

Colouring Graphs Without a Subdivided H-Graph: A Full Complexity Classification

Tala Eagling-Vose 

Department of Computer Science, Durham University, Durham, UK

Jorik Jooken 

Department of Computer Science, KU Leuven Campus Kulak-Kortrijk, 8500 Kortrijk, Belgium

Felicia Lucke 

ENS Lyon, France

Barnaby Martin 

Department of Computer Science, Durham University, Durham, UK

Daniël Paulusma 

Department of Computer Science, Durham University, Durham, UK

Abstract

We consider COLOURING on graphs that are H -subgraph-free for some fixed graph H , i.e., graphs that do not contain H as a subgraph. A variety of problems can be fully classified on H -subgraph-free graphs if they are (C1) polynomial-time solvable on graphs of bounded treewidth; (C2) NP-complete for subcubic graphs, and (C3) stay NP-complete under edge subdivision of subcubic graphs. However, COLOURING is polynomial-time solvable for subcubic graphs due to Brooks' Theorem. Hence, it does not satisfy C2 and is therefore not a so-called C123-problem.

Can we still classify the complexity of COLOURING on H -subgraph-free graphs?

It is known that even 3-COLOURING is NP-complete for H -subgraph-free graphs whenever H has a cycle; or a vertex of degree at least 5; or a component with two vertices of degree 4, while COLOURING is polynomial-time solvable for H -subgraph-free graphs if H is a forest of maximum degree at most 3, in which each component has at most one vertex of degree 3. For connected graphs H , this means that it remains to consider when H is tree of maximum degree 4 with exactly one vertex of degree 4, or a tree of maximum degree 3 with at least two vertices of degree 3. We let H be a so-called subdivided “H”-graph, which is either

- (i) a subdivided \mathbb{H}_0 : a tree of maximum degree 4 with exactly one vertex of degree 4 and no vertices of degree 3, or
- (ii) a subdivided \mathbb{H}_1 : a tree of maximum degree 3 with exactly two vertices of degree 3.

In the literature, only a limited number of polynomial-time and NP-completeness results for cases (i) and (ii) are known. We develop polynomial-time techniques that allow us to determine the complexity of COLOURING on H -subgraph-free graphs for all the remaining subdivided “H”-graphs, so we fully classify both cases (i) and (ii). As a consequence, the complexity of COLOURING on H -subgraph-free graphs has now been settled for all connected graphs H except when H is

- a tree of maximum degree 4 with exactly one vertex of degree 4 and at least one vertex of degree 3; or
- a tree of maximum degree 3 with at least three vertices of degree 3.

So far, for similar graph problems that are not C123, only partial classifications have been shown for H -subgraph-free graphs for cases (i) and (ii). Our goal was therefore to develop techniques of wider applicability. To illustrate this, we also employ our new techniques to obtain the same new polynomial-time results for another classic graph problem, namely STABLE CUT.

2012 ACM Subject Classification Mathematics of computing → Graph theory; Theory of computation → Graph algorithms analysis; Theory of computation → Problems, reductions and completeness

Keywords and phrases colouring, forbidden subgraph, complexity dichotomy

Funding *Jorik Jooken*: supported by a Postdoctoral Fellowship of the FWO (222524N).

Felicia Lucke: supported by EPSRC (EP/X01357X/1) and SNSF Postdoc Mobility Grant 230578.

Daniël Paulusma: supported by Leverhulme Trust (RPG-2024-182) and EPSRC (EP/X01357X/1).

1 Introduction

For an integer $k \geq 1$, a k -colouring c of a graph $G = (V, E)$ maps every vertex $u \in V$ to an integer $c(u) \in \{1, 2, \dots, k\}$ such that $c(v) \neq c(w)$ for every two vertices v and w with $vw \in E$. The corresponding decision problem COLOURING has as input a graph G and integer k and is to decide if G has a k -colouring. If k is not part of the input but a fixed constant, we denote this problem as k -COLOURING. It is well known that 3-COLOURING is NP-complete [18]. This led to an extensive study of the complexity of COLOURING for special graph classes.

