J.P. Morgan Quant Mentorship Program 2022

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Case Study A: Derivatives

Continuous Compounding

1.For Compound Interest, with principal amount P and rate r % p.a. when compounded annually,

Amount after 1 year, $A=P + P \times r/100=P(1+r/100)$

Amount after 2 years, $A=P(1+r/100) + P(1+r/100) \times r/100= P(1+r/100)^2$ (∵ principal amount for the 2nd year= final amount of the 1st year)

∴Amount after t years= $P(1+r/100)^t$

When compounded semi-annually: new rate= rate/2, new time = time $\times$ 2,

∴Amount after t years when compounded semi-annually= $P (1+r/ (2\times100))^{(2\times t)}$ ------------(1)

When compounded n times in 1 year: new rate= rate/n, new time= $n \times t$,

∴Amount after t years when compounded n times in 1 year = $P (1+r/ (n\times100))^{(n \times t)}$ ---------(2)

a. Given,

Principal amount=$10,000

Rate= 5% p.a. compounded semi-annually

Time= 10 years

From eqn. (1),

∴Amount after 10 years= $P (1+r/ (2\times100))^{(2\times t)} = \$10,000 (1+5/ (2\times100))^{(2\times 10)}$

$=\$10,000(1.025)^{20}=\$10,000\times1.63861644=\$16386.1644$

b. If $10,000 is compounded weekly for 10 years,

From eqn. (2),

∴Amount when compounded 52 times in 1 year = $\$10,000 (1+5/ (52\times100))^{(52\times 10)}$ (∵ 1 year has 52 weeks)

$=\$10,000(1.000962)^{520}=\$10,000\times1.64832525$

$=\$16483.2525$

c. If $10,000 is compounded daily for 10 years,

From eqn. (2),

∴Amount when compounded 52 times in 1 year = $\$10,000 (1+5/ (365\times100))^{(365\times 10)}$ (∵ 1 year has 365 days as 2022 is not a leap year)

$=\$10,000(1.000137)^{3650}=\$10,000\times1.64866481$

=\$16486.6481

d. As we noticed, the final amount is increasing with decreasing compounding time period.

If compounding is done continuously without any discrete time interval, then if n is the number of times compounding is done in 1 year, we can say that n tends to infinity.

From eqn. (2), we get:

∴Amount after t years when compounded n times in 1 year = P (1+r/ (n$\times$100)) ^ (n $\times$ t)

Amount = $\lim_{n\to\infty} P\left(1+\frac{r}{n\times100}\right)^{nT}$

We can see that the limit is of the form 1^{∞}

let $f(n) = \left(1+\frac{r}{n\times100}\right)$ & $g(n) = nT$

$\therefore \lim_{n\to\infty} f(x) = 1$ & $\lim_{n\to\infty} g(x) = \infty$.

The answer of the limit of this form is:

$e^{[f(x)-1]\,g(x)}$

Amount = $P \lim_{n\to\infty} \left(1+\frac{r}{n\times100}\right)^{nT}$

$= Pe^{\left(\frac{r}{n\times100}\right)nT} = Pe^{rT/100}$

∴From the above derivation, amount=P $\times$ e^ (r $\times$T/100) =\$10,000$\times$e^0.5=16487.2127

∴Again, with continuous compounding the amount further increases.

2.**Call/Put Option Pricing**

2.a) If I am looking for a call option at a strike of \$K at expiry in T years for a stock which is currently trading at \$S, I will calculate the intrinsic value which the difference of strike price and stock price at present. Taking into consideration that the stock price at present should be more as I will like to buy the stock at a price lower than the market price now after T years. So, the premium I would like to give at the expiry date for buying the call option with be less than \$(Stock price after T years-K) so that I suffer no loss. For example, if I buy a call option with a strike price of \$20 and the stock price after T years is \$30, premium paid by me at expiry date should be less than \$(30-20) =\$10 for buying the call option.

The principal amount which compounds to \$K in T years need to be calculated:

Final amount of continuous compounding =\$K (strike price)

Continuously compounded rate of interest=r%, Time to expiry =T years

Principal amount to invest can be calculated from the formula derived in (1.d)

Amount=Principal $\times$ e^ (r $\times$ T/100)

Principal=\$ K/e^ (r $\times$T/100)

Hence, the maximum amount we will be willing to pay today for the call option is:

\$(stock price today- amount at present which will be compounded to strike price after T years)

=\$(S- K $\times$ e^ (-r $\times$T/100)).

2.b) While looking for a put option of strike price \$K when the present stock price in the market is \$S, I will want to sell the stock at a strike price which is greater than the market price, so at the expiry date which is T years from now the difference in present market value and strike price will be \$(K-Stock price after T years). Also, while I am selling the put option, the buyer is buying a call option at a premium from me, let the price be \$C. Hence, at the expiry date, I can buy the put option at \$(Stock price after T years -K+C) without suffering any loss.

The principal amount which compounds to \$K in T years need to be calculated:

Final amount of continuous compounding after T years=\$K (strike price)

Continuously compounded rate of interest=r%, Time to expiry =T years

Principal amount to invest can be calculated from the formula derived in (1.d)

Amount=Principal $\times$ e^ (r $\times$ T/100)

Principal=\$ K/e^ (r $\times$T/100)

Hence, the maximum amount we will be willing to pay today for the put option is:

(Amount at present which will be compounded to strike price after T years- stock price today + premium received after selling the put option)

=\$(K $\times$ e^ (-r $\times$T/100) - S +C).

3. **BSM pricing**

$$
\begin{aligned}
C(S_t, t) &= N(d_1)S_t - N(d_2)Ke^{-r(T-t)} \\
d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right] \\
d_2 &= d_1 - \sigma\sqrt{T-t}
\end{aligned}
$$

$$
\begin{aligned}
P(S_t, t) &= Ke^{-r(T-t)} - S_t + C(S_t, t) \\
&= N(-d_2)Ke^{-r(T-t)} - N(-d_1)S_t
\end{aligned}
$$

3.a)

For call option:

BSM pricing for call option.

$C(S_t, t) = N(d_1) S_t - N(d_2) K e^{-r(T-t)}$

for $t \to T$, $d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln(S_t/K) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right]$

$\therefore \lim_{t \to T} d_1 \Rightarrow \infty$. (as denominator approaches 0).

$d_2 = d_1 - \sigma\sqrt{T-t}$

$\therefore \lim_{t \to T} d_2 = d_1$ $\quad \therefore d_2 \Rightarrow \infty$.

CDF of standard normal distribution:

$$N(d_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1} e^{-\frac{d_1^2}{2}}\, d(d_1)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{d_1^2}{2}}\, d(d_1) \quad [\because d_1 \to \infty]$$

$$= \int_{-\infty}^{\infty} \frac{\sqrt{2}\, e^{-\frac{d_1^2}{2}}}{2\sqrt{\pi}}\, d(d_1)$$

$$= \frac{\sqrt{2}}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{d_1^2}{2}}\, d(d_1)$$

let $t = \frac{\sqrt{2}\, d_1}{2}$, $dt = \frac{\sqrt{2}}{2}\, d(d_1)$,

$$= \frac{\sqrt{2}}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \sqrt{2}\, dt$$

$$= \int_{-\infty}^{\infty} \frac{e^{-t^2}}{\sqrt{\pi}}\, dt \quad --- \ (1)$$

Gaussian error function $\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-z^2}\, dz$

∴ from (1),

$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{2}{\sqrt{\pi}} e^{-t^2}\, dt = \frac{1}{2}\operatorname{erf}(t)$$

$$= \frac{\operatorname{erf}\left(\frac{\sqrt{2}\, d_1}{2}\right)}{2}$$

$$\therefore \int_{-\infty}^{\infty} \frac{\sqrt{2}\, e^{-\frac{d_1^2}{2}}}{2\sqrt{\pi}}\, d(d_1) = \lim_{d_1 \to \infty} \frac{\operatorname{erf}\left(\frac{\sqrt{2}\, d_1}{2}\right)}{2} - \lim_{d_1 \to -\infty} \frac{\operatorname{erf}\left(\frac{\sqrt{2}\, d_1}{2}\right)}{2} = 1.$$

Since the limits are from –inf to inf of the integral, any kind of probability density function will give the result 1. So, the value of N(d1) =1.

For t ⟶T, d2=d1, So, N(d1) =N(d2) =1.

$C(S_t, t) = N(d_1) S_t - N(d_2) K e^{(-r(T-t))}$

For t ⟶T, $C(S_t, t) = S_t - K e^{(0)} = S_t - K$

For put option:

$$N(d_1) = \int_{-\infty}^{d_1} e^{-\frac{d_1^2}{2}} d(d_1)$$

$$N(-d_1) = \int_{-\infty}^{-d_1} e^{-\frac{d_1^2}{2}} d(d_1)$$

at $t \rightarrow T$, $d_1 \longrightarrow \infty$.

$$N(-\infty) = \int_{-\infty}^{-\infty} e^{-\frac{d_1^2}{2}} d(d_1)$$

$$= 0$$

For t $\longrightarrow$T, d2=d1, So, N(-d1) =N(-d2) =0.

P (S_t, t) =N(-d_2) K e^(-r(T-t))- N(-d_1) S_t = 0

For t $\longrightarrow$T,

In question (2), we see that at expiry time period left is 0,

For the equations derived in (2.a), C=$(S- K $\times$ e^ (-r $\times$T/100)).

Putting T=0, C= $(S-K) which matches the call option pricing derived in (3.a)

For the equations derived in (2.b), P= =$(K $\times$ e^ (-r $\times$T/100) - S +C).

Putting T=0, P=K-S+C = $(S-K –S –K) = 0 (since, value of call option derived above is $(S-K)) which matches the Put option pricing derived in (3.a)

3.b) For this question, I have made the graphs using MATLAB.

