

MEROMORPHIC MAPS FROM \mathbb{C}^p INTO SEMI-ABELIAN VARIETIES AND GENERAL PROJECTIVE VARIETIES

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ABSTRACT. In [25], W. Stoll proposed a method of studying holomorphic functions of several complex variables by reducing them to one variable through fiber integration. In this paper, we use this method to extend some important Nevanlinna-type results for holomorphic curves into projective varieties to meromorphic maps from \mathbb{C}^p to projective varieties. This includes Bloch's theorem and Noguchi-Winklemann-Yamanoi's Second Main Theorem for holomorphic maps into semi-abelian varieties intersecting an effective divisor, as well as Huynh-Vu-Xie's Second Main Theorem for meromorphic maps into projective space intersecting with a generic hypersurface with sufficiently high degree.

1. INTRODUCTION

Recently, there have been several works in the study of meromorphic maps from \mathbb{C}^p to projective varieties, where $p \geq 1$ is an integer. Notably, A. Etesse [8] developed the theory of the Green-Griffiths jet differentials of p -germs, Q. Cai, M. Ru, and C. J. Yang [4] used the result of Etesse to establish several defect relations for meromorphic maps from \mathbb{C}^p to projective varieties. See also [16], and for earlier results, [10]. In this paper, we study meromorphic maps from \mathbb{C}^p into projective varieties by reducing the p -variables to one variable through fiber integration, following the method of W. Stoll [25].

We first focus on the meromorphic maps from \mathbb{C}^p into semi-abelian varieties. The key result we used is the following (see Corollary 5.1.9 in [13]): *A semi-torus A contains only countably many sub-semi-tori.* We also note that every meromorphic map from \mathbb{C}^p into a semi-abelian variety is indeed holomorphic. Therefore, we only state the results for holomorphic maps from \mathbb{C}^p into semi-abelian varieties.

Our first result is to establish Bloch's theorem for several complex variables.

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Theorem 1.1 (Bloch's theorem for several variables). Let A be a semi-abelian variety and let $f : \mathbb{C}^p \rightarrow A$ be a holomorphic map. Then the Zariski closure of $f(\mathbb{C}^p)$ is a translate of a semi-abelian subvariety of A .

Next we extend the result of Noguchi-Winklemann-Yamanoi's Second Main Theorem (see Theorem 6.4.1 in [13] or [14]; see also [15]) to the case of several complex variables.

Theorem 1.2. Let A be a semi-torus with equivariant compactification \overline{A} . Let $f : \mathbb{C}^p \rightarrow A$ be a holomorphic map with Zariski dense image in \overline{A} . Let D be an effective divisor on A which extends to a divisor \overline{D} on \overline{A} . Then, possibly after changing the compactification \overline{A} (depending only on D and independent of f), there exists a positive integer k_0 , depending on f and D , such that

$$T_f(r, \overline{D}) \leq N_f^{(k_0)}(r, D) + S_f(r, \overline{D}).$$

The second part of this paper is to study meromorphic maps from \mathbb{C}^p to projective varieties, where $p \geq 1$ is an integer. We first establish the following version of the Second Main Theorem with truncation level one (see Theorem 1.4 in [4]). For related results, see also [11].

Theorem 1.3. Let X be a smooth projective variety of dimension n . Let D be a normal crossing divisor on X and A be an ample line bundle on X . Let

$$\mathcal{P} \in H^0(X, E_{k,m}^{GG}(T_X^*(\log D)) \otimes A^{-\tilde{m}}).$$

Let $f : \mathbb{C}^p \rightarrow X$ be a meromorphic map with $f(\mathbb{C}^p) \not\subset \text{Supp}(D)$. Assume that $\mathcal{P}(j_k(f_{\vec{a}})) \not\equiv 0$ for some $\vec{a} \in S_p(1)$. Then,

$$\tilde{m}T_f(r, A) \leq mN_f^{(1)}(r, D) + S_f(r, A).$$

To apply Theorem 1.3 in the case $X = \mathbb{P}^n(\mathbb{C})$, where D is a generic hypersurface of sufficiently high degree d and $A = \mathcal{O}(1)$, following Siu's strategy (see [22], [23]), we combine the existence theorem for logarithmic jet differentials on $\mathbb{P}^n(\mathbb{C})$ vanishing on an ample divisor with the technique of slanted vector fields.

More precisely, let $S := \mathbb{P}H^0(\mathbb{P}^n(\mathbb{C}), \mathcal{O}(d))$ be the projective parameter space of homogeneous polynomials of degree d in $\mathbb{P}^n(\mathbb{C})$ and let $s \in S$ be the point corresponding to the above hypersurface D . We view $X = \mathbb{P}^n(\mathbb{C})$ as the fiber $\mathbb{P}^n(\mathbb{C}) \times \{s\} \subset \mathbb{P}^n(\mathbb{C}) \times S$. Let \mathcal{P}_s be a fixed nonzero logarithmic jet differential on

$\mathbb{P}^n(\mathbb{C}) \times \{s\}$. By the semi-continuity theorem ([9], Theorem 12.8), it extends to a holomorphic family \mathcal{P} of nonzero logarithmic jet differentials parametrized by points of a Zariski open neighborhood U_s of s in S . Applying slanted vector fields with low pole order to this holomorphic family then produces new jet differentials satisfying the hypotheses of Theorem 1.3. This leads to the following Second Main Theorem for meromorphic maps into projective space intersecting a generic hypersurface of sufficiently high degree (for details, see [23], [11], [21]).

Theorem 1.4. Let $D \subset \mathbb{P}^n(\mathbb{C})$ be a generic hypersurface having degree

$$d \geq 15(5n+1)n^n.$$

Let $f : \mathbb{C}^p \rightarrow \mathbb{P}^n(\mathbb{C})$ be a meromorphic map. Assume that f has the Zariski dense image. Then the following estimate holds:

$$T_f(r) \leq N_f^{(1)}(r, D) + S_f(r).$$

Note that in the above theorem, f is assumed to be algebraically non-degenerate (i.e. f has the Zariski dense image). However, according to Riedl and Yang [17], to derive the hyperbolicity of the complement of generic hypersurfaces in $\mathbb{P}^n(\mathbb{C})$, it can be reduced to studying the algebraic degeneracy of the image of f . For the progress on the hyperbolicity of the complement of generic hypersurfaces in $\mathbb{P}^n(\mathbb{C})$, see [23], [1], [2], and [21].

2. NOTATIONS AND PREPARATIONS

We recall some notations and definitions. For references, see [19] and [20]. Let $p \geq 1$ be an integer. For $z = (z_1, \dots, z_p) \in \mathbb{C}^p$, we define $|z| = (|z_1|^2 + \dots + |z_p|^2)^{1/2}$. We denote by $B_p(r) = \{z \in \mathbb{C}^p \mid |z| < r\}$ the open ball in \mathbb{C}^p of radius r centered at the origin, and $S_p(r) = \{z \in \mathbb{C}^p \mid |z| = r\}$ the boundary of $B_p(r)$. Let $\iota : S_p(r) \hookrightarrow \mathbb{C}^p$ be the natural inclusion. Then the pull-back of the form

$$\sigma_p = d^c \log |z|^2 \wedge (dd^c \log |z|^2)^{p-1}$$

on $\mathbb{C}^p \setminus \{0\}$ gives a positive measure on $S_p(r)$ with total measure 1, i.e. $\int_{S_p(r)} \iota^* \sigma_p = 1$. We define $\nu_p(z) = dd^c |z|^2$ on \mathbb{C}^p . It defines a Lebesgue measure on \mathbb{C}^p such that $B_p(r)$ has measure r^{2p} .

