Constant Function Market Makers and Friends

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Outline

Trading assets

- Many possible ways of buying/selling goods!
- Order books, auctions, *etc.*
- We will focus on a (computationally) simple mechanism
- (When dealing with blockchains: simple is better)

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- But we clearly can't allow any trade!

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- These are the *constant function market makers* (CFMMs)

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- Easy/inexpensive to check if a trade makes sense: evaluate ψ
- Most importantly, ψ is often simple to write down
- Leads to a very general, practical theory

A quick aside

- We will only discuss the 2 asset case
- No fees (‘path independent’)
- Everything generalizes beautifully to n assets
- But many fee (‘path deficient’) questions remain
- See: ‘Improved Price Oracles [...],’ Angeris and Chitra, 2020

Outline

Example: constant sum

- The simplest example of a trading function is the 'constant sum':
$$\psi(R_a, R_b) = R_a + R_b$$

- In other words, a trade Δ_a, Δ_b is accepted only when
$$\psi(R_a + \Delta_a, R_b + \Delta_b) = \psi(R_a, R_b)$$
or
$$(R_a + \Delta_a) + (R_b + \Delta_b) = R_a + R_b$$
i.e., if, and only if
$$\Delta_a = -\Delta_b.$$

Example: constant sum (cont.)

- To get out one unit of a, we have to put in one unit of b
- In other words, the *price* of a relative to b is always 1
- Q: if 1 unit of a is worth more than 1 unit of b, what happens?

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- Think about it, we will come back to this :)

Example: constant product

- A (very!) popular trading function is the 'constant product':

$$\psi(R_a, R_b) = R_a R_b$$

(originally Uniswap, later adopted throughout)

- A little bit of algebra gives

$$\Delta_a = -\frac{R_a \Delta_b}{R_b + \Delta_b}.$$

Example: constant product (cont.)

- More complicated...
- But, we can still compute the marginal price at some reserves!

$$-\frac{d\Delta_a}{d\Delta_b} = \frac{R_b}{R_a}$$

- We can see the marginal price is *adaptive*
- More of R_a relative to R_b: price of a vs. b goes down
- (And vice versa)

Examples: continued

- Many more!
- Constant mean market (Balancer):

  $$\psi(R_a, R_b) = R_a^w R_b^{1-w},$$

  where $0 < w < 1$ (with $w = 1/2$ equivalent constant product.)
- Curve:

  $$\psi(R_a, R_b) = (R_a + R_b) - \alpha \frac{1}{R_a R_b},$$

  where $\alpha \geq 0$ (with $\alpha = 0$ is equivalent to constant sum)

Examples: continued (cont.)

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- What can we say about them?
- Can we give geometric interpretations?
- (obviously, the answer is yes :)

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The level sets

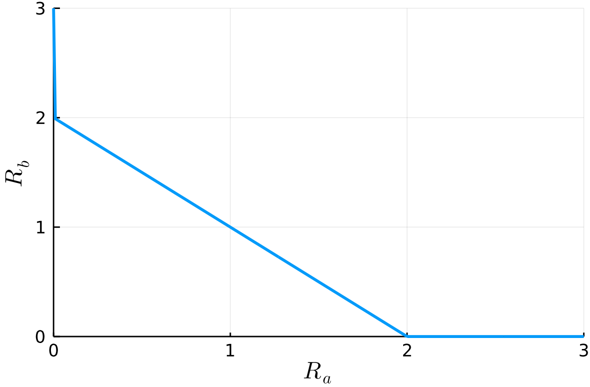
- We can view the trading function as its *level sets*
- *i.e.*, what are the reserves R_a, R_b that satisfy

$$\psi(R_a, R_b) = k$$

for some constant k?

Level sets (constant sum)

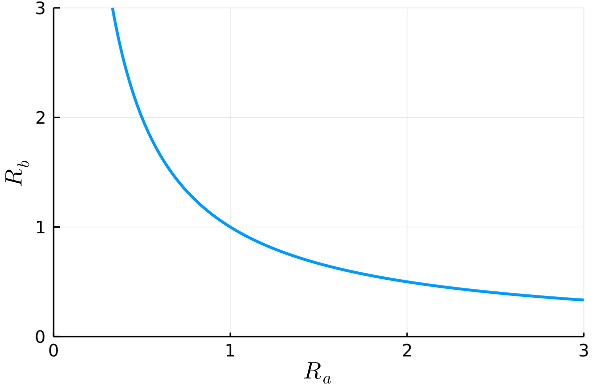
- The graph for $k = 2$, constant sum



- (Note that $R_a \geq 2$ means $R_b = 0$)

