Prove or Disprove: if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ is $a^c \equiv b^d \pmod{m}$? [Section 4.1 Book P245]

Not True: $m=10$. $c=0$. $d=10$. $a=b=5$ $\Rightarrow$ $5^0 = 1 \not\equiv 0 \pmod{10}$ $5^{10} \pmod{10}$

Or: $m=5$. $c=1$. $d=6$. $a=b=3$

Given $\gcd(a,m)=1$ Prove: inverse of a mod m is unique Section 4.4. 7. Page 284

method1:

Assume that $a^{-1}$ for m is not unique: b. and c. are both mult inverses of a module m

$ab \equiv 1 \bmod m \Rightarrow m \mid ab-1$

$ac \equiv 1 \bmod m \Rightarrow m \mid ac-1$

$\Rightarrow m \mid (ab-1)-(ac-1) = ab-ac$

$ab \equiv ac \pmod{m}$

$b \equiv c \pmod{m}$ ✓

只证同余
不能相同

method2:

$b \equiv b \cdot 1 \bmod m$

$\equiv b \cdot (c \cdot a) \bmod m$

$\equiv (b \cdot c) \cdot a \bmod m$

$\equiv c \cdot (b \cdot a) \bmod m$

$\equiv c \cdot 1 \bmod m$

Prove or disprove if $a \mid bc$. $a,b,c \in \mathbb{Z}^+$ then $a \mid b$ or $a \mid c$ P.244.

Not True: $a=6$ $b=3$ $c=2$.

深层原因: a.b.c 不互质

Prove: if $a \mid bc$. $\gcd(a,b)=1$. $a,b,c \in \mathbb{Z}^+$. Then $a \mid b$ or $a \mid c$

By Bezout, $\exists\, s,t \in \mathbb{Z}$ that $sa+tb=1 \Rightarrow sac+tbc=c$

$a \mid bc \Rightarrow bc = ka \Rightarrow sac+tka=c$

$a(sc+tk)=c \Rightarrow a \mid c$

Hard one: 5.1.72 P332.

Prove: It is possible to arrange $1,2,\cdots,n$ in a row so that the average of any two never appears between them.

Ex: 1234 → 2143

Prove: when $n=2^k$

$k=0$ $n=2^0=1$ 1 No average

$k=1$ $n=2^1=2$ 1.2

$k=2$ $n=2^2=4$ 1,2,3,4 → 1. 3. 2. 4

Did not finish

当出现两个 initial states 的时候，我们就会想到 Strong Induction

P 397. #33

How many strings of 8 English Letters are there

e) contains at least one vowel letters can be repeated

$26^8 - 21^8$

f) contains exact one vowel, ... can be .....

$5 \cdot 21^7 \cdot 8$

P391. #47

6 people in a row include ( bride & groom )

a) if b next to g

BG.绑一起: $P\binom{5}{5} \cdot P\binom{2}{2} = 5! \cdot 2!$

$\boxed{! : \text{factorial}}$

b) B not next to G

$6! - 2 \cdot 5!$

c) B is somewhere to left of G

B要么在左边 要么在右边 $6!/2$

Prove : In any group of 5 people, there are 2 who have an identical number of friends within the group.

朋友数 : 0, 1, 2, 3, 4.

If any person has 0 friends then no one will has 4 friends.
which means the number of the friends : 0, 1, 2, 3
or 1, 2, 3, 4.

Five people share 4 possibility. There must be two has the same ...

**21.** What is the probability that a fair die never comes up an even number when it is rolled six times?

P451

2 equally prob out comes : odd , even

$2^6$ parity outcomes when roll 6 times ; only one consist of 6 odd ;

$1/2^6$

P451 #25

Choose 6 of n. order does not matter

$\binom{n}{6}$ possible $\Rightarrow \frac{1}{\binom{n}{6}}$

P451 #27

choose 6 of n. order does not matter, exactly one of six is correct. Prob

# 6 : ways to choose correct num

# $\binom{n-6}{5}$ : ways to choose wrong number

# $\binom{n}{6}$ : All

Prob $\frac{6 \cdot \binom{n-6}{5}}{\binom{n}{6}}$

P452 37

Which is more likely: roll a total of 9 when two dice are rolled or rolling a total of 9 when three dice are rolled?

