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Lab Report 1

The report was started by downloading the Newton_sqrt.txt file, which is a file in Matlab that allows the user to input a number that the file will then find the square root of using the Newton square root technique.

1. **If the method works as desired, what would you expect to happen to the "log-errors"? Does the plot show this expected behaviour? Why did we choose to plot the "log-errors" and not the errors themselves?**

The errors show us how far we are away from the actual value, so after an iteration we are more likely to be closer to the actual value. This is why I would expect the errors and therefore the log errors to decrease and therefore the graph to show a decrease in the log errors. We log the errors as opposed to actually plotting the errors as they are easier to scale on a graph, because if an initial guess gives an errors that is a long way off then the graph will be poorly scaled.

2. **Experiment with different values of *a*. Does the number of iterations required vary significantly with respect to *a*?**

a	Number of Iterations
729	9
7290	11
72900	13
729000	15
7290000	16
72900000	18

The initial number, 729, was increased by a scale of 10 and then run using the Newton square root method. The number of iterations was then recorded in the table above. It can be seen that there is a clear increase in the number of iterations as the value of a tends to increase.

3. **Prove (algebraically) that, if *x_i* equals the square-root of *a*, then *x_(i+1) = x_i*.**

$$x^2 = a$$
$$x_{i+1} = \frac{1}{2}\left(x_i + \frac{a}{x_i}\right)$$
$$x = \frac{1}{2}\left(x + \frac{a}{x}\right)$$

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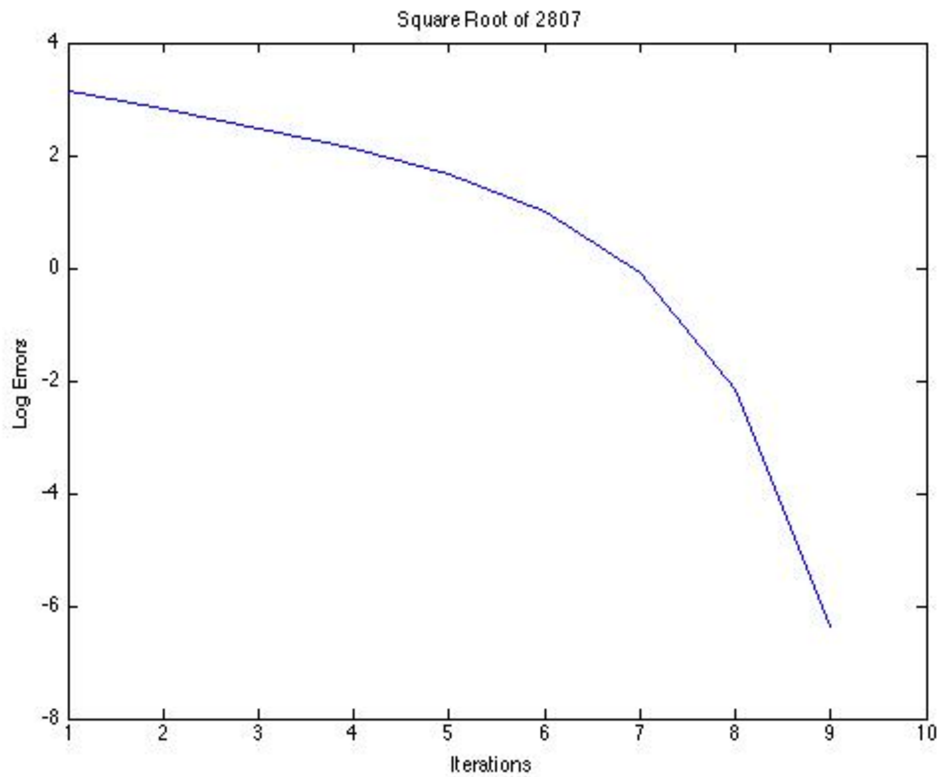
$$x = \frac{x}{2} + \frac{a}{2x}$$
$$2x = x + \frac{a}{x}$$
$$x = \frac{a}{x}$$
$$x^2 = a, x = \sqrt{a}$$

Therefore they are both equal.

4. **<u>Take the last five digits (in order) of your university registration number. Find the square-root of this five-digit number by using the "Newton_sqrt.m" program. Annotate and save the plot.</u>**

My last 5 reg number digits: 02807=2807



After a number of 10 iterations the program found that the square root was 52.981128715798420.

