# Week 9 - SciPy and (Mathematical) Optimization

# Optimization in Math/Economics

Many critical problems in analysis are framed as optimization problems.

- Classification Models
- Regression Models
- Utility Maximization
- General Equilibrium Problems
- Dynamic Programming

# Optimization

Framing a problem as a case of optimization provides an intuitive way to solve it, and (typically) clear indications of when a solution has been reached

- What mathematical conditions allow us to establish the existence of a minimum value?

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- What mathematical conditions allow us to establish the existence of a minimum value?
  - First derivative(s) equal to 0
  - Second derivatives positive

# Optimization in Economics - Utility Maximization

Consider an individual choosing consumption of two goods in order to maximize utility:

$$U(x, y) = x^{\alpha} y^{\beta}$$

The individual also has a budget constraint:

$$I = p_x \cdot x + p_y \cdot y$$

# Utility Maximization

The two equations represent a **constrained optimization** problem.

- Want to consume **all the things**, but can't afford to.
- How happy can we be, given a limited amount of money?

Stated mathematically:

$$max_{x,y} \;\; U(x,y) \;\; \text{subject to}$$

$$I = p_x \cdot x + p_y \cdot y$$

# Utility Maximization

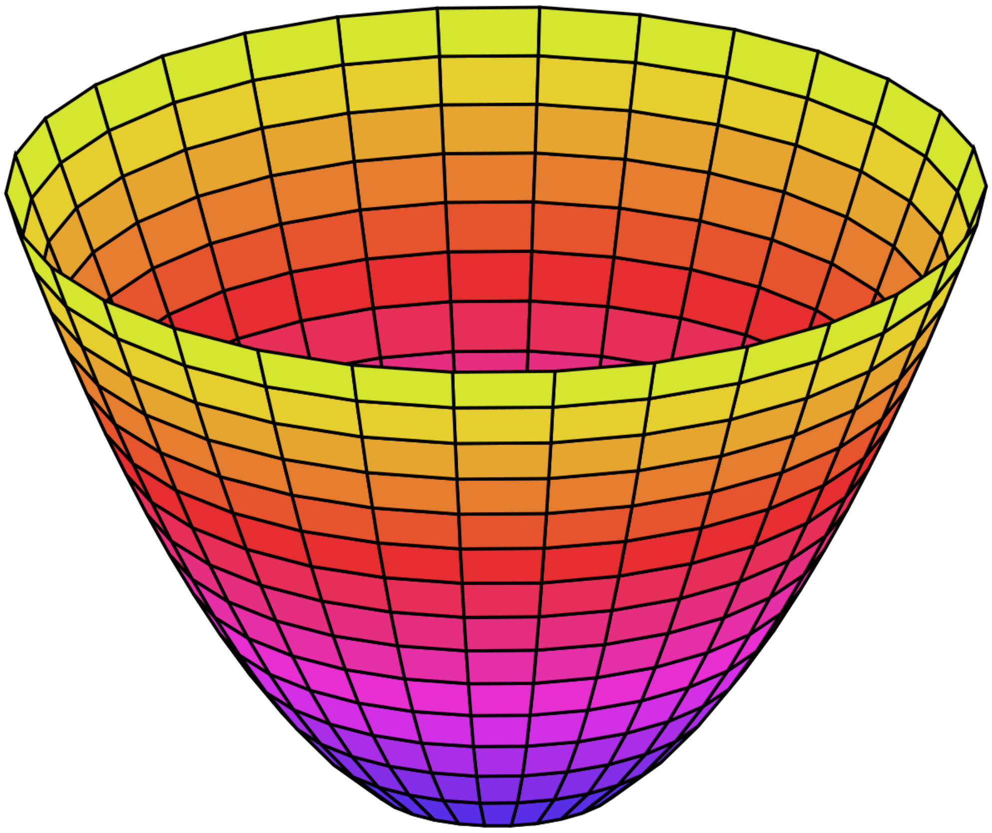
We can rewrite these two equations as a single problem in order to find the maximum:

$$\mathcal{L} = x^{\alpha} y^{\beta} + \lambda(I - p_x \cdot x - p_y \cdot y)$$

This is the **Lagrangian** expression of our optimization problem.

