1.1 $R(N)$: is the probability that you get your own sandwich when there are N customers.

$R(1)$: when there is only 1 customer, that means there is only one order is waiting for you to pick up. And the probability of taking your order is 100% which means $R(1) = 1$

1.2. $R(2)$: when there are 2 customers, that means there are 2 orders. The first customer has 2 options, take his own sandwich or wrong sandwich. The probability of taking his own sandwich is 50% which means $R(2) = \frac{1}{2}$

1.3 When $N \geq 2$, probability of me gets my own sandwich is $\frac{1}{N}$

Case 1: if first person get second person's sandwich, then probability of me to get my own is $\frac{1}{N}(R(N-1))$

Case 2. if first person get third person's sandwich, the probability : $\frac{1}{N}(R(N-2))$

⋮

if first person get $(N-1)$ person's sandwich the probability : $\frac{1}{N}(R(2))$

if first person get my sandwich, then the probability is 0

$\therefore$ Sum up all the cases :

$$R(N) = \frac{1}{N} + \frac{1}{N}R(N-1) + \frac{1}{N}R(N-2) + \cdots + \frac{1}{N}R(2) + 0$$

$$= \frac{1}{N} \cdot R(N-1) + \frac{1}{N}R(N-2) + \cdots + \frac{1}{N}R(2) + \frac{1}{N}R(1)$$

$$R(N) = \frac{1}{N}\sum_{i=1}^{N-1} R(i)$$

1.4 From 1.3 we know that when $N \geq 2$

$$R(N) = \frac{1}{N} \sum_{i=1}^{N-1} R(i)$$

$\therefore$ when $N \to N+1$:

$$R(N) \to N \cdot R(N)$$
$$= N \cdot \frac{1}{N} \sum_{i=1}^{N-1} R(i)$$
$$= \sum_{i=1}^{N-1} R(i)$$

$$R(N+1) = \frac{1}{N+1} \sum_{i=1}^{N} R(i)$$

$$= \frac{1}{N+1} R(N) + \frac{1}{N+1} \sum_{i=1}^{N-1} R(i)$$

$$= \frac{1}{N+1} \cdot \left[N R(N) + R(N) \right]$$

$$= \frac{(N+1) \cdot R(N)}{N+1} = R(N)$$

1.5 $R(100) = \frac{1}{100} \sum_{i=1}^{99} R(i) = \frac{1}{2}$

$R(1000) = \frac{1}{1000} \sum_{i=1}^{999} R(i) = \frac{1}{2}$

1.6 if there is a extra sandwich

$Q(2)$: there are 2 customers and 3 sandwichs
the first person has 3 options, that means
the probability of him getting his own sandwich
is $\frac{1}{3}$, which means that will leave
$\frac{2}{3}$ probability for second person to get his
own sandwich.

$\therefore Q(2) = P[X_1 = 1, X_2 = 3] + P[X_1 = 3, X_2 = 2]$

$= P[X_1 = 2 \mid X_1 = 1] \, P[X_1 = 1] + P[X_2 = 2 \mid X_1 = 3] \, P[X_1 = 1]$

$= 1 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = \frac{2}{3}$

1.7. when $N \geq 2$, and there are $N+1$ sandwich, and probability for the first person is $\frac{1}{N+1}$. if the first person takes his own sandwich, then everyone can get their sandwich.

if the first person takes the second person's sandwich then we know from 1.3 that

$$\frac{1}{N+1} Q(N-1)$$

In $(N-1)$ situtation, and probability is $\frac{1}{N+1}$

the first person takes $(N-1)$ person's sandwich will be $\frac{1}{N+1} Q(2)$

$$Q(N) = \frac{1}{N+1} + \frac{1}{N+1} Q(N-1) + \cdots + \frac{1}{N+1} Q(2)$$

$$= \frac{1}{N+1} + \frac{1}{N+1} \sum_{i=2}^{N-1} Q(i)$$

$$Q(N+1) = \frac{1}{N+2} + \frac{1}{N+2} \sum_{i=2}^{N} Q(i)$$

$$= \frac{1}{N+2} + \frac{1}{N+2} \left[\sum_{i=2}^{N-1} Q(i) + Q(N) \right]$$

$$= \frac{1}{N+2} + \frac{1}{N+2} \left[(N+1) Q(N) - 1 + Q(N) \right]$$

$$= \frac{1}{N+2} + \frac{1}{N+2} \left[(N+2) Q(N) - 1 \right]$$

$$= \frac{1}{N+2} + Q(N) - \frac{1}{N+2}$$

$$= Q(N)$$

$$\therefore Q(100) = Q(99) = \cdots Q(2) = \frac{2}{3}$$

2.1 when G is a complete graph, $\alpha(G) = 1$
because every vertex is connected to an adjacent edge

when G is a cycle on n vertices $\alpha(G) = \frac{n}{2}$

2.2. When a vertex is ~~added~~ about to add in the set, the greedy algorithm will check every vertex and edge, to make sure if any edge match any vertex that already in I, and then it will check the new vertex. Thus, the algorithm make sure if there are independents before and after the adding. Thus, the output is a maximal independent set