2 dice: $\dfrac{\#\text{ ways to get } 9}{6^2} = \dfrac{|(3,6)\ (6,3)\ (4,5)\ (5,4)|}{6^2} = \dfrac{4}{36} = \dfrac{1}{9}$

3 dice: $\dfrac{\#\text{ ways to get } 9}{6^3}$ ← (stars and bars)

Find all solutions: $x_1+x_2+x_3 = 9$ where $1 \le x_i \le 6$

$\Leftrightarrow$ $x_1+x_2+x_3 = 6$ $(x_i \le 6)$ 6 star 3 bar

$\binom{6+3-1}{3-1} = \binom{8}{2} = 28$

$28 - 3 = 25$

减去 $x_i=6$ 的情况，至多只有一个等于 6，不然总和超过 6

排列组合应怎么做

$9=1+8 \Rightarrow 8=2+6=3+5=4+4=5+3=6+2$

$9=2+7 \Rightarrow 7=1+6=2+5=3+4=4+3$

$9=3+6$

$9=4+5$

$9=5+4$

$9=6+3$

25种

7.2 P466 #1

biased coin Head 3× likely as Tails

$P(H)=3P(T)$ $P(H)+P(T)=1$

$3P(T)+P(T)=1$ $P(H)=\frac{3}{4}$

$P(T)=\frac{1}{4}$

P466 #3

biased dice rolling $\overset{\text{equal}}{2 \text{ or } 4}$ is 3× likely as any other #

$P(2)=P(6)=3P(1)=3P(3)=3P(5)=3P(6)$

?

$P(2 \text{ or } 4) = 3\cdot P(1,3,5,6)$

$P(2 \text{ or } 4) = x$ $P(\text{rest}) = 1-x$

$x = 3(1-x) \Rightarrow x=\frac{3}{4}$

$P(2)=P(4)=\frac{3}{8}$ $P(1)=P(3)=P(5)=P(6)=\frac{1}{16}$

**7.** What is the probability of these events when we randomly select a permutation of {1, 2, 3, 4}?

**a)** 1 precedes 4.
**b)** 4 precedes 1.
**c)** 4 precedes 1 and 4 precedes 2.
**d)** 4 precedes 1, 4 precedes 2, and 4 precedes 3.
**e)** 4 precedes 3 and 2 precedes 1.

P466 #7

a) b) 1 either precedes 4 or follows 4 a) = b) = $\frac{1}{2}$

c) $\underline{4}\ \_\ \_\ \_$ ($3!=6$) $\_\ \underline{3}\ \underline{4}\ \_$ ($2!=2$) $= \dfrac{8}{4!} = \dfrac{1}{3}$

d) $3! = 6 \Rightarrow \dfrac{6}{4!} = \dfrac{1}{4}$

e) $\underbrace{\text{4 precedes 3}}_{\frac{1}{2}}$ and $\underbrace{\text{2 precedes 1}}_{\frac{1}{2}}$ $\Rightarrow \frac{1}{2}\times\frac{1}{2} = \frac{1}{4}$

In bridge, 52 card standard deck dealt to 4 players: North, South, East, West
Each player receives 13 cards.

— What Prob North has all Ace? (Hands not indistinguishable, positions not interchangeable)

# possible distribution of hands: $\binom{52}{13}\cdot\binom{39}{13}\cdot\binom{26}{13}\binom{13}{13}$

P(North has all Aces) = P(one player has all Aces)

$= $ # distributions where one player holds all Aces / All

$= \binom{48}{9}\binom{39}{13}\binom{26}{13}\binom{13}{13}$ / All

— Each Player has one Ace

$= 4!\cdot\binom{48}{12}\cdot\binom{36}{12}\cdot\binom{24}{12}\cdot\binom{12}{12}$ / All

Sequence of 7 independent tosses of fair coin

A: # Head is even $P(H=0)+P(H=2)+P(H=4)+P(H=6) = \left(1+\binom{7}{2}+\binom{7}{4}+\binom{7}{6}\right)/2^7 = \frac{1}{2}$ P(A)

B: # Head is at least 4 $P(B) = P(H\geq 4) = P(H=4)+\cdots+P(H=7) = \frac{1}{2}$ P(B)

C: # 1st 3 tosses are Head $P(C) = \frac{2^4}{2^7} = \frac{1}{2^3} = \frac{1}{8}$ P(C)

$A\cap C$ P(# H's even $\cap$ 1st 3 tosses H's)

= P(1st 3 tosses H's $\cap$ exactly 1 more H out of 4) + P(…… $\cap$ … 3 more H ….)