Given,

K=$50, r=12%p.a., T=1 year, σ= 0.3

BSM price of call option C vs. S_t for S_t varying from 1 to 100:

For time t=0 years,

for $t = 0$,

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right]$$

$$= \frac{1}{0.3}\left[\ln\left(\frac{S_t}{50}\right) + \left(0.12 + \frac{0.3^2}{2}\right)\right]$$

$$= \frac{1}{0.3}\left[\ln\left(\frac{S_t}{50}\right) + 0.165\right]$$

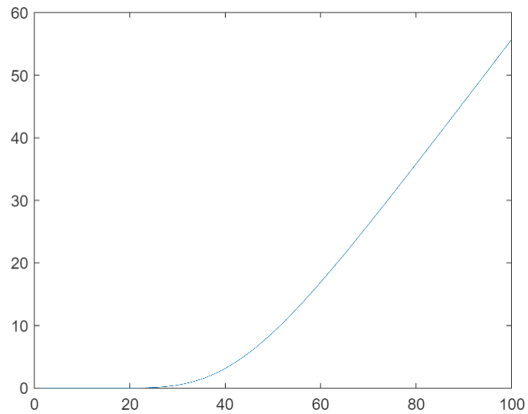
$$d_2 = d_1 - \sigma\sqrt{T}$$

$$= \frac{1}{0.3}\left[\ln\left(S_t/50\right) + 0.165\right] - 0.3$$

$C(S_t, t) = N(d_1)\, S_t - N(d_2) \times 50 \times e^{\wedge}(-0.12(T-0))$

$C(S_t, t) = N(d_1)\, S_t - N(d_2) \times 50 \times e^{\wedge}(-0.12)$

```
clc;
clear all;
St=linspace(1,100,10000);
d1=(1/0.3)*(log(St./50)+0.165);
d2=d1-0.3;
C=(normcdf(d1).*St)-(normcdf(d2)*50*exp(-0.12));
plot(St,C)
```



For time t=0.5 years,

for $t = 0.5$

$$d_1 = \frac{1}{\sigma\sqrt{T-0.5}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-0.5)\right]$$

$$= \frac{1}{0.3\sqrt{0.5}}\left[\ln\left(S_t/50\right) + \left(0.12 + \frac{0.3^2}{2}\right)0.5\right]$$

$$= \frac{1}{0.212}\left[\ln\left(\frac{S_t}{50}\right) + 0.0825\right]$$

$$d_2 = d_1 - \sigma\sqrt{T-0.5}$$

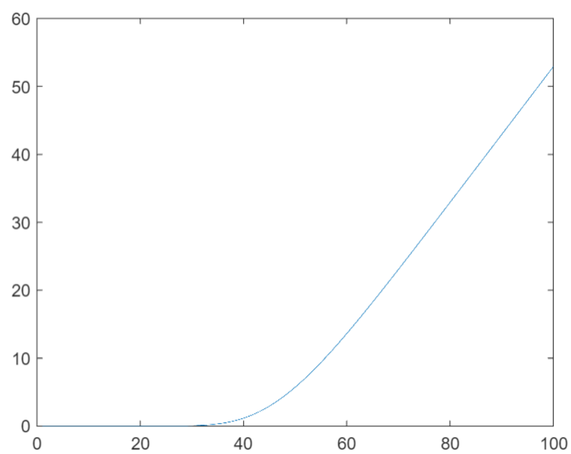
$$= \frac{1}{0.212}\left[\ln\left(S_t/50\right) + 0.0825\right] - 0.212$$

$C(S_t, t) = N(d_1)\, S_t - N(d_2) \times 50 \times e^{(-0.12(T-0.5))}$

$C(S_t, t) = N(d_1)\, S_t - N(d_2) \times 50 \times e^{(-0.12 \times 0.5)}$

$C(S_t, t) = N(d_1)\, S_t - N(d_2) \times 50 \times e^{(-0.06)}$

```
clc;
clear all;
St=linspace(1,100,10000);
d1=(1/0.212)*(log(St./50)+0.0825);
d2=d1-0.212;
C=(normcdf(d1).*St)-(normcdf(d2)*50*exp(-0.06));
plot(St,C)
```



For time t= 1 year,

for $t = 1$

$$d_1 = \frac{1}{\sigma\sqrt{T-1}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r+\frac{\sigma^2}{2}\right)(T-1)\right]$$

$d_1 \Rightarrow \infty$ (∵ denominator becomes 0)

∴ $d_2 \Rightarrow \infty$

as derived in (3.a), when $d_1 \rightarrow \infty$ & $d_2 \rightarrow \infty$

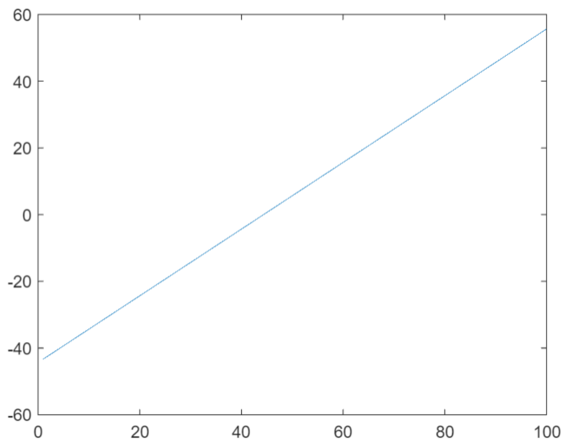
$N(d_1) = N(d_2) = 1$

$C(S_t, t) = N(d_1)\, S_t - N(d_2) \times 50 \times e^{\wedge}(-0.12(T-1))$

$C(S_t, t) = S_t - 50 \times e^{\wedge}(0)$ (since, $N(d_1) = N(d_2) = 1$)

$C(S_t, t) = S_t - 50$

```
clc;
clear all;
St=linspace(1,100,10000);
C=(normcdf(Inf).*St)-(normcdf(Inf)*50*exp(-0.12));
plot(St,C)
```



3.c) Given,

K=$50, r=12%p.a., T=1 year, σ= 0.3

BSM price of Put option P vs. S_t for S_t varying from 1 to 100:

For time t=0 years,

for $t = 0$,

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right]$$

$$= \frac{1}{0.3}\left[\ln\left(\frac{S_t}{50}\right) + \left(0.12 + \frac{0.3^2}{2}\right)\right]$$

$$= \frac{1}{0.3}\left[\ln\left(\frac{S_t}{50}\right) + 0.165\right]$$

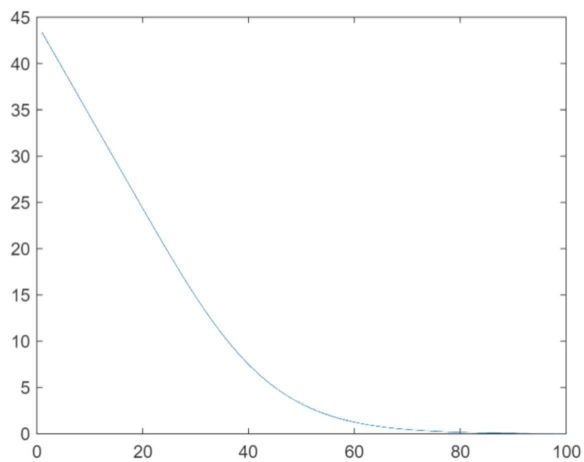
$$d_2 = d_1 - \sigma\sqrt{T}$$

$$= \frac{1}{0.3}\left[\ln\left(S_t/50\right) + 0.165\right] - 0.3$$

P (S_t, t) = N(-d_2) × 50× e^ (-0.12(T-0))- N(-d_1) S_t

P (S_t, t) = N(-d_2) × 50× e^ (-0.12)- N(-d_1) S_t

```
clc;
clear all;
St=linspace(1,100,10000);
d1=(1/0.3)*(log(St./50)+0.165);
d2=d1-0.3;
P=-(normcdf(-d1).*St)+(normcdf(-d2)*50*exp(-0.12));
plot(St,P)
```



For time t=0.5 years,

for $t = 0.5$

$$d_1 = \frac{1}{\sigma\sqrt{T-0.5}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-0.5)\right]$$

$$= \frac{1}{0.3\sqrt{0.5}}\left[\ln\left(S_t/50\right) + \left(0.12 + \frac{0.3^2}{2}\right)0.5\right]$$

$$= \frac{1}{0.212}\left[\ln\left(\frac{S_t}{50}\right) + 0.0825\right]$$

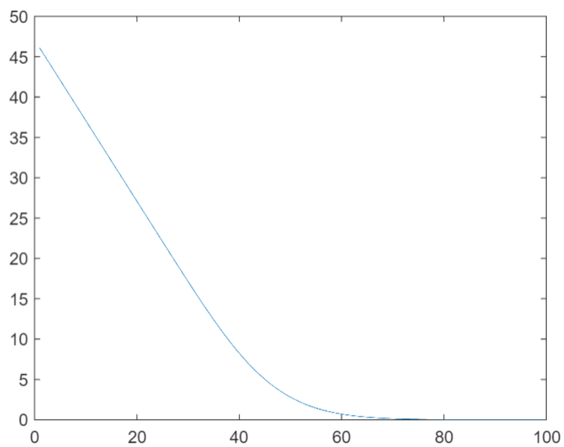
$$d_2 = d_1 - \sigma\sqrt{T-0.5}$$

$$= \frac{1}{0.212}\left[\ln\left(S_t/50\right) + 0.0825\right] - 0.212$$

P (S_t, t) = N(-d_2) × 50× e^ (-0.12(T-0.5))- N(-d_1) S_t

P (S_t, t) = N(-d_2) × 50× e^ (-0.06)- N(-d_1) S_t

```
clc;
clear all;
St=linspace(1,100,10000);
d1=(1/0.212)*(log(St./50)+0.0825);
d2=d1-0.212;
P=-(normcdf(-d1).*St)+(normcdf(-d2)*50*exp(-0.06));
plot(St,P)
```



For time t=1 year,

for $t = 1$

$$d_1 = \frac{1}{\sigma\sqrt{T-1}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-1)\right]$$

