Unofficial Solutions to "Linear Algebra: Theory and Applications" by Ward Cheney and David Kincaid

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For readers

This is an unofficial solution manual for “Linear Algebra: Theory and Applications” by Ward Cheney and David Kincaid. I wrote this when I was a teaching assistant of Professor Chun-Ching Lu’s course EECS205003 Linear Algebra at NTHU in 2021 fall. For those who find this, I hope it will be helpful. If you find any errors or have some suggestions, please feel free to contact me. My email address is a900955712@gmail.com.

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Chapter 1

Systems of Linear Equations

1.1

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The notation $A \sim B$ means that each of the matrices A and B can obtained from the other by applying one or more allowable row operations. Note that we have three elementary row operations: replacement, scale, and swap. Since an equivalence relation must be symmetric. Instead of showing that the left one is similar to the right one, we will show that the right one is similar to the left one.

$$
\begin{array}{r} \to \\ -2 \\ 2 \end{array}\begin{bmatrix} 0 & 1 & 3 & 15 & 13 & 4 \\ 0 & 2 & 6 & 35 & 28 & 3 \\ 0 & -2 & -6 & 20 & -6 & -3 \end{bmatrix} \sim \begin{array}{r} \\ \to \\ -10 \end{array}\begin{bmatrix} 0 & 1 & 3 & 15 & 13 & 4 \\ 0 & 0 & 0 & 5 & 2 & -5 \\ 0 & 0 & 0 & 50 & 20 & 5 \end{bmatrix}
$$

$$
\sim \begin{array}{r} \\ \\ \frac{1}{55} \end{array}\begin{bmatrix} 0 & 1 & 3 & 15 & 13 & 4 \\ 0 & 0 & 0 & 5 & 2 & -5 \\ 0 & 0 & 0 & 0 & 0 & 55 \end{bmatrix} \sim \begin{array}{r} -4 \\ 5 \\ \to \end{array}\begin{bmatrix} 0 & 1 & 3 & 15 & 13 & 4 \\ 0 & 0 & 0 & 5 & 2 & -5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\sim \begin{bmatrix} 0 & 1 & 3 & 15 & 13 & 0 \\ 0 & 0 & 0 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

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When we try to solve a linear equation system with an augmented matrix. We can see that the reduction of the augmented matrix doesn't depend on the right-hand side vector. Since these three systems have the same coefficient matrix, we can append the all three right-hand side vectors in the one augmented matrix, and it won't affect the result.

$$
\left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 4 & -1 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-2}{7} & \frac{11}{14} & \frac{-1}{14} \\ 0 & 1 & 0 & \frac{1}{7} & \frac{5}{14} & \frac{-3}{14} \\ 0 & 0 & 1 & \frac{3}{7} & \frac{-13}{14} & \frac{5}{14} \end{array}\right]
$$

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If we view the first matrix as a linear system of 3 equations in 3 unknowns, then the second one is the solution to the system if it is a reduced row echelon form of the first matrix. Therefore, all we need to do is to check whether $x_1 = 13$, $x_2 = 23$, and $x_3 = -17$ is the solution to the linear system

$$\begin{cases} 13x_1 + 17x_2 - 31x_3 = 1097 \\ 11x_1 - 19x_2 + 7x_3 = -413 \\ 5x_1 + 3x_2 + 29x_3 = -359 \end{cases}.$$

Since $13 \cdot 13 + 17 \cdot 23 - 31 \cdot (-17) = 1087$, the second matrix is not the reduced row echelon form of the first matrix.

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We can remove the second column of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, and it is still in reduced echelon form.

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(1) Let $A = [a_{ij}]$ be a matrix with integer entries only. To obtain the row echelon form of a matrix, what we need to do is move the row with the leading nonzero element closer to the right up, and eliminate the all nonzero elements below that leading nonzero element to 0. Since a matrix with integer entries only is a matrix with integer entries only after a swap operation. If we can show that we can eliminate the all nonzero elements below that leading nonzero element to 0 with elementary row operations and still let the matrix has only integer entries, then we are done.

Let a_{ik} be a leading nonzero element, and assume for all rows below the ith row, they don't have nonzero element in the first column to $(k-1)$th column. Let $a_{jk} \neq 0$ and $j > i$. Then

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{ik} & a_{i(k+1)} & a_{i(k+2)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{jk} & a_{j(k+1)} & a_{j(k+2)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \sim \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{ik} & a_{i(k+1)} & a_{i(k+2)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{jk}a_{ik} & a_{j(k+1)}a_{ik} & a_{j(k+2)}a_{ik} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$\sim \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{ik} & a_{i(k+1)} & a_{i(k+2)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{jk}a_{ik} - a_{ik}a_{jk} & a_{j(k+1)}a_{ik} - a_{i(k+1)}a_{jk} & a_{j(k+2)}a_{ik} - a_{i(k+2)}a_{jk} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Since the addition, subtraction, and multiplication of integers are still integers. It shows that a matrix with integer elements only is row equivalent to a matrix in row echelon form having only integer entries.

(2) No. A counter example is

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{1}{2} \end{bmatrix}.$$

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Every 1×5 matrix is in row echelon form. A big difference between row echelon form and reduced echelon form is whether the leading nonzero entries are required to be one. Therefore the matrix $\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in row echelon form, but it is not reduced row echelon form.

1.2

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First, we can eliminate all the nonzero elements in the upper triangle with backward substitution. Second, using scale operations, we can scale the all nonzero leading element to 1. Therefore, the left matrix is row equivalent to the right one.

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First, we suppose that $p \neq q$. Then the inverse of these four row operations are:

(1) $\mathbf{r}_p \leftarrow \mathbf{r}_p + (-\alpha)\mathbf{r}_q$

(2) $\mathbf{r}_q \leftarrow \frac{1}{\alpha}(\mathbf{r}_q - \mathbf{r}_p)$

(3) $\mathbf{r}_p \leftrightarrow \mathbf{r}_q$

(4) $\mathbf{r}_q \leftarrow \frac{1}{\beta}\mathbf{r}_p$

For the first two cases, we must assume $q \neq p$, otherwise when $\alpha = -1$, the new row will be a zero row and it is not invertible.

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Let $\mathbf{A} = \begin{bmatrix} 3 & 1 & 4 & 2 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$. By **Theorem 1.2.4**, since each row of the matrix $\mathbf{A}$ has a pivot position, the system of equations $\mathbf{Ax} = \mathbf{b}$ is consistent for all $\mathbf{b}$ in $\mathbb{R}^3$. By **Theorem 1.2.3**, since the system of equations $\mathbf{Ax} = \mathbf{b}$ is consistent for all $\mathbf{b}$ in $\mathbb{R}^3$, the columns of $\mathbf{A}$ span $\mathbb{R}^3$. Therefore $\mathrm{Col}(\mathbf{A}) = \mathbb{R}^3$.

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In example 8, we have the two equations:

$$\begin{cases} x_1 - x_3 = -2 \\ x_2 + 2x_3 = 11 \end{cases}$$

Since we are asked to use x_1 as the free variable, we can rearrange the two equations like this:

$$\begin{cases} x_3 = x_1 + 2 \\ x_2 + 2(x_1 + 2) = 11 \end{cases} \quad \Rightarrow \begin{cases} x_2 = -3 - 2x_1 \\ x_3 = 2 + x_1 \end{cases}$$

If we add $x_1 = x_1$ in the system, then we have

$$\begin{cases} x_1 = 0 + x_1 \\ x_2 = -3 - 2x_1 \\ x_3 = 2 + x_1 \end{cases} \quad \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix} + s \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \ s \in \mathbb{R}$$

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Since $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \sim \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}$, there exists some elementary row operations $\mathbf{E}_1, \mathbf{E}_2, \cdots, \mathbf{E}_m$ where $m \in \mathbb{N}$ such that

$$\mathbf{E}_1\mathbf{E}_2\cdots\mathbf{E}_m \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}.$$

Since $\mathbf{E}_1\mathbf{E}_2\cdots\mathbf{E}_m\mathbf{a}_i = \mathbf{b}_i \ \forall\ i \in \{1,2,\cdots,n\}$, we have

$$\mathbf{E}_1\mathbf{E}_2\cdots\mathbf{E}_m \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_k\end{bmatrix} = \begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k\end{bmatrix}.$$

Therefore $\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_k\end{bmatrix} \sim \begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k\end{bmatrix}$. Recall Ex. 1.1.15 will help you understand this exercise better.

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Check **Theorem 1.2.5** in the textbook.

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No. Let

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ and } [\mathbf{B}|\mathbf{c}] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

then $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\2\\0\end{bmatrix}$.
Note that if two linear systems are row equivalent to each other, they share the same set of solutions. Please check **Theorem 1.2.1** in the textbook.

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a. By some elementary row operations, we have

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 7 & -3 \\ 0 & 1 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 & 5 & -4 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 & 5 & -4 \end{array}\right].$$

We can rewrite the linear system as this

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix}.$$

If we choose x_3 and x_5 as free variables. Then the other three are dependent variables.

$$\begin{cases} x_1 = -1 & - x_5, \\ x_2 = 1 & + 3x_5, \\ x_4 = -4 & - 5x_5. \end{cases}$$

With these equations, we can write down the general solution of this linear system.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} -1 \\ 3 \\ 0 \\ -5 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \\ -4 \\ 0 \end{bmatrix}.$$

b. Similar to a.

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In this exercise, we need to use **Theorem 1.2.5**. By **Theorem 1.2.5**, if we can make sure there's that no pivot is in the last column of a linear system, then it is consistent. Since

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 2 & 1 & 5 & b \\ 1 & -1 & 1 & c \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & 1 & b-2a \\ 0 & 0 & 0 & b+c-2a \end{array}\right].$$

If $b+c-2a=0$, then this linear system is consistent. Please check **Theorem 1.2.5**.

1.3

4

Recall that the rank of a matrix is the number of pivots in its reduced row echelon form, so we first find the reduced row echelon form of this matrix

$$\begin{bmatrix} 1 & 4 & -5 & 10 \\ 3 & 1 & 7 & -3 \\ 2 & 2 & 2 & 2 \\ 1 & 3 & -3 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ The matrix has rank 2.

Now, we find the kernel of this matrix. Since the sets of solutions of two row equivalent augmented matrices are the same, we now find the kernel of the right matrix since it'd be easier.

$$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

Thus $\mathrm{Ker}(A) = \mathrm{Span}\{(-3,2,1,0),(2,-3,0,1)\}$.

6

Let $\mathbf{A}$ be a matrix with 3 columns such that

$$\mathbf{A} \begin{bmatrix} 1 & -2 \\ 3 & 0 \\ 2 & 4 \end{bmatrix} = \mathbf{0}.$$

Then for any row $\mathbf{a}_i$ of A, we have

$$\begin{bmatrix} a_{i1} & a_{i2} & a_{i3} \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 0 \\ 2 & 4 \end{bmatrix} = \mathbf{0}$$

$$\Rightarrow \begin{cases} a_{i1} = 2a_{i3} \\ a_{i2} = \frac{-4}{3} a_{i3} \end{cases}$$

$$\Rightarrow \mathbf{a}_i = s \begin{bmatrix} 2 & \frac{-4}{3} & 1 \end{bmatrix},\ s \in \mathbb{R}$$

If a nonzero matrix such that all of it's rows can be written in this form, then it is one of choices. Therefore, we can just choose $A = \begin{bmatrix} 2 & \frac{-4}{3} & 1 \end{bmatrix}$ to be our answer.

7

First, let's look at this augmented matrix. If we use row operations on this system to make the left part become the identity matrix, then the right part will the be the matrix of those row operations.

$$\mathbf{E}_1\mathbf{E}_2\cdots\mathbf{E}_m[\mathbf{A}|\mathbf{I}] = [\mathbf{I}|\mathbf{E}] \Rightarrow \mathbf{EA} = \mathbf{I}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & 0 \\ 2 & -4 & 1 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array}\right] \sim \frac{1}{7}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-22}{7} & \frac{-1}{7} & \frac{31}{7} \\ 0 & 1 & 0 & \frac{-9}{7} & \frac{-2}{7} & \frac{13}{7} \\ 0 & 0 & 1 & \frac{8}{7} & \frac{1}{7} & \frac{-10}{7} \end{array}\right]$$

We have $\mathbf{A}^{-1} = \frac{1}{7}\begin{bmatrix} -22 & -1 & 31 \\ -9 & -2 & 13 \\ 8 & 1 & -10 \end{bmatrix}$. Then $[\mathbf{A}|\mathbf{B}] \sim [\mathbf{I}|\mathbf{A}^{-1}\mathbf{B}]$.

13

Suppose $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_3 + \mathbf{v}_1\}$ is linear dependent, then there exists $a, b, c \in \mathbb{R}$ such that $a(\mathbf{v}_1 + \mathbf{v}_2) + b(\mathbf{v}_2 + \mathbf{v}_3) + c(\mathbf{v}_3 + \mathbf{v}_1) = \mathbf{0}$ where a, b, c are not all zeroes. Then we have

$$(a + c)\mathbf{v}_1 + (a + b)\mathbf{v}_2 + (b + c)\mathbf{v}_3 = \mathbf{0}.$$

Since $\{\mathbf{v}_1, \mathbf{v}_2 . \mathbf{v}_3\}$ is linearly independent, we have

$$\begin{cases} a + c = 0 \\ a + b = 0 \\ b + c = 0 \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = 0 \\ c = 0 \end{cases}.$$

$\Rightarrow \{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_3 + \mathbf{v}_1\}$ is linearly independent.

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By **Theorem**1.3.12 in the textbook, the column vectors of a matrix form a linearly dependent set if and only if there is a column having no pivot. So now let's take a look at this matrix.

$$\begin{bmatrix} 3 & 4 & 6 \\ 2 & 1 & -1 \\ 7 & -3 & -23 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the last column of this matrix doesn't have a pivot, so the given set is linearly dependent.

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Review Ex. 1.3.38 and Cor. 1.3.1 in the textbook.

24

By **Theorem 1.3.13** in the textbook, the column vectors of a matrix form a linearly independent set if and only if there is a pivot position in each column of the matrix. Since the matrix has n column and there is a pivot in each column, so its rank is n.

25

If the rank of $\mathbf{A}$ is n, then each column of $\mathbf{A}$ has a pivot position,. Then by **Theorem** 1.3.13, the columns of $\mathbf{A}$ form a linearly independent set, a contradiction.

30

Similar to Ex.4.

31

Since the columns of $\mathbf{A}$ form a linearly independent set, by **Theorem** 1.3.13, the rank of A is n.

33

For an RREF, all the zero rows should be moved to the bottom, so the RREF of $\mathbf{A}$ is like this

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$$

The two 1's in square boxes are the pivot positions of $\mathbf{A}$.

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It's obvious that the number of pivots can not exceed the number of rows and the number of columns. For this problem, please review **Theorem** 1.3.7 in the textbook.

38

No. Suppose $\mathbf{A}$ and $\mathbf{B}$ are not row equivalent to each other but have the same RREF $\mathbf{C}$, then

$$\begin{cases} \mathbf{A} \sim \mathbf{C} \\ \mathbf{B} \sim \mathbf{C} \end{cases} \Rightarrow \mathbf{A} \sim \mathbf{B} \ (\rightarrow\leftarrow)$$

41

If the rank of $[\mathbf{A}|\mathbf{b}]$ is greater than the rank of A, then there is a pivot in the last column of $[\mathbf{A}|\mathbf{b}]$. Then by **Theorem** 1.2.4 in the textbook, we can conclude that $\mathbf{Ax} = \mathbf{b}$ is inconsistent.

42

First we note that

$$\mathbf{a}_1 + \mathbf{a}_2 = \begin{bmatrix} 2n+1 & 2n+1 & \cdots & 2n+1 \end{bmatrix}.$$

Let $\mathbf{a}_i$ be the ith row of $\mathbf{A}_n$ where $i > 2$.

If $\mathbf{a}_i = \begin{bmatrix} (i-1)n+1 & (i-1)n+2 & \cdots & (i-1)n+n \end{bmatrix}$, then $\mathbf{a}_i = \dfrac{i-1}{2n+1}(\mathbf{a}_1 + \mathbf{a}_2) + \mathbf{a}_1$.

If $\mathbf{a}_i = \begin{bmatrix} (i-1)n+n & \cdots & (i-1)n+2 & (i-1)n+1 \end{bmatrix}$, then $\mathbf{a}_i = \dfrac{i-2}{2n+1}(\mathbf{a}_1 + \mathbf{a}_2) + \mathbf{a}_2$.

$$\Rightarrow \mathbf{Rank}(\mathbf{A}_n) = \begin{cases} 1 & , n = 1 \\ 2 & , n \geq 2 \end{cases}.$$

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Suppose $a\mathbf{v}_2 + b\mathbf{v}_2 = \mathbf{0}$ where $(a, b) \neq (0, 0)$,
then $(a - b\lambda)\mathbf{v}_1 + b(\mathbf{v}_2 + \lambda\mathbf{v}_1) = \mathbf{0}, (\rightarrow\leftarrow)$.

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If we let $\mathbf{x}$ be a zero vector, then no matter what $\mathbf{A}$ is, $\mathbf{Ax} = \mathbf{0}$. So we don't need any extra conditions to guarantee $\mathbf{Ax} = \mathbf{0}$ is consistent. $\mathbf{Ax} = \mathbf{0}$ is consistent for all $\mathbf{A}$.

60

Let $\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \end{bmatrix}$.
Then $\mathbf{Rank}(\mathbf{A}) \leq \min(4,5) = 4 \Rightarrow \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_5\}$ is linearly dependent. Since there are 5 columns in $\mathbf{A}$ but only 4 pivots, there is a column in $\mathbf{A}$ has no pivot.

62

Suppose $x_1 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}$, then $\begin{bmatrix} 1 & 4 \\ 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}$. Since $\left[\begin{array}{cc|c} 1 & 4 & 7 \\ 3 & 1 & -1 \\ 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$, $x_1 = -1$ and $x_2 = 2$.

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Similar to Ex.42. $\mathbf{Rank}(\mathbf{A}_n) = \begin{cases} 1 & , n = 1 \\ 2 & , n \geq 2 \end{cases}$.

67

No, we can't do that. If a linear system has no solution or has a unique solution, then it has no free variable.

68

If $\mathbf{Ax} = \mathbf{b}$ is consistent, then there is no pivot in the last column of $[\mathbf{A}|\mathbf{b}]$, so the rank of $[\mathbf{A}|\mathbf{b}]$ is k. Otherwise, if $\mathbf{Ax} = \mathbf{b}$ is inconsistent, then there is a pivot in the last column of $[\mathbf{A}|\mathbf{b}]$, so the rank of $[\mathbf{A}|\mathbf{b}]$ is $k + 1$.

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The answer is yes. We can use this row operation to create the 3 elementary row operations.