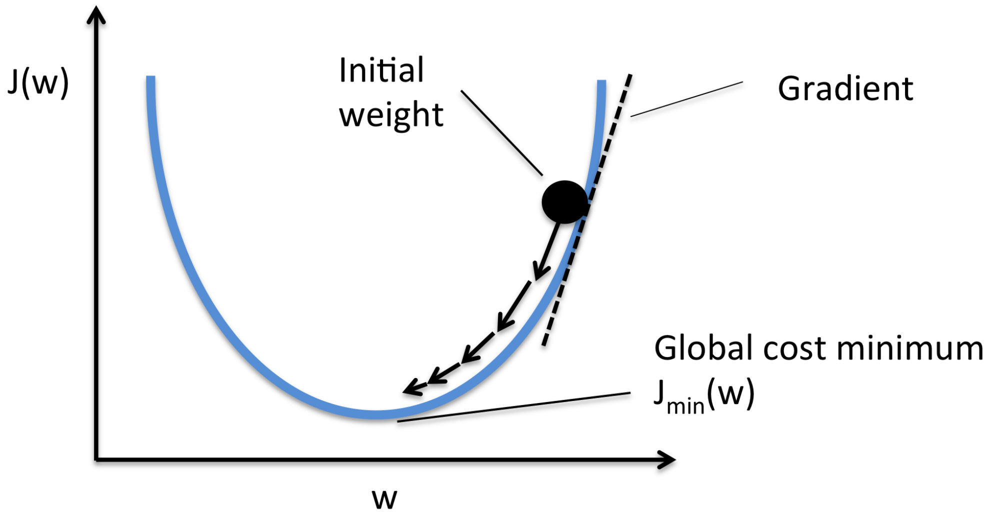
Let's walk through the solution.

# Geometrically



In a computational optimization problem, we need to find a way to get ourselves to the minimum, and to know when we get there.

# Geometrically




Typically, we use some form of **gradient descent** to find our way to the minimum value of a function.

# What is a gradient?

Consider a function, $f$, with two variables $x$ and $y$. This function has two partial derivatives:

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

Each partial derivative tells us how $f$ changes as we move in a particular dimension **all else constant**.

# What is a gradient?

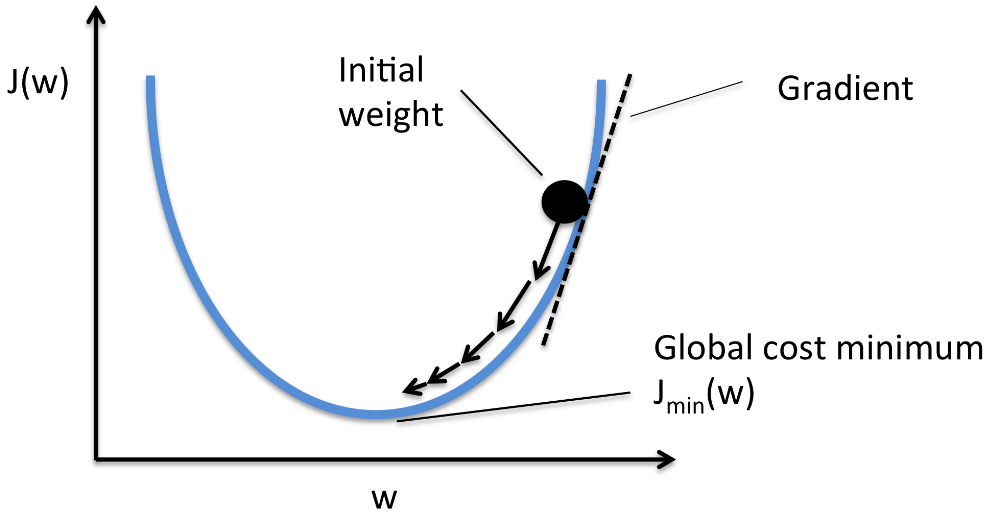
The **gradient**, then, is the vector of all partial derivatives of a given function at any point along the function:

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

We can use the gradient to determine the linear approximation of a function at any given point.