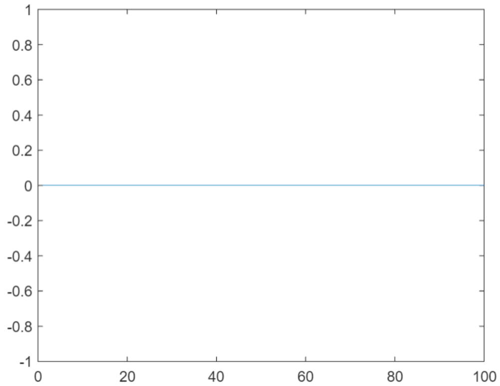
$d_1 \longrightarrow \infty$ ($\because$ denominator becomes 0)

$\therefore d_2 \longrightarrow \infty$

$N(-d_1) = N(-d_2) = 0$

P=N($-d_2$) K e^(-r(T-t))- N($-d_1$) S_t = 0

```
clc;
clear all;
St=linspace(1,100,10000);
P=(normcdf(-Inf)*50*exp(-0.12))-(normcdf(-Inf).*St);
plot(St,P)
```



4.Delta Calculation

4.a) Given, S=\$125, K=\$50, r=12%p.a., T=1 year, σ= 0.3

Delta Calculation for Call Option:

$$C(S_t,t) = N(d_1)S_t - N(d_2)Ke^{-r(T-t)}$$

For delta calculation, we differentiate the call option w.r.t. S_t

$$\frac{\partial C(S_t,t)}{\partial S_t} = \frac{dN(d_1)}{dS_t}S_t + N(d_1) - \frac{dN(d_2)}{dS_t}Ke^{-r(T-t)}$$

We know, $N(d_1) = \int_{-\infty}^{d_1} e^{-\frac{d_1^2}{2}}\, d(d_1)$

$$\frac{dN(d_1)}{dS_t} = \frac{dN(d_1)}{d(d_1)} \times \frac{d(d_1)}{dS_t}$$

$$= e^{-\frac{d_1^2}{2}} \times \frac{1}{\sigma\sqrt{T-t}}\left(\frac{1}{S_t}\right)$$

We know, $N(d_2) = \int_{-\infty}^{d_2} e^{-\frac{d_2^2}{2}}\, d(d_2)$

$$\frac{dN(d_2)}{dS_t} = \frac{dN(d_2)}{d(d_2)} \times \frac{d(d_2)}{dS_t} = e^{-\frac{d_2^2}{2}} \times \frac{d(d_1)}{dS_t}$$

$$= e^{-\frac{d_2^2}{2}} \times \frac{1}{\sigma\sqrt{T-t}}\left(\frac{1}{S_t}\right)$$

$$\frac{\partial C(S_t,t)}{\partial S_t} = e^{-\frac{d_1^2}{2}} \times \frac{1}{\sigma\sqrt{T-t}} \times \frac{1}{S_t} \times S_t + N(d_1) - e^{-\frac{d_2^2}{2}} \times \frac{1}{\sigma\sqrt{T-t}} \times \frac{1}{S_t} \times Ke^{-r(T-t)}$$

For time t=0 years,

d1=1/0.3[(ln (S_t /50)) +0.165] =1/0.3[ln (2.5) +0.165] =1/0.3(0.916+0.165) =3.604

d2=d1- 0.3

d2= 3.604-0.3=3.304

Using the derivation of N(d1) is (3.a), we get from scientific calculator, N (3.604) =0.999843 for $t = 0$ years,

$$\frac{dC(S_t, 0)}{dS_t} = e^{-\frac{3.604^2}{2}} \times \frac{1}{0.3} + 0.999843 - e^{-\frac{3.304^2}{2}} \times \frac{1}{0.3} \times \frac{1}{125} \times 50\, e^{-0.12}$$

$$= 5.04 \times 10^{-3} + 0.999843 - 5.04 \times 10^{-3}$$

$$\approx 0.999843$$

Hence, delta for t=0 for call option is 0.999843.

<u>For time t=0.5 years,</u>

d1=1/0.212[(ln (S_t /50)) +0.0825] =1/0.212[ln (2.5) +0.0825] =1/0.212(0.916+0.0825) =4.71

d2=d1- 0.212

d2= 4.71-0.212=4.498

Using the derivation of N(d1) is (3.a), we get from scientific calculator, N (4.71) =0.9999987

for $t = 0.5$ years

$$\frac{dC(S_t, 0.5)}{dS_t} = e^{-\frac{4.71^2}{2}} \times \frac{1}{0.3} + 0.9999987 - e^{-\frac{4.498^2}{2}} \times \frac{1}{0.3} \times \frac{1}{125} \times 50\, e^{-0.06}$$

$$= 5.08 \times 10^{-5} + 0.9999987 - 5.08 \times 10^{-5}$$

$$= 0.9999987 \approx 0.999999.$$

Hence, delta for t=0.5 years for call option is 0.999999.

<u>For time t=1 year,</u>

d1→infinity and d2→d1 or , d2→ infinity

Using the derivation of N(d1) is (3.a), we get from scientific calculator, N (infinity) =1

C (S_t , t) = S_t -K

Therefore, d C (S_t , t)/ d S_t =1

Hence, delta for t= 1 year for call option is 1.

Delta Calculation for put option:

$$P = N(-d_2)Ke^{-r(T-t)} - N(-d_1)S_t$$

differentiating $P(S_t, t)$ w.r.t. S_t

$$\frac{\partial P(S_t,t)}{\partial S_t} = \frac{d\,N(-d_2)}{d S_t}Ke^{-r(T-t)} - \frac{d\,N(-d_1)}{d S_t}S_t - N(-d_1)$$

We know, $N(-d_2) = \int_{-\infty}^{-d_2} e^{-\frac{d_2^2}{2}}\, d(d_2)$

$$\frac{d\,N(-d_2)}{d S_t} = \frac{d\,N(-d_2)}{d(d_2)} \times \frac{d(d_2)}{d S_t}$$

$$= -e^{-\frac{d_2^2}{2}} \times \frac{1}{\sigma\sqrt{T-t}}\left(\frac{1}{S_t}\right)$$

We know, $N(-d_1) = \int_{-\infty}^{-d_1} e^{-\frac{d_1^2}{2}}\, d(d_1)$

$$\frac{d\,N(-d_1)}{d S_t} = \frac{d\,N(-d_1)}{d(d_1)} \times \frac{d(d_1)}{d S_t}$$

$$= -e^{-\frac{d_1^2}{2}} \times \frac{1}{\sigma\sqrt{T-t}}\left(\frac{1}{S_t}\right)$$

$$\frac{\partial P(S_t,t)}{\partial S_t} = \left[-e^{-\frac{d_2^2}{2}} \times \frac{1}{\sigma\sqrt{T-t}} \times \frac{1}{S_t}\right]Ke^{-r(T-t)}$$

$$+ e^{-\frac{d_1^2}{2}} \times \frac{1}{\sigma\sqrt{T-t}} - N(-d_1)$$

<u>For time t=0 years,</u>

d1=1/0.3[(ln (S_t /50)) +0.165] =1/0.3[ln (2.5) +0.165] =1/0.3(0.916+0.165) =3.604

d2=d1- 0.3

d2= 3.604-0.3=3.304

Using the derivation of N(d1) is (3.a), we get from scientific calculator, N (-3.604) =0.0001567 for $t = 0$ years,

$$\frac{dP(S_t, 0)}{dS_t} = \left[-e^{-\frac{3.304^2}{2}} \times \frac{1}{0.3} \times \frac{1}{125}\right] 50e^{-0.12} + e^{-\frac{3.604^2}{2}} \times \frac{1}{0.3} - 0.0001567$$

$$= -5.04 \times 10^{-3} + 5.04 \times 10^{-3} - 0.0001567$$

$$= -0.0001567$$

Hence, the delta for t=0 years for put option is - 1.567* 10^ (-4)

<u>For time t=0.5 years,</u>

d1=1/0.212[(ln (S_t /50)) +0.0825] =1/0.212[ln (2.5) +0.0825] =1/0.212(0.916+0.0825) =4.71

d2=d1- 0.212

d2= 4.71-0.212=4.498

Using the derivation of N(d1) is (3.a), we get from scientific calculator, N (-4.71) =1.24*10^ (-6)

for $t = 0.5$ years

$$\frac{dP(S_t, 0.5)}{dS_t} = \left[-e^{-\frac{4.498^2}{2}} \times \frac{1}{0.212} \times \frac{1}{125}\right] 50e^{-0.06} + e^{-\frac{4.71^2}{2}} \times \frac{1}{0.212} - 1.24 \times 10^{-6}$$

$$= -7.18 \times 10^{-5} + 7.18 \times 10^{-5} - 1.24 \times 10^{-6}$$

$$= -1.24 \times 10^{-6}$$

Hence, the delta for t=0.5 years for put option is - 1.24* 10^ (-6)

<u>For time t=1 year,</u>

d1→infinity and d2→d1 or, d2→ infinity

Using the derivation of N(d1) is (3.a), we get from scientific calculator, N (-infinity) =0

P (S_t , t) =0

Therefore, d P (S_t , t)/ d S_t =0

Hence, delta for t= 1 year for put option is 0.