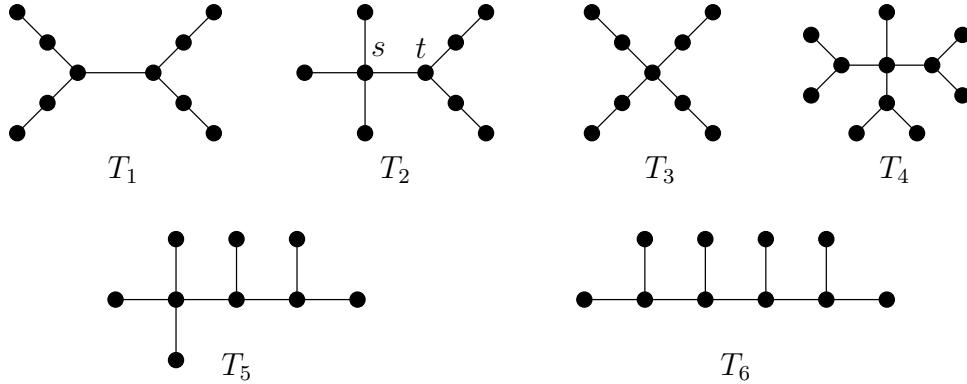
Most studied graph classes are *hereditary*, i.e., closed under vertex deletion. In particular, Král et al. [16] determined, for every graph H , the complexity of COLOURING for H -free graphs, i.e., graphs that do not contain H as an induced subgraph. For connected H , this classification implies that COLOURING is polynomial-time solvable if H is an induced subgraph of the 4-vertex path P_4 and NP-complete otherwise. The complexity classification becomes much more involved for (H_1, H_2) -free graphs, see [8], and is not yet settled. The same holds for the classification of k -COLOURING for H -free graphs; see e.g. [2, 10, 12] for recent progress.

We consider *monotone* graph classes, which are not only closed under vertex deletion but also under edge deletion. Complexity aspects of COLOURING have been less well studied for monotone graph classes than for hereditary graph classes, but see e.g. [24] for complexity results for graphs with only odd cycles of length 3 or 5 as a subgraph. Here, we focus on H -subgraph-free graphs, i.e., graphs that do not contain some graph H as a subgraph. Every H -subgraph-free graph is also H -free, but the reverse implication only holds if H is a complete graph. A range of graph problems on partitioning, covering, network design, width parameter computation etc. can be fully classified on H -subgraph-free graphs (even on \mathcal{H} -subgraph-free graphs for finite sets of forbidden subgraphs \mathcal{H}) if they are (C1) polynomial-time solvable on graphs of bounded treewidth; (C2) NP-complete for *subcubic* graphs (graphs of maximum degree at most 3), and (C3) stay NP-complete under edge subdivision of subcubic graphs [13]. Such problems are called C123. For a connected graph H , every C123-problem is polynomial-time solvable on H -subgraph-free graphs if H is a subcubic tree with at most one vertex of degree 3 and NP-complete otherwise. However, COLOURING does not satisfy C2, as it is polynomially solvable for subcubic graphs due to Brooks' Theorem. Our research question is:

Is it still possible to classify the complexity of COLOURING for H -subgraph-free graphs?

To get a handle on answering this question we restrict ourselves to *connected* H . Even the classification of COLOURING on H -subgraph-free graphs for connected H turns out to be much more problematic than the corresponding one for H -free graphs, as we now discuss.

Known Results. It follows from a result of Emden-Weinert, Hougardy and Kreuter [5] (see also [16, 19]) that 3-COLOURING is NP-complete for graphs of girth g for every $g \geq 4$, that is, for (C_3, \dots, C_{g-1}) -subgraph-free graphs, so in particular for H -subgraph-free graphs if H has a cycle C_s . Hence, we may assume H is a tree. Garey, Johnson and Stockmeyer [7] proved that 3-COLOURING is NP-complete for graphs of maximum degree 4, so for H -subgraph-free graphs if H is a tree of maximum degree at least 5. On the positive side, Kobler and Rotics [15] proved that COLOURING is polynomial-time solvable even for graphs of bounded clique-width and thus also for graphs of bounded treewidth (the latter also follows as the chromatic number of a graph is bounded by its treewidth thus allowing for a standard dynamic programming algorithm for colouring graphs of bounded treewidth). So, COLOURING satisfies C1. It follows from results of Robertson and Seymour [23] that for a connected graph H , the class of H -subgraph-free graphs has bounded treewidth if and

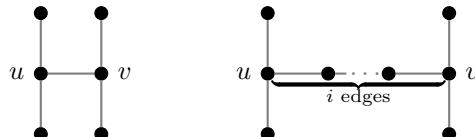


■ **Figure 1** The trees T_1, \dots, T_6 from [9]; note that $T_1 = \mathbb{H}_1^{2,2,2,2}$ and $T_3 = S_{2,2,2,2}$.

only if H is a subcubic tree with at most one vertex of degree 3. Hence, COLOURING is polynomial-time solvable for H -subgraph-free graphs if H is a subcubic tree with at most one vertex of degree 3.