Let h be a holomorphic function on an open domain $U \subset \mathbb{C}^p$. For every $a \in U$, we expand h around a as

$$h = \sum_m P_m(z - a),$$

where P_m are homogeneous polynomials of degree m . Then we define the divisor $\nu_h^0 : U \rightarrow \mathbb{Z}$ associated to h by

$$\nu_h^0(a) := \min\{m \mid P_m \not\equiv 0\}.$$

A map $\nu : U \rightarrow \mathbb{Z}$ is called a *divisor* if for every $z_0 \in U$, there exist two nonzero holomorphic functions h, g on a connected open neighborhood $W \subset U$ of z_0 such that $\nu(z) = \nu_h^0(z) - \nu_g^0(z)$ for each $z \in W$ outside an analytic subset of W of dimension $\leq p - 2$. We denote the support of ν by

$$|\nu| = \{z \in U \mid \nu(z) \neq 0\} \cap U,$$

which is a purely $(p - 1)$ -dimensional analytic subset of U if $|\nu| \neq \emptyset$. We view two divisors as the same if they are identical outside an analytic subset of dimension $\leq p - 2$.

We now recall the standard notation in Nevanlinna theory (see [19], [20]). Let X be a smooth projective variety of dimension n and D be an effective Cartier divisor on X . Denote by $[D]$ the line bundle associated to D , and fix a Hermitian metric on $[D]$. Let f be a meromorphic map of \mathbb{C}^p into X , with $f(\mathbb{C}^p) \not\subset \text{Supp } D$.

Fix $s > 0$. We define the *counting function* of f with respect to D by

$$N_f(r, s, D) = \int_s^r \frac{dt}{t^{2p-1}} \int_{B_p(t) \cap f^* D} v_p^{p-1}.$$

Let $f^* D = \sum_\lambda k_\lambda D_\lambda$ be its irreducible decomposition. For $1 \leq k \leq \infty$, denote $(f^* D)^{(k)} = \sum_\lambda \min\{k, k_\lambda\} D_\lambda$, the *truncated counting function of f to level k* with respect to D is defined by

$$N_f^{(k)}(r, s, D) = \int_s^r \frac{dt}{t^{2p-1}} \int_{B_p(t) \cap (f^* D)^{(k)}} v_p^{p-1}.$$

Note that $N_f^{(\infty)}(r, s, D) = N_f(r, s, D)$.

The *Weil function* on X with respect to D is defined by

$$\lambda_D(x) = -\log \|s_D(x)\|,$$

where s_D is the canonical section of $[D]$. The *proximity function* of f with respect to D is defined by

$$m_f(r, s, D) = \int_{S_p(r)} \lambda_D(f) \sigma_p - \int_{S_p(s)} \lambda_D(f) \sigma_p.$$

The *characteristic function* of f with respect to D is defined by

$$T_f(r, s, D) = \int_s^r \frac{dt}{t^{2p-1}} \int_{B_p(t)} f^* c_1[D] \wedge v_p^{p-1}.$$

Note that for $1 \leq k \leq \infty$, $T_f(r, s, D)$, $m_f(r, s, D)$, and $N_f^{(k)}(r, s, D)$ only depend on s up to a bounded term, so sometimes we denote them as $T_f(r, D)$, $m_f(r, D)$, and $N_f^{(k)}(r, s, D)$ when $s = 1$.

As is customary in Nevanlinna theory, we use the symbol $S_f(r, D)$ for a non-negative small term such that

$$S_f(r, D) \leq_{exc} O(\log^+ T_f(r, D)) + O(\log r) + O(1).$$

where \leq_{exc} stands for the validity of the inequality except for r in exceptional intervals with finite measure.

If $X = \mathbb{P}^n(\mathbb{C})$ and $H \subset \mathbb{P}^n(\mathbb{C})$ is a hyperplane, we write $T_f(r, H) = T_f(r)$, $S_f(r, H) = S_f(r)$ for simplicity. When $n = 1$, we also denote $N_f(r, a)$ by $N(f, r, a)$ for $a \in \mathbb{P}(\mathbb{C})$.

With the above notation, we state the following theorem which is one of the fundamental theorems in Nevanlinna theory.

Theorem 2.1 (First Main Theorem). Let X be a complex projective variety, and let D be an effective Cartier divisor. Let $f : \mathbb{C}^p \rightarrow X$ be meromorphic such that $f(\mathbb{C}^p) \not\subset \text{Supp}(D)$. Then

$$T_f(r, s, D) = m_f(r, s, D) + N_f(r, s, D).$$

Let $\vec{a} \in \mathbb{C}^p$ with $|\vec{a}| = 1$ and define the map $\iota_{\vec{a}} : \mathbb{C} \rightarrow \mathbb{C}^p$ by $\iota_{\vec{a}}(z) = z\vec{a}$. It is easy to see that for almost all $\vec{a} \in S_p(1)$, $f_{\vec{a}} := f \circ \iota_{\vec{a}} : \mathbb{C} \rightarrow X$ is holomorphic.

Lemma 2.2. Let X be a smooth projective variety of dimension n , and let D be an effective Cartier divisor on X . Suppose that $f : \mathbb{C}^p \rightarrow X$ is a meromorphic map such that

$$f(\mathbb{C}^p) \not\subset \text{Supp}(D).$$

Then, for almost all $\vec{a} \in S_p(1)$, the restriction map $f_{\vec{a}} : \mathbb{C} \rightarrow X$ also satisfies

$$f_{\vec{a}}(\mathbb{C}) \not\subset \text{Supp}(D).$$

Proof. Consider the meromorphic map

$$\tilde{f} : \mathbb{C}^p \times \mathbb{C} \rightarrow X, \quad \tilde{f}(v, z) = f(zv).$$

Then $\tilde{f}(\mathbb{C}^p \times \mathbb{C}) \not\subset \text{Supp}(D)$ and $\tilde{f}^{-1}(\text{Supp}(D))$ is a proper analytic subvariety of a Zariski open subset of $\mathbb{C}^p \times \mathbb{C}$. Let $K = \left\{ \vec{a} \in S_p(1) \mid f_{\vec{a}} : \mathbb{C} \rightarrow X, f_{\vec{a}}(\mathbb{C}) \subseteq \text{Supp}(D) \right\} \subseteq S_p(1)$. We claim that $m_{S_p(1)}(K) = 0$, where $m_{S_p(1)}$ is the induced Lebesgue measure from the standard Lebesgue measure m of \mathbb{C}^p . Indeed, assume $m_{S_p(1)}(K) > 0$ and denote by $\mathbb{C}K$ the set $\{av : a \in \mathbb{C}, v \in K\}$, then $m_{\mathbb{C}^p \times \mathbb{C}}(\mathbb{C}K \times \mathbb{C}) = +\infty$. On the other hand, since $\mathbb{C}K \times \mathbb{C} \subset \tilde{f}^{-1}(\text{Supp}(D))$, we get $m_{\mathbb{C}^p \times \mathbb{C}}(\mathbb{C}K \times \mathbb{C}) \leq m_{\mathbb{C}^p \times \mathbb{C}}(\tilde{f}^{-1}(\text{Supp}(D))) = 0$. This is a contradiction. \square

From the above Lemma, we know that $m_{f_{\vec{a}}}(r, s, D)$, $N_{f_{\vec{a}}}(r, s, D)$, and $N_{f_{\vec{a}}}^{(k)}(r, s, D)$ are well defined for $1 \leq k \leq \infty$, for almost all $\vec{a} \in S_p(1)$.

Lemma 2.3 ([25]). Let $r > 0$ and let $F \in L^1(S_p(r))$, then

$$\int_{S_p(r)} F \sigma_p = \int_{S_p(1)} \left(\int_0^{2\pi} F(re^{i\theta} \vec{a}) \frac{d\theta}{2\pi} \right) \sigma_p(\vec{a}).$$

Let $D \subset \mathbb{C}^n$ be an irreducible reduced divisor, defined by a holomorphic function h , i.e. $\{h = 0\} = D$.

Then $D = D_{\text{reg}} \cup D_{\text{sing}}$ with $\dim D_{\text{sing}} \leq n - 2$.

Consider $\pi : \mathbb{C}^n \setminus \{0\} \rightarrow \mathbb{P}^{n-1}(\mathbb{C})$ defined by $x \mapsto [x]$. Then $\dim \pi(D_{\text{sing}} \setminus \{0\}) \leq n - 2$, hence $\pi(D_{\text{sing}} \setminus \{0\})$ has zero measure in $\mathbb{P}^{n-1}(\mathbb{C})$.