4.b) For numerically calculating the deltas as a slope of the plots:

Given, S=$125, K=$50, r=12%p.a., T=1 year, σ= 0.3

For calculating slope in the following time periods, we take the reference point at S_t = $10:

For Call Option:

For time t=0 year,

d1=3.604 and d2=3.304 (derived above in 4.a)

C (125, 0) =N (3.604) *125– N (3.304) *50 *exp (-0.12) =0.999843*125 – 0.999523*50*0.887=80.65

C (10, 0) = N (3.604) *10-N (3.304) *50*exp (-0.12) =0.999843*10 - 0.999523*50*0.887=-34.33

Delta= slope= [C (125, 0)- C (10 ,0)] / (125-10) = (80.65+34.33)/115 =0.999826

For time t=0.5 year,

d1=4.71 and d2=4.498 (derived above in 4.a)

C (125, 0.5) =N (4.71) *125– N (4.498) *50 *exp (-0.06) =0.99999876*125 – 0.99999657*50*0.942=77.9

C (10, 0.5) = N (3.604) *10-N (3.304) *50*exp (-0.12) = 0.99999876*10 -0.99999657* 50*0.942 = -37.09999

Delta= slope= [C (125, 0.5)- C (10 ,0.5)] / (125-10) = (77.9+37.09999)/115 =0.999999

For time t=1 year,

d1→infinity and d2→d1 or, d2→ infinity

C (125, 1) =N (infinity) *125– N (infinity) *50 *exp (0) =125 – 50=75

C (10, 1) = N (infinity) *10– N (infinity) *50 *exp (0) =10-50=-40

Delta= slope= [C (125,1)- C (10,1)] / (125-10) = (75+40)/115 =1

For Put Option:

For time t=0 years,

d1=3.604 and d2=3.304 (derived above in 4.a)

P (125,0) = N (-3.304) *50*exp (-0.12)-N (-3.604) *125=0.000477*50*0.887-0.000157*125 =0.0015299

P (10,0) = N (-3.304) *50*exp (-0.12)-N (-3.604) *10=0.000477*50*0.887-0.000157*10=0.01958

Delta= slope= [P (125, 0)- P (10 ,0)]/ (125-10) = (0.0015299-0.01958)/115=-1.57*10^ (-4) =-0.000157.

For time t=0.5 years,

d1=4.71 and d2=4.498 (derived above in 4.a)

P (125,0.5) = N (-4.71) *50*exp (-0.06)-N (-4.498) *125=1.24*10^ (-6) *50*0.942-3.429*10^ (-6) *125 =-3.7* 10^ (-4)

P (10,0.5) = N (-4.71) *50*exp (-0.06)-N (-4.498) *10=1.24*10^ (-6) *50*0.942-3.429*10^ (-6) *10 =2.4114*10^ (-5) =0.241*10^ (-4)

Delta= slope= [P (125, 0.5)- P (10 ,0.5)]/ (125-10) = (-3.7* 10^ (-4) - 0.241*10^ (-4))/115=-3.42*10^ (-6)

<u>For time t=1 year,</u>

d1→infinity and d2→d1 or, d2→ infinity

P (125, 1) =N (-infinity) *125– N (-infinity) *50 *exp (0) =0

C (10, 1) = N (-infinity) *10– N (-infinity) *50 *exp (0) =0

Delta= slope= [C (125,1)- C (10,1)] / (125-10) = 0

5.a) For a stock the following relation holds between the prices of call and put options, both with strike price K and time T: $C - P = S(t) - K * exp(-r*T)$.

A portfolio whose value depends on the current stock price $S = S(t)$ and is hence denoted by $V(S)$.

Derivative of V(S) with respect to S which is called the delta of the portfolio.

For small price variations from S to $S + \Delta S$ the value of the portfolio will change by

$\Delta V(S) \approx \mathrm{d}\, V(S)/\mathrm{d}\, S \times \Delta S$.

We take a portfolio composed of stock, bonds and the hedged derivative security, its value given by

$V(S) = x\, S + y + z\, D(S)$, where the derivative security price is denoted by $D(S)$ and a bond with current value 1 is used. For $z = -1$,

$d\, V(S) / \mathrm{d}\, S = x - d\, D(S)/ d\, S$

The delta of the call option is given by, $d\, C(S)/ \mathrm{d}\, S = N(d1)$

The delta of the put option is given by, $d\, P(S)/ \mathrm{d}\, S = -N(-d1)$

The portfolio $(x, y, z) = (N(d1), y, -1)$, where the position in stock $N(d1)$ is computed for the initial stock price $S = S(t)$, has delta equal to zero for any money market position y.

It's value $V(S) = N(d1)\, S + y - C(S)$, does not vary much under small changes of the stock price about the initial value.

By the Black–Scholes formula for $C(S)$ this gives $y = -X$ *exp $(-T\, r)$ * $N(d2)$.

Given, r= 12% p.a., T=1 year, σ= 0.3, let strike price be $K.

At time t=0 years, $C(S_t, t) = N(d_1)\, S_t - N(d_2) \times K \times e$^ (-0.12(T-0))

For Digital Option,

i) **At S_t = $30**, we can see from the payoff plot that S_T = $30 gives $0 payoff, hence strike price is also $30.

d_1=1/0.3[ln (30/30) + 0.165) =0.55, d_2 = 0.55-0.3=0.25

C ($S_{t,}$ t) =N(d_1) S_t - N(d_2) $\times$ 30$\times$ e^ (-0.12)

=0.7088*30 – 0.5987* 30*0.887= 5.33326

Hence, price of call option is $5.3326.

We saw in (4.a) that delta for call option is d [C ($S_{t,}$ t)] / d S_t = N(d_1)

Delta=0.7088

If, we are long in 'n' units of call option, we short 0.7088*n units of the underlying stock, so the portfolio will have n- 0.7088*n units =0.2912 *n

For the above units, the total premium for call option is $5.3326 *n

To construct the hedge, we buy 0.2912*n stocks for 0.2912*n*30= $ 8.736*n

Borrowing=$ (5.3326 *n – 8.736 *n) = $ -3.4034 *n

Hence the portfolio is (0.2912*n, -3.4034*n, -n).

ii) **At S_t = $50**, it is a long call at $40.

Strike price is also $40.

d_1=1/0.3[ln (50/40) + 0.165) =1.294, d_2 = 1.294-0.3=0.994

C ($S_{t,}$ t) =N(d_1) S_t - N(d_2) $\times$ 40$\times$ e^ (-0.12) =0.9022*50-0.8399*40*0.887=$15.3103

Hence, price of call option is $15.3103.

We saw in (4.a) that delta for call option is d [C ($S_{t,}$ t)] / d S_t = N(d_1)

Delta=0.9022

If, we are long in 'n' units of call option, we short 0.7088*n units of the underlying stock, so the portfolio will have n- 0.9022*n units =0.0978 *n

For the above units, the total premium for call option is $15.1303*0.0978*n=1.4797*n

To construct the hedge, we buy 0.0978*n stocks for 0.0978*n*50= $ 4.89*n

Borrowing=$ (1.4797*n – 4.89*n) = $ -3.4102*n

Hence the portfolio is (0.0978*n, -3.4102*n, -n).

iii) **At S_t = $70,** it is a short call at $60.

Strike price is also $60.

d_1=1/0.3[ln (70/60) + 0.165) =1.0638, d_2 = 1.0638-0.3=0.7638

C ($S_{t,}$ t) =N(d_1) S_t - N(d_2) $\times$ 60$\times$ e^ (-0.12)

=0.8563*70-0.7775*60*0.887=$18.56245.

Hence, price of call option is $18.56245.

We saw in (4.a) that delta for call option is d [C ($S_{t,}$ t)] / d S_t = N(d_1)

Delta=0.8563

If, we are short in 0.8563 units of call option,

For the above units, the total premium for call option is $18.56245 *0.8563*n=$ 15.895*n

To construct the hedge, we sell 0.8563*n stocks for 0.8563*n*70= $ 59.941*n

Excess=$ (59.941*n –15.895*n) = $44.0597*n

Hence the portfolio is (0.8563*n, 44.0597*n, -n).

5.b) **For Corridor Option Payoff**, Given, r= 12% p.a., T=1 year, σ**=** 0.3, let strike price be $K.

At time t=0 years, C ($S_{t,}$ t) =N(d_1) S_t - N(d_2) $\times$ K$\times$ e^ (-0.12(T-0))

i) **At S_t = $10**, Strike price is also $10.

d_1=1/0.3[ln (10/10) + 0.165) =0.55, d_2 = 0.55-0.3=0.25

C ($S_{t,}$ t) =N(d_1) S_t - N(d_2) $\times$ 50$\times$ e^ (-0.12)

=0.7088*10 – 0.5987* 10*0.887= $1.77753

Hence, price of call option is $1.77753.

We saw in (4.a) that delta for call option is d [C ($S_{t,}$ t)] / d S_t = N(d_1)

Delta=0.7088

If, we are long in 'n' units of call option, we short 0.7088*n units of the underlying stock, so the portfolio will have n- 0.7088*n units =0.2912 *n

For the above units, the total premium for call option is $1.77753 *n

To construct the hedge, we buy 0.2912*n stocks for 0.2912*n*10= $ 2.912*n

Borrowing=$ (1.77753. *n –2.912*n) = $-1.13447 *n

Hence the portfolio is (0.2912*n, - 1.13447 *n, -n), whose value is 0, hence it is riskless.

ii) **At S_t = $30**, it is a long call at $20.

Strike price for long call option is $20.

d_1=1/0.3[ln (30/20) + 0.165) =1.9016, d_2 = 1.9016-0.3=1.6016

$C(S_t, t) = N(d_1) S_t - N(d_2) \times 20 \times e^{\wedge}(-0.12)$

$= 0.9714*30 – 0.9454* 20*0.887 = \$12.3706.$

Hence, price of call option is \$12.3706.

We saw in (4.a) that delta for call option is $d [C(S_t, t)] / d S_t = N(d_1)$

Delta=0.9714

If, we are long in 'n' units of call option, we short 0.9714*n units of the underlying stock, so the portfolio will have n- 0.9714*n units =0.0286 *n

For the above units, the total premium from selling call option is \$12.3706 *0.0286*n= 0.3538*n

To construct the hedge, we buy 0.0286*n stocks for 0.0286*n*30= \$0.858*n

Borrowing=\$ (0.3538*n –0.858*n) = \$-0.5042*n

Hence the riskless portfolio is (0.0286*n, - 0.5042*n, -n).

iii) **At S_t = \$50**, it is a short call \$40.