$= \frac{1}{8}\binom{4}{1}\frac{1}{2^4} + \frac{1}{8}\binom{4}{3}\cdot\frac{1}{2^4} = \frac{1}{16}$

$B\cap C$

Joseph. n个人分帽子，帽子有主人， X：每个人拿到自己的帽子

计算 E(X)：

Let $X_i = \begin{cases} 1 & \text{person } i \text{ got his hat} \\ 0 & \text{otherwise} \end{cases}$

$X = \sum_{i=1}^{n} X_i$ $E(X) = \sum E(X_i)$ 这很重要，加法性不要求相互独立

$= \sum E(X_i) = n\cdot\frac{1}{n} = 1$

7.4 #1 P492

X = # of successes in Bernoulli trials, success prob p : $E(X) = np$

$X = X_1+X_2+\cdots+X_n$ where $X_i = \begin{cases} 1 & \text{if success} \\ 0 & \text{o/w} \end{cases}$ $p=\frac{1}{2}$ $n=5$ $E(X)=\frac{5}{2}$

#3 $E(X)=np$ $p=\frac{1}{6}$ $n=10$ $E(X)=\frac{5}{3}$

P492 #7 The final exam of a discrete mathematics course consists of 50 true/false questions, each worth two points, and 25 multiple-choice questions, each worth four points. The probability that Linda answers a true/false question correctly is 0.9, and the probability that she answers a multiple-choice question correctly is 0.8. What is her expected score on the final?

#7. 50 T/F 2pts $p = 0.9$ $X := $ # successes
25 T/F 4pts $p = 0.8$ $Y := $ # successes

$$X_i = \begin{cases} 1 & \text{answer } Q_i \text{ correctly} \\ 0 & \text{o/w} \end{cases} \qquad Y_i = \begin{cases} 1 & \text{answer } Q_i \text{ correctly} \\ 0 & \text{o/w} \end{cases}$$

$X = X_1 + X_2 + \cdots + X_{50}$

$E(X_i) = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0)$ $\qquad$ $E(Y_i) = 1 \cdot P(Y_i = 1) + 0 \cdot P(Y_i = 0)$

$= 0.9$ $\qquad$ $= 0.8$

$E(X) = 50 \cdot \frac{9}{10} = 45$ $\qquad$ $E(Y) = 25 \times 0.8 = 20$ $\qquad$ $E(Y) \times 4 = 80$

$E(X) \times 2 = 90$

$E(\text{grade}) = E(X+Y) = E(X) + E(Y) = \cancel{65}$ 170

P492 #12 ☆ Geometric Distribution 几何分布

A. $P(A) = p$. 不停实验直到A成功 $E(A) = \frac{1}{p}$

roll a fair die until 6 comes up, roll X times $E(X) = ?$

$X :=$ # times to roll the die

$p(X=n) = \underbrace{(1-p)^{n-1}}_{\text{失败 } n-1\text{次}} \times p = \left(\frac{5}{6}\right)^{n-1} \cdot \frac{1}{6}$

$E(X) = \frac{1}{p} = 6$

$\boxed{\text{until}}$ keyword of Geometric Distribution

#13 roll pair of fair dice until sum is 7

$p(\text{sum} = 7) = \frac{1}{6}$ $\qquad$ $E(\#\text{ times to roll a sum } 7) = 6$

Linearity of expectation

Snowfall Record

- Start this year: want to find expected # of record show occur in next n year
- 1st year: necessarily a record

2nd year: will be record if snowball exceeds 1st year

Prob: ? $= \frac{1}{2}$ by symmetry

$$X_i = \begin{cases} 1 & \text{if it is a record} \\ 0 & \text{o/w} \end{cases}$$

