Week 5 - SciPy and (Mathematical) Optimization

Optimization in Math/Economics

Many critical problems in analysis are framed as optimization problems.

- Classification Models
- Regression Models
- Utility Maximization
- General Equilibrium Problems
- Dynamic Programming

Optimization

Stating a problem as a case of optimization provides an intuitive way to solve a problem, and (typically) clear indications of when a solution has been reached

- What mathematical conditions allow us to establish the existence of a minimum value?

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- What mathematical conditions allow us to establish the existence of a minimum value?
  - First derivative(s) equal to 0
  - Second derivatives positive

Optimization in Economics - Utility Maximization

Consider an individual choosing consumption of two goods in order to maximize utility:

$$U(x, y) = x^{\alpha} y^{\beta}$$

The individual also has a budget constraint:

$$I = p_x \cdot x + p_y \cdot y$$

Utility Maximization

The two equations represent a **constrained optimization** problem.

- Want to consume ∞, but can't afford to.
- How happy can we be, given a limited amount of money?

Stated mathematically:

$$max_{x,y} \;\; U(x, y) \;\; \text{subject to}$$

$$I = p_x \cdot x + p_y \cdot y$$

Utility Maximization

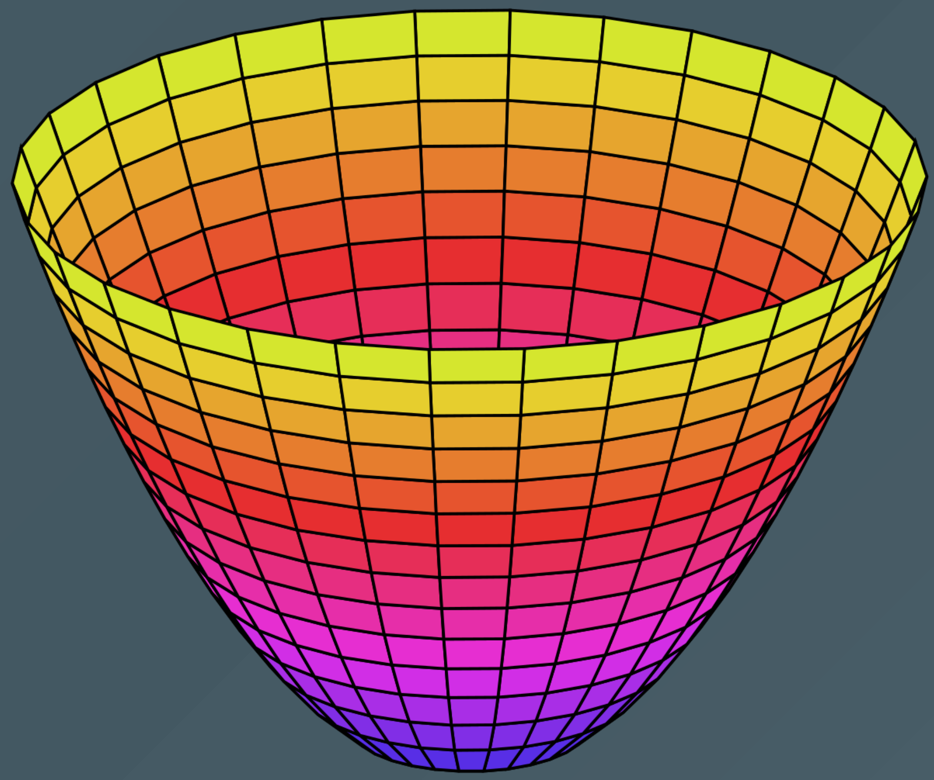
We can rewrite these two equations as a single problem in order to find the maximum:

$$\mathcal{L} = x^{\alpha} y^{\beta} + \lambda(I - p_x \cdot x - p_y \cdot y)$$

This is the **Lagrangian** expression of our optimization problem.