When the price becomes less than \$40, we can sell it at \$40.

Strike price is also \$40.

$d_1 = 1/0.3[\ln(50/40) + 0.165) = 1.294$, $d_2 = 1.294-0.3 = 0.994$

$C(S_t, t) = N(d_1) S_t - N(d_2) \times 40 \times e^{\wedge}(-0.12)$

$= 0.9022*50-0.8399*40*0.887 = \$15.3103$

Hence, price of call option is \$15.3103.

We saw in (4.a) that delta for put option is $d [P(S_t, t)] / d S_t = - N(d_1)$

Delta=0.9022

If, we are short in 0.9022 units of call option,

For the above units, the total premium for buying the call option is \$15.3103 *0.9022*n=\$13.813*n

To construct the hedge, we sell 0.9022*n stocks for 0.9022*n*50= \$45.11*n

Excess=\$ (45.11*n –13.813*n) = \$31.297*n

Hence the portfolio is (0.9022*n, 31.297*n, -n).

iv) **At S_t = \$70**, it is a short call at \$60

Strike price is also \$60.

$d_1 = 1/0.3[\ln(70/60) + 0.165) = 1.0638$, $d_2 = 1.0638-0.3 = 0.7638$

$C(S_t, t) = N(d_1) S_t - N(d_2) \times 60 \times e^{\wedge}(-0.12)$

=0.8563*70-0.7775*60*0.887=$18.56245.

Hence, price of call option is $18.56245.

We saw in (4.a) that delta for call option is d [C ($S_{t,}$ t)] / d S_t = N(d_1)

Delta=0.8563

If, we are short in 0.8563*n units of the call option,

For the above units, the total premium for buying the call option is $18.56254 *0.8563*n=15.895*n

To construct the hedge, we sell 0.8563*n stocks for 0.8563*n*70= $59.941*n

Excess=$ (59.941 *n –15.895*n) = $44.046*n

Hence the portfolio is (0.8563*n, 44.046 *n, -n).

v) **At S_t = $90**, it is at $80 long call.

Strike price is also $80.

d_1=1/0.3[ln (90/80) + 0.165) =0.9426, d_2 = 0.9426-0.3=0.6426

C ($S_{t,}$ t) =N(d_1) S_t - N(d_2) × 80× e^ (-0.12)

=0.8271*90 – 0.7398* 80*0.887= $21.943

Hence, price of call option is $21.943.

We saw in (4.a) that delta for call option is d [C ($S_{t,}$ t)] / d S_t = N(d_1)

Delta=0.8271

If, we are long in ‘n’ units of call option, we short 0.8271*n units of the underlying stock, so the portfolio will have n- 0.8271*n units =0.1729 *n

For the above units, the total premium for call option is $21.943 *0.1729*n=3.7939*n

To construct the hedge, we buy 0.1729*n stocks for 0.1729*n*90= $15.561*n

Borrowing=$ (3.7939*n –$15.561*n) = $-11.7671*n

Hence the portfolio is (0.1729*n, -11.7671*n, -n).

5.c) Let S_0 be the stock price at time t=0, K be the strike price.

Let S_T be the price at expiry then, payoff =max (S_T -K,0) = $(S_T - K)^+$

Portfolio Set up: i)A position in Δ shares, S

ii)A position in risk-free bond, B

C= S_0 Δ +B, ----------------------------(1)

When stock price is greater than S_0 , it is S_u and When stock price is greater than S_0 , it is S_b

Putting S_u and S_b in eqn. (1). We can get the value of Δ and B.

For S_t=$10, its already a riskless portfolio, since the graph is almost horizontal here and hence the strike price will be $10.

d_1=1/0.3[ln (10/10) + 0.165) =0.55, d_2 = 0.55-0.3=0.25

C (S_t, t) =N(d_1) S_t - N(d_2) × 50× e^ (-0.12)

=0.7088*10 – 0.5987* 10*0.887= $1.77753

Hence, price of call option is $1.77753.

We saw in (4.a) that delta for call option is d [C (S_t, t)] / d S_t = N(d_1)

Delta=0.7088

If, we are long in 'n' units of call option, we short 0.7088*n units of the underlying stock, so the portfolio will have n- 0.7088*n units =0.2912 *n

For the above units, the total premium for call option is $1.77753 *n

To construct the hedge, we buy 0.2912*n stocks for 0.2912*n*10= $ 2.912*n

Borrowing=$ (1.77753. *n –2.912*n) = $-1.13447 *n

Hence the portfolio is (0.2912*n, - 1.13447 *n, -n), whose value is 0, hence it is riskless.

For S_t=$30, it is a long call on $20,

Strike price =$20.

d_1=1/0.3[ln (30/20) + 0.165) =1.9016, d_2 = 1.9016-0.3=1.6016

C (S_t, t) =N(d_1) S_t - N(d_2) × 20× e^ (-0.12)

=0.9714*30 – 0.9454* 20*0.887= $12.3706.

Hence, price of call option is $12.3706.

We saw in (4.a) that delta for call option is d [C (S_t, t)] / d S_t = N(d_1)

Delta=0.9714

If, we are long in 'n' units of call option, we short 0.9714*n units of the underlying stock, so the portfolio will have n- 0.9714*n units =0.0286 *n

For the above units, the total premium from selling call option is $12.3706 *0.0286*n= 0.3538*n

To construct the hedge, we buy 0.0286*n stocks for 0.0286*n*30= $0.858*n

Borrowing=$ (0.3538*n –0.858*n) = $-0.5042*n

Hence the riskless portfolio is (0.0286*n, - 0.5042*n, -n).

For S_t=$50, it is a long call on $20,

Strike price =$20.

d_1=1/0.3[ln (50/20) + 0.165) =3.6043, d_2 =3.6043-0.3=3.3043

C ($S_{t,}$ t) =N(d_1) S_t - N(d_2) × 20× e^ (-0.12)

=0.99984*50 – 0.99952* 20*0.887= $32.2605

Hence, price of call option is $32.2605.

We saw in (4.a) that delta for call option is d [C ($S_{t,}$ t)] / d S_t = N(d_1)

Delta=0.99984.

If, we are long in 'n' units of call option, we short 0.99984*n units of the underlying stock, so the portfolio will have n- 0.99984*n units =1.6*10^(-4) *n

For the above units, the total premium from selling call option is $32.2605 *1.6*0^(-4)*n= 5.1616*10^(-3)*n

To construct the hedge, we buy 1.6*10^(-4)*n stocks for 1.6*10^(-4)*n *50= $8*10^(-3)*n

Borrowing=$ (5.1616*10^(-3)*n –8*10^(-3)*n) = $-5.1536*10^(-3)*n

Hence the riskless portfolio is (1.6*10^(-4) *n, -5.1536*10^(-3)*n, -n).

For S_t=$70, it is a short call on $20,

Strike price =$20.

d_1=1/0.3[ln (70/20) + 0.165) =4.726, d_2 =4.726-0.3=4.426

C ($S_{t,}$ t) =N(d_1) S_t - N(d_2) × 20× e^ (-0.12)

=0.9999988*70 – 0.9999952* 20*0.887= $52.26

Hence, price of call option is $52.26

We saw in (4.a) that delta for call option is d [C ($S_{t,}$ t)] / d S_t = N(d_1)

Delta=0.9999988.

If, we are short in 0.9999988 units of call option,

For the above units, the total premium from selling call option is $52.26*0.9999988*n= 52.2599*n

To construct the hedge, we sell 0.999988*n stocks for 0.9999988 *n *70= $69.9999*n

Excess=$ (69.9999*n –52.2599*n) = $17.74*n

Hence the riskless portfolio is (0.9999988*n, $17.74*n, -n).

v) **At S_t = $90**, it is at $80 short call.

Strike price is also \$80.

d_1=1/0.3[ln (90/80) + 0.165) =0.9426, d_2 = 0.9426-0.3=0.6426

C ($S_{t,}$ t) =N(d_1) S_t - N(d_2) $\times$ 80$\times$ e^ (-0.12)

=0.8271*90 – 0.7398* 80*0.887= \$21.943

Hence, price of call option is \$21.943.

We saw in (4.a) that delta for call option is d [C ($S_{t,}$ t)] / d S_t = N(d_1)

Delta=0.8271

If, we are short in 0.8271 units of call option,

For the above units, the total premium for buying call option is \$21.943. *0.8271*n=18.149*n

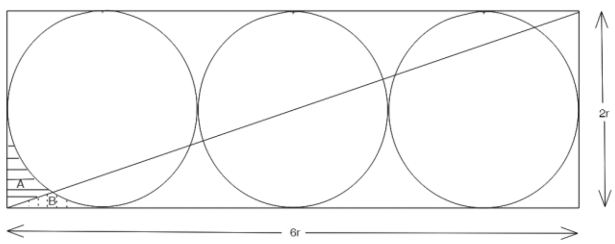
To construct the hedge, we sell 0.8271 *n stocks for 0.8271 *n*90= \$74.439*n

Excess=\$ (74.439*n –\$18.149*n) = \$56.29*n

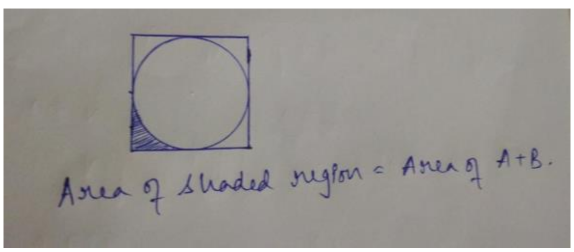
Hence the portfolio is (0.8271*n, \$56.29*n, -n).