3. Newton's Method for cube-roots

1. **<u>Write down (in algebraic form) the recurrence relation that is being used in this program.</u>**

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The recurrence relation being used in this method is:

$$x_{i+1} = \left(2x_i + \frac{a}{x_i^2}\right)\frac{1}{3}$$
$$x = \left(2x + \frac{a}{x^2}\right)\frac{1}{3}$$
$$x = \frac{2}{3}x + \frac{a}{3x^2}$$
$$3x^3 = 2x^3 + a$$
$$a = x^3$$

2. **<u>For any given (positive) value of *a*, suppose that (for some value of *i*), *x_i = cube_root(a)* + *h*, where *h* is small. Show (algebraically) that, then, the difference between *x_(i+1)* and *cube_root(a)* is approximately a multiple of *h^2*.</u>**

$$x_i = \sqrt[3]{a} + h \qquad\qquad x_{i+1} = \left(2x_i + \frac{a}{x_i^2}\right)\frac{1}{3}$$
$$\frac{1}{3}(2(\sqrt[3]{a} + h) + \frac{a}{(\sqrt[3]{a} + h)^2} - \sqrt[3]{a}$$
$$\frac{1}{3}\left(\frac{2(\sqrt[3]{a} + h)^3 + a}{(\sqrt[3]{a} + h)^2}\right) - \sqrt[3]{a}$$
$$\frac{2}{3}(\sqrt[3]{a} + h) + \frac{a}{3(\sqrt[3]{a} + h)^2} - \sqrt[3]{a}$$
$$-\frac{1}{3}\sqrt[3]{a} + \frac{2}{3}h + \frac{1}{3}\frac{a}{\sqrt[3]{a} + h^2}$$
$$\frac{1}{3}\left(2h - \sqrt[3]{a} + \frac{a}{(\sqrt[3]{a} + h)}\right)$$
$$\frac{1}{3}\left(\frac{(h + \sqrt[3]{a})^2(2h - \sqrt[3]{a}) + a}{(\sqrt[3]{a} + h)(\sqrt[3]{a} + h)}\right)$$
$$\frac{1}{3}\left(\frac{2h + (4\sqrt[3]{a^2} - \sqrt[3]{a})}{(h + \sqrt[3]{a})^2}\right)h^2$$

As h is small when we expand this using taylor series the h on the top and the bottom of the division can be taken out thus leaving the h^2 as the multiplier.

3. **<u>Experiment (using the program) with different values of *a*. Does the number of iterations required vary significantly with respect to *a*?</u>**

a	Number of Iterations
4096	17
40960	21
409600	24

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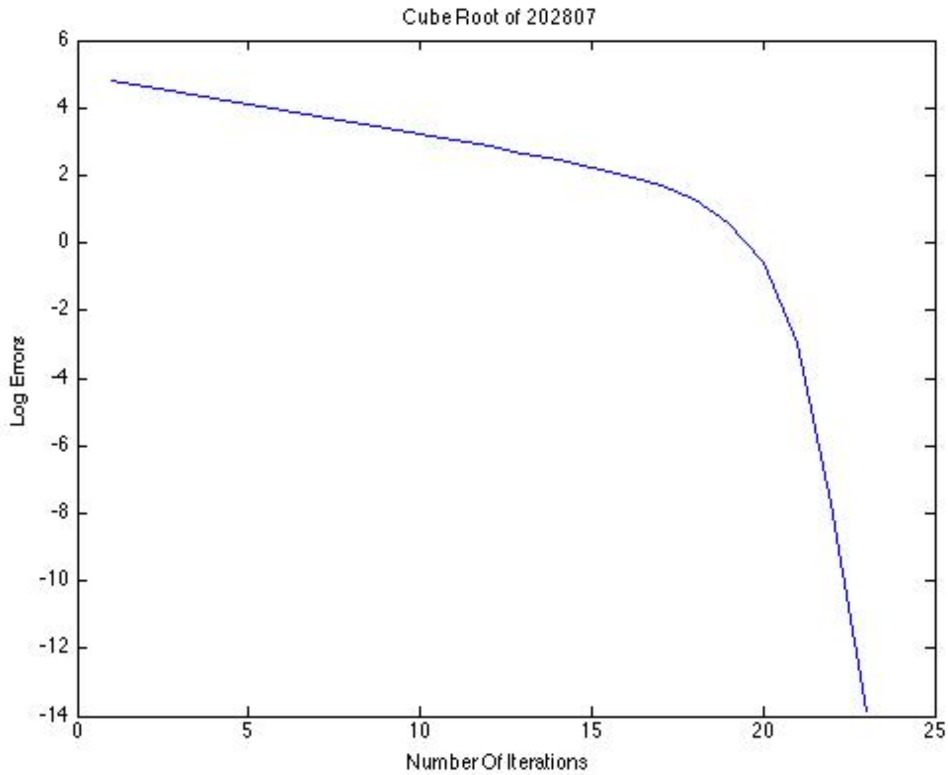
409	13
40	9

Using the same scaling as before we can see that the number of iterations increases as the value of a increases.