Golovach et al. [9] initiated a more systematic study in the complexity of COLOURING for H -subgraph-free graphs. To explain their results, we first consider the trees of maximum degree at most 4 displayed in Figure 1. The *subdivision* of an edge $e = uv$ in a graph replaces uv with a new vertex w made adjacent to u and v . For a graph F , a *subdivided* F is a graph obtained from F by subdividing each edge zero or more times. The “H”-graph $\mathbb{H} = \mathbb{H}_1$ is the graph displayed in Figure 2 that looks like the letter “H”. For $d \geq c \geq b \geq a \geq 1$ and $i \geq 1$, a subdivided “H”-graph $\mathbb{H}_i^{a,b,c,d}$ is obtained from \mathbb{H} by subdividing the horizontal edge $i - 1$ times and the vertical edges $a - 1, b - 1, c - 1$ and $d - 1$ times, respectively. For example, $\mathbb{H} = \mathbb{H}_1^{1,1,1,1}$ and $T_1 = \mathbb{H}_1^{2,2,2,2}$. For $p \geq 0$, let T_2^p be the tree obtained from T_2 by subdividing p times the edge st , so $T_2^0 = T_2$. Golovach et al. [9] proved that 3-COLOURING is NP-complete for H -subgraph-free graphs if H is a tree with at least two vertices of degree 4, or contains as a subgraph either a subdivided T_1 , i.e., a tree $\mathbb{H}_i^{a,b,c,d}$ with $i \geq 1$ and $a \geq 2$; or a tree T_2^p with $0 \leq p \leq 9$, or a tree from $\{T_3, T_4, T_5, T_6\}$. They also proved that COLOURING is polynomial-time solvable for H -free graphs if H is a forest of maximum degree at most 4 on at most seven vertices, which includes the cases where $H \in \{\mathbb{H}_1^{1,1,1,1}, \mathbb{H}_2^{1,1,1,1}, \mathbb{H}_1^{1,1,1,2}\}$.

Johnson et al. [14] focussed on the case where H is a subdivided star $S_{p,q,r,s}$ of maximum degree 4 for some $s \geq r \geq q \geq p \geq 1$, which is obtained from the 5-vertex star $K_{1,4}$ by subdividing each edge $p-1, q-1, r-1$ and $s-1$ times, respectively; for example, $K_{1,4} = S_{1,1,1,1}$ and $T_3 = S_{2,2,2,2}$. From the above it follows that COLOURING on $S_{p,q,r,s}$ -subgraph-free graphs is polynomial-time solvable if $(p, q, r, s) \in \{(1, 1, 1, 1), (1, 1, 1, 3), (1, 1, 2, 2)\}$ and NP-complete, even for $k = 3$, if $p \geq 2$. Johnson et al. [14] extended these results by showing polynomial-time solvability if $p = q = 1$ and $r \geq 1$. In fact, they showed this not only for COLOURING but for every graph problem that (i) can be solved in polynomial time on subcubic graphs; (ii) can be solved in polynomial time on graphs of bounded treedepth; and (iii) can be solved on



■ **Figure 2** The “H”-graph $\mathbb{H} = \mathbb{H}_1 = \mathbb{H}_1^{1,1,1,1}$ and for $i \geq 1$, the graph $\mathbb{H}_i = \mathbb{H}_i^{1,1,1,1}$ obtained from \mathbb{H} by subdividing $i - 1$ times the *horizontal* edge uv . The other edges of \mathbb{H} are the *vertical* edges.

graphs with proper bridges by using a polynomial-time reduction to a family of instances on graphs that are either of bounded treedepth or subcubic. Here, a *proper bridge* in a connected graph G is an edge e that is not incident to a vertex of degree 1 such that $G - e$ is disconnected. As other examples of such problems, they gave (INDEPENDENT) FEEDBACK VERTEX SET, CONNECTED VERTEX COVER and MATCHING CUT.

