Analiza I

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Važno pročitati

Zadaci možda nisu svi isparvi.

U slučaju da ima grešaka, izvinjavam se.

Neki zadaci su preskočeni jer mi je bilo mrsko da ih radim

1. $x^2-2mx+4m-1=0$ za koje vrijednosti m su sva realna rješenja kvadratne jednačine u intervalu od $(-1,1)$?

$x^2-2m+4m-1=0$ $\boxed{D>0}$

$D=b^2-4ac$

$(2m)^2-4(4m-1)>0$

$4m^2-16m+4>0$

$m^2-4m+1>0$

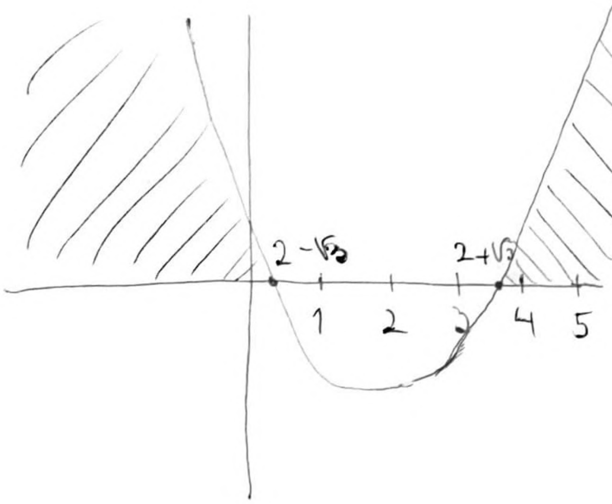
$m_{1/2}=\frac{4\pm\sqrt{16-4}}{2}$

$m_{1/2}=\frac{4\pm\sqrt{12}}{2}$

$m_{1/2}=\frac{4\pm2\sqrt{3}}{2}$

$m_1=2+\sqrt{3}$

$m_2=2-\sqrt{3}$



$x_{1/2}=\frac{2m\pm\sqrt{4m^2-4(4m-1)}}{2}$

$x_{1/2}=\frac{2m\pm\sqrt{4m^2-16m+4}}{2}$

$x_{1/2}=\frac{2m\pm\sqrt{4(m^2-4m+1)}}{2}$

$x_{1/2}=\frac{2m\pm\cancel{2}\sqrt{m^2-4m+1}}{\cancel{2}}$

$x_{1/2}=m\pm\sqrt{m^2-4m+1}$

1° $m-\sqrt{m^2-4m+1}>-1$ D.p. $m>-1$

$\sqrt{m^2-4m+1}<1+m\ /^2$

$\cancel{m^2}-4m+\cancel{1}<\cancel{1}+2m+\cancel{m^2}$

$-6m<0\ /\cdot -1/:6$

$m>0$ $m\in(0,+\infty)$

2° $m+\sqrt{m^2-4m+1}<1$ D.p. $1-m>0$, $m<1$

$\sqrt{m^2-4m+1}<1-m\ /^2$

$\cancel{m^2}-4m+\cancel{1}<\cancel{1}-2m+\cancel{m^2}$

$-2m<0$

$m>0$ $m\in(0,1)$

$m\in(0,+\infty)\cap(0,1)$

$m\in(0,1)$

$m\in(0,1)\cap(-\infty,2-\sqrt{3})$

$\boxed{m\in(0,2-\sqrt{3})}$

2. $0{,}75^x > \frac{\sqrt{3}}{2}$, $0{,}75 = \frac{3}{4}$

1° način $\left(\frac{3}{4}\right)^x > \frac{\sqrt{3}}{2} \Big/^2$ D.p. $\mathbb{R}$

$\left(\frac{3}{4}\right)^{2x} < \frac{3}{4}$

$2x < 1$

$\boxed{x < \frac{1}{2}}$

2° način $\left(\frac{3}{4}\right)^x > \frac{\sqrt{3}}{2}$

$\left(\frac{3}{4}\right)^x < \left(\frac{3}{4}\right)^{\frac{1}{2}}$

$\boxed{x < \frac{1}{2}}$

3. $\log_2 x + \log_4 x + \log_{16} x = 7$

1° način $\log_{2^1} x + \log_{2^2} x + \log_{2^4} x = 7$

$\log_2 x + \frac{1}{2}\log_2 x + \frac{1}{4}\log_2 x = 7$

$\frac{7}{4}\log_2 x = 7 \,/ : 7$

$\frac{1}{4}\log_2 x = 1 \,/ \cdot 4$

$\log_2 x = 4 \Rightarrow 2^4 = x \Rightarrow \boxed{x = 16}$

2° način $\log_{16} x = t$ $\quad 16^t = x$

$\log_4 x = 2t \leftarrow 4^{2t} = x$

$\log_2 x = 4t \leftarrow 2^{4t} = x$

$4t + 2t + t = 7$

$7t = 7$

$\boxed{t = 1}$

$\boxed{x = 16}$

3

4. Dati su skupovi $A=\{a,b,1\}$, $B=\{b,1,c\}$, $C=\{a,1\}$.

Odrediti: a) $(A \cup B) \cap C$

b) $(A \cap \overset{C}{\cancel{B}}) \cup (B \cap C)$

c) Odrediti da vrijedi da su tvrdnje ekvivalentne za sve skupove

a) $A \cup B = \{a, b, 1, c\}$

$(A \cup B) \cap C = \{a, 1\}$

b) $A \cap \overset{C}{\cancel{B}} = \{a, 1\}$

$B \cap C = \{1\}$

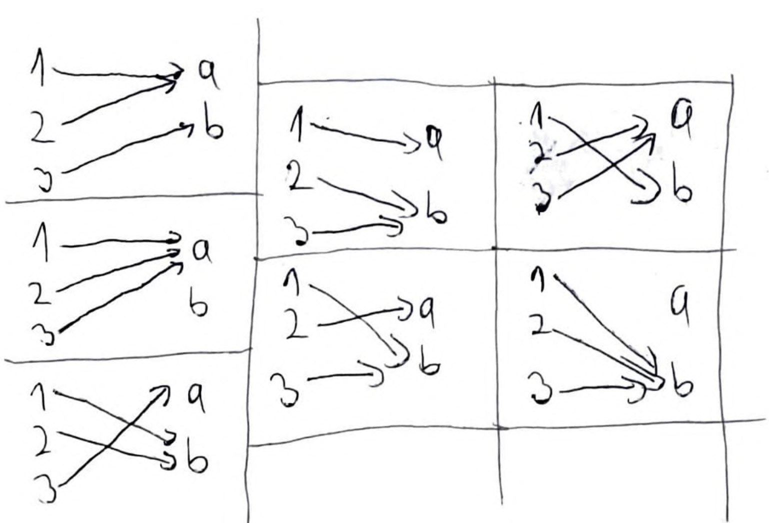
$(A \cap C) \cup (B \cap C) = \{a, 1\}$

c) $\underbrace{\underbrace{(A \vee B)}_{G} \wedge C}_{T} \Leftrightarrow \underbrace{\underbrace{(A \wedge C)}_{S} \vee \underbrace{}_{Z}(B \wedge C)}_{N}$

A	B	C	$A \vee B$	$G \wedge C$	$A \wedge C$	$B \wedge C$	$S \vee Z$	$T \Leftrightarrow N$
1	1	1	1	1	1	1	1	1
1	1	0	1	0	0	0	0	1
1	0	1	1	1	1	0	1	1
1	0	0	1	0	0	0	0	1
0	1	1	1	1	0	1	1	1
0	1	0	1	0	0	0	0	1
0	0	1	0	0	0	0	0	1
0	0	0	0	0	0	0	0	1

Tačno/Vrijedi

5. Naći sve funkcije koje preslikavaju $A=\{1,2,3\}$ u $B=\{a,b\}$.



$\boxed{2^3 = 8}$

1. $|x+2| = |x|$. Rješiti jednačinu

1° $x+2 \geq 0 \Rightarrow x \geq -2 \land x \geq 0$ $\quad x \in [0, +\infty)$

$x+2 = x$ Nema rj.

2° $x+2 < 0 \Rightarrow x < -2 \land x < 0$ $\quad x \in (-\infty, -2)$

$-x-2 = -x$ Nema rj.

3° $x+2 < 0 \Rightarrow x \leq -2 \land x \geq 0$ $\quad x \in \emptyset$

Nema rj.

4° $x+2 \geq 0 \Rightarrow x \geq 2 \land x < 0$ $\quad x \in (0, 2]$

$x+2 = -x$

$2 = -2x$

$\boxed{x = -1}$

2. a) Dokazati $|x|+|y| \geq |x+y|$

1° $x+y \geq 0$

$|x|+|y| \geq x+y$

$$\left.\begin{array}{l} |x| \geq x \\ |y| \geq y \end{array}\right\} + \quad \underline{\underline{|x|+|y| \geq x+y}}$$

2° $x+y < 0$

$|x|+|y| \geq -x-y$

$$\left.\begin{array}{l} |x| \geq -x \\ |y| \geq -y \end{array}\right\} + \quad |x|+|y| \geq -x-y$$

b) Dokazati: $|x-y| \geq ||x|-|y||$

$$\underbrace{-|x-y| \leq |x|-|y|}\ \underbrace{\leq |x-y|}$$

$-|x-y| \leq |x|-|y|$

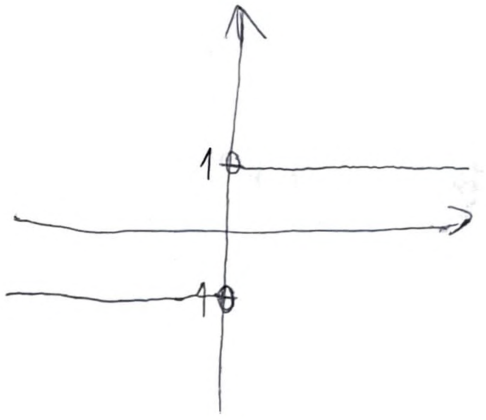
$-|x|+|y| \leq |x-y|$

$|y| \leq |y-x|+|x|$

$|x|-|y| \leq |x-y|$

$|x| \leq |x-y|+|y|$

3. Nacrtati graf: $g: \mathbb{R} \to \mathbb{R}$ $g(x) = sgn(x) = \begin{cases} 1, & x>0 \\ 0, & x=0 \\ -1, & x<0 \end{cases}$



4. a) Dokazati $1^2+2^2+3^2+\ldots+n^2 = \frac{1}{6}n(n+1)\cdot(2n+1)$

1° Pretpostavimo da vrijedi za neko $n=1$

$1 = \frac{1}{6}\cdot 2\cdot 3 \Rightarrow 1 = \frac{6}{6} \Rightarrow 1=1$ ✓

2° Pretpostavimo da vrijedi za neko $n \geq 1$

$1^2+2^2+\ldots+n^2 = \frac{1}{6}n(n+1)(2n+1)$

3° Dokažimo da tvrdnja vrijedi za neko $n+1$

3° Dokažimo da tvrdnja vrijedi za $n+1$

$$\underbrace{1^2+2^2+\dots+n^2}+(n+1)^2=\frac{1}{6}(n+1)(n+2)(2n+3)$$

$$\frac{1}{6}n(n+1)(2n+1)+(n+1)^2=\frac{1}{6}(n+1)(n+2)(2n+3) \,/ : (n+1)$$

$$\frac{1}{6}n(2n+1)+(n+1)=\frac{1}{6}(n+2)(2n+3) \,/ \cdot 6$$

$$n(2n+1)+6n+6=(n+2)(2n+3)$$

$$\cancel{2n^2}+\cancel{n}+\cancel{6n}+\cancel{6}=\cancel{2n^2}+\cancel{3n}+\cancel{4n}+\cancel{6}$$

$$0=0 \checkmark$$

4° Na osnovu PMI, tvrdnja vrijedi za svako $n\in\mathbb{N}$

b) $\sum_{k=1}^{n} k^3 = 1^3+2^3+\dots+n^3$. Dokazati: $1^3+2^3+\dots+n^3=(1+2+\dots+n)^2$

