Analiza I

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II Parcijala

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Važno pročitati

Zadaci možda nisu svi ispravno urađeni.
U slučaju da ima grešaka, izvinjavam se.
(Mislim da nisam preskakao zadatke ovaj put)

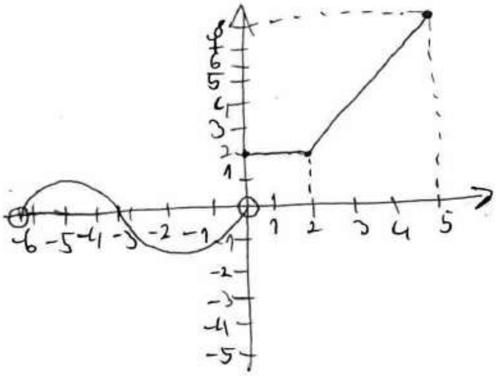
① $f(x) = \begin{cases} \sin(x), & -2\pi < x < 0 \\ 2, & 0 \leq x < 2 \\ 2x-2, & 2 \leq x \leq 5 \end{cases}$

Naći $f(-\pi), f(0), f(2)$ i skicirati grafik funkcije f.

$f(-\pi) = \sin(-\pi) = 0$

$f(0) = 2$

$f(2) = 2$



D.P. $x \in (-2\pi, 5]$

Rang: $[-1, 1] \cup [2, 8]$

② Odrediti D.P. funkcija:

a) $f(x) = (x^2-2) \cdot \sqrt{\frac{x^2+2}{x^3-3}}$

1° $x^3 - 3 \neq 0$

$x^3 \neq 3$

$\boxed{x \neq \sqrt[3]{3}}$

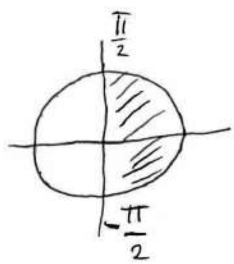
2° $\frac{x^2+2}{x^3-3} \geq 0$

x^2+2 je uvijek >0, gledamo $x^3-3>0$

$x^3 > 3$

$\boxed{x > \sqrt[3]{3}}$

D.P. $x \in (\sqrt[3]{3}, +\infty)$



b) $f(x) = \sqrt{\cos(\sqrt{x})}$

1° $x \geq 0$ 2° $\cos(\sqrt{x}) \geq 0$

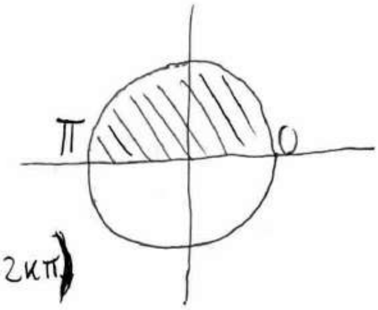
$\sqrt{x} \in [2k\pi - \frac{\pi}{2}, 2k\pi + \frac{\pi}{2}]$

D.P. $x \in [$

c) $f(x) = \ln(\sin(\frac{\pi}{x}))$

$1^\circ \; x \neq 0$

$2^\circ \; \sin(\frac{\pi}{x}) > 0$



$\frac{\pi}{x} \in (0 + 2k\pi, \pi + 2k\pi)$

$2k\cancel{\pi} < \frac{\cancel{\pi}}{x} < \cancel{\pi}(2k+1)$

$2k < \frac{1}{x} < 2k+1$

$\frac{1}{2k} > x > \frac{1}{2k+1}$

D.P.: $x \in \left(\frac{1}{2k+1}, \frac{1}{2k}\right) \; k \in \mathbb{Z} \setminus \{0\}$

d) $f(x) = \arcsin \frac{3x}{1+x}$

$1^\circ \; x = -1$

$2^\circ \; \underbrace{-1 \leq \overbrace{\frac{3x}{1+x}}^{2^\circ}}_{1^\circ} \leq 1$

$1^\circ \; -1 \leq \frac{3x}{1+x}$

$0 \leq \frac{3x}{1+x} + 1$

$0 \leq \frac{3x+1+x}{1+x}$

$0 \leq \frac{4x+1}{1+x}$

	$-\infty$ … -1	-1 … $-\frac{1}{4}$	$-\frac{1}{4}$ … $+\infty$
$4x+1$	−	−	+
$1+x$	−	+	+
	+	−	+

$x_1 \in (-\infty, -1] \cup [-\frac{1}{4}, +\infty)$

$2^\circ \; \frac{3x}{1+x} \leq 1$

$\frac{3x}{1+x} - 1 \leq 0$

$\frac{3x-1-x}{1+x} \leq 0$

$\frac{2x-1}{1+x} \leq 0$

	$-\infty$ … -1	-1 … $\frac{1}{2}$	$\frac{1}{2}$ … $+\infty$
$2x-1$	−	−	+
$1+x$	−	+	+
	+	−	+

$x_2 \in [-1, \frac{1}{2}]$

$x_1 \cap x_2 \Rightarrow$ D.P. $x \in [-\frac{1}{4}, \frac{1}{2}]$

e) $f(x)=ctg(\pi x)+\arccos(3^x)$

1° $\underbrace{-1 \le 3^x}_{\text{Ovo vrijedi } \forall x\in\mathbb{R}} \le 1$

$\underbrace{3^x \le 1}_{1^\circ}$

2° $\pi x \neq 0$

$\pi x \neq k\pi$

$x \neq k \quad k\in\mathbb{Z}$

1° $3^x \le 1$

$3^x \le 3^0$

$\boxed{x \le 0}$

D.P. $x\in(-\infty,0]\setminus\mathbb{Z}$

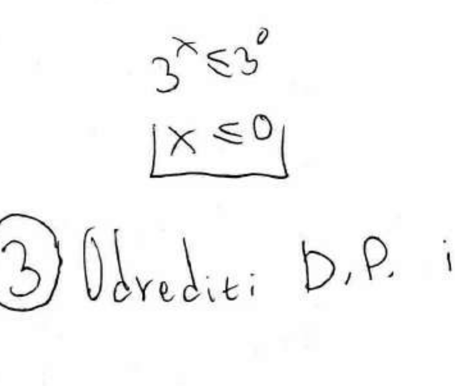
(3) Odrediti D.P. i skup vrijednosti funkcije:

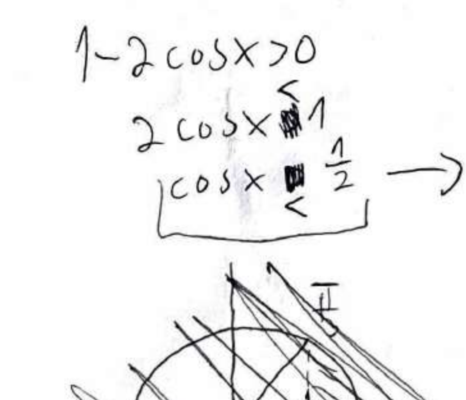
$$f(x)=\log(1-2\cos x)$$

$1-2\cos x>0$

$2\cos x < 1$

$\boxed{\cos x < \frac{1}{2}}$ → Ovo se ispituje





$\underline{x\in\left(2k\pi+\frac{\pi}{3},\ 2k\pi+\frac{5\pi}{3}\right)}$

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(4) Ispitati periodičnost i odrediti osnovni period ako on postoji sljedećih funkcija:

a) $f(x) = 3\cos(4x-2) + \cos^2 x + \sin^4 x + \dfrac{1}{\ln(\sin^2 x)}$

b) $g(x) = tg\sqrt{x}$

a) $3\cos(4x-2) \to \frac{2\pi}{4} = \frac{\pi}{2} \to$ osnovni period

$\cos^2 x \to \pi$ osnovni period

$\sin^4 x \to \pi$ osnovni period

$\dfrac{1}{\ln(\sin^2 x)} \to \pi$ osnovni period

$-\pi$ zajednički osnovni period

b) $g(x) = tg\sqrt{x}$

$tg x$ D.P. $x \in \mathbb{R} \setminus \{\frac{k\pi}{2}\}$

Pretp. da je $g(x)$ periodična sa periodom T

$g(x) = g(x+T)$

$g(0) = g(T) \Rightarrow g(T) = 0 = tg\sqrt{T} \Rightarrow \sqrt{T} = k\pi,\ k \in \mathbb{Z},\ k \neq 0$ jer $T > 0$

$g(T) \to g(2T) = 0 \Rightarrow tg\sqrt{2T} = 0$ $\quad \sqrt{2T} = h\pi$

$h \in \mathbb{Z},\ h \neq 0$

$\dfrac{\sqrt{2T}}{\sqrt{T}} = \dfrac{h\pi}{k\pi} \Rightarrow \underbrace{\sqrt{2} = \dfrac{h}{k}}_{}$

Kontradikcija jer $\sqrt{2} \in \mathbb{I}$ a $\frac{h}{k} \in \mathbb{Q}$

⑤ Ispitati periodičnost Dirichletove funkcije:

$$f(x)=\begin{cases}1, & x\in\mathbb{Q}\\ 0, & x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}$$

$f(x)=f(x+T)$

1° $T\in\mathbb{Q}$ $f(x)=f(x+T)$ $\forall x\in\mathbb{R}$

1) $x\in\mathbb{Q} \Rightarrow x+T\in\mathbb{Q}$

$f(x)=f(x+T)=1$

2) $x\in\mathbb{R}\setminus\{\mathbb{Q}\}\Rightarrow x+T\in\mathbb{R}\setminus\{\mathbb{Q}\}$

$f(x)=f(x+T)=0$

Vrijedi da je svaki racionalan broj period funkcije.

2° $T\in\mathbb{R}\setminus\{\mathbb{Q}\}$ $f(x)=f(x+T)$ $\forall x\in\mathbb{R}$

Uzmimo $x=0$: $\underset{1}{\underset{\parallel}{f(0)}}=\underset{0}{\underset{\parallel}{f(T)}}$ $\Rightarrow$ kontradikcija

Skup perioda funkcije je $\mathbb{Q}^+$, nema osnovnog perioda.

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(6) Dokazati da je bijektivna i odrediti inverznu funkciju hiperboličke sinusne funkcije $sh(x) = \frac{e^x - e^{-x}}{2}$.

1) Sirjektivnost

$\forall y \in \mathbb{R} \ \exists x \in \mathbb{R}$ t.d. $sh(x) = y$

$\frac{e^x - e^{-x}}{2} = y \ / \cdot 2$

$e^x - \frac{1}{e^x} = 2y \ / \cdot e^x$

$e^{2x} - 1 = 2e^x y$

$e^{2x} - 2e^x y - 1 = 0 \qquad t = e^x \quad t > 0$

$t^2 - 2ty - 1 = 0$

$t_{1/2} = \frac{2y \pm \sqrt{4y^2+4}}{2}$

$t_{1/2} = \frac{2y \pm 2\sqrt{y^2+1}}{2}$

$t_{1/2} = y \pm \sqrt{y^2+1}$

Jedino rješenje je $y + \sqrt{y^2+1} = e^x$

$e^x = y + \sqrt{y^2+1}$

$\boxed{x = \ln(y + \sqrt{y^2+1})} \to$ Ovo je inverz funkcije.

Vrijedi sirjektivnost.