$E(X_i) = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0)$

☆ $P(X_i = 1) = ? \Rightarrow \frac{1}{i}$ : 从 i 年中挑出一个作为 record $\Leftarrow$

$\therefore E(X_i) = \frac{1}{i}$

$\boxed{\text{Accurate approximation } \ln n + \gamma + \frac{1}{2n}, \ \gamma = 0.5772, \ n\text{越大,我们估的越准!!!}}$

$\boxed{\text{老师这儿写: record snowfall for 1st } i \text{ years is equally likely to fall in any of those years}}$

$X :=$ total num of record in n years

$= X_1 + X_2 + \cdots + X_n$

$E(X) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ $\qquad$ divergent (harmonic) series

$\approx \int_1^n \frac{dx}{x} = \ln n - \ln 1 = \ln n$

## Linear of Expectation

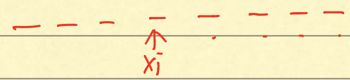
Document of length n: picking words at random from a small dictionary {a, in, the, cat, hat} each word equally likely to be picked

Expected # times phrase "a cat in the hat" appears

Let $X_i = \begin{cases} 1 & \text{if phrase appears starting at position } i \text{ (句子从位置i开始出现)} \\ 0 & \text{o/w} \end{cases}$

At any point in document, phrase has equally prob of appearing

$P(X_i=1) = (\frac{1}{5})^5$




$X$ = # times phrase appears

$= X_1 + X_2 + \cdots + \boxed{X_{n-4}}$ 因为句子有5个词

$E(X) = E(X_1 + \cdots + X_{n-4}) = \sum_{i=1}^{n} (\frac{1}{5})^5 = \frac{n-4}{5^5}$

## Linear of Expectation

A card drawn of random from standard deck of playing cards

If red player wins 1, if black player loses 2 what is the value of the game?

$\underset{\text{"red"}}{X_r} = \begin{cases} 1 & \text{if } i \text{ red} \\ 0 & \text{o/w} \end{cases}$

$X = X_1 + \cdots + X_{52}$

$E(X_r) = 1 \cdot P(X_r = 1) + 0 \cdot P(X_r = 0)$

$= P(X_r = 1) = \frac{1}{2}$

$E(X) = 52 \cdot \frac{1}{2} = 36$

$E(\text{winning}) = 1 \cdot P(X = \text{red}) - 2 \cdot P(X = \text{black})$

$= 1 \cdot \frac{1}{2} - 2 \cdot \frac{1}{2} = -\frac{1}{2}$

Roulette wheel : 0, 00, integers 1 through 36

P492. #16

2 fair coin flipped   $X :=$ total # of H's   $Y :=$ total # of T's   show $X, Y$ not independent

$S = \{HH, HT, TH, TT\}$

$p(i,j) = P(X=i, y=j) \neq P(X=i)\, P(Y=j)$

| $i \backslash j$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | $\frac{1}{4}$ |
| 1 | 0 | $\frac{1}{2}$ | 0 |
| 2 | $\frac{1}{4}$ | 0 | 0 |

} sum=1

To show independence: NTS: $\forall i,j$ $P(X=i, y=j) = P(X=i, Y=j)$
for all

To show not independence: For an $i,j$ pair $P(X=i, Y=j) \neq P(X=i)\, P(Y=j)$

P 493. # 37

$X$ on $S$  $s \in S$  $X(s) \geq 0$  $\forall s \in S$  Prove $P(X(s) \geq a) \leq \frac{E(X)}{a}$  $\forall a > 0$   Markov's inequality

Proof: $E(X) = \sum_i i\, P(X=i)$

$$\frac{E(X)}{a} = \underbrace{\sum_i \frac{i}{a} P(X=i)}_{\text{what know about this sum?}} = \sum_{i \geq a} \frac{i}{a} P(X=i) + \underbrace{\sum_{i<a} \frac{i}{a} P(X=i)}_{\Rightarrow\ \geq 0}$$

if $i \geq a$  $\frac{E(X)}{a} \geq \sum_{i \geq a} \frac{i}{a} P(X=i)$

$\geq \sum_{i \geq a} P(X \geq a) = P(X(s) \geq a)$

P493 # 38   7.4

装瓶场装汽水 (a bottling plant)   $X$: # of cans filled in a day   ($X \geq 0 \Rightarrow$ Markov good)

