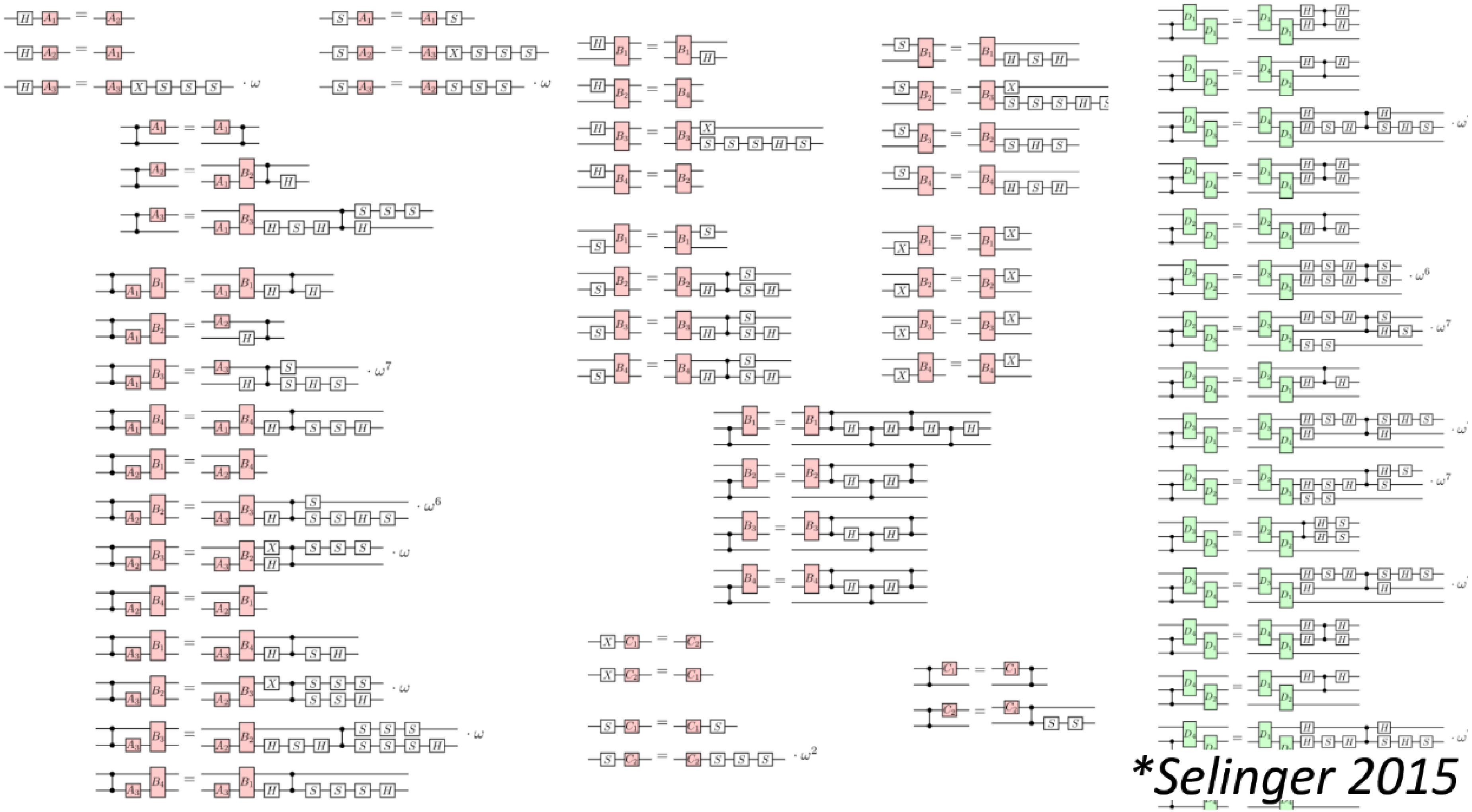
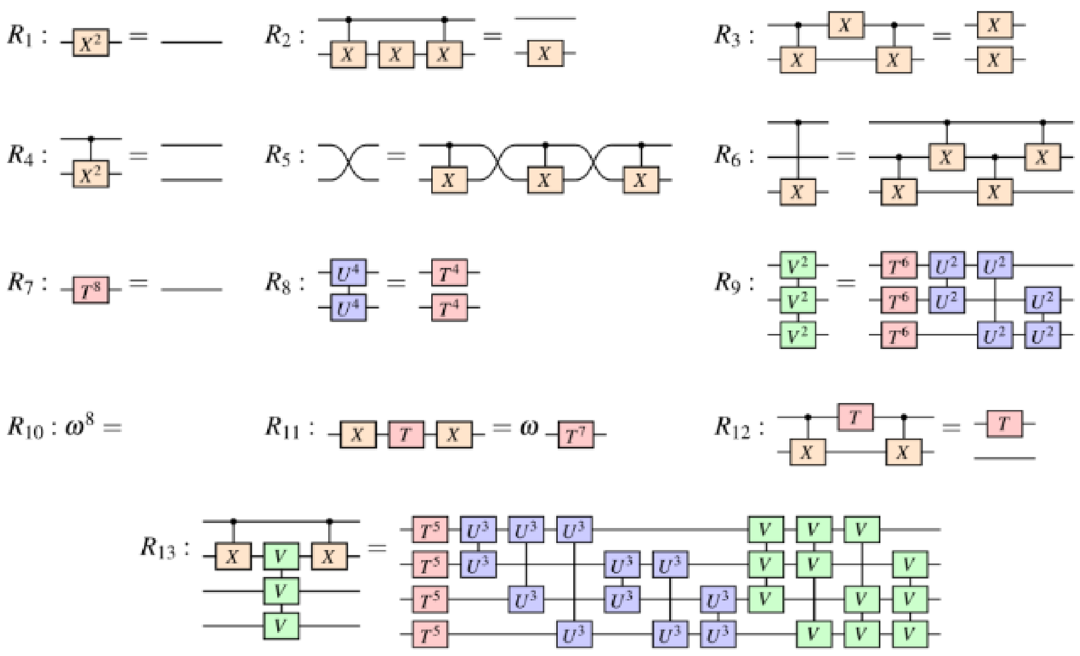
“Why not quantum circuits?”

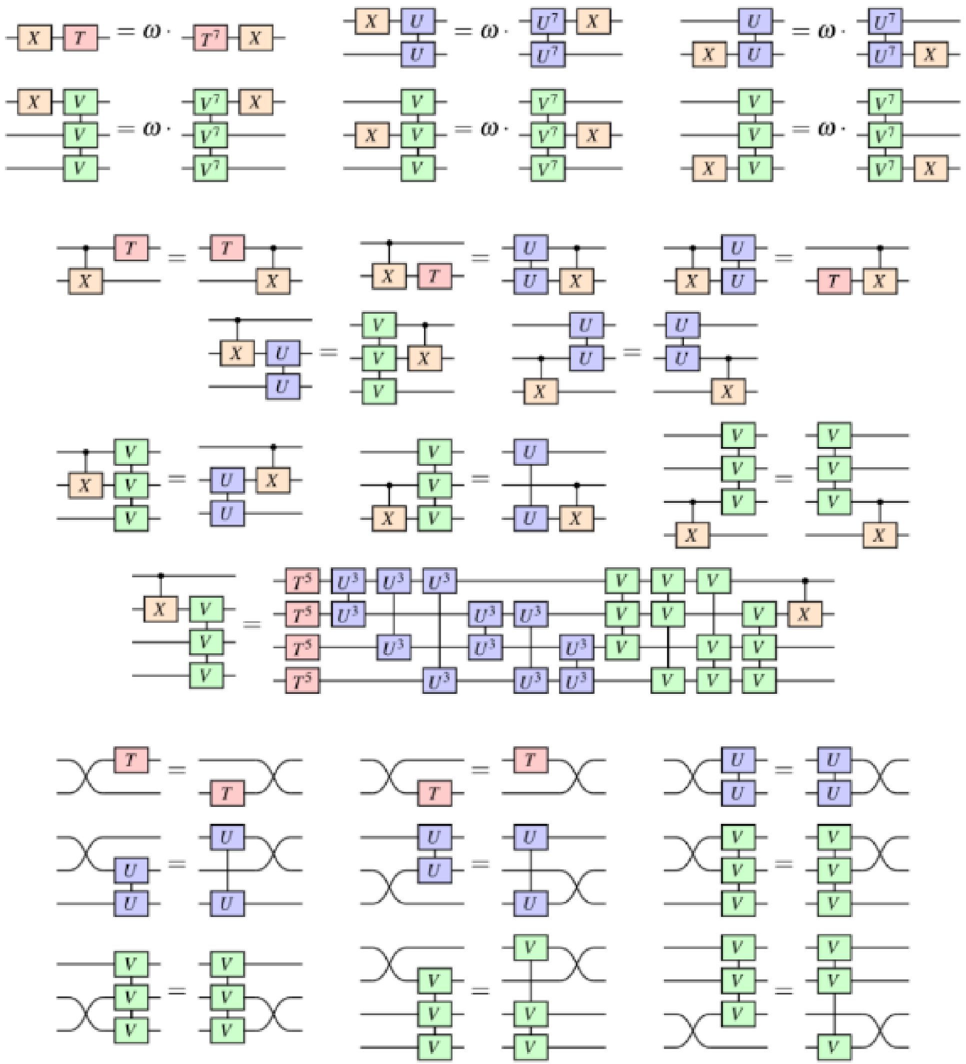


*Selinger 2015

[1]

“Why not quantum circuits?”





*Amy, Chen, & Ross 2018

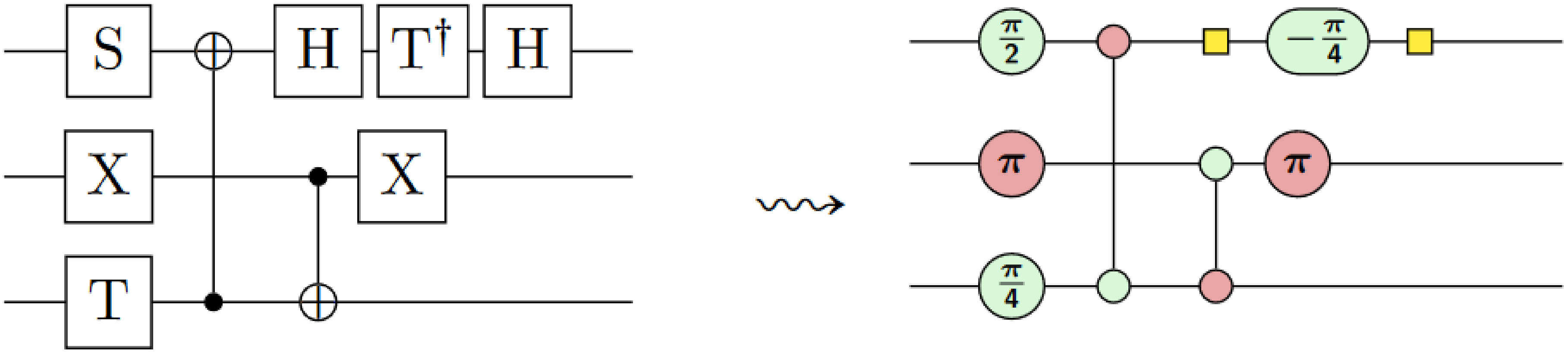
“Why not quantum circuits?”

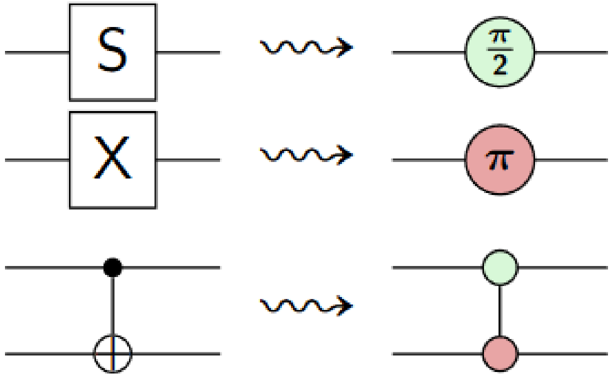
- No simple set of rewrite rules/identities
- Choice of gates is a bit arbitrary
- Restricted positioning

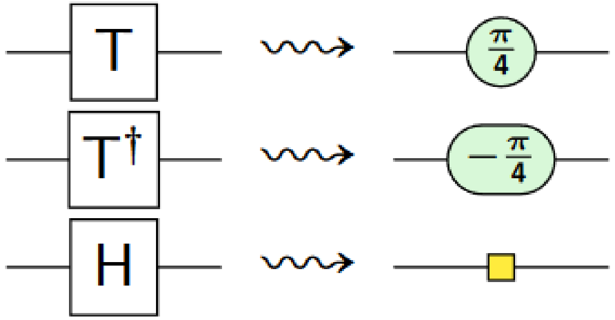
[1]

Remedy ZX-Calculus: Gates

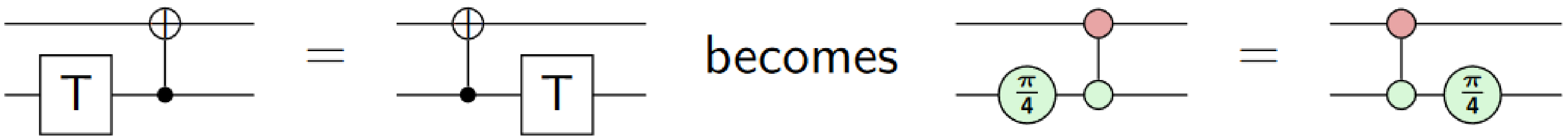
- ZX-Diagrams can be viewed as an alternative to quantum circuits (and much more!)