Think about it as the slope and direction of a hill you are hiking on.

# Gradient Descent




**Gradient Descent** is a technique in which we algorithmically choose to move toward what we believe to be the minimum value of our function based on the current slope of our function.

# Gradient Descent

**Steps of a gradient descent algorithm:**

- Evaluate the gradient of the function
- Find the direction of steepest descent
- Determine how far to move in that direction
- Move to new point
- Reevaluate the gradient
- Stop moving when gradient is within a margin of error from 0

# Exercise

Estimate the gradient of the following function (using Python!):

$$y = \sin(x) - (z+3)^2$$

where

1. $x = 1$ and $z = 10$
2. $x = -3$ and $z = 2$

Hint: try changing $x$ and $z$ by small amounts, and evaluating the difference

# Exercise Answer

```
import numpy as np

# Define function y in terms of x and z
def y(x, z):
  return np.sin(x) - (z+3)**2

# Approximate gradient in x and z at point 1
(y(1,10) - y(1.1,10))/.1 # x direction, value ~0.497
(y(1,10) - y(1,10.1))/.1 # z direction, value ~26.1

# Approximate gradient in x and z at point 2
(y(-3,2) - y(-2.9,2))/.1 # x direction, value ~0.981
(y(-3,2) - y(-3,2.1))/.1 # z direction, value ~10.1
```

# Gradients and Optimization in Scipy

When we need to optimize a function, we can easily do so using Scipy's built-in `optimize` module.

```
import numpy as np
from scipy.optimize import minimize
```

There is NO maximize function.

Why?

# Gradients and Optimization in Scipy

When we need to optimize a function, we can easily do so using Scipy's built-in `optimize` module.

```
import numpy as np
from scipy.optimize import minimize
```

There is NO maximize function.

Why? Because any maximization problem can be restated as a minimization problem!

# Optimizing our exercise function

```
def y(x, z):
  return np.sin(x) - (z+3)**2
```

There are two problems with our function that we need to resolve:

1. We need to be able to pass an array of values to our function when using `minimize`, and here we pass two numbers
2. Our problem is concave, and so has a maximum
    - We need to restate it as a minimization problem

# Optimizing our exercise function

```
def y(x, z):
  return np.sin(x) - (z+3)**2

def q(x):
  return -1*y(x[0], x[1])
```

We can wrap our function `y` in a new function `q` taking only an array of numbers, and have it return the negative of our original equation.

Now we are ready to minimize!

# Optimizing our exercise function

```
>>> res = minimize(q, [0,0]) # provide function and
>>> res                      # starting guess to minimize
      fun: -0.9999999999999999
 hess_inv: array([[  1.00056201e+00,   4.96640470e-04],
       [  4.96640470e-04,   5.00458796e-01]])
      jac: array([  0.00000000e+00,  -7.45058060e-09])
  message: 'Optimization terminated successfully.'
     nfev: 28
      nit: 6
     njev: 7
   status: 0
  success: True
        x: array([ 1.57079632, -3.00000001])
```

What happens if you try a different starting point?

## For Lab/Homework

You will add a Logistic Regression method to your `RegressionModel` class. You will need to do the following in order to implement Logistic Regression:

- Create a function to calculate the log-likelihood function
- Use gradient descent
- Calculate the model standard error (for use below)
- Calculate coefficient standard errors, z-statistics, and p-values