1 Obligations in Foreign Currencies

A European company has an obligation that it must pay in USD (*e.g.*, an insurance claim that is to be paid out in the US). The amount of money that must be paid is a nonlinear function $f(p)$ of the exchange rate from EUR to USD, p, which fluctuates over time. Specifically, 1 EUR $= p$ USD, and p can be interpreted as the price of 1 EUR in USD. This obligation may need to be paid at any moment, so the company must hold cash reserves with at least $f(p)$ in USD at all times.

The company could fulfill this obligation by simply always holding $\max_p f(p)$ USD in reserves, but it prefers to keep as much of its cash in EUR as possible. Thus, the company continuously rebalances its reserves between the two currencies, ensuring it always has $f(p)$ USD as p changes, while keeping the rest of its cash in EUR. For this problem, we assume that f is nonnegative, strictly increasing, and differentiable on its domain; *i.e.*, $f(p) \geq 0$ and $f'(p) > 0$ for $p \geq 0$. (Note that we do *not* assume that f is convex or concave.)

a) For a given price p, the company must hold $f(p)$ USD in reserves. Assume that the exchange rate is p_1 and the company holds both $f(p_1)$ USD and some amount of EUR. If the exchange changes from p_1 to p_2, show that the amount of EUR the company needs to buy (or sell) is given by
$$\int_{p_1}^{p_2} \frac{f'(p)}{p} dp.$$
Hint: if the price increases from p to $p+h$, we must sell some EUR to increase our USD position by $f(p+h) - f(p)$.

If the company trades using this strategy, it must have enough EUR in its reserves to buy USD as the exchange rate goes up. At some price p, it would then need to hold at least
$$g(p) = \int_p^\infty \frac{f'(q)}{q} dq$$
in EUR to execute this strategy, as p may—but hopefully does not—increase without bound in the future. (Assume that f is such that $g(p)$ is finite for $p \geq 0$.) If the company's reserves contain exactly $f(p)$ USD and $g(p)$ EUR, their value in USD is
$$V(p) = f(p) + pg(p).$$

b) Show that we can equivalently write V as
$$V(p) = V(0) + \int_0^p g(q) dq.$$

c) Show that $V(p)$ is a nonnegative, nondecreasing, concave function on $p > 0$.

Equipped with reserves $(f(p), g(p))$ for a given exchange rate p, it is natural to consider the set of reserves that allow the company to cover its obligation as the price p varies. For this, we will consider an object similar to the epigraph of a function, called the *dominating reserves*, defined as the set
$$S = \{x \in \mathbf{R}^2 \mid x_1 \geq f(p),\ x_2 \geq g(p) \text{ for some } p \geq 0\}.$$
(In finance, we sometimes say that a vector x *dominates* a vector y whenever $x \succeq y$.) We will show that the set S is a convex set and can therefore be easily optimized over.

d) Find a (simple) function φ such that $x_2 \geq \varphi(x_1)$ if, and only if, $x \in S$.

e) Show that φ is convex. Argue that this shows that S is a convex set.

This result allows the company to incorporate this constraint into standard portfolio allocation techniques across all its assets (*e.g.*, using Markowitz portfolio optimization). Interestingly, this new function φ and the portfolio value function V are intimately related.

f) Show that φ^{-1} is the conjugate of $-V$ with negated arguments, *i.e.*, $\varphi^{-1}(y) = (-V)^*(-y)$.