2) Injektivnost

Pretpostavimo da $f(x_1) = f(x_2) = y$, onda $x_1 = \ln(y + \sqrt{y^2+1}) = x_2$

Vrijedi injektivnost.

(7) Neka je $l>0$ realan broj. Dokazati da se svaka funkcija $f:(-l,l)\to\mathbb{R}$ može predstaviti u obliku zbira neke parne i neke neparne funkcije.

$f:(-l,l)\to\mathbb{R}$, $l>0$

$f(x)=\underbrace{p(x)}_{\text{parna}}+\underbrace{h(x)}_{\text{neparna}}$

$p(x)=p(-x)$ – Ovo nam treba

$p(x)=\dfrac{f(x)+f(-x)}{2}$

$h(x)=\dfrac{f(x)-f(-x)}{2}$

parna funkcija proguta minus

minus izleti ispred nepane funkcije

$f(-x)=\overbrace{p(x)}-\overbrace{h(x)}$

$f(x)=p(x)+h(x)$

$f(-x)=p(x)-h(x)$

$f(x)+f(-x)=p(x)+\cancel{h(x)}+p(x)-\cancel{h(x)}$

$\left.\begin{aligned}p(x)&=\frac{f(x)+f(-x)}{2}\\ h(x)&=\frac{f(x)-f(-x)}{2}\end{aligned}\right\}+ \Rightarrow p(x)+h(x)=f(x)$

$p(x)+h(x)=\dfrac{\cancel{2}f(x)}{\cancel{2}}=f(x)$ ✓ Ovo se i trebalo dokazati

(8) Data je funkcija $f:\mathbb{R}\to\mathbb{R}$ takva da postoji $T>0$ za koje vrijedi $f(x+T)=-f(x)$, $\forall x\in\mathbb{R}$. Dokazati da je $2T$ period funkcije f.

$T>0: f(x+T)=-f(x) \quad \forall x\in\mathbb{R}$

$f(x)=f(x+2T)$

$f(x+2T)=f((x+T)+T)=-f(x+T)=f(x)$

Dokazali smo da $f(x+2T)=f(x)$ ✓

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(1) Ako je $f(x)=4x-2$, dokazati da je $\lim_{x\to 1} f(x)=2$

Ovo znači sljedeće:

$\forall \varepsilon>0 \ \exists \delta=\delta(\varepsilon)>0$ t. d. $|x-1|<\delta \Rightarrow |f(x)-2|<\varepsilon$

$|f(x)-2|<\varepsilon = |4x-2-2|<\varepsilon = |4(x-1)|<\varepsilon = 4|x-1|<\varepsilon$

$= \underline{|x-1|<\frac{\varepsilon}{4}}$

Sljedi da je traženo $\delta=\frac{\varepsilon}{4}$

(2) Odrediti $\lim_{x\in\mathbb{N}\to\infty} x-\lfloor x\rfloor$, a zatim dokazati da ne postoji $\lim_{x\to\infty} x-\lfloor x\rfloor$.

$\lim_{x\in\mathbb{N}\to\infty} x-\lfloor x\rfloor = \lim_{x\in\mathbb{N}\to\infty} x-x=0$

Neka je sada: $a_n=n$

$b_n=n+\frac{1}{2}$ ($\frac{1}{2}$ je dodaje da ne bi bio cio broj)

$a_n\to\infty \ (n\to\infty)$

$b_n\to\infty \ (n\to\infty)$

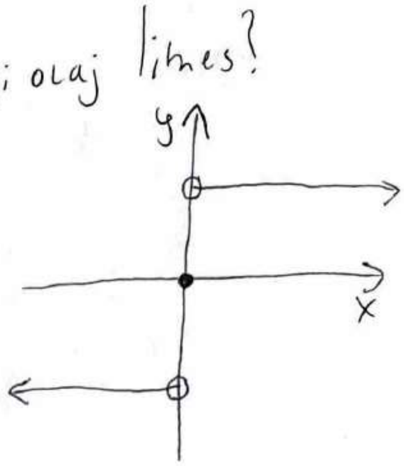
$f(a_n)=n-\lfloor n\rfloor=0, \ \lim_{n\to\infty} f(a_n)=0$

$f(b_n)=\cancel{n}+\frac{1}{2}-\underbrace{\lfloor n+\cancel{\tfrac{1}{2}}\rfloor}_{\cancel{n}}=\frac{1}{2}$

Po Heinovom teoremu $\lim_{x\to\infty} x-\lfloor x\rfloor$ ne postoji

(3) $\lim\limits_{x\to 0} sgn(x)$. Da li postoji ovaj limes?

$$sgn(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$



$\lim\limits_{x\to 0^-} sgn(x) = -1$
$\lim\limits_{x\to 0^+} sgn(x) = 1$
Ovaj limes nepostoji jer su lijevi i desni limesi različiti.

(4) Dokazati da ne postoji limes ni u jednoj tački Dirichletove funkcije.

Dirichletova funkcija: $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{I} \end{cases}$

Pretpostavimo suprotno.

$\exists \lim\limits_{x\to a} f(x) = y$

$\varepsilon = \frac{1}{3} \Rightarrow \left(y - \frac{1}{3}, y + \frac{1}{3}\right) \Rightarrow$ ne sadrži i 0 i 1 u isto vrijeme.

$\forall \delta > 0 : (a-\delta, a+\delta)$ Sadrži i racionalne i iracionale brojeve

$\exists x_1 \in (a-\delta, a+\delta) : f(x_1) = 0 \quad (x_1 \in \mathbb{I})$

$\exists x_2 \in (a-\delta, a+\delta) : f(x_2) = 1 \quad (x_2 \in \mathbb{Q})$

$\Rightarrow$ kontradikcija jer $\left(y - \frac{1}{3}, y + \frac{1}{3}\right)$ ne sadrži i $f(x_1) = 0$ i $f(x_2) = 1$.

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(5) Dokazati po definiciji da:

$$\lim_{x\to\infty} \frac{\sqrt{x^2-1}}{x-1} = 1$$

To znači da:

$\forall \varepsilon > 0 \ \exists M = M(\varepsilon)$ t.d. $x > M \Rightarrow |f(x)-1| < \varepsilon$

$$\left|\frac{\sqrt{x^2-1}}{x-1} - 1\right| < \varepsilon \Leftrightarrow$$

$$\left|\frac{\sqrt{(x-1)(x+1)}}{\sqrt{(x-1)^2}} - 1\right| < \varepsilon \Leftrightarrow \left|\sqrt{\frac{(x-1)(x+1)}{(x-1)^2}} - 1\right| < \varepsilon \Leftrightarrow \left|\underbrace{\sqrt{\frac{x+1}{x-1}}}_{>1} - 1\right| < \varepsilon$$

$$\sqrt{\frac{x+1}{x-1}} - 1 < \varepsilon \Leftrightarrow \sqrt{\frac{x+1}{x-1}} < \varepsilon + 1 \Big/^2$$

$$\frac{x+1}{x-1} < (\varepsilon+1)^2$$

$$\cancel{1}+\frac{2}{x-1} < \varepsilon^2 + 2\varepsilon \cancel{+1}$$

$$\frac{2}{x-1} < \varepsilon^2 + 2\varepsilon$$

$$\frac{x-1}{2} > \frac{1}{\varepsilon^2+2\varepsilon} \Big/ \cdot 2$$

$$x-1 > \frac{2}{\varepsilon^2+2\varepsilon}$$

$$x > \frac{2}{\varepsilon^2+2\varepsilon} + 1$$

Slijedi da je $M = \dfrac{2}{\varepsilon^2+2\varepsilon} + 1$

7) Dokazati po definiciji da je:

$$\lim_{x\to a} \sin x = \sin a$$

$\forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon)$ t.d. ako $|x-a| < \delta = |\sin x - \sin a| < \varepsilon$

$$\sin x - \sin a = 2\sin\frac{\cancel{x}-a}{2}\cos\frac{x+a}{2}$$

$$|\sin x - \sin a| = \left|\underbrace{2\sin\frac{x-a}{2}}_{\leq \frac{x-a}{2}}\underbrace{\cos\frac{x+a}{2}}_{\leq 1}\right| \leq \cancel{2}\frac{|x-a|}{\cancel{2}}\cdot 1 = |x-a| < \delta$$

jer $|\sin x| \leq |x|$

Slijedi $\varepsilon = \delta$.

8) a)

$$\lim_{x\to 0}\frac{x}{\sqrt{1+x}-\sqrt{1-x}}\cdot\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} = \lim_{x\to 0}\frac{\cancel{x}\sqrt{1+x}+\cancel{x}\sqrt{1-x}}{2\cancel{x}} =$$

$$= \lim_{x\to 0}\frac{\sqrt{1+x}+\sqrt{1-x}}{2} = 1$$

b)

$$\lim_{x\to 1}\frac{x^2-1}{2x^2-x-1} = \frac{(\cancel{x-1})(x+1)}{2(\cancel{x-1})(x+\frac{1}{2})} = \frac{x+1}{2x+1} = \frac{2}{3}$$

c)

$$\lim_{x\to -\infty}\frac{x^3+5x^2-2x-1 /:x^3}{x^2-1 /:x^3} = \text{"}\frac{1}{0^-}\text{"} = -\infty$$

d)

$$\lim_{x\to 0}\frac{\sqrt{x+1}-\sqrt{1-x}}{\sqrt[3]{x+1}-\sqrt[3]{1-x}}\cdot\frac{\sqrt{x+1}+\sqrt{1-x}}{\sqrt{x+1}+\sqrt{1-x}}\cdot\frac{(\sqrt[3]{x+1})^2+\sqrt[3]{x+1}\sqrt[3]{x-1}+(\sqrt[3]{1-x})^2}{-\|-}$$

$$= \lim_{x\to 0}\frac{\cancel{2x}\left((\sqrt[3]{x+1})^2+\sqrt[3]{x+1}\cdot\sqrt[3]{1-x}+(\sqrt[3]{1-x})^2\right)}{(\sqrt{x+1}+\sqrt{1-x})\underbrace{(x+1-1+x)}_{\cancel{2x}}} = \ldots = \frac{3}{2}$$

e) $\lim\limits_{x\to\pm\infty} (x - \ln|\operatorname{sh} x|)$

$\operatorname{sh} x = \frac{e^x - e^{-x}}{2}$

1° $\lim\limits_{x\to-\infty} x - \ln\left|\frac{e^x-e^{-x}}{2}\right| = -\infty - \infty = -\infty$

2° $\lim\limits_{x\to+\infty} x - \ln\left|\frac{e^x-e^{-x}}{2}\right| \overset{\lim_{x\to+\infty}}{=} x - \ln\left|\frac{e^x - \frac{1}{e^x}}{2}\right| \overset{\lim_{x\to+\infty}}{=} x - \ln\left|\frac{e^{2x}-1}{2\cdot e^x}\right| =$

$= \lim\limits_{x\to+\infty} \ln e^x - \ln\left(\frac{e^{2x}-1}{2e^x}\right) \overset{\lim_{x\to+\infty}}{=} \ln\left(\frac{e^x}{\frac{e^{2x}-1}{2e^x}}\right) = \lim\limits_{x\to+\infty}\left(\frac{2e^{2x}}{e^{2x}-1}\right) = \ln(2)$

f) $\lim\limits_{x\to 0} \frac{(1+x)^{10}-1}{x} =$ (Prepisati fino od Adija)

