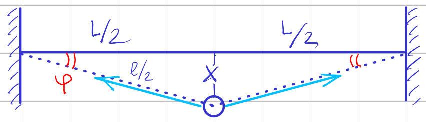
Svārstības

1

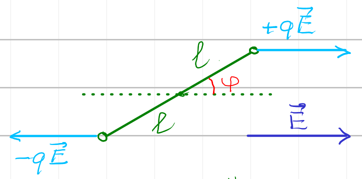


$F_{rez} = 2F\sin\varphi$; $\quad \sin\varphi \doteq \tan\varphi = \frac{2x}{L}$

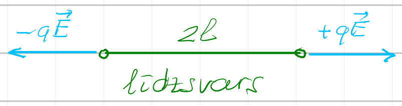
$m\ddot{x} = -4\frac{F}{L}x \rightsquigarrow \ddot{x} + \left(2\sqrt{F/mL}\right)^2 x = 0$

$x(t) = x_0 \cos 2t\sqrt{F/mL}$; $\quad v_{max} = x_0\omega = \underline{\underline{2x_0\sqrt{F/mL}}}$

2



$\rightsquigarrow$



$\vec{M} = J\ddot{\vec{\varphi}} \rightsquigarrow \cancel{2}(qE\ell\sin\varphi) = -\cancel{2}m\ell^2\ddot{\varphi}$

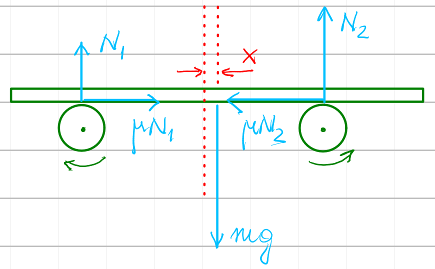
$\varphi \ll 1 \Rightarrow \sin\varphi \doteq \varphi$

$qE\varphi + m\ell\ddot{\varphi} = 0 \rightsquigarrow \ddot{\varphi} + \frac{qE}{m\ell}\varphi = 0$

$x = \ell\sin\varphi \doteq \ell\varphi \Rightarrow \ddot{x} + \frac{qE}{m\ell}x = 0$

$v_0 = x_0\omega = x_0\sqrt{\frac{qE}{m\ell}} \rightsquigarrow \underline{\underline{m = \left(\frac{x_0}{v_0}\right)^2\frac{qE}{\ell}}}$

3



$$\begin{cases} N_1 + N_2 = mg \\ \mu(N_1 - N_2) = m\ddot{x} \\ N_2 L = mg\left(L/2 + x\right) \end{cases}$$

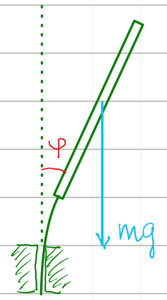
$N_2 = mg\left(\frac{1}{2} + \frac{x}{L}\right)$

$N_1 = mg\left(\frac{1}{2} - \frac{x}{L}\right)$

$\mu\cancel{m}g \cdot \frac{2x}{L} + \cancel{m}\ddot{x} = 0 \rightsquigarrow \ddot{x} + \frac{2\mu g}{L}x = 0$

$\underline{\underline{\omega = \sqrt{\frac{2\mu g}{L}}}}$

4



$$\vec{M} = J\ddot{\vec{\varphi}} \rightsquigarrow mg\frac{L}{2}\sin\varphi - k\varphi = \frac{mL^2}{3}\ddot{\varphi}$$

$$\varphi \ll 1 \Rightarrow \sin\varphi \doteq \varphi \Rightarrow \left(k - mg\frac{L}{2}\right)\varphi + \frac{mL^2}{3}\ddot{\varphi} = 0$$

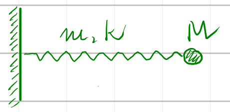
$$\ddot{\varphi} + \left(\frac{3k}{mL^2} - \frac{3g}{2L}\right)\varphi = 0 \Rightarrow \omega_{\uparrow} = \sqrt{\frac{3}{L}\left(\frac{k}{mL} - \frac{g}{2}\right)}$$

Ja plāksnīte ir iestiprināta otrādi, tad smaguma spēka radītais moments ir vērsts tajā pašā virzienā kā plāksnītes radītais moments, un

$$\omega_{\downarrow} = \sqrt{\frac{3}{L}\left(\frac{k}{mL} + \frac{g}{2}\right)}.$$

$$\frac{T_{\uparrow}}{T_{\downarrow}} = \frac{\omega_{\downarrow}}{\omega_{\uparrow}} = \underline{\underline{\sqrt{\frac{2k + mgL}{2k - mgL}}}}.$$