2.3 We know that any complete G will give us a set of size 1

∴ the node is the middle has the size 1, because he is connecting every other nodes.
The rest of nodes have size of $n-1$, because if we want to make those independent set of size 1, we need to add them to every other node, except for the middle one, because the middle node already has independent set of size 1

2.4. We know that ~~any independent set~~ because of second step two vertices cannot match edge ~~connected~~. Now when we are adding vertex to an empty set I, the second step of this algorithm will always check if there is any edge that with both vertices in I. If there are not, keep adding. If there is, we delete one of the two vertices from I at random. Because of the second step, it is impossible for this graph to have two connected vertices, which meets an independent set's requirement. Thus, the output is an independent set.
However, since we are deleting a vertex in random, so there is a chance that the independent vertices ~~deleted~~ deleted

Thus, the independent set will not necessarily become the maximal

2.5 Due to the Binomial distribution, the probability of a vertex being add is p, and probability of won't be add is $(1-p)$. We know that Binomial distribution is $E[x] = N \cdot p$. Because it is step 1, so $N = |V|$. Thus, expected number of vertices in I after step 1 is $p|V|$

2.6 To form an edge, we have to have 2 vertices that connects ~~not~~ in set I. The probability of one vertex is added is p. So the probability of 2 vertices are added in set I is p^2. We know that Binomial distribution $E[x] = N \cdot p$. in this case, N will be $|E|$, number of edges. and p ~~will~~ will be p^2.
Thus. the expect number of edges in I after Step 1 is $p^2|E|$

2.7 From 2.5 we know that expect number of vertices $= p|V|$
2.6 we know that expect number of edges $= p^2|E|$
Now, after Step 2, we can say that the expected size of I is expect number of vertices $-$ expected number of edges which is $= p|V| - p^2|E|$

But, since we need to delete a point in step 2. And when we delete a ~~point~~ vertex, multiple edges may also being deleted. Which means number of vertices that will be deleted $\leq p^2|E|$. Thus, expected size can be larger.

Thus, the expected ~~number~~ size of I after Step 2 $\geq p|V| - p^2|E|$

2-8 $\alpha(G) \geq \frac{|V|^2}{4|E|}$

we know from 2-7, expect size of set $I \geq p|V| - p^2|E|$

And when we solve it using derivative : $0 = |V| - 2p|E|$

$$p = \frac{|V|}{2|E|}$$

Thus, when $p = \frac{|V|}{2|E|}$, produces the largest expected independent set

When we plug in $p = \frac{|V|}{2|E|}$ in 2-7 equation, it will be

$\frac{|V|^2}{2|E|} - \frac{|V|^2}{4|E|}$, where $|E| = \frac{|V|^2}{4|E|}$.

We know that the algorithm will always give us independent set, but may be not maximal independent set.

$\therefore a(G) \geq \frac{|V|^2}{4|E|}$

3.1 $T(n+2) = \frac{1}{2}[T(n+1) + T(n)]$

$$F(x) = \sum_{n=0}^{\infty} T(n)x^n \qquad \therefore T(0) = a$$

$$T(1)\cdot x^1 = b \cdot x^1$$

$$T(2)\, x^2 = \frac{1}{2}(T(1) + T(0))\, x^2$$

$$\dots\dots$$

$$F(x) = \sum_{n=0}^{\infty} T(n)x^n = [a + bx + \underbrace{\tfrac{1}{2}(T(1)x^2 \dots)}_{H} + \underbrace{\tfrac{1}{2}(T(0)x^2 \dots)}_{I}]$$