① Dokazati da je $\lim\limits_{x\to 0} \frac{\ln(1+x)}{x} = 1$

$\frac{\ln(1+x)}{x} = \frac{1}{x}\cdot\ln(1+x) = \ln\left((1+x)^{\frac{1}{x}}\right) = \ln\left(\left(1+\frac{1}{\frac{1}{x}}\right)^{\frac{1}{x}}\right) = \ln e = 1$

② $\lim\limits_{x\to 0} \frac{e^x-1}{x} = 1$

$\lim\limits_{x\to 0} \frac{\ln(1+x)}{x} = 1 \Rightarrow \lim\limits_{t\to 0} \frac{t}{e^t-1} = \lim\limits_{t\to 0}\frac{e^t-1}{t} = 1$

$\ln(1+x) = t$

$1+x = e^t$

$\boxed{x = e^t - 1}$

$x\to 0,\ t\to 0$

③ $\lim\limits_{x\to a}\frac{a^x-1}{x}=\lim\limits_{x\to 0}\frac{e^{x\cdot\ln(a)}-1\cdot\ln(a)}{x\cdot\ln(a)}=\lim\limits_{x\to 0}1\cdot\ln(a)=\ln(a)$

$a^x=(e^{\ln(a)})^x=e^{x\cdot\ln(a)}$

④ Izračunati sljedeće limese:

a) $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}=\lim\limits_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)}{x^2}=\lim\limits_{x\to 0}2\cdot\left(\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\right)^2\cdot\frac{1}{4}=$

$=\left[\frac{x}{2}\to 0\right]=2\cdot 1^2\cdot\frac{1}{4}=\frac{1}{2}$

b) $\lim\limits_{x\to 0}\frac{\sin x}{\sin\frac{x}{2}}=\lim\limits_{x\to 0}\frac{\overbrace{\frac{\sin x}{x}}^{1}}{\underbrace{\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}}_{1}}\cdot 2=2$

c) $\lim\limits_{x\to\infty}\left(\frac{x^2+1}{x^2-1}\right)^{x^2-1}=\lim\limits_{x\to\infty}\left(1+\frac{2}{x^2-1}\right)^{x^2-1}=\lim\limits_{x\to\infty}e^{\frac{\cancel{2}\cancel{(x^2-1)}}{\cancel{2}}\cdot\frac{2}{\cancel{x^2-1}}}=e^2$

d) $\lim\limits_{x\to 1}\frac{\sin(x-1)}{x^2-1}=\lim\limits_{x\to 1}\frac{\sin(x-1)}{(x-1)(x+1)}=\lim\limits_{x\to 1}\frac{\sin(x-1)}{(x-1)}\cdot\lim\limits_{x\to 1}\frac{1}{x+1}=\frac{1}{2}$

e) $\lim\limits_{x\to 0}\frac{e^x-1}{\sin x}=\lim\limits_{x\to 0}\frac{\overbrace{\frac{e^x-1}{x}}^{1}}{\underbrace{\frac{\sin x}{x}}_{1}}=1$

f) $\lim\limits_{x\to 0}(1+\operatorname{tg}x)^{\operatorname{ctg}x}=\lim\limits_{x\to 0}\left(1+\frac{\sin x}{\cos x}\right)^{\frac{\cos x}{\sin x}}=\lim\limits_{x\to 0}\left(1+\frac{1}{\frac{\cos x}{\sin x}}\right)^{\frac{\cos x}{\sin x}}=e$

g) $\lim\limits_{x\to 0}\frac{e^{-2x}-1}{x}=\lim\limits_{x\to 0}\frac{e^{-2x}-1}{-2x}\cdot(-2)=-2$

h) $\lim\limits_{x\to 1}\frac{x^m-1}{x^n-1}=\lim\limits_{x\to 1}\frac{e^{\ln x\cdot m}-1}{e^{\ln x\cdot n}-1}=\frac{\overbrace{\frac{e^{\ln x\cdot m}-1}{\ln x\cdot m}}^{1}}{\underbrace{\frac{e^{\ln x\cdot n}-1}{\ln x\cdot n}}_{1}}\cdot\frac{m}{n}=\frac{m}{n}$

i) $\lim_{x\to 0} \frac{1}{x}\cdot \ln\left(\sqrt{\frac{1+x}{1-x}}\right) = \lim_{x\to 0} \frac{1}{2}\cdot \ln\left(1+\frac{1}{\frac{1-x}{2x}}\right)$ ooo

ooo $)^{\frac{1-x}{2x}\cdot\frac{2x}{1-x}\cdot\frac{1}{x}} = \frac{1}{2}\ln e^2 = \frac{1}{2}\cdot 2 = 1$

Drugi način

$\lim_{x\to 0} \frac{1}{2}\cdot \frac{\ln\left(1+\frac{2x}{1-x}\right) /\cdot \frac{2}{1-x}}{x \; /\cdot \frac{2}{1-x}} = \frac{1}{2}\cdot 2 = 1$

j) $\lim_{x\to e} \frac{\ln x - 1}{x-e} = \lim_{x\to e} \frac{\ln x - \ln e}{x-e} = \lim_{x\to e} \frac{\ln(\frac{x}{e})}{x-e} = \lim_{x\to e} \frac{\ln(1+\frac{x-e}{e}) /:e}{x-e /:e} =$

$= \lim_{x\to e} \frac{\ln(1+\frac{x-e}{e})}{\frac{x-e}{e}}\cdot\frac{1}{e} = \frac{1}{e}$

5) Izračunati sljedeće limese:

a) $\lim_{x\to 0} \frac{e^{10x}-e^{5x}}{\sin 10x - \sin 5x} = \frac{10x+\cancel{1}-5x-\cancel{1}}{10x-5x} = \frac{5x}{5x} = 1$

$e^{5x} \sim 5x+1 \qquad \sin 10x \sim 10x$

$e^{10x} \sim 10x+1 \qquad \sin 5x \sim 5x$

b) $\lim_{x\to 0} \frac{\ln(1+4x)}{5x} = \frac{4\cancel{x}}{5\cancel{x}} = \frac{4}{5}$

$\ln(1+4x) \sim 4x$

Neprekidnost f-ije

① sgn(x). Ispitati neprekidnost funkcije.

$$\operatorname{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

$\forall a > 0 \quad \lim\limits_{x \to a} f(x) = 1$ Ovo je prekid druge vrste.

$\forall a < 0 \quad \lim\limits_{x \to a} f(x) = -1$

② Dokazati da je funkcija

$$f(x) = \begin{cases} x \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

neprekidna u svakoj tački $x \in \mathbb{R}$

D.P. ove funkcije je $\forall x \in \mathbb{R}$

1° $a \neq 0$, $a \in \mathbb{R}$

$$\lim_{x \to a} f(x) = \lim_{x \to a} x \cdot \sin\frac{1}{x} = a \cdot \sin\frac{1}{a} = f(a)$$

2° $a = 0$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x \cdot \overbrace{\sin\frac{1}{x}}^{\text{ogr. } -1 \text{ i } 1} = \lim_{x \to 0} x \cdot \text{ograničeno} = 0 \cdot \text{ograničeno} = 0 = f(0)$$

(3) Ispitati neprekidnost funkcije:

$$f(x) = \begin{cases} -\frac{1}{x}, & \text{za } x<0 \\ 1, & \text{za } 0 \le x \le 1 \\ x, & \text{za } 1 < x < 2 \\ 3, & \text{za } 2 < x < 3 \end{cases}$$

D.P.

$x \in (-\infty, 2) \cup (2, 3)$

1° $a<0$ $\lim\limits_{x\to a} -\frac{1}{x} = -\frac{1}{a} = f(a)$

2° $a=0$ $\lim\limits_{x\to 0^+} f(x) = 1 = f(0)$

3° $a \in (0, 1]$

$\lim\limits_{x\to a} f(x) = 1 = f(a) \to$ f. nepr. u a

$\lim\limits_{x\to 1^-} f(x) = 1$

$\lim\limits_{x\to 1^+} f(x) = 1 = f(1) \to$ f. nepr. u $a=1$

4° $a \in (1,2)$

$\lim\limits_{x\to a} f(x) = x = a = f(a)$

5° $a=3$, $3 \notin$ D.P. ali je 3 tačka gomilanja f

$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} = 3 = f(a)$. Limes $\lim\limits_{x\to 2^+}$ nepostoji

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(4) Kako treba definisati ~~konstantu~~ konstantu a kako bi funkcija:

$$f(x)=\begin{cases} 1-x^2, & za \ x<0 \\ a, & za \ x=0 \\ 1+x, & za \ x>0 \end{cases}$$

bila neprekidna?

Za $a<0$ $f(x)=1-x^2$ je neprekidna f-ija jer je elementarna

za $a>0$ $f(x)=1+x$ je neprekidna f-ija jer je elementarna

Tražimo lijevi i desni limes.

$\lim\limits_{x\to0^+} f(x)=\lim\limits_{x\to0^+} 1+x=1$

$\lim\limits_{x\to0^-} f(x)=\lim\limits_{x\to0^-} 1-x^2=1$

Dakle $\lim\limits_{x\to a} f(x)=f(a)$, pa $a=1$ za $x=0$

(5) Dodefinisati funkciju $f(x)=x\cdot\ln^2(x)$ u tački $x=0$ tako da funkcija bude neprekidna sa desne strane u $x=0$

D.P. ove funkcije je $x\in(0,+\infty)$

Traži se da je $\lim\limits_{x\to0^+} f(x)=f(0)$.

$\lim\limits_{x\to0^+} f(x)=\lim\limits_{x\to0^+} x\cdot\ln^2(x)=\left|\text{,,}0\cdot+\infty\text{''}\right|$

$f(x)=x\cdot\ln^2(x)$

Uvedimo smjenu: $x=e^{-t}$ $\left|\begin{matrix} x\to0^+ \text{ kada} \\ t\to+\infty \end{matrix}\right|$

$$e^{-t} \cdot \ln^2(e^{-t}) = f(t)$$

$$\frac{1}{e^t} \cdot (-t)\underbrace{\ln(e)}_{1} \cdot (-t)\underbrace{\ln(e)}_{1} = f(t)$$

$$\frac{t^2}{e^t} = f(t)$$

$$\lim_{t\to\infty} \frac{t^2 /: e^t}{e^t /: e^t} = \frac{\frac{\cancel{t}^2}{e^t} \to 0}{1} = \frac{0}{1} = 0$$

$$f(0) = 0$$

$$f(x) = \begin{cases} x \cdot \ln^2(x), & x > 0 \\ 0, & x = 0 \end{cases}$$

(1) Ispitati uniformnu neprekidnost funkcije $f(x)=\frac{1}{x}$ na skupu $(0,1)$

Dokazati ćemo da nije uniformno neprekidna na tom intervalu.

$$x_n' = \frac{1}{n} \in (0,1)$$

$$x_n'' = \frac{1}{n+1} \in (0,1)$$

$n \in \mathbb{N}$

$$|x_n' - x_n''| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \left|\frac{1}{n(n+1)}\right| \xrightarrow{n\to+\infty} 0$$

$$|f(x_n') - f(x_n'')| = \left|f\left(\frac{1}{n}\right) - f\left(\frac{1}{n+1}\right)\right| = |\cancel{n} - (\cancel{n}+1)| = 1 \not\xrightarrow{n\to+\infty} 0$$

Po Heine-ovom teoremu funkcija nije uniformno neprekidna na $(0,1)$.