4. **Take the last six digits (in order) of your university registration number. Find the cube-root of this six-digit number by using the "Newton_cube.m" program. Annotate and save the plot.**

My last six reg digits: 202807



Halley's Method for square-roots

1. **Write down (in algebraic form) the recurrence relation that is being used in this program.**

$$x_{i+1} = \frac{x_i(x_i^2 + 3a)}{(3x_i^2 + a)}$$
$$x = \frac{x^3 + 3ax}{3x^2 + a}$$
$$3x^3 + ax = x^3 + 3ax$$

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$$\boldsymbol{2x^3 = 2ax}$$
$$\boldsymbol{x^2 = a}$$
$$\boldsymbol{x = \sqrt{a}}$$

2. **For any given (positive) value of *a*, suppose that (for some value of *i*), *x_i = sqrt(a)* + *h*, where *h* is small. Show (algebraically) that, then, the difference between *x_(i+1)* and *sqrt(a)* is approximately a multiple of *h^3*.**

$$x_i = \sqrt{a} + h \qquad x_{i+1} = \frac{x_i(x_i^2 + 3a)}{(3x_i^2 + a)}$$

$$\frac{(\sqrt{a} + h)((\sqrt{a} + h)^2 + 3a)}{(3(\sqrt{a} + h)^2 + a)} - \sqrt{a}$$

$$\frac{(\sqrt{a} + h)((\sqrt{a} + h)^2 + 3a) - \sqrt{a}(3(\sqrt{a} + h)^2 + a)}{(3(\sqrt{a} + h)^2 + a)}$$

$$\frac{(\sqrt{a} + h)(4a + h^2 + 2\sqrt{a}h) - \sqrt{a}(3(a + h^2 + 2\sqrt{a}h) + a)}{(3(\sqrt{a} + h)^2 + a)}$$

$$\frac{4a^{\frac{3}{2}} + \sqrt{a}h^2 + 2ah + 4ah + h^3 + 2\sqrt{a}h^2 - 4a^{\frac{3}{2}} - 3\sqrt{a}h^2 - 6ah}{(3(\sqrt{a} + h)^2 + a)}$$

$$\frac{h^3}{(3(\sqrt{a} + h)^2 + a)}$$

The denominator can then be expanded using Taylor series and then the small h values can be eliminated thus leaving it a multiple of h^3.

3. **For three significantly different values of *a*, compare the number of iterations required by Halley's Method and Newton's Method to find the square-root. Can you draw any conclusions?**

Value of a	**Number of Iterations using Newton's Method**	**Number of Iterations using Halley's Method**
729	9	7
72900	13	9
7290000	16	11

From the table above it is clear to see that the number of iterations using Halley's Method is consistently less than the number of iterations for Newton's Method.

4. **Take the last five digits (in order) of your university registration number. Find the square-root of this number by using the "Halley_sqrt.m" program. Annotate and save the plot.**

