Master Theorem for Recurrence Relations

The master theorem is a powerful technique for solving divide and conquer recurrence relations.

General Form

The master theorem applies to recurrences of the form:

$t(n) = a \times t(\frac{n}{b}) + f(n)$

Components

In the recurrence relation $t(n) = a \times t(\frac{n}{b}) + f(n)$:

- a = number of subproblems
- n/b = size of each subproblem
- $f(n)$ = work done outside the recursive calls

Master Theorem Cases

The master theorem has three cases, which depend on comparing the growth rates of $f(n)$ and $n^{\log_b a}$.

Case	Condition	Result
Case 1	If $f(n)$ grows polynomially slower than $n^{\log_b a}$	Then $t(n) = \theta(n^{\log_b a})$
Case 2	If $f(n)$ has the same growth rate as $n^{\log_b a}$ (possibly with logarithmic factors)	Then $t(n) = \theta(n^{\log_b a} \times \log n^{\text{appropriate power}})$
Case 3	If $f(n)$ grows polynomially faster than $n^{\log_b a}$ and satisfies regularity condition	Then $t(n) = \theta(f(n))$

Example Problem Walkthrough

Let's analyze the recurrence relation:

$t(n) = 2 \times t(\frac{n}{2}) + n^{\frac{1}{2}}$

Step 1: Identify Parameters

- $a = 2$
- $b = 2$
- $f(n) = n^{\frac{1}{2}}$

Step 2: Calculate $\log_b a$

$\log_2 2 = 1$

Step 3: Compare Growth Rates

Compare $f(n) = n^{\frac{1}{2}}$ with $n^{\log_b a} = n^1 = n$.

$n^{\frac{1}{2}}$ grows slower than n. Specifically, $f(n) = O(n^{\log_b a - \epsilon})$, where $\epsilon = \frac{1}{2}$.

Step 4: Apply Master Theorem

Since $f(n)$ grows polynomially slower than $n^{\log_b a}$, we are in **Case 1**.

Therefore, $t(n) = \theta(n^{\log_b a}) = \theta(n^1) = \theta(n)$.

Step 5: Verification

Expanding the recurrence a few times:

$t(n) = 2t(\frac{n}{2}) + n^{\frac{1}{2}}$

Expanding $t(\frac{n}{2})$:

$t(n) = 2 \times [2t(\frac{n}{4}) + (\frac{n}{2})^{\frac{1}{2}}] + n^{\frac{1}{2}}$

Simplifying:

$t(n) = 4t(\frac{n}{4}) + 2(\frac{n}{2})^{\frac{1}{2}} + n^{\frac{1}{2}}$

$t(n) = 4t(\frac{n}{4}) + n^{\frac{1}{2}} \times 2^{\frac{1}{2}} + n^{\frac{1}{2}}$

Continuing this expansion confirms that the solution is indeed $\theta(n)$.

Step 6: Determine Correct Answer

Given the options:

- $t(n) = \theta(\log n)$
- $t(n) = \theta(n^{\frac{1}{2}})$
- $t(n) = \theta(n)$
- $t(n) = \theta(n \log n)$
- $t(n) = \theta(n^{\frac{3}{2}})$
- $t(n) = \theta(n^2)$
- The recurrence cannot be solved with the master theorem.

The correct answer is: $t(n) = \theta(n)$.