(2) Ispitati uniformnu neprekidnost funkcija:

a) $f(x) = \frac{3x}{10-x^2}$ $(-2 \leq x \leq 3)$

funkcija je definisana na datom intervalu, pa je po Kantorovom teoremu ona i uniformno neprekidna na tom intervalu.

b) $f(x) = \ln(x)$ $(0 < x < 1)$

$$x_n' = e^{-n} \in (0,1)$$

$$x_n'' = e^{-(n+1)} \in (0,1)$$

Nije uniformno neprekidna. Treba samo dokazati tu tvrdnju.

$$|x_n' - x_n''| = |e^{-n} - e^{-(n+1)}| = \left|\frac{1}{e^n} - \frac{1}{e^{n+1}}\right| \xrightarrow{n\to+\infty} 0$$

$$|f(x_n') - f(x_n'')| = |\ln(e^{-n}) - \ln(e^{-(n+1)})| = |\cancel{-n} + \cancel{n} + 1| = 1 \not\xrightarrow{n\to+\infty} 0$$

Po Heine-ovom teoremu, f-ja nije uniformno neprekidna na $(0,1)$

(3) Dokazati da je funkcija uniformno neprekidna $f(x)=\sqrt{x}$ na intervalu $[1,+\infty)$

$\forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon)$ t.d. $\forall x_1, x_2 \in [1,+\infty)$

$|x_1 - x_2| < \delta \Rightarrow |\sqrt{x_1} - \sqrt{x_2}| < \varepsilon$

$$|\sqrt{x_1} - \sqrt{x_2}| = \left|\frac{x_1 - x_2}{\sqrt{x_1}+\sqrt{x_2}}\right| = \frac{|x_1 - x_2|}{\underbrace{|\sqrt{x_1}+\sqrt{x_2}|}_{\text{Ovo je pozitivno}}} = \frac{\varepsilon}{2} \Rightarrow \boxed{\delta = 2\varepsilon}$$

$\sqrt{x_1} + \sqrt{x_2} \geq 1+1 = 2$

(Uzima se 1 zbog D.P.-a)

(4) Dokazati da $f(x) = \sin\left(\frac{1}{x}\right)$ nije uniformno neprekidna na intervalu $x \in \left(0, \frac{2}{\pi}\right)$

$$x_n' = \frac{1}{2n\pi} \qquad f(x_n') = \sin(2n\pi) = 0$$

$$x_n'' = \frac{1}{2n\pi + \frac{\pi}{2}} \quad f(x_n'') = \sin\left(2n\pi + \frac{\pi}{2}\right) = 1$$

$$|x_n' - x_n''| = \left|\frac{1}{2n\pi} - \frac{1}{2n\pi + \frac{\pi}{2}}\right| = \left|\frac{\frac{\pi}{2}}{2n\pi\left(2n\pi + \frac{\pi}{2}\right)}\right| \xrightarrow{n \to +\infty} 0$$

$$|f(x_n') - f(x_n'')| = \left|\sin(2n\pi) - \sin\left(2n\pi + \frac{\pi}{2}\right)\right| = |0-1| = 1 \ \not\xrightarrow{n \to +\infty} 0$$

Data funkcija nije uniformno neprekidna na intervalu $\left(0, \frac{2}{\pi}\right)$

① Dokazati:

a) $c' = 0$

$$\lim_{h\to 0} \frac{f(x_0-h)-f(x_0)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \frac{0}{h} = 0$$

b) $x' = 1$

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0} \frac{(\cancel{x_0}+h)-\cancel{x_0}}{h} = \frac{h}{h} = 1$$

c) $(x^n)' = n x^{n-1}$

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0} \frac{(x_0+h)^n - x_0^n}{h} = [\text{Po NBF}] =$$

$$= \lim_{h\to 0} \frac{\cancel{\binom{n}{0} x_0^n} + \binom{n}{1} x_0^{n-1} h + \ldots + \binom{n}{n} x_0^0 \cdot h^n \cancel{- x_0^n}}{h} = \lim_{h\to 0} \frac{n \cdot x_0^{n-1} \cdot h + \ldots + x_0^0 h^n}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{h}(n \cdot x_0^{n-1} + \ldots + h^{n-1})}{\cancel{h}} = \lim_{h\to 0} n \cdot x_0^{n-1} + \ldots + h^{n-1} = n \cdot x_0^{n-1}$$

d) $(a^x)' = a^x \cdot \ln(a)$ $a > 0$

$$f'(x_0) = \lim_{h\to 0} \frac{a^{x_0+h} + a^{x_0}}{h} = \lim_{h\to 0} \frac{a^{x_0}(a^h+1)}{h} = a^{x_0} \cdot \lim_{h\to 0} \frac{(a^h+1)}{h} = a^{x_0} \cdot \ln(a)$$

e) $(e^x)' = e^x$ analogno kao pod d)

f) $(\sin x)' = \cos x$

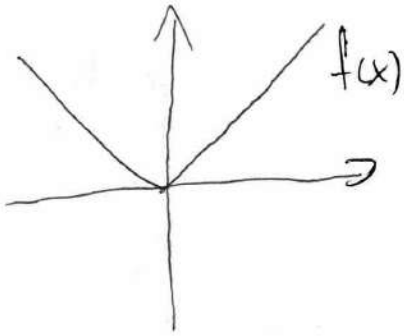
$$\lim_{h\to 0} \frac{\sin(x_0+h) - \sin(x_0)}{h} = \lim_{h\to 0} \frac{2 \cos\left(\frac{x_0+h+x_0}{2}\right) \sin\left(\frac{x_0+h-x_0}{2}\right)}{h} =$$

$$= \cancel{2} \lim_{h\to 0} \frac{\cos\left(x_0+\frac{h}{2}\right) \overbrace{\sin\left(\frac{h}{2}\right)}^{\to 1}}{\frac{h}{2}} \cdot \frac{1}{\cancel{2}} = \cos x_0$$

(2) Ispitati u kojim tačkama je funkcija $f(x)=|x|$ neprekidna, a u kojim je diferencijabilna.

$f(x)=|x|$ D.P. $\forall x \in \mathbb{R}$

funkcija $f(x)$ je neprekidna na svom domenu



$$f'(x) = \begin{cases} 1, & x_0 > 0 \\ -1, & x_0 < 0 \\ \text{nema}, & x_0 = 0 \end{cases}$$

$$\lim_{h \to 0^-} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h \to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h \to 0^-} \frac{-h-0}{h} = -1$$

$$\lim_{h \to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$$

Lijevi i desni se razlikuju, pa izvod u $x_0=0$ nepostoji.

(3) Izračunati po definiciji $f'(2)$ ako je

a) $f(x)=x^2-3x+4$

$$f'(2) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{2} = \lim_{h \to 0} \frac{f(2+h)-f(2)}{2} = \lim_{h \to 0} \frac{h^2+4+4h-6-3h+4-2}{h}$$

$$= \lim_{h \to 0} \frac{h^2+h}{h} = \lim_{h \to 0} \frac{\cancel{h}(h+1)}{\cancel{h}} = \lim_{h \to 0} h+1 = 1$$

b) $f(x)=\ln(x)-x$

$$f'(2)=\lim_{h\to 0}\frac{\ln(2+h)-(2+h)-\ln(2)+2}{h}=$$

$$=\lim_{h\to 0}\frac{\ln(2+h)-\cancel{2}+h-\ln(2)+\cancel{2}}{h}=\lim_{h\to 0}\frac{\ln(2+h)-\ln(2)+h}{h}=$$

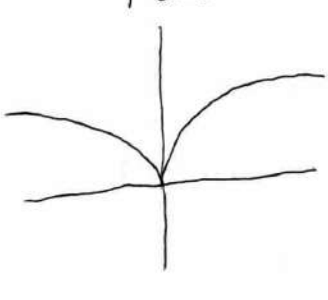
$$=\lim_{h\to 0}\frac{\ln\left(\frac{2+h}{h}\right)}{h}-1=\lim_{h\to 0}\underbrace{\frac{\ln\left(1+\frac{h}{2}\right)}{\frac{h}{2}}}_{=1 \text{ kada } \frac{h}{2}\to 0}\cdot\frac{1}{2}-1=-\frac{1}{2}$$

4) Izračunati ako postoji izvod u tački $x=0$ funkcija:

a) $f(x)=\sqrt[3]{x}$

$$f'(x)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{\sqrt[3]{h}}{h}=\lim_{h\to 0}\frac{1}{h^{\frac{2}{3}}}=+\infty$$

b) $f(x)=\sqrt[3]{x^2}$



$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{h^{\frac{2}{3}}}{h}=$$

$$=\lim_{h\to 0}\frac{1}{h^{\frac{1}{3}}}=\pm\infty \text{ zato što:}$$

$$f'_-(0)=\lim_{h\to 0-}\frac{1}{\sqrt[3]{h}}=-\infty$$

$$f'_+(0)=\lim_{h\to 0+}\frac{1}{\sqrt[3]{h}}=+\infty$$

$$f'_-(0)\neq f'_+(0)\Rightarrow \nexists\, f'(0)$$

⑤ Neka je $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Da li postoji $f'(0)$?

Prvo, za ovako zadane funkcije provjeravamo neprekidnost.

Neprekidnost u 0:

$\lim_{x \to 0} f(x) = f(0)$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} x \cdot \sin \frac{1}{x} = 0$ (zbog ograničenosti sinusa)

Dakle, f je neprekidna u 0.

$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h \cdot \sin \frac{1}{h} - 0}{h} =$

$= \lim_{h \to 0} \sin \frac{1}{h}$ (Ovaj limes ne postoji jer sinus ne teži ičemu)

$$\left.\begin{aligned} x_n' &= \frac{1}{2n\pi} \to 0 \quad (n \to +\infty) \\ x_n'' &= \frac{1}{2n\pi + \frac{\pi}{4}} \to 0 \quad (n \to +\infty) \end{aligned}\right\} |x_n' - x_n''| \xrightarrow{n \to +\infty} 0$$

$f(x_n') = f\left(\frac{1}{2n\pi}\right) = \sin\left(\frac{1}{\frac{1}{2n\pi}}\right) = \sin(2n\pi) = 0$

$f(x_n'') = f\left(\frac{1}{2n\pi + \frac{\pi}{4}}\right) = \sin\left(\frac{1}{\frac{1}{2n\pi + \frac{\pi}{4}}}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

Slijedi da $|f(x_n') - f(x_n'')| \overset{n \to +\infty}{\not\to} 0$

Po Heineovom teoremu N.P. $f'(0)$.