Now denote $D_{\text{reg}} \setminus \{0\} = D_{\text{reg}}^*$ and consider $\pi|_{D_{\text{reg}}^*} : D_{\text{reg}}^* \rightarrow \mathbb{P}^{n-1}(\mathbb{C})$. Note that D_{reg}^* is a complex manifold in $\mathbb{C}^n \setminus \{0\}$ of dimension $n - 1$.

Lemma 2.4. Let $v \in D_{\text{reg}}^* \subset \mathbb{C}^n \setminus \{0\}$. Then $(d\pi|_{D_{\text{reg}}^*})_v$ is isomorphic if and only if $T_v^{1,0}(D_{\text{reg}}^*) \oplus \mathbb{C}\{v\} = \mathbb{C}^n$.

Proof. $T_v^{1,0}(D_{\text{reg}}^*)$ and $\mathbb{C}\{v\}$ either satisfy

$$T_v^{1,0}(D_{\text{reg}}^*) \oplus \mathbb{C}\{v\} = \mathbb{C}^n, \quad \text{or} \quad \mathbb{C}\{v\} \subseteq T_v^{1,0}(D_{\text{reg}}^*).$$

Therefore, if we have $\mathbb{C}\{v\} \subseteq T_v^{1,0}(D_{\text{reg}}^*)$ and because $\ker(d\pi)_v = \mathbb{C}\{v\}$, we get $\mathbb{C}\{v\} = \ker(d\pi|_{D_{\text{reg}}^*})_v$. Hence $(d\pi|_{D_{\text{reg}}^*})_v$ is not an isomorphism.

Conversely, if $T_v^{1,0}(D_{\text{reg}}^*) \oplus \mathbb{C}\{v\} = \mathbb{C}^n$, then again by $\ker(d\pi)_v = \mathbb{C}\{v\}$, we conclude that $(d\pi|_{D_{\text{reg}}^*})_v$ is isomorphic. \square

By Sard's theorem, the set of critical values of $\pi|_{D_{\text{reg}}^*}$ has zero measure in $\mathbb{P}^{n-1}(\mathbb{C})$. Denote this set by K . Then for all $y \in \mathbb{P}^{n-1}(\mathbb{C}) \setminus (K \cup \pi(D_{\text{sing}} \setminus \{0\}))$, set $(\pi|_{D_{\text{reg}}^*})^{-1}(y) = \{v_1, v_2, \dots\}$. We have $T_{v_i}^{1,0}(D_{\text{reg}}^*) \oplus \mathbb{C}\{v\} = \mathbb{C}^n$ for $i = 1, 2, \dots$

Suppose $T_v^{1,0}(D_{\text{reg}}^*) \oplus \mathbb{C}\{v\} = \mathbb{C}^n$, then the intersection number of $\mathbb{C}\{v\}$ and D_{reg}^* at v is 1, i.e. $i_v(\mathbb{C}\{v\}, D_{\text{reg}}^*) = 1$. Indeed, let $v \in D_{\text{reg}}^* \subset \mathbb{C}^n \setminus \{0\}$, then there exists $h \in \mathcal{O}_{\mathbb{C}^n, v}$ such that $(Z(h))_v = (D_{\text{reg}}^*)_v$ and $\nabla h(v) \neq 0$. The intersection number is given by $i_v(\mathbb{C}\{v\}, D_{\text{reg}}^*) = \text{ord}_1 h_v(z)$, where $h_v(z) := h(zv)$. We have $h'_v(1) = v \cdot \nabla h(v)$, and assume $h'_v(1) = 0$, by $T_v^{1,0}(D_{\text{reg}}^*) = \{x \in \mathbb{C}^n \mid x \cdot \nabla h(v) = 0\}$. We have $v \in T_v^{1,0}(D_{\text{reg}}^*)$, which implies $\mathbb{C}\{v\} \subseteq T_v^{1,0}(D_{\text{reg}}^*)$. But this contradicts $T_v^{1,0}(D_{\text{reg}}^*) \oplus \mathbb{C}\{v\} = \mathbb{C}^n$. Therefore, $h'_v(1) \neq 0$, hence $\text{ord}_1 h_v(z) = 1$, and consequently $i_v(\mathbb{C}\{v\}, D_{\text{reg}}^*) = 1$.

By the above lemma, we have the following proposition.

Proposition 2.5. With the above notations, we have

$$\begin{aligned} \int_{S_p(1)} m_{f_{\vec{a}}}(r, s, D) \sigma_p(\vec{a}) &= m_f(r, s, D), \\ \int_{S_p(1)} T_{f_{\vec{a}}}(r, s, D) \sigma_p(\vec{a}) &= T_f(r, s, D) + O(1), \\ \int_{S_p(1)} N_{f_{\vec{a}}}^{(k)}(r, D) \sigma_p(\vec{a}) &= N_f^{(k)}(r, D) + O(\log r), \text{ for any } 1 \leq k \leq \infty. \end{aligned}$$

Proof. The first equality holds by Lemma 2.3 and the definition of proximity function. For the second equality, without loss of generality, we assume that D is very ample, since any Cartier divisor can be written as the difference of two ample divisors. Then D induces a canonical embedding $\iota : X \hookrightarrow \mathbb{P}^N$ for some N , such that $\iota^* \mathcal{O}_{\mathbb{P}^N}(1) \cong [D]$. Denote by ω_{FS} the Fubini-Study metric on \mathbb{P}^N . Then:

$$\begin{aligned} \int_s^r \frac{dt}{t^{2p-1}} \int_{B_p(t)} f^* \iota^* \omega_{FS} \wedge v_p^{p-1} &= \int_s^r \frac{dt}{t^{2p-1}} \int_{B_p(t)} f^* c_1[D] \wedge v_p^{p-1} \\ &= T_f(r, s, D) + O(1). \end{aligned}$$

By the Green-Jensen formula (see [19], [20]), we have

$$\begin{aligned}
& \int_s^r \frac{dt}{t^{2p-1}} \int_{B_p(t)} (\iota \circ f)^* \omega_{FS} \wedge \nu_p^{p-1} \\
&= \frac{1}{2} \int_{S_p(r)} \log \|\iota \circ f\| \sigma_p - \frac{1}{2} \int_{S_p(s)} \log \|\iota \circ f\| \sigma_p \\
&= \int_{S_p(1)} \frac{1}{2} \left(\int_{S_1(r)} \log \|\iota \circ f_{\vec{a}}\| \sigma_1 - \int_{S_1(s)} \log \|\iota \circ f_{\vec{a}}\| \sigma_1 \right) \sigma_p(\vec{a}) \\
&= \int_{S_p(1)} \left(\int_s^r \frac{dt}{t} \int_{B_1(t)} (\iota \circ f_{\vec{a}})^* \omega_{FS} \right) \sigma_p(\vec{a}) \\
&= \int_{S_p(1)} \left(\int_s^r \frac{dt}{t} \int_{B_1(t)} f_{\vec{a}}^* c_1[D] \right) \sigma_p(\vec{a}) = \int_{S_p(1)} T_{f_{\vec{a}}}(r, s, D) \sigma_p(\vec{a}) + O(1).
\end{aligned}$$

Hence

$$T_f(r, s, D) = \int_{S_p(1)} T_{f_{\vec{a}}}(r, s, D) \sigma_p(\vec{a}) + O(1).$$

By the First Main Theorem, it follows that

$$(1) \quad \int_{S_p(1)} N_{f_{\vec{a}}}(r, s, D) \sigma_p(\vec{a}) = N_f(r, s, D) + O(1).$$

We now prove the third identity. Since f^*D is an effective divisor on \mathbb{C}^p with the irreducible decomposition $f^*D = \sum_{\lambda \in I} k_\lambda D_\lambda$ and $\sum_{\lambda \in I} \min\{k, k_\lambda\} D_\lambda$ is a principal divisor on \mathbb{C}^p , there is a holomorphic function $g : \mathbb{C}^p \rightarrow \mathbb{P}^1(\mathbb{C})$ such that $g^*(0) = \sum_{\lambda \in I} \min\{k, k_\lambda\} D_\lambda$, by (1) we have,

$$\int_{S_p(1)} N_{g_{\vec{a}}}(r, s, (0)) \sigma_p(\vec{a}) = N_g(r, s, (0)) + O(1) = N_f^{(k)}(r, s, D) + O(1).$$

For almost all $\vec{a} \in S_p(1)$, the punctured complex line $\mathbb{C}^* \vec{a}$ intersects with $\text{Supp } D_\lambda$ transversally. There are only finitely many $\lambda \in I$ such that $0 \in D_\lambda$, denote them by D_1, D_2, \dots, D_n .