1° Pretpostavimo da vrijedi za neko $n=1$

$$1^3=1^2 \Rightarrow 1=1$$

2° Pretpostavimo da vrijedi za $n\geq 1$

$$1^3+2^3+\dots+n^3=(1+2+\dots+n)^2$$

3° Dokazati da vrijedi $n+1$

$$\underbrace{1^3+2^3+\dots+n^3}+(n+1)^3=(1+2+\dots+n+(n+1))^2$$

$$\underbrace{(1+2+\dots+n)}_{a}{}^2+(n+1)^3=(\underbrace{1+2+\dots+n}_{a}+(n+1))^2$$

$$a^2+(n+1)^3=(a+(n+1))^2$$

$$(n+1)^3=(a+(n+1))^2-a^2$$

3° Nastavak:

$(n+1)^3=(a+(n+1))^2-a^2$

$(n+1)^3=((a+(n+1))-a)((a+(n+1))+a)$

$(n+1)^3=(\cancel{a}+n+1-\cancel{a})(a+n+1+a)$

$(n+1)^3=(n+1)(2a+n+1) \,/ : (n+1)$

$(n+1)^2=2a+n+1$

$n^2+\cancel{2}n+\cancel{1}=2a+\cancel{n}+\cancel{1}$

$2a=n^2+n$

$2(1+2+\ldots+n)=n(n+1) \,/ : 2$

$\boxed{1+2+\ldots+n=\frac{n(n+1)}{2}}$ → Dokažimo ovo $\checkmark$

1° $n=1$

$1=1$

2° $n\geq 1$

$1+2+\ldots+n=\frac{n(n+1)}{2}$

3° $n+1$

$\underbrace{1+2+\ldots+n}+(n+1)=\frac{(n+1)(n+2)}{2}$

$\frac{n(n+1)}{2}+(n+1)=\frac{(n+1)(n+2)}{2} \,/ : 2$

$\underbrace{n(n+1)}_{\cancel{n^2}+\cancel{n}}+\cancel{2n}+\cancel{2}=\cancel{n^2}+\cancel{2n}+\cancel{n}+\cancel{2}$

$0=0$ $\checkmark$

4° Po PMI, tvrdnja vrijedi za svako $n\in\mathbb{N}$

4° Po PMI, tvrdnja vrijedi za svako $n\in\mathbb{N}$

5. Dokazati: $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n}} \geq \sqrt{n} \quad \forall n \in \mathbb{N}$

1° Pretpostavimo da vrijedi za $n=1$

$\sqrt{1} \geq \sqrt{1} \Rightarrow T$

2° Pretpostavimo da vrijedi za neko $n \geq 1$

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n}} \geq \sqrt{n} \quad \forall n \in \mathbb{N}$$

3° Dokažimo da vrijedi za $n+1$

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n+1}} \geq \sqrt{n+1}$$

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n}} \geq \sqrt{n+1}-\frac{1}{\sqrt{n+1}}$$

Dokažimo sljedeće:

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n}} \geq \underline{\sqrt{n} \geq \sqrt{n+1}-\frac{1}{\sqrt{n+1}}}$$

Ako dokažemo drugu polovicu koja je podvučena onda smo gotovi.

$$\sqrt{n} \geq \sqrt{n+1}-\frac{1}{\sqrt{n+1}} \Big/ \cdot \sqrt{n+1}$$

$$\sqrt{n(n+1)} \geq n+\cancel{1}-\cancel{1} \Big/^2$$

$n^2+n \geq n^2$ ✓ Ovo je tačno

4° Po PMI, tvrdnja vrijedi za $\forall n \in \mathbb{N}$

6. Dokazati: $\sqrt[n+1]{n+1} < \sqrt[n]{n} \quad n \geq 3, n \in \mathbb{N}$

1° Pretpostavimo da $n=3$

$$\sqrt[4]{4} < \sqrt[3]{3} \quad /^{12}$$

$$4^3 < 3^4$$

2° Pretpostavimo da vrijedi za $n \geq 3$

$$\sqrt[n+1]{n+1} < \sqrt[n]{n} \quad /^{n(n+1)}$$

$$(n+1)^n < n^{n+1}$$

$$(n+1)^n < n \cdot n^n \quad /:n^n$$

$$\left(\frac{(n+1)^n}{n^n}\right) < n \Rightarrow \left(\frac{n+1}{n}\right)^n < n \Rightarrow \boxed{\left(1+\frac{1}{n}\right)^n < n}$$

Ovo će trebati kasnije

3° Dokažimo da vrijedi za $n+1$

$$\sqrt[n+2]{n+2} < \sqrt[n+1]{n+1} \quad /^{(n+2)(n+1)}$$

$$(n+2)^{n+1} < (n+1)^{n+2}$$

$$(n+2)^{n+1} < (n+1)^{n+1}(n+1) \quad /:(n+1)^{(n+1)}$$

$$\left(\frac{n+2}{n+1}\right)^{n+1} < n+1$$

$$\underbrace{\left(1+\frac{1}{n+1}\right)^{n+1}} < n+1$$

$$\left(1+\frac{1}{n+1}\right)^{n+1} < \underbrace{\left(1+\frac{1}{n}\right)^{n+1}}$$

$$\underbrace{\left(1+\frac{1}{n}\right)^{n}}_{<n}\left(1+\frac{1}{n}\right) = n\left(1+\frac{1}{n}\right) = \boxed{n+1}$$

$$\boxed{\text{Dakle: } \left(1+\frac{1}{n+1}\right)^{n+1} < n+1}$$

4° Po PMI, tvrdnja vrijedi za $\forall n \in \mathbb{N}$ gdje $n \geq 3$

7. $(1+x_1)(1+x_2)\dots(1+x_n) \geqslant 1+x_1+x_2+\dots+x_n$ $\quad \boxed{x_1, x_2, \dots, x_n > -1}$

brojevi istog znaka

1° $n=1$

$1+x_1 \geqslant 1+x_1$ ✓

2° Pretpostavimo da vrijedi za $n \geqslant 1$

$(1+x_1)(1+x_2)\dots(1+x_n) \geqslant 1+x_1+x_2+\dots+x_n$

3° Dokažimo za $n+1$

$x_{n+1} > -1$, pa slijedi da je $1+x_{n+1} > 0$

Pošto to vrijedi, možemo množiti nejednakost sa tim:

$(1+x_1)(1+x_2)\dots(1+x_n)(1+x_{n+1}) \geqslant (1+x_1+x_2+\dots+x_n)(1+x_{n+1})$

Posmatrajmo desnu stranu:

$(1+x_1+\dots+x_n)(1+x_{n+1}) = 1+x_1+x_2+\dots+x_n+((1+x_1+x_2+\dots+x_n)^{x_{n+1}}$

$= 1+x_1+x_2+\dots+x_n+x_{n+1}+\underline{(x_1+x_2+\dots+x_n)x_{n+1}}$

Pošto su $x_1, x_2, \dots, x_{n+1}$ istog znaka

$(x_1+x_2+\dots+x_n)x_{n+1} \geqslant 0$

Dakle

$(1+x_1+x_2+\dots+x_n)(1+x_{n+1}) \geq 1+x_1+x_2+\dots+x_{n+1}$

Zaključujemo

$(1+x_1)(1+x_2)\dots(1+x_{n+1}) \geqslant 1+x_1+x_2+\dots+x_{n+1}$

4° Po PMI, tvrdnja vrijedi

8. $\sin x + \sin 2x + \dots + \sin nx = \dfrac{\sin\left(\frac{nx}{2}\right)\sin\left(\frac{(n+1)x}{2}\right)}{\sin\left(\frac{x}{2}\right)}$

Za svako $n \in \mathbb{N}$ i svako $x \in \mathbb{R} \setminus 2k\pi$ $k \in \mathbb{Z}$

1° $n=1$ $\quad \sin x = \dfrac{\cancel{\sin\left(\frac{x}{2}\right)}\sin x}{\cancel{\sin\left(\frac{x}{2}\right)}} \Rightarrow \sin x = \sin x$ ✓

2° Pretpi da vrijedi za neko $n \geq 1$

$$\sin x + \sin 2x + \dots + \sin nx = \frac{\sin\left(\frac{nx}{2}\right)\sin\left(\frac{(n+1)x}{2}\right)}{\sin\left(\frac{x}{2}\right)}$$

3° Dokažimo da vrijedi za $n \geq 1$

Odma smjena!

$$\frac{\sin\left(\frac{nx}{2}\right)\sin\left(\frac{(n+1)x}{2}\right)}{\sin\frac{x}{2}} + \sin(n+1)x = \frac{\sin\left(\frac{(n+1)x}{2}\right)\sin\left(\frac{(n+2)x}{2}\right)}{\sin\left(\frac{x}{2}\right)}$$

$$\frac{\sin\left(\frac{nx}{2}\right)\sin\cancel{\left(\frac{(n+1)x}{2}\right)}}{\sin\frac{x}{2}} + 2\sin\cancel{\frac{(n+1)x}{2}}\cos\frac{(n+1)x}{2} =$$

$$= \frac{\sin\cancel{\frac{(n+1)x}{2}}\sin\frac{(n+2)x}{2}}{\sin\left(\frac{x}{2}\right)}$$

$$\frac{\sin\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)} + 2\cos\frac{(n+1)x}{2} = \frac{\sin\frac{(n+2)x}{2}}{\sin\left(\frac{x}{2}\right)} \Big/ \cdot \sin\frac{x}{2}$$

$$\sin\left(\frac{nx}{2}\right) + 2\cos\frac{(n+1)x}{2}\sin\frac{x}{2} = \frac{\sin\frac{(n+2)x}{2}}{\cancel{\sin x}}$$

$$\sin\left(\frac{nx}{2}\right)+2\cos\left(\frac{(n+1)x}{2}\right)\sin\left(\frac{x}{2}\right)=\sin\left(\frac{(n+2)x}{2}\right)$$

$$\boxed{\frac{(n+2)x}{2}=\frac{nx+2x}{2}=\frac{nx+x+x}{2}=\frac{nx+x}{2}+\frac{x}{2}=\frac{(n+1)x}{2}+\frac{x}{2}}$$

$$\sin\left(\frac{nx}{2}\right)+2\cos\left(\frac{(n+1)x}{2}\right)\sin\left(\frac{x}{2}\right)=$$

$$=\sin\left(\frac{(n+1)x}{2}+\frac{x}{2}\right)$$

$$\sin\left(\frac{nx}{2}\right)+\cancel{2}\cos\left(\frac{(n+1)x}{2}\right)\sin\left(\frac{x}{2}\right)=$$

$$=\sin\left(\frac{(n+1)x}{2}\right)\cos\left(\frac{x}{2}\right)+\cancel{\sin\left(\frac{x}{2}\right)\cos\left(\frac{(n+1)x}{2}\right)}$$

$$\sin\left(\frac{nx}{2}\right)+\cos\left(\frac{(n+1)x}{2}\right)\sin\frac{x}{2}=\sin\left(\frac{(n+1)x}{2}\right)\cos\left(\frac{x}{2}\right)$$

$$\sin\left(\frac{nx}{2}\right)=\sin\left(\frac{(n+1)x}{2}\right)\cos\left(\frac{x}{2}\right)-\cos\left(\frac{(n+1)x}{2}\right)\sin\left(\frac{x}{2}\right)$$

$$\sin\left(\frac{nx}{2}\right)=\sin\left(\frac{(n+1)x}{2}-\frac{x}{2}\right)$$

$$\sin\left(\frac{nx}{2}\right)=\sin\left(\frac{nx}{2}\right)$$ ✓

4° Po PMI, tvrdnja vrijedi za $\forall n\in\mathbb{N}, \forall x\in\mathbb{R}\setminus 2k\pi$

13

1. Dokazati da $133 \mid 11^{n+2} + 12^{2n+1} \ \forall n \in \mathbb{N}_0$

1° Baza: $n=0$

$133 \mid 11^2 + 12 = 133$ ✓

2° Pretpostavimo da vrijedi za neko n

$133 \mid 11^{n+2} + 12^{2n+1}$

3° Dokažimo da vrijedi za $n+1$

$133 \mid 11^{n+3} + 12^{2n+3}$

$11^{n+3} + 12^{2n+3} =$

$= 11^{n+2} \cdot 11 + 12^{2n+1} \cdot 12^2 = 11^{n+2} \cdot 11 + 12^{2n+1} \cdot 11 + 133 =$

$= \underbrace{11\left(11^{n+2} + 12^{2n+1}\right)}_{\text{Po pretpostavci ovo je djeljivo sa } 133} + \underbrace{133}_{\text{Očigledno}}$ ✓

4° Po PMI, tvrdnja vrijedi $\forall n \in \mathbb{N}_0$

2) Neka je S skup koji ima n elemenata.
Dokazati da njegov partitivni skup ima 2^n elemenata.
(Intuicija: uzimaš 2^n jer za svaki element imaš dvije opcije, ili ga uključiti u $P(S)$ ili ne. Razmišljati binarno.)