1. Replacement: Let $\alpha = 1$.
2. Scale: $\begin{cases} \mathbf{r}_i \leftarrow \alpha \mathbf{r}_i + \beta \mathbf{r}_j \\ \mathbf{r}_i \leftarrow \mathbf{r}_i - \beta \mathbf{r}_j \end{cases}$
3. Swap: $\begin{cases} \mathbf{r}_i \leftarrow \mathbf{r}_i + \mathbf{r}_j \\ \mathbf{r}_j \leftarrow \mathbf{r}_i - \mathbf{r}_j \\ \mathbf{r}_i \leftarrow \mathbf{r}_i - \mathbf{r}_j \end{cases}$

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a. To show these 6 properties are equivalent to each other, we only need to show that a implies b, b implies c,..., and f implies a, then we are done. Here we will only show that f implies a.

f$\rightarrow$a: Suppose each column of $\mathbf{A}$ has a pivot, then the RREF of $\mathbf{A}$ must be in this form

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Then the only solution of $\mathbf{Ax} = \mathbf{0}$ is $\mathbf{0}$, $(\rightarrow\leftarrow)$.

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Since the rank is less than the number of the rows, there are zero rows in its reduced row echelon form, so yes, its rows form a linearly dependent set. Please review **Theorem** 1.3.11 in the textbook.

Chapter 2

Vector Spaces

2.1

12

For all $s \in \text{Span}(S)$, there exists some $s_i \in S$ and $b_j \in \mathbb{R}$ such that $s = b_1 s_1 + b_2 s_2 + \cdots + b_m s_m \in \text{Span}(S)$. Since $S \subseteq \text{Span}(T)$, for all s_i, there exists an $n_i \in \mathbb{N}$ and some $t_{i1}, \cdots, t_{in_i} \in T$ such that $s_i = a_{i1}t_{i1} + a_{i2}t_{i2} + \cdots + a_{in_i}t_{in_i}$ for some $a_{ij} \in \mathbb{R}$. Hence, we have

$$s = \sum_{i=1}^{m} b_i \left(\sum_{j=1}^{n_i} a_{ij} \right) t_{ij} \in \text{Span}(T).$$

Therefore, $\text{Span}(S) \subseteq \text{Span}(T)$.

21

The 5th and 10th properties will be false.

24

$\text{Span}(U) \subseteq \text{Span}(\text{Span}(U))$: It's trivial.
$\text{Span}(\text{Span}(U)) \subseteq \text{Span}(U)$:
Since $\text{Span}(U) \subseteq \text{Span}(U)$, by ex.12, we have $\text{Span}(\text{Span}(U)) \subseteq \text{Span}(U)$.

25

a. Yes. The span of the empty set is empty.

b. Yes. $\text{Span}(\{\mathbf{0}\}) = \{\mathbf{0}\}$.

c. No. $\text{Span}(\{(1,0)\}) = \text{Span}(\{2,0\})$.

d. No. $\text{Span}(\{(1,0)\}) \subseteq \text{Span}(\{2,0\})$.

e. Yes.

30

This means $\left[\begin{array}{cc|c} 1 & -5 & 3 \\ 3 & 5 & -1 \\ -1 & 2 & \alpha \end{array}\right]$ is consistent. To find out what values of α can make this linear system consistent, you can use the similar method in exercise 50 in section 1.2.

45 and 47

Similar to ex.30.

51

Almost all properties in finite-dimensional vector space also works in infinite-dimensional vector space

2.2

1

Let $\mathbf{u} = (7,3)$ and $\mathbf{v} = (-5,6)$.

$$\begin{aligned} L &= \{\mathbf{v} + t(\mathbf{u} - \mathbf{v}) | t \in \mathbb{R}\} \\ &= \{(-5,6) + t(12,-3) | t \in \mathbb{R}\} \\ &= \{\mathbf{u} + s(\mathbf{v} - \mathbf{u}) | s \in \mathbb{R}\} \\ &= \{(7,3) + s(-12,3) | s \in \mathbb{R}\} \end{aligned}$$

We can see that the answer is not unique.

2

If these two lines intersect at some point, then there exists t_1 and t_2 such if you substitute t_1 into the first line, it will be the same as we substitute t_2 into the second line.

$$\begin{aligned} &\begin{bmatrix} 3 \\ 4 \\ -5 \\ 6 \\ 2 \end{bmatrix} + t_1 \begin{bmatrix} 2 \\ -2 \\ 1 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 17 \\ -10 \\ 2 \\ 27 \\ 44 \end{bmatrix} + t_2 \begin{bmatrix} -3 \\ 2 \\ -5 \\ 1 \\ 4 \end{bmatrix} \\ \Rightarrow &\begin{bmatrix} 2 & 3 \\ -2 & -2 \\ 1 & 5 \\ 3 & -1 \\ 6 & -4 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 14 \\ -14 \\ 7 \\ 21 \\ 42 \end{bmatrix} \end{aligned}$$

Since $\left[\begin{array}{cc|c} 2 & 3 & 14 \\ -2 & -2 & -14 \\ 1 & 5 & 7 \\ 3 & -1 & 21 \\ 6 & -4 & 42 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$, we have $t_1 = 7$ and $t_2 = 0$.

These two line intersect at $(17, -10, 2, 27, 44)$.

5

By **Theorem** 2.2.1 in the textbook, if $\mathbf{u} - \mathbf{w}$ and $\mathbf{v}$ are multiples of $\mathbf{z}$, then the two lines are the same.

$$\begin{aligned} \mathbf{u} - \mathbf{w} &= (3,7) - (18,-5) = (-15,12) = \frac{3}{4}\mathbf{z} \\ \mathbf{v} &= -\frac{1}{4}\mathbf{z} \\ \Rightarrow &L_1 = L_2. \end{aligned}$$

6

We can let s be the x_2 and t be x_3. Then we have

$$\begin{aligned}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} &= \begin{bmatrix}5/3\\0\\0\end{bmatrix} + x_2\begin{bmatrix}-7/3\\1\\0\end{bmatrix} + x_3\begin{bmatrix}3\\0\\1\end{bmatrix}\\
\Rightarrow x_1 &= \frac{5}{3} - \frac{7}{3}x_2 + 3x_3\\
\Rightarrow 3x_1 &+ 7x_2 - 9x_3 = 5\end{aligned}$$

7

We rewrite the equation as

$$\begin{aligned}x_2 &= \frac{7+5x_3}{3}\\
\Rightarrow \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} &= \begin{bmatrix}0\\\frac{7}{3}\\0\end{bmatrix} + x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_3\begin{bmatrix}0\\\frac{5}{3}\\1\end{bmatrix}.\end{aligned}$$

8

Review Ex.47 in section 1.2.

26

Two lines in $\mathbb{R}^3$ have a point in common only if they are the same line or they are on the same plane and not parallel to each other. So the answer is no, two lines in $\mathbb{R}^3$ will likely have no intersection.

31

Similar to Ex.5.

35

Similar to example 8 in the textbook.

37

You can apply some row operations on the augmented matrix of this linear system and the set of solutions will not change. For example,

$$\begin{cases}2x_1 + 2x_2 + 3x_3 + 4x_4 = -1\\ 2x_2 - 9x_3 - 12x_4 = -4\end{cases}.$$

39

Since these two lines are not parallel to each other, if we want to find a plane containing two different lines, we first need to make sure they have a point in common. By ex.2, we can find that $(2,-2,3)$ is these two lines' intersection. With $(1,-2,3), (-2,5,-7)$, and $(2,-2,3)$, we can use the same method in ex.35 to find this plane.

42

Substitute (c,c,c) in this equation, then it's obvious that (c,c,c) is on this plane for all c.

47

Let $t=1$ and $s=0$, then $4\times 3-4+5\times 0=8\neq 0$. Hence, this is not a parametric form of the plane.

49

To find three points on this plane, you can just try three different (s,t), then you'll have three different points of this plane. About the equation of this plane, it's very similar to exercise 6.

2.3

1

Let $T(\mathbf{x})=A\mathbf{x}$ where A is a 3×3 matrix. Then we have

$$A\underbrace{\begin{bmatrix}1&2&4\\3&-1&-9\\2&3&5\end{bmatrix}}_{B}=\begin{bmatrix}5&11&23\\2&-6&-22\\2&3&6\end{bmatrix}=\mathbf{C}$$

Since $(AB)^t=B^tA^t$,

$$\begin{aligned}&C^t=B^tA^t\\ \Rightarrow&\begin{bmatrix}1&3&2\\2&-1&3\\4&-9&-5\end{bmatrix}A^t=\begin{bmatrix}5&2&2\\11&-6&3\\23&-22&6\end{bmatrix}\end{aligned}$$

Then we can solve A as in previous exercises.

2

Let A be the matrix form of T. Since

$$\begin{aligned}T(\mathbf{x})=&x_1\underbrace{\left(2\begin{bmatrix}-2\\5\\-3\end{bmatrix}+\begin{bmatrix}1\\-3\\2\end{bmatrix}\right)}_{\mathbf{a}_1}+x_2\underbrace{\left(3\begin{bmatrix}3\\-4\\2\end{bmatrix}-3\begin{bmatrix}1\\-3\\2\end{bmatrix}\right)}_{\mathbf{a}_2}+x_3\underbrace{\left(-2\begin{bmatrix}3\\-4\\2\end{bmatrix}+4\begin{bmatrix}-2\\5\\-3\end{bmatrix}\right)}_{\mathbf{a}_3}\\ =&\begin{bmatrix}\mathbf{a}_1&\mathbf{a}_2&\mathbf{a}_3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\end{aligned}$$

We have $A=\begin{bmatrix}\mathbf{a}_1&\mathbf{a}_2&\mathbf{a}_3\end{bmatrix}$.

10

(1) Since Ker$\mathbf{A}=\{\mathbf{0}\}$, T is one-to-one. Let $\mathbf{y}=(0,0,1)$, then it's obvious that $\mathbf{A}\mathbf{x}=\mathbf{y}$ is inconsistent, so T is not surjective.

(2) Since Ker$(\mathbf{A}^T)\neq\{\mathbf{0}\}$, $\mathbf{A}^T$ is not injective. For all $\mathbf{y}\in\mathbb{R}^2$, since there is no pivot in the last column of the system $[\mathbf{A}^T|\mathbf{y}]$, $\mathbf{A}\mathbf{x}=\mathbf{y}$ is consistent. Therefore, $\mathbf{A}^T$ is surjective.

17

Since you have only 2 equations with 3 unknowns, there is one free variable. So no matter what a,b,c,d,e,f you choose, Ker(A) contains not only zero and A can not be one-to-one.

18

Let $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$. Suppose $\mathbf{Ax} = \mathbf{y}$, then

$$\left[\begin{array}{cc|c} a & b & y_1 \\ c & d & y_2 \\ e & f & y_3 \end{array}\right] \sim$$

21

There is a mistake in part d. This is not a reflection but a rotation of 90° counterclockwise around the origin.

26

Please review **Theorem** 7 in the textbook.

29

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, then $(0, 0, 1)$ is in $\mathrm{Ker}(A)$ and $\mathrm{Range}(A)$.

32

(1) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a finite linearly dependent indexed subset of $\mathbb{R}^n$. Then there is a linear relation in S

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k = 0$$

with $\alpha_1, \alpha_2, \ldots, \alpha_k$ not all zeros. Now the image $T(S)$ of S under T is the indexed subset $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)\}$ of $\mathbb{R}^m$. Since

$$\mathbf{0} = T(\mathbf{0}) = T(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k) = \alpha_1 T(\mathbf{v}_1) + \alpha_2 T(\mathbf{v}_2) + \cdots + \alpha_k T(\mathbf{v}_k),$$

$T(S)$ is a linearly dependent indexed set.

(2) Consider the zero map.

33

Let $\mathbf{A}$ be the matrix form of T, then we have, $A\begin{bmatrix}\mathbf{r} & \mathbf{u} & \mathbf{w}\end{bmatrix} = \begin{bmatrix}\mathbf{s} & \mathbf{v} & \mathbf{z}\end{bmatrix}$. Now we take the transpose of both sides of the equation, then we have

$$\begin{bmatrix}\mathbf{r}^T \\ \mathbf{u}^T \\ \mathbf{w}^T\end{bmatrix} \mathbf{A}^T = \begin{bmatrix}\mathbf{s}^T \\ \mathbf{v}^T \\ \mathbf{z}^T\end{bmatrix}.$$

Let $\mathbf{A}' = \begin{bmatrix}\mathbf{r}^T \\ \mathbf{u}^T \\ \mathbf{w}^T\end{bmatrix}$ and $\mathbf{B}' = \begin{bmatrix}\mathbf{s}^T \\ \mathbf{v}^T \\ \mathbf{z}^T\end{bmatrix}$, then we can find $\mathbf{A}^T$ by solving the system $\mathbf{AX} = \mathbf{B}$, and we can have $\mathbf{A}$. Note that if $\mathbf{AX} = \mathbf{B}$ is inconsistent, then such linear transformation doesn't exist.

34

Look at the hangout.

40

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

41

(1) Since $\mathbf{u} \neq \mathbf{v}$, they are not both $\mathbf{0}$, then $(\alpha\mathbf{u}, \alpha\mathbf{v})$ is such a pair for all $\alpha \in \mathbb{R}$.

(2) Use mean value theorem and intermediate value theorem.

44

You can ignore the second part, since the domain of f is $\mathbb{R}^3$, this line segment is not in the domain of f.

51

Suppose that $F(G(a)) = F(G(b))$. Because F is injective, $G(a) = G(b)$. And because G is injective, $a = b$. We have $F \circ G$ is injective.

52

By **Theorem** 2, let $\mathbf{A}$ be the matrix form of L where $\mathbf{A}$ is a $p \times k$ martix. We will show that if $p < k$, then L is not one-to-one. If $p < k$, then we have at least $k - p$ free variables. Since $\mathrm{Ker}(\mathbf{A})$ contains not only zero vector, L is not one-to-one.

53

With proof by contrapositive, we show that if $m > n$, then any linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is not a surjective.

Consider a linear transformation $T(\mathbf{x}) = A\mathbf{x}$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ with

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

T is surjective if and only if $T(\mathbb{R}^n) = \mathbb{R}^m$, i.e., given any vector $\mathbf{y} = (y_1, y_2, \cdots, y_m)$ in $\mathbb{R}^m$, there is a vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ in $\mathbb{R}^n$ such that $T(\mathbf{x}) = A\mathbf{x} = \mathbf{y}$. Since $m > n$, there are at most n pivot positions in A. By elementary row operations on the augmented matrix, we must have

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & y_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & y_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & y_m \end{array}\right] \Rightarrow \left[\begin{array}{c|c} & y'_1 \\ U & \vdots \\ & y'_n \\ \hline & y'_{n+1} \\ \mathbf{0} & \vdots \\ & y'_m \end{array}\right]$$

where $\begin{bmatrix} U \\ \mathbf{0} \end{bmatrix}$ is in the reduced row echelon form of A with U an $n \times n$ matrix and $\mathbf{0}$ the $(m-n) \times n$ zero matrix and $y'_1, y'_2, \cdots, y'_m$ are linear combinations of $y_1, y_2, \cdots, y_m$. It is clear that the linear system $A\mathbf{x} = \mathbf{y}$ is not always consistent for any given $\mathbf{y}$ in $\mathbb{R}^m$. Thus we conclude that T is not surjective.

2.4

3

Suppose $af(t) + bg(t) = 0, \ \forall\ t \in \mathbb{R}$.
Choose $t_1 = 0$ and $t_2 = \frac{\pi}{2}$, then we have

$$\begin{cases} a \cdot 0 + b \cdot 1 = 0 \\ a \cdot 1 + b \cdot 0 = 0 \end{cases} \quad \Rightarrow (a, b) = (0, 0) \Rightarrow \{f, g\} \text{ is linearly independent.}$$

4

Recall that $\cos 2t = \cos^2 t - \sin^2 t = (1 - \sin^2 t) - \sin^2 t = 1 - 2\sin^2 t$.
$\because -f(t) + g(t) + 2h(t) = 0\ \forall\ t \in \mathbb{R}$,
$\therefore \{f, g, h\}$ is linearly dependent.

8

For this exercise, the answer is no. The set of these three matrix is linear dependent. To find the linear relation of these three matrices, you can use the same method in example 11 in the textbook.

9

$\begin{bmatrix} 7 & 3 & 6 \\ 1 & -2 & 13 \\ 2 & 4 & -14 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix}$
$\Rightarrow 3\mathbf{u} - 5\mathbf{v} = \mathbf{w}$.
$\because \operatorname{Span}(\mathbf{w}) \subset \operatorname{Span}(\mathbf{u}, \mathbf{v})$,
$\therefore \operatorname{Span}\{\mathbf{u}, \mathbf{v}\} \bigcap \operatorname{Span}\{\mathbf{w}\} = \operatorname{Span}\{\mathbf{w}\}$ and
$\operatorname{Span}\{\mathbf{u}, \mathbf{v}\} + \operatorname{Span}\{\mathbf{w}\} = \operatorname{Span}\{\mathbf{u}, \mathbf{v}\}$.

11

No, since $1 \otimes x \neq x\ \forall\ x \in \mathbb{R}$.

13

a. Let $p(t) = at^3 + bt^2 + ct + d$ where $a, b, c, d \in \mathbb{R}$.
   Then

$$\begin{aligned} (Lp)(t) =& 2(3at^2 + 2bt + c) + 3t(6at + 2b) \\ =& 24at^2 + 10bt + 2c. \end{aligned}$$

   Since $(Lp)(t) = 0\ \forall t \in \mathbb{R}$ if and only if $(a, b, c) = (0, 0, 0)$, $\operatorname{Ker}(L) = \mathbb{R}$.

b. Similar to part a.

14

No, since $3p_1(t) - 2p_2(t) + p_3(t) = 0$.

17

In this exercise, we will try to use vectors with exactly two zero entries to construct standard vectors in $\mathbb{R}^4$. Let

$$T = \left\{ \begin{bmatrix}2\\1\\0\\0\end{bmatrix} - \begin{bmatrix}1\\1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\2\\1\\0\end{bmatrix} - \begin{bmatrix}0\\1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\2\\1\end{bmatrix} - \begin{bmatrix}0\\0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\2\end{bmatrix} - \begin{bmatrix}1\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\}$$

and $S = \{\mathbf{v} \in \mathbb{R}^4 | \text{There are exact two zero entries in } \mathbf{v}\}$.
$\because T \subseteq \text{Span}(S)$, $\therefore \text{Span}(T) = \mathbb{R}^4 \subseteq \text{Span}(\text{Span}(S)) = \text{Span}(S) \subseteq \mathbb{R}^4 \Rightarrow \text{Span}(S) = \mathbb{R}^4$.

