Tensor Product of Modules

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***Abstract*—This paper provides an explanation to what tensor product is and why it exists. One of the main reasons for this is plainness and neatness it provides when dealing with commutative algebra. We also give a brief treatment of rings and modules including submodules.**

***Index Terms*—algebra, rings, modules, tensor product, bilinear mappings, homomorphism**

Introduction

The first part of this paper is devoted to a quick review of basic terminology, definitions, and useful facts. After this we proceed to the second part which consists of tensor product definition and theorem of its existence and uniqueness.

Main Part

Notation and Terminology

A **commutative ring** A **with an identity element** is a set with two binary operations (addition and multiplication) such that:

1. A is an abelian group with respect to addition:
   a) $a + b \in A \quad \forall a, b \in A$.
   b) $a + (b + c) = (a + b) + c \quad \forall a, b, c \in A$.
   c) $\exists 0 \in A\colon a + 0 = 0 + a = a \quad \forall a \in A$.
   d) $\forall a \in A \quad \exists b \in A\colon a + b = b + a = 0 \quad (b = -a)$.
   e) $a + b = b + a \quad \forall a, b \in A$.
2. Multiplication is associative and distibutive over addition:
   a) $(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \forall a, b, c \in A$.
   b) $a \cdot (b + c) = a \cdot b + a \cdot c \quad \forall a, b, c \in A$.
3. Multiplication is commutative:
   $a \cdot b = b \cdot a \quad \forall a, b \in A$.
4. There exists an identity element:
   $\exists 1 \in A\colon 1 \cdot a = a \cdot 1 = a \quad \forall a \in A$.

Throughout this paper the word **ring** shall mean a commutative ring with and indentity element.

A **ring homomorphism** is a mapping $f\colon A \to B$, where A and B are rings such that:

1. f respects addition: $f(x + y) = f(x) + f(y)$.
2. f respects multiplication: $f(x \cdot y) = f(x) \cdot f(y)$.
3. f respects identity element: $f(1_A) = 1_B$.

Let A be a ring. An $\boldsymbol{A}$**-module** is a pair (M, μ) that contatins an abelian group M and a mapping $\mu\colon A \times M \to M$ (we shall write $\mu(a, x) = ax$ for short) such that:

1. $a(x + y) = ax + ay \quad \forall a \in A \quad \forall x, y \in M$.
2. $(a + b)x = ax + bx \quad \forall a, b \in A \quad \forall x \in M$.
3. $(ab)x = a(bx) \quad \forall a, b \in A \quad \forall x \in M$.
4. $1x = x \quad \forall x \in M$.

It is often written as M instead of (M, μ).

The following examples are of some help in order to understand the concept of modules:

1. If K is a field, then a vector space over K is identical to K-module.
2. If $K[x]$ is a polynomial ring, then $K[x]$-module is a vector space over K with a linear transformation.
3. Every abelian group is $\mathbb{Z}$-module with nx being equal to $\underbrace{x + \cdots + x}_{n} \quad (0x = 0, -nx = -(nx))$.

A **submodule** $\boldsymbol{M'}$ of M is a subgroup of M which is closed under multiplication by elements of A.

A **quotient of** $\boldsymbol{M'}$ by M is an A-module M/M' defined by $a(x + M') = ax + M'$. Notice that $ax + M' = bx + M' \Leftrightarrow ax - bx \in M'$.

Let M, N, P be A-modules. A mapping $f\colon M \to N$ is an $\boldsymbol{A}$**-module homomorphism** (or is $\boldsymbol{A}$**-linear**) if

1. $f(x + y) = f(x) + f(y) \quad \forall x, y \in M$.
2. $f(ax) = a \cdot f(x) \quad \forall a \in A \quad \forall x \in M$.

A mapping $g\colon M \times N \to P$ is a $\boldsymbol{A}$**-bilinear** if

1. $y \mapsto g(x, y)$ is A-linear $\forall x \in M$.
2. $x \mapsto g(x, y)$ is A-linear $\forall y \in N$.

A **free** $\boldsymbol{A}$**-module** $\boldsymbol{A^{(M \times N)}}$ is A-module such that:

1. Its elements can be represented as formal finite linear combinations $\sum\limits_{k=1}^{n} a_k \cdot (x_k, y_k)$ for $a_k \in A, x_k \in M, y_k \in N$.
2. There is a function $i\colon A \to A^{M \times N}$ such that for any module B and any set map $\varphi\colon A \to B$, there exists a unique module homomorphism $\Phi\colon A^{M \times N} \to B$ such that $\varphi = \Phi \circ i$.

Now we can proceed to stating the theorem.

Tensor Product

Theorem. *Let M, N be A-modules. Then there exists a pair (T, g) consisting of an A-module T and an A-bilinear mapping $g\colon M \times N \to T$ such that:*

1. *Given any A-module and any A-bilinear mapping $f\colon M \times N \to P$, there exists a unique A-linear $f'\colon T \to P$ such that $f = f' \circ g$.*
2. *If (T, g) and (T', g') are two pairs with this property, then there exists a unique isomorphism (bijective homomorphism) $j\colon T \to T'$ such that $j \circ g = g'$.*

The A-module T satisfying above properties is called the **tensor product** of M and N, and is denoted by $M \otimes_A N$.