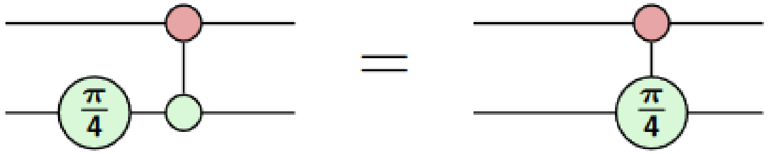




Remedy ZX-Calculus: Rewrite Rules for Gates



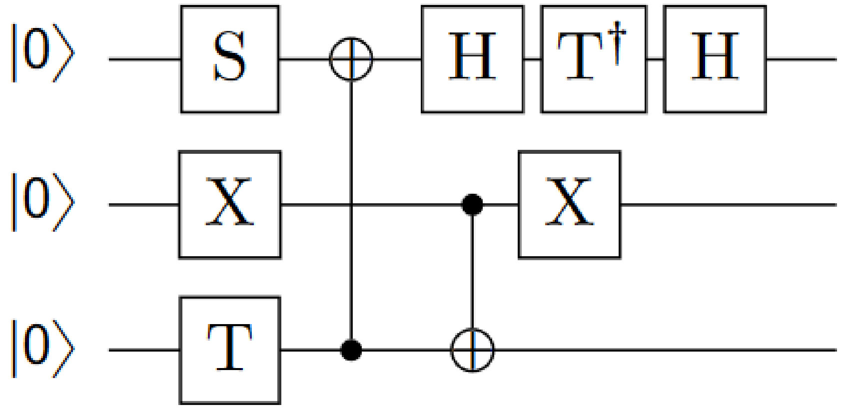
→ Spiders of same color commute through each other

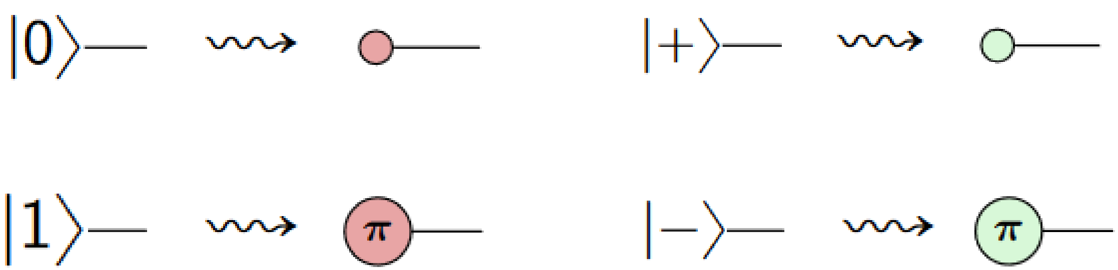


→ More fundamental rule: Spiders of same color fuse!

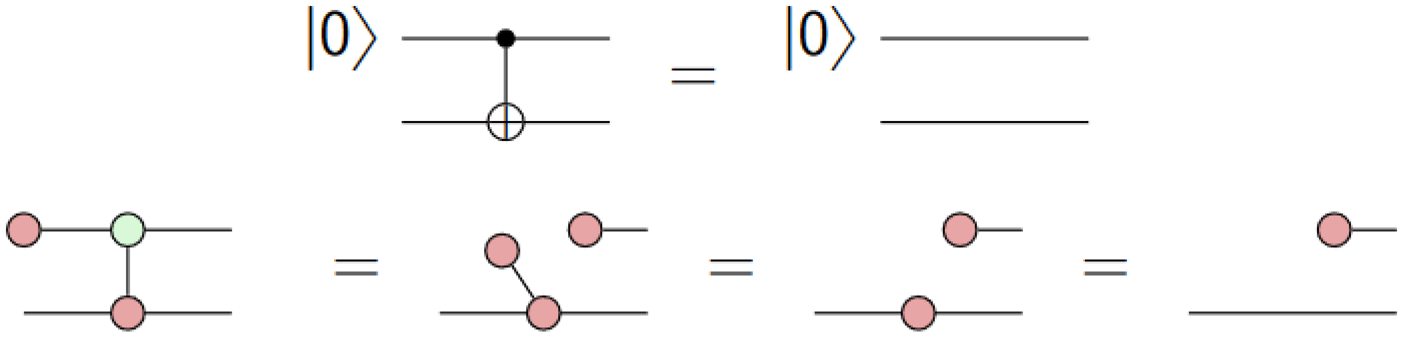
[1]

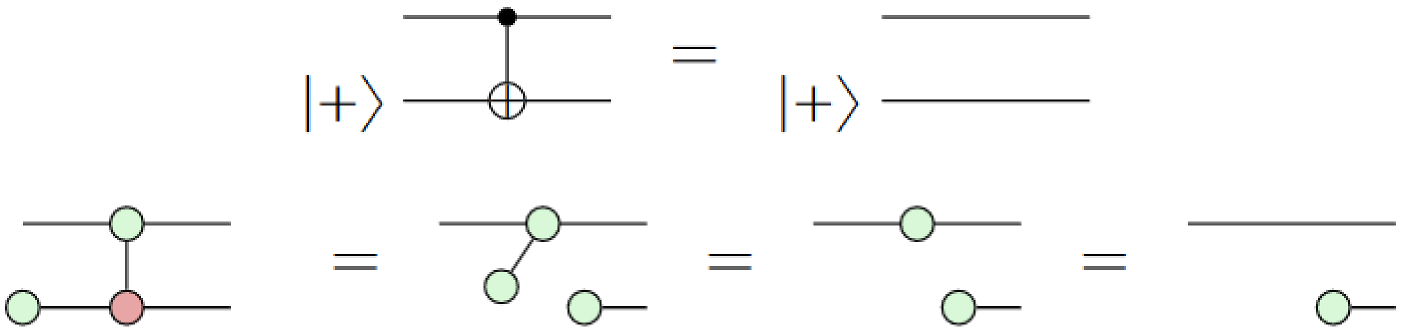
Remedy ZX-Calculus: States





Remedy ZX-Calculus: Rewrite Rules for States



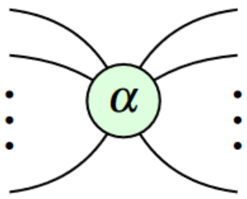


→ States copy through opposite-colored spiders

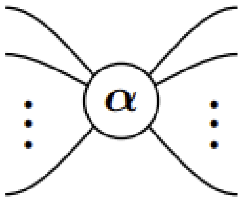
[1]

Formal Introduction

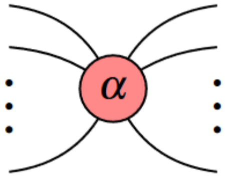
- Green- / Z-Spider



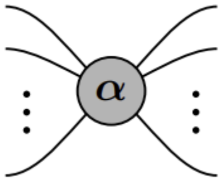
$$:= \quad |0\ldots0\rangle\langle0\ldots0| + e^{i\alpha}\,|1\ldots1\rangle\langle1\ldots1| \quad =:$$



- Red- / X-Spider

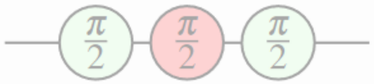


$$:= \quad |+\ldots+\rangle\langle+\ldots+| + e^{i\alpha}\,|-\ldots-\rangle\langle-\ldots-| \quad =:$$



- Hadamard

$$\equiv \quad e^{-i\frac{\pi}{4}}$$

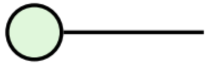


$$\equiv$$

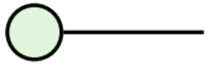
Formal Introduction

$$= \quad |0\rangle + |1\rangle = \sqrt{2}\,|+\rangle$$

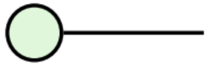
- Different Conventions:



$$= \quad \sqrt{2}\,|+\rangle$$



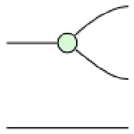
$$\approx \quad |+\rangle$$



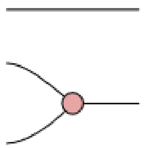
$$= \quad |+\rangle$$

Formal Introduction

- Vertical composition ⇔ Tensor product



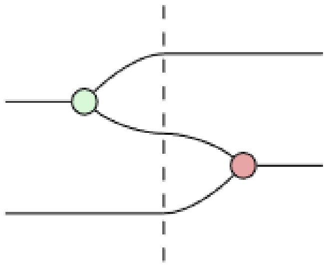
$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

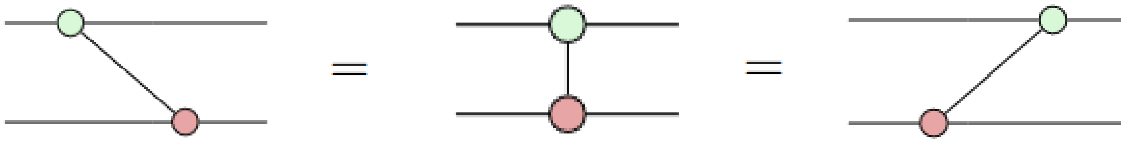
- Horizontal composition ⇔ Matrix multiplication



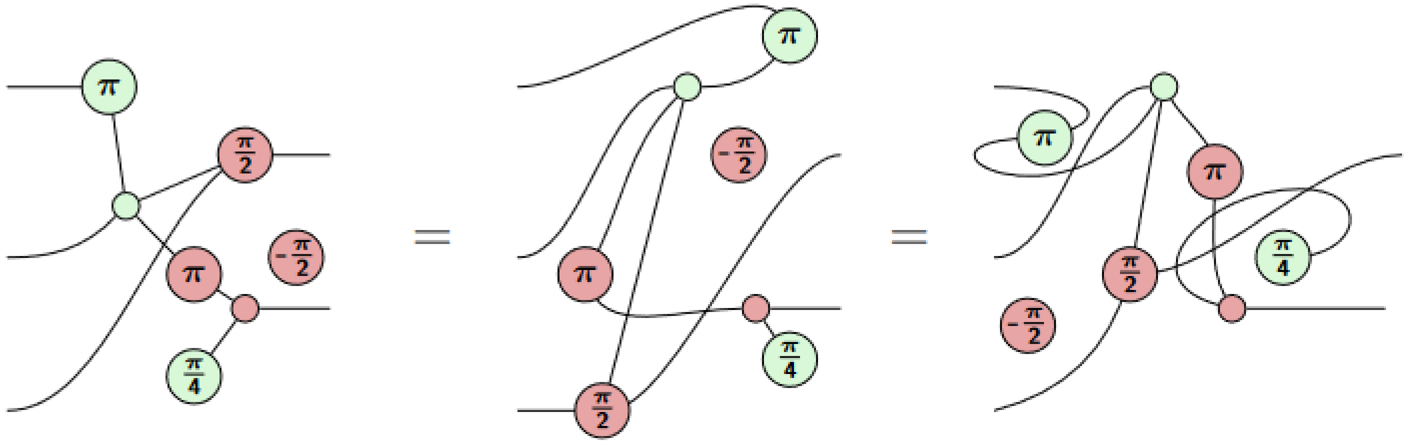
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Formal Introduction

- Symmetries



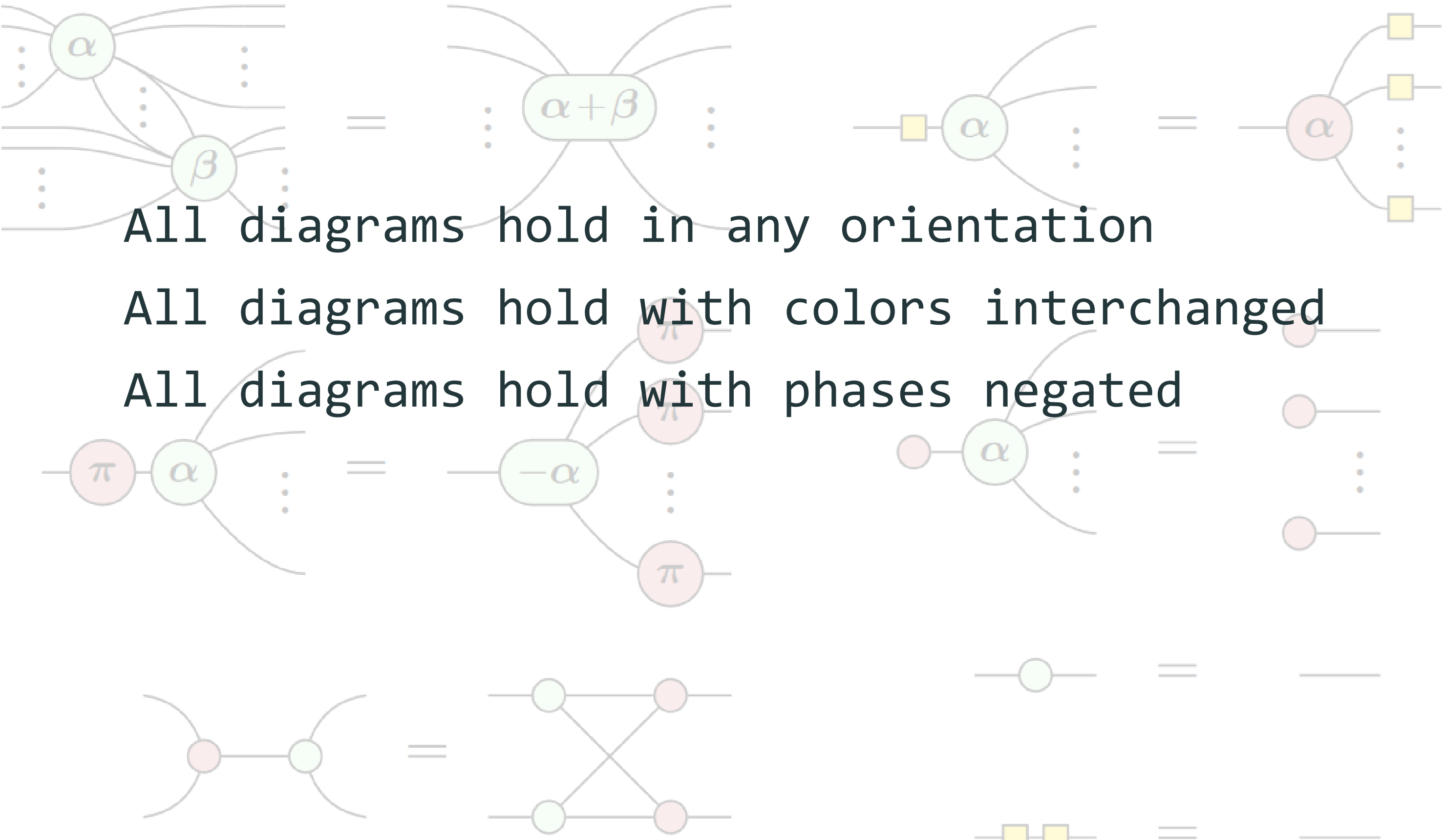
- → Only connectivity matters!