20

Similar to Ex.3.

21

No. Let $f(x)$ be a discontinuous function defined on $[-1, 1]$, then $f - f = 0$ is continuous on $[-1, 1]$.

24

Similar to Example 17 in the textbook.

27

Let the vector space be V.

a. Suppose G is linearly dependent. There exists $\mathbf{g}_1, \mathbf{g}_2, \cdots, \mathbf{g}_m \in G$ such that

$$a_1\mathbf{g}_1 + a_2\mathbf{g}_2 + \cdots + a_m\mathbf{g}_m = 0.$$

Since $G \subseteq H$, it means H is linearly dependent, $(\rightarrow\leftarrow)$.

b. Since H is subset of V, we have $\text{Span}(H) \subseteq V$.
Since $G \subseteq H$, we have $V = \text{Span}(G) \subseteq \text{Span}(H)$.
$\Rightarrow \text{Span}(H) = V$.

c. $G = \left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\}, H = \left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \right\}$

d. $G = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}, H = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}, V = \mathbb{R}^3$

28

We will show that Axiom 2,3 $\Leftrightarrow (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{v} + (\mathbf{u} + \mathbf{w})$.

"$\Rightarrow$" $(\mathbf{u} + \mathbf{v}) + \mathbf{w} \underbrace{=}_{A.2} (\mathbf{v} + \mathbf{u}) + \mathbf{w} \underbrace{=}_{A.3} \mathbf{v} + (\mathbf{u} + \mathbf{w})$.

"$\Leftarrow$" A.2 $\mathbf{u}+\mathbf{v} \underbrace{=}_{A.4} (\mathbf{u}+\mathbf{v})+\mathbf{0} = \mathbf{v}+(\mathbf{u}+\mathbf{0}) \underbrace{=}_{A.4} \mathbf{v}+\mathbf{u}$.

A.3 $(\mathbf{u}+\mathbf{v})+\mathbf{w} \underbrace{=}_{A.2} (\mathbf{v}+\mathbf{u})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$.

29

Suppose $m > n$. Since $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_m\} \subseteq \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m\}$, for all $\mathbf{u}_i$, there exist some scalars $a_{i1}, a_{i2}, \cdots, a_{in}$ such that

$$\mathbf{u}_i = a_{i1}\mathbf{v}_i + a_{i2}\mathbf{v}_2 + \cdots + a_{in}\mathbf{v}_n.$$

Let

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_m\mathbf{u}_m = \mathbf{0}.$$

Then we have,

$$\begin{aligned}\mathbf{0} =&(a_{11}x_1 + a_{21}x_2 + \cdots + a_{m1}x_m)\mathbf{v}_1 + \\ &(a_{12}x_1 + a_{22}x_2 + \cdots + a_{m2}x_m)\mathbf{v}_2 + \\ &\qquad\vdots \\ &(a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{mn}x_m)\mathbf{v}_n.\end{aligned}$$

It gives us the following homogeneous linear system,

$$\begin{cases} a_{11}x_1 + a_{21}x_2 + \cdots + a_{m1}x_m = 0 \\ a_{12}x_1 + a_{22}x_2 + \cdots + a_{m2}x_m = 0 \\ \quad\vdots \\ a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{mn}x_m = 0 \end{cases}.$$

Since there are more unknowns than equations, there exists non trivial solutions to this linear system. But $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_m\}$ is linearly independent, so we have a contradiction. Therefore, $m \le n$.

30

Let $V = \{\mathbf{0}, \mathbf{1}\}$.
Define $\oplus$ as logical OR and $\alpha \otimes x = x \forall x \in V, \alpha \in \mathbb{R}$.

A.1 $x \oplus y = y \oplus x \forall x, y \in V$.

A.2 $x \oplus (y \oplus z) = x \oplus (y \oplus z) \forall x, y$ and $z \in V$.

A.3 $\mathbf{0}$ is the additive identity since $\mathbf{0} \oplus \mathbf{0} = \mathbf{0}$ and $\mathbf{1} \oplus \mathbf{0} = \mathbf{1}$.

A.5 $\alpha \otimes (x \oplus y) = x \oplus y = \alpha \otimes x \oplus \alpha \otimes y$ and $(\alpha + \beta) \otimes x = x = x \oplus x = \alpha \otimes x \oplus \beta \otimes x \forall x, y \in V$ and $\alpha \in \mathbb{R}$.

A.6 $(\alpha\beta) \otimes x = x = \beta \otimes x = \alpha \otimes (\beta \otimes x) \forall x \in V$ and $\alpha, \beta \in \mathbb{R}$.

A.7 $1 \otimes x = x \forall x \in V$.

We can see that $\mathbf{0}$ is the additive identity, but $\mathbf{1}$ doesn't have an additive inverse. Therefore the other 6 axioms don't imply the axiom 4, and $\mathbf{1} \oplus (-1) \otimes \mathbf{1} = \mathbf{1} \neq \mathbf{0}$.

31

A.1 $\forall u, v \in V$, $u \oplus v = uv \in V$.

A.2 $\forall u, v \in V$, $u \oplus v = uv = vu = v \oplus u$.

A.3 $\forall u, v, w \in V$, $(u \oplus v) \oplus w = (uv) \oplus w = (uv)w = u(vw) = u \oplus (vw) = u \oplus (v \oplus w)$.

A.4 $\forall u \in V$, $u \oplus 1 = u \cdot 1 = u$, so 1 is a zero vector.

A.5 $\forall u \in V$, $u \oplus \frac{1}{u} = 1 = \mathbf{0}$, so $\frac{1}{u}$ is an additive inverse of u.

A.6 $\forall u \in V$ and $\forall \alpha \in \mathbb{R}$, $\alpha \otimes u = u^{\alpha} \in V$.

A.7 $\forall u, v \in V$ and $\forall \alpha \in \mathbb{R}$, $\alpha \otimes (u \oplus v) = (uv)^{\alpha} = u^{\alpha} v^{\alpha} = (\alpha \otimes u) \oplus (\alpha \otimes v)$.

A.8 $\forall u \in V$ and $\forall \alpha, \beta \in \mathbb{R}$, $(\alpha + \beta) \otimes u = u^{\alpha+\beta} = u^{\alpha} \cdot u^{\beta} = (\alpha \otimes u) \oplus (\beta \otimes u)$.

A.9 $\forall u \in V$ and $\forall \alpha, \beta \in \mathbb{R}$, $\alpha \otimes (\beta \otimes u) = (u^{\beta})^{\alpha} = u^{\alpha\beta} = (\alpha\beta) \otimes u$.

A.10 $\forall u \in V$, $1 \otimes u = u^1 = u$.

Since the set V with the two algebraic operations fulfills all the axioms of vector spaces, $(V, \oplus, \otimes)$ is a vector space.

32

A.1 $\forall (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, $(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1 + a, x_2 + y_2 + b) \in \mathbb{R}^2 = V$.

A.2 $\forall (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, $(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1 + a, x_2 + y_2 + b) = (y_1 + x_1 + a, y_2 + x_2 + b)$ $= (y_1, y_2) \oplus (x_1, x_2)$.

A.4 $\forall (x_1, x_2) \in \mathbb{R}^2$, $(x_1, x_2) \oplus (-a, -b) = (x_1, x_2)$, so $(-a, -b)$ is a zero vector.

34

"$\subseteq$" Let $x \in \text{Span}(S) + \text{Span}(T)$, then there exist $s \in \text{Span}(S)$ and $t \in \text{Span}(T)$ such that $x = s + t$. Since $s \in \text{Span}(S)$, there exist $s_1, s_2, \cdots, s_m \in S$ such that

$$s = a_1 s_1 + a_2 s_2 + \cdots + a_m s_m, \text{ where } a_i \in \mathbb{R}.$$

Similarly, there exist $t_1, t_2, \cdots, t_n \in T$ such that

$$t = b_1 t_1 + b_2 t_2 + \cdots + b_n t_n, \text{ where } b_i \in \mathbb{R}.$$

Therefore, $x = s + t \in \text{Span}(S \bigcup T)$.
$\Rightarrow \text{Span}(S) + \text{Span}(T) \subseteq \text{Span}(S \bigcup T)$.

"$\supseteq$" We leave this part to you.

36

(1) Let $S = \{\mathbf{0}\}$, then there is no such $\mathbf{v}$.

(2) No. Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$,
then $\text{Span}(S) \neq \text{Span}(\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\})$

40

Let $S = \{p_1, p_2, \cdots, p_n\}$ where $p_i = \sum_{j=0}^{\deg(p_i)} c_j t^j$ is a nonzero polynomial and $0 < \deg(p_1) < \deg(p_2) < \cdots < \deg(p_n)$.
Suppose S is linearly dependent, then there exists $a_1, a_2, \cdots, a_n \in \mathbb{R}$ such that

$$p = a_1 p_1 + a_2 p_2 + \cdots + a_n p_n = 0 \ \forall \ t \in \mathbb{R} \text{ and } \sum_{i=1}^{n} |a_i| > 0.$$

Let a_k be the last nonzero coefficient, then $\deg(p) = \deg(p_k)$ which has a most $\deg(p_k)$ roots, $(\rightarrow\leftarrow)$.

46

Consider A^T and use **Theorem** 7 in the textbook.

47

1. No. Of course $\mathbb{R}^n$ is not inside $\mathbb{R}^m$. The elements of $\mathbb{R}^n$ are vectors with n tuples while the elements of $\mathbb{R}^m$ are vectors with m tuples.
2. As long as n is less than or equal to m, $\mathbb{P}_n$ is a subspace of $\mathbb{P}_m$

49

A vector space is defined with a set and two algebraic operations. It will not be a vector space if any one of these is not defined. Since there is no algebraic operation defined, it is not a vector space.

51

a. No, $i(1, 0) = (i, 0) \notin \mathbb{R}^2$.

b. Yes, it is similar to Ex.35.

Chapter 3

Matrix Operations

3.1

7

a. $\mathbf{P}_{12}$

b. $\mathbf{S}_2(3)$

c. $\mathbf{S}_2(2)$

d. It's not an elementary row operation, since you can not multiply an row by zero.

e. $E_{21}(3)$

f. $E_{12}(2)$

18

From Example 10, we have $A = \underbrace{\frac{1}{2}(A + A^T)}_{\text{symmetric}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{skew-symmetric}}$

21

Please review the proof of **Theorem** 4 in the textbook.

26

For this exercise, you can check all the axioms of vector spaces for these three cases. Of course it's correct. But for a vector space V, to check whether a subset of V is a vector space, we don't need to check all the axioms, since V already guarantees some properties. Here we introduce the subspace test.

Let V be a vector space and $S \subseteq V$. S is a vector space if S pass the subspace test.

The Subspace Test:

(1) If $s_1, s_2 \in S$, then $s_1 + s_2 \in S$.

(2) If $s_1 \in S$ and $\alpha \in \mathbb{R}$, then $\alpha s_1 \in S$.

Since we have known that $n \times n$ matrices form a vector space, now we only need to show whether they pass the subspace test. Here we show the case of symmetric matrices and leave the rest to you.

(1) Let $D = A + B$, $D_{ij} = A_{ij} + B_{ij} = A_{ji} + B_{ji} = D_{ji}$, so D is symmetric.

(2) Let $D = \alpha A$, $D_{ij} = \alpha A_{ij} = \alpha A_{ji} = D_{ji}$, so D is symmetric.

Therefore, the symmetric matrices in $\mathbb{R}^{n\times n}$ form a vector space.

29

Here we provide an example where the product of two symmetric matrices is not symmetric.

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$$

30

It's similar to Exercise 26. You can use subspace test to solve it.

31

Since $A \cdot A^n = A^{n+1} = A^n \cdot A$,

$$\begin{aligned} &A(\alpha_0 I + \alpha_1 A + \cdots + \alpha_m A^m) \\ =&(\alpha_0 A + \alpha_1 A^2 + \cdots + \alpha_m A^{m+1}) \\ =&(\alpha_0 I + \alpha_1 A + \cdots + \alpha_m A^m)A. \end{aligned}$$

33

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\begin{aligned} A^2 =& \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} \\ =&(a + d)A + (bc - ad)I. \end{aligned}$$

38

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \nsim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, but $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$.

41

Let $\mathbf{r}_i$ be the ith row of A, then

$$\mathbf{r}_i = \begin{bmatrix} x_i y_1 & x_i y_2 & \cdots & x_i y_n \end{bmatrix} = x_i \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}.$$

We can see that each row of A is a scalar multiple of $\mathbf{y}^T$. Therefore, $\text{rank}(A) = \begin{cases} 0, & \text{if } \mathbf{y} = \mathbf{0}, \\ 1, & \text{otherwise.} \end{cases}$

49

(1) (i) $A \oplus B = AB + BA = BA + AB = B \oplus A$. $\oplus$ is commutative.

(ii) Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = C\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $C = I$, then

$$(A \oplus B) \oplus C = \mathbf{0}_{2\times 2}$$

and

$$A \oplus (B \oplus C) = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}.$$

Therefore, $\oplus$ is not associative.

(2) $\boxplus$ is neither commutative nor associative.

51

Let $\mathbf{x}$ be a nontrivial solution of $A^2\mathbf{x} = A(A\mathbf{x}) = \mathbf{0}$.
Suppose $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x}$ is a nontrivial solution of $A\mathbf{y} = \mathbf{0}$.
Suppose $A\mathbf{x} \neq \mathbf{0}$, then $A\mathbf{x}$ is a nontrivial solution of $A\mathbf{y} = \mathbf{0}$.

56

a. False. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$.

b. False. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

c. False. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$.

d. False. Let $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

e. True. $C(A + B) = CA + CB = CB + CA$.

f. False. $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

g. False. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$.

h. True. $(A + B)C = AC + BC = BC + AC$.

61

(1) Let A be a $m \times n$ matrix, then A^TA is $n \times n$ and AA^T is $m \times m$. Since $A^TA = AA^T$, $m = n$ and A is square.

(2) No. Consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

70

With some observation, we can see that this operation actually means $A + B^T$. $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is a counter example.

71

Let this mapping be T, A, B be two matrices with the same size, and $\alpha, \beta \in \mathbb{R}$. Then

$$T(\alpha A + \beta B) = (\alpha A + \beta B)^T = (\alpha A)^T + (\beta B)^T = \alpha A^T + \beta B^T.$$

T is linear.

3.2

6

Let these three matrices be A, B and C, we have $AB = C$.
Let $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{bmatrix}$ where $\mathbf{b}_i$ is the ith column of B. Then for all $\mathbf{v} = (a, b, c, d)^T \in \mathbb{R}^4$, $A\mathbf{x} = \mathbf{v}$ is consistent since $A \begin{bmatrix} a\mathbf{b}_1 & b\mathbf{b}_2 & c\mathbf{b}_3 & d\mathbf{b}_4 \end{bmatrix} = (a, b, c, d)^T$. Then by **Theorem** 1.2.4, each row of A has a pivot. Therefore, this equation is not true.

8

Similar to Example 11.

11

Similar to Example 10.

12

Please read the proof of **Theorem** 4 in the textbook.

14

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & e & f \\ 0 & 1 & g & h \end{array}\right]$$

Here we skip the detailed computation, and we have

$$\Rightarrow \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

18

Let the polynomial be $p(t) = a_0 + a_1 t + \cdots + a_n t^n$, then

$$\begin{bmatrix} 1 & 0 & \cdots & 0^n \\ 1 & 1 & \cdots & 1^n \\ 1 & 2 & \cdots & 2^2 \\ 1 & -1 & \cdots & (-1)^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ 1 \\ -11 \end{bmatrix}.$$

It can be checked that this linear system is consistent only if $n \geq 3$.

20

Similar to Example 11.
Let this matrix be A. If $BA = I$, then $A^T B^T = I$. Therefore, the left inverse of A is the transpose of the right inverse of A^T.

23

We will show that each row and each column of A has a pivot.

(i) If A has a right inverse B.
$\Rightarrow A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$ since $A(B\mathbf{b}) = \mathbf{b}$.
$\underbrace{\Rightarrow}_{\text{Thm 1.2.4}}$ Each row of A has a pivot.

(i) If A has a left inverse B.
$\Rightarrow A^T\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$ since $A^T(B^T\mathbf{b}) = \mathbf{b}$.
$\underbrace{\Rightarrow}_{\text{Thm 1.2.4}}$ Each row of A^T has a pivot.
$\Rightarrow$ Each column of A has a pivot.

With (i) and (ii), we can conclude that $A \sim I$.

24

Use method of contradiction on Ex.23.

26

Let A be a $m \times n$ matrix, B be a right inverse of A and $\mathbf{b}_i$ be the ith column of B. Since B is A's inverse, we have $A\mathbf{b}_i = \mathbf{e}_i$. If A has a right inverse, then each row of A has a pivot. Since $\text{rank}(A) = m \leq \min(m, n)$, we have $n \geq m$. If $n = m$, then by **Theorem** 4, $BA = I$, so $n > m$. $\because n > m$, $\therefore$ the linear system $A\mathbf{b}_i = \mathbf{e}_i$ has free variables, $\therefore$ the right inverse is not unique.

28

Suppose A is invertible. $AB = \mathbf{0} \Rightarrow A^{-1}AB = A^{-1}\mathbf{0} = \mathbf{0} = IB = B, (\rightarrow\leftarrow)$.

29

False. Let $A = \begin{bmatrix} 4 & -9 \\ -2 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$.

30

Let the inverse of A^{-1} be B, then

$$AA^{-1}B = A\underbrace{(A^{-1}B)}_{I} = A = \underbrace{(AA^{-1})}_{I}B = B.$$

We have $(A^{-1})^{-1} = A$.

34

Let A be a symmetric matrix and B be the inverse of A
$AB = I \Rightarrow B^TA^T = I = B^TA$.
By **Theorem** 4, $B = B^T$.

35

Let A be an invertible matrix, then

$$\begin{aligned} &AA^{-1} = I \\ \Rightarrow &(A^{-1})^T A^T = I \\ \Rightarrow &(A^T)^{-1} = (A^{-1})^T. \end{aligned}$$

36

$\because A$ is invertible, by Ex.23, $A \sim I$. $[A|B] \sim [\underbrace{I}_{C} | \underbrace{A^{-1}B}_{E}]$.

37

A is a $m \times n$ matrix with right inverse.
$\underbrace{\Longrightarrow}_{\text{Thm.3}}$ $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$.
$\underbrace{\Longrightarrow}_{\text{Thm.1.2.4}}$ Each row of A has a pivot.
$\Rightarrow m \leq n$ $(\rightarrow\leftarrow)$.

38

The statement of this exercise is too vague, so you can ignore this one.