Level sets (constant product)

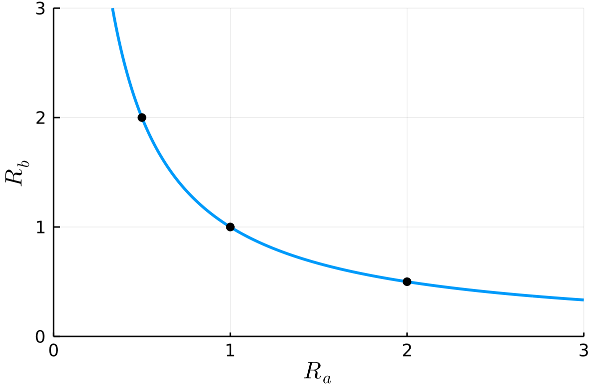
- The graph for $k = 1$, constant product/Unsiwap



- Simply a hyperbola

Why is this useful?

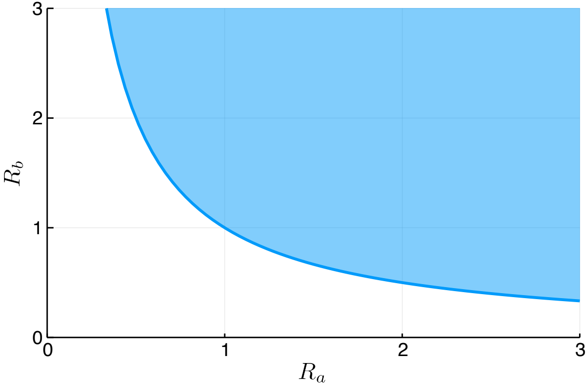
- Traders can trade against reserves, so long as reserves *after the trade* remain on the level set



- (An alternate characterization of ψ)

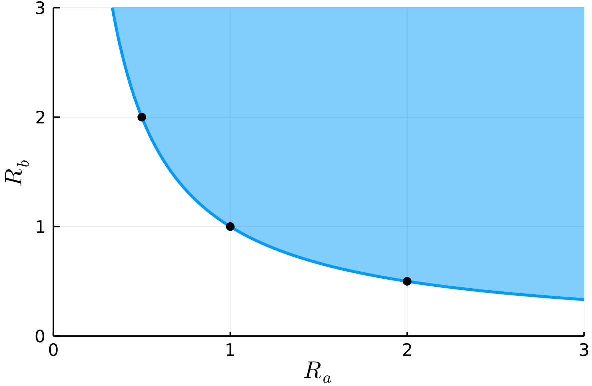
The reachable set

- We define the *reachable set* S as all points above/to the right



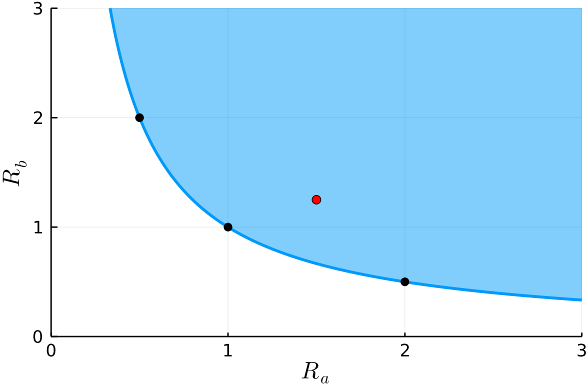
The reachable set (cont.)

- Since we've only added points, all of the reserves that were reachable are still 'reachable'



The reachable set (cont.)

- But, no rational trader will ever pick a point inside of the set!



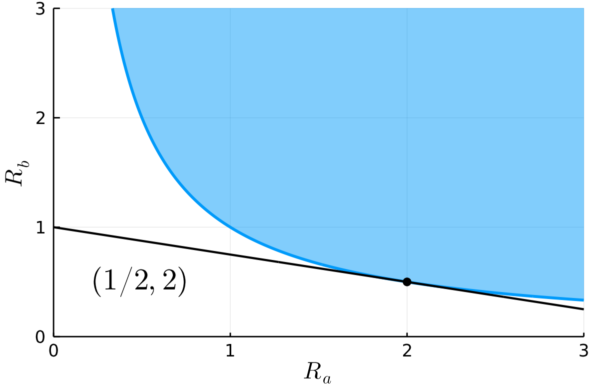
- (The set of all such points is called the *dominated interior*)

Properties of the reachable set

- The resulting set S is convex ($\approx$ easy to optimize over)
  In fact, all known CFMMs have convex reachable sets
- Essentially unique over all trading functions for rational agents
- Easy to write for most CFMMs
- Leads to simple definitions and proofs!