5



Assumption $m \ll M$ tells us that stretching of the spring is still linear. That means that velocity of an element of the spring is proportional to its position from the fixed end: $v = v_0\, x/L$. Kinetic energy of the spring is then

$$\int_0^L \frac{m\,dx}{L} \cdot \frac{v_0^2 x^2}{2L^2} = \frac{m v_0^2}{2L^3} \cdot \frac{L^3}{3} = \frac{m}{3} \cdot \frac{v_0^2}{2}$$

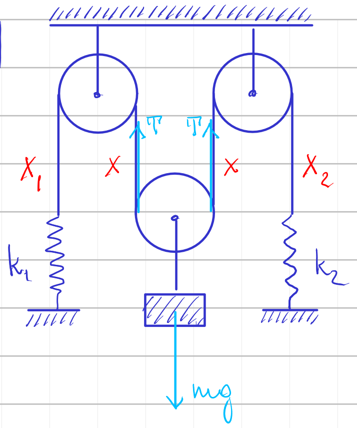
and the total kinetic energy of the system is

$$\left(M + \frac{m}{3}\right)\frac{v_0^2}{2}.$$

That means that the system has an effective mass $\mu = M + \frac{1}{3}m$ and so the frequency of its oscillations

$$\omega = \sqrt{\frac{k}{\mu}} = \underline{\underline{\sqrt{\frac{k}{M + \frac{1}{3}m}}}}.$$

6



Diega saglabāšanās: $\ddot{x}_1 + 2\ddot{x} + \ddot{x}_2 = 0$

II NL: $2T - mg = m\ddot{x}$

Huka l.: $T = k_1 x_1 = k_2 x_2$

$$\begin{cases} 2k_1 x_1 - mg = m\ddot{x} \\ 2k_2 x_2 - mg = m\ddot{x} \end{cases}, \quad \begin{cases} x_1 - \frac{mg}{2k_1} = \frac{m}{2k_1}\ddot{x} \\ x_2 - \frac{mg}{2k_2} = \frac{m}{2k_2}\ddot{x} \end{cases}$$

$$\xi \stackrel{def}{=} x_1 + x_2 \implies \ddot{x} = -\frac{1}{2}\ddot{\xi}$$

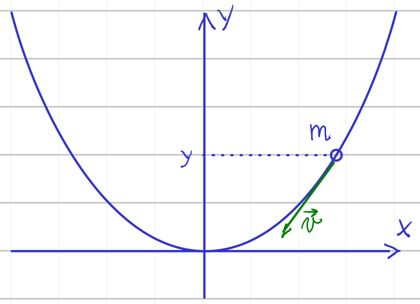
$$\xi - \frac{mg}{2}\left(\frac{1}{k_1} + \frac{1}{k_2}\right) = \frac{m}{2}\left(\frac{1}{k_1} + \frac{1}{k_2}\right)\cdot\left(-\frac{1}{2}\ddot{\xi}\right)$$

$$\ddot{\xi} + \left(\frac{4}{m}\frac{k_1 k_2}{k_1 + k_2}\right)\xi = 2g$$

The cable will always be taught if $|\ddot{x}| < g$, so $a_{max} = \omega^2 A = g$ and hence

$$\underline{\underline{A = \frac{g}{\omega^2} = \frac{mg}{4}\cdot\frac{k_1 + k_2}{k_1 k_2}}}.$$

7



As velocity is always directed along the curve, it is natural to parametrise this curve with an arc length.
Kinetic energy of the particle will then be $W_k = \frac{1}{2}m\dot{s}^2$ and potential energy $W_p = mgy$.
On the other hand, taking into account that the motion is harmonic, $W_p = \frac{1}{2}ks^2$. Thus,

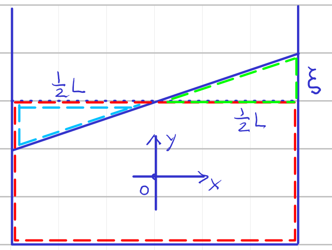
$$mgy = \frac{ks^2}{2} \leadsto s = \sqrt{\frac{2mgy}{k}}.$$

Differentiating with respect to x, we get $mgy' = kss'$ and, taking into account that $ds^2 = dx^2 + dy^2$ or, equivalently, $s' = \sqrt{1 + y'^2}$, we get that $mgy' = \sqrt{2kmgy}\sqrt{1+y'^2}$.

$$y'^2 = \underbrace{\frac{2k}{mg}}_{=\varkappa} y(1 + y'^2) \leadsto y'^2 = \frac{\varkappa y}{1 - \varkappa y}.$$

This equation describes a cycloid.