1° Baza $S = \emptyset$ $\quad P(S) = \{\emptyset\}$ $\quad |P(S)| = 1 = 2^0$ ✓

2° Pretpostavka da za svaki skup n elemenata vrijedi da $|P(S)| = 2^n$.

$|\ |$ → modul ili veličina skupa

3° Dokazati da vrijedi za sve skupove $n+1$ elemenata

$S = \{a_1, a_2, \dots, a_{n+1}\}$

$P(S) = \{A \mid A \subseteq S\}$

1° $a_{n+1} \in A \subseteq S$

2° $a_{n+1} \notin A \subseteq S$ → Ovaj slučaj slijedi iz pretpostavke

$A = a_{n+1} \cup A'$ gdje je $A' = \{a_1, a_2, \dots, a_n\}$ → modul ovoga je 2^n

Po pretpostavci ponovo ima 2^n A'-ova. Onda isto to vrijedi za A. Dakle,
$|P(S)| = |1^\circ| + |2^\circ| = 2^n + 2^n = 2^{n+1}$

4° Po PMI, tvrdnja vrijedi $\forall n \in \mathbb{N}$.

③ Koristeći se formulom;

$$\operatorname{arctg}(\alpha)+\operatorname{arctg}(\beta)=\operatorname{arctg}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$$

i PMI, naći sumu:

$$S_n=\operatorname{arctg}\left(\frac{1}{2}\right)+\operatorname{arctg}\left(\frac{1}{8}\right)+\ldots+\operatorname{arctg}\left(\frac{1}{2n^2}\right)$$

$$S_1=\operatorname{arctg}\left(\frac{1}{2}\right)$$

$$S_2=\operatorname{arctg}\left(\frac{1}{2}\right)+\operatorname{arctg}\left(\frac{1}{8}\right)=\left(\frac{\frac{5}{8}}{\frac{15}{16}}\right)=\operatorname{arctg}\left(\frac{2}{3}\right)$$

$$S_3=\operatorname{arctg}\left(\frac{2}{3}\right)+\operatorname{arctg}\left(\frac{1}{18}\right)=\operatorname{arctg}\left(\frac{\frac{13}{18}}{\frac{52}{54}}\right)=\operatorname{arctg}\left(\frac{3}{4}\right)$$

Dakle, možemo uočiti da:

$S_n=\operatorname{arctg}\left(\frac{n}{n+1}\right)$. Trebamo ispitati ovo.

1° Baza $n=1$

$S_1=\operatorname{arctg}\frac{n}{n+1}=\operatorname{arctg}\left(\frac{1}{2}\right)$. Vidjeli smo da ovo vrijedi

2° Pretpostavimo da vrijedi za svako n

$$S_n=\operatorname{arctg}\left(\frac{1}{2}\right)+\ldots+\operatorname{arctg}\left(\frac{1}{2n^2}\right)$$

3° Dokažimo da vrijedi za neko $n+1$

$$S_{n+1}=\operatorname{arctg}\left(\frac{1}{2}\right)+\ldots+\operatorname{arctg}\left(\frac{1}{2n^2}\right)+\operatorname{arctg}\left(\frac{1}{2(n+1)^2}\right)$$

Treba dokazati $\Rightarrow S_{n+1}=\left(\frac{n+1}{n+2}\right)$.

Po pretpostavci imamo:

$$S_n=\operatorname{arctg}\left(\frac{n}{n+1}\right)$$

$S_{n+1}=\operatorname{arctg}\left(\frac{n}{n+1}\right)+\operatorname{arctg}\left(\frac{1}{2(n+1)^2}\right)$. Iskoristiti formulu!

$$\operatorname{arctg}(\alpha)+\operatorname{arctg}(\beta) = \operatorname{arctg}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$$

Prvo brojnik:

$$\frac{n}{n+1}+\frac{1}{2(n+1)^2}=\frac{2n(n+1)n+1}{2(n+1)^2}$$

Nazivnik:

$$1-\frac{n}{2(n+1)^3}=\frac{2n^3+6n^2+5n+2}{2(n+1)^3}$$

$$\operatorname{arctg}\left(\frac{\frac{2n^2+2n+1}{2(n+1)^2}}{\frac{2n^3+6n^2+5n+2}{2(n+1)^3}}\right)=$$

$$\operatorname{arctg}\left(\frac{(2n^2+2n+1)(n+1)}{2n^3+6n^2+5n+2}\right)=\operatorname{arctg}\left(\frac{n+1}{n+2}\right)$$

Kada imaš ovako nešto, slobono rastavi ako znaš koji oblik tražiš. U 99% slučajeva je tačno.

4° Po PMI, tvrdnja vrijedi za $\forall n \in \mathbb{N}$

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(4) Dokazati da za svaki prirodan broj n, i sve realne brojeve x i y vrijedi NBF

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Test: $n=3$

$$(x+y)^3 = \binom{3}{0} x^3 y^0 + \binom{3}{1} x^2 y^1 + \binom{3}{2} x^1 y^2 + \binom{3}{3} x^0 y^3$$

$$= x^3 + 3x^2 y + 3xy^2 + y^3$$

1° Baza $n=0$

$$(x+y)^0 = \sum_{k=0}^{0} \binom{0}{k} x^{0-k} y^k = \binom{0}{0} x^0 y^0 = 1 \checkmark$$

2° Pretpostavka $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

3° Dokaz za $n+1$

$$(x+y)^{n+1} = (x+y)(x+y)^n = (\text{po pretpostavci}) =$$

$$= (x+y)\left(\binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \dots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n\right)$$

$$= \left[\binom{n}{0} x^{n+1} y^0 + \binom{n}{1} x^n y^1 + \dots + \binom{n}{n-1} x^2 y^{n-1} + \binom{n}{n} x^1 y^n\right] +$$

$$+ \left[\binom{n}{0} x^n y^1 + \binom{n}{1} x^{n-1} y^2 + \dots + \binom{n}{n-1} x^1 y^n + \binom{n}{n} x^0 y^{n+1}\right] =$$

$$= \binom{n}{0} x^{n+1} y^0 + x^n y^1 \underbrace{\left(\binom{n}{1} + \binom{n}{0}\right)}_{\binom{n+1}{1}} + x^{n-1} y^2 \underbrace{\left(\binom{n}{2} + \binom{n}{1}\right)}_{\binom{n+1}{2}} + \dots + x^1 y^n \underbrace{\left(\binom{n}{n} + \binom{n}{n-1}\right)}_{\binom{n+1}{n}}$$

$$+ \binom{n}{n} x^0 y^{n+1} = \binom{n+1}{0} x^{n+1} y^0 + \binom{n+1}{1} x^n y^1 + \dots + \binom{n+1}{n} x^1 y^n + \binom{n+1}{n+1} x^0 y^{n+1}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} y^k \checkmark$$

4° Po PMI, tvrdnja vrijedi $\forall n \in \mathbb{N}$

(5) U razvoju izraza $\left(x\sqrt{x}+\frac{1}{x^4}\right)^n$ po NBF, koeficient trećeg člana je za 44 veći od koef. drugog člana.

$$\left(x\sqrt{x}+\frac{1}{x^4}\right)^n=\sum_{k=0}^{n}\binom{n}{k}\left(x^{\frac{3}{2}}\right)^{n-k}\cdot\left(x^{-4}\right)^k=$$

$$=\sum_{k=0}^{n}\binom{n}{k}\cdot x^{\frac{3}{2}n-\frac{11}{2}k}$$

$$C_3=[\text{za } k=2]=\binom{n}{2}$$

$$C_2=[\text{za } k=1]=\binom{n}{1}$$

$$C_3=C_2+44$$

$$\binom{n}{2}=\binom{n}{1}+44$$

$$\binom{n}{2}=\frac{n!}{2!\cdot(n-2)!}=\frac{n(n-1)\cancel{(n-2)!}}{2!\cdot\cancel{(n-2)!}}=\frac{n(n-1)}{2}$$

$$\binom{n}{1}=n$$

$$\frac{n(n-1)}{2}=n+44\ /\cdot 2$$

$$n^2-n=2n+88$$

$$n^2-3n-88=0$$

$$n_{1/2}=\frac{3\pm\sqrt{9+4\cdot 88}}{2}$$

$$n_{1/2}=\frac{3\pm 19}{2}$$

$\boxed{n_1=-8}$ ✗

$\boxed{n_2=11}$ ✓

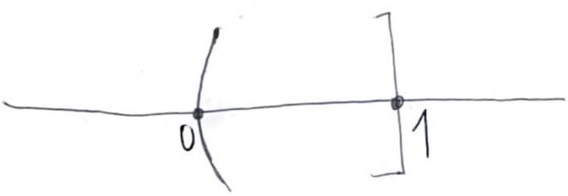
$\min S = a \quad a \in S, \forall x \in S \ a \leq x$

$\max S = b \quad b \in S, \forall x \in S \ b \geq x$

Najmanja gornja i najveća donja granica.
Supremum je najmanja gornja granica.
Infimum je najveća donja granica.

Primjer: $I = (0, 1]$

$\min I = /$
$\max I = 1$
$\inf I = 0$
$\sup I = 1$



1) Ispitati da li skupovi A i B imaju inf, sup, min i max.

a) $A = \{x \in \mathbb{R} \mid |x-4| \leq 5\}$

b) $B = \{x \in \mathbb{R} \mid 5|x| - |x^2| - 6 > 0\}$

a) $|x-4| \leq 5$

$-5 \leq x - 4 \leq 5 \ / +4$

$-1 \leq x \leq 9$

$\min A = -1$ jer $\forall x \in A \ x \geq -1$
$\inf A = -1 = \min A$
$\max A = 9$ jer $\forall x \in A \ x \leq 9$
$\sup A = 9 = \max A$

b) $B = \{x \in \mathbb{R} \mid 5|x| - |x^2| - 6 > 0\}$

$|x^2| = |x|^2$, neka $|x| = t$

Onda imamo: $5t - t^2 - 6 > 0$

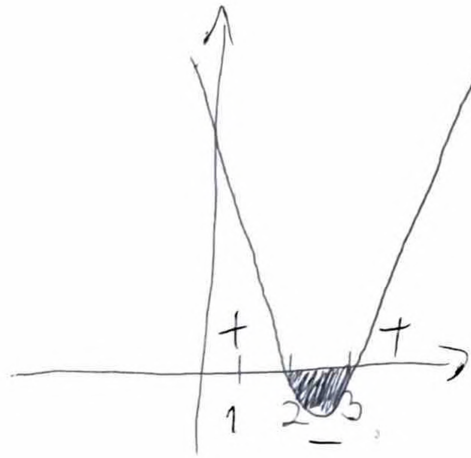
$-t^2 + 5t - 6 > 0 \,/\cdot(-1)$

$t^2 - 5t + 6 < 0$

$t^2 - 3t - 2t + 6 < 0$

$(t-3)(t-2) < 0$

$\boxed{t_1 = 3,\ t_2 = 2}$



Iz grafika vidimo da je $t \in (2,3)$, dakle $|x| \in (2,3)$.
Na kraju imamo da $x \in (-2,-3) \cup (2,3)$.