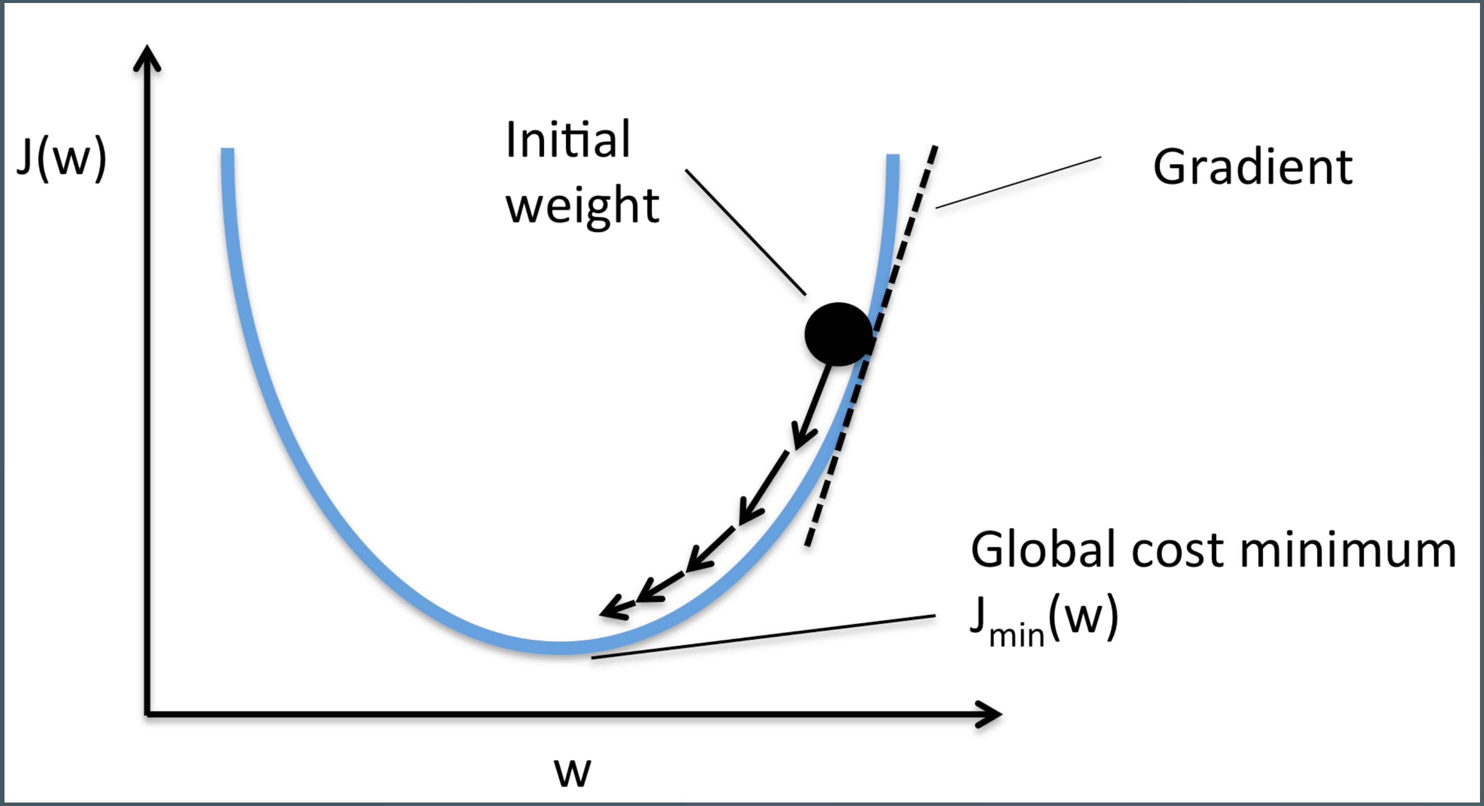
Let's walk through the solution.

Geometrically



In an optimization problem, we need to find a way to get ourselves to the minimum, and to know when we get there.

Geometrically



Typically, we use some form of **gradient descent** to find our way to the minimum value of a function.

What is a gradient?

Consider a function, f, with two variables x and y. This function has two partial derivatives:

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

Each partial derivative tells us how f changes as we move in a particular dimension **all else constant**.