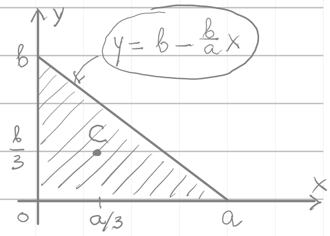
8



As we are interested in the motion of the whole bulk of water, we might start with considering the centre of mass. To find its coordinates, we split the bulk into three parts (red, green and blue) and choose the origin to be at the com of the unperturbed (red) bulk. The blue prism is removed from the bulk, so its mass is negative.

Theorem The com of a triangle is located at the intersection of the medians. Medians are split by this point in 2:1 ratio measured from the vertex.

! Corrolary For a right-angle triangle (see fig.) the com has coordinates $(\frac{1}{3}a; \frac{1}{3}b)$.



Proof For a uniform planar object, its com has coordinates

$$x_{com} = \frac{\iint x\,dx\,dy}{\iint dx\,dy} \quad \text{and} \quad y_{com} = \frac{\iint y\,dx\,dy}{\iint dx\,dy}.$$

Consider x coordinate of com of the triangle.

$$x_{com} = \frac{\iint x\,dx\,dy}{ab/2} = \frac{2}{ab}\int_{x=0}^{x=a} dx\, x \int_{y=0}^{y=b-\frac{b}{a}x} dy$$

$$= \frac{2}{ab}\int_0^a x\left(b - \frac{b}{a}x\right)dx = \frac{2}{ab}\left(\frac{bx^2}{2} - \frac{b}{a}\frac{x^3}{3}\right)_0^a$$

$$= 2a\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{a}{3}.$$

Along the same lines, due to symmetry,

$$y_{com} = \frac{b}{3}.$$

∎

According to the theorem, the com of the green and blue triangles have coordinates $(\pm\frac{1}{3}L; \frac{1}{2}h \pm \frac{1}{3}\xi)$. The mass of the bulk of water (red) is $\varrho w h L$ and that of the blue and the green prisms is $\pm\frac{1}{4}\varrho w \xi L$.

$$x_{com} = \frac{2 \cdot 1/4\, \varrho w \xi L \cdot 1/3\, L}{\varrho w h L} = \frac{\xi L}{6h}; \quad y_{com} = \frac{2 \cdot 1/4\, \varrho w \xi L \cdot 1/3\, \xi}{\varrho w h L} = \frac{\xi^2}{6h}$$

$$W_p(\xi) = m g y_{com} = \frac{\varrho g w h L}{3h}\cdot\frac{\xi^2}{2} = \frac{\varkappa \xi^2}{2}$$

$$W_k(\xi) = \frac{m(\dot{x}_{com}^2 + \dot{y}_{com}^2)}{2} = \frac{m\dot{\xi}^2}{72h^2}\left(L^2 + \cancel{4\xi^2}^{\to 0}\right) = \left[\xi \ll L\right]$$

$$= \frac{\varrho w L^3}{36h}\cdot\frac{\dot{\xi}^2}{2} = \frac{\mu\dot{\xi}^2}{2}$$

$$\omega = \sqrt{\frac{\varkappa}{\mu}} = \sqrt{\frac{\cancel{\varrho} g \cancel{w} L}{\cancel{3}}\cdot\frac{36h}{\cancel{\varrho w} L^3}} = \sqrt{\frac{12gh}{L^2}} \; ; \quad \underline{\underline{T = \pi\sqrt{\frac{L^2}{3gh}}}}$$

Linearisation gives us $T^{-2} = \frac{3g}{\pi^2 L^2} h$.

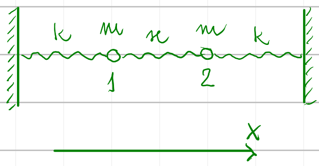
h/mm	30	50	69	88	107	124	142
T^{-2}/s^{-2}	0,316	0,510	0,718	0,857	1,00	1,21	1,49

Calculated equation is $T^{-2} = \underbrace{(9{,}93\ \mathrm{s}^{2}\cdot\mathrm{m}^{-1})}_{A}\, h$

$$\frac{3g}{\pi^2 L^2}\cdot\frac{1}{A} = 1{,}3 \sim 1.$$

Relative error of T^{-2} is about 30% so for T it will be about 15%.