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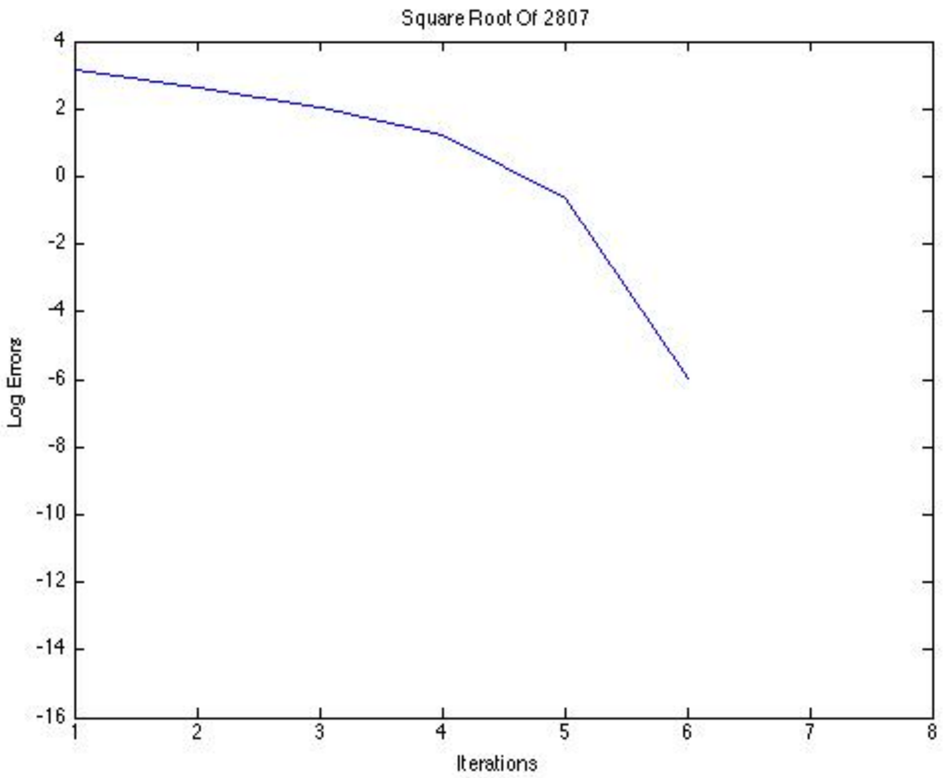
My last 5 reg number digits: 02807=2807



5. Halley's Method for cube-roots

1. **<u>Write down (in algebraic form) the recurrence relation that is being used in this program</u>**

$$x_{i+1} = \frac{x_i(x_i^3 + 2a)}{(2x_i^3 + a)}$$
$$x = \frac{x(x^3 + 2a)}{(2x^3 + a)}$$

2. **<u>Prove (algebraically) that, if *x_i* equals the cube-root of *a*, then *x_(i+1)* = *x_i*.</u>**

$$x_i = \sqrt[3]{a} \qquad x = \frac{x(x^3 + 2a)}{(x^3 + a)}$$
$$2x^4 + xa = x^4 + 2xa$$
$$x^4 = xa$$
$$x^3 = a$$
$$x = \sqrt[3]{a}$$

Therefore they are both equal

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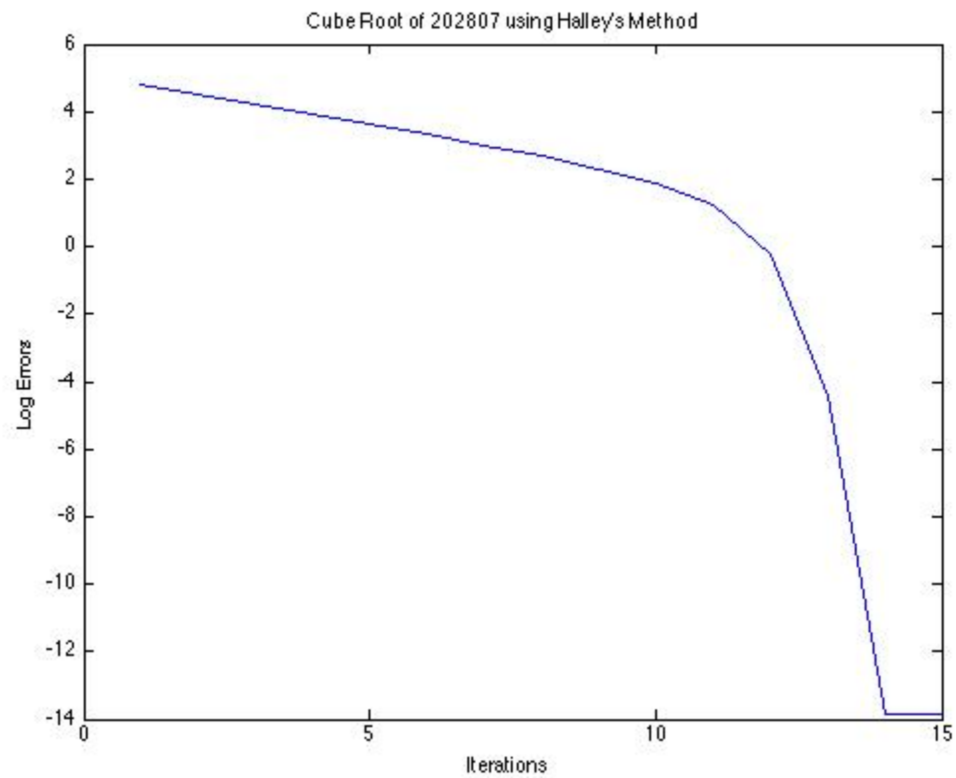
3. **<u>For three significantly different values of *a*, compare the number of iterations required by Halley's Method and Newton's Method to find the cube-root. Can you draw any conclusions?</u>**