$$H = \frac{x}{2}(T(1)x^1 \dots)$$

$$= \frac{x}{2}\left[\sum_{n=0}^{\infty} T(n)x^n - T(0)\right]$$

$$= \frac{x}{2}[F(x) - a]$$

$$I = \frac{x^2}{2}(T(0) + T(1)x \dots)$$

$$= \frac{x^2}{2}\left[\sum_{n=0}^{\infty} T(n)x^n\right]$$

$$= \frac{x^2}{2}F(x)$$

$$F(x) = a + bx + \frac{x}{2}[F(x) - a] + \frac{x^2}{2}F(x)$$

$$2F(x) = 2a + 2bx + xF(x) - ax + x^2F(x)$$

$$ax - 2bx - 2a = F(x)[x^2 + x - 2]$$

3.2 From 3.1, we know that

$$ax - 2bx - 2a = F(x)[x^2 + x - 2]$$

$$\therefore F(x) = \frac{ax - 2bx - 2a}{x^2 + x - 2}$$

3.3 We know that $F(x) = \dfrac{ax-2bx-2a}{x^2+x-2}$

$$\therefore F(x) = \frac{A}{x-1} + \frac{B}{x+2}$$

Since $(x-1)(x+2) = x^2+x-2$

$\therefore$ we need to make $A(x+2)+B(x-1) = ax-2bx-2a$

$\cancel{\therefore ax-2bx-2a = Ax+2A+Bx-B}$

when $x=-2$ $\qquad 0 + -3B = -2a+4b-2a$

$$B = \frac{4b-4a}{-3} = \frac{-4b+4a}{3}$$

when $x=1$ $\qquad 3A+0 = a-2b-2a$

$$A = \frac{(-a-2b)}{3}$$

For A, $\dfrac{\frac{-a-2b}{3}}{x-1} = \dfrac{-a-2b}{3} \cdot \dfrac{1}{x-1} = \dfrac{a+2b}{3} \cdot \dfrac{1}{1-x}$

$$= \frac{a+2b}{3} \sum_{n=0}^{\infty} x^n$$

For B, $\dfrac{\frac{4b-4a}{-3}}{x+2} = \dfrac{4b-4a}{-3} \cdot \dfrac{1}{x+2} = -\dfrac{4b+4a}{3} \cdot \dfrac{1}{x+2}$

$$= \frac{2a-2b}{3} \cdot \frac{1}{\frac{1}{2}x+1} \rightarrow \frac{1}{1-(-\frac{1}{2}x)}$$

$$= \frac{2a-2b}{3} \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n$$

3.4 $F(x) = \frac{A}{x-1} + \frac{B}{x+2}$

$$= \left(\frac{a+2b}{3}\right)\sum_{n=0}^{\infty} x^n + \left(\frac{2a-2b}{3}\right)\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n$$

$$= \left(\frac{a+2b}{3}\right)\sum_{n=0}^{\infty} x^n + \left(\frac{2a-2b}{3}\right)\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n \cdot x^n$$

$$= \sum_{n=0}^{\infty}\left(\frac{a+2b}{3} + \left(\frac{2a-2b}{3}\right)\cdot\left(-\frac{1}{2}\right)^n\right)\cdot x^n$$

$$= \sum_{n=0}^{\infty} T(n)\, x^n$$

3.5 $T(n)$, for $n \geq 2$

$$T(n) = -A + \frac{B}{2}\left(-\frac{1}{2}\right)^n$$

$$= \frac{a+2b}{3} + \frac{2a-2b}{3}\left(-\frac{1}{2}\right)^n$$

3.6 $\lim_{n\to\infty} T(n) = \frac{a+2b}{3} + \frac{2a-2b}{3}\left(-\frac{1}{2}\right)^n$

since $\left(-\frac{1}{2}\right)^n$ goes to ∞ $= 0$

$$\lim_{n\to\infty} T(n) = \frac{a+2b}{3} + \frac{2a-2b}{3}\cdot 0$$

$$= \frac{a+2b}{3}$$

Bouns problem 1:

In the original setup, assume we have 5 people. The first person have all the sandwich options, including his. Thus, he should have 100% chance of getting his own sandwich. For the second person, he should have 50% chance of getting his sandwich, since the first person took one sandwich from all sandwichs. For the third person, he should have 1/3 chance of getting his sandwich, since he has one less option compare with the second person. For the fourth person, he have 1/4 chance, because the same reason as before. Fifth person have 1/5 chance. Thus, in the line of 5, the expect number of people who should get their own sandwich is 1 + 1/2 + 1/3 + 1/4 + 1/5. Now, for the line of 100 people, the expect number of people who should get their own sandwich is 1 + 1/2 + 1/3 + 1/4 + 1/5 + … + 1/99 + 1/100

Bouns problem 3:

$T(n) = 1/n(T(n)+T(n-1)+T(n-2)+\ldots+T(1)+T(0))$

$$\lim_{n \to \infty} 1/n(T(n)+T(n-1)+T(n-2)+\ldots+T(1)+T(0)) = (T(n)+T(n-1)+T(n-2)+\ldots+T(1)+T(0))/n = \infty/\infty = 1$$