Our Focus. The graph $S_{1,1,1,1}$ is also known as the graph \mathbb{H}_0 , which is part of an infinite sequence $\mathbb{H}_0, \mathbb{H}_1, \mathbb{H}_2, \dots$, where $\mathbb{H}_i = \mathbb{H}_i^{1,1,1,1}$ for $i \geq 1$; see also Figure 2. We note that the infinite set $\mathcal{M} = \{C_3, C_4, \dots, \mathbb{H}_0, \mathbb{H}_1, \mathbb{H}_2, \dots\}$ is a maximal antichain in the poset of connected graphs under the subgraph relation. Conditions C2 and C3 ensure that for every finite set $\mathcal{M}' \subseteq \mathcal{M}$, C123-problems are NP-complete on \mathcal{M}' -subgraph-free graphs. If C2 or C3 is not satisfied, as is the case with COLOURING, we must consider graphs H from \mathcal{M} . For example, this has been done by Lozin et al. [20] for HAMILTON CYCLE, k -INDUCED DISJOINT PATHS, STAR 3-COLOURING and C_3 -COLOURING, which all satisfy C2 but not C3. For each of these four problems they left some challenging open cases of graphs \mathbb{H}_i . As COLOURING is NP-complete for (C_3, \dots, C_{g-1}) -subgraph-free graphs for every $g \geq 3$ [5], we only need to focus on graphs from $\{\mathbb{H}_0, \mathbb{H}_1, \mathbb{H}_2, \dots\}$. However, in order to generalize the known results, we will consider *every* subdivided \mathbb{H}_0 and *every* subdivided \mathbb{H}_1 .

Our Results. We fully classify the complexity of COLOURING on H -subgraph-free graphs if H is a subdivided \mathbb{H}_0 (subdivided $S_{1,1,1,1}$), which is a subdivided star of maximum degree 4, or subdivided \mathbb{H}_1 , which is a subcubic tree with exactly two vertices of degree 3. We do this by proving that all the remaining cases are polynomial-time solvable. Combining this with the results from [9, 14] yields:

► **Theorem 1.** *For $d \geq c \geq b \geq a \geq 1$ and $i \geq 1$, COLOURING on $\mathbb{H}_i^{a,b,c,d}$ -subgraph-free graphs is polynomial-time solvable if $a = 1$ and NP-complete, even for $k = 3$, if $a \geq 2$.*

► **Theorem 2.** *For $s \geq r \geq q \geq p \geq 1$, COLOURING on $S_{p,q,r,s}$ -subgraph-free graphs is polynomial-time solvable if $p = 1$ and NP-complete, even for $k = 3$, if $p \geq 2$.*

Combining Theorems 1 and 2 with the other aforementioned known results [9, 5, 23] shows that the following two cases are still unresolved:

- H is a tree of maximum degree 4 with exactly one vertex of degree 4 and at least one vertex of degree 3; or
- H is a subcubic tree with at least three vertices of degree 3.

We discuss the open cases in more detail in Section 8 after first, in Section 7, discussing the broader applications of our techniques for other problems. In particular, we employ our new techniques, in Section 7, to obtain the same new polynomial-time results as for COLOURING for another classic graph problem, namely STABLE CUT. This problem is to decide if a connected graph G has an independent set I such that $G - I$ is disconnected.

Proof Ideas behind Our New Technique. As COLOURING is polynomial-time solvable for subcubic graphs by Brook's Theorem, we may assume the input graph G has a vertex of degree at least 4. We may also assume that G is 2-connected; else we can consider its 2-connected components separately. Hence, G has no proper bridge. The approach of Johnson et al. [14] was to show that a connected $S_{1,1,s,s}$ -subgraph-free graph G with a vertex of degree at least 4 and no proper bridges has bounded treedepth (note that we may assume $r = s$, as every $S_{1,1,r,s}$ -subgraph-free graph is $S_{1,1,s,s}$ -free if $s \geq r$). Their idea is to take a vertex u of degree at least 4. If $G - u$ has unbounded treedepth, $G - u$ must contain a

long path P . Due to the 2-connectivity u can be connected to a middle vertex of P by two vertex-disjoint paths, and this yields an $S_{1,1,s,s}$, a contradiction.