For almost all $\vec{a} \in S_p(1)$,

$$(\mathbb{C}\vec{a}) \cap \sum_{\lambda \in I} \min\{k, k_\lambda\} D_\lambda = \sum_{i=1}^n \text{ord}_0(D_i) \min\{k, k_i\} [0] + \sum_{\lambda} \min\{k, k_\lambda\} (P_{\lambda,1}^{\vec{a}} + P_{\lambda,2}^{\vec{a}} + \dots)$$

and

$$((\mathbb{C}\vec{a}) \cap \sum_{\lambda \in I} k_\lambda D_\lambda)^{(k)} := \left(\sum_{i=1}^n \text{ord}_0(D_i) k_i [0] + \sum_{\lambda} k_\lambda (P_{\lambda,1}^{\vec{a}} + P_{\lambda,2}^{\vec{a}} + \dots) \right)^{(k)}$$

$$= \min \left\{ \sum_{i=1}^n \text{ord}_0(D_i) k_i, k \right\} [0] + \sum_{\lambda} \min\{k_{\lambda}, k\} (P_{\lambda,1}^{\vec{a}} + P_{\lambda,2}^{\vec{a}} + \dots)$$

where $\mathbb{C}^* \vec{a}$ intersects with $\text{Supp } D_{\lambda}$ at $P_{\lambda,1}^{\vec{a}}, P_{\lambda,2}^{\vec{a}}, \dots$

Then we have

$$\begin{aligned} \int_{S_p(1)} N_{g_{\vec{a}}}(r, s, 0) \sigma_p(\vec{a}) &= \int_{S_p(1)} \int_s^r \frac{dt}{t} \left(\int_{B_1(t) \cap \sum \min\{k, k_{\lambda}\} D_{\lambda}} 1 \right) \sigma_p(\vec{a}) \\ &= \int_{S_p(1)} \int_s^r \frac{dt}{t} \left(\int_{B_1^*(t) \cap \sum \min\{k, k_{\lambda}\} (P_{\lambda,1}^{\vec{a}} + P_{\lambda,2}^{\vec{a}} + \dots)} 1 \right) \sigma_p(\vec{a}) \\ &\quad + \sum_{i=1}^n \text{ord}_0(D_i) \min\{k, k_i\} \log\left(\frac{r}{s}\right). \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_{S_p(1)} N_{f_{\vec{a}}}^{(k)}(r, s, D) \sigma_p(\vec{a}) &= \int_{S_p(1)} \int_s^r \frac{dt}{t} \left(\int_{(f_{\vec{a}}^* D)^{(k)} \cap B_1(t)} 1 \right) \sigma_p(\vec{a}) \\ &= \int_{S_p(1)} \int_s^r \frac{dt}{t} \left(\int_{B_1(t) \cap \left(\min\{\sum_{i=1}^n \text{ord}_0(D_i) k_i, k\} [0] + \sum_{\lambda} \min\{k, k_{\lambda}\} (P_{\lambda,1}^{\vec{a}} + P_{\lambda,2}^{\vec{a}} + \dots) \right)} 1 \right) \sigma_p(\vec{a}) \\ &= \int_{S_p(1)} \int_s^r \frac{dt}{t} \left(\int_{B_1^*(t) \cap \sum_{\lambda} \min\{k, k_{\lambda}\} (P_{\lambda,1}^{\vec{a}} + P_{\lambda,2}^{\vec{a}} + \dots)} 1 \right) \sigma_p(\vec{a}) \\ &\quad + \min \left\{ \sum_{i=1}^n \text{ord}_0(D_i) k_i, k \right\} \log \frac{r}{s}. \end{aligned}$$

Therefore

$$\left| \int_{S_p(1)} N_{f_{\vec{a}}}^{(k)}(r, s, D) \sigma_p(\vec{a}) - \int_{S_p(1)} N_{g_{\vec{a}}}(r, s, 0) \sigma_p(\vec{a}) \right| = O(\log \frac{r}{s}).$$

This implies that

$$N_f^{(k)}(r, D) = \int_{S_p(1)} N_{f_{\vec{a}}}^{(k)}(r, D) \sigma_p(\vec{a}) + O(\log r).$$

□

3. GREEN-GRIFFITHS JET DIFFERENTIALS AND LOGARITHMIC JET DIFFERENTIALS

We recall the concept of the Green-Griffiths jet differentials and the Green-Griffiths logarithmic jet differentials. For references, see [5], [6].

• **Jet bundle.** Let X be a complex manifold of dimension n . Let $x \in X$ and let $J_k(X)_x$ be the set of equivalence classes of holomorphic maps $f : (\Delta, 0) \rightarrow (X, x)$, where Δ is a disc of unspecified positive radius, with the equivalence relation $f \sim g$ if and only if $f^{(j)}(0) = g^{(j)}(0)$ for $0 \leq j \leq k$, when computed in some local coordinate system of X near x . The equivalence class of f is denoted by $j_k(f)$, which is called the k -jet of f . A k -jet $j_k(f)$ is said to be *regular* if $f'(0) \neq 0$. Set

$$J_k(X) = \cup_{x \in X} J_k(X)_x$$

and consider the natural projection

$$\pi_k : J_k(X) \rightarrow X.$$

Then $J_k(X)$ is a complex manifold which carries the structure of a holomorphic fiber bundle over X , which is called the *k -jet bundle over X* . When $k = 1$, $J_1(X)$ is canonically isomorphic to the holomorphic tangent bundle TX of X . On $J_k X$, there is a natural \mathbb{C}^* action defined by, for any $\lambda \in \mathbb{C}^*$ and $j_k(f) \in J_k X$, set

$$\lambda \cdot j_k(f) = j_k(f_\lambda)$$

where f_λ is given by $t \mapsto f(\lambda t)$.

• **Jet Differential.** A *jet differential of order k* is a holomorphic map $\omega : J_k X \rightarrow \mathbb{C}$, and a *jet differential of order k and degree m* is a holomorphic map $\omega : J_k X \rightarrow \mathbb{C}$ such that

$$\omega(\lambda \cdot j_k(f)) = \lambda^m \omega(j_k(f)).$$

The Green-Griffiths sheaf $\mathcal{E}_{k,m}^{GG}$ of order k and degree m is defined as follows: for any open set $U \subset X$,

$$\mathcal{E}_{k,m}^{GG}(U) = \{\text{jet differentials of order } k \text{ and degree } m \text{ on } U\}.$$

It is a locally free sheaf, we denote its vector bundle on X as $E_{k,m}^{GG}\Omega_X$. Let $\omega \in \mathcal{E}_{k,m}^{GG}(U)$, the differentiation $d\omega \in \mathcal{E}_{k+1,m+1}^{GG}(U)$ is defined by

$$d\omega(j_{k+1}(f)) = \frac{d}{dt} \omega(j_k(f)(t)) \Big|_{t=0}.$$

Note that, formally, we have $d(d^p z_j) = d^{p+1} z_j$ for the local coordinates (z_1, \dots, z_n) .

