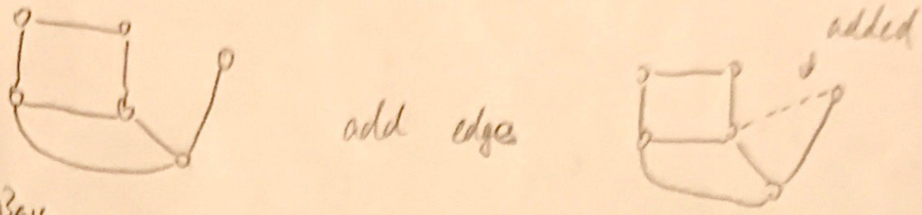
1-1



Base: If $V=2$, $e=1$, $f=1$

Euler formula: $v-e+f = 2-1+1 = 2$ ✓

IH: $V-e+f=2$ is true for every planar graph

let's say we have a tree, G

$V = e+1$, $f=1$

$V-e+f = (e+1)-e+1 = 2$

Let e_1 be the leaf of G

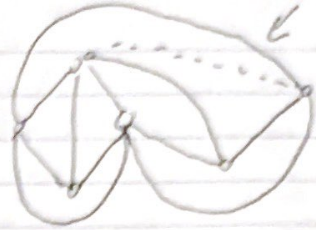
Then $(G-e_1)$ has $(V-1)$ edges, $(f-1)$ faces

Put it in IH.

$$V-e+f \to V-(e-1)+(f-1)$$
$$= V-e\cancel{+1}+f\cancel{-1}$$
$$= V-e+f = 2 \checkmark$$

$\therefore$ $V-e+f$ is true for every planar graph

1.2 For Figure 3



this area should add an edge

Euler formular:

$V - e + f = 2$

$f = 2 + e - v$

We know that a triangulation looks like this: 

which has $1f$, $3v$, $3e$

By contradiction, assume we have 

which has $1f$, $4v$, $4e$

if this is an edge maximum planar

then, $f = 2 + e - v$

$1 = 2 + 4 - \cancel{4}$

$1 = 2$ X

$\therefore$  is not an edge maximum planar

and miss 1 edge.

$\therefore$ By contridiction, any planar triangulation, every face is a triangle and surround by 3 edges

Also, $f = 2 + e - v$

Since we have connected planar graph

$2e \geq 3f$

$2e \geq 3(2 + e - v)$

$2e \geq 6 + 3e - 3v$ $\therefore$ surround by 3 edge

$e \leq 3v - 6$

1.3 $v - e + f = 2$

$f = 2 + e - v$

for every edge, we have 2 face
for every face, we have 3 vertex

$\therefore 2e \geq 3f$

$2e \geq 3(2 + e - v)$

$2e \geq 6 + 3e - 3v$

$e \leq -6 + 3v$

$e \leq 3v - 6$

1.4 First, we say ave deg of vertex in triangulation greater or equal to 6.

Ave deg : $\frac{2|E|}{|V|}$

assume: $\frac{2|E|}{|V|} \geq 6$

we know from 1.3, $|E| = 3|V| - 6$

$\therefore \frac{2(3|V| - 6)}{|V|} \geq 6$

$\frac{6|V| - 12}{|V|} \geq 6$

$6 - \frac{12}{|V|} \geq 6$

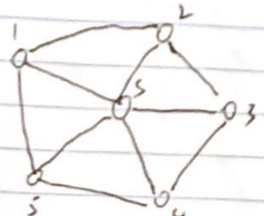
In this case, $\frac{12}{|V|}$ must be negative. However, $|V|$ cannot be negative.

$\therefore$ By contradiction, ave deg of vertex in triangulation will always < 6

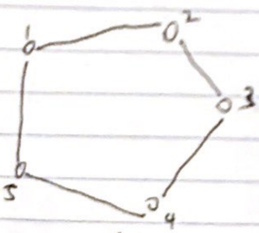
1.5 Base case:

~~Stas~~ since graph has 1, 2, 3, 4, 5 or 6 vertices the result is immediate, then we only need to prove when we have 7 or more vertices

let's say we have a graph G has $V = k+1$ vertices and G' that been removed a ~~vertex~~ vertex s, which means G' only have k vertices.



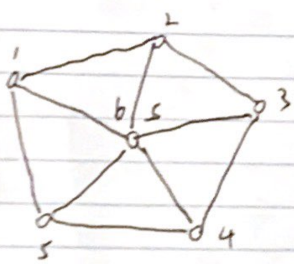
G



G'

Since all graphs with k vertices can be colored in 6 colors, ~~so~~ then G' can also colored using only 6 colors

By putting the vertex s back to G'



and we can color vertex s to color 6

It is clear that we can color S using different color than its neighbors.