Given: $E(X) = 10{,}000$   $Var(X) = 1{,}000$

a) compute a upper bound on the Prob that $X \geq 11{,}000$   Markov  $P(X \geq a) \leq \frac{E(X)}{a}$

b) compute a lower bound on the Prob that $9{,}000 \leq X \leq 11{,}000$   Chebyshev  $P(|X - E(X)| \geq r) \leq \frac{Var(X)}{r^2}$

Solution: (a) $P(X \geq 11{,}000) \leq \frac{E(X)}{11000} = \frac{10}{11}$ (upper bound)

(b) $r = 1000$ (between 9000 and 11,000 where $E(X) = 10{,}000$)

$P(|X - 10{,}000| \geq 1000) \leq \frac{1000}{(1000)^2} = \frac{1}{1000} = 0.001$

$P(|X - 10{,}000| < 1000) = 1 - P(|X - 10{,}000| \geq 1000) > 0.999$

#29. P498.

$X$ on $S$   Prove: $Var(aX+b) = a^2 Var(X)$   $a, b \in \mathbb{R}$

$E(aX+b) = E(aX) + E(b) = aE(X) + b$

$$\begin{aligned}
Var(aX+b) &= E((aX+b)^2) - (E(aX+b))^2 \\
&= E(a^2X^2 + 2abX + b^2) - (aE(X)+b)^2 \\
&= a^2 E(X^2) + 2ab E(X) + b^2 - [a^2(E(X))^2 + 2abE(X) + b^2] \\
&= a^2 E(X^2) - a^2 (E(X))^2 \\
&= a^2 (E(X^2) - (E(X))^2) = a^2 Var(X)
\end{aligned}$$

#29 P493

$X_n$ R.V. $X_n$ = # of tails − # of heads  n fair coins flipped

a) $E(X_n)$  b) $Var(X_n)$

① E(# of tails) $X_i = \begin{cases} 1 & \text{if i-th is tail} \\ 0 & \text{o/w} \end{cases}$  $X = \sum_{i=1}^{n} X_i$  $E(X) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} 1 \cdot P = \frac{n}{2}$

Why can't we use:
n- bernoulli trails
$E(X) = np$
$Var(X) = np(1-p)$

不确定

a) $E(T) = \frac{n}{2}$  $H = n - T$

$E(T-H) = E(T-(n-T)) = E(2T-n) = 2E(T) - n = 2 \cdot \frac{n}{2} - n = 0$

b) $Var(2T-h) = 4Var(T)$ → $Var(T) = E(T^2) - (E(T))^2$

$E(T^2) = E((\sum T_i)^2)$ $\xrightarrow{\text{~~更简单的~~}}$ $\cancel{E}((nT_i)^{\cancel{2}}) = \cancel{n^2} E(\cancel{T_i}^2) = \cancel{\frac{n^2}{2}}$ can't

这只是个巧合. 假设 $P(T)=\frac{1}{3}$

老师法: $\frac{n}{3} + 2 \cdot \frac{n(n-1)}{9} = \frac{2n^2+n}{9}$

这种法: $\frac{n^2}{3}$  不再相等

$\therefore Var(T) = \frac{n^2}{2} - \frac{n^2}{4} = \frac{n^2}{4}$

$= E[(\sum_{i=1}^{n} T_i^2) + (\sum_{i \neq j} 2T_iT_j)]$

$Var(2T-h) = 4Var(T) = n^2$

$\begin{cases} E(T_i^2) = \frac{1^2}{2} + \frac{0^2}{2} = \frac{1}{2} \\ E(T_iT_j) = \frac{1}{4} \end{cases}$ (只有 $T_i=1, T_j=1$ contribute to E)

$= \sum_{i=1}^{n} E(T_i^2) + 2\sum_{i \neq j} E(T_iT_j)$

$= \sum_{i=1}^{n} \frac{1}{2} + 2 \cdot \sum_{i \neq j} \frac{1}{4} = \frac{n}{2} + 2 \cdot C(n,2) \cdot \frac{1}{4}$