[1]

Formal Introduction

- Rewrite rules



All diagrams hold in any orientation

All diagrams hold with colors interchanged

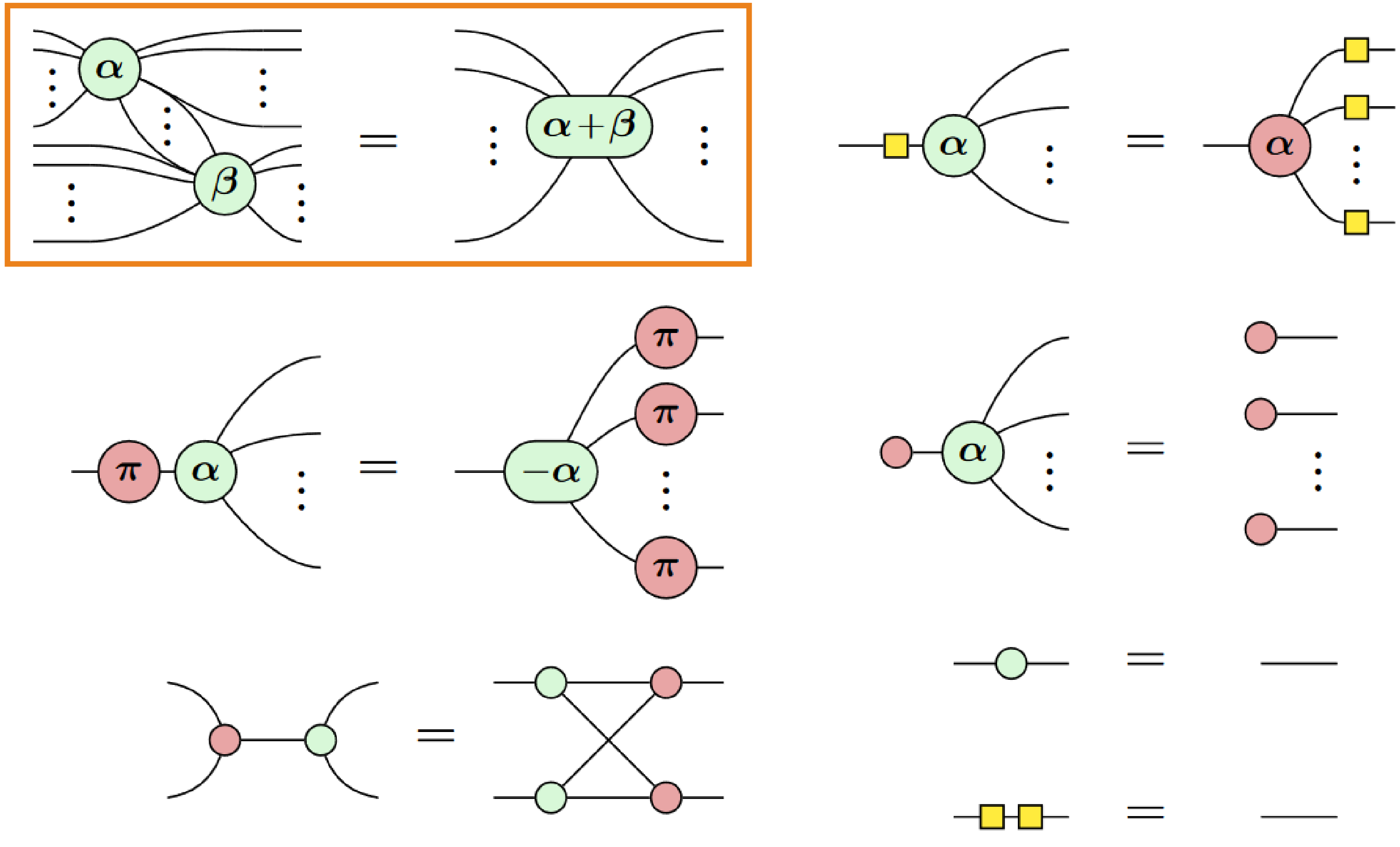
All diagrams hold with phases negated

$\alpha, \beta \in [0, 2\pi]$

[1]

Formal Introduction

- Spider fusion rule

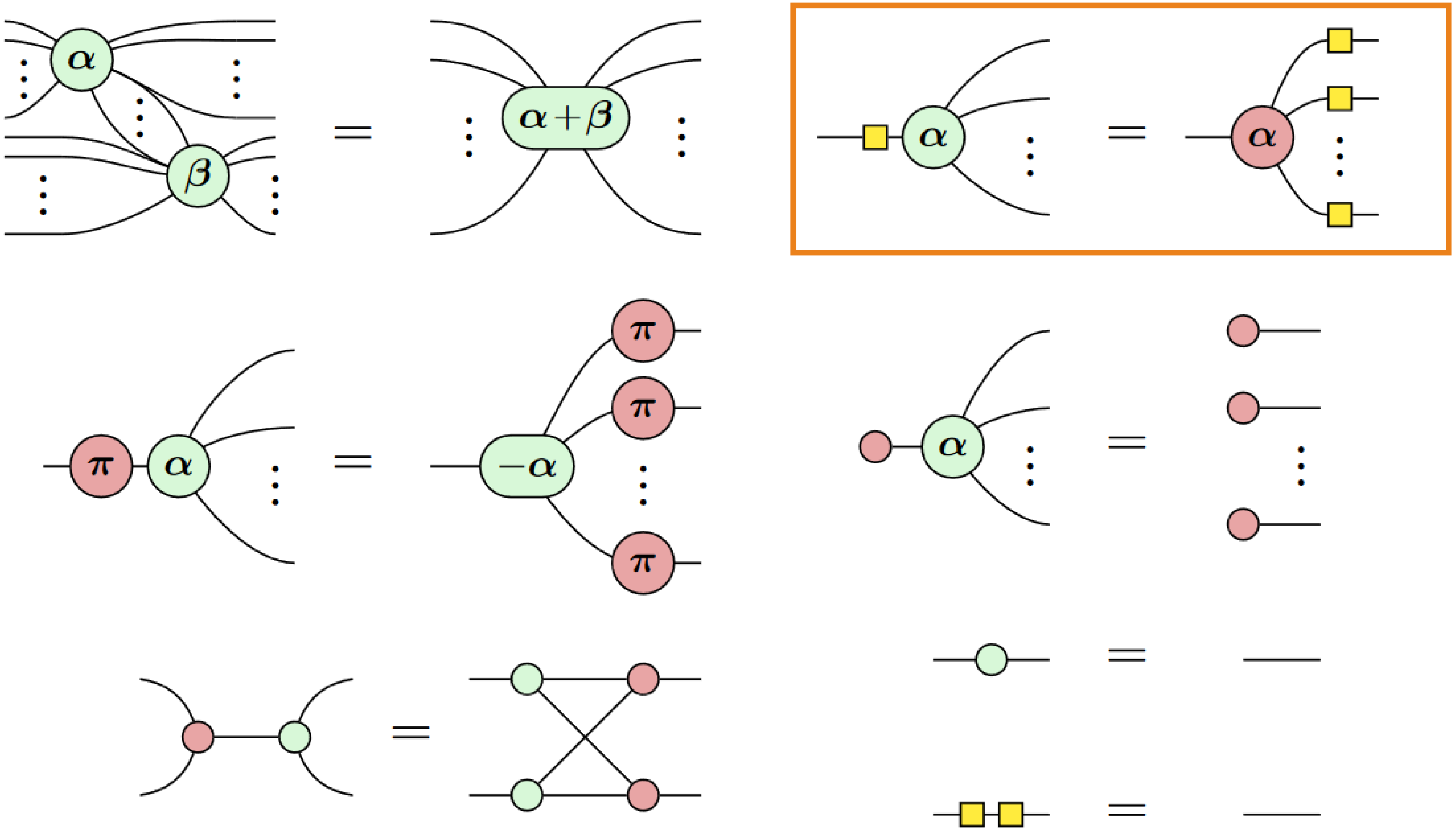


$\alpha, \beta \in [0, 2\pi]$

[1]

Formal Introduction

- Color change rule

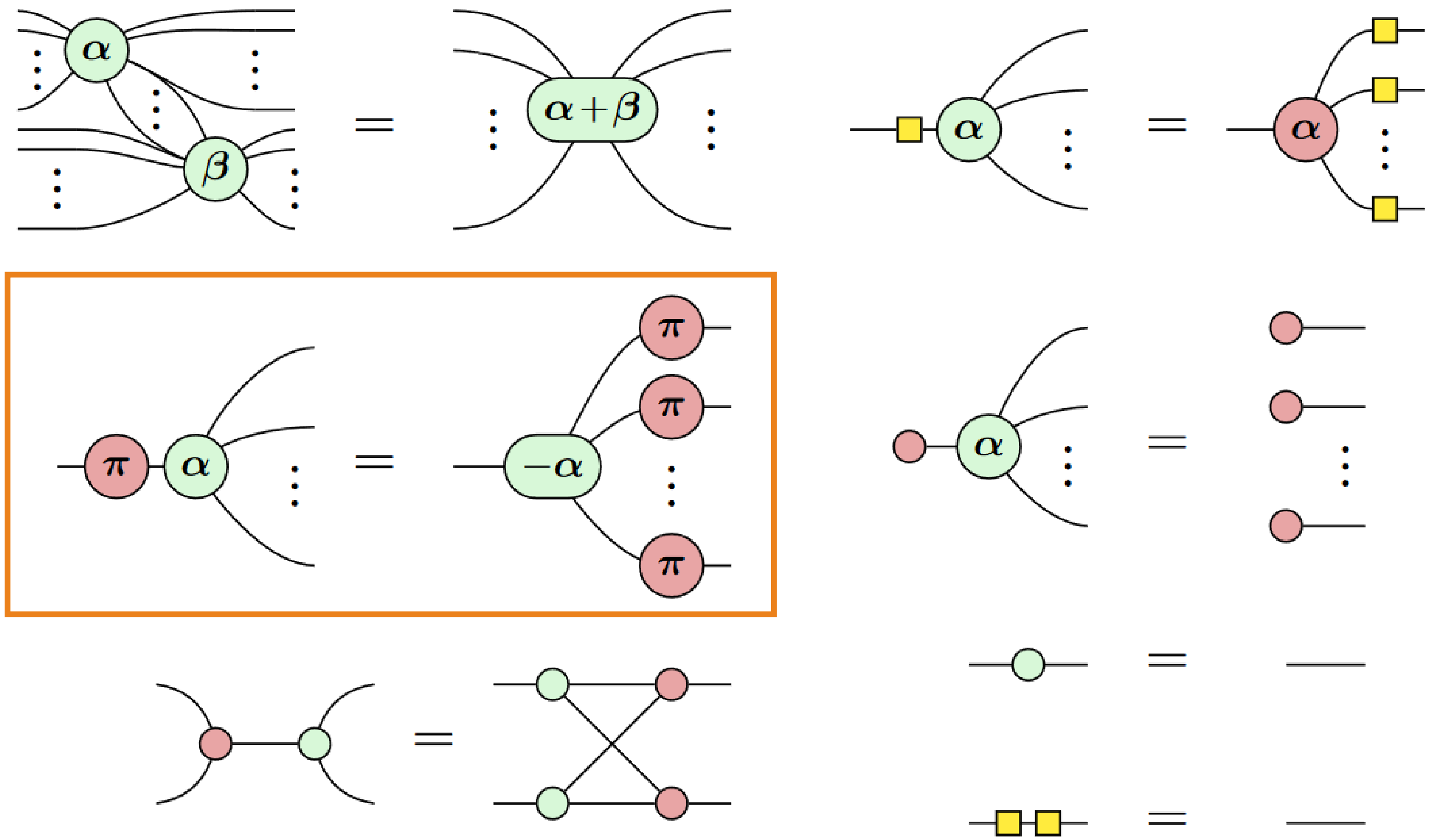


$\alpha, \beta \in [0, 2\pi]$

[1]