$B = \{x \in \mathbb{R} \mid 2 < x < 3 \ \vee\ -3 < x < -2\}$

1) B nema min. Pretpostavimo suprotno, da $\exists \min B = a$, $\forall x \in B\ a \leq x$.

1° $a \in (2,3)$: $-\frac{5}{2} < a$, pošto $-\frac{5}{2} \in B$, to je #.

2° $a \in (-2,-3)$: $\frac{-3+a}{2} = x \rightarrow -3 < \frac{-3+a}{2} < -2 \,/\cdot 2 \Rightarrow -6 < -3+a < -4 \,/+3$

$\Rightarrow -3 < a < -1$

$\frac{-3+a}{2} < a \Rightarrow -3 + a < 2a \Rightarrow \boxed{-3 < a}$

sa $a = \min B$, pa $\min B$ nepostoji.

-> znači kontradikcija

Nastavak:

2) B nema max. Pretpostavimo suprotno, $\exists \max B = b, \forall x \in B, x \leq b$.

1° $b \in (-2,-3)$: $\frac{5}{2}$ pripada B pa #

2° $b \in (2,3)$: $\frac{3+b}{2} = x \Rightarrow 2 < \frac{3+b}{2} < 3 / \cdot 2 \Rightarrow 4 < 3+b < 6 \overset{-3}{\Rightarrow} 1 < b < 3$

$\Downarrow$ $\frac{3+b}{2} > b \Rightarrow 3+b > 2b \Rightarrow 3 > b$

sa $\max B = b$, pa B nema max.

3) Dokazati da $\inf B = -3$

I Dokaz da je -3 donja granica $\Leftrightarrow \forall x \in B$ $x \geq 3$

II Dokaz da je -3 najveća donja granica

$-3+\varepsilon$, $\varepsilon > 0$, $\exists x \in B: -3+\varepsilon > x$ $\forall \varepsilon > 0$

1° $-3+\varepsilon \in (-3,-2)$: $x = \frac{-3+(-3+\varepsilon)}{2} \Rightarrow -3+\frac{\varepsilon}{2}$

$x \in B \Leftrightarrow -3 < -3+\frac{\varepsilon}{2} < -2 \Leftrightarrow 0 < \varepsilon < 2$

$-3+\varepsilon > x \Leftrightarrow \cancel{-3}+\varepsilon > \cancel{-3}+\frac{\varepsilon}{2} \Leftrightarrow \varepsilon > \frac{\varepsilon}{2}$

2° $-3+\varepsilon \geq -2$: $x = -\frac{5}{2} \in B$,

$-3+\varepsilon > x \Leftrightarrow -3+\varepsilon > -\frac{5}{2}$

4) Dokazati da $\sup B = 3$

I Dokaz da je 3 gornja granica $\Leftrightarrow \forall x \in B$ $x \leq 3$

II Dokaz da je 3 najmanja gornja granica $\Leftrightarrow \forall \varepsilon > 0$ $\exists x \in B: 3-\varepsilon < x$

1° $3-\varepsilon \geq 2$: $x = \frac{5}{2} \in B$

$x > 3-\varepsilon \Leftrightarrow \frac{5}{2} > 3-\varepsilon \Leftrightarrow 2\varepsilon > \frac{1}{2} \Leftrightarrow \varepsilon \geq 1$

2° $3 > 3-\varepsilon > 2$; $x = \frac{3+3-\varepsilon}{2} \rightarrow 3-\frac{\varepsilon}{2} \Rightarrow x \in B \Rightarrow 2 < 3-\frac{\varepsilon}{2} < 3 \Rightarrow$

$2 > \varepsilon > 0$

$\Downarrow$

$3-\varepsilon < x \Leftrightarrow \cancel{3}-\varepsilon < \cancel{3}-\frac{\varepsilon}{2} \Rightarrow \varepsilon > \frac{\varepsilon}{2}$

1 a) $S_1 = \{x \in \mathbb{N} : 1 \leq 3^x < 25\}$

$S_1 \in \{1, 2\}$

$\min S_1 = \inf S_1 = 1$

$\max S_1 = \sup S_1 = 2$

b) $S_2 = \{x \in \mathbb{Q} : 1 \leq 3^x < 25\}$

$x = 1 \in S_1$

$x = 2 \in S_2$

$x = -\frac{m}{n} \in \mathbb{Q}$

$$3^{-\frac{m}{n}} = \frac{1}{3^{\frac{m}{n}}} < 1 \iff 3^{\frac{m}{n}} > 1 = 3^0$$

$\frac{m}{n} > 0 \Rightarrow x \notin S_2$

$\Rightarrow x = 0 \in S_2$ minimum

$\min S_2 = \inf S_2 = 0$

$3^x = 25$

$x = \log_3 25$

Pretpostavimo da $\frac{m}{n} > 0$ (jer $0 = \min S_2$)

$\frac{m}{n} + 1 \quad x = \frac{m}{n} + \varepsilon, \ \varepsilon > 0$

$\varepsilon = \frac{1}{k}, k \in \mathbb{N} \Rightarrow x \in \mathbb{Q}$

$\boxed{x > \frac{m}{n}}$

$$\underbrace{1 \leq 3^{\frac{m}{n} + \frac{1}{k}}} < 25$$

Ovo automatski vrijedi jer $\frac{m}{n} + \frac{1}{k} > 0$

Nastavak:

$$1 \leq 3^{\frac{m}{n}+\frac{1}{k}} < 25$$

$$3^{\frac{m}{n}+\frac{1}{k}} < 25 = 3^{\log_3 25}$$

$$\frac{m}{n}+\frac{1}{k} < \log_3 25$$

$$\frac{1}{k} < \log_3 25 - \frac{m}{n}$$

$$\frac{m}{n} < \log_3 25$$

$$k > \frac{1}{\log_3 25 - \frac{m}{n}} = t > 0$$

$$k = \lceil t \rceil + 1$$

$$S_2 = \{ x \in \mathbb{Q} \mid 1 \leq 3^x < 25 \} = \{ x \in \mathbb{Q} \mid 0 \leq x < \log_3 25 \}$$

③ a) Naći supremum skupa $S_2 = \{x \in \mathbb{Q} : 1 \leq 3^x < 25\}$

Odredimo mogući supremum.

$3^x < 25$

$3^x < 3^{\log_3 25}$

$x < \log_3 25$

Dakle $s = \log_3 25$.

1° s je gornja granica

$\forall x \in S_2$ t.d. $3^x < 25$ vrijedi da $x < \log_3 25 = s$ pa je s gornja granica skupa S_2

2° s je najmanja gornja granica

s je najmanja gornja granica, odnosno treba dokazati da za svako $\forall \varepsilon > 0$ $s - \varepsilon \leq x$, ε proizvoljno mal broj postoji x koje je veće od $s-\varepsilon$, gdje je ε proizvoljno mal broj postoji x koje je veće od $s-\varepsilon$, tako da $s-\varepsilon$ nije tog $s-\varepsilon$, a i dalje vrijedi da $x < s$, tako da $s-\varepsilon$ nije najmanja gornja granica, već je to s.

Dokažimo to lemom o gustoći skupa $\mathbb{Q}$ u $\mathbb{R}$ skupu.

Lema govori da $\forall x, y \in \mathbb{R} ; x < y \ \exists q \in \mathbb{Q} : x < q < y$.

Sada, po navedenoj lemi, možemo dokazati:

$\forall \varepsilon > 0, \exists x \in \mathbb{Q}$ t.d. $s - \varepsilon < x < s = \log_3 25$.

Po tome, vrijedi da je $s = \sup S_2$.

③ b) Odrediti min, max, sup, inf skupa $S_3 = \{x^2 \in \mathbb{R} : -2 < x < 2\}$

$S_3 = \{x \in \mathbb{R} : 0 \leq x < 4\}$

$\min S_3 = \inf S_3 = 0$ jer vrijedi da $\forall x \in S_3 \; x \geq 0$.

$\max S_3$ ne postoji. Pretpostavimo suprotno, da $M = \max S_3$.

To bi značilo da $\forall x \in S_3 \; M \geq x$. Po lemi o gustoći $\mathbb{Q}$ u $\mathbb{R}$ $\exists n \in \mathbb{Q}$ t.d. $M < n < 4$, onda je $n > M$ pa kontradikcija sa činjenicom da je $M = \max S_3$.

Ostaje dokazati da $s = 4 = \sup S_3$.

1° s je gornja granica

$\forall x \in S_3$ vrijedi da $x < s = 4$

2° s je najmanja gornja granica

$\forall \varepsilon > 0 \; \exists x \in S_3$ t.d. $s - \varepsilon < x$.

Po lemi o gustoći skupa $\mathbb{Q}$ u skupu $\mathbb{R}$, vrijedi da $\exists q \in \mathbb{Q}$ t.d. $\forall x \in \mathbb{R}, y \in \mathbb{R}, x < y$, vrijedi $x < q < y$.

Sada, $\exists x \in S_3$ t.d. vrijedi $s - \varepsilon < x < s = 4$. Ovo znači da $s - \varepsilon$ nije najmanja gornja granica, jer postoji $x \in S_3$ koje je veće. Tako da $s = 4 = \sup S_3$.

(4) Pronaći min, inf, max, sup od $S_4 = \left\{ \frac{1}{x^2+1} \mid x \in \mathbb{R} \right\}$

$$0 < \frac{1}{x^2+1} \leq 1$$

1) $\max S_4 = \sup S_4 = 1$, jer $\forall x \in \mathbb{R}$ $\frac{1}{x^2+1} \leq 1$.

2) Nema $\min S_4$.
Pretpostavimo da ima $m = \min S_4$. $0 < m \leq 1$.
Onda po pretp. $\forall x \in \mathbb{R}$, $m \leq \frac{1}{x^2+1}$. Dalje:

$m(x^2+1) \leq 1$

$x^2+1 \leq \frac{1}{m}$

$\forall x \in \mathbb{R} : x^2 \leq \frac{1-m}{m}$ (*)

Ako uzmemo sada $x^2 = \frac{1-m}{m} + 1$ onda je $\frac{1-m}{m} + 1 > \frac{1-m}{m}$
a ranije u (*) bilo je da je $x^2 \leq \frac{1-m}{m}$, pa kontradikcija.

3) Dokažimo da $\inf S_4 = 0$

1° 0 ~~[illegible]~~ donja granica
$\forall x \in S_4$ vrijedi da $\frac{1}{x^2+1} > 0$, pa 0 donja granica

2° 0 najveća donja granica
$\forall \varepsilon > 0$, $\exists x \in \mathbb{R}$ t.d. $\frac{1}{x^2+1} \in S_4$ t.d. $0 + \varepsilon > \frac{1}{x^2+1}$
$\iff x^2+1 > \frac{1}{\varepsilon} \iff x^2 > \frac{1}{\varepsilon} - 1$. Uzmimo $x^2 = \frac{1}{\varepsilon}$.

5) $S_6 = \{ \frac{1}{n} \mid n \in \mathbb{N} \}$. Naći min, max, sup, inf.

$0 < \frac{1}{n} \leq 1$, dakle S_6 je ograničen.

1) $\max S_6 = \sup S_6 = 1$ jer $\forall n \in \mathbb{N}$ vrijedi da $\frac{1}{n} \leq 1$

2) $\min S_6$ ne postoji

pretp. da $\exists m \in \mathbb{N} : \frac{1}{m} = \min S_6 \Leftrightarrow \frac{1}{m} \leq \frac{1}{n} \; \forall n \in \mathbb{N}$

$\Leftrightarrow n \leq m$. Uzmimo $n = m+1$. Onda slijedi kontradikcija.