① Izvesti realacije za izvod funkcija:

a) $(tgx)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\sin x' \cos x - \sin x \cos x'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} =$

$= \frac{1}{\cos^2 x}$

b) $(chx)' = \left(\frac{e^x - e^{-x}}{2}\right)' = \frac{1}{2}(e^x + e^{-x})' = \frac{1}{2}(e^x)' - \frac{1}{2}(e^{-x})' =$

$= \frac{1}{2}e^x - \frac{1}{2}\left(\frac{1}{e^x}\right)' = \frac{e^x}{2} - \frac{1}{2e^x} = \frac{e^x - e^x}{2} = shx$

c) $(cth\, x)' = \left(\frac{chx}{shx}\right)' = \frac{ch'x \cdot shx - chx \cdot shx}{sh^2 x} = \frac{sh^2 x - ch^2 x}{sh^2 x} = -\frac{1}{sh^2 x}$

$sh^2 x - ch^2 x = \frac{(e^x - e^{-x})^2 - (e^x + e^{-x})^2}{4} = \frac{(-2e^{2x})(2e^{2x})}{4} = \frac{-4}{4} = -1$

② Izračunati izvode funkcija:

a) $(\ln(x))'$

$f(x) = e^x$

$\ln(x) = f^{-1}(x)$

$f(x) = e^x = y = \ln(y)$

$x = f^{-1}(y)$

$x'y = \frac{1}{y'x} \Leftrightarrow x'y = \frac{1}{(e^x)'} = \frac{1}{e^x}$

$x'y = \frac{1}{y}$, dakle

$(\ln(y))' = \frac{1}{y}$

b) $(\arcsin(x))'$

$f(x) = \sin x$

$\arcsin y = f^{-1}(y) = x \Rightarrow x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$x'_y = \frac{1}{y'_x} = \frac{1}{(\sin x)'}$

$x'_y = \frac{1}{\cos x}$ $\quad \cos x = \pm\sqrt{1-\sin^2 y}$ } Uzima se + radi restrikcije

$= \sqrt{1-y^2}$

$x'_y = \frac{1}{\sqrt{1-y^2}}$

$(\arcsin y)' = \frac{1}{\sqrt{1-y^2}}$

(3) Izračunati izvode funkcija:

a) $y = \frac{3}{4}x^4 - \frac{5}{3}x^3 + 3x - 10^{13}$

$y' = 3x^3 - 5x^3 + 3$

b) $y = \frac{1}{\sqrt{x}}$

$y' = x^{-\frac{1}{2}\,'} = -\frac{1}{2}x^{-\frac{3}{2}}$

c) $y = \frac{4}{\sqrt[3]{x^2}} - \frac{3}{x\sqrt[3]{x}} = 4 \cdot x^{-\frac{2}{3}} - 3 \cdot x^{-\frac{4}{3}} = 4 \cdot (-\frac{2}{3}) x^{-\frac{5}{3}} -$

$3 \cdot (-\frac{4}{3}) \cdot x^{-\frac{7}{3}} = -\frac{8}{3}x^{-\frac{5}{3}} + 4x^{-\frac{4}{3}}$

d) $y = (\sqrt{a} - \sqrt{x})^2 = 2(\sqrt{a} - \sqrt{x}) \cdot \frac{1}{-2\sqrt{x}} = \frac{1}{\sqrt{x}}(\sqrt{x} - \sqrt{a})$

e) $y = \left(1 - \frac{1}{\sqrt[3]{x}}\right)^2$

$y' = 2\left(1 - \frac{1}{\sqrt[3]{x}}\right)\left(1 - \frac{1}{\sqrt[3]{x}}\right)'$

$y' = \frac{2}{3}\left(1 - \frac{1}{\sqrt[3]{x}}\right) \cdot x^{-\frac{4}{3}}$

f) $y = \frac{\sqrt[3]{x} - 1}{\sqrt[3]{x} + 1}$

$y' = \frac{\frac{2}{3}x^{-\frac{2}{3}}}{(\sqrt[3]{x}+1)^2}$

(4) Izračunati izvode funkcija:

a) $y = \sin^2 x$

$y' = 2\sin x \cdot (\sin x)' = 2\sin x\cos x = \sin 2x$

b) $y = x \cdot \arcsin x$

$y' = \arcsin x + x \cdot \frac{1}{\sqrt{1+x^2}} = \arcsin x \cdot \frac{x}{\sqrt{1+x^2}}$

c) $y = e^{7^x} = e^{7^x} \cdot (7^x)' = e^{7^x} \cdot 7^x \cdot \ln 7$

(5) Izračunati izvode funkcija:

a) $y = \ln(e^x + e^{-x} + 2)$

$\frac{1}{e^x + e^{-x+2}} \cdot (e^x + e^{-x} + 2)' = \frac{1}{e^x + e^{-x} + 2} \cdot (e^x + e^{-x} + (-x)') = \frac{e^x + e^{-x}}{e^x + e^{-x} + 2}$

b) $y = x^x \,/\ln$

$\ln y = \ln(x^x)$

$\ln y = x \ln x \,/'$

$\frac{y'}{y} = x' \cdot \ln x + x \cdot \ln x' = \ln x + \underbrace{\frac{\cancel{x}}{\cancel{x}}}_{} \}\, 1$

$\frac{y'}{y} = \ln x + 1$

$y' = y(\ln x + 1)$

$y' = x^x(\ln x + 1)$

(1) Odrediti $y'_x = \frac{dy}{dx}$ za funkciju zadanu parametarski:

$$x = 2\cos t, \quad y = t - \sin t$$

$x = 2\cos t$

$y = t - \sin t$

$y'_x = \frac{dy}{dx} = ?$

$$y'_x = \frac{y'_t}{x'_t} = \frac{(t-\sin t)'}{(2\cos t)'} = \frac{1-\cos t}{-2\sin t}$$

(2) Odrediti $y'_x = \frac{dy}{dx}$ implicitno zadanih funkcija:

a) $xy - \operatorname{tg} y = 0$

b) $x^y = y^x$

a) $xy - \operatorname{tg} y = 0 \,/'$

$(xy - \operatorname{tg} y)'_x = 0$

$x' \cdot y + x \cdot y' - \frac{1}{\cos^2 y} \cdot y' = 0$

$y + xy' - \frac{y'}{\cos^2 y} = 0$

$$y' = \frac{y \cdot \cos^2 y}{1 - x\cos^2 y}$$

b) $x^y = y^x \,/'$

$(x^y)' = (y^x)'$

$(x^y)'_x = ?$

$x^y = u \,/\ln$

$y \ln x = \ln u \,/'$

$\frac{y}{x} + y' \ln x = \frac{u'}{u} \,/\cdot u$

$\boxed{x^y\left(\frac{y}{x} + y'\ln x\right) = u'}$

$y^x = v \,/\ln$

$(x \ln y)' = (\ln v)'$

$1 + \ln y = \frac{v'}{v} \,/\cdot v$

$\boxed{v' = y^x\left(\ln y + \frac{xy'}{y}\right)}$

$$u' = v' \Rightarrow x^y\left(\frac{y}{x} + y'\ln x\right) = y^x\left(\ln y + \frac{xy'}{y}\right) \Rightarrow y' = \frac{y^x \ln y - x^y \cdot \frac{y}{x}}{x^y \cdot \ln x - y^{x-1} \cdot x}$$

① Pokazati da funkcija $f(x)=\frac{e^{5x}+2}{e^x}$

zadovoljava diferencijalnu jednačinu $y'''-13y'-12y=0$

$f(x)=\frac{e^{5x}+2}{e^x}$

$y=\left(\frac{e^{5x}+2}{e^x}\right)'=(e^{4x}+2\cdot e^{-x})' \Rightarrow$

$y'=e^{4x}\cdot 4+2\cdot e^{-x}\cdot(-1)$

$y'=4\cdot e^{4x}-2\cdot e^{-x}\cdot(-1)/'$

$y''=(4\cdot e^{4x}-2\cdot e^{-x}\cdot(-1))'$

$y''=16e^{4x}+2e^{-x}/'$

$y'''=64e^{4x}-2e^{-x}$

$64e^{4x}-2e^{-x}-52e^{4x}+26e^{-x}-12e^{4x}-24e^{-x}=0$ ✓

② Odrediti n-ti izvod funkcija, pri čemu je n prirodan broj:

a) $y=x^n$

$y'=n\cdot x^{n-1}$

$y''=n(n-1)x^{n-2}$

$y'''=n(n-1)(n-2)x^{n-3}$

$\vdots$

$y^{(n)}=n(n-1)(n-2)\dots(n-(n-1))x^{n-n}=n!$

b) $y = \cos x$

$y' = -\sin x$

$y'' = -\cos x$

$y''' = \sin x$

$y^{IV} = \cos x$

$$y^{n}(x) = \begin{cases} \cos x, & \text{ако } 4 \mid n \\ -\sin x, & \text{ако } 4 \mid n-1 \\ -\cos x, & \text{ако } 4 \mid n-2 \\ \sin x, & \text{ако } 4 \mid n-3 \end{cases}$$

$y' = -\sin x \; (\cos(x + \frac{\pi}{2}))$

$y'' = -\cos x \; (\cos(x + \pi))$

c) $y = x^2 \cdot e^{-x}$

$$y^{(n)} = \sum_{k=0}^{n} (x^2)^{(n-k)} (e^{-x})^{k} \cdot \binom{n}{k}$$

$$y^{(n)} = (x^2)^{(n)} \cdot (e^{-x})^0 \binom{n}{0} + (x^2)^{(n-1)} \cdot (e^{-x}) \cdot \binom{n}{1} + \dots +$$

$$+ (x^2)^{(3)} \cdot (e^{-x})^{(n-3)} \cdot \binom{n}{n-3} + \dots + (x^2)^0 \cdot (e^{-x})^{(n)} \cdot \binom{n}{n}$$

$$y^{(n)} = 2 \cdot (e^{-x})^{(n-2)} \cdot \binom{n}{n-2} + 2 \cdot x \cdot (e^{-x})^{(n-1)} \cdot n + x^2 \cdot (e^{-x})^{(n)}$$

$(e^{-x})' = -e^{-x}$

$(e^{-x})'' = e^{-x}$

$\vdots$

$(e^{-x})^{(n)} = (-1)^n \cdot e^{-x}$

(3) Odrediti $y'_x, y''_{xx}, y'''_{xxx}$ za funkciju definisanu sa

$x=\ln(1+t^2),\ y=t-\operatorname{arctg} t$

$$y'_x=\frac{y'_t}{x'_t}=\frac{1-\frac{1}{1+t^2}}{\frac{2t}{1+t^2}}=\frac{t^2}{2t}=\frac{t}{2}$$

$$y''_{xx}=\frac{\left(\frac{t}{2}\right)'}{\frac{2t}{1+t^2}}=\frac{\frac{1}{2}}{\frac{2t}{1+t^2}}=\frac{t^2+1}{4t}=\frac{t+t^{-1}}{4}$$

$$y'''_{xxx}=\frac{(y''_{xx})_t}{x'_t}=\frac{\left(\frac{t+t^{-1}}{4}\right)'}{\frac{2t}{1+t^2}}=\frac{\frac{1}{4}\left(1+(-1)\cdot t^{-2}\right)}{\frac{2t}{1+t^2}}=$$

$$=\frac{1}{8}\cdot\frac{\frac{t^2-1}{t^2}}{\frac{t}{t^2+1}}=\frac{(t^2+1)(t^2-1)}{t^3}$$