• **Logarithmic Jet bundle.** In the logarithmic setting, let $D \subset X$ be a normal crossing divisor on X . Recall that $D \subset X$ is *of normal crossing* if, at each $x \in X$, there exist some local coordinates $z_1, \dots, z_r, z_{r+1}, \dots, z_n$ centered at x such that D is locally defined by

$$D = \{z_1 \cdots z_r = 0\}.$$

A holomorphic section $s \in H^0(U, J_k(X))$ over an open subset $U \subset X$ is said to be a *logarithmic k-jet field* if $\tilde{\omega} \circ s$ are holomorphic for all sections $\omega \in H^0(U', T_X^*(\log D))$, for all open subsets $U' \subset U$, where $\tilde{\omega}$ are induced maps defined as in [13]. Such logarithmic k -jet fields define a subsheaf of $J_k(X)$, and this subsheaf is itself a sheaf of sections of a holomorphic fiber bundle over X , called the *logarithmic k-jet bundle over X along D* , denoted by $J_k(X, -\log D)$.

The group \mathbb{C}^* acts fiberwise on the jet bundle as follows. For local coordinates

$$z_1, \dots, z_\ell, z_{\ell+1}, \dots, z_n \quad (\ell = \ell(x))$$

centered at x in which $D = \{z_1 \dots z_\ell = 0\}$, for any logarithmic k -jet field along D represented by some germ $f = (f_1, \dots, f_n)$, if $\varphi_\lambda(z) = \lambda z$ is the homothety with ratio $\lambda \in \mathbb{C}^*$, the action is given by

$$\begin{cases} (\log(f_i \circ \varphi_\lambda))^{(j)} = \lambda^j (\log f_i)^{(j)} \circ \varphi_\lambda & (1 \leq i \leq \ell), \\ (f_i \circ \varphi_\lambda)^{(j)} = \lambda^j f_i^{(j)} \circ \varphi_\lambda & (\ell + 1 \leq i \leq n). \end{cases}$$

- **Logarithmic Jet differential.** A *logarithmic jet differential* of order k and degree m at a point $x \in X$ is a polynomial $Q(f^{(1)}, \dots, f^{(k)})$ on the fiber over x of $J_k(X, -\log D)$ enjoying weighted homogeneity:

$$Q(j_k(f \circ \varphi_\lambda)) = \lambda^m Q(j_k(f)).$$

Denote by $E_{k,m}^{GG} T_X^*(\log D)_x$ the vector space of such polynomials and set

$$E_{k,m}^{GG} T_X^*(\log D) := \bigcup_{x \in X} E_{k,m}^{GG} T_X^*(\log D)_x.$$

By Faà di Bruno's formula [12], $E_{k,m}^{GG} T_X^*(\log D)$ carries the structure of a vector bundle over X , called the *logarithmic Green–Griffiths vector bundle*.

A global section \mathcal{P} of $E_{k,m}^{GG} T_X^*(\log D)$ locally is of the following type in any local coordinates (z_1, \dots, z_n) on U with $D \cap U = \{z_1 \dots z_r = 0\}$,

$$\begin{aligned} \mathcal{P} = \sum_{\nu} \omega_{\nu_{1,1} \dots \nu_{1,k} \dots \nu_{n,1} \dots \nu_{n,k}} & \left(\frac{dz_1}{z_1} \right)^{\nu_{1,1}} \dots \left(\frac{d^k z_1}{z_1} \right)^{\nu_{1,k}} \dots \left(\frac{dz_r}{z_r} \right)^{\nu_{r,1}} \\ & \dots \left(\frac{d^k z_r}{z_r} \right)^{\nu_{r,k}} (dz_{r+1})^{\nu_{r+1,1}} \dots (d^k z_{r+1})^{\nu_{r+1,k}} \\ & \dots (dz_n)^{\nu_{n,1}} \dots (d^k z_n)^{\nu_{n,k}}, \end{aligned}$$

$$\nu = (\nu_{1,1}, \dots, \nu_{1,k}, \dots, \nu_{n,1}, \dots, \nu_{n,k})$$

is a kn -tuple of non-negative integers with

$$(\nu_{1,1} + 2\nu_{1,2} + \dots + k\nu_{1,k}) + \dots + (\nu_{n,1} + 2\nu_{n,2} + \dots + k\nu_{n,k}) = m,$$

and $\omega_{\nu_{1,1}\cdots\nu_{1,k}\cdots\nu_{n,1}\cdots\nu_{n,k}}$ is a locally defined holomorphic function.

Proposition 3.1. Let X be a complex projective manifold of dimension n , and let D be a reduced effective divisor on X . Let L be a line bundle on X with a Hermitian metric h . Let $\mathcal{P} \in H^0(X, E_{k,m}^{GG}(T_X^*(\log D)) \otimes L)$ be a twisted logarithmic k -jet differential. Let $f : \mathbb{C}^p \rightarrow X$ be a holomorphic map such that $f(\mathbb{C}^p) \not\subset D$. Then

$$\int_{S^{2p-1}(1)} \int_0^{2\pi} \log^+ \|\mathcal{P}(j_k(f_{\vec{a}}))(re^{i\theta})\|_h \frac{d\theta}{2\pi} \sigma(\vec{a}) \leq S_f(r, E)$$

where E is an (arbitrarily) ample divisor on X .

Proof. Our proof is partly based on Lemma 4.7.1 in [13]. By virtue of the Hironaka desingularization, we may assume that D is of normal crossing type. We take an affine covering $\{U_\alpha\}$ of M and rational holomorphic functions $(x_{\alpha 1}, \dots, x_{\alpha n})$ on U_α such that

$$dx_{\alpha 1} \wedge \cdots \wedge dx_{\alpha n}(x) \neq 0, \quad \forall x \in U_\alpha,$$

and

$$D \cap U_\alpha = \{x_{\alpha 1} \cdots x_{\alpha s_\alpha} = 0\}.$$

On every U_α one gets

$$\mathcal{P}|_{U_\alpha} = P_\alpha \left(\frac{d^i x_{\alpha j}}{x_{\alpha j}}, d^h x_{\alpha l} \right), \quad 1 \leq i, h \leq k, 1 \leq j \leq s_\alpha, s_\alpha + 1 \leq l \leq n,$$

where P_α is a polynomial in the variables described above whose coefficients are rational holomorphic functions on U_α .

Let $f(z_1, \dots, z_p) = (f_{\alpha 1}(z_1, \dots, z_p), \dots, f_{\alpha n}(z_1, \dots, z_p))$. Then

$$\mathcal{P}(j_k(f_{\vec{a}}))(z) = P_\alpha \left(\frac{\frac{\partial^i f_{\alpha j, \vec{a}}}{\partial z^i}}{f_{\alpha j, \vec{a}}}, \frac{\partial^h f_{\alpha l, \vec{a}}}{\partial z^h} \right).$$

We have

$$\begin{aligned} \frac{\partial f_{\alpha t, \vec{a}}}{\partial z}(z) &= \sum_{i=1}^p a_i \frac{\partial f_{\alpha t}}{\partial z_i}(\vec{a}z), & \frac{\partial^2 f_{\alpha t, \vec{a}}}{\partial z^2}(z) &= \sum_{i,j=1}^p a_i a_j \frac{\partial^2 f_{\alpha t}}{\partial z_i \partial z_j}(\vec{a}z) \dots, \\ \frac{\partial^k f_{\alpha t, \vec{a}}}{\partial z^k}(z) &= \sum_{|\beta|=k} \frac{k!}{\beta!} (\vec{a})^\beta \frac{\partial^{|\beta|} f_{\alpha t}}{\partial z^\beta}(\vec{a}z), & 1 \leq t \leq n. \end{aligned}$$

Hence, on each local chart U_α ,

$$\log^+ \|\mathcal{P}(j_k(f_{\vec{a}}))\|_h(z)$$

$$\begin{aligned}
&\leq C_{U_\alpha} \sum_{\ell=1}^k \sum_{|\beta|=\ell} \left(\sum_{j=1}^{s(\alpha)} \log^+ \left| \frac{\partial^{|\beta|} f_{\alpha j}}{\partial z^\beta} (\vec{a}z) \right| + \sum_{h=1}^n \log^+ \left| \frac{\partial^{|\beta|} f_{\alpha h}}{\partial z^\beta} (\vec{a}z) \right| \right) \\
&\leq C'_{U_\alpha} \sum_{\ell=1}^k \sum_{|\beta|=\ell} \sum_{j=1}^n \log^+ \left| \frac{\partial^{|\beta|} f_{\alpha j}}{\partial z^\beta} (\vec{a}z) \right| + O(1).
\end{aligned}$$

Integrating the above inequality over the sphere $S_p(r)$ and applying Theorem A8.1.5 in [19] and Proposition 2.5.20 in [13], we obtain

$$\int_{S^{2p-1}(1)} \int_0^{2\pi} \log^+ \|\mathcal{P}(j_k(f_{\vec{a}}))\|_h(re^{i\theta}) \frac{d\theta}{2\pi} \sigma(\vec{a}) \leq S_f(r, E).$$

□

4. HOLOMORPHIC MAPS FROM \mathbb{C}^p INTO SEMI-ABELIAN VARIETIES

Proof of Bloch's theorem for several variables.