3) $\inf S_6 = 0$

1° Donja granica je 0

$\forall n \in \mathbb{N}: \frac{1}{n} > 0$ pa 0 donja granica

2° Najveća donja granica je 0

$\forall \varepsilon > 0 \; \exists n \in \mathbb{N}$ t.d. $0 + \varepsilon > \frac{1}{n} \Leftrightarrow \varepsilon > \frac{1}{n} \Leftrightarrow n > \frac{1}{\varepsilon}$

$n = \lfloor \frac{1}{\varepsilon} \rfloor + 1$

6) $X = \left\{ 1 + \frac{(-1)^n}{n} \,\middle|\, n \in \mathbb{N} \right\}$

Raspišimo prvo malo da vidimo kako se ponašaju elementi

$n=1$ $\quad$ $n=2$ $\quad$ $n=3$ $\quad$ $n=4$ $\ldots$

$1+\left(-\frac{1}{1}\right)=0$ $\quad$ $1+\frac{1}{2}$ $\quad$ $1-\frac{1}{3}$ $\quad$ $1+\frac{1}{4}$ $\ldots$

$$0 \leq 1+\frac{(-1)^n}{n} \leq 1+\frac{1}{2} = \frac{3}{2}$$

1) $\min X = \inf X = 0$

$\forall n \in \mathbb{N} \quad 0 \leq 1+\frac{(-1)^n}{n} \; / \cdot n$

$0 \leq n+(-1)^n \Leftrightarrow -n \leq (-1)^n \; / \cdot (-1) \Leftrightarrow n \geq -1 \cdot (-1)^n \Leftrightarrow n \geq (-1)^{n+1}$

Iz ovoga vidimo da $n \geq 1 \; \forall n \in \mathbb{N}$, pa $0 = \min X = \inf X$

2) $\max X = \sup X = \frac{3}{2}$

$\forall n \in \mathbb{N} \quad 1+\frac{(-1)^n}{n} \leq \frac{3}{2} \; / \cdot 2n$

$2n+2(-1)^n \leq 3n \Leftrightarrow 2n-3n \leq -\left(2 \cdot (-1)^n\right) \; / \cdot (-1) \Leftrightarrow n \geq 2 \cdot (-1)^n$

1° $n=1$ $\quad 1 \geq -2$, W, tako da prvi slučaj vrijedi

2° $n \geq 2$ $\quad$ W, ovo također vrijedi

Iz ovoga vidimo da vrijedi $\forall n \in \mathbb{N}$, pa $\frac{3}{2} = \max X = \sup X$

Bijektivna funkcija $f: \overline{X} \to \overline{Y}$

1) Injektivna $\forall x_1, x_2 \quad x_1 \neq x_2$ vrijedi da $f(x_1) \neq f(x_2)$

2) Sirjektivna $\forall y \in \overline{Y}$, $\exists x \in \overline{X}$ tako da $f(x) = y$

$|\overline{X}| = |\overline{Y}|$ ako su konačni

$\overline{X} \sim \overline{Y}$ ako postoji bijekcija $f: \overline{X} \to \overline{Y}$.
Tada kažemo da su $\overline{X}$ i $\overline{Y}$ ekvipotentni.

(1) Dokazati da je $\mathbb{N} \sim 2\mathbb{N}$

$2\mathbb{N} = \{2 \cdot n \mid n \in \mathbb{N}\}$. Neki prikaz bi bio:

$$\begin{matrix} 1 & 2 & \dots & n \\ \downarrow & \downarrow & & \downarrow \\ 2 & 4 & \dots & 2n \end{matrix}$$

Dokazati da je $f: \mathbb{N} \to 2\mathbb{N}$ bijekcija.
$f(n) = 2n$

1) Injekcija

$\forall n_1, n_2 \quad n_1 \neq n_2 \Rightarrow f(n_1) \neq f(n_2)$

$$2n_1 \neq 2n_2$$
$$n_1 \neq n_2$$

2) Sirjekcija

$\forall n \in 2\mathbb{N} \ \exists m \in \mathbb{N}$ t.d. $f(m) = n$

n je oblika $n = 2a \quad m = \frac{n}{2} = a$. Slijedi da $f(m) = 2a = n$

Vrijedi bijekcija, pa je $\mathbb{N} \sim 2\mathbb{N}$

② Dokazati da $\mathbb{Z} \sim \mathbb{N}$

~~Injektivnost~~ $-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$

$\qquad\qquad\qquad\qquad\qquad\qquad 1 \quad 2 \quad 3 \quad 4 \ldots$

$$f(n) = \begin{cases} 1, & n = 0 \\ 2n, & n > 0 \\ -2n+1, & n < 0 \end{cases}$$

1) Injektivnost

$\forall x_1, x_2$ t.d. $x_1 = x_2$ $\quad f(x_1) = f(x_2)$

1° $x_1 = x_2 = 0$ $\quad f(x_1) = f(x_2) = 1$

2° $x_1 = x_2 = n$ $\quad f(x_1) = f(x_2) = 2n \;, n > 0$

3° $x_1 = x_2 = n$ $\quad f(x_1) = f(x_2) = -2n+1 \;, n < 0$

2) Sirjektivnost

$\forall n \in \mathbb{N} \; \exists m \in \mathbb{Z}$ t.d. $f(m) = n$

1° $n = 1 \Rightarrow m = 0$ t.d. $f(0) = 1 = n$ vrijedi ✓

2° $n = 2k \Rightarrow m = k$ t.d. $f(m) = 2k = n$ vrijedi ✓

3° $n = 2k+1 \Rightarrow m = -k$ t.d. $f(m) = -2(-k)+1 = 2k+1 = n$ vrijedi ✓

Bijekcija je, pa je $\mathbb{Z} \sim \mathbb{N}$.

- Konačan i beskonačan skup
- Prebrojiv skup: ako postoji bijekcija sa skupom prirodnih brojeva.

Npr. Skupovi $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$

- Najviše prebrojiv skup: ako je konačan ili prebrojiv
- Neprebrojivi skupovi: $\mathbb{R}$, $\mathbb{I}$

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$$

- algebarski prebrojiv
- trancedentni neprebrojiv

(1) Dokazati da ako su A i B prebrojivi skupovi, da je onda skup $A \times B = \{(a,b) \mid a \in A, b \in B\}$ prebrojiv skup.

$$A \sim \mathbb{N} = \{a_1, a_2, \dots\}$$

$$B \sim \mathbb{N} = \{b_1, b_2, \dots\}$$

$$A \times B = \{(a,b) \mid a \in A, b \in B\} \sim \mathbb{N}$$

$$\begin{matrix} (a_1,b_1) & (a_1,b_2) & (a_1,b_3) & \cdots \\ (a_2,b_1) & (a_2,b_2) & (a_2,b_3) & \cdots \\ (a_3,b_1) & (a_3,b_2) & (a_3,b_3) & \cdots \\ \vdots & \vdots & \vdots & \end{matrix}$$

	a_1	a_2	a_3	a_4	$a_5 \dots$
b_1	(a_1,b_1)	(a_2,b_1)	(a_3,b_1)	$\dots$	
b_2	(a_1,b_2)	(a_2,b_2)	(a_3,b_2)	$\dots$	
b_3	(a_1,b_3)	(a_2,b_3)	(a_3,b_3)	$\dots$	
b_4	$\vdots$	$\vdots$	$\vdots$		
b_5					
$\vdots$					

Prateći ovu liniju možemo pokupiti svaki element, time dokazati ekvipotentnos sa $\mathbb{N}$.

$(a_1,b_1) = x_1$

$(a_2,b_1) = x_2$

$(a_1,b_2) = x_3$

$\vdots$

(2) $\mathbb{Q}$ je prebrojiv skup: $\mathbb{Q} = \left\{ \frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{Z} / \{0\} \right\}$

	1	2	3	4	5 ...
1	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$
2	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$
3	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$
4	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$
5	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{5}$
⋮					

- Nakon niza $\frac{a}{b}$, dodajemo $-\frac{a}{b}$
- 0 je dodana na početak

(3) Skup $\mathbb{R}$ nije prebrojiv - Kantorova dijagonalizacija

(0, 1) - interval nije prebrojiv

$x_1 = 0, a_1 a_2 a_3 a_4 \ldots$

$x_2 = 0, b_1 b_2 b_3 b_4 \ldots$

$x_3 = 0, c_1 c_2 c_3 c_4 \ldots$

$x_4 = 0, d_1 d_2 d_3 d_4 \ldots$

⋮

$x = 0.\, f(a_1) f(b_2) f(c_3) f(d_4) \ldots$

$$f(x) = \begin{cases} 1, \text{ako } x \neq 1 \\ 2, \text{ako } x = 1 \end{cases}$$

$x \neq x_1$, $x \neq x_2$ ⋮

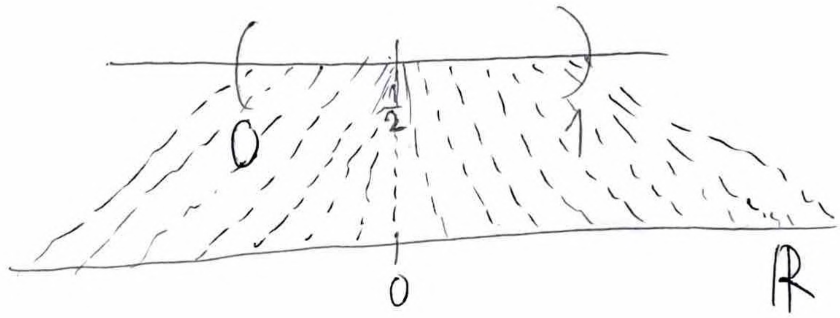
$(a_1 \neq f(a_1))$ $(b_2 \neq f(b_2))$

4) Dokazati da $(0,1) \sim \mathbb{R}$

$f: (0,1) \to \mathbb{R}$

f: bijekcija

$f\left(\frac{1}{2}\right) = 0$



$\left(0, \frac{1}{2}\right]: -\left(\frac{1}{x} - 2\right) = \frac{-1+2x}{x} = \frac{2x-1}{x}$

$\left(\frac{1}{2}, 1\right) \quad \frac{1}{1-x} - 2 = \frac{2x-1}{1-x}$

$$f(x) = \begin{cases} \frac{2x-1}{x}, & x \in \left(0, \frac{1}{2}\right] \\ \frac{2x-1}{1-x}, & x \in \left(\frac{1}{2}, 1\right) \end{cases}$$

INJEKTIVNOST:

Pretp. da $f(x_1) = f(x_2)$

1° $f(x_1) = \frac{2x_1-1}{x_1}$ $\qquad x_1, x_2 \in \left(0, \frac{1}{2}\right]$

$f(x_2) = \frac{2x_2-1}{x_2}$

$\frac{2x_1-1}{x_1} = \frac{2x_1-1}{\cancel{x_2}} \; / \, x_1x_2$

$(2x_1-1)x_2 = (2x_2-1)x_1$

$\cancel{2x_1x_2} - x_2 = \cancel{2x_1x_2} - x_1$

$\boxed{x_1 = x_2}$ ✓

2° $f(x_1)=\frac{2x_1-1}{1-x_1}$ $\quad x_1,x_2\in(\frac{1}{2},1)$

$f(x_2)=\frac{2x_2-1}{1-x_2}$

$$\frac{2x_1-1}{1-x_1}=\frac{2x_2-1}{1-x_2}\Big/(1-x_1)(1-x_2)$$

$$(2x_1-1)(1-x_2)=(2x_2-1)(1-x_1)$$

$$\cancel{2x_1}-2x_1x_2-\cancel{1}+\cancel{x_2}=\cancel{2x_2}-2x_1x_2-\cancel{1}+\cancel{x_1}$$

$$\boxed{x_1=x_2}$$

3° Naglašava se da je nemoguće da su x_1,x_2 u različitim intervalima jer je u 1° f negativno, a u 2° f pozitivno.

SIRJEKTIVNOST:

$\forall a\in\mathbb{R}\ \exists b\in(0,1)$ t.d. $f(b)=a$

1° $a\leq 0$ $\quad b\in(0,\frac{1}{2}]$

$$\frac{2b-1}{b}=a\Big/b$$

$$2b-1=ab$$

$$\boxed{b=\frac{1}{2-a}},\ 2-a\geq 0\Rightarrow a\leq 2$$

$$0<\frac{1}{2-a}\leq\frac{1}{2}\Big/2(2-a)$$

$$0<\underbrace{2\leq 2-a}\ \rightarrow\ 2\leq 2-a\Leftrightarrow 0\leq a\Leftrightarrow a\leq 0\ \checkmark$$

Posmatrajući pretpostavku, ovo vrijedi.