$= \frac{n}{2} + \frac{n^2-n}{4} = \frac{n^2}{2}$

---

#29 P493

$X_n$ R.V. $X_n$ = # of tails − # of heads  n fair coins flipped

a) $E(X_n)$  b) $Var(X_n)$

① E(# of tails) $X_i = \begin{cases} 1 & \text{if i-th is tail} \\ 0 & \text{o/w} \end{cases}$  $X = \sum_{i=1}^{n} X_i$  $E(X) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} 1 \cdot P = \frac{n}{2}$

Why can't we use:
n- bernoulli trails
$E(X) = np$
$Var(X) = np(1-p)$

a) $E(T) = \frac{n}{2}$  $H = n - T$

$E(T-H) = E(T-(n-T)) = E(2T-n) = 2E(T) - n = 2 \cdot \frac{n}{2} - n = 0$

b) $Var(2T-h) = 4Var(T)$ → $Var(T) = E(T^2) - (E(T))^2$

$E(T^2) = E((\sum T_i)^2)$

② $\boxed{\therefore Var(T) = \frac{n^2}{2} - \frac{n^2}{4} = \frac{n^2}{4}}$

$= E[(\sum_{i=1}^{n} T_i^2) + (\sum_{i \neq j} 2T_iT_j)]$

$Var(2T-h) = 4Var(T) = n^2$

wrong ⇒ $Var(T) = \frac{n^2+n}{4} - \frac{n^2}{4}$

$\begin{cases} E(T_i^2) = \frac{1^2}{2} + \frac{0^2}{2} = \frac{1}{2} \\ E(T_iT_j) = \frac{1}{4} \end{cases}$ (only $T_i=1, T_j=1$ contribute to E)

③ $npq = n \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{n}{4}$

$= \sum_{i=1}^{n} E(T_i^2) + 2\sum_{i \neq j} E(T_iT_j)$

Does this mean we can use n-bernoulli trails?

$= \sum_{i=1}^{n} \frac{1}{2} + 2 \cdot \sum_{i \neq j} \frac{1}{4} = \frac{n}{2} + 2 \cdot C(n,2) \cdot \frac{1}{4}$

$\boxed{= \frac{n}{2} + \frac{n^2-n}{4} = \frac{n^2}{2}}$ wrong,

suppose unfair coin $P(T)=\frac{1}{3}$ $P(H)=\frac{2}{3}$

① $\frac{n}{2} + \frac{n^2-n}{4} = \frac{n^2+n}{4}$

$E(T^2) = \sum_{i=1}^{n} \frac{1}{3} + 2 \cdot \binom{n}{2} \cdot \frac{1}{9} = \frac{n^2+2n}{9}$

$(E(T))^2 = \frac{n^2}{9}$  $E(T^2) - (E(T))^2 = \frac{2n}{9}$

$= Var(T) = n \cdot pq = n \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{2n}{9}$ (Still hold)

10.2 P 668 #70

A bipartite graph $G=(V,E)$ is n-regular graph (every vertex has degree n) for $n \in N^*$. $(V_1, V_2)$ is a bipartition of $V$.

Prove : $|V_1| = |V_2|$

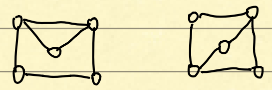
Proof : let $|V_1| = n_1$, $|V_2| = n_2$

Count edge in 2 ways : #edge $\begin{cases} \text{Count edges by endpoint in } V_1: & n_1 \cdot n \\ \text{Count edges by endpoint in } V_2: & n_2 \cdot n \end{cases}$

$n_1 \cdot n = n_2 \cdot n \Rightarrow n_1 = n_2$

Find a pair of nonisomorphic graphs with the same degree sequence (defined in the preamble to Exercise 36 in Section 10.2) such that one graph is bipartite, but the other graph is not bipartite.