Proof. Without loss of generality, we assume $f(0) = 0$. Let X be the Zariski closure of $f(\mathbb{C}^p)$. By Corollary 5.1.9 in [13], X contains only countably many distinct proper semi-abelian subvarieties, say $A_1, A_2, \dots, A_n, \dots$, then by Lemma 2.2, there exist $K_i \subset S_p(1)$, $i = 1, 2, \dots$, such that $m_{S_p(1)}(K_i) = 0$, and for $\vec{a} \notin K_i$, one has $f_{\vec{a}}(\mathbb{C}) \not\subset A_i$. Thus, for $\vec{a} \notin \bigcup_{i=1}^{\infty} K_i$, we have $f_{\vec{a}}(\mathbb{C}) \not\subset A_i$, $i = 1, 2, \dots$, it follows that $X = \overline{f_{\vec{a}}(\mathbb{C})}^{\text{Zar}}$ for such (fixed) \vec{a} .

However, by Bloch's theorem for holomorphic maps from \mathbb{C} into semi-abelian varieties (one-variable-version of Bloch's theorem), X is a semi-abelian subvariety of A . This proves our theorem. □

Proof of Theorem 1.2.

Proof. We follow the arguments of the proof of Theorem 6.4.1 in [13]. We may assume from the beginning that

- (i) A is a semi-abelian variety admitting an algebraic presentation

$$0 \longrightarrow (\mathbb{C}^*)^p \longrightarrow A \longrightarrow A_0 \longrightarrow 0,$$

and \overline{A} is a smooth projective equivariant compactification of A ;

- (ii) $\text{St}(D)^0 = \{0\}$;
- (iii) the closure \overline{D} is big on \overline{A} and is in good position.

Since ∂A is of normal crossing, consider the logarithmic k -th jet bundle

$$J_k(\overline{A}; \log \partial A)$$

over \overline{A} along ∂A , and a morphism

$$\psi_k : J_k(\overline{A}; \log \partial A) \rightarrow J_k(\overline{A}).$$

Because of the flat structure of the logarithmic tangent bundle $T(\overline{A}; \log \partial A)$,

$$J_k(\overline{A}; \log \partial A) \cong \overline{A} \times \mathbb{C}^{nk}.$$

Let

$$\pi_1 : J_k(\overline{A}; \log \partial A) \cong \overline{A} \times \mathbb{C}^{nk} \rightarrow \overline{A}, \quad \pi_2 : J_k(\overline{A}; \log \partial A) \cong \overline{A} \times \mathbb{C}^{nk} \rightarrow \mathbb{C}^{nk}$$

be the first and second projections. For a k -jet $y \in J_k(\overline{A}; \log \partial A)$ we call $\pi_2(y)$ the *jet part* of y . We define

$$J_k(\overline{D}; \log \partial A) = \psi_k^{-1}(J_k(\overline{D})).$$

Note that $J_k(\overline{D}; \log \partial A)$ is a subspace of $J_k(\overline{A}; \log \partial A)$, which depends in general on the embedding $\overline{D} \hookrightarrow \overline{A}$ (cf. Sect. 4.6.3 in [13]). We also note that $\pi_2(J_k(\overline{D}; \log \partial A))$ is an algebraic subset of \mathbb{C}^{nk} , because π_2 is proper.

Without loss of generality, we assume $f(0) = 0$. Since f is non-degenerate, $\overline{f(\mathbb{C}^p)}^{Zar} = A$. By the proof of Bloch's theorem above, we have $\overline{f_{\vec{a}}(\mathbb{C})}^{Zar} = A$, for almost every $\vec{a} \in S_p(1)$. That is, $f_{\vec{a}} : \mathbb{C} \rightarrow A$ is algebraically non-degenerate for almost every $\vec{a} \in S_p(1)$.

Fix an $\vec{a}_0 \in S_p(1) \setminus \bigcup_{i=1}^{\infty} K_i$, and let

$$j_k(f_{\vec{a}_0}) : \mathbb{C} \rightarrow J_k(\overline{A}; \log \partial A) \cong \overline{A} \times \mathbb{C}^{nk}$$

be the k -th jet lift of $f_{\vec{a}_0}$. Let $Y_{\vec{a}_0, k} = \pi_2(X_k(f_{\vec{a}_0}))$, it follows from Lemma 6.4.5 in [13] that there exists an integral k_0 and a polynomial function $R_0(w)$ in $w \in \mathbb{C}^{nk_0}$ such that

$$R_0|_{\pi_2(J_k(\overline{D}; \log \partial A))} \equiv 0, \quad R_0|_{Y_{\vec{a}_0, k_0}} \not\equiv 0.$$

Note here that R_0 depends on the vector $\vec{a}_0 \in S_p(1)$.

Denote by $\pi : \mathbb{C}^n \rightarrow A$ the universal covering and let $\tilde{f} : z \in \mathbb{C}^p \mapsto (\tilde{f}_1(z), \dots, \tilde{f}_n(z)) \in \mathbb{C}^n$ be a lift of f with $\pi(\tilde{f}) = f$. Let $\tilde{f}_{\vec{a}}(z) = (\tilde{f}_{\vec{a}, 1}(z), \dots, \tilde{f}_{\vec{a}, n}(z))$. Notice that $R_0|_{Y_{\vec{a}_0, k_0}} \not\equiv 0$ is equivalent to $R_0(\tilde{f}'_{\vec{a}_0}, \dots, \tilde{f}_{\vec{a}_0}^{(k_0)}) \not\equiv 0$. Let $H(\vec{a}, z) = R_0(\tilde{f}'_{\vec{a}}, \dots, \tilde{f}_{\vec{a}}^{(k_0)})$. Then there exists $z_0 \in \mathbb{C}$ with $H(\vec{a}_0, z_0) \neq 0$. Thus the zero set $Z := \{\vec{a} \in \mathbb{C}^p : H(\vec{a}, z_0) = 0\}$ is a proper analytic subset, so if we

restrict it to the unit sphere $S_p(1) \subset \mathbb{C}^p$, then $Z|_{S_p(1)}$ is of measure zero in $S_p(1)$. Hence, for almost all $\vec{a} \in S_p(1)$, we have $R_0(\tilde{f}'_{\vec{a}}, \dots, \tilde{f}_{\vec{a}}^{(k_0)}) \not\equiv 0$.

