Exercise 2.16 (Martingale formulation of Bellman's optimality principle) Suppose your winning per unit stake on game n are ϵ_n, where the ϵ_n are i.i.d. r.v.s with

$$\mathsf{P}\{\epsilon_n = 1\} = p = 1 - \mathsf{P}\{\epsilon_n = -1\},$$

with $p > 1/2$. Your bet $\cancel{\alpha_n}$ β_n on game n must lie between 0 and Z_{n-1}, your capital at time $n-1$. Your object is to maximise your 'interest rate' $\mathsf{E}\log(Z_N/Z_0)$, where $N =$ *length of the game* is finite and Z_0 is a given constant. Let $\mathcal{F}_n = \sigma(\epsilon_1, \dots, \epsilon_n)$ be your 'history' upto time n. Let $\{\alpha_n\}_n$ be an admissible strategy, i.o.w. a predictable sequence. Show that $\log(Z_n) - n\alpha$ is a supermartingale with α the entropy given by

$$\alpha = p\log p + (1-p)\log(1-p) + \log 2.$$

Hence $\mathsf{E}\log(Z_n/Z_0) \le N\alpha$. Show also that for some strategy $\log(Z_n) - n\alpha$ is a martingale. What is the best strategy?

$0 \le \beta_n \le Z_{n-1}$. By definition $\log(Z_n) - n\alpha$ is a supermartingale iff

$$\mathbb{E}\left[\log(Z_n) - n\alpha \mid Z_{n-1}, \dots, Z_0\right] \le \log(Z_{n-1}) - (n-1)\alpha.$$

$$\iff \mathbb{E}\left[\log(Z_n) \mid Z_{n-1}, \dots, Z_0\right] \le \log(Z_{n-1}) + \alpha$$

$$\iff \text{(1)}\ \mathbb{E}\left[\log\left(\frac{Z_n}{Z_{n-1}}\right) \mid Z_{n-1}, \dots, Z_0\right] \le \alpha.$$

We know that $0 \le Z_n = \epsilon_n\beta_n + Z_{n-1} \le 2Z_{n-1}$, let γ_n be the proportion of the portfolio being bet, then:

$$\mathbb{E}\left[\log\left(\frac{Z_n}{Z_{n-1}}\right) \mid \sigma_{n-1}\right] = p\log\left(\overbrace{\frac{\beta_n + Z_{n-1}}{Z_{n-1}}}^{1+\gamma}\right) + (1-p)\log\left(\overbrace{\frac{-\beta_n + Z_{n-1}}{Z_{n-1}}}^{1-\gamma}\right)$$

where $0 \le \beta_n \le Z_{n-1}$.

$$= p\log(1+\gamma) + (1-p)\log(1-\gamma)$$

$$= p\log\left(\frac{1+\gamma}{2}\right) + (1-p)\log\left(\frac{1-\gamma}{2}\right) + \log 2$$

Max of cross entropy.

$$\le p\log p + (1-p)\log(1-p) + \log 2.$$

(achieved by picking $2p = 1 + \gamma = \frac{\beta_n + Z_{n-1}}{Z_{n-1}} \Rightarrow \boxed{\beta_n = Z_{n-1}(2p-1)}$).