P678 #67



10.4 #28 P691 (induction + contradiction)

28. show that every connected graph with n vertices has at least n-1 edges

Proof : induction on $n$, # of vertices

Basis : $n=1$ single vertex 0 edge = n-1

IH : For $n$ vertices $\Rightarrow$ n-1 edge

For contradiction : Assume IH is true

p: Let G be connected with n+1 vertices and less then n edges, $n \geq 1$

since $\underbrace{\sum_{v \in V} \deg(v) = 2|E|}_{\text{handshake}}$ $\sum_{v \in V} \deg(v) < 2n < 2(n+1)$

$\Rightarrow$ some vertex $V_0$ has degree $< 2$

还有什么没用上吗？ $\Rightarrow$ connected

since G is connected $\Rightarrow$ $V_0$ is not isolated $\Rightarrow \deg(V_0) \neq 0$

$\therefore \deg(V_0) = 1$ Remove $V_0$ and its edge from G. get $G'$, $G'$ is still connected

$\Downarrow$

p $\Rightarrow$ $G'$ has n vertices and has less than n-1 edge

p 和 IH 冲突

10.2 #49 P667

# subgraph of $K_3$? at least one vertex

- 1 vertex $\binom{3}{1} = 3$
- 2 vertex $\binom{3}{2} = 3 \Rightarrow$ then choose edge $\Rightarrow$ 2 way in each case to include an edge

  $\Rightarrow 6$ subgraph
- 3 vertex $\binom{3}{3} = 1 \Rightarrow$ edge $\begin{cases} \text{exist} \\ \text{not exist} \end{cases} \Rightarrow 2^3 = 8$ way

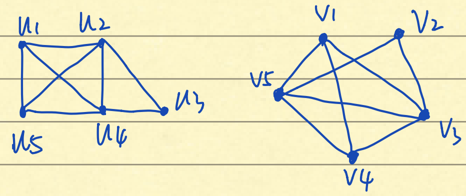
total : $3+6+8 = 17$

P667 #55 10.2

How many vertices ? regular graph of degree 4 with 10 edges

$\sum_{v \in V} \deg(v) = 2|E| \Rightarrow 4 \cdot n = 2 \cdot 10 \Rightarrow n=5 \Rightarrow K_5$

10.3 #38 P676    同构? X




10.4 #42 P691

G. connected  G is union of $G_1, G_2$  Prove $G_1, G_2$ have at least one common vertex

Proof: for contradiction  Assume no common vertex

Then there can be no path from $G_1$ to $G_2$ $\Rightarrow\Leftarrow$ contradiction with G is connected

Prove: Every tree is bipartite

Proof: Induction on # vertices n.

Basis: $n=2$ o–o  $n=1$ o  bipartite

IH: every tree on n vertices is bipartite

NTS: every tree on n+1 vertices is bipartite

Inductiv Step: let T be a tree on n+1 vertices

By HW8, has vertex of degree 1 $\Rightarrow$ leaf $\Rightarrow$ call it V $\Rightarrow$ let U be the neighbour $(v,u)$

Remove v from T $\Rightarrow$ T-V is a tree (By verbal argument)

T-V $\Rightarrow$ has n vertices and is bipartite (By IH)

$\Rightarrow$ $T-V = A \cup B$ (一个划分) (u in $A \cup B$, v not in $A \cup B$)

if $u \in A$ then $A \cup \{B \cup \{v\}\}$ is a bipartite for T

o/w GB ... $B \cup \{A \cup \{v\}\}$ is ........ T

10.4 #35 P691

v. endpoint of a cut edge. Show v is a cut vetex iff v is not pendant

Prove: v cut vertex iff $deg(v) \neq 1$

Proof: ① $(\Rightarrow)$ contraposition $deg(v)=1 \Rightarrow v$ is not a cut vertex

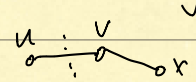
Let $(u,v)$ be a cut edge in G. v endpoint when $(u,v)$ removed G is disconnected into 2 CC's $C_1$ and $C_2$

WLOG $u \in C_1$ and $v \in C_2$ If $deg(v)=1$ in G $deg(v)$ is now 0, and v is isolated
∴ removing v does not disconnected $C_1$, so v is not a cut vertex ■