Let $\{U_j\}$ be an affine open covering of \overline{A} such that the line bundle $[\overline{D}]$ is locally trivial over every U_j . We take a regular function σ_j on each U_j such that σ_j is a defining function of $\overline{D} \cap U_j$, i.e. $(\sigma_j) = \overline{D}|_{U_j}$. Fix a Hermitian metric $\|\cdot\|$ on $[\overline{D}]$. Then there are positive smooth functions h_j on U_j such that

$$\|\sigma(x)\| = \frac{|\sigma_j(x)|}{h_j(x)}, \quad x \in U_j,$$

which is a well-defined function on \overline{A} . We regard R_0 as a regular function on each $U_j \times \mathbb{C}^{n k_0}$. Then we have the following equation on every $U_j \times \mathbb{C}^{n k_0}$

$$(2) \quad b_{j0}\sigma_j + b_{j1}d\sigma_j + \dots + b_{jk_0}d^{k_0}\sigma_j = R_0.$$

Here each b_{ji} is of the form

$$b_{ji} = \sum_{\text{finite}} b_{jil\beta_l}(x) w_l^{\beta_l},$$

where $b_{jil\beta_l}(x)$ are regular functions on U_j , and w_l are the coordinate functions of $\mathbb{C}^{n k_0}$. Thus, we infer that, in every U_j

$$\frac{1}{\|\sigma\|} = \frac{h_j}{|\sigma_j|} = \left| h_j b_{j0} + h_j b_{j1} \frac{d\sigma_j}{\sigma_j} + \dots + h_j b_{jk_0} \frac{d^{k_0}\sigma_j}{\sigma_j} \right| \frac{1}{R_0}.$$

Take relatively compact open subsets $U'_j \Subset U_j$ such that $\bigcup U'_j = \overline{A}$. For every j there is a constant $C_j > 0$ such that, for $x \in U'_j$,

$$h_j |b_{ji}| \leq \sum_{\text{finite}} h_j |b_{jil\beta_l}(x)| |w_l|^{\beta_l} \leq C_j \sum_{\text{finite}} |w_l|^{\beta_l}.$$

Thus, after enlarging C_j if necessary, there is $d_j > 0$ such that, for $z \in \mathbb{C}$ and $f_{\vec{a}}(z) \in U'_j$,

$$h_j(f_{\vec{a}}(z)) |b_{ji}(j_{k_0}(f_{\vec{a}})(z))| \leq C_j \left(1 + \sum_{1 \leq l \leq n, 1 \leq k \leq k_0} |\tilde{f}_{\vec{a},l}^{(k)}(z)| \right)^{d_j}$$

holds for almost all $\vec{a} \in S_p(1)$.

Therefore

$$\begin{aligned} \frac{1}{\|\sigma(f_{\vec{a}}(z))\|} &= \frac{1}{|R_0(\tilde{f}'_{\vec{a}}(z), \dots, \tilde{f}_{\vec{a}}^{(k_0)}(z))|} \left| h_j b_{j0} + h_j b_{j1} \frac{d\sigma_j}{\sigma_j} + \dots + h_j b_{jk_0} \frac{d^{k_0}\sigma_j}{\sigma_j} \right| \\ &\leq \frac{1}{|R_0(\tilde{f}'_{\vec{a}}(z), \dots, \tilde{f}_{\vec{a}}^{(k_0)}(z))|} \sum_{j'=1}^N C_{j'} \left(1 + \sum_{1 \leq l \leq n, 1 \leq k \leq k_0} |\tilde{f}_{\vec{a},l}^{(k)}(z)| \right)^{d_{j'}} \end{aligned}$$

$$\times \left(1 + \left| \frac{d\sigma_{j'}}{\sigma_{j'}}(j_1(f_{\vec{a}})(z)) \right| + \cdots + \left| \frac{d^{k_0}\sigma_{j'}}{\sigma_{j'}}(j_{k_0}(f_{\vec{a}})(z)) \right| \right).$$

It follows that

$$\begin{aligned} m_{f_{\vec{a}}}(r, \overline{D}) &= \int_0^{2\pi} \log \frac{1}{\|\sigma(f_{\vec{a}}(re^{i\theta}))\|} \frac{d\theta}{2\pi} - \int_0^{2\pi} \log \frac{1}{\|\sigma(f_{\vec{a}}(e^{i\theta}))\|} \frac{d\theta}{2\pi} \\ &\leq \int_0^{2\pi} \log \frac{1}{|R(\tilde{f}'_{\vec{a}}, \dots, \tilde{f}_{\vec{a}}^{(k_0)})(re^{i\theta})|} \frac{d\theta}{2\pi} \\ &\quad + O\left(\sum_{\substack{1 \leq l \leq n \\ 1 \leq k \leq k_0}} \int_0^{2\pi} \log^+ |\tilde{f}_{\vec{a},l}^{(k)}(re^{i\theta})| \frac{d\theta}{2\pi} + \sum_{\substack{1 \leq j \leq N \\ 1 \leq k \leq k_0}} \int_0^{2\pi} \log^+ \left| \frac{(\sigma_j \circ f_{\vec{a}})^{(k)}}{\sigma_j \circ f_{\vec{a}}}(re^{i\theta}) \right| \frac{d\theta}{2\pi} \right) \\ &\quad - \int_0^{2\pi} \log \frac{1}{\|\sigma(f_{\vec{a}}(e^{i\theta}))\|} \frac{d\theta}{2\pi} + O(1). \end{aligned}$$

Since $R_0(\tilde{f}'_{\vec{a}}, \dots, \tilde{f}_{\vec{a}}^{(k_0)})$ is a holomorphic function on $\mathbb{C}^p \times \mathbb{C}$, it can be written as $z^{n_0} \sum_{i=0}^{\infty} a_i(\vec{a}) z^i$, with $a_0 \not\equiv 0$ on \mathbb{C}^p .

Therefore

$$\begin{aligned} \int_0^{2\pi} \log \frac{1}{|R_0(\tilde{f}'_{\vec{a}}, \dots, \tilde{f}_{\vec{a}}^{(k_0)})(re^{i\theta})|} \frac{d\theta}{2\pi} &= \int_0^{2\pi} \log \frac{1}{|\sum_{i=0}^{\infty} a_i(\vec{a}) z^i|} (re^{i\theta}) \frac{d\theta}{2\pi} - n_0 \log r \\ &\leq \int_0^{2\pi} \log \frac{1}{|a_0(\vec{a})|} \frac{d\theta}{2\pi} - n_0 \log r \\ &= \log \frac{1}{|a_0(\vec{a})|} - n_0 \log r. \end{aligned}$$

Integrating the above inequality over $S_p(1)$, and applying Proposition 2.5, Proposition 3.1, Theorem A8.1.5 in [19], and Lemma 3.8 in [14], we get

$$\begin{aligned} \int_{S_p(1)} \log \frac{1}{|a_0(\vec{a})|} \sigma(\vec{a}) &< \infty, \\ \int_{S_p(1)} \int_0^{2\pi} \log^+ \left| \frac{(\sigma_j \circ f_{\vec{a}})^{(k)}}{\sigma_j \circ f_{\vec{a}}}(re^{i\theta}) \right| \frac{d\theta}{2\pi} \sigma(\vec{a}) &\leq S_f(r, \overline{D}), \end{aligned}$$

and

$$\int_{S_p(1)} \int_0^{2\pi} \log^+ |\tilde{f}_{\vec{a},l}^{(k)}(re^{i\theta})| \frac{d\theta}{2\pi} \sigma(\vec{a}) \leq S_f(r, \overline{D}).$$

Observe that $\text{ord}_z f_{\vec{a}}^* D > k$ if and only if $j_k(f_{\vec{a}})(z) \in J_k(\overline{D}; \log \partial A)$, from this and (2), we infer that

$$\text{ord}_z f_{\vec{a}}^* D - \min\{\text{ord}_z f_{\vec{a}}^* D, k_0\} \leq \text{ord}_z (R_0(\tilde{f}'_{\vec{a}}, \dots, \tilde{f}_{\vec{a}}^{(k_0)})).$$

Thus, we have, after taking integration, that

$$N_{f_{\vec{a}}}(r, D) - N_{f_{\vec{a}}}^{(k_0)}(r, D) \leq N(r, R_0(\tilde{f}'_{\vec{a}}, \dots, \tilde{f}_{\vec{a}}^{(k_0)}), 0).$$