Case Study B

Circles



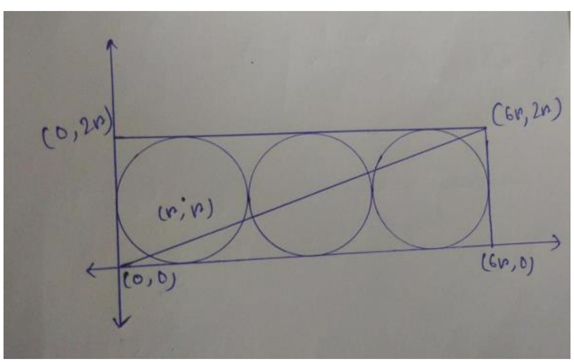
1.1 If we take one square or dimensions 2r ╳ 2r and a circle carved inside it with radius r with their centers coinciding:



The area A+B=¼ (area of square- area of circle)

$= ¼ (4r^2 - \pi r^2) = r^2/4(4 - \pi) = 0.215 r^2$

If we place the rectangle and the circle in the coordinate axis in the following manner:



Equation of the diagonal: $y=x/3$

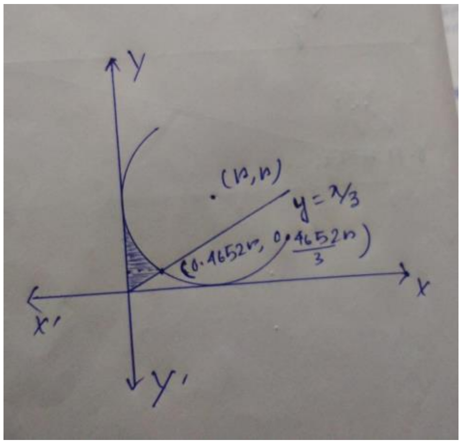
Equation of the circle: $(x-r)^2 + (y-r)^{2=}r^2$

Intersection point of the diagonal and the circle are:

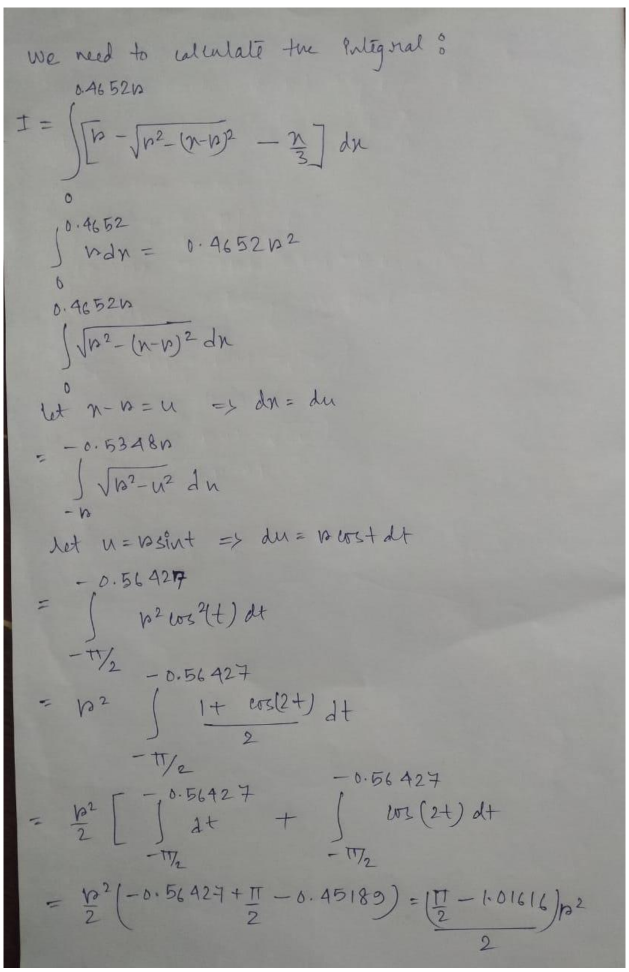
$(x-r)^2 + (x/3-r)^{2=}r^2$

or,$10x^2 - 24xr + 9r^2 = 0$

Using Sridhar Acharya, we get: x1=0.4652 and x2= 1.935 where the line intersects the circle.



For finding the area of the above shaded region A,



We need to calculate the integral :

$$I = \int_0^{0.4652r} \left[r - \sqrt{r^2-(x-r)^2} - \frac{x}{3}\right] dx$$

$$\int_0^{0.4652} r\,dx = 0.4652r^2$$

$$\int_0^{0.4652r} \sqrt{r^2-(x-r)^2}\,dx$$

let $x - r = u \Rightarrow dx = du$

$$= \int_{-r}^{-0.5348r} \sqrt{r^2-u^2}\,du$$

let $u = r\sin t \Rightarrow du = r\cos t\,dt$

$$= \int_{-\pi/2}^{-0.56427} r^2\cos^2(t)\,dt$$

$$= r^2 \int_{-\pi/2}^{-0.56427} \frac{1+\cos(2t)}{2}\,dt$$

$$= \frac{r^2}{2}\left[\int_{-\pi/2}^{-0.56427} dt + \int_{-\pi/2}^{-0.56427}\cos(2t)\,dt\right]$$

$$= \frac{r^2}{2}\left(-0.56427+\frac{\pi}{2} - 0.45189\right) = \frac{\left(\frac{\pi}{2}-1.01616\right)r^2}{2}$$

$$\int_0^{0.4652r} \frac{x}{3}\,dx = 0.03606\,r^2$$

$$\therefore I = 0.4652\,r^2 - \frac{\left(\frac{\pi}{2} - 1.01616\right)r^2}{2} - 0.03606\,r^2$$

$$\therefore I = 0.15181\,r^2$$

Area of A = 0.15181 r^2

Area of A / Area of (A+ B) =0.1518/0.215=0.7061

1.2) For ‘n’ circles arranged inside a rectangle of dimensions ‘2nr’ by ‘2r’ in a similar fashion,

As n is a natural number, we see that n can take the least value of 1, for n=1:

Area of (A+B) =¼ ($4r^2$- π r^2) = r^2/4(4 - π) = 0.215 r^2

Area of A= Area of (A+ B)/2 (since figure is symmetric)

=0.1075 r^2

Area ratio= area of A/ area of (A+B)

Area ratio=½.

n keeps increasing with increasing number of circles,

When n tends to infinity, Area of B tends to 0.

Area ratio= area of A/ area of (A+B)

Or, area ratio tends to 1.

Hence, the range of area ratio is [½ ,1).

For n circles,

Equation of circle : $(x-r)^2+(y-r)^2=r^2$

Equation of diagonal line : $y=\frac{x}{n}$

Intersection point :

$$(x-r)^2+\left(\frac{x}{n}-r\right)^2=r^2$$

$$\Rightarrow (n^2+1)x^2-2xrn(n+1)+r^2n^2=0$$

By Sridhar Acharya method —

$$x_1=\frac{rn\left(n+1-\sqrt{2n}\right)}{n^2+1}$$

$$y_1=\frac{r\left(n+1-\sqrt{2n}\right)}{n^2+1}$$

Also, for n circles,

$$\text{Area of } A=\int_0^{\frac{rn(n+1-\sqrt{2n})}{n^2+1}}\left[r-\sqrt{r^2-(x-r)^2}-\frac{x}{n}\right]dx$$

Using integral calculator,

$$=\frac{r^2n\left(n-\sqrt{2n}+1\right)}{n^2+1}-\frac{r^2}{2}\left[\sin^{-1}\left(\frac{n-\sqrt{2n}\,(n)-1}{n^2+1}\right)+\frac{\pi}{2}\right.$$

$$\left.+\frac{1}{2}\sin\left(2\sin^{-1}\left(\frac{n-n\sqrt{2n}-1}{n^2+1}\right)\right)\right]-\frac{r^2n\left(n-\sqrt{2n}+1\right)^2}{2(n^2+1)^2}$$

When $n\longrightarrow\infty$,

$$\text{Area of } A=r^2-\frac{r^2}{2}\left(\frac{\pi}{2}\right)=r^2\left(1-\frac{\pi}{4}\right)$$

$$=0.215r^2$$

$$\therefore \frac{\text{Area of } A}{\text{Area of } (A+B)}\Rightarrow 1.$$

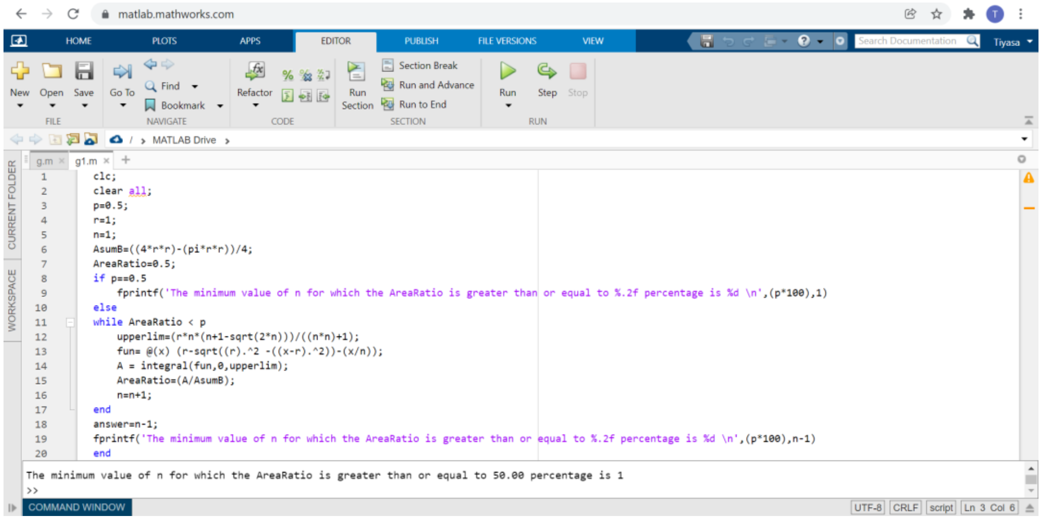
1.3) I have calculated the range using the integral used above with the required limits .