9



$$\begin{cases} 1: & m\ddot{x}_1 = -kx_1 - \varkappa(x_1 - x_2) \\ 2: & m\ddot{x}_2 = -kx_2 - \varkappa(x_2 - x_1) \end{cases}$$

$(1)+(2)$: $m(\ddot{x}_1 + \ddot{x}_2) = -k(x_1 + x_2) \Rightarrow m\ddot{\xi}_1 + k\xi_1 = 0$

$(1)-(2)$: $m(\ddot{x}_1 - \ddot{x}_2) = -k(x_1 - x_2) - 2\varkappa(x_1 - x_2) \Rightarrow m\ddot{\xi}_2 + (k + 2\varkappa)\xi_2 = 0$

$$\begin{cases} \xi_1 = A_1\cos\omega_1 t + B_1\sin\omega_1 t & \left[\omega_1^2 = k/m\right] \\ \xi_2 = A_2\cos\omega_2 t + B_2\sin\omega_2 t & \left[\omega_2^2 = \frac{k+2\varkappa}{m} = \omega_1^2 + \frac{2\varkappa}{m}\right] \end{cases}$$

$$\left.\begin{aligned} \xi_1(0) = A &\Rightarrow A_1 = A \\ \dot{\xi}_1(0) = 0 &\Rightarrow \omega_1 B_1 = 0 \Rightarrow B_1 = 0 \end{aligned}\right\} \xi_1(t) = A\cos\omega_1 t$$

$$\left.\begin{aligned} \xi_2(0) = A &\Rightarrow A_2 = A \\ \dot{\xi}_2(0) = 0 &\Rightarrow B_2 = 0 \end{aligned}\right\} \xi_2(t) = A\cos\omega_2 t$$

$$\varkappa \ll k \Rightarrow \omega_2 = \omega_1\sqrt{1 + \frac{2\varkappa}{k}} \doteq \omega_1\left(1 + \frac{\varkappa}{k}\right) = \omega_1 + 2\varepsilon.$$

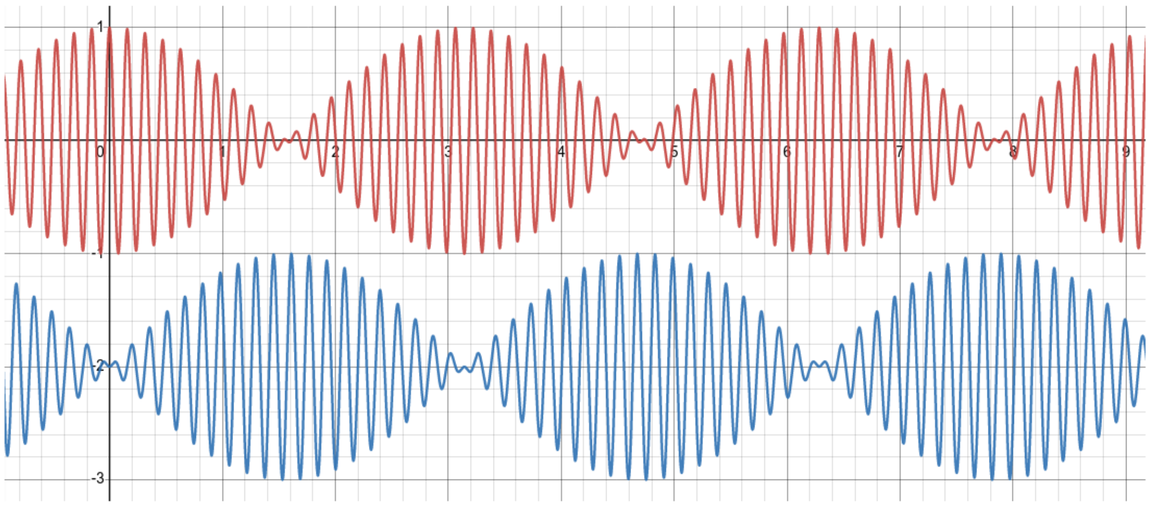
$$x_1(t) = \frac{\xi_1(t) + \xi_2(t)}{2} = A\cos\frac{(\omega_1 + \omega_2)t}{2}\cos\frac{(\omega_1 - \omega_2)t}{2}$$

$$\underline{\underline{= A\cos\left[(\omega_1 + \varepsilon)t\right]\cos\varepsilon t.}}$$