Nastavak zadatka:

$2^\circ \rightarrow$ sirjektivnost

$a>0 \quad b\in\left(\frac{1}{2},1\right)$

$f(b)=a$

$$\frac{2b-1}{1-b}=a$$

$$\boxed{b=\frac{a+1}{a+2}}, \quad a+2>0$$

$$\frac{1}{2}<\frac{a+1}{a+2}<1$$

$$\frac{1}{2}<1+\frac{1}{a+2}<1 \Big/ -1$$

$$-\frac{1}{2}<\frac{1}{a+2}<0 \Big/ (-1)$$

$$\frac{1}{2}>\frac{1}{a+2}>0$$

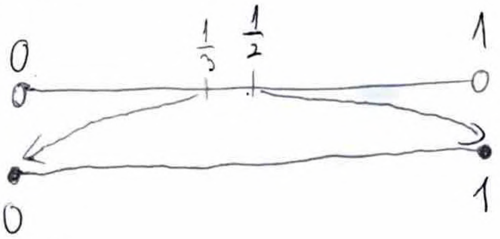
$$0<\frac{1}{a+2}<\frac{1}{2} \qquad \frac{1}{a+2}<\frac{1}{2} \Rightarrow 2<a+2$$

$$0<a$$

Po pretpostavci, vrijedi i 2° slučaj.

Dokazana je bijekcija, pa je $(0,1)\sim\mathbb{R}$

(5) Dokazati da $(0,1) \sim [0,1]$



$\frac{1}{2} \to 1$

$\frac{1}{3} \to 0$

$\frac{1}{4} \to \frac{1}{2}$

$\frac{1}{5} \to \frac{1}{3}$

$\frac{1}{6} \to \frac{1}{4}$

$\vdots$

$$f(x) = \begin{cases} x, & x \in (0,1)\ x \neq \frac{1}{k}, k \in \mathbb{N} \\ 1, & x = \frac{1}{2} \\ 0, & x = \frac{1}{3} \\ \frac{1}{k-2}, & x = \frac{1}{k}, k \geq 4, k \in \mathbb{N} \end{cases}$$

Injektivnost:

$f(x_1) = f(x_2)$

$1^\circ \quad x_1 = x_2$

$2^\circ \quad x_1 = x_2 = \frac{1}{2} \qquad f(x_1) = f(x_2) = 1$

$3^\circ \quad x_1 = x_2 = \frac{1}{3} \qquad f(x_1) = f(x_2) = 0$

$4^\circ \quad x_1 = x_2 \Rightarrow \frac{1}{x_1} = \frac{1}{x_2} \qquad f(x_1) = f(x_2) \Rightarrow \frac{1}{x_1 - 2} = \frac{1}{x_2 - 2} \qquad x_1, x_2 \geq 4$

Nastavak zadatka:

Sirjektivnost:

$\forall a \in [0,1] \; \exists b \in (0,1)$ t.d. $f(b) = a$

1° $a \neq \frac{1}{k}, k \in \mathbb{N}$

$b = a, \; f(b) = b = a$

2° $a = 1$

$b = \frac{1}{2}$

3° $a = 0$

$b = \frac{1}{3}$

4° $a = \frac{1}{k}, k \geq 4$

$b = \frac{1}{k-2}$ $\quad\quad 0 < \underbrace{\frac{1}{k-2} < 1}$

$f(b) = a$ $\quad\quad \frac{1}{k-2} < 1$

$1 < k-2$

$\boxed{k > 3}$ Pošto $k \in \mathbb{N}$, po pretpostavci ovo vrijedi.

1) $x_n = \frac{1}{n} \cdot \cos(n\pi), n \in \mathbb{N}$ ograničen, te odrediti $\max(x_n)$, $\min(x_n)$, $\inf(x_n)$ i $\sup(x_n)$.

$$x_n = \frac{1}{n} \cos(n\pi)$$

$\exists M \in \mathbb{R} > 0$ t.d. $|x_n| \leq M$

$$x_n = \frac{1}{n} \cdot (-1)^n$$

$$\left| \frac{1}{n} \cdot \cos(n\pi) \right| \leq M$$

$$\left| \frac{1}{n} \right| \cdot \left| \cos(n\pi) \right| \leq 1$$

$\boxed{M=1}$, ograničen odozgo sa $\boxed{M=1}$

$x_2 = \frac{1}{2}$, $x_4 = \frac{1}{4}$, $x_6 = \frac{1}{6} \dots$

$x_1 = -1$, $x_3 = \frac{1}{-3}$, $x_5 = \frac{1}{-5} \dots$

$$x_{2k} = \frac{1}{2k} \qquad k \in \mathbb{N}$$

$$x_{2k-1} = -\frac{1}{2k-1}$$

1° $x_2 > x_4 > x_6 \dots$

$$x_{2k} > x_{2k+2}$$

$$\frac{1}{2k} > \frac{1}{2k+2}$$

Vidimo da za $k=1$, $\max x_n = x_2 = \frac{1}{2}$ ✓

2° $x_1 < x_3 < x_5 \dots$

$$x_{2k-1} < x_{2k+1}$$

$$-\frac{1}{2k-1} < -\frac{1}{2k+1} \Big/ \cdot (-1)$$

$$\frac{1}{2k-1} > \frac{1}{2k+1}$$

Vidimo da za $k=1$ vrijedi, pa $\min x_n = x_1 = -1$ ✓

② Odrediti maximum niza $x_n = \frac{n^2}{2^n}$

$\max x_n = ?$

2^n - raste brže

n^2 - raste sporije

$x_n > x_{n+1}$

$\frac{n^2}{2^n} > \frac{(n+1)^2}{2^{(n+1)}} \quad / \cdot 2^{n+1}$

$2n^2 > (n+1)^2$

$2n^2 > n^2 + 2n + 1$

$n(n-2) > 1 \quad \forall n \geq 3$

Radi ovoga, znamo da se maximum nalazi u prva tri člana niza.

Dakle $\max x_n = \{x_1, x_2, x_3\}$?

$x_1 = \frac{1}{2}$

$x_2 = 1$

$x_3 = \frac{9}{8}$

$x_4 = 1$

$x_5 = \ldots$ neki još manji izraz.

(1) Dokazati da $\lim_{n\to\infty} \frac{1}{n} = 0$

$n_0 \in \mathbb{N}$

Ovo znači da za $\forall \varepsilon > 0$ $\exists n_0 = n_0(\varepsilon)$ t.d. $\forall n \geq n_0$ vrijedi $\left|\frac{1}{n} - 0\right| < \varepsilon$. Dokažimo to:

$$\left|\frac{1}{n} - 0\right| < \varepsilon$$

$$\left|\frac{1}{n}\right| < \varepsilon$$

$$\frac{1}{n} < \varepsilon \quad / \cdot n$$

$$1 < \varepsilon n \quad / : \varepsilon$$

$$\frac{1}{\varepsilon} < n \Leftrightarrow n > \frac{1}{\varepsilon}$$

$\cancel{n > \frac{1}{\varepsilon}}$ Sada možemo odabrati n_0, jer po definiciji tražimo $n \geq n_0$, pa pritome $n \geq n_0 > \frac{1}{\varepsilon}$.

$n_0 = \left\lfloor \frac{1}{\varepsilon} \right\rfloor + 1$, jer tražimo cjeli broj i mora biti $n \geq n_0$, zato dodajemo 1 da ne bi bilo strogo manje zbog cio dio dpk.

2) Dokazati da $\lim_{n\to\infty} x_n = 1$ za $x_n = \frac{n}{n+1}$

Ovo znači da za $\forall \varepsilon > 0 \ \exists n_0 = n_0(\varepsilon) \in \mathbb{N}$ gdje $\forall n \geq n_0$ vrijedi: $\left| \frac{n}{n+1} - 1 \right| < \varepsilon$.

$$\left| \frac{n}{n+1} - 1 \right| < \varepsilon$$

$$\left| \frac{\cancel{n} - \cancel{n} - 1}{n+1} \right| < \varepsilon$$

$$\left| -\frac{1}{n+1} \right| < \varepsilon \iff \frac{1}{n+1} < \varepsilon \,/\cdot(n+1) \iff 1 < \varepsilon(n+1) \,/:\varepsilon \iff$$

$$\frac{1}{\varepsilon} - 1 < n$$

$\boxed{n > \frac{1-\varepsilon}{\varepsilon}}$ Odaberimo $n_0 = n_0(\varepsilon) = \left\lfloor \frac{1-\varepsilon}{\varepsilon} \right\rfloor + 1$

Ponovo uzimamo cio dio da bi bio prirodan broj, i uzimamo +1 da bi se osigurali da nije negativan.

(3) Dokazati da $\lim_{n\to\infty} \frac{2n+2}{7n+4} = \frac{2}{7}$

$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ t.d. $\forall n \geq n_0$ $\left|\frac{2n+2}{7n+4} - \frac{2}{7}\right| < \varepsilon.$

$\left|\frac{14n+14-14n-8}{7(7n+4)}\right| < \varepsilon \Leftrightarrow \left|\frac{6}{7(7n+4)}\right| < \varepsilon$

$6 < 7\varepsilon(7n+4)$

$49n\varepsilon > 6 - 28\varepsilon$

$\boxed{n > \frac{6-28\varepsilon}{49\varepsilon}}$

$\boxed{n_0 = \left\lfloor \frac{6-28\varepsilon}{49\varepsilon} \right\rfloor + 1}$

(4) $\lim_{n\to\infty} 3^{\sqrt{n}} = +\infty$. Dokazati da vrijedi.

$\forall n > 0$ t.d. $\forall n \geq n_0$

implicira jer nije ekvivalentno

$3^{\sqrt{n}} > n = 3^{\log_3 n} \Leftrightarrow \sqrt{n} > \log_3 n \overset{2}{\not\Leftarrow}\Rightarrow n > \log_3^2 n$

Ponovo n_0 odabir $n_0 = \lfloor \log_3^2 n \rfloor + 1$

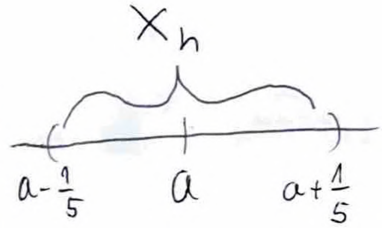
⑤ Dokazati da $\lim\limits_{n\to\infty} (-1)^n$ divergira.

$x_n = (-1)^n, \; n \in \mathbb{N}$

Pretpostavimo suprotno, da x_n konvergira.

$\exists a \in \mathbb{R} : \lim\limits_{n\to\infty} x_n = a$ (Ovako se generalno vodi do kontradikcije)

$\varepsilon = \frac{1}{5}$ $\exists n_0 \in \mathbb{N}$ t.d. $\forall n \geq n_0$ $\left|(-1)^n - a\right| < \frac{1}{5}$



$$\left.\begin{aligned} |x_n - a| &< \frac{1}{5} \\ |x_{n+1} - a| &< \frac{1}{5} \end{aligned}\right\} \text{vrijedi?}$$

$$|x_n - x_{n+1}| = |(x_n - a) - (x_{n+1} - a)| = |(x_n - a) + (a - x_{n+1})| \leq$$

$$\leq |x_n - a| + |a - x_{n+1}| < \frac{1}{5} + \frac{1}{5} < \frac{2}{5} \qquad |x_n - x_{n+1}| = 2$$

$$\left|(-1)^n - (-1)^{n+1}\right| = \left|(-1)^n + (-1)^n\right| = \left|2 \cdot (-1)^n\right| = 2$$

Kontradikcija sa $\lim\limits_{n\to\infty} x_n = a$, pa niz divergira.

(6) Dokazati da niz x_n zadan sa $x_n = n^{(-1)^n}$ nije ograničen, a ipak nije beskonačno velik kad $n \to +\infty$.