43

$$\begin{aligned} &(A^T B^T)(A^{-1}B^{-1})^T \\ =&A^T B^T (B^{-1})^T (A^{-1})^T = A^T (B^{-1}B)^T (A^{-1})^T \\ =&A^T I (A^{-1})^T = (A^{-1}A)^T = I \\ \therefore &(A^T B^T)^{-1} = (A^{-1}B^{-1})^T \end{aligned}$$

44

Use Ex.23 and **Theorem** 7 in the textbook.

46

$$\begin{aligned} &T \text{ is injective} \\ \Longleftrightarrow\ &\mathrm{Ker}(A) \\ \Longleftrightarrow\ &\text{Every column of } A \text{ has a pivot.} \\ \underbrace{\Longleftrightarrow}_{\text{A is } n\times n}\ &\text{Every row of } A \text{ has a pivot.} \\ \underbrace{\Longleftrightarrow}_{\text{Thm. 1.2.4}}\ &\mathrm{Col}(A) = \mathbb{R}^n \end{aligned}$$

51

Similar to Ex. 44.

56

If BA is invertible, then $[(BA)^{-1}B]A = I$, A is invertible. Multiply each side of the equation by A^{-1}, we have $BAA^{-1} = CAA^{-1} \Rightarrow B = C$.

57

Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$.

63

The rows of A form a linearly independent set.
$\Rightarrow$Every row in the RREF of A has a pivot.
(Since A is square)
$\Rightarrow$Every column in the RREF of A has a pivot.
$\Rightarrow$Columns of A form a linearly independent set.

79

Let $AB = D$. $\because (D^{-1}A)B = I$ and $A(BD) = I$, $\therefore$ A and B are invertible.

83

$A^k(A^k)^{-1} = I = A[A^{k-1}(A^k)^{-1}]$

84

$$\begin{aligned}
&\forall \mathbf{b} \in \mathbb{R}^m, AA^T(AA^T)^{-1}\mathbf{b} = \mathbf{b} \\
\Rightarrow &A\mathbf{x} = \mathbf{b} \text{ is consistent } \forall \mathbf{b} \in \mathbb{R}^m \\
\Rightarrow &\text{Each row of } A \text{ has a pivot.} \\
\Rightarrow &\text{The rows of } A \text{ form a linearly indep. set.}
\end{aligned}$$

85

(1) First we note that let $\mathbf{y} \in \mathbb{R}^n$, if $\mathbf{y}^T\mathbf{y} = \mathbf{0}$, then $\mathbf{y} = \mathbf{0}$. Now we will show that the kernel of ABB^TA contains only zero vector.

$$\begin{aligned}
&ABB^TA^T\mathbf{x} = \mathbf{0}. \\
\Rightarrow &\mathbf{x}^TABB^TA^T\mathbf{x} = (B^TA^T\mathbf{x})^T(B^TA^T\mathbf{x}) = \mathbf{x}^T\mathbf{0} = \mathbf{0} \\
\Rightarrow &B^TA^T\mathbf{x} = \mathbf{0} \\
&\text{(Since } B \text{ is invertible.)} \\
\Rightarrow &A^T\mathbf{x} = \mathbf{0} \\
&\text{(Since each row of } A \text{ has a pivot, the row of } A \text{ form a linearly independet set.} \\
&\text{Therefore each column of } A^T \text{ has a pivot. Since each column of } A^T \text{ has a pivot,} \\
&\text{we have } \mathbf{x} = \mathbf{0}) \\
\Rightarrow &\mathbf{x} = \mathbf{0}
\end{aligned}$$

Since $\text{Ker}(ABB^TA^T) = \{\mathbf{0}\}$, by **Theorem** 9, ABB^TA^T is invertible.

(2) No. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Chapter 4

Determinants

4.1

4

$$\text{Det}\begin{bmatrix} 3 & 7 & 2 \\ -2 & 4 & 1 \\ 6 & -2 & 4 \end{bmatrix} = \frac{1}{2}\frac{1}{3}\text{Det}\begin{bmatrix} 6 & 14 & 4 \\ -6 & 12 & 3 \\ 6 & -2 & 4 \end{bmatrix} = \frac{1}{2}\frac{1}{3}\text{Det}\begin{bmatrix} 6 & 14 & 4 \\ 0 & 26 & 7 \\ 0 & -16 & 0 \end{bmatrix} = -\frac{1}{2}\frac{1}{3}\text{Det}\begin{bmatrix} 6 & 14 & 4 \\ 0 & -16 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$
$= -\frac{1}{2}\frac{1}{3}6(-16)7 \neq 0 \Rightarrow A$ is invertible.

15

By **Theorem 3**, the determinant of the matrix is $(12t+1)(12t+7) - (12t+3)(12t+2) = 144t^2 + 96t + 7 - 144t^2 - 60t - 6 = 36t + 1$. Therefore, if $t = \frac{-1}{36}$, then this matrix is non-invertible.

20

Use **Theorem 6** and **Theorem 7** in the textbook.

23

Here we provide two methods:

(1) Review Example 6 and use Euclidean Algorithm. This one is a little bit troublesome, but since this problem is in this section, we still mention this method.

(2) The second method is based on the concept of cofactor expansion which will be mentioned in section 4.2. Let A_n be a $n \times n$ matrix with only integer entries. Apparently, $\text{Det}(A_1) = A_1 \in \mathbb{Z}$. Suppose $\text{Det}(A_k) \in \mathbb{Z}$, then $\text{Det}(A_{k+1}) = \sum_{j=1}^{k+1} a_{1j}(-1)^{1+j}\text{Det}(\mathbf{M}^{1j}) \in \mathbb{Z}$.

34

$$\mathrm{Det}(\alpha A) = \mathrm{Det}\begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha a_{n1} & \alpha a_{n2} & \cdots & \alpha a_{nn} \end{bmatrix} = \alpha \mathrm{Det}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha a_{n1} & \alpha a_{n2} & \cdots & \alpha a_{nn} \end{bmatrix}$$

$$= \cdots = \alpha^n \mathrm{Det}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \alpha^n \mathrm{Det}(A).$$

37

By **Theorem** 4, the columns of the matrix are linear dependent. So if we change the order of the column, by **Theorem** 4 again, the determinant is still 0. Hence we have

$$\mathrm{Det}\begin{bmatrix} x_1 & x_2 & 1 \\ v_1 & v_2 & 1 \\ w_1 & w_2 & 1 \end{bmatrix} = 0 \Leftrightarrow \mathrm{Det}\begin{bmatrix} 1 & x_1 & x_2 \\ 1 & v_1 & v_2 \\ 1 & w_1 & w_2 \end{bmatrix} = 0.$$

Since $\mathbf{v}$ and $\mathbf{w}$ are distinct point, without loss of generality, we assume that $v_1 \neq w_1$.

$$\mathrm{Det}\begin{bmatrix} 1 & x_1 & x_2 \\ 1 & v_1 & v_2 \\ 1 & w_1 & w_2 \end{bmatrix} = \mathrm{Det}\begin{bmatrix} 1 & v_1 & v_2 \\ 1 & w_1 & w_2 \\ 1 & x_1 & x_2 \end{bmatrix} = \mathrm{Det}\begin{bmatrix} 1 & v_1 & v_2 \\ 0 & w_1 - v_1 & w_2 - w_2 \\ 0 & x_1 - v_1 & x_2 - v_2 \end{bmatrix}$$

$$= \mathrm{Det}\begin{bmatrix} 1 & v_1 & v_2 \\ 0 & w_1 - v_1 & w_2 - v_2 \\ 0 & 0 & (x_2 - v_2) - \frac{(w_2 - v_2)(x_1 - v_1)}{w_1 - v_1} \end{bmatrix} = (w_1 - v_1)(x_2 - v_2) - (w_2 - v_2)(x_1 - v_1) = 0$$

This is a line in $\mathbb{R}^2$ passing through $\mathbf{v}$ and $\mathbf{w}$. If $v_1 = w_1$, then you can just swap the second column and the third column and you will have the similar result.

46

a. First, we multiply the first row by (-1) and add it to the 2nd row and the 3rd row.

$$\mathrm{Det}\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = \mathrm{Det}\begin{bmatrix} 1 & a & a^2 \\ 0 & b - a & b^2 - a^2 \\ 0 & c - a & c^2 - a^2 \end{bmatrix}.$$

Now, we have two cases:

(i) If $b - a \neq 0$, $\mathrm{Det}\begin{bmatrix} 1 & a & a^2 \\ 0 & b - a & b^2 - a^2 \\ 0 & c - a & c^2 - a^2 \end{bmatrix} = \mathrm{Det}\begin{bmatrix} 1 & a & a^2 \\ 0 & b - a & b^2 - a^2 \\ 0 & 0 & c^2 - a^2 - \frac{(c-a)(b^2-a^2)}{b-a} \end{bmatrix} = (a - b)(a - c)(c - b)$.

(ii) If $b - a = 0$, then the second row is a zero row and $\mathrm{Det}\begin{bmatrix} 1 & a & a^2 \\ 0 & b - a & b^2 - a^2 \\ 0 & c - a & c^2 - a^2 \end{bmatrix} = 0 = (a - b)(a - c)(c - b)$.

b. We first multiply the first row by (-1) and add it to the second row.

$$\text{Det}\begin{bmatrix}1 & x & y\\ 1 & x_0 & y_0\\ 0 & 1 & m\end{bmatrix} = \text{Det}\begin{bmatrix}1 & x & y\\ 0 & x_0-x & y_0-y\\ 0 & 1 & m\end{bmatrix} = -\text{Det}\begin{bmatrix}1 & x & y\\ 0 & 1 & m\\ 0 & x_0-x & y_0-y\end{bmatrix}$$
$$= -\text{Det}\begin{bmatrix}1 & x & y\\ 0 & 1 & m\\ 0 & 0 & (y_0-y)-m(x_0-x)\end{bmatrix} = (y-y_0)-m(x-x_0)=0.$$

c. Similar to Ex.37.

50

First we swap the first row and second row and we have

$$\text{Det}\begin{bmatrix}3 & x & 5\\ 2 & 1 & 0\\ y & 4 & 7\end{bmatrix} = -\text{Det}\begin{bmatrix}2 & 1 & 0\\ 3 & x & 5\\ y & 4 & 7\end{bmatrix} = -\text{Det}\begin{bmatrix}2 & 1 & 0\\ 0 & x-\frac{3}{2} & 5\\ 0 & 4-\frac{y}{2} & 7\end{bmatrix} = 0.$$

We know that the determinant of a square matrix is 0 if and only if its columns form a linearly dependent set, so if we change the order of the columns, the determinant still remains 0.

$$-\text{Det}\begin{bmatrix}2 & 1 & 0\\ 0 & 5 & x-\frac{3}{2}\\ 0 & 7 & 4-\frac{y}{2}\end{bmatrix} = 0 = \text{Det}\begin{bmatrix}2 & 1 & 0\\ 0 & 5 & x-\frac{3}{2}\\ 0 & 0 & (4-\frac{y}{2})-\frac{7}{5}(x-\frac{3}{2})\end{bmatrix}$$
$$\Leftrightarrow 5\left(4-\frac{y}{2}\right)-7\left(x-\frac{3}{2}\right)=0$$

4.2

1

First we find the explicit form of f. We will use the second row to do cofactor expansion.

$$f(\mathbf{x}) = x_1(-1)^3\text{Det}\begin{bmatrix}7 & 2\\ 4 & -3\end{bmatrix} + x_2(-1)^4\text{Det}\begin{bmatrix}3 & 2\\ 5 & -3\end{bmatrix} + x_3(-1)^5\text{Det}\begin{bmatrix}3 & 7\\ 5 & 4\end{bmatrix}$$
$$= 29x_1 - 19x_2 + 23x_3 = \begin{bmatrix}23 & 19 & 23\end{bmatrix}\mathbf{x}.$$

Since $\begin{bmatrix}23 & 19 & 23\end{bmatrix}$ is a matrix, apparently, f is a linear function of $\mathbf{x}$.

7

Similar to Example 10 in the textbook.

8

First, we apply a replacement operation on this matrix.

$$\text{Det}\begin{bmatrix} -2 & 1 & 2 & 5 \\ 2 & 0 & 4 & 5 \\ 3 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix} = \text{Det}\begin{bmatrix} -2 & 1 & 2 & 5 \\ 2 & 0 & 4 & 5 \\ 3 & 1 & 0 & 0 \\ -5 & 0 & 0 & 0 \end{bmatrix}$$

Now, we use the 4th row to do a cofactor expansion.

$$=(-5)(-1)^{(4+1)}\text{Det}\begin{bmatrix} 1 & 2 & 5 \\ 0 & 4 & 5 \\ 1 & 0 & 0 \end{bmatrix}$$

And then we use the 3rd row to do a cofactor expansion.

$$=5 \cdot 1(-1)^{(3+1)}\text{Det}\begin{bmatrix} 2 & 5 \\ 4 & 5 \end{bmatrix} = -50.$$

15

Use **Theorem 6** and **Example 11** in the textbook.

25

Similar to Example 4.

28

First, we find the explicit form of this function. We find the cofactor expansion of $\text{Det}(A)$ using the pth row.

$$\text{Det}(A)(x) = (-1)^{p+q}x\text{Det}(\mathbf{M}^{pq}) + \sum_{j=1, j\neq q}^{n} (-1)^{p+j}a_{pj}\text{Det}(\mathbf{M}^{pj}).$$

Now, we find the derivative of $\text{Det}(A)$. Since the derivative of a constant is 0, so the derivative of this part is 0.

$$\frac{d}{dx}\text{Det}(A)(x) = (-1)^{p+q}\text{Det}(\mathbf{M}^{pq}).$$

33

Consider the cofactor expansion of $\text{Det}(A)$ of an $n \times n$ matrix A using row i,

$$\text{Det}(A) = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}\text{Det}(M^{ij}).$$

There is a j_0 such that $\text{Det}(M^{ij_0}) \neq 0$, otherwise $\text{Det}(A) = 0$, which is a contradiction to the invertibility of A. If we change the (ij_0)-th entry a_{ij_0} of A to

$$a'_{ij_0} = -\frac{(-1)^{i+j_0}}{\text{Det}(M^{ij_0})} \sum_{j=1, j\neq j_0}^{n} (-1)^{i+j}a_{ij}\text{Det}(M^{ij}),$$

then the new matrix A' has

$$\begin{aligned}\mathrm{Det}(A') &= \sum_{j=1, j\neq j_0}^{n} (-1)^{i+j} a_{ij}\mathrm{Det}(M^{ij}) + (-1)^{i+j_0} a'_{ij}\mathrm{Det}(M^{ij_0}) \\ &= \sum_{j=1, j\neq j_0}^{n} (-1)^{i+j} a_{ij}\mathrm{Det}(M^{ij}) - \sum_{j=1, j\neq j_0}^{n} (-1)^{i+j} a_{ij}\mathrm{Det}(M^{ij}) \\ &= 0.\end{aligned}$$

Thus A' is non-invertible.

36

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. $\mathrm{Det}(AB) = \mathrm{Det}(BA) = 0$, but $AB \neq BA$.

41

Let A be a square matrix. If change the entry a_{ij} to a different a'_{ij} and call this new matrix A', then we have $\mathrm{Det}(A) - \mathrm{Det}(A') = (-1)^{i+j}\mathrm{Det}(\mathbf{M}^{ij})(a_{ij} - a'_{ij})$.
Therefore, if $\mathrm{Det}(\mathbf{M}^{ij}) \neq 0$, then we can change a_{ij} to a different a'_{ij} without changing the determinant.

42

(i) Since $1 = \mathrm{Det}(\mathbf{I}) = \mathrm{Det}(\mathbf{A}\mathbf{A}^{-1}) = \mathrm{Det}(\mathbf{A})\mathrm{Det}(\mathbf{A}^{-1})$, $[\mathrm{Det}(\mathbf{A}^{-1})] = [\mathrm{Det}(\mathbf{A})]^{-1}$.

(ii) Let $f(x) = x^2$. $\mathrm{Det}[f(A)] = \mathrm{Det}(A^2) = \mathrm{Det}(A)\mathrm{Det}(A) = [\mathrm{Det}(A)]^2 = f[\mathrm{Det}(A)]$.
In this case, f is not limited to invertible matrices.

44

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. $\mathrm{Det}(A + B) = 0 \neq \mathrm{Det}(A) + \mathrm{Det}(B) = 2$.

45

(1) $\mathrm{Det}(AB) = \mathrm{Det}(A)\mathrm{Det}(B) = 17 \times 2 = 34$.

(2) $\mathrm{Det}(BA^T) = \mathrm{Det}(B)\mathrm{Det}(A^T) \underbrace{=}_{\text{Thm 3}} \mathrm{Det}(B)\mathrm{Det}(A) = 34$.

(3) Use Exercise 4.1.34.

(4) Use Exercise Exercise 42.

(5) Use Exercise 4.1.34.

53

$\mathrm{Det}(C) = \mathrm{Det}(AB) = \mathrm{Det}(A)\mathrm{Det}(B) = 6 \times (-6) = -36$.

Chapter 5

Vector Subspaces

5.1

1

Use **Theorem** 5.1.1 in the lecture note. Remind you that you need to check that it's not an empty set.

4

No. You can prove it by the failure of the closure property of vector addiction. Let this set be S. Define $\mathbf{v}(\alpha, \beta) = (3\alpha - 2\beta, \alpha + \beta + 2, -2\beta + \alpha, 5\alpha + \pi\beta, \alpha + \beta)$. Then $\mathbf{v}(0,0) + \mathbf{v}(1,0) \notin S$.

8

By **Theorem 5.1.9**, if two matrices are row equivalent to each other, then they have the same row space. So here, you just need to show that these two matrices are row equivalent to each other.

9

From the definition of U, we have $x_1 = -x_2 + x_3$. The description of the image of U by the mapping T can be written as

$$T[U] = \{3(-x_2 + 2x_3) - 2x_2 + 5x_3, 2(-x_2 + 2x_3) + x_2 - x_3 | x_2, x_3 \in \mathbb{R}\} = \mathbb{R}^2.$$

15

No. Let $U = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \middle| a \in \mathbb{R} \right\}$ and $W = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \middle| b \in \mathbb{R} \right\}$. We can see that $U \cup W$ fails the closure property of vector addition.

16

Before we find this exact condition, we need to mention a lemma first.
Lemma 1. Let U be a subspace of a vector space V and $u, u' \in V$. If $u \in U$ and $u' \notin U$, then $u + u' \notin U$.
Proof. Suppose that $u + u' \in U$, then by the closure property of vector addition $(u + u') + (-u) = u' \in U$ $(\rightarrow\leftarrow)$.
Now we can solve exercise 16. Let $u \in U$ and $w \in W$. If $U \cup W$ is a subspace, then $u + w \in U$ or $u + w \in W$ by **Lemma 1**. We have $w \in U$ or $u \in W \Rightarrow W \subseteq U$ or $U \subseteq W$. For the converse, if

$W \subseteq U$ or $U \subseteq W$, trivially, $U \cup W$ is a vector subspace. Therefore, $U \cup W$ is a vector subspace if and only if $W \subseteq U$ or $U \subseteq W$.