By induction, all planar graph with k vertices can be colored using 6 colors, then K+1 vertices can also be colored at the same way

2.1 if N is number of vertices

~~$N \leq 4$~~ $N = 4$

Every time, you make a match you cannot match the same one for the next vertice

$\therefore\ 4 \cdot (4-1) \cdot (4-1-1) \cdot (4-1-1-1)$

$= 4!$

$\therefore\ N!$

~~to maximum~~

~~m = number of edges in A~~

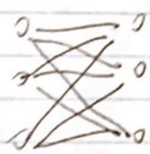
~~n = number of edges in B~~

~~m must equal n~~

or ~~m as close as possible to n~~

2.2 If N is number of vertice on each side of A and B

if $N = 3$, each vertice on one side will have N edges and vertices on the other side share the same edge with first side



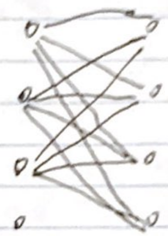
$\therefore$ maximum edges : $N \times N$

$= N^2$

2.3 $N^2 - N$ edges

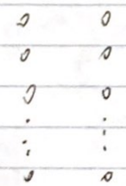
let's make $N = 4$

$N^2 - N = 12$



Clearly, this is not a perfect matching.

2.4 Starting at



if we randomly put edges.

probability of having a perfect match will be $\dfrac{N!}{\binom{N^2}{N}}$

if $N = 4$

total match: 16

$\binom{N^2}{N} = \binom{16}{4} = 1820$, perfect match: $\dfrac{4!}{1820}$

2.5 ~~$E[X^3]$~~ ~~$E[X]^2$~~

2.5 If $|E|$ edges added between A & B, the perfect match is: $\binom{|E|}{N}$

~~[illegible]~~ From 2.4, we know perfect match is $\dfrac{N!}{\binom{N^2}{N}}$

$\therefore$ expect number of perfect match is

$$\binom{|E|}{N} \cdot \frac{N!}{\binom{N^2}{N}} = \frac{N!\binom{|E|}{N}}{\binom{N^2}{N}}$$

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$$\binom{|E|}{N} \cdot \frac{N!}{\binom{N^2}{N}} = \frac{N!\binom{|E|}{N}}{\binom{N^2}{N}}$$

2.6. if $|E| = 3N$

$$E[X_S] = \frac{N!\binom{3N}{N}}{\binom{N^2}{N}}$$

$$= \frac{\cancel{\sqrt{2\pi N}}\left(\frac{N}{e}\right)^N \frac{1}{\cancel{\sqrt{2\pi N}}} \sqrt{\frac{3}{2}}\left(\frac{3^3}{2^3}\right)^N}{\frac{1}{\sqrt{2\pi}Ne}(eN)^N}$$

$$= \frac{\left(\frac{N}{e}\right)^N \cdot \sqrt{\frac{3}{2}}\left(\frac{27}{4}\right)^N \cdot \sqrt{2\pi}Ne}{(eN)^N}$$

$$\lim_{N\to\infty} \frac{\left(\frac{N}{e}\right)^N \sqrt{\frac{3}{2}}\left(\frac{27}{4}\right)^N \cdot \sqrt{2\pi}Ne}{(eN)^N} = \frac{\sqrt{\frac{3}{2}}\left(\frac{27}{4}\right)^N \cdot \sqrt{2\pi} \cdot Ne}{e^{2N}}$$

$$= \lim_{N\to\infty} \sqrt{\frac{3}{2}}\left(\frac{27}{4e^2}\right)^N \cdot \sqrt{2\pi} \cdot Ne$$

$\because 4e^2 > 27 \quad \therefore \left(\frac{27}{4e^2}\right)^\infty$ goes to 0

$\therefore \sqrt{\frac{3}{2}} \cdot 0 \cdot \sqrt{2\pi}Ne$ goes to 0 $\quad\therefore$ As $N \to \infty$ expect matching goes to ∞

2.7 Same as 2.6.

if $|E| = 4N$ $\qquad \dfrac{N! \binom{4N}{N}}{\binom{N^2}{N}}$

$$= \frac{\cancel{\sqrt{2\pi N}} \left(\frac{N}{e}\right)^N \frac{1}{\cancel{\sqrt{2\pi N}}} \sqrt{\frac{4}{3}} \left(\frac{4^4}{3^3}\right)^N}{\frac{1}{\sqrt{2\pi} N e} (eN)^N}$$

$$= \frac{\left(\frac{N}{e}\right)^N \cdot \sqrt{\frac{4}{3}} \left(\frac{256}{27}\right)^N \cdot \sqrt{2\pi}\, Ne}{(eN)^N}$$

$$= \sqrt{\frac{4}{3}} \left(\frac{256}{27}\right)^N \cdot \sqrt{2\pi}\, Ne \cdot \frac{1}{e^{2N}}$$

$$= \sqrt{\frac{4}{3}} \left(\frac{256}{27e^2}\right)^N \cdot \sqrt{2\pi} \cdot Ne$$

$$\lim_{N\to\infty} \sqrt{\frac{4}{3}} \left(\frac{256}{27e^2}\right)^N \cdot \sqrt{2\pi} \cdot Ne$$