Ako bi ovaj niz bio ograničen, to bi značilo da $\exists M$ konstanta takva da $\forall n \in \mathbb{N}$ vrijedi $|x_n| \leq M$. Pošto je M konstanta, jasno je da će za dovoljno veliko n da vrijedi sljedeće: $M < 2n = |2n^{(-1)^{2n}}| = |2n| = |x_{2n}|$. Dakle niz nije ograničen jer je došlo do kontradikcije sa tim da $|x_n| \leq M$ $\forall n \in \mathbb{N}$.

Da bi $x_n \to +\infty$ kada $n \to +\infty$ onda $\forall c \geq 0$ $\exists N \in \mathbb{N}$ takvo da za $\forall n \geq N$, $n \in \mathbb{N}$ vrijedi $x_n > c$. Odaberimo $c = 2$. Također, posmatrajmo $n \geq N$ takvo da je n neparan broj iz $\mathbb{N}$. Tada imamo $x_n = n^{(-1)^n} = n^{-1} = \frac{1}{n} < c = 2$. Ovo je kontradikcija, pa ne vrijedi da $x_n \to +\infty$ kada $n \to +\infty$.

Da bi $x_n \to -\infty$ kada $n \to -\infty$ onda $\forall c \geq 0$ $\exists N \in \mathbb{N}$ takvo da za $\forall n \geq N, n \in \mathbb{N}$ vrijedi $x_n < c$. Odaberimo $c = 0$. Posmatrajmo također proizvoljno $n \in \mathbb{N}$ takvo da $n \geq N$. Očigledno je da $x_n = n^{(-1)^n} > 0$ za svako $n \in \mathbb{N}$ pa je to kontradikcija. Vrijedi da nije tačno da $x_n \to -\infty$ kada $n \to +\infty$.

Finalno, da bi $x_n \to \infty$ kada $n \to +\infty$ onda $\forall c \geq 0$ $\exists N \in \mathbb{N}$ takvo da za $\forall n \geq N, n \in \mathbb{N}$ vrijedi $|x_n| > c$. Odaberimo ponovo $c = 2$. Posmatrajmo ponovo $n \geq N, n \in \mathbb{N}$ koji su neparni. Onda vidimo da vrijedi: $|x_n| = |n^{(-1)^n}| = |n^{-1}| = \frac{1}{n} < c = 2$, što je kontradikcija.

Dakle niz x_n nije ograničen, a nije ni beskonačno velik.

1) Dokazati da je:

a) $\lim_{n\to\infty} \sqrt[n]{n} = 1$

Ovo znači da $\forall \varepsilon>0$ $\exists n_0 = n_0(\varepsilon), n_0 \in \mathbb{N}$ takvo da $\forall n \geq n_0$ vrijedi $|\sqrt[n]{n}-1|<\varepsilon$. Dalje:

$|\sqrt[n]{n}-1|<\varepsilon \Rightarrow \sqrt[n]{n}-1<\varepsilon \Rightarrow \sqrt[n]{n}<1+\varepsilon \,/^n \Rightarrow n<(1+\varepsilon)^n$.

Posmatrajmo binomni razvoj za $(1+\varepsilon)^n$:

$(1+\varepsilon)^n = \binom{n}{0}\varepsilon^0+\binom{n}{1}\varepsilon^1+\binom{n}{2}\varepsilon^2+\dots+\binom{n}{n}\varepsilon^n$. Uočimo treći sabirak.

$\binom{n}{2}\varepsilon^2 = \frac{n!}{2(n-2)!}\varepsilon^2 = \frac{n(n-1)\cancel{(n-2)}!}{2\cancel{(n-2)}!}\varepsilon^2 = \frac{n(n-1)}{2}\varepsilon^2$. Vrijedi da je

$\frac{n(n-1)}{2}\varepsilon^2 \leq \binom{n}{0}\varepsilon^0+\binom{n}{1}\varepsilon^1+\dots+\binom{n}{n}\varepsilon^n$. Sada posmatrajmo:

$\cancel{n} < \frac{\cancel{n}(n-1)}{2}\varepsilon^2$

$1<\frac{\varepsilon^2 n - \varepsilon^2}{2} \,/\cdot 2$

$2<\varepsilon^2 n-\varepsilon^2$

$2+\varepsilon^2<\varepsilon^2 n \,/:\varepsilon^2$

$\frac{2+\varepsilon^2}{\varepsilon^2}<n \Rightarrow \underbrace{n>1+\frac{2}{\varepsilon^2}}$. Odaberimo $n_0 = \lfloor \frac{2}{\varepsilon^2}+1\rfloor+1 = \lfloor\frac{2}{\varepsilon^2}\rfloor+2$

Sada ovo znači da $\forall\varepsilon>0$ $\exists n_0=\lfloor\frac{2}{\varepsilon^2}+2\rfloor$ takvo da $\forall n\geq n_0$ vrijedi $|\sqrt[n]{n}-1|<\varepsilon$ pa vrijedi da $\lim_{n\to\infty}\sqrt[n]{n}=1$.

b) $\lim\limits_{n\to\infty} \frac{n^k}{a^n} = 0 \quad \forall a, k \in \mathbb{R},\ a > 1$

Trebamo dokazati da $\forall \varepsilon > 0\ \exists n_0 = n_0(\varepsilon) \in \mathbb{N}$ takvo da $\forall n \geq n_0$ vrijedi da $\left|\frac{n^k}{a^n} - 0\right| < \varepsilon$. Dakle, $\left|\frac{n^k}{a^n} - 0\right| < \varepsilon \Rightarrow \frac{n^k}{a^n} < \varepsilon \Rightarrow \left(\frac{n}{\sqrt[k]{a^n}}\right)^k < \varepsilon \Rightarrow \frac{n}{\sqrt[k]{a^n}} < \sqrt[k]{\varepsilon}$. Sada, pošto $a > 1$, možemo uvesti smjenu $b = \sqrt[k]{a}$, $b > 1$. Sada vrijedi $\frac{n}{b^n} < \sqrt[k]{\varepsilon}$. Sada, pošto je $b > 1$, možemo napisati b u obliku $b = 1 + f$, gdje je $f > 0$. Dobijamo sada $\frac{n}{(1+f)^n} < \sqrt[k]{\varepsilon}$. Po Newtonovoj binomnoj formuli vrijedi:

$(1+f)^n = \binom{n}{0} f^0 + \binom{n}{1} f^1 + \binom{n}{2} f^2 + \dots + \binom{n}{n} f^n$. Uočimo treći sabirak:

$\binom{n}{2} f^2 = \frac{n!}{2!(n-2)!} f^2 = \frac{n(n-1)\cancel{(n-2)!}}{2\cancel{(n-2)!}} f^2 = \frac{n(n-1)}{2} f^2$. Pošto $f > 0$ i $n \in \mathbb{N}$, to znači da vrijedi: $\binom{n}{0} f^0 + \binom{n}{1} f^1 + \dots + \binom{n}{n} f^n > \frac{n(n-1)}{2} f^2$.

Posmatrajući sada nejednakost $\frac{n}{(1+f)^n} < \sqrt[k]{\varepsilon}$, vidimo sljedeće:

$$\frac{n}{\binom{n}{0} f^0 + \binom{n}{1} f^1 + \dots + \binom{n}{n} f^n} < \frac{n}{\frac{n(n-1)}{2} \cdot f^2} < \sqrt[k]{\varepsilon}.$$ Dalje, imamo da vrijedi:

$\frac{2}{(n-1) f^2} < \sqrt[k]{\varepsilon} \Rightarrow n > \frac{2}{\sqrt[k]{\varepsilon}\, f^2} + 1$. Odaberimo sada $n_0 = \left\lfloor \frac{2}{\sqrt[k]{\varepsilon}\, f^2} + 1 \right\rfloor + 1$

odnosno $n_0 = \left\lfloor \frac{2}{\sqrt[k]{\varepsilon}\, f^2} \right\rfloor + 2$. Vraćajući nazad smjene, imamo da je

$f = b - 1$ pa $n_0 = \left\lfloor \frac{2}{\sqrt[k]{\varepsilon}\,(b-1)^2} \right\rfloor + 2$, i $b = \sqrt[k]{a}$ pa $n_0 = \left\lfloor \frac{2}{\sqrt[k]{\varepsilon}\,(\sqrt[k]{a}-1)^2} \right\rfloor + 2$.

Ovo znači da $\forall \varepsilon > 0\ \exists n_0 = n_0(\varepsilon) = \left\lfloor \frac{2}{\sqrt[k]{\varepsilon}\,(\sqrt[k]{a}-1)^2} \right\rfloor + 2$ za koje vrijedi da $\forall n \geq n_0$ vrijedi $\left|\frac{n^k}{a^n} - 0\right| < \varepsilon$ pa je $\lim\limits_{n\to\infty} \frac{n^k}{a^n} = 0$.

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(5) Dokazati da niz $x_n = \frac{\cos(1)}{3} + \frac{\cos(2)}{3^2} + \dots + \frac{\cos(n)}{3^n}$ konvergira.

To znači da za x_n vrijedi da je košijev, pa je dovoljno to dokazati. Dakle, $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ takvo da $\forall m, n \geq N$ vrijedi da $|x_m - x_n| < \varepsilon$. Možemo pretpostaviti da $m > n$.

Posmatrajmo sada:

$$|x_m - x_n| = \left| \cancel{\frac{\cos(1)}{3}} + \cancel{\frac{\cos(2)}{3^2}} + \cancel{\dots} + \cancel{\frac{\cos(n)}{3^n}} + \frac{\cos(n+1)}{3^{n+1}} + \dots + \frac{\cos(m)}{3^m} - \left(\cancel{\frac{\cos(1)}{3}} + \cancel{\frac{\cos(2)}{3^2}} + \cancel{\dots} + \cancel{\frac{\cos(n)}{3^n}} \right) \right| =$$

$$\left| \frac{\cos(n+1)}{3^{n+1}} + \dots + \frac{\cos(m)}{3^m} \right| \leq \left| \frac{\cos(n+1)}{3^{n+1}} \right| + \left| \frac{\cos(n+2)}{3^{n+2}} \right| + \dots + \left| \frac{\cos(m)}{3^m} \right|$$

$$= \frac{|\cos(n+1)|}{3^{n+1}} + \frac{|\cos(n+2)|}{3^{n+2}} + \dots + \frac{\cos(m)}{3^m} =$$

$$= \frac{1}{3^{n+1}} + \frac{1}{3^{n+2}} + \dots + \frac{1}{3^m} < \frac{1}{3^{n+1}} + \frac{1}{3^{n+2}} + \dots = \frac{1}{3^{n+1}} \left(1 + \frac{1}{3} + \dots\right)$$

$$= \frac{1}{3^{n+1}} \cdot \frac{1}{1 - \frac{1}{3}} = \frac{1}{3^{n+\cancel{1}} \cdot \frac{2}{\cancel{3}}} = \boxed{\frac{1}{3^n \cdot 2}}$$

Zahtjevamo da je $\frac{1}{3^n \cdot 2} < \varepsilon$. Dalje, $1 < \varepsilon \cdot 3^n \cdot 2 \; /:2\varepsilon \Rightarrow$ $\frac{1}{2\varepsilon} < 3^n \Rightarrow \log \frac{1}{2\varepsilon} < \log 3^n$ (Gdje je log bilo koji sa bazom većom od 1) $\Rightarrow n \log 3 > \log \frac{1}{2\varepsilon} \Rightarrow n > \log \frac{1}{2\varepsilon} \cdot \log 3^{-1}$. Neka $N = \lfloor \log \frac{1}{2\varepsilon} \cdot \log 3^{-1} \rfloor + 1$.

Zaključujemo da je niz košijev. (Previše dugo da pišem)

(6) Dokazati da niz $x_n = \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n}$, $n \in \mathbb{N}$ divergira.

Dovoljno je dokazati da x_n nije Košijev. To znači da $\exists \varepsilon > 0$ tako da $\forall N \in \mathbb{N}$ $\exists m, n \geq N$ takvi da vrijedi $|x_m - x_n| \geq \varepsilon$. Neka je $\varepsilon = \frac{1}{2}$ i $N \in \mathbb{N}$ proizvoljan.