Johnson et al. [14] explained that 3-edge connectivity is needed for their approach to work for $S_{1,r,r,r}$ -subgraph-free graphs and that it is not clear if a suitably modified graph-structural result can be obtained in this way. They left the case where $H = S_{1,r,r,r}$ as an open problem. In our paper we solve this open problem by showing a new structural decomposition theorem. Section 5, is dedicated to this theorem which states that if an $S_{1,r,r,r}$ -subgraph-free graph G of maximum degree at least 4 contains no proper bridge alongside some other *special substructure*, then it must have bounded treedepth. To prove this statement we assume G has unbounded treedepth. By a result of Galvin, Rival, and Sands [6], G must have a long induced path (as it contains no large balanced bicliques because it is $S_{1,r,r,r}$ -subgraph-free). We take a vertex on this path and argue the existence of three long vertex-disjoint paths, giving an $S_{1,r,r,r}$, which is the desired contradiction. It then remains to show that we can preprocess instances of COLOURING in polynomial time to obtain a polynomial number of instances that satisfy the conditions of our new structural decomposition theorem.

Paper Organisation. To prove that COLOURING is polynomial-time solvable for $\mathbb{H}_i^{1,d,d,d}$ -subgraph-free graphs we use a similar technique, but the details are different. For this case, we need the $\mathbb{H}_i^{1,d,d,d}$ -subgraph-free input graph to have minimum degree at least 3 while it may not contain some fan (path with dominating vertex) and another special type of substructure. We note that in both proofs the connections of the fan and other special substructures to the rest of the graph must be in some controlled way, as else the preprocessing to obtain instances satisfying the structural decomposition theorems will not preserve the H -subgraph-freeness.

The additional notation and concepts that we need to show our structural results are introduced in Sections 2 and 3. Our structural results for $\mathbb{H}_i^{1,d,d,d}$ -subgraph-free graphs and for $S_{1,r,r,r}$ -subgraph-free graphs are proven in Sections 4 and 5, respectively. The algorithms to solve COLOURING on both graph classes are presented in Section 6. Finally, as mentioned, we show in Section 8 that there exist other problems to which our general techniques apply.

2 Preliminaries

All graphs considered here are finite, simple, and undirected. That is, a *graph* $G = (V, E)$ consists of a finite set V of *vertices* and a set $E \subseteq V^{(2)}$ of *edges*, where $V^{(2)}$ is the set of 2-element subsets of V . We may also denote the set of vertices and edges of G by $V(G)$ and $E(G)$ respectively. Let G be a graph. We denote the *neighbourhood* of a vertex $v \in V(G)$ by $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$. Note that we may also write $N(v)$ if the graph is clear from the context. For a subset $S \subseteq V(G)$, we write $N_G(S) = \bigcup_{v \in S} N_G(v)$. A graph H is a *subgraph* of G if H can be obtained from G by a sequence of vertex deletions and edge deletions. If H is a subgraph of G , we write $H \subseteq G$. A graph H is an *induced subgraph* of G if H can be obtained from G by a sequence of vertex deletions. For a vertex set $S \subseteq V(G)$, we write $G[S]$ to denote the subgraph of G *induced by* S , that is, the graph obtained from G after deleting the vertices not in S . We further let $G - S$ denote the graph $G[V \setminus S]$. For $S = \{u\}$, for some $u \in V(G)$, we also write $G - u$ instead of $G - \{u\}$. The *contraction* of an edge $e = uv$ in G replaces u and v by a new vertex w that is adjacent to every vertex in $(N_G(u) \cup N_G(v)) \setminus \{u, v\}$ (without creating parallel edges).

We may refer to a path P with vertices p_1, \dots, p_n and edges $p_{i-1}p_i$ for $2 \leq i \leq n$ by the sequence (p_1, p_2, \dots, p_n) . The *length* of P is its number of edges $n - 1$. The *distance* $\text{dist}_G(u, v)$ between two vertices u and v of a graph G is the length of a shortest path from u to v . An edge $uv \in E$ is a *bridge* if its removal increases the number of connected components.



Figure 3 An illustration of the notation regarding paths. The subpath $P[p_1 : p_2] = P[1 : 2]$ is highlighted by the purple box, $P[p_4 : p_7] = P[4 : 7]$ is highlighted by the teal box and $P[p_8 : p_{10}] = P[8 : 10]$ is highlighted by the orange box. Note that negative indexing may also be used i.e., $P[8 : -1] = P[-3 : -1] = P[8 : 10]$.

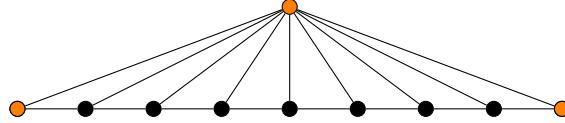


Figure 4 The fan graph F_{10} with poles drawn in orange.