$\because 27e^2 < 256$

$\therefore \left(\frac{256}{27e^2}\right)^\infty$ goes to ∞

$\therefore \sqrt{\frac{4}{3}} \cdot \infty \cdot \sqrt{2\pi} \cdot Ne$ goes to ∞

$\therefore$ As $N \to \infty$, the expect matching numbers $\to \infty$

3.0) $P(E_1 \text{ or } E_2 \text{ or } \dots \text{ or } E_m) \le \sum_{n=1}^{m} P(E_n)$

For $m=1$, $P(E_1) = P(E_1)$ ✓

For $m=2$, $P(E_1 \text{ or } E_2) = P(E_1) + P(E_2) - P(E_1 \text{ and } E_2)$

$\le P(E_1) + P(E_2)$ As $P(E_1 \text{ and } E_2)$ ✓

Hypothesis. $P(E_1 \text{ or } E_2 \text{ or } \dots \text{ or } E_m) \le \sum_{m=1}^{m} P(E_m)$

if $m = k+1$

$P(E_1 \text{ or } E_2 \text{ or} \dots E_k \text{ or } E_{k+1}) = P(E_1 \text{ or } E_2 \dots \text{or } E_k) + P(E_{k+1}) - P((E_1 \text{ or} \dots P E_k) \text{ And } P(E_{k+1}))$

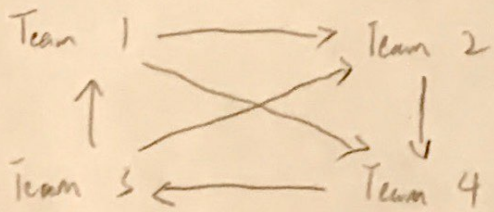
$\therefore P(E_1 \text{ or } E_2 \text{ or} \dots E_k \text{ or } E_{k+1}) \le P(E_1) + P(E_2) + \dots P(E_k) + P(E_{k+1})$ ✓

$\le \sum_{m=1}^{k+1} P(E_{k+1})$

$\therefore P(E_1 \text{ or } E_2 \text{ or} \dots E_n) \le \sum_{m=1}^{m} P(E_m)$

3.1) $\frac{N(N-1)}{2}$

3.2)



3.3) $p = \frac{1}{2}$, since game is decided by coin

$\sum_{k}^{n-1} \binom{n-1}{k} \cdot p^k \cdot (1-p)^{n-1-k}$

3.4 $P(\exists \text{ k-winner}) = N \cdot \sum_{k}^{n-1} \binom{n-1}{k} \cdot p^k \cdot (1-p)^{n-1-k}$

$= N \cdot \sum_{k}^{n-1} \binom{n-1}{k} \cdot \frac{1}{2}^k \left(\frac{1}{2}\right)^{n-1-k}$

3.5 By using the code, $k = 62$

if $N = 100$, $P(x_1 \ldots x_{100}) = 100 \cdot \sum_{k}^{99} \binom{99}{k} \cdot \left(\frac{1}{2}\right)^{99} < 1$

when $k = 62$, the result will be 0.772

which is < 1

$\therefore$ smallest $k = 62$

3.6 From 3.5, we know that when $N = 100$, $k = 62$

the probability of having k-winner is 0.772.

That means the probability of not having k-winner

is $1 - 0.772 = 0.228$

$\therefore$ we can argue that there exist possible tournaments with no k-winner, since the probaility of no k-winner is 0.228.

Bonus

Question 2 Bonus
1. we know that 3<α<4, and I assume α = 3.5

2. From 3.6, we know that when |E| = 3N, the expected number of matchings goes to 0.
so the the probability that has single line is very close to 0, but not 0, because our expect number is 0, and sometimes things get out of our expectation.

Question 3 Bonus

From 3.4, we know that $p(\text{exist k-winner}) = N \sum_{k}^{N-1} \binom{N-1}{k} * (1/2)^{\wedge}(N-1)$

if N goes to ∞, that mean N-1 also goes to ∞

which means $p(\text{exist k-winner}) = \infty * \sum_{k}^{\infty} \binom{\infty}{k} * (1/2)^{\wedge} \infty$

we know that $(1/2)^{\wedge} \infty = 1^{\wedge} \infty / 2^{\wedge} \infty = 1/2^{\wedge}\infty$ which goes to 0

so the equation = $\infty * \sum_{k}^{\infty} \binom{\infty}{k} * 0$ which = 0

so there exist tournaments without αN-winners, for all sufficiently large N