19

The hint of this problem in the back of the textbook has a typo. You should use **Theorem** 9 instead of 7 to solve this exercise.

24

A remark: A subspace W of a vector space V is called the smallest subspace containing a nonempty subset X of V if $X \subseteq W$ and for any subspace U of V such that $X \subseteq U$, we have $W \subseteq U$.

27

(1) Here we need to show that $D(\mathbb{P}_n) \subseteq \mathbb{P}_{n-1}$ and $D(\mathbb{P}_n) \supseteq \mathbb{P}_{n-1}$.

"$\subseteq$" We leave it to you.

"$\supseteq$" Let $\mathbf{u} = a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \cdots + a_0 \in \mathbb{P}_{n-1}$ where $a_i \in \mathbb{R}$. Now we want to find an element in $\mathbb{P}_n$ such that its derivative is $\mathbf{u}$. Let $\mathbf{v} = \dfrac{a_{n-1}}{n}t^n + \dfrac{a_{n-2}}{n-1}t^{n-1} + \cdots + a_0 t + c \in \mathbb{P}_n$ where $c \in \mathbb{R}$. Since $D(\mathbf{v}) = \mathbf{u}$, we have $\mathbf{u} \in D(\mathbb{P}_n)$. Therefore, $\mathbb{P}_{n-1} \subseteq D(\mathbb{P}_n)$.

(2) Similar to (1).

31

Recall that for two matrices A and B, $A = B \Leftrightarrow [A]_{ij} = [B]_{ij}, \forall i, j$. Then

$$\left(\sum_{k=1}^{n} \mathbf{u}_k \mathbf{v}_k^T\right)_{ij} = \sum_{k=1}^{n} \left(\mathbf{u}_k \mathbf{v}_k^T\right)_{ij} = \sum_{k=1}^{n} \left(\mathbf{u}_k\right)_i \left(\mathbf{v}_k\right)_j = \sum_{k=1}^{n} A_{ik} B_{kj} = (AB)_{ij}.$$

Hence, AB is equal to this matrix.

32

This exercise contains three parts.

(1) In the first part, you are asked to show that the restriction of L to U is linear. This part is very easy, so I leave it to the readers.

(2) For the second part, its domain is apparently just U by the definition.

(3) For the third part, the answer is no, L and the restriction of L to U does not necessarily have the same range. Here is a counter example. Let $\begin{array}{rccc} L: & \mathbb{R}^2 & \to & \mathbb{R} \\ & (x, y) & \mapsto & x + y \end{array}$ and $U = \{(x, y) \mid x + y = 0\}$. Then $\text{Range}(L) = \mathbb{R}$ but $\text{Range}(L|U) = \{0\}$.

33

No. Since an empty set is not a vector space.

34

Let A be $m \times n$ and B be $n \times k$.
Let $\mathbf{a}_i$ be the ith column of A and $\mathbf{b}_j$ be the jth column of B.
Note that $\text{Col}(A) \subseteq \mathbb{R}^m$, $\text{Col}(B) \subseteq \mathbb{R}^n$, and $\text{Col}(AB) \subseteq \mathbb{R}^m$

(1) Apparently, $\text{Col}(AB) \nsubseteq \text{Col}(B)$ if $m \neq n$.

(2) Note that $AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_k \end{bmatrix}$.
Let $\mathbf{v} = c_1(A\mathbf{b}_1) + c_2(A\mathbf{b}_2) + \cdots + c_k(A\mathbf{b}_k) \in \text{Col}(AB)$, where $c_i \in \mathbb{R}$.
Then $\mathbf{v} = \sum_{i=1}^{k} c_i \sum_{j=1}^{n} B_{jk}\mathbf{a}_j \in \text{Col}(A)$. Therefore, $\text{Col}(AB) \subseteq \text{Col}(A)$.

35

If $A = B = I$, then $\text{Col}(AB) = \text{Col}(A) = \text{Col}(B)$.
If $A = I$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $\text{Col}(AB) \neq \text{Col}(A)$.

36

This problem is similar to Exercise 34. If you take the transpose of AB, then you can just use exercise 34 to prove it.

44, 48, and 49

These three exercises are out of the scope of section 5.1. The concept of bases will be covered in section 5.2.

5.2

1

Similar to Example 2.

2

Similar to Example 7.

11

By rank-nullity theorem, we have $6 = \text{Dim}(\text{Ker}(A)) + \text{Dim}(\text{Rank}(A))$. Since the all solutions of $\mathbf{Ax} = \mathbf{0}$ are multiples of one nonzero vector, the dimension of the kernel of A is 1. Therefore, $\text{Dim}(\text{Rank}(A)) = 6 - 1 = 5$. Since each row of A has a pivot, A is onto by **Theorem 1.2.4**.

7

For this exercise, just find the general solution of $A\mathbf{x} = \mathbf{0}$, then you are done. The independent calculation here means that you need to multiply A by the elements in the basis you found and make sure it's 0.

9

For this exercise, you need to make sure L is injective and surjective. Since it a little bit troublesome, so I will just leave some hints.

(i) (Injective)
Let $p_1(t) = a_0 + a_1 t + \cdots + a_n t^n$ and $p_2(t) = b_0 + b_1 t + \cdots + b_n t^n$. You need to show that if $L(p_1(t)) = L(p_2(t))$, then $p_1(t) = p_2(2)$.

(ii) (Surjective)
Show that for all $p_2(t) = b_0 + b_1 t + \cdots + b_n t^n$, there exists an $p_1(t) \in \mathbb{P}_n$ such that $L(p_1(t)) = L(p_2(t))$.

10

Similar to Example 6. you can just take the coefficients of those polynomials as vectors and collect those vectors as a matrix, then you can find a basis by the similar method for matrices.

14

For this exercise, if you can show that the degrees of $p, p', \cdots, p^{(n)}$ are $n, (n-1), \cdots, 0$, respectively, then by Exercise 2.4.24, this set is a basis of $\mathbb{P}_n$.

26

Similar to Example 17.

27

Similar to Example 13.

31,33

Use **Theorem 5.2.11** and **Theorem 5.2.12**.

34

(i) First, we find the dimension of $\mathbb{R}^{m\times n}$. Here, we list a basis of $\mathbb{R}^{m\times n}$. Note that

$$\left\{ A \in \mathbb{R}^{m\times n} \middle| [A]_{ij} = \begin{cases} 1, \text{if } i = s \text{ and j=t} \\ 0, \text{ otherwise.} \end{cases} , \forall\ 1 \le s \le m, 1 \le t \le n \right\}$$

is a basis of $\mathbb{R}^{m\times n}$. This set collects all the matrices such that all entries are zeros except a one at some position. Since there are mn different matrices in this set, $\text{Dim}(\mathbb{R}^{m\times n}) = mn$.

(ii) For the second part, again, we list a basis of such subspace. Note that

$$\left\{ A \in \mathbb{R}^{n\times n} \middle| [A]_{ij} = [A]_{ji} = \begin{cases} 1, \text{if } i = s \text{ and j=t} \\ 0, \text{ otherwise.} \end{cases} , \forall\ 1 \le s \le t \le n \right\}$$

is a basis of this subspace. The dimension of this subspace is $\dfrac{n(n+1)}{2}$.

35

Here we consider the trace function

$$\begin{aligned} \mathrm{Tr}: \mathbb{R}^{n\times n} &\to \quad \mathbb{R} \\ A \quad &\mapsto \sum a_{ii} \end{aligned}.$$

Let the space of $n \times n$ matrices having zero trace be V. We can see that $V = \mathrm{Ker}(\mathrm{Tr})$. It's easy to show that the trace function is a linear transformation. By **Theorem** 5.2.17, we have $\mathrm{Dim}(\mathrm{Ker}(\mathrm{Tr})) = \mathrm{Dim}(\mathrm{Domain}(\mathrm{Tr})) - \mathrm{Dim}(\mathrm{Range}(\mathrm{Tr})) = n^2 - 1 \Rightarrow \mathrm{Dim}(V) = n^2 - 1$.

36

Use Exercise 5.1.27.

37

First we find a basis of V. Let S be a basis of $\mathrm{Null}(\mathbf{B})$. Note that $|S| = \mathrm{Nullity}(\mathbf{B})$. Since

$$\left\{ A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \in \mathbb{R}^{n\times n} \,\middle|\, \begin{cases} \mathbf{a}_i \in S, \text{ if } i = t \\ \mathbf{a}_i = \mathbf{0}, \text{ otherwise} \end{cases}, \forall\, 1 \le t \le n \right\}$$

is a basis of V. The dimension of V is $n \cdot \mathrm{Nullity}(\mathbf{B})$.

40

Similar to Ex. 35.

43

Let V be this vector space and $U = \{\mathbf{u}_1, \mathbf{u}_1, \cdots, \mathbf{u_k}, \cdots\}$. You need to show that:

(i) U is linearly independent.

(ii) $\mathrm{Span}(U) = V$. A reminder: any linear combination contains only finitely many terms.

45

By **Theorem** 5.2.19, since the dimension of the row space is equal to the dimension of the column space, the columns also span $\mathbb{R}^n$.

46

Use **Theorem 5.2.7** and **Theorem 5.2.19**.

51

Use **Theorem 5.2.11** and **Theorem 5.2.12**.

53,54

Use method of contradiction.

55

Use **Theorem 2.4.10**.

56

Use **Theorem 5.2.11**.

57

Theorem 5.2.6 states that for a finite-dimensional vector space, every bases contains the same number of elements. Since the dimension of $\mathbb{R}^m$ is m, by **Theorem 5.2.6**, if the columns form a basis of $\mathbb{R}^m$, then $n = m$.

58

Use **Theorem 5.2.14**.

62

Let $\mathbf{x}, \mathbf{y}$, and $\mathbf{u} \in X$.

(i) First, we check the reflexive property. Since U is a subspace, it contains $\mathbf{0}$, so the reflexive property holds.

(ii) Secondly, we check the symmetric property. If $\mathbf{x} - \mathbf{y} \in U$, then $\mathbf{y} - \mathbf{x} = -(\mathbf{x} - \mathbf{y}) \in U$.

(iii) Lastly, we will check the transitive property. If $\mathbf{x} - \mathbf{y} \in U$ and $\mathbf{y} - \mathbf{u} \in U$, then $\mathbf{x} - \mathbf{u} = (\mathbf{x} - \mathbf{y}) + (\mathbf{y} - \mathbf{u}) \in U$.

Since all three conditions are satisfied, it is an equivalence relation on X.

65

By observing the rule, we can see that once x_i and x_{i-1} is determined, the whole sequence is determined. Therefore the subspace can be spanned by the linearly independent set $\{(\cdots, x_{i-1} = a, x_i = 0, \cdots), (\cdots, x_{i-1} = 0, x_i = b, \cdots)\}$ where $a, b \neq 0$, and the dimension of the subspace is 2.

66

Let $\mathbf{v}$ be the vector in V such that has a unique expression as a linear combination of vectors in S. Suppose S is linearly dependent, then $\mathbf{0}$ can be expressed in at least two different ways of as a linear combination of vectors in S. Since $\mathbf{v} = \mathbf{v} + \mathbf{0}$, $\mathbf{v}$ also has at least two different expressions as linear combinations of vectors in S, a contradiction.

67

Similar to Ex.66.

72

If V spans W, then any linearly independent set can has at most 7 elements by **Theorem 2.4.10**. In this case, U cannot be linearly independent. Therefore, U spans W and V is linearly independent.

$$\Rightarrow \begin{cases} U \text{ spans } W \underbrace{\Rightarrow}_{\text{Thm. 11}} \operatorname{Dim}(W) \leq 8 \\ V \text{ is linearly independent } \underbrace{\Rightarrow}_{\text{Thm. 12}} 7 \leq \operatorname{Dim}(W) \end{cases} \Rightarrow \operatorname{Dim}(W) = 7 \text{ or } 8.$$

74

Suppose $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ is linearly dependent and $\sum_{i=1}^{n} \alpha_i \mathbf{x}_i = \mathbf{0}$ where $\sum_{i=1}^{k} |\alpha_i| > 0$.
Then $T(\sum_{i=1}^{n} \alpha_i \mathbf{x}_i) = \sum_{i=1}^{n} \alpha_i \mathbf{y}_i = \mathbf{0}$, a contradiction.
By the above argument, we can see that as long as $\{\mathbf{y}_1, \cdots, \mathbf{y}_n\}$ is a linearly independent set, then $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ is linearly independent.

77

For this problem, you don't need to find the range of T. You only need to show that kernel of T contains only zero, then you can use **Theorem** 5.2.17 to find the dimension of the range of T.

82

With the dimension of the kernel space, we can use rank-nullity theorem to find the dimension of the column space. By **Theorem 5.2.19** the row space and the column space have the same dimensions, so we have the dimension of the row space. And the range of a matrix is its column space, so the dimension of the range is equal to dimension of the column space.

87

Let $\mathbf{A}$ be a matrix. If $\mathbf{x}$ is in Col($\mathbf{A}^T$), there exists a $\mathbf{y}$ such that $\mathbf{A}^T\mathbf{y} = \mathbf{x}$ and $\mathbf{x}^T = (\mathbf{y}^T\mathbf{A})$. And if $\mathbf{x}$ is in Ker($\mathbf{A}$), then $\mathbf{A}\mathbf{x} = \mathbf{0}$. Therefore, we have $\mathbf{x}^T\mathbf{x} = \mathbf{y}^T\mathbf{A}\mathbf{x} = \mathbf{0}$ which implies $\mathbf{x} = \mathbf{0}$.

88

The equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ is consistent $\forall$ $\mathbf{b} \in \mathbb{R}^m$.
$\Leftrightarrow$ Col($\mathbf{A}$)$= \mathbb{R}^m$.
$\Leftrightarrow$ Dim(Col($\mathbf{A}$))$= m$
$\Leftrightarrow$ Dim(Row($\mathbf{A}^T$))$= m$
$\Leftrightarrow$ Dim(Ker($\mathbf{A}^T$))$= m - m = 0$ (By Rank-nullity Theorem)
$\Leftrightarrow$ Ker($\mathbf{A}^T$)$= \{\mathbf{0}\}$

89

For every given k,n,m, there exists more than one such matrices. Here I will give you an example satisfies the requirements. Let

$$A_{n\times m} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_k & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}.$$

Since the rank of this matrix is k, the dimension of its kernel is $n - k$.

5.3

1

Similar to Example 2.

4

Similar to Example 3.

6

Similar to Example 6.

9

Similar to Example 10.

11

Use **Theorem 5.3.1**.

14 and 17

Use **Theorem 5.3.4**.

19

Let $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = A$ and $\begin{bmatrix} 2 & \alpha \\ 0 & 2 \end{bmatrix} = B$. Find the solution of $XA - BX = 0$, then you will have $\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & \alpha \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}$.

22

S is linearly independent.
$\Rightarrow$ There exists some $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n \in S$ and $\alpha_1, \alpha 2 \cdots, \alpha_n \in \mathbb{R}^n$ such that $\sum_{i=1}^n \alpha_i \mathbf{x}_i = \mathbf{0}$, where $\sum_{i=1}^n |\alpha_i| > 0$.
$\Rightarrow \left[\sum_{i=1}^n \alpha_i \mathbf{x}_i \right]_{\mathcal{B}} = [\mathbf{0}]_{\mathcal{B}}$
$\Rightarrow \sum_{i=1}^n \alpha_i [\mathbf{x}_i]_{\mathcal{B}} = 0$
$\Rightarrow \{[\mathbf{x}]_{\mathcal{B}} \mid \mathbf{x} \in S\}$ is linearly dependent.
You can show that the opposite direction is also correct, so the coordinate doesn't matter whether a set is linear independent or not.

24

(1)Use **Theorem 4.2.2**.
(2)For the converse, you can find a counter example in exercise 19. If you let $\alpha = 0$, then you will have two matrices with the same determinants be not similar to each other.

27

(1) If A is invertible, then $AB = (AB)T = (AB)(AA^{-1}) = A(BA)A^{-1}$, so $AB \simeq BA$.

(2) The invertibility hypothesis is not necessary. Let $A = B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $AB \simeq BA$.

28

No. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, and $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $PA = BP$, but $A \not\simeq B$.

29

The hint of this exercise in the back of the textbook is wrong.

32

Let P be the transition matrix for $\mathcal{B}$ to $\mathcal{C}$ where $\mathcal{B}$ and $\mathcal{C}$ are two bases of a vector space $\mathbb{R}^n$. Let $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u_2}, \cdots, \mathbf{u}_n\}$. First we note that $P = \begin{bmatrix}[\mathbf{u}_1]_\mathcal{C} & [\mathbf{u}_2]_\mathcal{C} & \cdots & [\mathbf{u}_n]_\mathcal{C}\end{bmatrix}$. By Ex.22, since $\mathcal{B}$ is linearly independent, columns of P also form a linearly independent set. Then by Theorem 3.2.10, P is invertible.

34

Let $A = \begin{bmatrix}2 & 1\\0 & 2\end{bmatrix}, B = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$, and $P = \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$. Then you will find that $AP = PB$, but A is not similar to B.

35

The hint of this exercise in the back of the textbook is wrong.

43

a. Let $a : y = 0, b : x = 0$, and $c : x = 1$. bRa and aRc, but b and c don't have a point in common, so it fails the transitive property.

b. We leave it to you.

44

Similar to Example 2.

46

By Theorem 5.3.4, we have

$$\begin{aligned}[T(\mathbf{A})]_\mathcal{E} &= \begin{bmatrix}[T(\mathbf{E}_{11})]_\mathcal{E} & [T(\mathbf{E}_{12})]_\mathcal{E} & \cdots & [T(\mathbf{E}_{22})]_\mathcal{E}\end{bmatrix}[\mathbf{A}]_\mathcal{E}\\ &= \begin{bmatrix}a & c & 0 & 0\\ b & d & 0 & 0\\ 0 & 0 & a & c\\ 0 & 0 & b & d\end{bmatrix}[\mathbf{A}]_\mathcal{E}.\end{aligned}$$

This is the matrix representation of T.

49

Similar to Example 10.

50.c

If $A \simeq B$, there exists an invertible matrix P such that $A = P^{-1}BP$.
$\Rightarrow A^k = (P^{-1}BP)^k = \underbrace{(P^{-1}BP)(P^{-1}BP)\cdots(P^{-1}BP)}_{k} = P^{-1}B^kP$

$\Rightarrow A^k \simeq B^k$.

54

Use **Theorem 5.3.4**.

55

Similar to Example 10.

Chapter 6

Eigensystems

6.1

1 and 8

Similar to Example 2.