Marginal price

- For example, the *price* at some reserves is the tangent line



- No-arbitrage tells us these are the reserves at a given price

Equivalence

- It also gives us a useful way of characterizing equivalence!
- Two CFMMs are equivalent if, and only if, their reachable sets are equal
- Any rational agent will perform exactly the same trades on either
- (They will always choose points on the boundary!)

One more thing...

- It will additionally give us one more (very important) thing :)

Outline

Arbitrageurs

- We will introduce one more agent: the arbitrageur
- An external market has some price p_a and p_b for assets a, b
- The arbitrageur seeks to maximize profit by trading between:
  1. The CFMM
  2. The market
- The agent is locked in a zero-sum battle against liquidity providers!

Liquidity providers

- Given the market prices p_a, p_b, and reserves R_a, R_b, liquidity providers' value in reserves is:

$$V(p) = p_a R_a + p_b R_b$$

- But, note that no-arbitrage implies that the CFMM price needs to be equal to p!
- In other words, for every price p, there corresponds a value V
- How do we get this?

Liquidity provider value

- More generally, arbitrageurs maximize their profits when liquidity providers' profits are minimized, *i.e.*,

$$\begin{array}{ll} \text{minimize} & p_a R_a + p_b R_b \\ \text{subject to} & (R_a, R_b) \in S, \end{array}$$

  where S is the reachable set.

- The optimal objective value is $V(p_a, p_b)$

- (Exercise for the reader: show that this is indeed equivalent to maximizing arbitrage profit, answer in [AC20] :)

Liquidity provider value (examples)

- Some simple examples!
- Constant product:

$$V(p_a, p_b) = 2\sqrt{p_a p_b k}$$

- Constant mean:

$$V(p_a, p_b) = k \left(\frac{p_a}{w}\right)^w \left(\frac{p_b}{1-w}\right)^{1-w}$$

- Most are well known special cases of *Fenchel conjugates*
- But this need not have a closed form....

The punchline

- What if you wanted a specific payoff? *I.e.*, given a V, can we come up with a trading function ψ that *replicates* V?
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  *Subject to (some) terms and conditions

The punchline (continued)

- Clearly we can't do everything, but we can do anything that is *consistent*; *i.e.*,
  - Concave
  - Increasing
  - Nonnegative
  - 1-homogeneous (this one isn't too important, actually)

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- Clearly we can't do everything, but we can do anything that is *consistent*; *i.e.*,
  - Concave
  - Increasing
  - Nonnegative
  - 1-homogeneous (this one isn't too important, actually)
- In fact, CFMMs are *exactly equivalent* to the set of payoffs that satisfy these conditions!
- *i.e.*, every CFMM has a payoff of this form and vice versa

The punchline (continued)

- The proof isn't immediately obvious
- (In fact, it follows from a slightly tricky construction)
- But the proof is short! See "Replicating Market Makers," Angeris, Evans, and Chitra, 2021.
- The main thing is that, given any consistent V, we can find a ψ that gives the payoff, by solving:

$$\psi(R_a, R_b) = \sup_{p_a, p_b} \left(V(p_a, p_b) - (R_a p_a + R_b p_b)\right)$$

The punchline (examples)

- We can replicate a number of important instruments
- Black-Scholes covered call price:

$$\psi(R_a, R_b) = \begin{cases} 0 & R_b \leq K\Phi(\Phi^{-1}(1 - R_a) - \sigma\sqrt{\tau}) \\ +\infty & \text{otherwise.} \end{cases}$$

- Perpetual American puts:

$$\psi(R_a, R_b) = \begin{cases} 0 & K \leq R_b + KR_a^{\frac{2r}{2r+\sigma^2}} \\ +\infty & \text{otherwise.} \end{cases}$$

- *i.e.*, we can *engineer* instruments into CFMMs!

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- Leads to simple, general theory
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- Properties and behavior are well defined, somewhat studied

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- Though unstated, everything is just (basic!) convex analysis
- Properties and behavior are well defined, somewhat studied
- But there's always more to do :)

Thanks! (And references)

- “Improved Price Oracles: Constant Function Market Makers,” Angeris and Chitra, 2020
- “Replicating Market Makers,” Angeris, Evans, and Chitra, 2021