(4) Odrediti y' i y'' funkcije $y=y(x)$ zadate sa $x^2+y^2=1$

$2x+2y\cdot y'=0$

$2y\cdot y'=-2x$

$y'=-\frac{\cancel{2}x}{\cancel{2}y}$

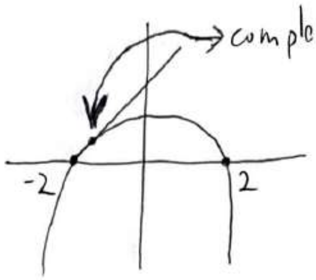
$y'=-\frac{x}{y}\ /'$

$$y''=\frac{-y+xy'}{y^2}=\frac{-y-\frac{x^2}{y}}{y^2}$$

① U kojoj tački $y=4-x^2$ paralelna tetivi AB, $A(-2,0)$
$B(1,3)$

$$K=\frac{y_2-y_1}{x_2-x_1}=\frac{3}{3}=1$$

completely wrong ali $\left(-\frac{1}{2}, 4-\left(\frac{1}{2}\right)^2\right)$



$K(y'(x_0))=1$

$y'=-2x \qquad -2x_0=K_{AB}=1$

$$\boxed{x_0=-\frac{1}{2}}$$

② Da li funkcija $f(x)=1-\sqrt[3]{x^2}$ ispunjava uslove Roll-ovog teorema na intervalu $[-1,1]$

$f(-1)=f(1) \rightarrow$ ovaj uslov vrijedi

$f(x)$ je neprekidna na $[-1,1] \rightarrow$ ovaj uslov vrijedi jer je elementarna funkcija

$f'(x)=-\frac{2}{3}\cdot x^{-\frac{1}{3}}=-\frac{2}{3}\cdot\sqrt[3]{\frac{1}{x}} \rightarrow$ funkcija nije diferencijalna u $x=0$

Ne ispunjava uslov Roll-ovog teorema.

③ Data je realna funkcija $f(x)=(x-1)(x-2)\dots(x-100)$. Dokazati da jednačina $f'(x)=0$ ima bar 99 različitih rješenja i odrediti intervale u kojim se ta rješenja nalaze.

1) Funkcija je neprekidna

2) Funkcija je polinom pa fija je dif na $\mathbb{R}$

$f(1)=f(2)=f(3)=\dots=f(100)=0$

1) Primjenom roll-a na $[1,2] \Rightarrow \exists x_1 \in (1,2)$ t.d. $f'(x_1)=0$

2) Primjenom roll-a na $[2,3] \Rightarrow \exists x_2 \in (2,3)$ t.d. $f'(x_2)=0$

$\vdots$

99) Primjenom roll-a na $[99,100] \Rightarrow \exists x_{99} \in (99,100)$ t.d. $f'(x_{99})=0$

(4) Dokazati da $\forall a>0$ vrijedi $\frac{1}{a+1} < \ln\left(1+\frac{1}{a}\right) < \frac{1}{a}$

$$\frac{1}{a+1} < \ln(a+1) - \ln(a) < \frac{1}{a}$$

$f(x) = \ln(x) \quad [a, a+1]$

1) $f(x)$ je nepr. kao elem. funkcija

2) $f'(x) = \frac{1}{x}$, postoji na segmentu $[a, a+1]$ $(a>0)$ $\exists c \in (a, a+1)$ t.d. $f'(c) = \frac{f(a+1) - f(a)}{a+1-a}$

po Lagranžovom teoremu

$$\frac{1}{c} = \ln(a+1) - \ln(a)$$

$$\frac{1}{a+1} < \frac{1}{c} < \frac{1}{a} \quad \text{jer je} \quad a+1 > c > a$$

$$\frac{1}{a+1} < \ln(a+1) - \ln(a) < \frac{1}{a}$$

(5) Dokazati nejednakost $\frac{xy+1}{y} < \frac{\ln\frac{x}{y}}{\operatorname{arctg}x - \operatorname{arctg}y} < \frac{xy+1}{x}$ za sve $0<x<y$

$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

$f(x) = \ln(x)$

$g(x) = \operatorname{arctg}(x)$

na $[x, y]$

1) $\ln(x)$ i $\operatorname{arctg}(x)$ su nepr. na $[x-y]$

2) $f'(x) = \frac{1}{x} \quad g'(x) = \frac{1}{1+x^2}$ oba izvoda postoje na $[x,y]$

3) $f'(x) \neq 0 \quad g'(x) \neq 0 \quad (x>0)$

4) $g(x) \neq g(y)$ jer $x \neq y$ a $\operatorname{arctg}(x)$ je injektivan

Po Košijevom teoremu $\exists c \in (x,y)$: $\frac{f'(c)}{g'(c)} = \frac{f(y) - f(x)}{g(y) - g(x)} = \frac{\ln y - \ln x}{\operatorname{arctg}y - \operatorname{arctg}x} = \frac{\ln x - \ln y}{\operatorname{arctg}x - \operatorname{arctg}y}$

$$\frac{f'(c)}{g'(c)} = \frac{\frac{1}{c}}{\frac{1}{1+c^2}} = \frac{1+c^2}{c}$$

$$\frac{xy+1}{x} > \frac{1+c^2}{c} > \frac{xy+1}{y}$$

$$\underbrace{y + \frac{1}{x} > \underbrace{c + \frac{1}{c}}_{}}_{\text{I}\!\!\text{F}}\; \underbrace{> x + \frac{1}{y}}_{\text{I}\!\!\text{F}} \quad \text{Ovo vrijedi iz } x < c < y$$

(6) Izračunati limese upotrebom L'Hospitalovog pravila:

a) $\lim\limits_{x\to 0}\frac{\sin x}{x} \overset{L.P.}{=} \lim\limits_{x\to 0}\cos x = 1$

b) $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2} \overset{L.P.}{=} \lim\limits_{x\to 0}\frac{\sin x}{2x} \overset{L.P.}{=} \lim\limits_{x\to 0}\frac{\cos x}{2} = \frac{1}{2}$

c) $\lim\limits_{x\to 0}\frac{\ln(1+x)}{x} \overset{L.P.}{=} \lim\limits_{x\to 0}\frac{1}{1+x}\cdot 1 = 1$

d) $\lim\limits_{x\to +\infty}\frac{e^x}{x^3+3x^2-4} = +\infty$

e) $\lim\limits_{x\to 0}\frac{\ln x}{ctg x} = \lim\limits_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{\sin^2 x}} = \lim\limits_{x\to 0}\frac{-\sin^2 x}{x} \overset{L.P.}{=} \lim\limits_{x\to 0}\frac{2\sin x\cos x}{1} = 0$

f) $\lim\limits_{x\to 0}\frac{x\cdot\ln x : x}{1 : x} = \frac{\ln x}{\frac{1}{x}} = „\frac{\infty}{\infty}" \overset{L.P.}{=} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -x = 0$

g) $\lim\limits_{x\to 0} x^x = „0^0"$

$\lim\limits_{x\to 0}\frac{x^2\cdot\sin\frac{1}{x}}{\sin x} = „\frac{0}{0}" = \lim\limits_{x\to 0}\frac{2x\cdot\sin\frac{1}{x} + x^2\cdot\cos\frac{1}{x}\cdot\left(-\frac{1}{x^2}\right)}{\cos x} =$

$= 2x\cdot\sin\frac{1}{x} - \cos\frac{1}{x}$, limes za $\cos\frac{1}{x}$ kada $x\to 0$ ne postoji, pa ćemo rastaviti drugačije jer L.P. nije moguć iskoristiti na izraz.

$\lim\limits_{x\to 0}\frac{x^2\cdot\sin\frac{1}{x}}{\sin x} = \lim\limits_{x\to 0}\frac{x}{\sin x}\cdot\sin\frac{1}{x}\cdot x = 1\cdot 0 = 0$

(7) Može li se primjeniti L'Hospitalovo pravilo na limes $\lim\limits_{x\to 0}\frac{x^2\sin\frac{1}{x}}{\sin x}$ i koliko iznosi taj limes.

(Zaboravio sam napisati zadatak prije rješenja)

1) $f(x) = x \cdot \ln(x)$. Odrediti intervale monotonosti funkcije $f(x)$ te lokalne ekstreme te funkcije.

D.P. $x > 0$

$f'(x) = \ln x + 1$

1) $f'(x) > 0$

$\ln x > -1$

$x > e^{-1}$

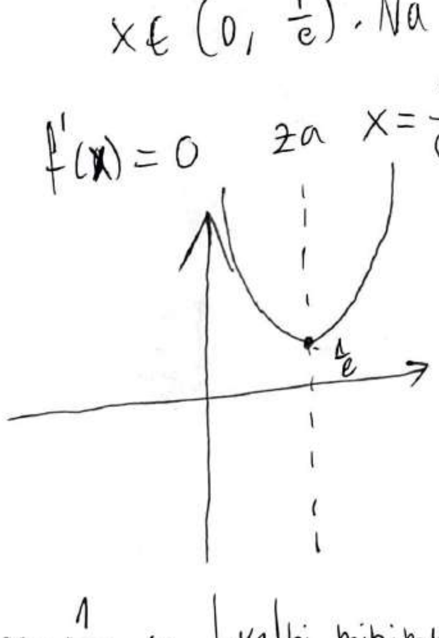
$x > \frac{1}{e}$

$x \in \left(\frac{1}{e}, +\infty\right)$. Na ovom intervalu funkcija je rastuća

2) $f'(x) < 0$

$x \in \left(0, \frac{1}{e}\right)$. Na ovom intervalu funkcija je opadajuća

$f'(x) = 0$ za $x = \frac{1}{e}$



$x = \frac{1}{e}$ je lokalni minimum

3 [redacted] 6

(2) $f(x)=x^{1-\ln x}$. Naći lokalne ekstreme funkcije $f(x)$.