11

Let this matrix be A and $\mathbf{u} = (1, 1, \cdots, 1)$. Since each column of this matrix sums to 3. We have $A^T\mathbf{u} = 3\mathbf{u}$. By Exercise 32, since 3 is an eigenvalue of A^T, 3 is also an eigenvalue of A.

14

Since $(3-\lambda)(2-\lambda) - 7\alpha = \lambda^2 - 5\lambda + (6 - 7\alpha)$, $\lambda = \dfrac{5 \pm \sqrt{25 - 4(6 - 7\alpha)}}{2} = \dfrac{5 \pm \sqrt{1 + 28\alpha}}{2}$. If λ is real, then $\alpha \geq \dfrac{-1}{28}$.

16

Similar to Example 2.

18

Similar to Example 4.

23

a. Let λ be an eigenvalue of the matrix, then $(a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad - bd) = 0$. We have $\lambda = \dfrac{a + b \pm \sqrt{(a-d)^2 + 4bc}}{2}$. λ is real if only if $(a-d)^2 + 4bc \geq 0$.

b. Since $(a-d)^2$ is already non-negative.

c. If the matrix is symmetric, then $b = c$, we have $bc = b^2 \geq 0$.

27

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $\text{Det}(A - 1 \cdot I) = 0 = (a-1)(d-1) - bc$, then 1 is one of A's eigenvalues.

28

Similar to exercise 27.

29

Let A be a square matrix. 0 is one of A's eigenvalues if and only if the determinant of A is 0. Therefore, by **Theorem 4.1.5**, a square matrix is invertible if and only if 0 is not one of its eigenvalues.

31

Since λ is an eigenvalue of A, $\exists$ a nonzero $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.
$\Rightarrow A^{-1}A\mathbf{x} = \lambda A^{-1}\mathbf{x}$ (Since A is invertible)
$\Rightarrow A^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}$ (divide both side of the equaiton by λ)
$\Rightarrow \lambda^{-1}$ is an eigenvalue of A^{-1}.

32

Let A be a square matrix. Since $\det(A - \lambda I) = \det((A - \lambda I)^T) = \det(A^T - \lambda I)$, A and A^T have the same characteristic polynomial. Therefore, A and A^T have the same eigenvalues.

33

Let A and B be two square matrices and $A = PBP^{-1}$ where P is invertible. Then $\mathrm{Det}(A - \lambda I) = \mathrm{Det}(PBP^{-1} - \lambda I) = \mathrm{Det}(P^{-1})\mathrm{Det}(PBP^{-1} - \lambda I)\mathrm{Det}(P) = \mathrm{Det}(B - \lambda I)$. Therefore A and B have the same eigenvalues.

35

Let $\mathbf{x}^{(r-1)} = (a, b)$, then $\mathbf{x}^{(r)} = (\frac{1}{2}a + \frac{1}{3}b, \frac{1}{2}a + \frac{2}{3}b))$. The sum of the two components of $\mathbf{x}^r$ is $a + b$ which is independent of r.

37

Let B be a square matrix with $\mathbf{u}_i$ as its ith column and D be a diagonal matrix such that $d_{ii} = \lambda_i \neq 0$ for all i. Let $A = BDB^{-1}$, then $\mathrm{Det}(A) = \mathrm{Det}(B)\mathrm{Det}(D)\mathrm{Det}(B^{-1}) \neq 0$, so A is invertible. Since $AB = BD$, $A\mathbf{u}_i = \lambda_i\mathbf{u}_i$, $\forall\ i$. A has $\mathbf{u}_i$ as eigenvectors. As long as we don't choose 0 as an eigenvalue of A, the eigenvalues can be assigned freely.

38

Let $\mathbf{v}$ be an eigenvector of A with the eigenvalue λ, then $(A\mathbf{v})^* = (\lambda\mathbf{v})^*$. Since A is real, $A\mathbf{v}^* = \lambda^*\mathbf{v}^*$.

$$
\begin{aligned}
&(A\mathbf{v}^*)^T\mathbf{v} = \lambda^*(\mathbf{v}^*)^T\mathbf{v} \\
\Rightarrow &(\mathbf{v}^*)^T \underbrace{A^T}_{=A} \mathbf{v} = \lambda^*||\mathbf{v}||^2 \\
\Rightarrow &\lambda||\mathbf{v}||^2 = \lambda^*||\mathbf{v}||^2 \\
&\text{(Since eigenvectors are nonzero)} \\
\Rightarrow &\lambda = \lambda^*(\lambda \text{ is real})
\end{aligned}
$$

39

Similar to exercise 38.

40

Use **Theorem 5.2.10** and **Theorem 6.1.3**.

41

Similar to Example 4.

45

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $A\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$. We have

$$\begin{cases} a+b=3 \\ 3a=3 \\ c+d=0 \\ 3c=-3 \end{cases} \Rightarrow \begin{cases} a=1 \\ b=2 \\ c=-1 \\ d=1 \end{cases} \Rightarrow A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}.$$

46

Similar to Example 2.

50

Since $\mathrm{Det}(A-\lambda I) = (a-\lambda)(c-\lambda)$, a and c are the two eigenvalues for A. The eigenvectors correspond to the eigenvalues a and c are $(1,0)^T$ and $(-b, a-c)^T$ respectively. Let $P = \begin{bmatrix} 1 & -b \\ 0 & a-c \end{bmatrix}$ and $D = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}$. Since $A = PDP^{-1}$, we have

$$A^k = PD^kP^{-1} = \begin{bmatrix} 1 & -b \\ 0 & a-c \end{bmatrix} \begin{bmatrix} a^k & 0 \\ 0 & c^k \end{bmatrix} \frac{1}{a-c} \begin{bmatrix} a-c & b \\ 0 & 1 \end{bmatrix}.$$

51

Let λ be an eigenvalue of A and $\mathbf{v}$ be the corresponding eigenvector. Then $\underbrace{A^2}_{=O}\mathbf{v} = \lambda^2\mathbf{v} = \mathbf{0}$. Since eigenvectors are nonzero, $\lambda = 0$.

52

Since $\mathbf{P}$ is invertible, every column of $\mathbf{P}$ is nonzero. Let $\mathbf{A}$ be a $n \times n$ matrix and $D_{ii} = \lambda_i$ and $\mathbf{u}_i$ be the ith column of $\mathbf{P}$, then $\mathbf{AP} = \mathbf{A}[\mathbf{u}_1\ \mathbf{u}_2\ \cdots\ \mathbf{u}_n] = [\lambda_1\mathbf{u}_1\ \lambda_2\mathbf{u}_2\ \cdots\ \lambda_n\mathbf{u}_n] = \mathbf{PD}$. Since $\mathbf{Au}_i = \lambda_i\mathbf{u}_i$ for all $1 \le i \le n$, the columns of $\mathbf{P}$ are eigenvectors of $\mathbf{A}$ and the diagonal elements of $\mathbf{D}$ are eigenvalues of $\mathbf{A}$.

54

Similar to exercise 37.

64

(i) Here we find the first eigenvalue. Since the rows of $\mathbf{A}$ are dependent, $\mathrm{Det}(\mathbf{A}) = \mathrm{Det}(\mathbf{A} - 0I) = 0$. 0 is an eigenvalue of $\mathbf{A}$.

(ii) Now we find the second eigenvalue of A. Since all of A's row are the same, we can express A as $\mathbf{u}\mathbf{v}^T$ where $\mathbf{u} = (1, 1, \cdots, 1)$ and $\mathbf{v} = (a_1, a_2, \cdots, a_n)$. Since

$$\mathbf{Au} = \mathbf{u}\mathbf{v}^T\mathbf{u} = \left(\sum_{i=1}^{n} a_i\right)\mathbf{u},$$

$\displaystyle\sum_{i=1}^{n} a_i$ is an eigenvalue of $\mathbf{A}$ and $\mathbf{u}$ is the corresponded eigenvector.

66

Use the method of induction.

67

Assume that L has an eigenvalue λ and a corresponding nonzero eigenvector $p(t)$. Then $(Lp)(t) = tp(t) = \lambda p(t)$. Since eigenvector are nonzero, we have $\lambda = t$, a contradiction.

69

Since $\mathbf{A}^T\mathbf{u} = \mathbf{u}$, 1 is an eigenvalue of $\mathbf{A}^T$. By exercise 32, 1 is also an eigenvalue of $\mathbf{A}$. Therefore $\mathbf{Ax} = \mathbf{x}$ has a nontrivial solution.

70 and 76

Similar to Example 4 and 5.

Chapter 7

Inner-Product Vector Spaces

7.1

2

Similar to Example 7.

8

Use **Theorem 7.1.1**.

10

Similar to Example 12.

11

a. Yes, since $\langle \alpha \mathbf{x}, \beta \mathbf{y} \rangle = \alpha \bar{\beta} \langle \mathbf{x}, \mathbf{y} \rangle = 0$ for all scalars α and β.

b. No. Let $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{z} = \mathbf{x}$.

c. Yes, since $\langle \alpha \mathbf{x}, \beta \mathbf{y} + \gamma \mathbf{z} \rangle = \alpha \bar{\beta} \langle \mathbf{x}, \mathbf{y} \rangle + \alpha \bar{\gamma} \langle \mathbf{x}, \mathbf{z} \rangle = 0$ for all scalars α, β, and γ.

13

Use **Theorem 7.1.11**.

15

$||\mathbf{x} + \mathbf{y}||^2 = \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle = ||\mathbf{x}||^2 + ||\mathbf{y}||^2$. The converse is not true. Here is an example. Let $\mathbf{x} = (1, 0)$ and $\mathbf{y} = (i, 0)$. You can see that the inner product of $\mathbf{x}$ and $\mathbf{y}$ is not zero, but they satisfy the given equation.

17

$f(t) = \sqrt{1^2 + 3^+ t^2 + 7^2} = \sqrt{59 + t^2} \geq \sqrt{59}$. It is similar for g.

18

If $\langle \mathbf{x}, \mathbf{y} \rangle > 0$, then $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$. $\Rightarrow \cos \theta = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{x}|| \cdot ||\mathbf{y}||} > 0 \Rightarrow \theta$ is acute.

19

Suppose that $\mathbf{x} \neq \mathbf{0}$. Let $\mathbf{y} = \mathbf{x}$, then $\langle \mathbf{x}, \mathbf{y} \rangle = ||\mathbf{x}||^2 > 0$, a contradiction.

20

Since $\mathbf{y} \neq \mathbf{0}$, $\text{Proj}_{\mathbf{x}}\mathbf{y} = \dfrac{\langle \mathbf{x}, \mathbf{y} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle}\mathbf{y} = \mathbf{0} \Leftrightarrow \langle \mathbf{x}, \mathbf{y} \rangle = 0$. Similarly, $\text{Proj}_{\mathbf{y}}\mathbf{x} = \dfrac{\langle \mathbf{y}, \mathbf{x} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle}\mathbf{x} = \mathbf{0} \Leftrightarrow \langle \mathbf{y}, \mathbf{x} \rangle = 0$. Therefore, $\text{Proj}_{\mathbf{x}}\mathbf{y} = \mathbf{0} \Leftrightarrow \text{Proj}_{\mathbf{y}}\mathbf{x} = \mathbf{0}$.

24

Since $||\mathbf{x}||^2$ is real, so $\langle \mathbf{y}, \mathbf{x} \rangle = \overline{\langle \mathbf{x}, \mathbf{y} \rangle} = \langle \mathbf{x}, \mathbf{y} \rangle$. Now we check whether the norm of $\mathbf{x} - \mathbf{y}$ is zero or not.

$$||\mathbf{x} - \mathbf{y}|| = ||\mathbf{x}||^2 - \langle \mathbf{x}, \mathbf{y} \rangle - \langle \mathbf{y}, \mathbf{x} \rangle + ||\mathbf{y}||^2 = 0 \Rightarrow \mathbf{x} = \mathbf{y}.$$

27

Since Col($\mathbf{A}$) =Row($\mathbf{A}^T$), by **Theorem 7.1.8**, $[\text{Col}(\mathbf{A})]^{\perp} = \text{Null}(\mathbf{A}^T)$.

34

For the part a and part c of this exercise, we can see that the left hand side of the equation is equal to the right hand side with some simple expansions. For part b, the equation doesn't always hold. Here is a simple counter example.

b. Consider $\mathbb{C}^2$ over $\mathbb{R}$.Let $\mathbf{x} = (i, 1)$ and $\mathbf{y} = (1, 0)$, then the equation doesn't hold.

47

Show that $\displaystyle\int_{-1}^{1} P_i(t)P_j(t) = 0$ if $i \neq j$.

50

$$\begin{aligned}
&||\mathbf{x} - \hat{\mathbf{y}}||^2 - ||\mathbf{x} - \mathbf{y}||^2 \\
=&(||\mathbf{x}||^2 - 2\text{Re}(\langle \mathbf{x}, \hat{\mathbf{y}} \rangle) + ||\hat{\mathbf{y}}||^2) - (||\mathbf{x}||^2 - 2\text{Re}(\langle \mathbf{x}, \mathbf{y} \rangle) + ||\mathbf{y}||^2) \\
=&(1 - ||\mathbf{y}||^2) - (\frac{1}{||\mathbf{y}||} - 1)2\text{Re}(\langle \mathbf{x}, \mathbf{y} \rangle) \qquad \because \hat{\mathbf{y}} \text{ is a unit vector} \\
\leq&(||\mathbf{y}||^2 + ||\mathbf{y}||)(\frac{1}{||\mathbf{y}||} - 1) - (\frac{1}{||\mathbf{y}||} - 1)2||\mathbf{y}|| \qquad \because \text{Re}(\langle \mathbf{x}, \mathbf{y} \rangle) \leq \text{Re}(|\langle \mathbf{x}, \mathbf{y} \rangle|) \leq ||\mathbf{x}|| \cdot ||\mathbf{y}|| = ||\mathbf{y}|| \\
=&(\frac{1}{||\mathbf{y}||} - 1)(||\mathbf{y}||^2 - ||\mathbf{y}||) < 0 \qquad \because ||\mathbf{y}|| > 1
\end{aligned}$$

51

Let $\text{Proj}_{\mathbf{y}}\mathbf{x} = \mathbf{p}$,

$$\begin{aligned}
&||\mathbf{x}||^2 = ||\mathbf{p} + (\mathbf{x} - \mathbf{p})||^2 = ||\mathbf{p}||^2 + \langle \mathbf{p}, \mathbf{x} - \mathbf{p} \rangle + \langle \mathbf{x} - \mathbf{p}, \mathbf{p} \rangle + ||\mathbf{x} - \mathbf{p}||^2 \\
\Rightarrow &||\mathbf{x}|| \geq ||\mathbf{p}|| \qquad \because \mathbf{p} \text{ is perpendicular to } \mathbf{x} - \mathbf{p}
\end{aligned}$$

54

Suppose that $\mathbf{y} \neq \mathbf{0}$. Let $\mathbf{p} = \text{Proj}_{\mathbf{y}}\mathbf{x}$ and $\mathbf{w} = \mathbf{x} - \mathbf{p}$, we have

$$||\mathbf{x}||^2 = ||\mathbf{p}||^2 + ||\mathbf{w}||^2 = \frac{|\langle \mathbf{x}, \mathbf{y} \rangle|^2}{||\mathbf{y}||^2} + ||\mathbf{w}||^2 \geq \frac{|\langle \mathbf{x}, \mathbf{y} \rangle|^2}{||\mathbf{y}||^2}.$$

Therefore, the equality holds iff $||\mathbf{w}|| = 0$ iff $\mathbf{x}$ is a multiple of $\mathbf{y}$. Here, you can see that if $\mathbf{y} = \mathbf{0}$, then $\mathbf{x}$ is not a multiple of $\mathbf{y}$. Therefore, we need to change the phrasing a little bit. To cover the case where $\mathbf{y} = \mathbf{0}$, the Cauchy-Schwarz inequality is an equality if and only if $\mathbf{x}$ is a scalar multiple of $\mathbf{y}$ or $\mathbf{y}$ is a scalar multiple of $\mathbf{x}$.

62

We check the four axioms of an inner product.

1. It is clear that when $f \equiv 0$,

$$\langle f, f \rangle = \int_a^b f(x)\overline{f(x)}w(x)dx = \int_a^b 0\ w(x)dx = 0.$$

   To have $\langle f, f \rangle > 0$ whenever $f \neq 0$, i.e.,

$$\langle f, f \rangle = \int_a^b f(x)\overline{f(x)}w(x)dx = \int_a^b |f(x)|^2 w(x)dx > 0 \quad \forall f \neq 0,$$

   we must have $w(x) \geq 0\ \forall x \in [a, b]$ and $w(x)$ not identical to the zero function in any subinterval $[c, d]$ of $[a, b]$ with $c < d$.

2. $\langle f, g \rangle = \overline{\langle g, f \rangle}$.

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}w(x)dx = \int_a^b \overline{g(x)\overline{f(x)}w(x)}dx = \overline{\langle g, f \rangle}$$

   since $w(x) \geq 0$ for all $x \in [a, b]$.

3. $\langle f + h, g \rangle = \langle f, g \rangle + \langle h, g \rangle$.

$$\begin{aligned} \langle f + h, g \rangle &= \int_a^b \big(f(x) + h(x)\big)\overline{g(x)}w(x)dx \\ &= \int_a^b f(x)\overline{g(x)}w(x)dx + \int_a^b h(x)\overline{g(x)}w(x)dx \end{aligned}$$

4. $\langle \alpha f, g \rangle = \alpha \langle f, g \rangle$

$$\langle \alpha f, g \rangle = \int_a^b \alpha f(x)\overline{g(x)}w(x)dx = \alpha \int_a^b f(x)\overline{g(x)}w(x)dx = \alpha \langle f, g \rangle.$$

We conclude that $w(x)$ should be non-negative on $[a, b]$ and cannot be identical to the zero function in any subinterval $[c, d]$ of $[a, b]$ with $c < d$.

70

Let $\mathbf{v}$ be a unit vector such that $\mathbf{v} \perp \mathbf{u}$. Let $x = \sqrt{1 - c^2}\mathbf{v} + c\mathbf{u}$. First, we show that $\mathbf{x}$ is a unit vector.

$$||\mathbf{x}||^2 = (1 - c^2)||\mathbf{v}||^2 + c^2||\mathbf{u}||^2 = 1.$$

Now, we show that $\langle \mathbf{x}, \mathbf{u} \rangle = c$.

$$\langle \mathbf{x}, \mathbf{u} \rangle = \sqrt{1 - c^2}\langle \mathbf{v}, \mathbf{u} \rangle + c||\mathbf{u}||^2 = c.$$

71

Let θ be the angle between $\mathbf{x}$ and $\mathbf{y}$. Since θ is acute, $\cos\theta > 0$.
$\Rightarrow \langle \mathbf{x}, \mathbf{y}\rangle = \cos\theta \cdot ||\mathbf{x}|| \cdot ||\mathbf{y}|| \geq 0$.
$\Rightarrow ||\mathbf{x}+\mathbf{y}||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + 2\langle \mathbf{x}, \mathbf{y}\rangle \geq ||\mathbf{x}||^2 + ||\mathbf{y}||^2$.