A bridge is *proper* if neither incident vertex has degree 1. A graph is *quasi-bridgeless* if it contains no proper bridge.

An *elimination forest* of a graph G is a rooted forest T such that $V(G) = V(T)$ and for every $uv \in E(G)$ both u and v are on the same root-to-leaf path of T . The *treedepth* $\text{td}(G)$ is the minimum height of an elimination forest of G .

We further highlight the following two results from the literature regarding treedepth. Let $K_{r,s}$ denote the complete bipartite graph with partitions of size r and s respectively.

► **Theorem 3** ([22]). *Let G be a graph of treedepth at least d . Then G has a subgraph isomorphic to a path of length at least d .*

► **Theorem 4** ([6]). *For all $r, s, \ell \geq 1$, there is a number $c(r, s, \ell)$ such that every $K_{r,s}$ -subgraph-free graph of treedepth $c(r, s, \ell)$ has an induced P_ℓ .*

We also consider ordered lists. For lists $A = [a_1, \dots, a_n]$ and $B = [b_1, \dots, b_m]$, let $A + B = [a_1, \dots, a_n, b_1, \dots, b_m]$. For paths $P = (p_1, \dots, p_n) \subseteq G$ and $Q = (q_1, \dots, q_m) \subseteq G - (V(P) \setminus \{p_n\})$ such that $p_n = q_1$, we let $P + Q$ denote the path $(p_1, \dots, p_n, q_2, \dots, q_m)$.

For a path $P = (p_1, \dots, p_n)$, for every $i \in \{-n, \dots, -1\}$, we refer to p_{n+i+1} as p_i , i.e., $p_{-1} = p_n$. See Figure 3. For $i, j \in \{-n, \dots, n\} \setminus \{0\}$, let $i' = i$, if $i \geq 1$, and $i' = n + i + 1$ otherwise. Likewise, if $j \geq 1$, let $j' = j$, and $j' = n + j + 1$ otherwise. If $i' \leq j'$, let $P[i : j] = (p_{i'}, p_{i'+1}, \dots, p_{j'})$, else let $P[i : j] = (p_{j'}, p_{j'+1}, \dots, p_{i'})$. For vertices $u, v \in V(P)$, if $\text{dist}_P(p_1, u) = i - 1$ and $\text{dist}_P(p_1, v) = j - 1$, then let $P[u : v] = P[i : j]$. Further, for every integer $i \in \{-n, \dots, n\} \setminus \{0\}$ and vertex $u \in V(P)$, let $P[: i] = P[1 : i]$, $P[i :] = P[i : -1]$, $P[: u] = P[p_1 : u]$ and $P[u :] = P[u : p_{-1}]$.

The fan graph F_n is obtained by taking some path P of length $n - 1$ and adding a new vertex adjacent to every vertex of P . We call this new vertex the *centre* of the fan and those vertices at the ends of P the *ends* of the fan. We also say that the centre and ends are the three *poles* of the fan, see Figure 4. We say G contains some *protected fan* of order n , if there is some $Q \subseteq V(G)$ such that $G[Q]$ is isomorphic to F_{n-1} and for all $v \in Q$, either v is a pole or $N(v) \subseteq Q$.

3 T-type and L-type subgraphs, Jumps and Chain Extensions

The following definitions allow us to formalise the structural results which will be developed in Sections 4 and 5. In particular, the key results of those sections show that if a graph does not contain one of those ‘ H ’-type graphs we are concerned with, i.e., H -subgraph-free with

$H \in \{\mathbb{H}_m^{1,k,k,k}, S_{1,k,k,k}\}$, for some $m, k \geq 1$, then either the graph contains a large protected fan (see *Preliminaries*), or there exists a pair of vertices whose removal leaves a component of bounded treedepth. As we explain in Section 6, when such a component also satisfies additional properties, we can preprocess the input graph to an equivalent instance which is also H -subgraph-free but contains no such cut vertices. Towards this we define *T-type* and *L-type* subgraphs.

► **Definition 5** (*T-type* subgraph). A subgraph $G' \subseteq G$ is a *T-type* subgraph, for some treedepth bound $c \geq 1$, if G' has treedepth at most c and there exists some witness set $S \subseteq V(G) \setminus V(G')$ of size at most 2 such that $N(V(G')) \setminus V(G') = S$ and for each $v \in S$, $|N(v) \cap (V(G') \cup S)| \geq 2$.