Odaberimo $m = N+N \geq N$ i $n = N \geq N$. Sada imamo:

$$|x_m - x_n| = |x_{2N} - x_N| = \left| \cancel{\frac{1}{1}} + \cancel{\frac{1}{2}} + \cancel{\dots} + \cancel{\frac{1}{N}} + \frac{1}{N+1} + \dots + \frac{1}{N+N} - \left(\cancel{\frac{1}{1}} + \cancel{\frac{1}{2}} + \cancel{\dots} + \cancel{\frac{1}{N}} \right) \right| =$$

$$= \left| \frac{1}{N+1} + \dots + \frac{1}{N+N} \right| = \frac{1}{N+1} + \frac{1}{N+2} + \dots + \frac{1}{N+N}$$

$$\geq \frac{1}{N+N} + \frac{1}{N+N} + \dots + \frac{1}{N+N} = \frac{N}{N+N} = \frac{\cancel{N}}{2\cancel{N}} = \frac{1}{2} \geq \varepsilon$$

Postoji dakle $\varepsilon = \frac{1}{2} > 0$ tako da $\forall N \in \mathbb{N}$ postoje $m = 2N \geq N$ i $n = N \geq N$ takvi da je $|x_m - x_n| \geq \varepsilon$. Dakle vrijedi da niz x_n nije Košijev, pa samim tim nije ni konvergentan.

1) Dokazati da niz $x_n = 1+\frac{1}{2}+\dots+\frac{1}{n}-\ln(n)$ konvergira.

Dokazati ćemo da je niz x_n opadajući i ograničen odozdo. Imamo:

$$x_{n+1}-x_n = \cancel{1}+\cancel{\frac{1}{2}}+\dots+\cancel{\frac{1}{n}}+\frac{1}{n+1}-\ln(n+1)-\cancel{1}-\cancel{\frac{1}{2}}-\dots-\cancel{\frac{1}{n}}+\ln(n)$$

$$=\frac{1}{n+1}-\left(\ln(n+1)-\ln(n)\right)=\frac{1}{n+1}-\ln\left(\frac{n+1}{n}\right).$$

(Dokaže se da je $\ln\left(\frac{n+1}{n}\right) > \frac{1}{n+1}$ sa $1-\frac{1}{x} < \ln(x) < x-1$)

Dakle niz je opadajući. Ostaje dokazati da je niz ograničen odozdo.

Sada:

$$x_n = 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\ln(n) > \ln\left(1+\frac{1}{1}\right)+\ln\left(1+\frac{1}{2}\right)+\dots+\ln\left(1+\frac{1}{n}\right)-\ln(n)$$

$$=\ln(2)+\ln\left(\frac{3}{2}\right)+\ln\left(\frac{4}{3}\right)+\dots+\ln\left(\frac{n+1}{n}\right)-\ln(n)=$$

$$=\ln\left(\cancel{2}\cdot\frac{\cancel{3}}{\cancel{2}}\cdot\frac{\cancel{4}}{\cancel{3}}\cdot\dots\cdot\frac{n+1}{\cancel{n}}\cdot\frac{1}{n}\right)=\ln\left(\frac{n+1}{n}\right)>\frac{1}{n+1}>0$$

Dokazali smo da je niz ograničen odozdo i da je opadajući, pa je on konvergentan.

③ Dokazati da je niz $x_n = \left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\cdot\ldots\cdot\left(1+\frac{1}{2^n}\right)$ konvergentan.

Dokazati ćemo da je niz rastući i ograničen odozgo.

$$\frac{x_{n+1}}{x_n} = \frac{\cancel{\left(1+\frac{1}{2}\right)}\cancel{\left(1+\frac{1}{4}\right)}\cdot\ldots\cdot\left(1+\frac{1}{2^{n+1}}\right)}{\cancel{\left(1+\frac{1}{2}\right)}\cancel{\left(1+\frac{1}{4}\right)}\cdot\ldots\cdot\cancel{\left(1+\frac{1}{2^n}\right)}} = 1+\frac{1}{2^{n+1}} > 1,$$ pa je

niz monotono rastući. Ostaje dokazati da je on ograničen odozgo. Posmatrajmo x_n.

$$\ln x_n = \ln\left(1+\frac{1}{2}\right)+\ln\left(1+\frac{1}{4}\right)+\ldots+\ln\left(1+\frac{1}{2^n}\right) <$$

$$< \frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n} < \frac{1}{2}+\frac{1}{2^2}+\ldots = \frac{1}{2}\left(1+\frac{1}{2}+\ldots\right) =$$

$$= \cancel{\frac{1}{2}}\cdot\frac{1}{\cancel{1-\frac{1}{2}}} = 1,$$

$\ln x_n = 1$, pa je $x_n = e^1 = e$. Niz je ograničen odozgo sa

brojem e.

4) Dokazati da niz $x_n = \frac{n!}{(2n-1)!!}$, $n \in \mathbb{N}$ konvergira, i naći limes niza x_n.

Prvo, ispitajmo monotonost.

$$\frac{x_{n+1}}{x_n} = \frac{\frac{(n+1)!}{(2n+1)!!}}{\frac{n!}{(2n-1)!!}} = \frac{\frac{(n+1)\cancel{n!}}{(2n+1)\cancel{(2n-1)!!}}}{\frac{\cancel{n!}}{\cancel{(2n-1)!!}}} = \frac{n+1}{2n+1} < 1$$ pa niz je monotono opadajući.

Primjetimo da niz $x_{n+1} = \frac{n+1}{2n+1} \cdot x_n$

Ovo je rekurzija jer buduće stanje zavisi od prošlog.

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = L$$

$$L = \lim_{n\to\infty} \frac{n+1}{2n+1} \cdot L$$

$$L = \frac{1}{2} L$$

$$L - \frac{1}{2} L = 0$$

$$\frac{1}{2} L = 0$$

$$L = 0$$

Pa je limes niza $x_n = \frac{n!}{(2n-1)!!} = 0$.

(5) Niz a_n zadan je sa $a_1=1$, $a_{n+1}=a_n+n+1$ za $n\geq 1$. Izračunati limes $\frac{a_n+a_{n+1}}{n^2}$

Ovo je niz zadan rekurzivno. Ispišimo par članova:

$a_1=1$

$a_2=3$

$a_3=6$

$\vdots$

$a_n=1+2+\ldots+n$. Ovo je naša pretpostavka. Dokažimo:

PMI:

1° Baza $n=1$

$a_1=1$ ✓

2° ~~Korak~~ Pretpostavka da $a_n=1+2+\ldots+n$

3° Dokaz za $a_{n+1}=1+2+\ldots+n+n+1$

$a_{n+1}=1+2+\ldots+n+n+1$

$a_{n+1}=a_n+n+1$ ✓

4° Po PMI, tvrdnja je tačna za svako $n\in\mathbb{N}$

$$\lim_{n\to\infty}\frac{a_n+a_{n+1}}{n^2}=\frac{\frac{n(n+1)}{2}+\frac{(n+1)(n+2)}{2}}{n^2}=\frac{\frac{n^2+n+n^2+3n+2}{2}}{\frac{n^2}{1}}=$$

$$=\frac{2n^2+4n+2\ /:n^2}{2n^2/:n^2}=\frac{2}{2}=1.$$

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Točan zadatak

⑥ Neka je p proizvoljan prirodan broj.

Izračunati $\lim\limits_{n\to\infty} \frac{\overbrace{1^p+2^p+\ldots+n^p}^{x_n}}{\underbrace{n^{p+1}}_{y_n}}$

Na ovaj zadatak se primjenjuje Štolcov teorem.

Uslovi Štolcovog teorema:

1° $y_{n+1} > y_n$ (monotono rastući)

2° $\lim\limits_{n\to\infty} y_n = +\infty$

(Uslovi se lahko ispitaju, te oba vrijede) Slijedi:

$$\lim_{n\to\infty}\frac{x_{n+1}-x_n}{y_{n+1}-y_n} = \frac{\cancel{1^p}+\cancel{\ldots}+\cancel{n^p}+(n+1)^p-\cancel{1^p}+\cancel{\ldots}+\cancel{n^p}}{(n+1)^{p+1}-n^{p+1}} = \frac{(n+1)^p}{(n+1)^{p+1}-n^{p+1}}$$

Sada, po Newtonovoj binomnoj formuli, vrijedi:

$$= \frac{\binom{p}{0}n^p+\binom{p}{1}n^{p-1}+\ldots+\binom{p}{p}n^0}{\cancel{\binom{p+1}{0}n^{p+1}}+\binom{p+1}{1}n^p+\ldots+\binom{p+1}{p+1}n^0-\cancel{n^{p+1}}}.$$

Uočimo prvi član zbira u nazivniku: $\binom{p+1}{0}n^{p+1} = \frac{(p+1)!}{(p+1)!}n^{p+1} = n^{p+1}$. Kraćenje je dozvoljeno, pa imamo:

$$\lim_{n\to\infty}\frac{\binom{p}{0}n^p+\binom{p}{1}n^{p-1}+\ldots+\binom{p}{p}n^0}{\binom{p+1}{1}n^p+\ldots+\binom{p+1}{p+1}n^0} = \frac{n^p+p\cdot n^{p-1}+\ldots+1\ /:n^p}{(p+1)\cdot n^p+\ldots+1\ /:n^p} = \frac{1+\underbrace{\frac{p}{n}}_{\to 0}+\underbrace{\ldots}_{\to 0}+\underbrace{\frac{1}{n^p}}_{\to 0}}{p+1+\underbrace{\ldots}_{\to 0}+\underbrace{\frac{1}{n^p}}_{\to 0}} =$$

$$= \boxed{\frac{1}{p+1}}$$

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① $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$. Ispitati konvergenciju reda.

$$S_n = \sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)} = \frac{A}{2k-1} + \frac{B}{2k+1}$$

$$1 = A(2k+1) + B(2k-1)$$

$$0k + 1 = A2k + A + B2k - B$$

$$0k + 1 = 2k(A+B) + A - B$$

$$A + B = 0$$

$$A - B = 1$$

$$A = 1 + B$$

$$-2B = 1$$

$$B = -\frac{1}{2}$$

$$A = \frac{1}{2}$$

$$S_n = \sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)} = \frac{\frac{1}{2}}{2k-1} - \frac{\frac{1}{2}}{2k+1}$$

$$S_n = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{2k-1} - \frac{1}{2k+1}$$

$$= \frac{1}{2}\left(\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \dots + \frac{1}{2k+1}\right)$$

$$= \frac{1}{2}\left(1 - \cancel{\frac{1}{2k+1}}_{0}\right) \qquad \boxed{S_n \xrightarrow{n\to\infty} \frac{1}{2}}$$

Parcijalna suma konvergira, pa konvergira i Σ.

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 + 2n - 2n + 1} \sim \frac{1}{n^2} \quad 2 > 0 \text{ pa } (K)$$

Po poredbenom 2, red konvergira.

(3) Ispitati konvergenciju reda:

$$\sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2$$

Gauss: $$\frac{a_n}{a_{n+1}}=\frac{(2n-1)!!\cdot(2n-1)!!\cdot(2n+2)!!\cdot(2n+2)!!}{(2n)!!\cdot(2n)!!\cdot(2n+1)!!\cdot(2n+1)!!}=$$

$$=\frac{(2n+2)^2}{(2n+1)^2}=\frac{4n^2+8n+4}{4n^2+4n+1}=\frac{4n^2+4n+1+4n+3}{4n^2+4n+1}=$$

$$=1+\frac{4n+3}{4n^2+4n+1}=1+\frac{1}{n}\,\frac{4n+3}{4n^2+4n+1}-\frac{1}{n}=$$

$$=1+\frac{1}{n}+\frac{\cancel{4n^2}+3n-\cancel{4n^2}-4n-1}{n(4n^2+4n+1)}=$$

$$=1+\frac{1}{n}+\frac{-(n+1)\cdot n}{n^2(4n^2+4n+1)}$$

$\lambda=1,\ \gamma=1,\ \varepsilon=1,\ \theta_n=\frac{-(n+1)\cdot n}{4n^2+4n+1}.$

θ_n je ograničen niz, pa je niz po Gauss-u divergentan.