73

If $\mathbf{x} = \mathbf{0}$, then apparently this sequence converges to 0. If $\mathbf{x} \neq \mathbf{0}$.Let $\hat{\mathbf{u}} = \mathbf{u}/||\mathbf{u}||$, $\hat{\mathbf{v}} = \mathbf{v}/||\mathbf{v}||$ and $\dfrac{\langle \mathbf{x}, \mathbf{u}\rangle}{||\mathbf{x}|| \cdot ||\mathbf{u}||} = \alpha$, then $P(\mathbf{x}) = ||\mathbf{x}||\alpha\hat{\mathbf{u}}$. Let $\dfrac{\langle \mathbf{u}, \mathbf{v}\rangle}{||\mathbf{u}|| \cdot ||\mathbf{v}||} = \beta$, then $(QP)(\mathbf{x}) = ||\mathbf{x}||\alpha\dfrac{\langle \hat{\mathbf{u}}, \mathbf{v}\rangle}{||\mathbf{v}||}\hat{\mathbf{v}} = ||\mathbf{x}||\alpha\beta \cdot \hat{\mathbf{v}}$. The sequence can be rewritten as $\mathbf{x}$, $||\mathbf{x}||\alpha\hat{\mathbf{u}}$, $||\mathbf{x}||\alpha\beta\hat{\mathbf{v}}$, $||\mathbf{x}||\alpha^2\beta\hat{\mathbf{u}}, \cdots$. By Ex 54, since $\mathbf{u}$ and $\mathbf{v}$ are not multiples of each other, $|\beta| < 1$. Thus, this sequence converges to $\mathbf{0}$.

74

(1) If $\mathbf{u} = (2, 0)$ and $\mathbf{v} = (1, 1)$, then $\langle \mathbf{u}, \mathbf{v}\rangle = 2 = ||\mathbf{v}||^2 = 2$.

(2) Let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{v} = (v_1, v_2)$.

   (i) $\langle \mathbf{x}, \mathbf{v}\rangle = x_1v_1 + x_2v_2 = \sqrt{v_1^2 + v_2^2} = ||\mathbf{v}||$. Thus, this set is a line in $\mathbb{R}^2$.

   (ii) From $\langle \mathbf{x}, \mathbf{v}\rangle = x_1v_1 + x_2v_2 = x_1^2 + x_2^2 = ||\mathbf{x}||^2$, we have $\left(x_1 - \dfrac{v_1}{2}\right)^2 + \left(x_2 - \dfrac{v_2}{2}\right)^2 = \left(\dfrac{v_1}{2}\right)^2 + \left(\dfrac{v_2}{2}\right)^2$. Thus, this set is a circle centered at $(\dfrac{v_1}{2}, \dfrac{v_2}{2})$ with radius $\dfrac{||\mathbf{v}||}{2}$.

76

"$\Rightarrow$" $\langle \mathbf{x}, \mathbf{y}\rangle \geq 0$, then it means $\langle \mathbf{x}, \mathbf{y}\rangle$ is real and $\langle \mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{y}, \mathbf{x}\rangle$
$\Rightarrow \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{x}\rangle + 2\langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle \geq \langle \mathbf{x}, \mathbf{x}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle$ (Since $||\mathbf{x}|| > 0$)
$\Rightarrow ||\mathbf{x}+\mathbf{y}|| > ||\mathbf{y}||$

"$\Leftarrow$" Here is a counter example to show that the converse is not always true. Consider the inner product space $\mathbb{R}^2$ with the standard inner product. Let $\mathbf{x} = (-1, 2)$ and $\mathbf{y} = (1, 0)$. $||\mathbf{x}+\mathbf{y}|| = 2 > ||\mathbf{y}|| = 1$, but $\langle \mathbf{x}, \mathbf{y}\rangle = -1$.

77

Similar to Ex.76.

78

Use Ex.54.

79

$$||\mathbf{x}|| = ||(\mathbf{x}+\mathbf{y}) - \mathbf{y}|| \leq ||\mathbf{x}-\mathbf{y}|| + ||\mathbf{y}|| \quad \Rightarrow ||\mathbf{x}|| - ||\mathbf{y}|| \leq ||\mathbf{x}-\mathbf{y}||.$$
$$||\mathbf{y}|| = ||(\mathbf{y}+\mathbf{x}) - \mathbf{x}|| \leq \underbrace{||\mathbf{y}-\mathbf{x}||}_{=||\mathbf{x}-\mathbf{y}||} + ||\mathbf{x}|| \quad \Rightarrow ||\mathbf{y}|| - ||\mathbf{x}|| \leq ||\mathbf{x}-\mathbf{y}||.$$
$$\Rightarrow \big|||\mathbf{x}|| - ||\mathbf{y}||\big| \leq ||\mathbf{x}-\mathbf{y}||.$$

80

By Cauchy–Schwarz inequality, $||\mathbf{x}|| = \sqrt{\displaystyle\sum_{i=1}^{n} x_i^2} \leq \displaystyle\sum_{i=1}^{n} \sqrt{x_i^2} = \displaystyle\sum_{i=1}^{n} |x_i|$.

84

Here we will show that if $\langle \mathbf{x}, \mathbf{y} \rangle \neq 0$, then we can always choose an α such that the inequality doesn't hold. If $\langle \mathbf{x}, \mathbf{y} \rangle \neq 0$, let $\alpha = -\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{y}||^2 \cdot |\langle \mathbf{x}, \mathbf{y} \rangle|^2}$, then

$$\begin{aligned} ||\mathbf{x} + \alpha \mathbf{y}||^2 =& ||\mathbf{x}||^2 + \overline{\alpha}\langle \mathbf{x}, \mathbf{y} \rangle + \alpha \langle \mathbf{y}, \mathbf{x} \rangle + |\alpha|^2 ||\mathbf{y}||^2 \\ =& ||\mathbf{x}||^2 - 2\frac{1}{||\mathbf{y}||^2} + \frac{1}{||\mathbf{y}||^2} < ||\mathbf{x}||^2. \end{aligned}$$

85

First, there is an error in this exercise. The inequality should be $\langle \mathbf{u}, \mathbf{x} \rangle \langle \mathbf{v}, \mathbf{x} \rangle \geq 0$. Here we will show that, if $\mathbf{u} \neq \mathbf{v}$, then we can always find an $\mathbf{x}$ to make this inequality fail.

(i) If $\mathbf{u} = -\mathbf{v}$, let $\mathbf{x} = \mathbf{u}$, then the inequality will fail.

(ii) If $\mathbf{u} \neq \pm \mathbf{v}$. Let $\mathbf{w} = \mathbf{u} - \mathrm{Proj}_{\mathbf{v}} \mathbf{u} \neq \mathbf{0}$. Choose an α such that $\frac{1}{\alpha} > \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{w}||^2}$ and $\alpha > 0$. Let $\mathbf{x} = \mathbf{w} - \alpha \mathbf{v}$, then

$$\begin{cases} \langle \mathbf{u}, \mathbf{x} \rangle = & \langle \mathbf{w} + \mathrm{Proj}_{\mathbf{v}} \mathbf{u}, \mathbf{w} - \alpha \mathbf{v} \rangle = ||\mathbf{w}||^2 - \frac{\alpha \langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{v}||^2} ||\mathbf{v}||^2 > 0 \\ \langle \mathbf{v}, \mathbf{x} \rangle = & \langle \mathbf{v}, \mathbf{w} - \alpha \mathbf{v} \rangle = -\alpha ||\mathbf{v}||^2 < 0. \end{cases} \Rightarrow \langle \mathbf{u}, \mathbf{x} \rangle \langle \mathbf{v}, \mathbf{x} \rangle < 0.$$

By (i) and (ii), we have $\mathbf{u} = \mathbf{v}$.

88

By the Cauchy-Schwarz inequality in an inner product space, we have

$$|\langle \mathbf{x}, \mathbf{y} \rangle_o| \leq \|\mathbf{x}\|_o \|\mathbf{y}\|_o$$

with equality holds if and only if $\{\mathbf{x}, \mathbf{y}\}$ is a linearly dependent set. Note that

$$\|\mathbf{y}\|_o = \sqrt{3y_1^2 + 5y_2^2 + 2y_3^2}.$$

Since $\langle \mathbf{x}, \mathbf{x} \rangle_o \leq 7$, we have

$$|\langle \mathbf{x}, \mathbf{y} \rangle_o| \leq \sqrt{7} \cdot \sqrt{3y_1^2 + 5y_2^2 + 2y_3^2}.$$

7.2

1 and 3

Similar to Example 5. and 6.

2 and 7

Similar to Example 1.

13.

Similar to Example 8.

17.

No. Since $(-5,4,3)^T$ is not in Span $\left(\left\{\begin{bmatrix}3\\2\\1\end{bmatrix},\begin{bmatrix}-2\\3\\-2\end{bmatrix}\right\}\right)$, when the second equation is changed, the vector and the hyperplane onto which the vector projects are both changed.

18.

No. It's similar to Ex.17.

21.

This problem can be solved by using induction. First you suppose that $\mathbf{u}_1$ to $\mathbf{u}_k$ form an orthogonal set, and then use this inductive hypothesis to show that $\mathbf{u}_1$ to $\mathbf{u}_{k+1}$ also form an orthogonal set.

22.

First, we have a correction to this problem. The algorithm you should use is the algorithm in exercise 21 instead of 1.

(1) The first part is to find an orthogonal basis by using the algorithm in Exercise 21. It's just some simple calculation, so we will just leave it to readers.

(2) For the second part of this exercise, the answer is no, since the inner product of integer-valued vectors is not necessarily an integer. Here is a counter example. For $\mathbf{x},\mathbf{y}\in\mathbb{R}^2$, let $\langle\mathbf{x},\mathbf{y}\rangle = \mathbf{y}^T\begin{bmatrix}1 & 0\\0 & \pi\end{bmatrix}\mathbf{x}$. It can be shown that $\langle,\rangle$ is an inner product. Let $\mathbf{v}_1=(1,1)$ and $\mathbf{v}_2=(1,0)$. It's obvious that $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis of $\mathbb{R}^2$. Now let's apply this algorithm on $\mathbf{v}_1$ and $\mathbf{v}_2$. By this algorithm, we have $\mathbf{u}_1=\mathbf{v}_1$ and $\mathbf{u}_2=\mathbf{v}_2-\dfrac{\langle\mathbf{v}_2,\mathbf{u}_1\rangle}{\langle\mathbf{u}_1,\mathbf{u}_1\rangle}\mathbf{u}_1=(1,0)-\dfrac{1}{1+\pi}(1,1)\notin\mathbb{Z}^2$. You can see that even with scaling, $\mathbf{u}_2$ still can not be an integer vector.

24.

Suppose that A has n columns. Let $\mathbf{p}$ be the projection of $\mathbf{b}$ onto the column space of $\mathbf{A}$, then since the columns of A is orthogonal, $\mathbf{p}$ can be expressed as

$$\mathbf{p}=\sum_{j=1}^{n}\frac{\langle\mathbf{b},\mathbf{a}_j\rangle}{\langle\mathbf{a}_j,\mathbf{a}_j\rangle}\mathbf{a}_j=\mathbf{A}\begin{bmatrix}\frac{\langle\mathbf{b},\mathbf{a}_1\rangle}{\langle\mathbf{a}_1,\mathbf{a}_1\rangle}\\ \vdots\\ \frac{\langle\mathbf{b},\mathbf{a}_n\rangle}{\langle\mathbf{a}_n,\mathbf{a}_n\rangle}\end{bmatrix}=\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\begin{bmatrix}\langle\mathbf{b},\mathbf{a}_1\rangle\\ \vdots\\ \langle\mathbf{b},\mathbf{a}_n\rangle\end{bmatrix}=\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}.$$

This is the formula of $\mathbf{p}$ and the least-squares solution is $\mathbf{x}=(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$.

35.

We know the inner product of x and any $\mathbf{u}_i$ is 0 and so obviously, the inner product of $\mathbf{x}$ with any linear combination of $\mathbf{u}_i$ is also 0.

36.

Choose a vector out of the span of $\mathbf{v}_1$ to $\mathbf{v}_n$, and then apply Gram-Schmidt algorithm, then you are done.

38.

$\mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} = (\mathbf{A}\mathbf{x})^T\mathbf{A}\mathbf{x} = \langle \mathbf{A}\mathbf{x}, \mathbf{A}\mathbf{x}\rangle \geq 0.$

39.

The statement is wrong. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$, then A has complex eigenvalues.

40.

Let $\mathbf{p}$ be the projection of $\mathbf{b}$ onto the column space of $\mathbf{A}$,

$$\mathbf{b}^T\mathbf{A}\mathbf{x} = [(\mathbf{b}-\mathbf{p})+\mathbf{p}]^T\mathbf{p} = (\mathbf{b}-\mathbf{p})^T\mathbf{p} + \mathbf{p}^T\mathbf{p} \geq 0.$$

44.

"$\subseteq$" $\forall\, \mathbf{x} \in \ker(\mathbf{A}), \mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{0} = \mathbf{0} \Rightarrow \quad \mathbf{x} \in \ker(\mathbf{A}^T\mathbf{A}).$

"$\supseteq$" $\forall\, \mathbf{x} \in \ker(\mathbf{A}^T\mathbf{A}), \mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} = (\mathbf{A}\mathbf{x})^T(\mathbf{A}\mathbf{x}) = \mathbf{0} \Rightarrow \quad \mathbf{A}\mathbf{x} = \mathbf{0}, \mathbf{x} \in \ker(\mathbf{A}).$

$\Rightarrow \ker(\mathbf{A}) = \ker(\mathbf{A}^T\mathbf{A}).$

46.

Let $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $V = \text{Span}\,\{\mathbf{u}, \mathbf{v}\}$, then $\mathbf{u} \perp \mathbf{v}$ but $\mathbf{u} \not\perp V$.

50.

Let W be a subspace of $\mathbb{R}^n$. Let $S = \{\mathbf{v_1}, \mathbf{v}_2, \cdots, \mathbf{v}_m\}$ be a basis of $W^\perp$. Let $A = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{bmatrix}$, then

$$W \underbrace{=}_{\text{Thm.7.1.15}} (W^\perp)^\perp = (\text{Span}(S))^\perp = (\text{Row}(A))^\perp \underbrace{=}_{\text{Thm. 7.1.8}} \text{Ker}(A).$$

So A's kernel is W.

54

Note that $S = \{1, t, t^2\}$ is a basis of $\mathbb{P}_2$. Apply Gram-Schmidt Algorithm on S then we are done.

55

(a) When $n = 1$, $\text{Det}(A) = |a_{11}| \leq \prod_{i=1}^{1} \sum_{j=1}^{1} |a_{ij}|$. Suppose the inequality holds when $n = k$. When $n = k + 1$,

$$|\text{Det}(A)| = \left| \sum_{j=1}^{n} (-1)^{1+j} a_{1,j} \text{Det}(M^{1j}) \right| \leq \sum_{j=1}^{n} |a_{1j}| \cdot |\text{Det}(M^{1j})|$$

$$\leq \sum_{j=1}^{n} |a_{1j}| \cdot \left(\prod_{i=2}^{n} \sum_{j'=1, j' \neq j}^{n} |a_{ij'}| \right) \text{ (by the inductive hypothesis)}$$

$$\leq \sum_{j=1}^{n} |a_{1j}| \cdot \left(\prod_{i=2}^{n} \sum_{j'=1}^{n} |a_{ij'}| \right) = \prod_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|.$$

By induction, $|\text{Det}(A)| \leq \prod_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|$ for any $n \times n$ matrix A.

(b) (i) If $\text{Det}(A) = 0$, the inequality holds.

(ii) If $\text{Det}(A) \neq 0$. Let

$$A = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{bmatrix} \text{ and } N = \begin{bmatrix} \frac{\mathbf{r}_1}{\|\mathbf{r}_1\|} \\ \frac{\mathbf{r}_2}{\|\mathbf{r}_2\|} \\ \vdots \\ \frac{\mathbf{r}_n}{\|\mathbf{r}_n\|} \end{bmatrix} \text{, and then } \text{Det}(A) = \text{Det}(N) \prod_{i=1}^{n} \|\mathbf{r}_i\|.$$

By the linearity of the determinant function at each row. To prove this inequality is equivalent to proving $|\text{Det}(N)| \leq 1$. Let $P = NN^H$. By 2.(f) and 2.(g) in additional exercises, we have

$$\text{Det}(P) \leq \left(\frac{\text{tr}(P)}{n} \right)^n = \left(\frac{1}{n} \sum_{i=1}^{n} \frac{r_{i1}\overline{r_{i2}} + r_{i2}\overline{r_{i2}} + \cdots + r_{in}\overline{r_{in}}}{\|r_i\|} \right)^n = \left(\frac{1}{n} \sum_{i=1}^{n} 1 \right)^n = 1.$$

Therefore, $\text{Det}(P) = \text{Det}(N)\text{Det}(N^H) \leq 1$ and the inequality holds.

56

No. $\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$, but $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is invertible.

Chapter 8

Additional Topics

8.1

5

No. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Apparently there is no such θ can make $\sin\theta$ be both 1 and -1.

16

Similar to Ex.6.1.38 and Ex.6.2.39.

17

$$\mathbf{A} = \underbrace{\frac{\mathbf{A}+\mathbf{A}^H}{2}}_{\text{Hermitian}} + \underbrace{\frac{\mathbf{A}-\mathbf{A}^H}{2}}_{\text{skew}-\text{Hermitian}}$$

18

$\langle L(\mathbf{x}), \mathbf{y}\rangle = \int_{-1}^{1} p\mathbf{x}\bar{y}dt = \int_{-1}^{1} \mathbf{x}\overline{\bar{p}\mathbf{y}}dt = \langle \mathbf{x}, \bar{p}\mathbf{y}\rangle \Rightarrow L^*(\mathbf{y}) = \bar{p}\mathbf{y}$.

20

If the rows of $\mathbf{U}$ form an orthonormal set, then

$$\mathbf{U}\mathbf{U}^T = \mathbf{I} \Rightarrow \mathbf{U}^T\mathbf{U} = \mathbf{I}.$$

$\Rightarrow$ Columns of $\mathbf{U}$ form an orthonormal set.

21

If A is not square then of course we can not have the same conclusion. Consider $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$, then you can see that A's rows form an orthonormal set but A's columns don't.

24.