It follows that

$$\begin{aligned} & N(r, R_0(\tilde{f}'_{\vec{a}}, \dots, \tilde{f}_{\vec{a}}^{(k_0)}), 0) \\ &= \int_0^{2\pi} \log |R_0(\tilde{f}'_{\vec{a}}, \dots, \tilde{f}_{\vec{a}}^{(k_0)})| (re^{i\theta}) \frac{d\theta}{2\pi} - \int_0^{2\pi} \log |R_0(\tilde{f}'_{\vec{a}}, \dots, \tilde{f}_{\vec{a}}^{(k_0)})| (e^{i\theta}) \frac{d\theta}{2\pi} \\ &\leq \int_0^{2\pi} \log^+ |R_0(\tilde{f}'_{\vec{a}}, \dots, \tilde{f}_{\vec{a}}^{(k_0)})| (re^{i\theta}) \frac{d\theta}{2\pi} - \log |a_0(\vec{a})| \\ &\leq O \left(\sum_{\substack{1 \leq l \leq n \\ 1 \leq k \leq k_0}} \int_0^{2\pi} \log^+ |\tilde{f}_{\vec{a},l}^{(k)}(re^{i\theta})| \frac{d\theta}{2\pi} \right) - \log |a_0(\vec{a})|. \end{aligned}$$

Integrating the above inequality over $S_p(1)$, we get

$$N_f(r, D) \leq N_f^{(k_0)}(r, D) + S_f(r, \overline{D}).$$

Combining these together, it gives

$$T_f(r, \overline{D}) \leq N_f^{(k_0)}(r, D) + S_f(r, \overline{D}).$$

□

5. HOLOMORPHIC MAPS FROM \mathbb{C}^p INTO PROJECTIVE VARIETIES

Proposition 5.1. Let c be a positive integer with $c \geq 5n - 1$, and let $D \subset \mathbb{P}^n(\mathbb{C})$ be a generic smooth hypersurface of degree $d \geq 15(c+2)n^n$. Let $f : \mathbb{C}^p \rightarrow \mathbb{P}^n(\mathbb{C})$ be a meromorphic, algebraically nondegenerate map. Then, for jet order $k = n$ and for weighted degrees $m \gg d$, there exist an integer $0 \leq \ell \leq m$ and a global logarithmic jet differential

$$\mathcal{P} \in H^0 \left(\mathbb{P}^n(\mathbb{C}), E_{n,m}^{\text{GG}} T_{\mathbb{P}^n(\mathbb{C})}^*(\log D) \otimes \mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(-cm + \ell(5n-2)) \right)$$

such that $\mathcal{P}(j_n(f_{\vec{a}})) \not\equiv 0$ for almost every $\vec{a} \in S_p(1)$.

Proof. Since f is algebraically nondegenerate, its radial derivative

$$\frac{\partial f}{\partial r}(z) = \sum_{i=1}^p \frac{z_i}{\|z\|} \frac{\partial f}{\partial z_i}(z), \quad z = (z_1, \dots, z_p) \in \mathbb{C}^p \setminus \{0\},$$

does not vanish identically; hence the set $\{z \in \mathbb{C}^p \setminus \{0\} : \frac{\partial f}{\partial r}(z) = 0\}$ has zero measure in \mathbb{C}^p .

By Proposition 2.1 in [11], there exists a nonzero logarithmic jet differential

$$\mathcal{P}_s \in H^0\left(\mathbb{P}^n(\mathbb{C}) \times \{s\}, E_{n,m}^{\text{GG}} T_{\mathbb{P}^n(\mathbb{C})}^*(\log D) \otimes \mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(-cm)\right).$$

This implies that $f^{-1}(\{\mathcal{P}_s = 0\} \cup D)$ has zero measure in \mathbb{C}^p . Consequently $f^{-1}(\{\mathcal{P}_s = 0\} \cup D) \cup \{\frac{\partial f}{\partial r} = 0\}$ has zero measure in \mathbb{C}^p , and we may choose $z_0 \in \mathbb{C}^p \setminus \{0\}$ with $f(z_0) \notin \{\mathcal{P}_s = 0\} \cup D$ and $\frac{\partial f}{\partial r}(z_0) \neq 0$.

Let $\vec{a}_0 := z_0/\|z_0\| \in S^{2p-1}(1)$ and consider the restriction $f_{\vec{a}_0} : \mathbb{C} \rightarrow \mathbb{P}^n(\mathbb{C}) \times \{s\}$, $f_{\vec{a}_0}(z) = f(z\vec{a}_0)$. By the semi-continuity theorem, \mathcal{P}_s extends to a holomorphic family \mathcal{P} of nonzero logarithmic jet differentials parametrized by points of a Zariski open neighborhood U_s of s in S . Following the argument in [11] (p. 668), there exist global slanted vector fields v_1, \dots, v_ℓ (for some $0 \leq \ell \leq m$) such that

$$((v_1 \cdots v_\ell) \mathcal{P})(j_n(f_{\vec{a}_0})) \not\equiv 0.$$

Hence

$$\tilde{\mathcal{P}}_s := (v_1 \cdots v_\ell) \mathcal{P}|_{\mathbb{P}^n(\mathbb{C}) \times \{s\}} \in H^0\left(\mathbb{P}^n(\mathbb{C}), E_{n,m}^{\text{GG}} T_{\mathbb{P}^n(\mathbb{C})}^*(\log D) \otimes \mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(-cm + \ell(5n - 2))\right)$$

satisfies $\tilde{\mathcal{P}}_s(j_n(f_{\vec{a}_0})) \not\equiv 0$. Similar with the argument in the proof of the Theorem 1.2, we have

$$\tilde{\mathcal{P}}_s(j_n(f_{\vec{a}})) \not\equiv 0 \quad \text{for almost every } \vec{a} \in S_p(1).$$

□

Proof of Theorem 1.3

Proof. Since $\mathcal{P}(j_k(f_{\vec{a}})) \not\equiv 0$ for some $\vec{a} \in S_p(1)$, it follows that $\mathcal{P}(j_k(f_{\vec{a}})) \not\equiv 0$ for almost all $\vec{a} \in S_p(1)$. Choose an \vec{a} with $f_{\vec{a}}(\mathbb{C}) \not\subset \text{Supp}(D)$ and $\mathcal{P}(j_k(f_{\vec{a}})) \not\equiv 0$. Following the proof of Theorem 1.6 in [3], we obtain, in the sense of currents,

$$dd^c \log \|P(j_k(f_{\vec{a}}))\|_{h^{-1}}^2 \geq \tilde{m} f_{\vec{a}}^* c_1(A) - m(f_{\vec{a}}^* D)^{(1)}$$

where h is a Hermitian metric on $A^{\tilde{m}}$. Hence, by applying the Green–Jensen formula, we obtain

$$(3) \quad \tilde{m} T_{f_{\vec{a}}}(r, A) - m N_{f_{\vec{a}}}^{(1)}(r, D) \leq \int_0^{2\pi} \log \|P(j_k(f_{\vec{a}})(re^{i\theta}))\|_{h^{-1}} \frac{d\theta}{2\pi} + O(1).$$

Finally, integrating (3) over $S_p(1)$, we get

$$\begin{aligned} \tilde{m} T_f(r, A) - m N_f^{(1)}(r, D) &\leq \int_{S_p(1)} \left(\int_0^{2\pi} \log \|\mathcal{P}(j_k(f_{\vec{a}})(re^{i\theta})\|_{h^{-1}} \frac{d\theta}{2\pi} \right) d\sigma_p(\vec{a}) \\ &\quad + O(\log r). \end{aligned}$$

By Proposition 3.1, we have

$$\tilde{m} T_f(r, A) - m N_f^{(1)}(r, D) \leq_{\text{exc}} S_f(r, A).$$

□

Proof of Theorem 1.4

Proof. Choose $\tilde{m} = mc - \ell(5n - 2) \geq m(c - 5n + 2) \geq m$, where c is a positive integer satisfying $c \geq 5n - 1$. Applying Proposition 1.3 in the case $X = \mathbb{P}^n(\mathbb{C})$, with D a generic hypersurface and $A = \mathcal{O}(1)$, we have

$$\tilde{m} T_f(r) - m N_f^{(1)}(r, D) \leq_{\text{exc}} S_f(r).$$

Then,

$$T_f(r) \leq_{\text{exc}} \frac{1}{c - 5n + 2} N_f^{(1)}(r, D) + S_f(r).$$

If we choose $c = 5n - 1$, we get

$$T_f(r) \leq_{\text{exc}} N_f^{(1)}(r, D) + S_f(r).$$

□

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