The general code I used for this question is written in MATLAB:

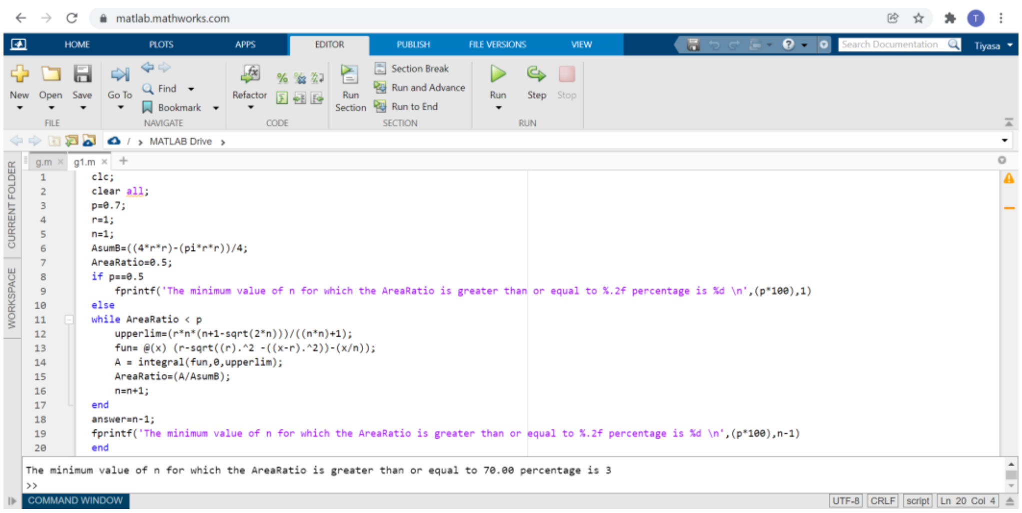
```
clc;

clear all;

x=input ('enter values in percentage:  ');

p=x/100;

r=1;

n=1;

AsumB=((4*r*r)-(pi*r*r))/4;

AreaRatio=0.5;

if p==0.5

   fprintf('The minimum value of n for which the AreaRatio is greater than or equal to %.2f percentage is
%d \n',(p*100),1)

else

while AreaRatio < p

   upperlim=(r*n*(n+1-sqrt(2*n)))/((n*n)+1);

   fun= @(x) (r-sqrt((r).^2 -((x-r).^2))-(x/n));

   A = integral(fun,0,upperlim);

   AreaRatio=(A/AsumB);

   n=n+1;

end

answer=n-1;

fprintf('The minimum value of n for which the AreaRatio is greater than or equal to %.2f percentage is
%d \n',(p*100),n-1)

end
```

Making the value of p to 0.5, 0.7,0.9,0.999,0.9999 will give the minimum value of ‘n’ for which the Area Ratio is greater than or equal to 50%, 70%,90%,99.9%,99.99% respectively.

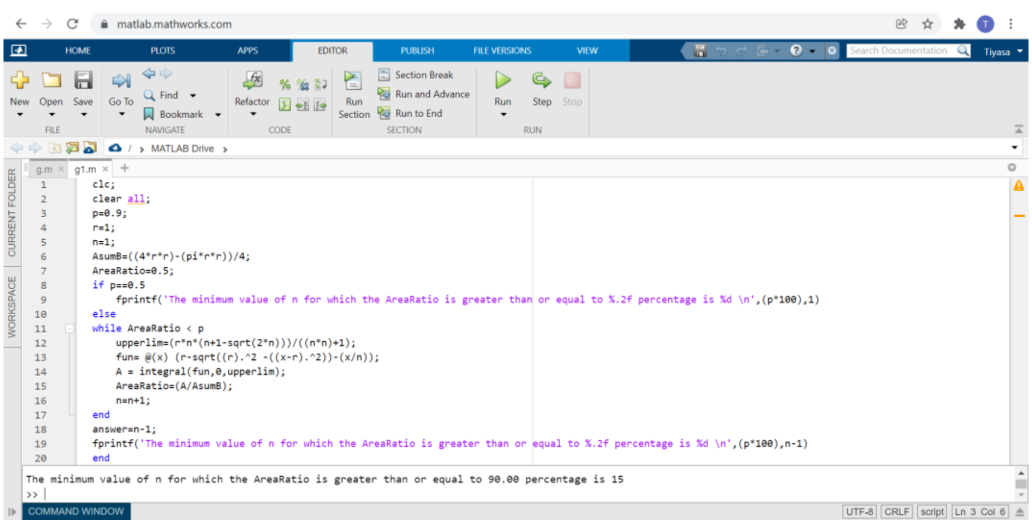
a) Minimum value of ‘n’ for which the Area Ratio is greater than or equal to 0.5 is 1 since the least value for the area ratio is ½ and it keeps increasing after that.



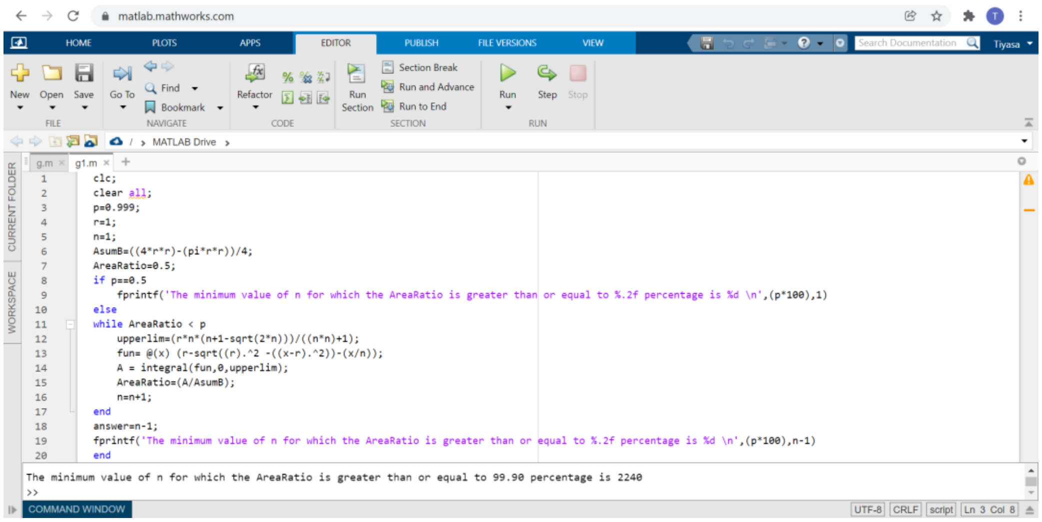
b) Minimum value of 'n' for which the Area Ratio is greater than or equal to 0.7 is 3.



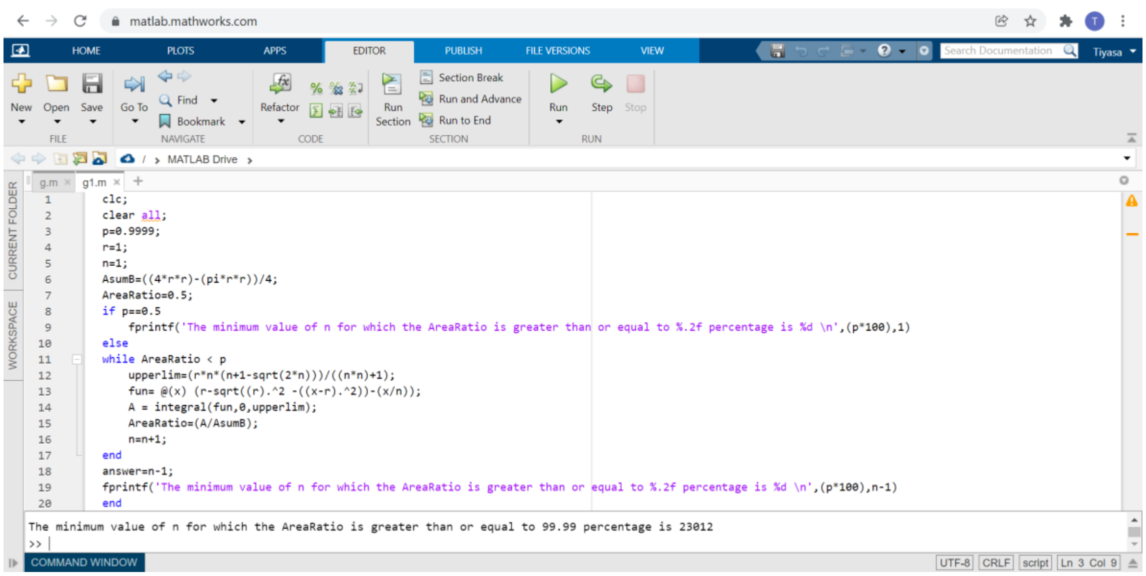
c) Minimum value of 'n' for which the Area Ratio is greater than or equal to 0.9 is 15.



d) Minimum value of ‘n’ for which the Area Ratio is greater than or equal to 0.999 is 2240.



e) Minimum value of ‘n’ for which the Area Ratio is greater than or equal to 0.9999 is 23012.



f) There is no value of n for which the area ratio becomes equal to or greater than 1.

Value of ‘n’ for which the Area Ratio is greater than or equal to 1 is tending to infinity since for area ratio to be 1, area of B should be equal to 0. But with increase in number of circles, area of B tends to 0, but doesn’t achieve 0. Also, area ratio can never be greater than 1 since, area of A <= area of (A+B) as area is a positive quantity.

2) For questions (2.1) and (2.2), I have used a general code written in JAVA:

To explain the code, I have solved (2.1) (c) , mathematically.

Here I have used Viterbi and Forward Algorithm.

For (2.1) (c), the string given to us is “ Easy Con”:

We will find most likely language transition String using "Viterbi Algorithm" and the probability for the language string using "Forward Algorithm".