Formal Introduction

- π commutation rule

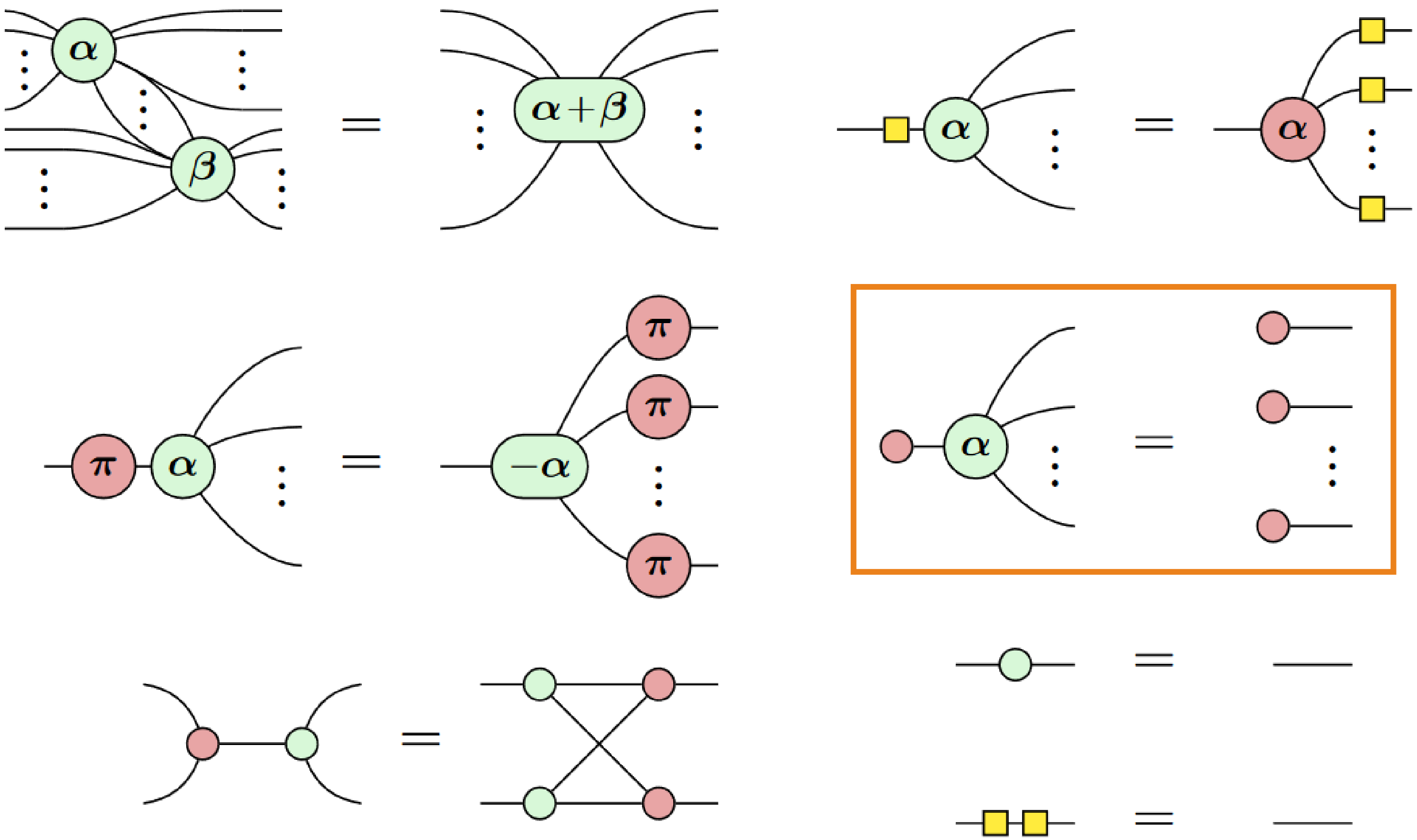


$$\alpha, \beta \in [0, 2\pi]$$

[1]

Formal Introduction

- State copy rule

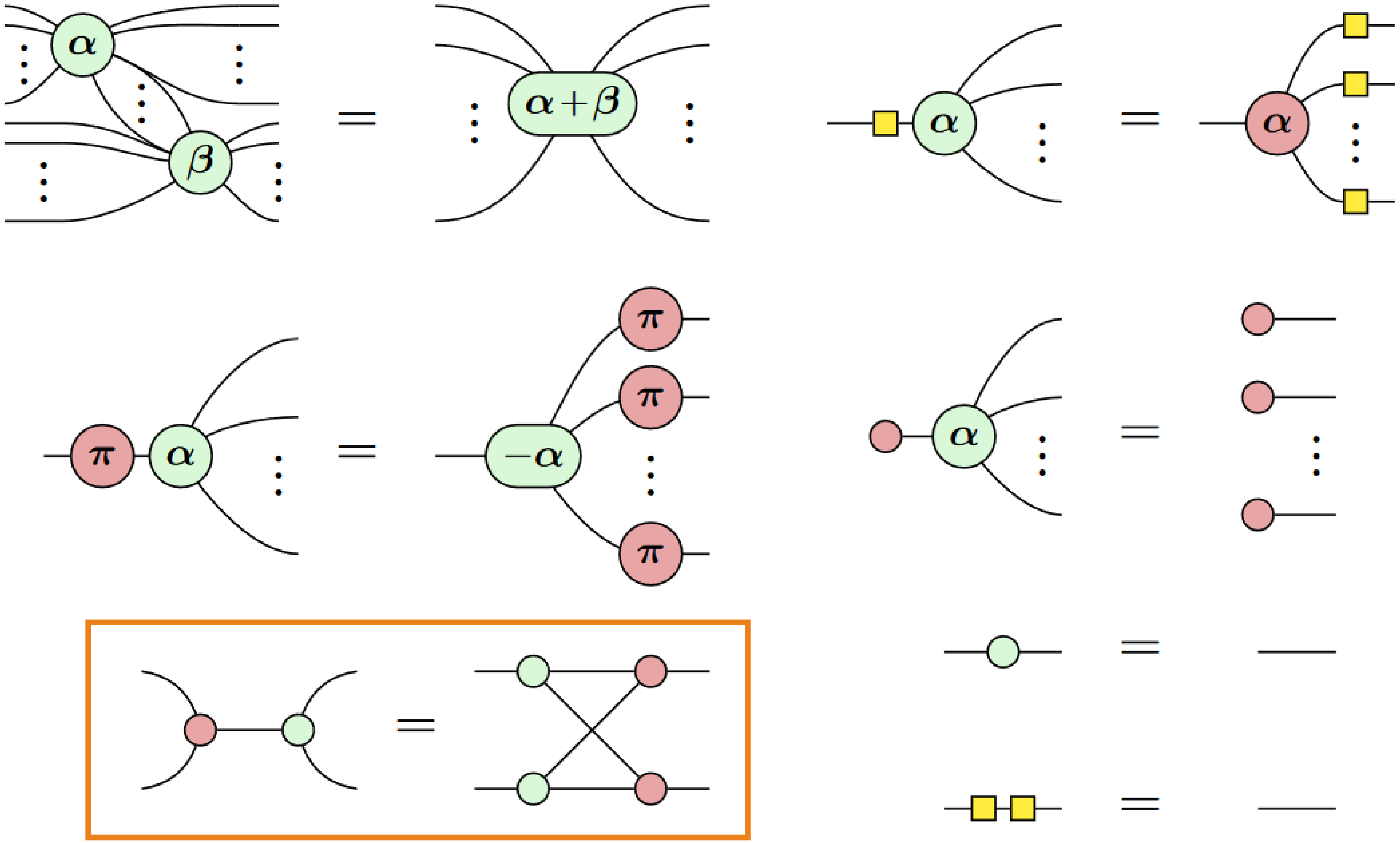


$\alpha, \beta \in [0, 2\pi]$

[1]

Formal Introduction

- Bialgebra rule

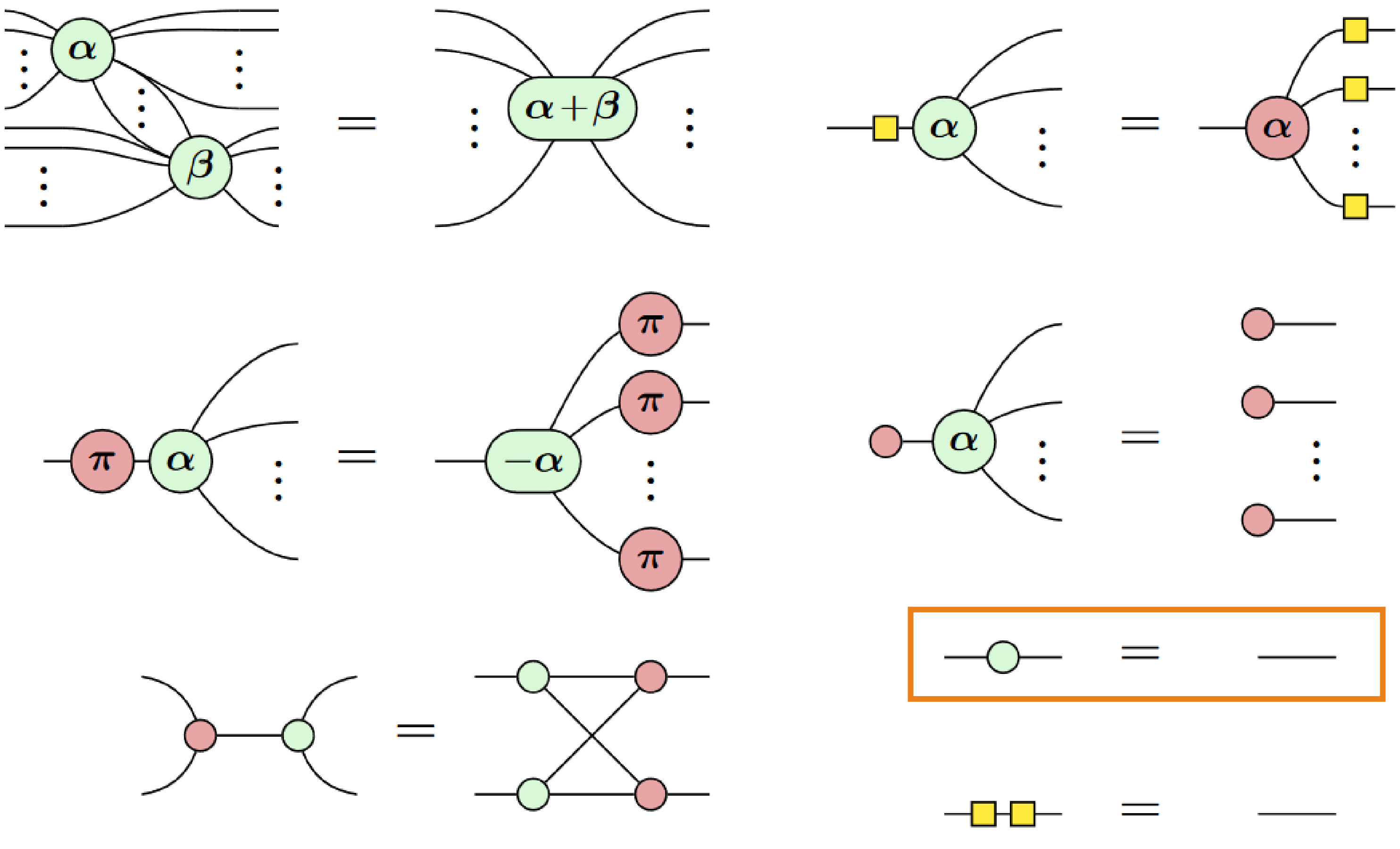


$$\alpha, \beta \in [0, 2\pi]$$

[1]

Formal Introduction

- Identity removal rule

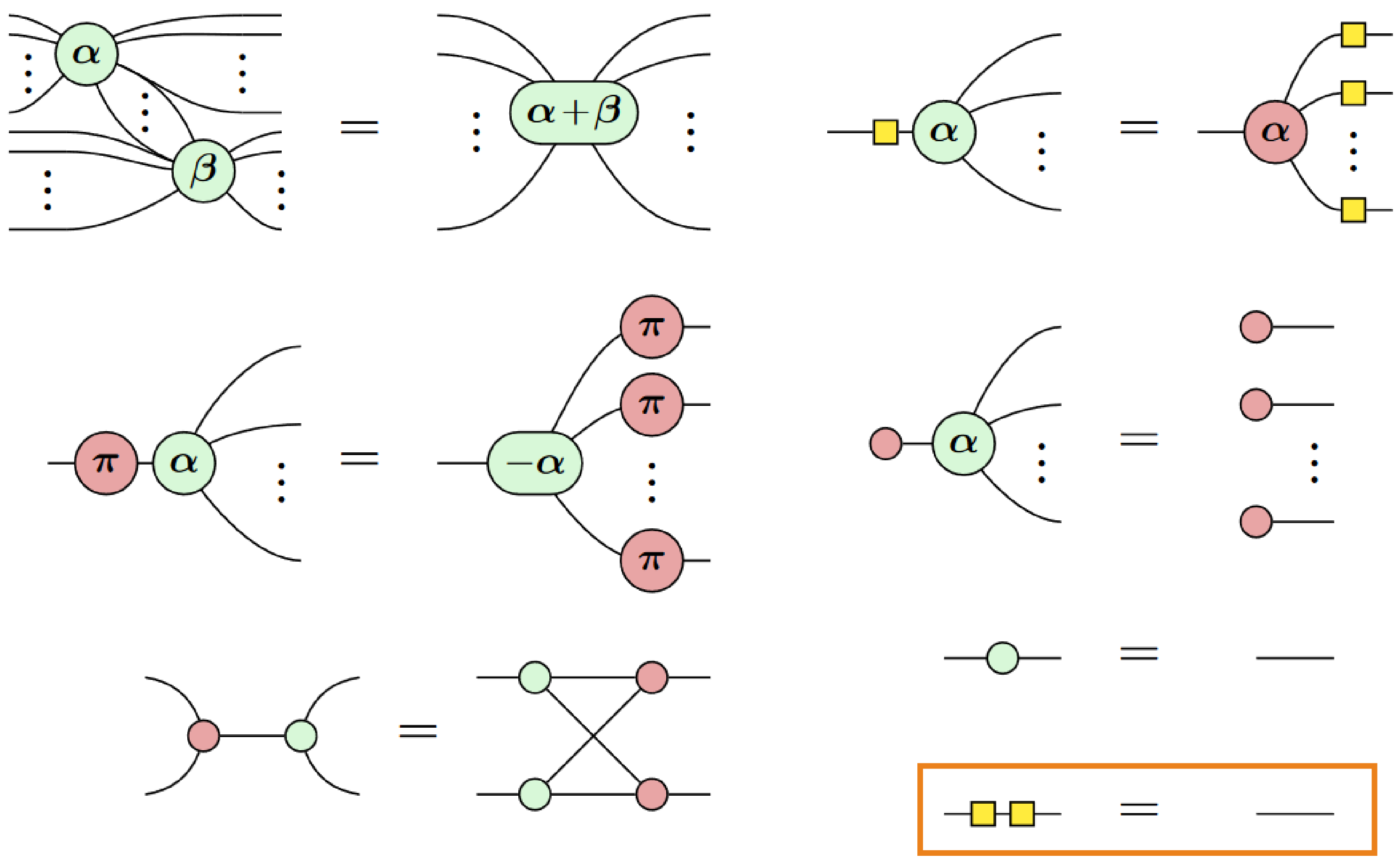


$$\alpha, \beta \in [0, 2\pi]$$

[1]