Value of a	**Number of Iterations using Newton's Method**	**Number of Iterations using Halley's Method**
40	9	6
4096	17	11
409600	24	15

Just like the methods for finding the square root, the cube root follows a pattern of Halley's method having a lower number of iterations than Newton's Method.

4. **<u>Take the last six digits (in order) of your university registration number. Find the cube-root of this number by using the "Halley_cube.m" program. Annotate and save the plot.</u>**



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1. **<u>Using only the software provided for this lab and no other (i.e. you are not allowed to modify any programs except to change the value of *a*), find the eighteenth root of the number you obtain by raising</u>**

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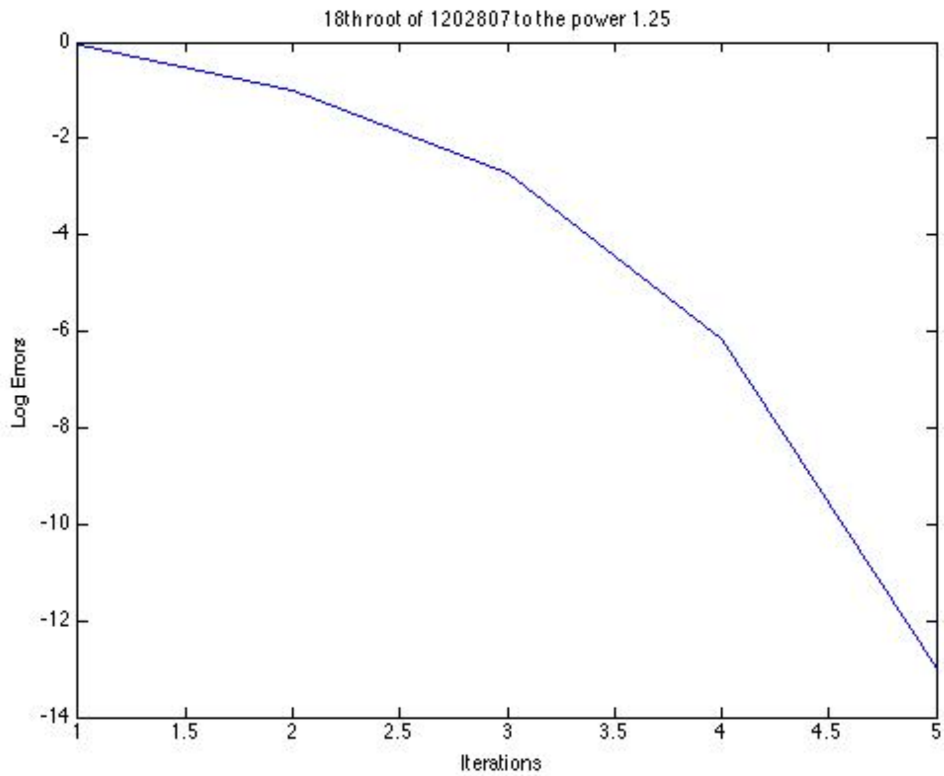
your entire university registration number to the power 1.25, and explain how you found the eighteenth root.

The input, a, can be changed to find the 18th root of a certain number for both the square root as well as the cubic root if we first take the square root:

$$x^{18} = a$$
$$\sqrt[9]{x^{18}} = \sqrt[9]{a}$$
$$x^2 = a^{\frac{1}{9}}$$

So the a that will be used in this section is:

$$a = 1202807$$
$$x^2 = 1202807^{\frac{1.25}{9}}$$



This gives a final answer of 2.643844022038778 after 5 iterations.

A similar alteration can be performed on the cube root function to get a similar answer:

$$x^{18} = a$$

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$$\sqrt[6]{x^{18}} = \sqrt[6]{a}$$
$$x^3 = a^{\frac{1}{6}}$$

This would give an a value of

$$a = 1202807$$
$$x^3 = 1202807^{\frac{1.25}{6}}$$