② $(\Leftarrow)$ $deg(v) \neq 1 \Rightarrow v$ cut edge

Suppose $deg(v) \neq 1$ Then after removing cut edge $(u,v)$

$v \in C_2$ and there are other vertices in $C_2$ and at least one, say x, such that $(v,x)$ is an cut edge in $C_2$




∴ removing v leaves at least 2 CC's in G, one containing u and the other containg x ∴ v is cut vertex ■

11.1 # 11.15b P756

G simple graph with n vertices

Prove : G is a tree iff G undirected, no simple circuit, $|E|=|V|-1$

What's missing? ⇒ G is connected

Suppose G has k CC's through $G_1 \ldots G_k$ Goal : prove $k=1$
Since G has no simple circuit, each CC has no simple circuit
∴ every CC is a tree. (no circuit + connected)

Suppose each CC $G_i$ $n_i$ vertices, for each i, $i=1,\ldots,k$
Sum the # of edges from each CC :

$$|E| = \text{sum} = \sum_{i=1}^{k} n_i - 1 = n-k$$

$n-k = n-1 \Rightarrow k=1$ so we only have one CC. that is G
∴ G is connected ■

10.5 #28 P705

$K_{m,n}$ Euler Circuit ?
if m,n even
$m>0$, $n>0$

Euler Path ?
$K_{2,n}$ , n: odd
$K_{m,2}$ , m: odd
$K_{1,1}$
$K_{m,n}$ m,n even

#26 a)
$K_n$ Euler Circuit ? every vertex has even deg when n is odd.
#26 b)
$C_n$ : $n \geq 3$ all $n>3$
#27 Euler Path?
a) $K_n$ $n=2$
b) $C_n$

10.5 #34 P705

No. why not ? 8 vertices of degree 2 in G (a,c,e,g,i,k,n,l)
edge (a,d) (a,b) etc.

听不懂

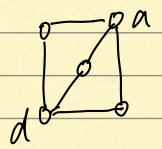
in G , 17 vertices
22 edges 24

b,d,h,f degree 3. not use one edge 4

p degree 4. not use two edge 2

P706 #48

Find a simple graph n vertices ($n \geq 3$) no Hamiton circuit but degree of each vertex $\geq \frac{n-1}{2}$

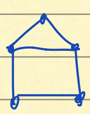



去 a,d 连通分支:3. $|a,d| = 2$ $3>2$.

∴ Dirac定理才把 lower bound 定义为 $\frac{n}{2}$

P706 #47

Dirac: $deg(v) \geq \frac{n}{2}$
Ore: $deg(v) + deg(u) \geq n$ u,v非邻接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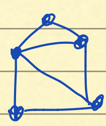
O: no
real: Yes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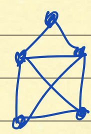
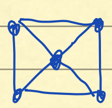
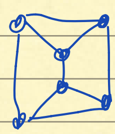
real: Yes


D: No
O: Yes
real: Yes


D: Yes
O: Yes
real: Yes


D: Yes
O: Yes
real: Yes

Prove $G=(V,E)$ is a tree with $n \geq 2$ vertices
Then $G$ has at least 2 vertices of degree 1

Proof: let $k$ be # of leaves in $G$ want to prove $k \geq 2$
By handshake: $2|E| = \sum_{v \in V} deg(v)$

$$2(n-1) = \sum_{v \in leaf} deg(v) + \sum_{v \notin leaf} deg(v)$$

$$2(n-1) \geq k + 2(n-k)$$

$$k \geq 2 \quad \blacksquare$$

$G=(V,E)$ simple, undirected $|V| = n$

Prove: if every vertex of $G$ has degree $\geq \frac{n}{2}$ Then $G$ is connected

Proof: Assume the contrary: $G$ is not connected
There is at least 2 non-empty CC's. Let them be $A_1$ and $A_2$
$G = A_1 \cup A_2$

- One of them has to have $\leq \frac{n}{2}$ vertices (o/w the total $> n$)
- Suppose $A_1$ has $\leq \frac{n}{2}$ vertices
- Since $A_1$ is not empty $\exists v \in A_1$ Since $A_1$ has $< \frac{n}{2}$ other than $V$

$\therefore$ $V$ can not has a degree $\geq \frac{n}{2}$