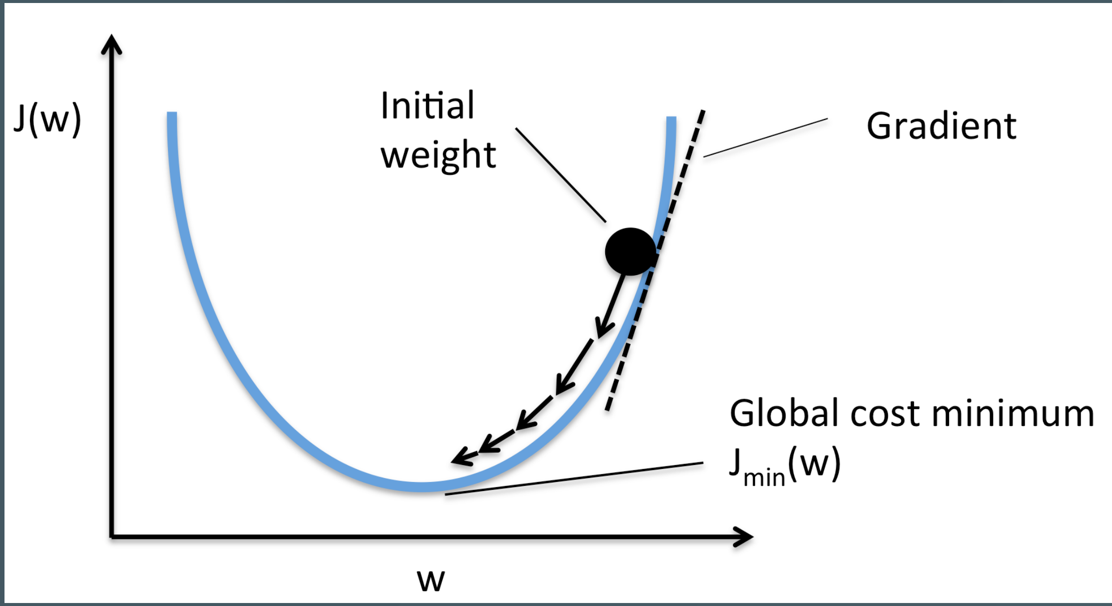
What is a gradient?

The **gradient**, then, is the vector of all partial derivatives of a given function at any point along the function:

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

We can use the gradient to determine the linear approximation of a function at any given point.

Gradient Descent



Gradient Descent is a technique in which we algorithmically choose to move toward what we believe to be the minimum value of our function based on the current slope of our function.

Gradient Descent

Steps to gradient descent algorithm:

- Evaluate the gradient of the function
- Find the direction of steepest descent
- Determine how far to move in that direction
- Move to new point
- Reevaluate the gradient
- Stop moving when gradient is within tolerance from 0

Exercise

Estimate the gradient of the following function (using Python!):

$$y = \sin(x) - (z + 3)^2$$

where

1. $x = 1$ and $z = 10$
2. $x = -3$ and $z = 2$

Hint: try changing x and z by small amounts, and evaluating the difference

Exercise Answer

```
import numpy as np

# Define function y in terms of x and z
def y(x, z):
  return np.sin(x) - (z+3)**2

# Approximate gradient in x and z at point 1
(y(1,10) - y(1.1,10))/.1 # x direction, value ~0.497
(y(1,10) - y(1,10.1))/.1 # z direction, value ~26.1

# Approximate gradient in x and z at point 2
(y(-3,2) - y(-2.9,2))/.1 # x direction, value ~0.981
(y(-3,2) - y(-3,2.1))/.1 # z direction, value ~10.1
```

Gradients and Optimization in Scipy

When we need to optimize a function, we can easily do so using Scipy's built-in `optimize` module.

```
import numpy as np
from scipy.optimize import minimize
```

There is NO maximize function.

Why?

Gradients and Optimization in Scipy

When we need to optimize a function, we can easily do so using Scipy's built-in `optimize` module.

```
import numpy as np
from scipy.optimize import minimize
```

There is NO maximize function.

Why? Because any maximization problem can be restated as a minimization problem!

Optimizing our exercise function

```
def y(x, z):
  return np.sin(x) - (z+3)**2
```

There are two problems with our function that we need to resolve:

1. We need to be able to pass an array of values to our function when using `minimize`, and here we pass two numbers
2. Our problem is concave, and so has a maximum
    - We need to restate it as a minimization problem

Optimizing our exercise function

```
def y(x, z):
  return np.sin(x) - (z+3)**2

def q(x):
  return -1*y(x[0], x[1])
```

We can wrap our function `y` in a new function `q` taking only an array of numbers, and have it return the negative of our original equation.

Now we are ready to minimize!

Optimizing our exercise function

```
>>> res = minimize(q, [0,0]) # provide function and
>>> res                      # starting guess to minimize
      fun: -0.9999999999999999
 hess_inv: array([[  1.00056201e+00,   4.96640470e-04],
       [  4.96640470e-04,   5.00458796e-01]])
      jac: array([  0.00000000e+00,  -7.45058060e-09])
  message: 'Optimization terminated successfully.'
     nfev: 28
      nit: 6
     njev: 7
   status: 0
  success: True
        x: array([ 1.57079632, -3.00000001])
```

What happens if you try a different starting point?

For Lab Today

Using a functional representation of the Lagrangian and the `minimize` function from `scipy.optimize`, solve the following specific utility maximization problem:

$$U(x, y) = x^{0.3}y^{0.7}$$

Income is \$500, and the price of x is \$3, while the price of y is \$20.

See the Lagrangian slide for help setting up the problem.