Formal Introduction

- Hadamard cancellation rule



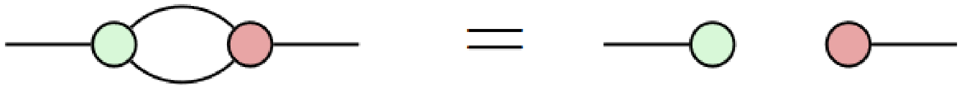
$$\alpha, \beta \in [0, 2\pi]$$

[1]

Formal Introduction

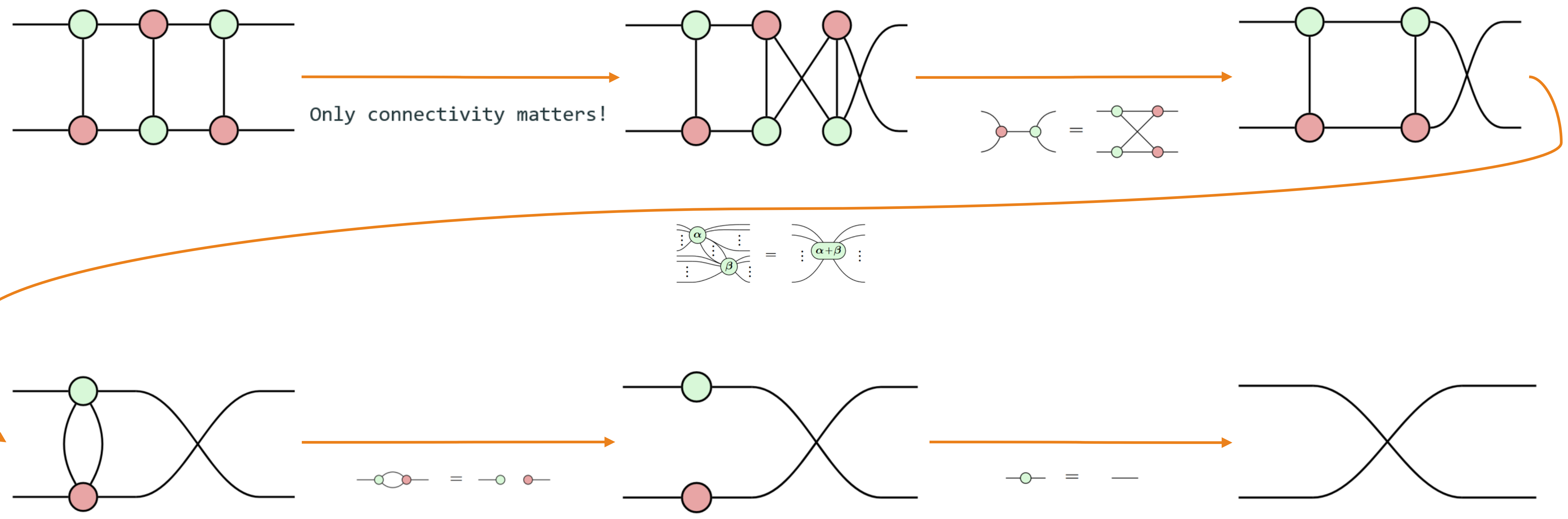
- Derived rules:

Hopf rule:



Formal Introduction

- Rewrite rules in action: Small example



Assessing graphical calculi

- Universality:

  Every linear map can be expressed within its framework.

  Categorically, this means that the interpretation functor is full.

- Completeness:

  Any equation involving linear maps derivable in multilinear algebra should also be derivable within the graphical language through the process of rewriting.

  Categorically, the interpretation functor is faithful.

- Soundness:

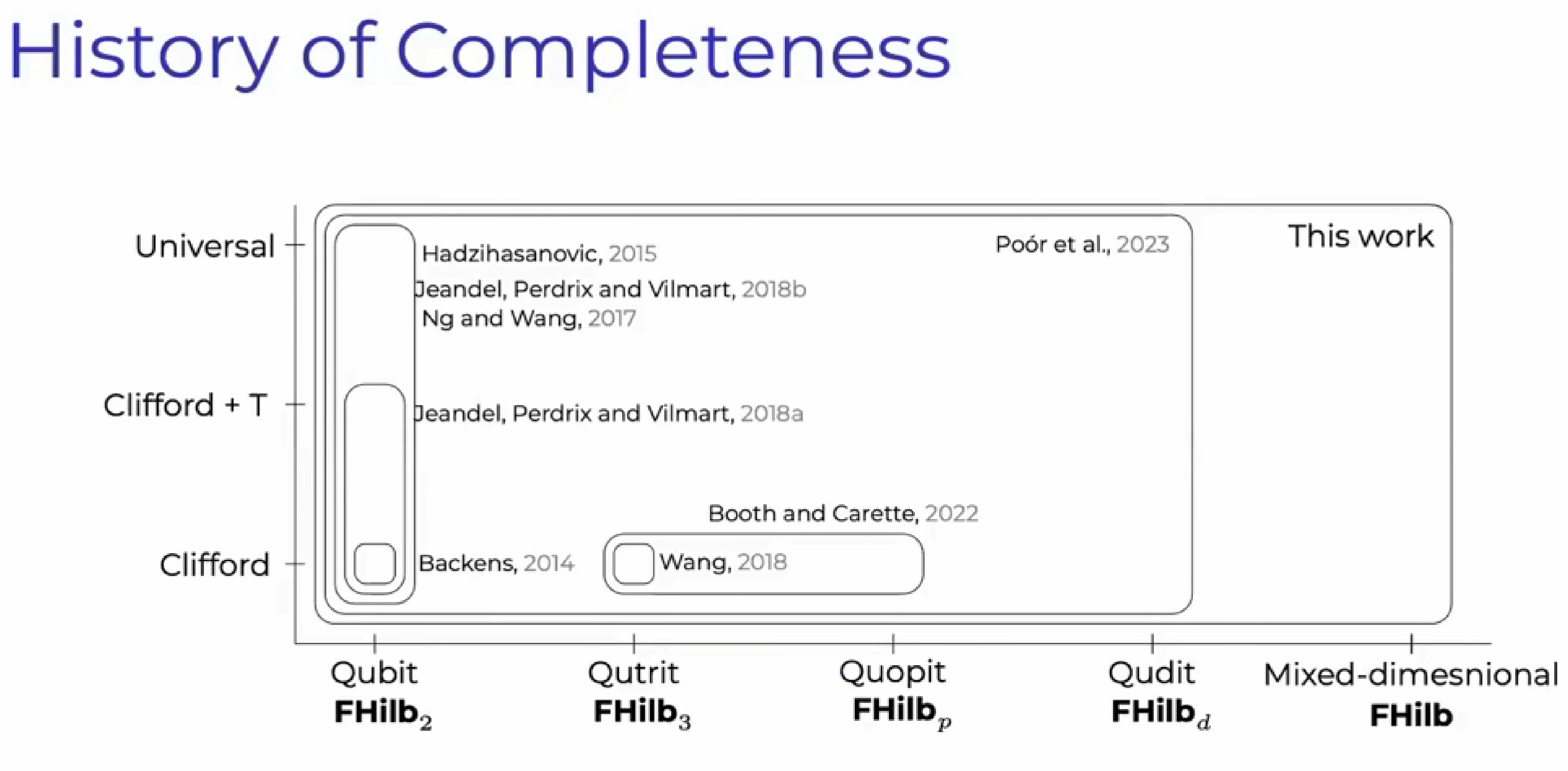
  The interpretation of any equality of diagrams is a valid equality of linear maps in **FHilb** (category consisting of all finite-dimensional Hilbert spaces and linear maps between them).

  Categorically, this interpretation from the graphical calculus category to its semantic category is a symmetric monoidal functor.

ZX-Calculus proven to have soundness and universality property in [4]

[3], [4]

Assessing graphical calculi



“Completeness of qufinite ZXW calculus, a graphical language for finite-dimensional quantum theory”

→ Different Fragments

[5]

Clifford-Fragment

- ZX-Diagrams with phases integer multiples of $\frac{\pi}{2}$ are Clifford
- ZX-Diagrams with phases integer multiples of $\frac{\pi}{4}$ are Clifford+T

→ Gottesman-Knill Theorem:

“Any quantum computer performing only: a) Clifford group gates, b) measurements of Pauli group operators, and c) Clifford group operations conditioned on classical bits, which may be the results of earlier measurements, can be perfectly simulated in polynomial time on a probabilistic classical computer.”

~ The Heisenberg Representation of Quantum Computers, 1998

[1]

Interlude

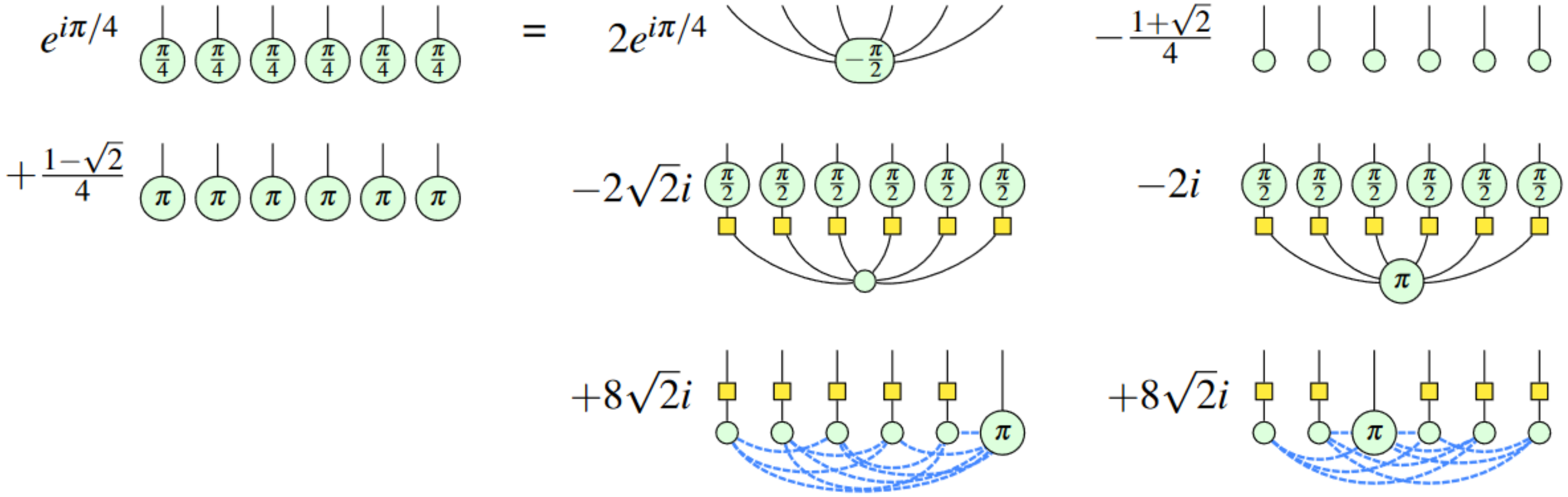
Applications

- Measurement based quantum computing [11]
- Quantum circuit optimization/simulation [12]
- Error-correction [13]
- Quantum natural processing [14]
- Quantum machine learning / Barren plateaus [15]
- Photonic quantum computing [16]
- SU(2) representation theory / LQG / Spin-Networks [8]
- Condensed matter physics [7] [17]
- Security proofs for quantum protocols (e.g. QKD) [20]

3. Application 1: Clifford-Diagram Decompositions

T-Spiders

- T-Spiders are Non-Clifford, but can be decomposed into $2^{\alpha t}$ Clifford summands
- Expressing decomposition presented by Bravyi, Smith, and Smolin as ZX-Diagrams yields “BSS-Decomposition”:

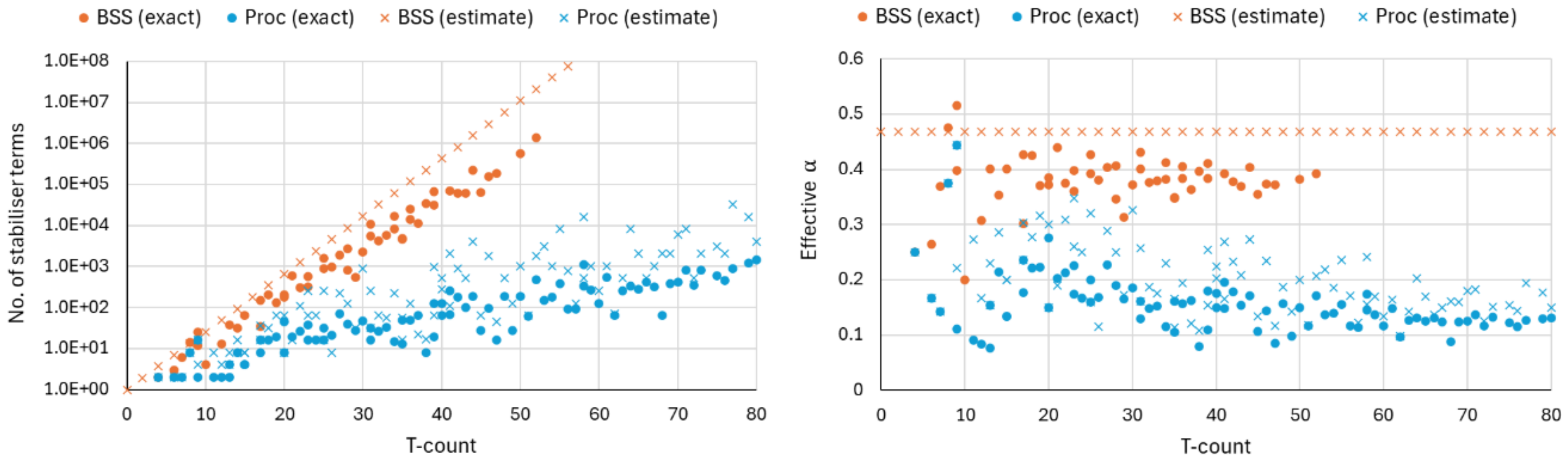


$$\alpha = \frac{\log_2 7}{6} \approx 0.468$$

[19]

T-Spiders

- *"Procedurally Optimised ZX-Diagram Cutting for Efficient T-Decomposition in Classical Simulation"* introduces "ZX-Calculus specific tricks" that allow for even better T-Spider decompositions:

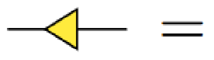


(a) Number of stabiliser terms versus T-count

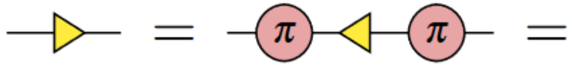
(b) Effective efficiency α versus T-count

Triangles

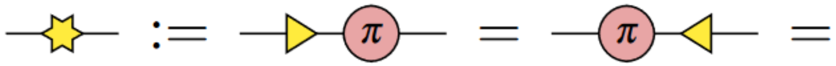
- Triangles are also Non-Clifford



$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$



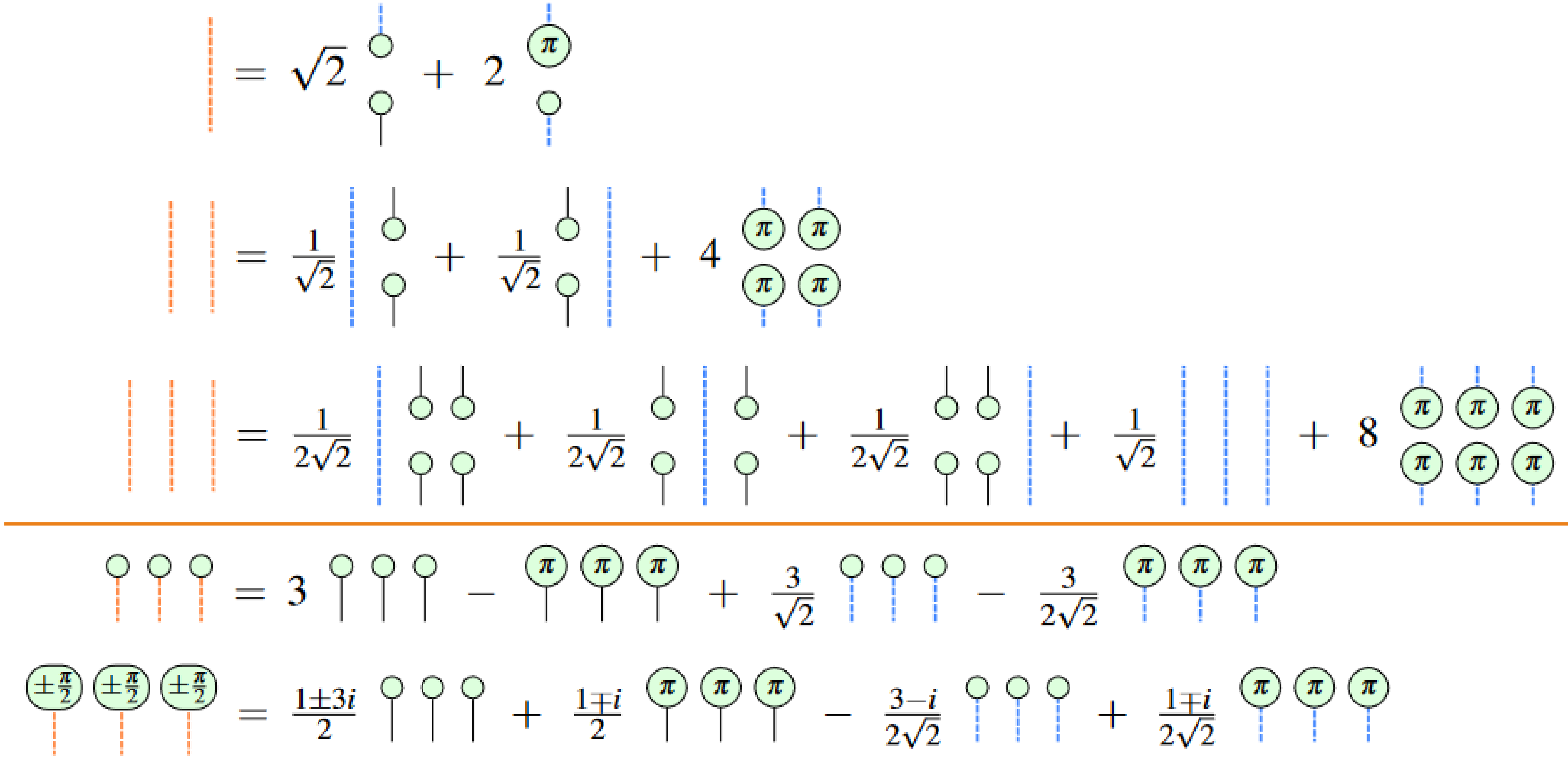
$$= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$



$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Triangles

- Decompositions into $2^{\beta t}$ Clifford summands:



$\beta = 1$

$\beta = \frac{\log_2 3}{2} \approx 0.792$

$\beta = \frac{\log_2 5}{3} \approx 0.774$

$\beta = \frac{\log_2 4}{3} \approx 0.667$

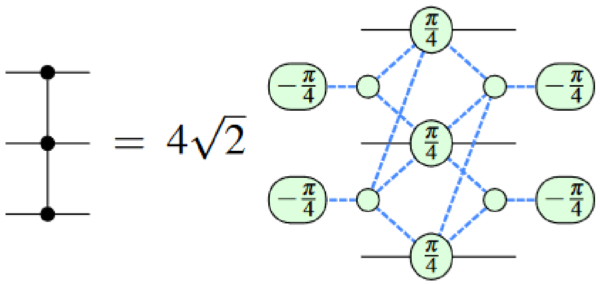
$\beta = \frac{\log_2 4}{3} \approx 0.667$

[9]

Triangles

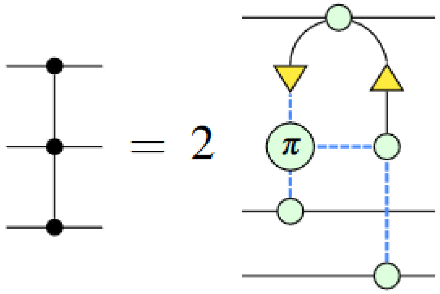
- CCZ:

7 T-spiders

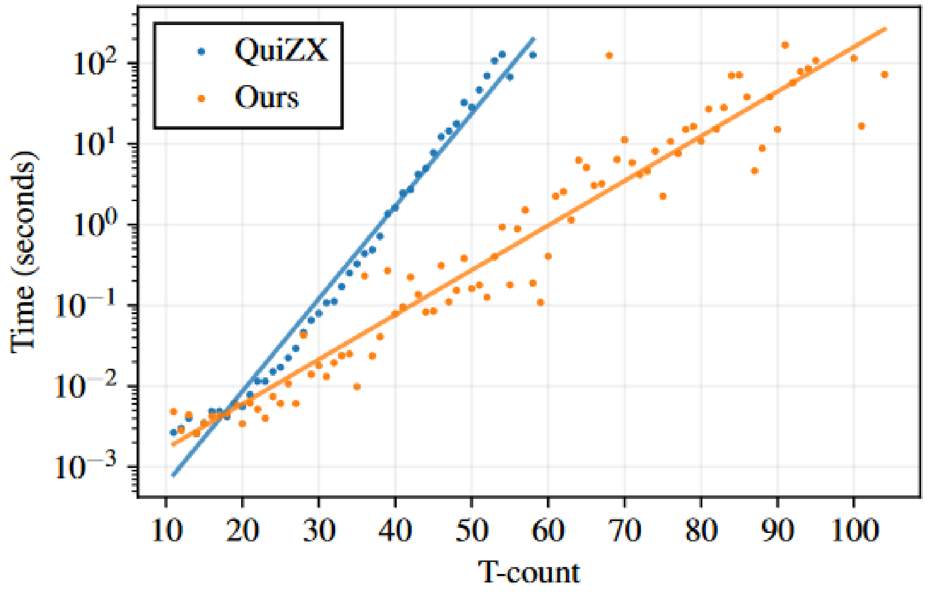


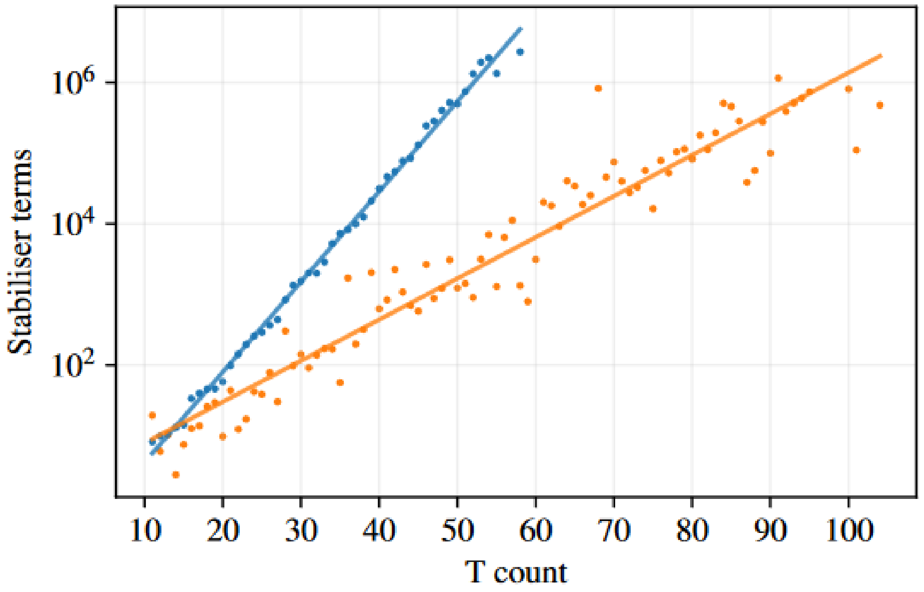
vs.

2 Triangles



Triangles





Runtime and total number of stabilizer terms for random 50-qubit Clifford+T+CCZ simulations

Comparison against:

QuiZX: a quick Rust port of PyZX

PyZX is a Python library for quantum circuit optimisation and compiling using the ZX-calculus. It's great for hacking, learning, and trying things out in Jupyter notebooks. However, it's written to maximise clarity and fun, not performance.

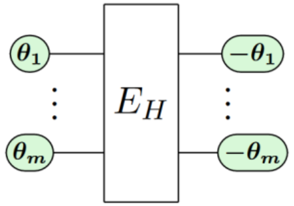
This is a port of some of the core functionality of PyZX to the Rust programming language. This is a modern systems programming language, which enables writing software that is very fast and memory efficient.