$\ln(f(x)) = (1-\ln x)\cdot\ln(x) = \ln x-\ln^2 x \,/'$

$$\frac{f'(x)}{f(x)} = \frac{1}{\ln x} - \frac{2\ln x}{x}$$

$$f'(x) = x^{1-\ln x}\cdot\frac{1}{x}(1-2\ln x) = x^{-\ln x}(1-2\ln x) = \frac{1-2\ln x}{x^{\ln x}}$$

D.P. $x>0$

$x=\sqrt{e}$ je stacionarna tačka

$x>\sqrt{e}$ $\quad \frac{\overbrace{1-2\ln x}^{<0}}{\underbrace{x^{\ln x}}_{>0}} < 0$ $\qquad$ $x<\sqrt{e}$ $\left(\ln x < \frac{1}{2}\right)$

$$\frac{1}{x^{\ln x}}\cdot(1-\ln x) > 0$$

(3) $x=0$. Ispitati sljedeće funkcije u tački x.

a) $f(x)=e^{x}+e^{-x}+2\cos x$, D.P. $x\in\mathbb{R}$

$f'(x)=e^{x}-e^{-x}-2\sin x$ $\qquad f'(0)=0$

$f''(x)=e^{x}+e^{-x}-2\cos x$ $\qquad f''(0)=0$

$f'''(x)=e^{x}-e^{-x}+2\sin x$ $\qquad f'''(0)=0$

$f^{IV}(x)=e^{x}+e^{-x}+2\cos x$ $\qquad f^{IV}(0)=4>0$ pa je 4 lokalni minimum

b) $f(x)=\begin{cases} e^{-\frac{1}{x^2}}, & \text{za } x\neq 0 \\ 0, & \text{za } x=0 \end{cases}$

$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{e^{-\frac{1}{x^2}}}{x}="\frac{0}{0}"=$ | Smjena $\frac{1}{x}=t$ | $\lim_{t\to\infty}\frac{e^{-t}}{\frac{1}{t}}=\lim_{t\to\infty}\frac{\frac{1}{e^t}}{\frac{1}{t}}=\lim_{t\to\infty}\frac{t}{e^t}=$

$=0$

$f''(0)=0$

$f'''(0)=0$

$\vdots$

za sve

za $x<0$ $\quad f'(x)=\frac{2\cdot e^{-\frac{1}{x^2}}}{x^3}<0$

za $x>0$ $\quad f'(x)=\frac{2\cdot e^{-\frac{1}{x^2}}}{x^3}>0$

Kasnije ću uraditi ovaj izvod kako treba.

$x=0$ je lokalni minimum

(4) Dokazati da jednačina $\ln x=\frac{2(x-1)}{x+1}$ nema realnih rješenja za $x>1$

$f(x)=\ln x-\frac{2(x-1)}{x+1}$

D.P. $x>0$

$f'(x)=\frac{1}{x}-2\cdot\frac{\cancel{x}+1-(\cancel{x}-1)}{(x+1)^2}=\frac{1}{x}-\frac{4}{(x+1)^2}=\frac{(x-1)^2}{x(x+1)^2}$

$f'(x)>0$ za $x>1$

$f(1)=0$ nema nula jer je graf uvijek iznad x ose pa nema realnih rješenja

(5) Odrediti intervale konveksnosti i konkavnosti $f(x)=x^2\ln x$ i naći prevoje

$f(x)=x^2\ln(x)$ D.P. $x>0$

$f'(x)=2x\ln x+x$

$f''(x)=2\ln x+2+1=2\ln x+3$

1) $f''(x)>0 \Leftrightarrow 2\ln x>-3 \Rightarrow \ln x>-\frac{3}{2} \Rightarrow x>\frac{1}{\sqrt{e^3}}$

2) $f''(x)<0 \Leftrightarrow 2\ln x<-3 \Rightarrow \ln x<-\frac{3}{2} \Rightarrow x<\frac{1}{\sqrt{e^3}}$

za $x\in(0,e^{-\frac{3}{2}}]$ funkcija je konkavna

za $x\in(e^{-\frac{3}{2}},+\infty)$ funkcija je konveksna

$x=e^{-\frac{3}{2}}$ je prevojna tačka.

(6) Odrediti asimptote krivih

$y=\sqrt[3]{x^3-3x}+\frac{x}{1-x^2}$

V.A. Ima smisla posmatrati samo u $x=-1$ i $x=1$

$\lim\limits_{x\to1^+}\sqrt[3]{x^3-3x}+\frac{x}{1-x^2}=-\infty$

$\lim\limits_{x\to1^-}\sqrt[3]{x^3-3x}+\frac{x}{1-x^2}=+\infty$

$x=1$ V.A. sa obje strane

$\lim\limits_{x\to-1^+}=-\infty$

$\lim\limits_{x\to-1^-}=+\infty$

$x=-1$ je V.A. sa obje strane

b) $y = x \cdot e^{\frac{1}{x}}$

D.P. $x \in (-\infty, 0) \cup (0, +\infty)$

V: D.V.A.

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x \cdot e^{\frac{1}{x}} = {}_{''}\, 0 \cdot \infty '' = \lim_{x \to 0^+} \frac{e^{\frac{1}{x}}}{\frac{1}{x}} =$$

$$= \left| \begin{array}{l} \text{Smjena: } \frac{1}{x} = t \\ t \xrightarrow{x \to 0^+} +\infty \end{array} \right| = \lim_{t \to +\infty} \frac{e^t}{t} = \infty$$

Desna vertikalna asimptota postoji.

L.V.A.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x \cdot e^{\frac{1}{x}} = {}_{''} \, 0 \cdot \infty '' = \lim_{x \to 0^-} \frac{e^{\frac{1}{x}}}{\frac{1}{x}} = \left| \begin{array}{l} \frac{1}{x} = t \\ t \xrightarrow{x \to 0^-} -\infty \end{array} \right| =$$

$$= \lim_{t \to -\infty} \frac{e^t}{t} = {}_{''} \frac{0}{\infty} '' = 0$$

Nema lijeve vertikale asimptote.

H: D.H.A.

$$\lim_{x \to +\infty} x \cdot e^{\frac{1}{x}} = {}_{''} +\infty \cdot 1 '' = +\infty$$

L.H.A.

$$\lim_{x \to -\infty} x \cdot e^{\frac{1}{x}} = {}_{''} -\infty \cdot 1 '' = -\infty$$

Ne postoji ni lijeva ni desna horizontalna

K: $k = \lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{x \cdot e^{\frac{1}{x}}}{x} = \lim_{x \to +\infty} e^{\frac{1}{x}} = 1$

$$n = \lim_{x \to +\infty} x \cdot e^{\frac{1}{x}} \cdot kx = x \cdot e^{\frac{1}{x}} - x = x\left(e^{\frac{1}{x}} - 1\right) = \frac{e^{\frac{1}{x}} - 1}{\frac{1}{x}} = 1$$

$y = x + 1$ je kosa asimptota funkcije.

① Odrediti Maclaurinovu formulu za funkcije:

a) $f(x) = e^x$

Imamo da je $a=0$,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} \cdot (x-a)^k + \frac{f^{(n+1)}(a+\theta(x-a))}{(n+1)!} \cdot (x-a)^{n+1}$$

$$f^{(k)}(x) = e^x \quad \forall k \in [0, n+1], k \in \mathbb{N}$$

$$f(0) = 1$$

$$f^{(k)}(0) = 1 \quad \forall k \in [0, n+1], k \in \mathbb{N}$$

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + \frac{e^{\theta x}}{(n+1)!} \cdot x^{n+1}$$

b) $f(x) = \sin x$

$f(x) = \sin x$	$f(0) = 0$
$f'(x) = \cos x$	$f'(0) = 1$
$f''(x) = -\sin x$	$f''(0) = 0$
$f'''(x) = -\cos x$	$f'''(0) = -1$
$f^{IV}(x) = \sin x$	$f^{IV}(0) = 0$
$f^{V}(x) = \cos x$	$f^{V}(0) = 1$
$\vdots$	$\vdots$

$$f(x) = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \frac{f^{(2n+2)}(\theta x)}{(2n+2)!} \cdot x^{2n+2}$$

② Napisati Taylorov polinom u tački $a=2$ funkcije

$f(x)=2x^3-4x^2+x-2$

$$P_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}\cdot(x-a)^k$$

Ovdje je $a=2$.

$f(x)=2x^3-4x^2+x-2$

$f'(x)=6x^2-8x+1$

$f''(x)=12x-8$

$f'''(x)=12$

$f^{IV}(x)=0$

$\vdots$

$f(2)=0$

$f'(2)=9$

$f''(2)=16$

$f'''(2)=12$

$f^{IV}(2)=0$

$\vdots$

$f^{(k)}=0$ za $k\geq 4$, $k\in\mathbb{N}$

$$P_n(x)=\frac{9(x-2)}{1!}+\frac{16(x-2)^2}{2!}+\frac{12(x-2)^3}{3!}$$

$R_n(x)=0$ jer $f^{IV}(\cancel{x})=0$

$P_n(x)=9x-18+8x^2-32x+32+2x^3-12x^2+24x-16$

$P_n(x)=2x^3-4x^2+x-2$

③ Napisati Maclaurinov polinom 6. stepena funkcije chx.

$a=0$

$$P_n(x)=\sum_{k=0}^{n=6}\frac{f^{(k)}(a)}{k!}\cdot(x-a)^k \quad chx=\frac{e^x-e^{-x}}{2}$$

$$shx=\frac{e^x+e^{-x}}{2}$$

$f(x)=chx \quad f(0)=0$

$f'(x)=shx \quad f'(0)=1$

$f''(x)=chx \quad f''(0)=0$

$f'''(x)=shx \quad f'''(0)=1$

$f^{IV}(x)=chx \quad f^{IV}(0)=0$

$f^{V}(x)=shx \quad f^{V}(0)=1$

$f^{VI}(x)=chx \quad f^{VI}(0)=0$

$$P_n(x)=\frac{x}{1!}+\frac{x^3}{3!}+\frac{x^5}{5!}$$

④ Da li postoji interval $[-a,a]$ promjenjive x u kome će Maclaurinov polinom 4. stepena aproksimirati funkciju $f(x)=\cos x$ sa greškom manjom od $\varepsilon=10^{-5}$

$f(x)=\cos x \quad f(0)=1$

$f'(x)=-\sin x \quad f'(0)=0$

$f''(x)=-\cos x \quad f''(0)=-1$

$f'''(x)=\sin x \quad f'''(0)=0$

$f^{IV}(x)=\cos x \quad f^{IV}(0)=1$

$$P_n(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}$$

$$R_n(x)=\frac{f^{(5)}(\theta x)}{5!}\cdot x^5=\frac{-\sin(\theta x)\cdot x^5}{5!}$$

$$|R_n(x)|<10^{-5}$$

$$\left|\frac{-\sin(\theta x)}{5!}\cdot x^5\right|<10^{-5}$$

$$\left|\frac{-\sin(\theta x)}{5!}\cdot x^5\right|\leq\left|\frac{x^5}{5!}\right|<10^{-5}$$

$$|x|^5<120\cdot10^{-5} \;/\sqrt[5]{\;}$$

$$|x|<\sqrt[5]{\frac{120}{10^5}}=\frac{\sqrt[5]{120}}{10} \qquad [-a,a]\in\left[-\frac{\sqrt[5]{120}}{10},\frac{\sqrt[5]{120}}{10}\right]$$

① Nacrtati grafik funkcija:

a) $f(x) = |x^3 - 2x^2| - x$

1) D.P. $x \in (-\infty, +\infty)$

2) Parnost/Neparnost i periodičnost

P: $f(-x) = f(x)$ ne vrijedi

N.P.: $f(-x) = -f(x)$ ne vrijedi

3) Nule funkcije

$f(x) = x^3 - 2x^2$

$x^3 - 2x^2 \geq 0$

$\underbrace{x^2}_{\text{uvijek} \geq 0}\underbrace{(x-2)}_{x \geq 2} \geq 0$

Sada funkciju možemo napisati drugačije:

$$f(x) = \begin{cases} x^3 - 2x^2 - x, & \text{za } x \geq 2 \\ -x^3 + 2x^2 - x, & \text{za } x < 2 \end{cases} = \begin{cases} x(x-1+\sqrt{2})(x-1-\sqrt{2}), & \text{za } x \geq 2 \\ -x(x-1)^2, & \text{za } x < 2 \end{cases}$$