(1) $\because A$ is unitary, $\therefore\ AA^H = I \Rightarrow A^H = A^{-1}$.
$A^{-1}(A^{-1})^H = A^H(A^H)^H = I \Rightarrow\ A^{-1}$ and A^H are unitary.

(2) Let A, B be two unitary matrices. $(AB)(AB)^H = ABB^HA^H = I \Rightarrow AB$ is unitary.

(3) The sum of two Hermitian matrices is Hermitian, but the sum of two unitary matrices is not necessarily unitary. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are both unitary, but $A + B$ is not unitary.

(4) $\frac{(A+A^H)}{2}$ is Hermitian.

25

$$\begin{aligned} \langle \mathbf{Ax}, \mathbf{y} \rangle &= \mathbf{y}^H \mathbf{Ax} \\ &= \mathbf{y}^H (\mathbf{A}^H)^H \mathbf{x} \\ &= (\mathbf{A}^H \mathbf{y})^H \mathbf{x} \\ &= \langle \mathbf{x}, \mathbf{A}^H \mathbf{y} \rangle \end{aligned}$$

27

This exercise is similar to the generalization of Example 8.1.2 in the textbook which is right down below example 8.1.2. Here is an example to show that if the basis is not orthonormal, then the matrix is not necessarily symmetric. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ and $L(\mathbf{x}) = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} \mathbf{x}$. Then $[L]_{\mathcal{B}}$ is not symmetric.

28

For this exercise, you just need to show that $\langle T(x_1, x_2), (y_1, y_2) \rangle_D = \langle (x_1, x_2), T(y_1, y_2) \rangle_D$.

30

Let T_1 and T_2 be two self-adjoint operators from V to V and $\alpha, \beta \in \mathbb{R}$. Show that $\langle (\alpha T_1 + \beta T_2)(\mathbf{x}), \mathbf{y} \rangle = \langle \mathbf{x}, (\alpha T_1 + \beta T_2)(\mathbf{y}) \rangle$.

33

Similar to Ex.33 and 28.

38.

A and B are real orthogonally diagonalizable.
$\Rightarrow$ A and B are real and symmetric. (by Ex.37)
$\Rightarrow$ $A + B$ is real and symmetric.
$\Rightarrow$ $A + B$ is orthogonally diagonalizable (By **Theorem** 8).

39.

Let $p(\lambda) = (-1)^n \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$ be A's characteristic polynomial.
$\because$ A is invertible $\therefore$ $p(0) = c_0 \neq 0$. By Cayley-Hamilton Theorem, we have

$$(-1)^n A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I = \mathbf{O}.$$

Since $c_0 \neq 0$ $A^{-1} = \frac{-1}{c_0}((-1)^n A^{n-1} + c_{n-1}A^{n-2} + \cdots + c_1 I)$.

42.

Let $q(t) = c_n t^n + \cdots + c_1 t + c_0$. Suppose that $\mathbf{v}$ is the corresponding eigenvector, then
$q(A)\mathbf{v} = (c_n A^n + \cdots + c_1 A + c_0 I)\mathbf{v} = (c_n \lambda^n + \cdots + c_1 \lambda + c_0)\mathbf{v} = q(\lambda)\mathbf{v}$
$\Rightarrow q(\lambda)$ is an eigenvalue of $q(A)$.

43.

By Cayley-Hamilton Theorem, for a square matrix A, $A, A^2, \cdots A^n$ form an linearly dependent set, so the dimension of the span of an $n \times n$ matrix's power is at most $n - 1 < \mathrm{Dim}(\mathbb{R}^{n\times n}) = n^2$. Therefore, no matrix's powers can span $\mathbb{R}^{n\times n}$.

45.

Let $p(\lambda) = (-1)^n \lambda^n + \displaystyle\sum_{i=0}^{n-1} c_i \lambda^i$, then $A^n = (-1)^{n+1} \sum_{i=0}^{n-1} c_i A^i$. For simplicity let

$$S = \mathrm{Span}(A^0, A^1, \cdots, A^{n-1}).$$

Here we start the induction. When $k = 0$, $A^0 \in S$. Suppose that $A^m \in S$ and

$$A^m = \sum_{i=0}^{n-1} \alpha_i A^i, \text{ where } \alpha_i \text{ are scalars.}$$

When $k = m + 1$,

$$\begin{aligned} A^{m+1} &= A \cdot A^m \\ &= A \sum_{i=0}^{n-1} \alpha_i A^i = \alpha_{n-1} A^n + \sum_{i=1}^{n-1} \alpha_{i-1} A^i \qquad \text{(by the Cayley-Hamilton Theorem)} \\ &= \alpha_{n-1} \left((-1)^{n+1} \sum_{i=0}^{n-1} c_i A^i \right) + \sum_{i=1}^{n-1} \alpha_{i-1} A^i \in S \end{aligned}$$

$\Rightarrow A^k \in S$ for all $k \in \mathbb{N}_0$.

46.

For this exercise, we just need to express B and C as linear combinations of A's powers. Since A's powers commute with each other, it will be obvious that B commutes with C.

48

(1) Let $\mathbf{r}_i$ be the ith row of $\mathbf{A}$ and $\mathbf{B} = \mathbf{A}\mathbf{A}^H$, then
$b_{ii} = \mathbf{e}_i^H \mathbf{A}\mathbf{A}^H \mathbf{e}_i = (\mathbf{A}^H \mathbf{e}_i)^H (\mathbf{A}^H \mathbf{e}_i) \geq 0.$

(2) $(\mathbf{A}\mathbf{A}^H)^H = (\mathbf{A}^H)^H \mathbf{A}^H = \mathbf{A}\mathbf{A}^H$.

53

Suppose that $\mathbf{x} \in \mathrm{Ker}(\mathbf{A}\mathbf{A}^H)$.

$$\left.\begin{array}{c} \mathbf{x}^T \mathbf{A}\mathbf{A}^T \mathbf{x} = (\mathbf{A}^T\mathbf{x})^T (\mathbf{A}^T \mathbf{x}) \\ \mathbf{x} \cdot \mathbf{0} = \mathbf{0} \end{array}\right\} \Rightarrow (\mathbf{A}^T\mathbf{x})(\mathbf{A}^T\mathbf{x}) = \mathbf{0} \Rightarrow \mathbf{A}^T\mathbf{x} = \mathbf{0}$$

$\Rightarrow \mathbf{x} \in \mathrm{Ker}(\mathbf{A}^T) \because \mathrm{Rank}(\mathbf{A}^T) = m \quad \therefore \mathbf{x} = \mathbf{0}$
$\Rightarrow \mathrm{Ker}(\mathbf{A}\mathbf{A}^T) = \{\mathbf{0}\} \Rightarrow \mathbf{A}\mathbf{A}^T$ is invertible.

54

The same as Ex. 3.2.85.

55

No. $\begin{bmatrix} 1 & -1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. $\begin{bmatrix} 1 & -1 \\ 0 & 4 \end{bmatrix}$ is similar to this diagonal $\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$, but $\begin{bmatrix} 1 & -1 \\ 0 & 4 \end{bmatrix}$ is not symmetric.

56

Suppose $\mathbf{u} \neq \mathbf{v}$, then $\mathbf{v} = \mathbf{u} + \mathbf{y}$ for some nonzero $\mathbf{y}$. Let $\mathbf{x} = \mathbf{y}$, then

$$\langle \mathbf{v}, \mathbf{x} \rangle = \langle \mathbf{v}, \mathbf{y} \rangle = \langle \mathbf{u}, \mathbf{y} \rangle + \underbrace{\langle \mathbf{y}, \mathbf{y} \rangle}_{>0} \neq \langle \mathbf{u}, \mathbf{y} \rangle = \langle \mathbf{u}, \mathbf{x} \rangle, \ (\rightarrow\!\times\!\leftarrow).$$

59.

No. Let $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, you can see that $2B \neq A + A^T$.

64

$$\begin{aligned} & \langle T(u), v \rangle \\ =& \langle wu, v \rangle \\ =& \int_{-1}^{1} w(t)u(t)v(t)dt \\ =& \int_{-1}^{1} u(t)w(t)v(t)dt = \langle u, wv \rangle = \langle u, T(v) \rangle \\ \Rightarrow& T \text{ is self-adjoint.} \end{aligned}$$

66

No. Let $u(t) = t^2 - 1$ and $v(t) = (t^2 - 1)(t + 1)$. Then $\langle L(u), v \rangle \neq \langle u, L(v) \rangle$.

8.2

18.

Let A be a $n \times n$ symmetric matrix such that all A's diagonal entries are positive and A has n distinct real eigenvalue. Let $\text{Det}(A - \lambda I) = p(\lambda)$, then $\text{Det}(A + cI - \lambda I) = p(\lambda - c)$. Let λ_m be A's smallest eigenvalue. Since $p(\lambda - c)$ shifts $p(\lambda)$ right c units, if $c > -\lambda_m$ and $-c \neq a_{ii}\ \forall\ i$, then $A + cI$ has n distinct positive eigenvalues and is positive definite .

21.

Let G be the Gram matrix arising from $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ which is linearly independent. For all $\mathbf{x} \neq 0$,

$$\mathbf{x}^T G\mathbf{x} = \mathbf{x}^T \sum_{i=1}^{n} x_i \begin{bmatrix} \langle \mathbf{v}_1, \mathbf{v}_i \rangle \\ \langle \mathbf{v}_2, \mathbf{v}_i \rangle \\ \vdots \\ \langle \mathbf{v}_n, \mathbf{v}_i \rangle \end{bmatrix} = \mathbf{x}^T \begin{bmatrix} \sum_{i=1}^{n} \langle \mathbf{v}_1, x_i\mathbf{v}_i \rangle \\ \sum_{i=1}^{n} \langle \mathbf{v}_2, x_i\mathbf{v}_i \rangle \\ \vdots \\ \sum_{i=1}^{n} \langle \mathbf{v}_n, x_i\mathbf{v}_i \rangle \end{bmatrix}$$

$$= \sum_{j=1}^{n} x_j \left(\sum_{i=1}^{n} \langle \mathbf{v}_j, x_i\mathbf{v}_i \rangle \right) = \langle \sum_{j=1}^{n} x_j\mathbf{v}_j, \sum_{i=1}^{n} x_i\mathbf{v}_i \rangle > 0$$

$\Rightarrow G$ is definite positive.

24.

Let A be a 2×2 symmetric positive definite matrix. Since A is symmetric, $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ for some $a, b, c \in \mathbb{R}$. Since A is positive definite, A's diagonal entries and eigenvalues are positive. we have
(i) $a, c > 0$.
(ii)

$$\begin{aligned} &\mathrm{Det}(A - \lambda I) = (a - \lambda)(c - \lambda) - b^2 = \lambda^2 - (a + c)\lambda + (ac - b^2) \\ \Rightarrow&\lambda = \frac{a + c \pm \sqrt{(a - c)^2 + (4b)^2}}{2} > 0 \\ \Rightarrow&a + c > \sqrt{(a - c)^2 + (4b)^2} \\ \Rightarrow&ac > b^2. \end{aligned}$$

Therefore, $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is symmetric positive definite if and only if $a, c > 0$ and $ac > b^2$.

25.

Let A be an $n \times n$ matrix and the $m \times m$ submatrix lying in the upper left-hand corner of A be P_m, $m \leq n$. Suppose that P_m is not positive definite. then$\exists \mathbf{x}' \in \mathbb{C}^m$ s.t. $\mathbf{x}'^H P_m \mathbf{x}' < 0$. Let $\mathbf{x} = \begin{bmatrix} \mathbf{x}' \\ \mathbf{0} \end{bmatrix} \in \mathbb{C}^n$, then $\mathbf{x}^H A\mathbf{x} = \mathbf{x}'^H P_m \mathbf{x}' < 0$, $(\rightarrow\leftarrow)$.

32.

The statement of this exercise is wrong. Here is a counter example. Let $A = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$. Note that A has two eigenvalues 6 and 4. By **Theorem 8.2.B**, A is positive definite. Let $D = \begin{bmatrix} 0.1 & 0 \\ 0 & 10 \end{bmatrix}$ and $\mathbf{x} = (10, 1)$, then

$$\mathbf{x}^T DA\mathbf{x} = \begin{bmatrix} 10 & 1 \end{bmatrix} \begin{bmatrix} 0.1 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 10 \end{bmatrix} \begin{bmatrix} 49 \\ -5 \end{bmatrix} = -1 < 0.$$

You can see that DA is not positive definite.

41.

Since L^T is a upper triangular matrix with positive diagonal entries, $\mathrm{Ker}(L^T) = \{\mathbf{0}\}$. For all $\mathbf{x} \neq \mathbf{0} \Rightarrow L^T\mathbf{x} \neq \mathbf{0} \Rightarrow \mathbf{x}^T LL^T\mathbf{x} = (L^T\mathbf{x})^T(L^T\mathbf{x}) > 0$. Therefore, LL^T is positive definite.

Appendix A

9

1. The $n = 2$ case is proven on p.571.
   Suppose it's true when $n = k$, when $n = k + 1$,

$$\begin{aligned} U \setminus \bigcup_{i=1}^{k+1} A_i =& U \setminus (\bigcup_{i=1}^{k} A_i \bigcup A_{k+1}) \\ =& (U \setminus \bigcup_{i=1}^{k} A_i) \bigcap (U \setminus A_{k+1}) \\ =& \left[\bigcap_{i=1}^{k} (U \setminus A_i) \right] \bigcap (U \setminus A_{k+1}) \\ =& \bigcap_{i=1}^{k+1} (U \setminus A_i) \end{aligned}$$

2. It's similar to part a.

10

(1) Is $f(\bigcap_i A_i) \subseteq \bigcap_i f(A_i)$?
If $f(\bigcap_i A_i) = \varnothing$, then trivially, $f(\bigcap_i A_i) \subseteq \bigcap_i f(A_i)$.
If $f(\bigcap_i A_i) \neq \varnothing$. For all $y \in f(\bigcap_i A_i)$, there is a $x \in \bigcap_i A_i$ such that $f(x) = y$. Since $x \in \bigcap_i A_i$, $x \in A_i$ for all i. This implies that $y \in f(A_i)$ for all i and then $y \in f(\bigcap_i A_i)$. Therefore, we can conclude that $f(\bigcap_i Ai) \subseteq \bigcap_i f(A_i)$.

(2) Is $\bigcap_i f(A_i) \subseteq f(\bigcap_i A_i)$?
Let $A = \{1, 2, 3\}$ and $B = a, b, c$. We construct a function f from A to B such that $f(1) = a$, $f(2) = b$, and $f(3) = a$. Let $A_1 = \{1, 2\}$ and $A_2 = \{3\}$. Then $f(A_1 \bigcap A_2) = f(\varnothing) = \varnothing$. But we have $f(A_1) \bigcap f(A_2) = \{a, b\} \bigcap \{a\} = \{a\}$. Therefore, $\bigcap_i f(A_i)$ is not necessarily a subset of $f(\bigcap_i A_i)$.

17

There exists a nonzero real number x, $x^4 \leq 0$.

22

"$\Rightarrow$" Suppose $A \nsubseteq B$ or $B \nsubseteq A$.

1. $A \nsubseteq B$: There is an a such that $a \in A$ but $a \notin B$. $(\rightarrow\leftarrow)$
2. $B \nsubseteq A$: Similar to above.

$\Rightarrow$ If $A = B$, then $A \subseteq B$ and $B \subseteq A$

"$\Leftarrow$" Suppose $A \neq B$, then there are 3 cases:

1. There exists an a such that $a \in A$ and $a \notin B$. But we have $A \subseteq B$, so it's impossible.
2. There exists a b such that $b \in B$ and $b \notin A$. Since we also have $B \subseteq A$, it's also impossible.

$\Rightarrow$ If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

23

If C is an empty set, then no matter whether E is an empty set or not, $C \subseteq E$.
If $C \neq \varnothing$, $\forall c \in C$, since C is a subset of D, $c \in D$. And also since $D \subseteq E$, $c \in E$. Therefore $C \subseteq E$.

39

No, its denial is $(\exists\, x \in \mathbb{R})(x^2 < 0)$

40

Substitute 9 and 16 into $\sqrt{x} + x = 12$, we can find that only $x = 9$ is the solution to this equation.

43

When we use induction, the inductive step should be able to be applied at any k. But in this case, the inductive step is not valid on $k = 1$.

44

P is true and Q is false and P is false.

45

Yes.

P	Q	R	$(P$ or $Q)$ and $(P$ or $R)$ and [not $(Q$ or $R)$]	$(P$ or $Q)$ and $(P$ or $R)$ and [not $(Q$ or $R)$]$\Rightarrow$ P
T	T	T	F	T
T	T	F	F	T
T	F	T	F	T
T	F	F	T	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

50

No.

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \Rightarrow (Q \Rightarrow P)$
F	F	T	T	T
F	T	T	F	F
T	F	F	T	T
T	T	T	T	T

49

First, please change the "consecutive integers" to "consecutive positive integers", otherwise, for any $n \in \mathbb{Z}$, n can be expressed as

$$n = [(-n-1) + \cdots + (-1) + 0 + 1 + \cdots + (n-1)] + n.$$

Consider a sequence $n, n+1, \cdots, n+(2k-1)$ where $n \geq 1$ and $k \geq 1$. Then sum of this sequence is

$$\begin{aligned}\sum_{i=n}^{n} +(2k-1)i =& \frac{[n+n+(2k-1)]\,(2k)}{2} \\ =& k(2n+2k-1).\end{aligned}$$

We can see that if $n \neq 1$ or $k \neq 1$, then $(2n+2k-1)$ is odd. Therefore for any 2^m where $m \in \mathbb{N}$, 2^m cannot be expressed as a sum of an even number of consecutive positive integers.

51

(1) We can prove it by the method of contradiction. Suppose $\neg q$ doesn't imply $\neg p$. Then p and $\neg q$ can happen at the same time. But we already have p implies q beforehand, a contradiction.

(2)

$\neg Q$	$\neg P$	$\neg Q$ implies $\neg P$
T	T	T
T	F	F
F	T	T
F	F	T

61

a True. It's an easy one. Please prove it by yourself.

b False. $(1-\pi) + \pi = 1$.

c True. It's an easy one. Please prove it by yourself.

d False. $\pi \times \frac{1}{\pi} = 1$

e Let a be a rational number and b be a irrational number. Suppose $a + b = c$ is a rational number, then from (a.) $b = c - a$ is also rational, a contradiction.

f False. $0 \times \pi = 0$.

80

For $n = 1$, $1^2 - 1 = 0$ is even.
Suppose statement is true when $n = k$, then when $n = k+1$,

$$(k+1)^2 - (k+1) = l^2 + 2k + 1 + k - 1 = \underbrace{(k^2 - k)}_{\text{is even}} + 2k.$$

Therefore, $n^2 - n$ is even for all $n \in \mathbb{N}$.