[9]

Triangles

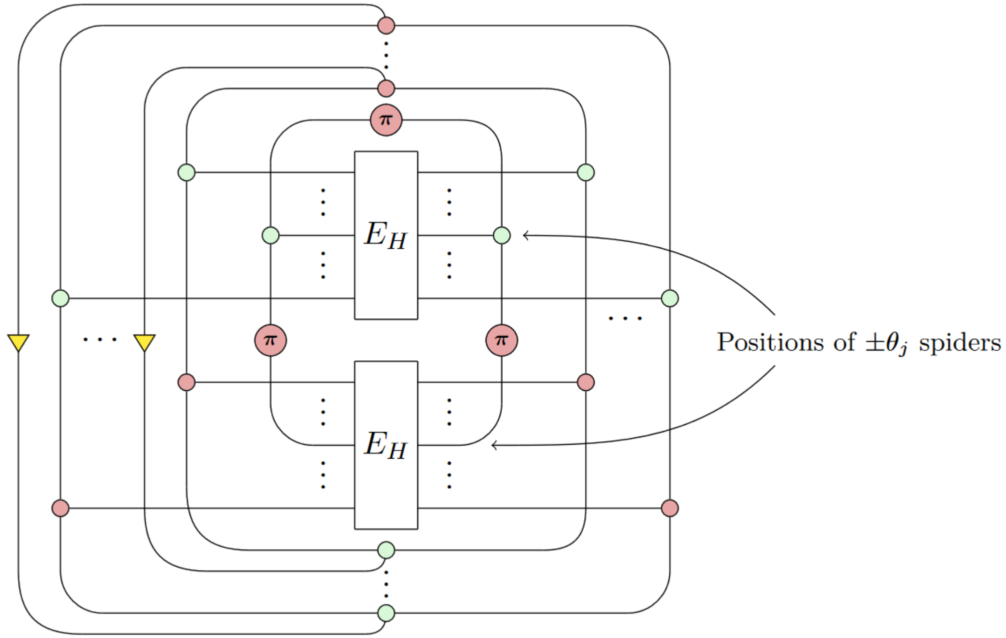
- Application: Diagrammatic variance calculation
  - Given an ansatz $U(\boldsymbol{\theta})$ and a Hamiltonian H the expectation value is given by:

$$\langle H \rangle = \langle 0 | U^{\dagger}(\boldsymbol{\theta}) H U(\boldsymbol{\theta}) | 0 \rangle =$$



  - The gradient variance is then given by:

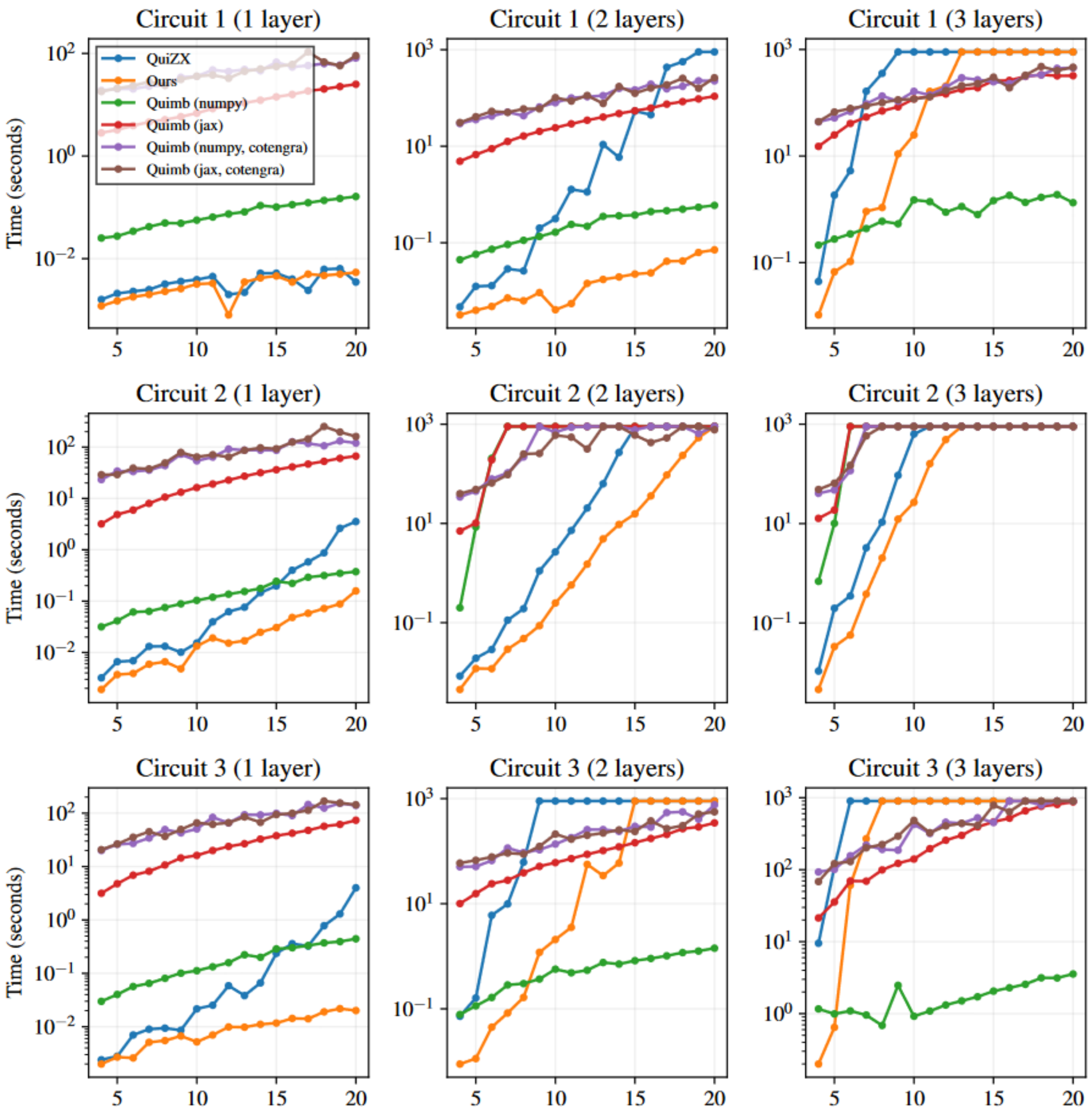
$$\mathbf{Var}\left(\frac{\partial \langle H \rangle}{\partial \theta_j}\right) =$$



→ If $\mathbf{Var}\left(\frac{\partial\langle H\rangle}{\partial \theta_j}\right) \approx 0$, barren plateau is likely going to start when using this ansatz

[9], [18]

Triangles



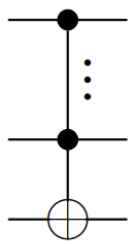
Circuit 1 (1 layer)
Circuit 1 (2 layers)
Circuit 1 (3 layers)
Circuit 2 (1 layer)
Circuit 2 (2 layers)
Circuit 2 (3 layers)
Circuit 3 (1 layer)
Circuit 3 (2 layers)
Circuit 3 (3 layers)
Time (seconds)
QuiZX
Ours
Quimb (numpy)
Quimb (jax)
Quimb (numpy, cotengra)
Quimb (jax, cotengra)

[9]

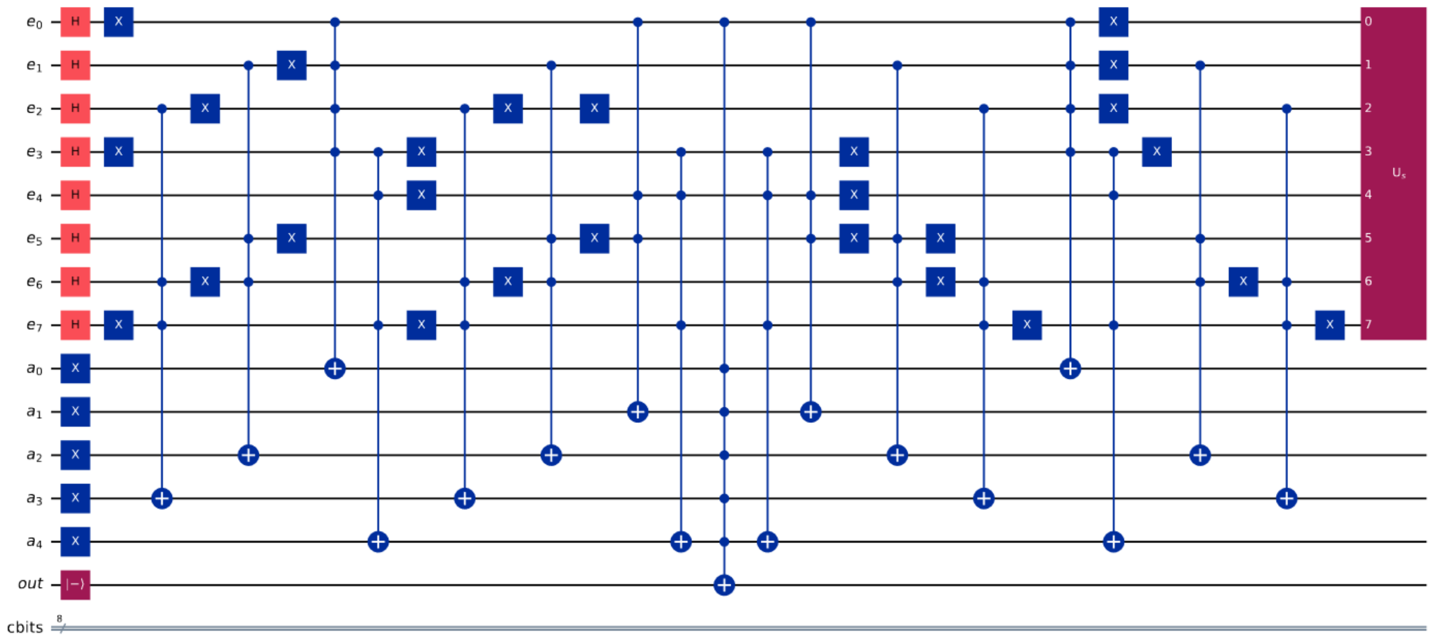
4. Application 2: Feynman Loop Configurations (Maturaarbeit)

Backstory

- Analyzing patterns of multi-controlled Toffoli gates using ZX-Calculus



- Search for applications of observed patterns:



[6]

Quantum querying based on multicontrolled Toffoli gates for causal Feynman loop configurations and directed acyclic graphs

Selomit Ramírez-Uribe,[b] **Andrés E. Rentería-Olivo**[a] **and Germán Rodrigo**[a]

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[b]*Facultad de Ciencias Físico-Matemáticas, Universidad Autónoma de Sinaloa, Ciudad Universitaria, CP 80000 Culiacán, Mexico.*

[6]

Feynman Loop Configurations

Regarding the LTD framework, its more remarkable feature is the existence of a manifestly causal representation. The direct LTD representation of Eq. (2.1) is computed by applying the Cauchy's residue theorem through the evaluation of nested residues [63]. To obtain the causal dual representation we sum over all the nested residues, explicitly cancelling all the noncausal contributions. Furthermore, a more suitable dual causal representation is found by cleverly reinterpreting it in terms of entangled thresholds [65, 67]. The analytical reconstruction is achieved by matching all combinations of $n - L$ thresholds that are causally compatible to each other, leading to the LTD causal representation,

$$\mathcal{A}_D^{(L)} = \int_{\vec{\ell}_1 \dots \vec{\ell}_L} \frac{1}{x_n} \sum_{\sigma \in \Sigma} \frac{\mathcal{N}_{\sigma(i_1, \dots, i_{n-L})}}{\lambda_{\sigma(i_1)}^{h_{\sigma(i_1)}} \cdots \lambda_{\sigma(i_{n-L})}^{h_{\sigma(i_{n-L})}}} + (\lambda_p^+ \leftrightarrow \lambda_p^-) \, , \tag{2.3}$$

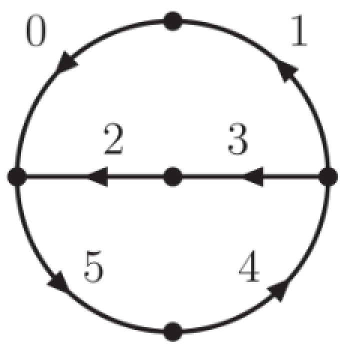
with $x_n = \prod_{i=1}^{n} 2q_{i,0}^{(+)}$ and $h_{\sigma(i)} = \pm$. The Feynman propagators from Eq. (2.1) are substituted in Eq. (2.3) by causal propagators $1/\lambda_p^{\pm}$, with

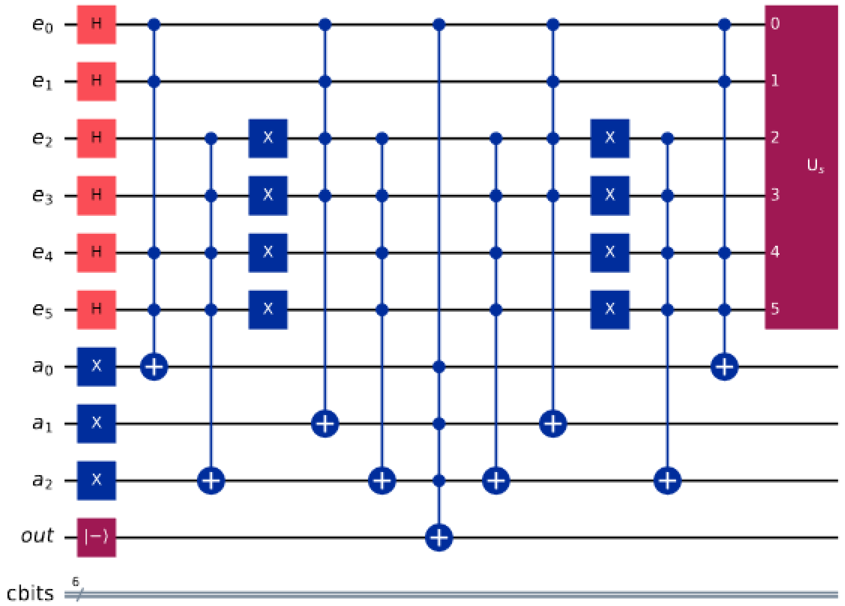
$$\lambda_p^{\pm} = \sum_{i \in p} q_{i,0}^{(+)} \pm k_{p,0} \, , \tag{2.4}$$