$f(x) \stackrel{?}{=} 0$

1° $-x(x-1)^2 = 0$

1) $x = 0$, pripada D.P.-u ✓

2) $x = 1$, pripada D.P.-u ✓

2° $x(x-1+\sqrt{2})(x-1-\sqrt{2})$

1) $x = 0$, ne plipada D.P.-u ✗

2) $x = 1+\sqrt{2}$, pripada D.P.-u ✓

3) $x = 1-\sqrt{2}$, ne pripada D.P.-u

$x = 1$ je dodirna tačka jer ima kvadrat

$x = 0$ i $x = 1+\sqrt{2}$ su presječne tačke

4) Znak funkcije

Posmatrajmo funkciju u ovom obliku:

$$f(x)=\begin{cases} x(x-1+\sqrt{2})(x-1-\sqrt{2}), & \text{za } x\geq 2 \\ -x(x-1)^2, & \text{za } x<2 \end{cases}$$

$f(x)\overset{?}{>}0$ i $f(x)\overset{?}{<}0$

	2 – $1+\sqrt{2}$	$1+\sqrt{2}$ – $+\infty$
x	$+$	$+$
$x-1+\sqrt{2}$	$+$	$+$
$x-1-\sqrt{2}$	$-$	$+$
$f(x)$	$-$	$+$

	$-\infty$ – 0	0 – 1	1 – 2
$-x$	$+$	$-$	$-$
$(x-1)^2$	$+$	$+$	$+$
$f(x)$	$+$	$-$	$-$

$f(x)>0$ za $x\in(-\infty,0)\cup(1+\sqrt{2},+\infty)$

$f(x)<0$ za $x\in(0,1+\sqrt{2})$

5) Asimptote

1° Vertikalnih asimptota nema jer nema prekida

2° Horizontalne asimptote

$$\left.\begin{array}{l}\lim\limits_{x\to+\infty} f(x)=+\infty \\ \lim\limits_{x\to-\infty} f(x)=-\infty\end{array}\right\}$$ Nema horizontalnih asimptota

3° Kose asimptote

$k=\lim\limits_{x\to\pm\infty}\dfrac{f(x)}{x}=\pm\infty$ Nema ni kosih asimptota

6) Prvi izvod, monotonost i stacionarne tačke

$$f'(x) = \begin{cases} 3x^2-4x-1, & x>2 \\ \text{za } x=2 \text{ treba po definiciji} \\ -3x^2+4x-1, & x<2 \end{cases}$$

D.P. je ostao isti.

$$\lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = 000 = 3$$

$$\lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} = 000 = -5$$

Oba izvoda su konačna pa je u $x=2$ šiljak. (jer su različita)

$$f'(x) = \begin{cases} 3\left(x-\frac{2-\sqrt{7}}{3}\right)\left(x-\frac{2+\sqrt{7}}{3}\right), & \text{za } x>2 \\ -3(x-1)\left(x-\frac{1}{3}\right)=, & \text{za } x<2 \\ =(-3x+3)\left(x-\frac{1}{3}\right) \end{cases}$$

	$-\infty$ — $\frac{2-\sqrt{7}}{3}$	$\frac{2-\sqrt{7}}{3}$ — $\frac{1}{3}$	$\frac{1}{3}$ — 1	1 — $\frac{2+\sqrt{7}}{3}$	$\frac{2+\sqrt{7}}{3}$ — 2	2 — $+\infty$
$x-\frac{2-\sqrt{7}}{3}$						+
$x-\frac{2+\sqrt{7}}{3}$						+
$-3x+3$	+	+	+	−	−	
$x-\frac{1}{3}$	−	−	+	+	+	
$f'(x)$	−	−	+	−	−	+
$f(x)$	↘	↘	↗	↘	↘	↗

$f'(0)=0$

$-3(x-1)\left(x-\frac{1}{3}\right)=0$

$\left.\begin{array}{l} x_1=1 \\ x_2=\frac{1}{3} \end{array}\right\}$ Stacionarne tačke

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7) Drugi izvod, konveksnost/konkavnost i prevojne tačke

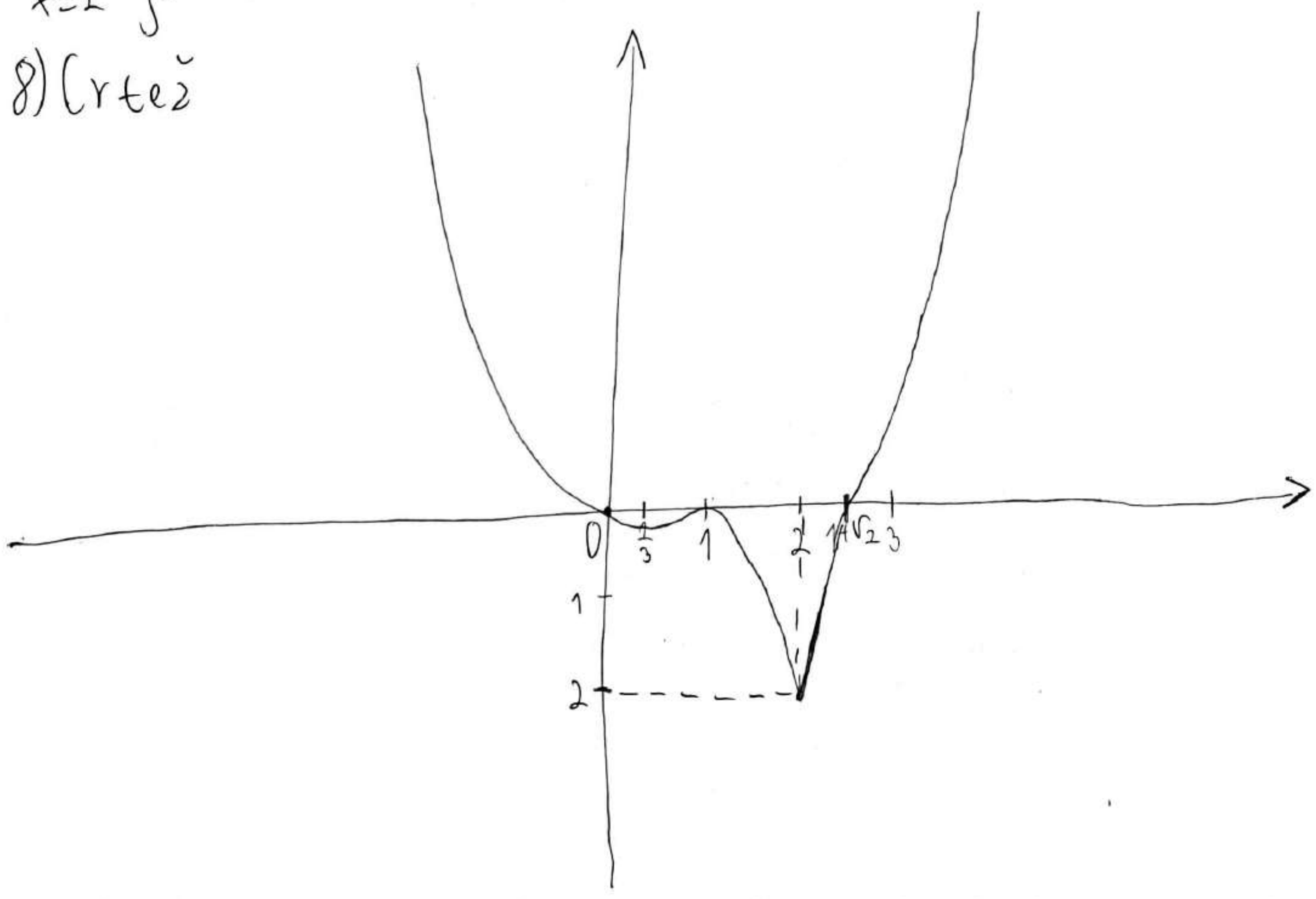
$$f''(x) = \begin{cases} 6x-4, & za\ x>2 \\ -6x+4, & za\ x<2 \end{cases}$$

	$-\infty$ … $\frac{4}{6}$	$\frac{4}{6}$ … 2	2 … $+\infty$
$6x-4$			+
$-6x+4$	+	−	
$f''(x)$	+	−	+
$f(x)$	∪	∩	∪

$x=\frac{4}{6}$ je prevojna tačka

$x=2$ je ugaona (špic)

8) Crtež



b) $f(x)=x\cdot e^{\frac{1}{x}}$

1) D.P. $x\in(-\infty,0)\cup(0,+\infty)$

2) (Ne)parnost i periodičnost

F-ija nije periodična

1) $f(-x)=f(x)$ ne vrijedi

2) $f(-x)=-f(x)$ ne vrijedi

3) Nule funkcije

Funkcija nikad nije nula radi D.P.-a

4) Znak funkcije

$-\infty$	0	$+\infty$
x	$-$	$+$
$e^{\frac{1}{x}}$	$+$	$+$
$f(x)$	$-$	$+$

5) Asimptote

$\underline{H}$: $\left.\begin{array}{l}\lim\limits_{x\to+\infty} x\cdot e^{\frac{1}{x}}=+\infty \\ \lim\limits_{x\to-\infty} x\cdot e^{\frac{1}{x}}=-\infty\end{array}\right\}$ Nema horizontalnih.

$\underline{V}$: $\lim\limits_{x\to 0^+} x\cdot e^{\frac{1}{x}}=000=+\infty$ Ima desna vertikalna.

$\lim\limits_{x\to 0^-} x\cdot e^{\frac{1}{x}}=0$ Nema lijeva vertikalna.

$\underline{K}$: $k=\lim\limits_{x\to+\infty}\frac{f(x)}{x}=1$ $\quad n=\lim\limits_{x\to\pm\infty}(f(x)-kx)=000=1$

$\boxed{y=x+1}$ je kosa asimptota.

6) Prvi izvod, stacionarne tačke, monotonost

$f(x) = x \cdot e^{\frac{1}{x}}$

$f'(x) = e^{\frac{1}{x}} + x \cdot e^{\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right) = e^{\frac{1}{x}}\left(1 - \frac{1}{x}\right) = e^{\frac{1}{x}}\left(\frac{x-1}{x}\right)$

D. P. je ostao isti.

	$-\infty$ 0	0 1	1 $+\infty$
$e^{\frac{1}{x}}$	+	+	+
$x-1$	-	-	+
x	-	+	+
$f'(x)$	+	-	+
$f(x)$	↗	↘	↗

$f'(x) = 0 \Rightarrow e^{\frac{1}{x}}\left(1 - \frac{1}{x}\right) = 0$

$x = 1$ je stacionarna tačka

7) Drugi izvod, konkavnost/konveksnost i prevojne tačke

$f'(x) = e^{\frac{1}{x}}\left(\frac{x-1}{x}\right)$

$f''(x) = \frac{e^{\frac{1}{x}}}{x^3}$ D. P. je ostao isti

	$-\infty$ 0	0 1	1 $+\infty$
$e^{\frac{1}{x}}$	+	+	+
x^3	-	+	+
$f''(x)$	-	+	+
$f(x)$	∩	∪	∪

$x=0$ nije u domenu

8) Crtež