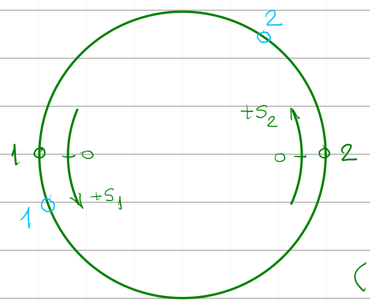
$$x_2(t) = \frac{\xi_1-\xi_2}{2} = A\sin\frac{(\omega_1+\omega_2)t}{2}\sin\frac{(\omega_2-\omega_1)t}{2}$$

$$= \underline{\underline{A\sin[(\omega_1+\varepsilon)t]\sin\varepsilon t}}\,.$$

The first term in the product represents high-frequency oscillations, while the second one modulates these oscillations by slowly changing the amplitude. Descriptively, oscillations are gradually transferred from mass 1 to mass 2 and backwards.



$\boxed{10}$ $\boxed{a}$



$$\begin{cases} m\ddot{s}_1 = k(s_2-s_1)+k(s_2-s_1) \\ m\ddot{s}_2 = -k(s_2-s_1)-k(s_2-s_1) \end{cases}$$

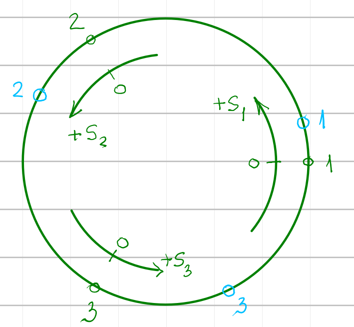
$$\begin{cases} m\ddot{s}_1 = 2k(s_2-s_1) & (1) \\ m\ddot{s}_2 = -2k(s_2-s_1) & (2) \end{cases}$$

$(1)+(2):\quad (\ddot{s}_1+\ddot{s}_2) = 0 \Rightarrow \ddot{\xi}_0 = 0$

$(1)-(2):\quad (\ddot{s}_1-\ddot{s}_2) = -4\frac{k}{m}(s_1-s_2) \Rightarrow \ddot{\xi}_1 + 4\frac{k}{m}\xi_1 = 0$

$$\underline{\underline{\begin{cases} \omega_0 = 0 \\ \omega_1 = 2\sqrt{k/m} \end{cases}}}$$

$\boxed{b}$



$$\begin{cases} m\ddot{s}_1 = -k(s_1-s_2)-k(s_1-s_3) \\ m\ddot{s}_2 = -k(s_2-s_1)-k(s_2-s_3) \\ m\ddot{s}_3 = -k(s_3-s_2)-k(s_3-s_1) \end{cases}$$

$$\begin{cases} m\ddot{s}_1 + k(2s_1-s_2-s_3) = 0 & (1) \\ m\ddot{s}_2 + k(2s_2-s_3-s_1) = 0 & (2) \\ m\ddot{s}_3 + k(2s_3-s_1-s_2) = 0 & (3) \end{cases}$$

$$(1)+(2)+(3):\ (\ddot{s}_1+\ddot{s}_2+\ddot{s}_3)=0 \Rightarrow \ddot{\xi}_0=0$$

$$(1)-(2):\ (\ddot{s}_1-\ddot{s}_2)+3\frac{k}{m}(s_1-s_2)=0 \Rightarrow \ddot{\xi}_1+3\frac{k}{m}\xi_1=0$$

$$(1)-(3):\ (\ddot{s}_1-\ddot{s}_3)+3\frac{k}{m}(s_1-s_3)=0 \Rightarrow \ddot{\xi}_2+3\frac{k}{m}\xi_2=0$$

$$(2)-(3):\ (\ddot{s}_2-\ddot{s}_3)+3\frac{k}{m}(s_2-s_3)=0 \Rightarrow \ddot{\xi}_3+3\frac{k}{m}\xi_3=0$$

Note that $\xi_1+\xi_2+\xi_3 \equiv \xi_4$ so only three out of four modes are independent.