Step 1:

Calculating probabilities for "E" & "S" using prior probabilities

$P(E) = P_s(E) \cdot P_e(Easy/E) = 0.6 \times 0.2 = 0.12$

$P(S) = P_s(S) \cdot P_e(Easy/S) = 0.4 \times 0.1 = 0.04$

Here, $\alpha(E) = 0.12 = P_s(E) \cdot P_e(Easy/E)$

$\alpha(S) = 0.04 = P_s(S) \cdot P_e(Easy/S)$

We are calculating $\alpha(E)$ & $\alpha(S)$ according to "Forward Algorithm" to find the probability of the Language string

Here. $P(E) > P(S)$.

Hence, 1st element of the language string will be "E".

Step 2

In step 2 we will use the $P(E)$, $P(S)$ $\alpha(E)$ & $\alpha(S)$ of step 1

$P(E) = P(E) \cdot P_e(Con/E) \cdot P_t(E/E)$

$= 0.12 \times 0.2 \times 0.3 = 0.0072$

$P(E) = P(S) \cdot P_e(Con/E) \cdot P_t(E/S) = 0.04 \times 0.2 \times 0.6$

$= 0.0048$

$P(S) = P(E) \cdot P_e(Con/S) \cdot P_t(S/E) = 0.12 \times 0.3 \times 0.7$

$= 0.0252$

$P(S) = P(S) \cdot P_e(Con/S) \cdot P_t(S/S) = 0.04 \times 0.3 \times 0.4$

$= 0.0048$

max. of two $P(E)$ is 0.0072

$\therefore P(E) = 0.0072$

Similarly, $P(S) = 0.0252$

Since, $P(S) > P(E)$

Hence, 2nd element of the language string will be "S"

Now, $\alpha(E) = \alpha(E) \cdot P_e(Con/E) P_t(S/E) + \alpha(S) P_e(Con/S) \cdot P_t(E/S)$

$= (0.12 \times 0.2 \times 0.3) + (0.04 \times 0.2 \times 0.6)$

$= 0.012$

$\alpha(S) = \alpha(E) \cdot P_e(Con/S) \cdot P_t(S/E) + \alpha(S) \cdot P_e(Con/S) \cdot P_t(S/S)$

$= (0.12 \times 0.3 \times 0.7) + (0.04 \times 0.3 \times 0.4)$

$= 0.03$

Ans: Probability for the language string is $\alpha(E) + \alpha(S)$

$= 0.042$

Ans: The most likely language transition string for "Easy Con" is "E S"

Final Answer: $\boxed{\text{"E S"} : 0.042}$

The code is :

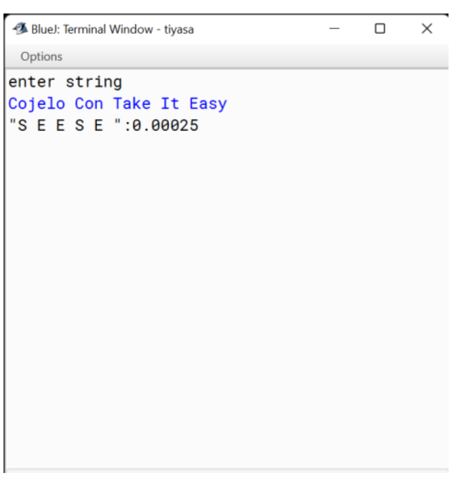
```
import java.util.Scanner;
public class jp_final
{
   Scanner sc=new Scanner (System.in);
   void main()
   {
      int l=0;
      int k=-1;
      System.out.println("enter string");
      String e=sc.nextLine();
      e=e.trim();
      for(int i=0;i<e.length();i++)
      {
         char cha=e.charAt(i);
         if(cha==' ')
         l++;                          // counting the number or words in the string
      }
      l=l+1;
      String[] s=new String[l];
      e=" "+e;
      for(int i=0;i<e.length();i++)
      {
         char ch=e.charAt(i);
           if(ch!=' ')
         s[k]=s[k]+ch;
         else
         { i++;
            k++;
```

```
            if(s[k]==null)
            s[k]=String.valueOf(e.charAt(i));       // storing the words in a string array


        }
    }
      String[] n=new String[5];
      n[0]="Cojelo";
      n[1]="Con";
      n[2]="Take";
      n[3]="It";
      n[4]="Easy";
      int[] input=new int[l];
      int c=0;
      for(int j=0;j<l;j++)
      {
         for(int i=0;i<5;i++)
         {
            if(s[j].equals(n[i]))       // checking whether the string array contains any or the words from the
given set.
            {
            input[c]=i;                // preparing an integer array  with the position of the matching words.
            c++;
         }
         }
         }
       double[][] Pt={{0.3,0.6},{0.7,0.4}}; //transition probability matrix
       double[] Ps={0.6,0.4};              // sentence start probability matrix
       double[][] Pe={{0.1,0.3},{0.2,0.3},{0.3,0.15},{0.2,0.15},{0.2,0.1}}; // emission probability  matrix
       double[][] dmP=new double[2][2];
```

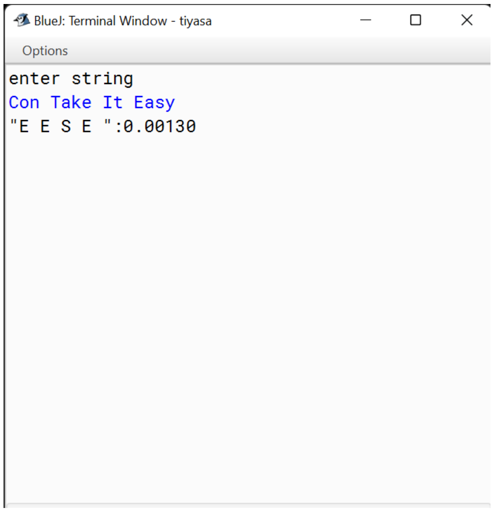
```
double[] stng=new double[l];        //stng- is most likely language transition string
int a=input[0];
double[] p=new double[2];
 p[0]=Ps[0]*Pe[a][0];//using the prior(sentence start) probability and the emission
probability,probability of english is calculated
 p[1]=Ps[1]*Pe[a][1];//using the prior(sentence start) probability and the emission
probability,probability of spanish is calculat
 double[] alpha=new double[2];
 alpha[0]=p[0];
 alpha[1]=p[1];
 if(p[0]>p[1])
 stng[0]=0;
 if(p[1]>p[0])
 stng[0]=1;
 //Here,we use Viterbi Algorithm to calculate the most likely language transition string
 //and Forward Algoritm to calculate the probability of the language string
 for(int i=1;i<l;i++)
  {
     a=input[i];
     dmP[0][0]=p[0]*Pe[a][0]*Pt[0][0];
     dmP[0][1]=p[1]*Pe[a][0]*Pt[0][1];
     dmP[1][0]=p[0]*Pe[a][1]*Pt[1][0];
     dmP[1][1]=p[1]*Pe[a][1]*Pt[1][1];
     p[0]=Math.max(dmP[0][0],dmP[0][1]);        // storing best probability for English
     p[1]=Math.max(dmP[1][0],dmP[1][1]);        // storing best probability for Spanish
     double  dummy=alpha[0]*Pe[a][0]*Pt[0][0]+alpha[1]*Pe[a][0]*Pt[0][1];
     alpha[1]=alpha[0]*Pe[a][1]*Pt[1][0]+alpha[1]*Pe[a][1]*Pt[1][1];
     alpha[0]=dummy;
     if(p[0]>p[1])
```

```
            stng[i]=0;
            if(p[1]>p[0])
            stng[i]=1;
           }
          double finalprob=alpha[0]+alpha[1]; // calculating the probability of language string according to Forward Algorithm
        System.out.print("\"") ;
          for ( int i=0;i<l;i++)
         {
           if (stng[i]==0)
      System.out.print("E ");
      else
       System.out.print("S ");
   }
   System.out.print("\"") ;
  System.out.println( ":"+ String.format("%.5f",finalprob));
}
}
```

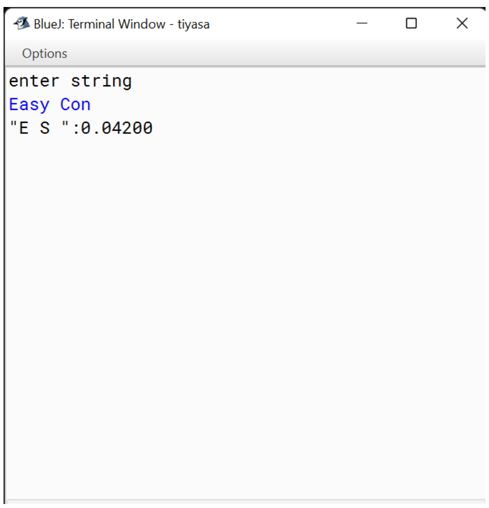
2.1.a) Output for the string “Cojelo Con Take It Easy” is “S E E S E” :0.00025



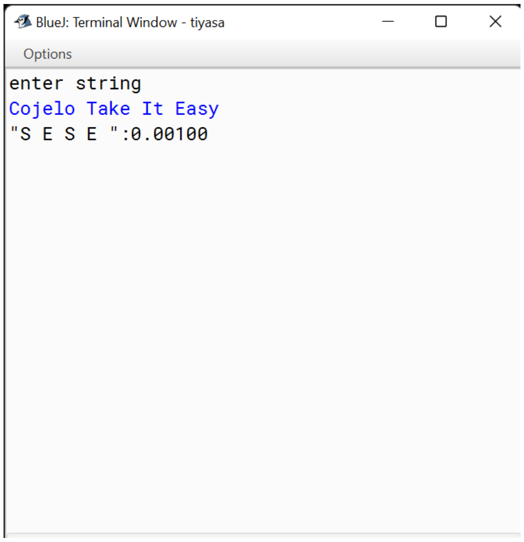
2.1.c) Output for the string “Con Take It Easy” is “E E S E” :0.00130



2.1.a) Output for the string “Easy Con” is “E S ” :0.04200



2.1.d) Output for the string “Cojelo Take It Easy” is “S E S E ” :0.00100


BlueJ: Terminal Window - tiyasa
Options
enter string
Cojelo Take It Easy
"S E S E ":0.00100