Proof. (Existence.) Let C be a free A-module $A^{M\times N}$. The naive approach to contruct such pair (T, g) would be to define T as C and $g(m, n) = 1\cdot(m, n)$. In the strict sense, g would not be A-bilinear, since

$$\begin{aligned} g(m+m', n) &= 1\cdot(m+m', n) \\ &\quad + 0\cdot(m, n) + 0\cdot(m', n), \\ g(m, n) + g(m', n) &= 0\cdot(m+m', n) \\ &\quad + 1\cdot(m, n) + 1\cdot(m', n). \end{aligned}$$

$$\begin{tikzcd} & C \arrow[dd, dotted, "\exists! f'"] \\ M\times N \arrow[ur, "g"] \arrow[dr, "f"'] & \\ & P \end{tikzcd}$$

Thus we consider a submodule of C which is linearly generated by all of its elements such as

$$\begin{aligned} (x+x', y)&-(x, y)-(x', y), \\ (x, y+y')&-(x, y)-(x, y'), \\ (ax, y)&-a\cdot(x, y), \\ (x, ay)&-a\cdot(x, y). \end{aligned}$$

We shall denote it as D. Let $T = C/D$. Now let $g = \otimes$, where $\otimes\colon M\times N \to T$ is defined as follows:

$$m\otimes n = (m, n) + D.$$

It can be easily shown that $\otimes$ is A-linear:

$$\begin{aligned} m+m'\otimes n &= (m+m', n) + D, \\ m\otimes n + m'\otimes n &= (m, n) + D + (m', n) + D \\ &= (m, n) + (m', n) + D. \end{aligned}$$

By construction, $(m+m', n) - (m, n) - (m', n) \in D \Leftrightarrow$
$(m+m', n) + D = (m, n) + (m', n) + D \Leftrightarrow$
$m+m'\otimes n = m\otimes n + m'\otimes n$.
Similarly, $m\otimes n+n' = m\otimes n + m\otimes n'$. Now,

$$\begin{aligned} a\cdot m\otimes n &= (a\cdot m, n) + D, \\ a\cdot(m\otimes n) &= a\cdot(m, n) + D. \end{aligned}$$

Notice that $(am, n) - a\cdot(m, n) \in D \Leftrightarrow$
$(am, n) + D = a\cdot(m, n) + D \Leftrightarrow am\otimes n = a\cdot(m\otimes n)$.
Likewise, $m\otimes an = a\cdot(m\otimes n)$. Thus, $\otimes$ is A-linear:

$$\begin{tikzcd} & C \arrow[dr, "\pi"] \arrow[dd, dotted, "\exists! f'"] & \\ M\times N \arrow[ur, "g"] \arrow[rr, "\otimes"] \arrow[dr, "f"'] & & T = C/D \\ & P & \end{tikzcd}$$

By the fundamental homomorphism theorem, homomorphisms from C/D to P are isomorphic to homomorphisms $f'\colon C\to P$ as long as $D \subseteq \ker f'$. It can be easily seen that

$$\begin{aligned} f'((m+m', n) &- (m, n) - (m', n)) = \\ &= f(m+m', n) - f(m, n) - f(m', n) \\ &= 0 \\ f'((a\cdot m, n) &- a\cdot(m, n)) = \\ &= f(a\cdot m, n) - a\cdot f(m, n) \\ &= 0 \end{aligned}$$

Similar can be done for the second argument. Hence there exists a unique A-linear homomorphism $\Phi\colon C/D \to P$ such that $f = \Phi\circ\otimes$.

$$\begin{tikzcd} & C \arrow[dr, "\pi"] \arrow[dd, dotted, "\exists! f'"] & \\ M\times N \arrow[ur, "g"] \arrow[rr, "\otimes"] \arrow[dr, "f"'] & & T = C/D \arrow[dl, "\Phi"] \\ & P & \end{tikzcd}$$

(Uniqueness.) Keeping in mind the second property, we get a unique $j\colon T \to T'$ such that $g' = j\circ g$. On the other hand, we have $j'\colon T'\to T$ such that $g = j'\circ g'$. Each of the compositions $j\circ j'$ and $j'\circ j$ must be the identity, therefore j is an isomorphism.

$$\begin{tikzcd} & T \arrow[dd, bend right, "j"'] \\ M\times N \arrow[ur, "g"] \arrow[dr, "g'"'] & \\ & T' \arrow[uu, bend right, "j'"'] \end{tikzcd}$$

□

Conclusion

We introduced a concept of tensor product. It is a multitool in many areas of commutative algebra capable of allowing us to study certain non linear mappings by transforming them first into linear ones, to which, for instance, we can apply linear algebra. S

References

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