► **Definition 6** (*L-type* subgraph). A subgraph $G' \subseteq G$ is an *L-type* subgraph, for some treedepth bound $c \geq 1$ and some length bound $\ell \geq 1$, if G' is a *T-type* subgraph with treedepth at most c , the witness set S has size 2 and there exists an induced path of length at least ℓ between the two vertices of S in the graph $G[V(G') \cup S]$.

The following definition and lemma regarding *minimal T-type* and *L-type* subgraphs will be especially useful in Section 6. The reason is that they ensure that *T-type* and *L-type* subgraphs can be preprocessed in polynomial time. This will then allow us to assume that they do not exist in our input graph G anymore.

► **Definition 7** (*Minimal T-type (L-type)* subgraph). $G' \subseteq G$ is a *minimal T-type (L-type)* subgraph if there is no $G'' \subsetneq G'$ such that G'' is a *T-type (L-type)* subgraph, with respect to the same treedepth bound (and length).

Note that *T-type* and *L-type* graphs may be disconnected. We now prove the following lemma.

► **Lemma 8.** For any $c \geq 1$ and $G' \subseteq G$, if G' is either a minimal *T-type* or *L-type* subgraph with treedepth bound c and witness set S , then $\text{td}(G[V(G') \cup S]) \leq 3c + 2$.

Proof. We note that by definition $\text{td}(G') \leq c$. We first suppose, for contradiction, that G' contains at least 4 connected components. As $|S| \leq 2$, there must exist components $C_1, C_2, C_3 \subseteq G'$, such that either both $N(V(C_1)) \setminus V(C_1) = S$ and $N(V(C_2)) \setminus V(C_2) = S$ or there is some $v \in S$ such that $N(V(C_1)) \setminus V(C_1) = \{v\}$ and $N(V(C_2)) \setminus V(C_2) = \{v\}$. Further, if G' is an *L-type* subgraph then we assume that for some component $C \in \{C_1, C_2, C_3\}$, there is an induced path meeting that length bound in $G[V(C) \cup S]$.

We let G'' be the disjoint union of C_1, C_2 and C_3 . Since G' contains at least 4 connected components, $G'' \subsetneq G$. In the first case it holds by definition that $N(V(G'')) \setminus V(G'') = S$. Given each vertex in S has some neighbour in C_1 and some neighbour in C_2 , it follows that for each $v \in S$, $|N(v) \cap (V(G'') \cup S)| \geq 2$. It follows that G'' is a *T-type (L-type)* subgraph, with respect to the same treedepth bound (and length) and witness set S . As $G'' \subsetneq G$ this contradicts the minimality of G' . Likewise, in the second case there is some $v \in S$ such that $N(V(G'')) \setminus V(G'') = \{v\}$ and $|N(v) \cap (V(G'') \cup S)| \geq 2$, that is G'' is a *T-type* subgraph, with respect to the same treedepth bound and with the witness set $\{v\}$. Again this contradicts the minimality of G' .

That is we now assume that G' contains at most 3 connected components. Recall that each of these components has treedepth at most c and $|S| \leq 2$. As adding a new vertex increases the treedepth by at most one, it follows that $\text{td}(G[V(G') \cup S]) \leq 3c + 2$. ◀

Further, the core of both of the key structural theorems (Theorems 20 and 19) will be in finding several long paths with a single common vertex.

Towards this, we identify a path P in our graph such that, for specific vertices in P , if removing these vertices disconnects P , then we obtain either a T -type or an L -type subgraph. That is, if no T -type or L -type subgraph exists, we conclude that there must exist additional paths between specific vertices which are disjoint from P .

These additional disjoint paths will then be used to construct long, internally disjoint paths between designated vertices. In turn, the existence of such paths will be used to show that our graph contains either an $S_{1,k,k,k}$ or an $\mathbb{H}_m^{1,k,k,k}$ subgraph, for $k, m \geq 1$.

The following definitions and corresponding notation will help us to formalise this reasoning and ensure that the constructed paths are indeed internally disjoint.