$$\begin{cases} \omega_0 = 0 \\ \omega_1=\omega_2=\sqrt{3}\sqrt{\frac{k}{m}} \end{cases}$$

c

N masses $\left\{ m\ddot{s}_n + k(2s_n - s_{n-1} - s_{n+1}) = 0 \right\}_{n=1}^{N}$

$$-m\omega_\ell^2 \exp i\left(\cancel{\omega_\ell t} + \frac{2\pi \ell n}{N}\right) + 2k \exp i\left(\cancel{\omega_\ell t} + \frac{2\pi\ell n}{N}\right)$$
$$-k\exp i\left(\cancel{\omega_\ell t}+\frac{2\pi\ell(n-1)}{N}\right) - k\exp i\left(\cancel{\omega_\ell t}+\frac{2\pi\ell(n+1)}{N}\right) \overset{?}{=} 0$$

$$\cancel{\exp i\left(\frac{2\pi\ell n}{N}\right)}\left[-m\omega_\ell^2+2k-k\exp\left(-i\frac{2\pi\ell}{N}\right)-k\exp\left(i\frac{2\pi\ell}{N}\right)\right]=0$$

$$-m\omega_\ell^2+2k-2k\cos\frac{2\pi\ell}{N}=0$$

$$\omega_\ell^2=\frac{2k}{m}\left(1-\cos\frac{2\pi\ell}{N}\right)=\frac{2k}{m}\cdot 2\sin^2\frac{\pi\ell}{N}$$

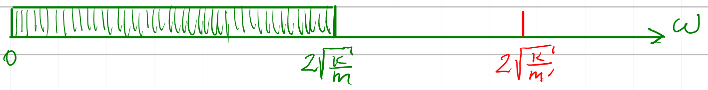
$$\omega_\ell = 2\sqrt{\frac{k}{m}}\sin\frac{\pi\ell}{N}.$$

d

Adding an extra mass would add one extra normal mode. As $m' \ll m$, the motion of m' will hardly affect the motion of neighbouring heavier masses, thus this new normal mode will be „local" in a sense that it involves only the motion of m', and none of the heavier masses. The corresponding frequency will be

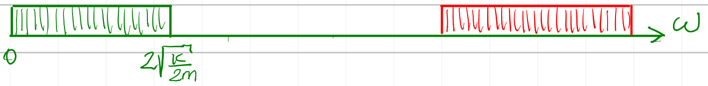
$$\omega' = \sqrt{\frac{2k}{m'}} \gg \omega_\ell.$$

The spectrum will look like this:

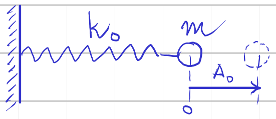


e

More light masses will increase the number of frequencies in the region of ω'. The spectrum will consist of two bands separated by a gap:



11



When a problem contains two widely separate timescales, one can solve for the fast motion neglecting the slow motion and then solve for slow motion by replacing the fast motion with an appropriate average.

Suppose at some moment k changes by Δk. Denote kinetic, potential and total energy before the change as W_k, W_p and W; and those after the change as W_k', W_p' and W'. Then

$$W_k' = W_k, \qquad W_p' = \frac{k+\Delta k}{2} x^2 = W_p\left(1+\frac{\Delta k}{k}\right)$$

$$W' = W_k' + W_p' = \underbrace{W_k + W_p} + W_p\frac{\Delta k}{k} = W + \underbrace{\left(W_p \frac{\Delta k}{k}\right)}_{=\Delta W}$$

Now to the slow motion. During one cycle, average kinetic energy is equal to average potential energy, meaning that $\langle W\rangle = \langle W_k + W_p\rangle = 2\langle W_p\rangle = W$ as total energy changes slowly in time. The change in total energy during an cycle is then

$$\Delta W = \langle W_p\rangle \frac{\Delta k}{k} = \frac{W}{2}\frac{\Delta k}{k} \rightsquigarrow \frac{\Delta W}{W} = \frac{1}{2}\frac{\Delta k}{k}.$$

Transforming from discrete to continuous changes,

$$\frac{dW}{W} = \frac{1}{2}\frac{dk}{k} \overset{\int}{\rightsquigarrow} \frac{W}{W_0} = \sqrt{\frac{k}{k_0}}.$$

As $W = \frac{1}{2}kA^2$, we finally get that

$$\frac{k_0 A_0^2}{\left(\frac{1}{2}k_0\right)A^2} = \sqrt{\frac{\frac{1}{2}k_0}{k_0}} \Rightarrow \underline{\underline{A = A_0 \sqrt[4]{2}}}.$$