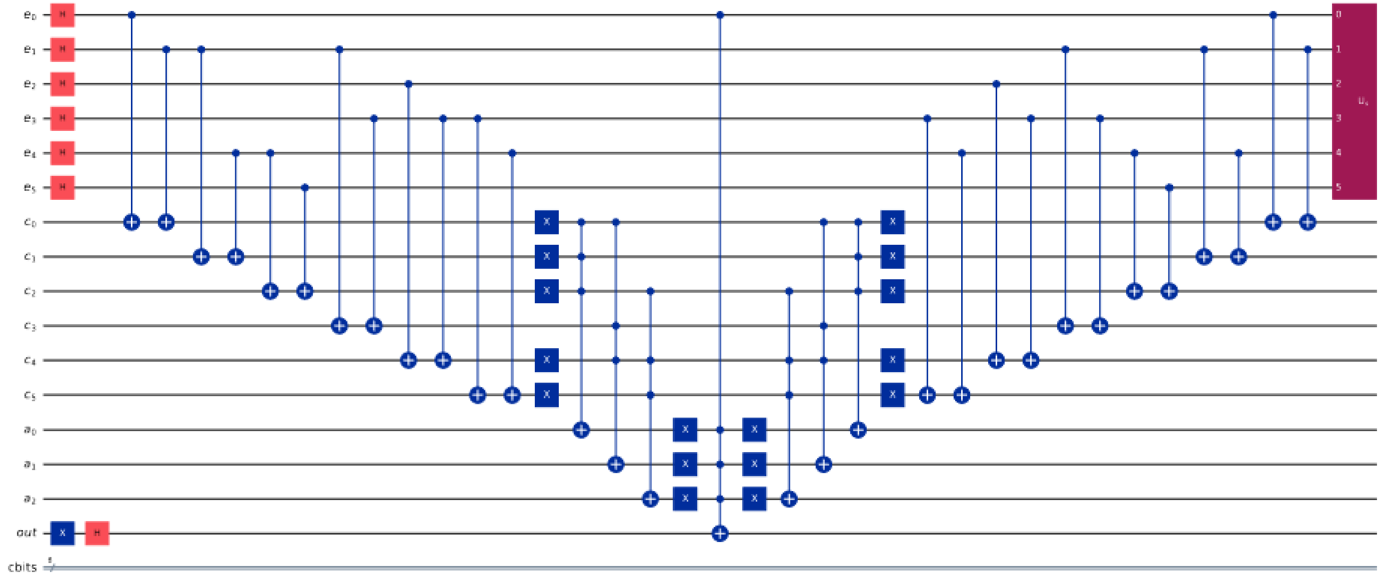
where p is a partition of the on-shell energies, and $k_{p,0}$ is a linear combination of the external momenta energy components. Given the sign of $k_{p,0}$, either λ_p^- or λ_p^+ becomes singular after all the propagators in p are set on shell. The combinations of entangled causal propagators represent causal thresholds that can occur simultaneously which are collected in the set Σ.

[6]

Example: Two eloops, six edges



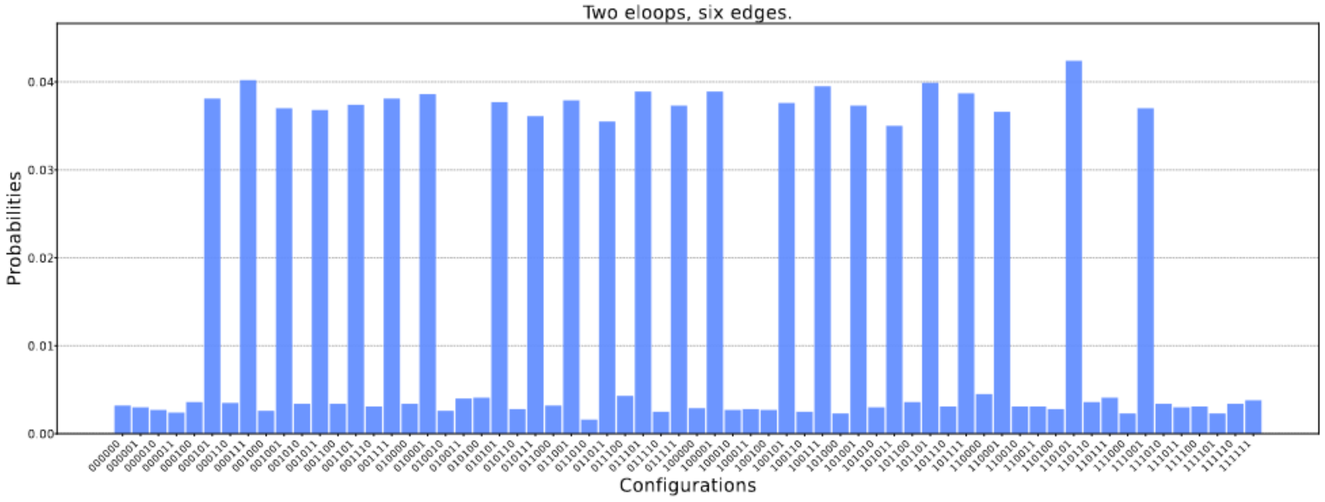




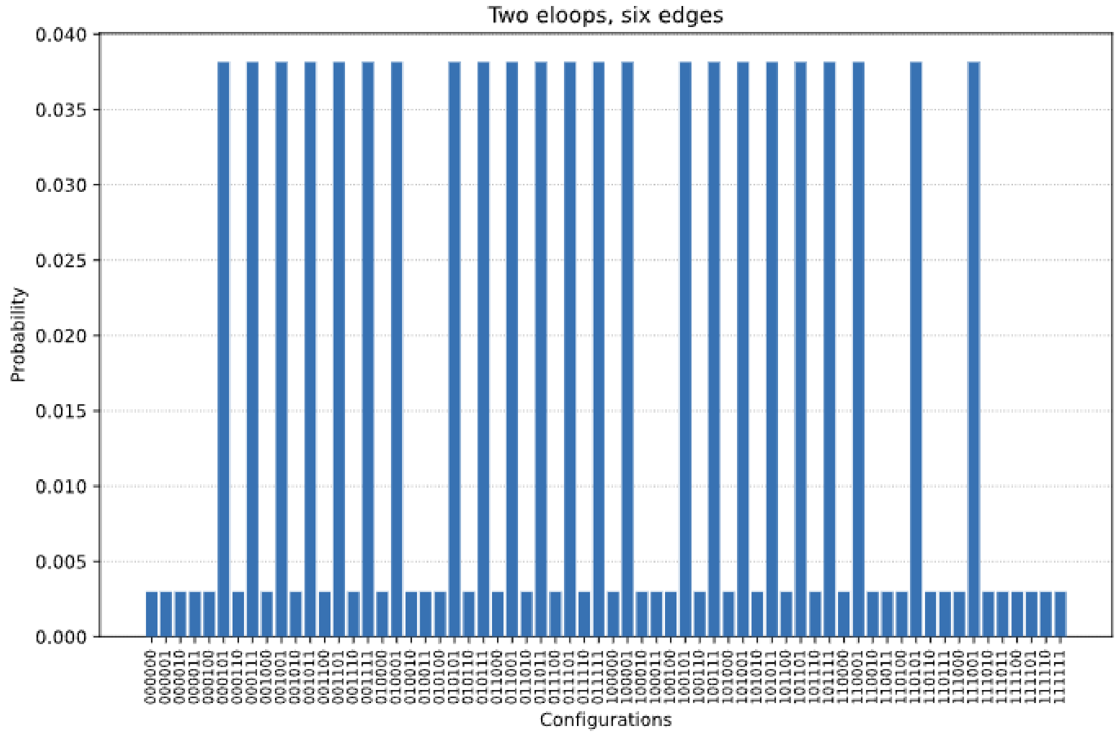
[6]

Example: Two eloops, six edges

Paper



My ZX-Replication



[6]

4. Bonus / Paper Recommendation:
AKLT-States & Spin-Networks for LQG

AKLT-States

AKLT-states as ZX-diagrams: diagrammatic reasoning for quantum states

Richard D. P. East,[1,2] John van de Wetering,[3] Nicholas Chancellor,[4] and Adolfo G. Grushin[5]

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[2] *Univ. Grenoble Alpes, CNRS, Grenoble INP, Institut Néel, 38000 Grenoble, France*
[3] *Radboud University Nijmegen, The Netherlands*
[4] *Durham University physics department and Durham-Newcastle Joint Quantum Centre, South Road, Durham UK*
[5] *Univ. Grenoble Alpes, CNRS, Grenoble INP, Institut Néel, 38000 Grenoble, France*

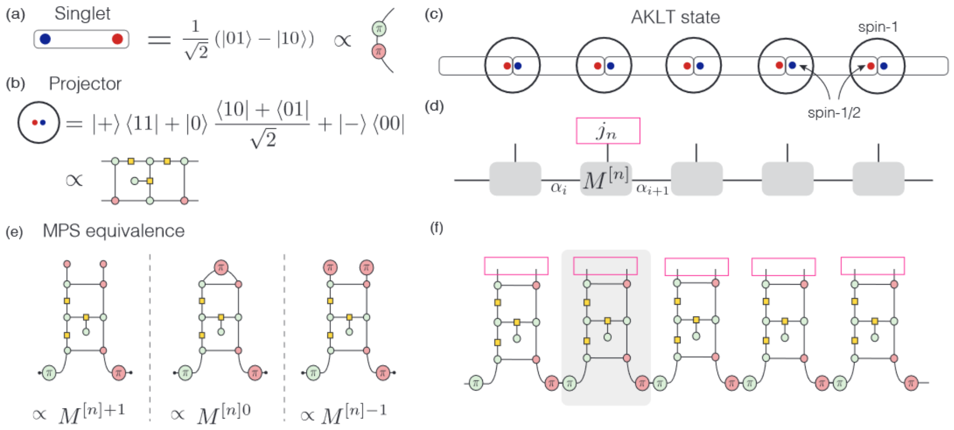
[7]

AKLT-States

- *Representation of spins higher than ½ within the ZXH-Calculus*
- *Expression of 1D AKLT state*
- *Recovering AKLT matrix-product state representation*
- *Recovering existence of topologically protected edge states*
- *Recovering non-vanishing of a string order parameter*

directly from representation

- *Analytically derive that the Berry phase of any finite-length 1D AKLT chain is π*
- *Alternative proof that the 2D AKLT state on a hexagonal lattice can be reduced to a graph state*
- *Illustrate a symmetry-breaking phase transition using diagrammatic 2D higher-order topological phases*



[7]

Spin-networks in the ZX-calculus

Richard D.P. East[1], Pierre Martin-Dussaud[2], and John Van de Wetering[3]

[1]LIG & Institut Néel, Grenoble, France

[2]Institute for Gravitation and the Cosmos, The Pennsylvania State University, University Park, Pennsylvania 16802, USA

[3]University of Oxford, United Kingdom

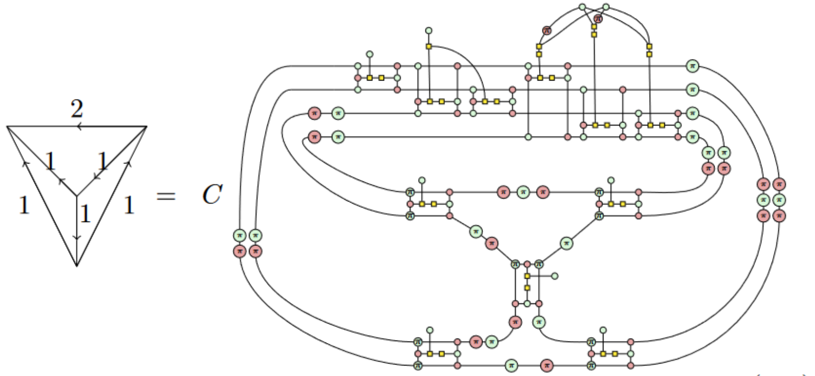
[8]

Spin-Networks

- SU(2) representation theory using ZXH-Calculus
- Representation on Yutsis-diagrams and Penrose binor calculus
- SU(2) invariance up to a phase of the Wigner symbols trivially provable diagrammatically
- Explicitly diagrammatically calculate *3jm*, *4jm* and *6j* symbols

- Follow up papers expected

$$\left\{\begin{matrix}2&1&1\\1&1&1\end{matrix}\right\} =$$



(150)

with

$$C = \frac{1}{48} * \left(\frac{1}{\sqrt{2}}\right)^5 * \left(\sqrt{\frac{2!2!2!}{4!1!1!1!}}\right)^2 * \left(\sqrt{\frac{4!2!2!}{5!2!2!0!}}\right)^2.$$

C captures the scalar corrections for the symmetrisers λ and by the four normalisations from the binor calculus N. The diagram evaluates to $480\sqrt{2}$ (as calculated in PyZX). We get the final answer

$$\left\{\begin{matrix}2&1&1\\1&1&1\end{matrix}\right\} = \frac{1}{6}.$$

Questions?

Thanks!

References

- [1] https://vdwetering.name/pdfs/presentation-zx-compilation.pdf
- [2] https://youtu.be/U9Lq0QUImQ8?si=pcMwk3j2jPMvew63
- [3] https://arxiv.org/abs/2309.13014
- [4] http://dx.doi.org/10.1088/1367-2630/13/4/043016
- [5] https://youtu.be/AhEEOqpzADY?si=kOn3z9p7JZRPa2iU
- [6] https://arxiv.org/abs/2404.03544
- [7] https://arxiv.org/abs/2012.01219
- [8] https://arxiv.org/abs/2111.03114
- [9] https://arxiv.org/abs/2307.01803
- [10] https://arxiv.org/abs/2403.10964
- [11] https://quantum-journal.org/papers/q-2019-04-26-134/
- [12] https://arxiv.org/abs/2109.01076
- [13] https://arxiv.org/abs/1812.01238
- [14] https://arxiv.org/abs/2012.03755
- [15] https://arxiv.org/abs/2102.01828
- [16] https://arxiv.org/pdf/2204.12985
- [17] https://arxiv.org/pdf/2404.02975
- [18] https://arxiv.org/abs/2201.13250
- [19] https://arxiv.org/abs/1506.01396
- [20] https://www.sciencedirect.com/science/article/pii/S1571066111000351