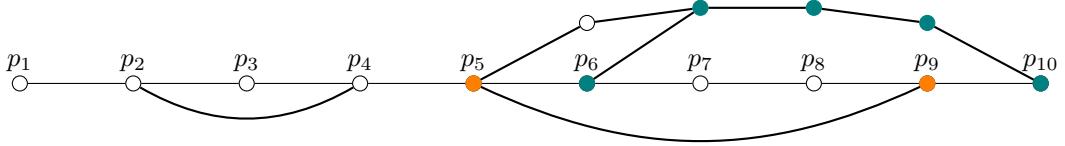


Figure 5 An illustration of Definitions 9, 10, and 11. The jump $Z^P(p_6, p_{10})$ is highlighted in teal and the jump $Z^P(p_5, p_9)$ in orange. There is a negative jump out of (p_4, p_7) . There are three positive jumps out of (p_4, p_7) and these have endpoints (p_5, p_9) , (p_5, p_{10}) and (p_6, p_{10}) , respectively. Here the maximum positive jump out of (p_4, p_7) has endpoints (p_6, p_{10}) , that is $(x^+(p_4, p_7), y^+(p_4, p_7)) = (p_6, p_{10})$.

► **Definition 9 (Jump).** Let G be a graph, and let $P = (p_1, \dots, p_n) \subseteq G$ be a path in G . For a pair of vertices $a, b \in V(P)$, a jump between a and b (with respect to P) is a path from a to b that is both edge-disjoint from P and internally vertex-disjoint from P , see Figure 5.

While there does not necessarily exist a unique jump between any pair of vertices, for a graph G and path $P \subseteq G$, for every $u, v \in V(P)$, we fix an arbitrary jump between u and v and call this $Z^P(u, v)$, if such a jump exists.

► **Definition 10 (Positive/negative jump out of an interval).** Let G be a graph, and let $P = (p_1, \dots, p_n) \subseteq G$ be a path in G .

For $u, v \in V(P)$, suppose $u \in V(P[:v])$. We say that the pair (a, b) are the endpoints of some jump out of the interval (u, v) if there is some jump $Z^P(a, b)$ between a and b , with $a \in V(P[u:v]) \setminus \{u, v\}$, $b \in V(P) \setminus V(P[u:v])$. If $b \in V(P[v:])$ then we say this is a positive jump out of (u, v) , else this is a negative jump out of (u, v) , see Figure 5.

► **Definition 11 (Maximum positive/negative jump out of an interval).** Let G be a graph, and let $P = (p_1, \dots, p_n) \subseteq G$ be a path in G . For $u, v \in V(P)$, without loss of generality $u \in V(P[:v])$. We consider jumps with respect to P .

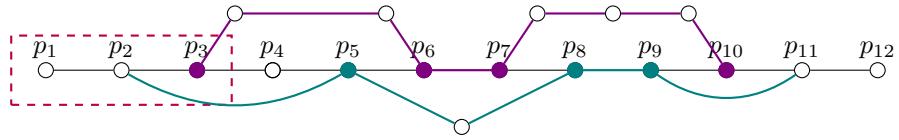
Suppose there is some positive (negative) jump out of the interval (u, v) with endpoints (a, b) such that $a \in V(P[u:v]) \setminus \{u, v\}$ and $b \in V(P[v:])$ ($b \in V(P[:u])$ if this is a negative jump). This jump is the maximum positive (negative) jump out of the interval (u, v) , if b maximises $\text{dist}_P(v, b)$ ($\text{dist}_P(u, b)$ if this is a negative jump) and a minimises $\text{dist}_P(a, b)$. See Figure 5 for an example.

We note that every maximum positive jump out of an interval has the same endpoints. For a path P and vertices $u, v \in V(P)$, we let $x^+(u, v)$ and $y^+(u, v)$ denote the endpoints of a

maximum positive jump out of (u, v) , with $x^+(u, v) \in V(P[u : v]) \setminus \{u, v\}$ and $y^+(u, v) \in V(P) \setminus V(P[u : v])$. Likewise, let $x^-(u, v)$ and $y^-(u, v)$ be the endpoints of a maximum negative jump out of (u, v) with $x^-(u, v) \in V(P[u : v]) \setminus \{u, v\}$ and $y^-(u, v) \in V(P) \setminus V(P[u : v])$.

► **Definition 12** (Jump sequence). *For a graph G and a path subgraph $P = (p_1, \dots, p_n)$ we say the list $I = [(x_1, y_1), \dots, (x_{|I|}, y_{|I|})]$ is a jump sequence if there is a jump between x_i and y_i (with respect to P) for every $i \in \{1, \dots, |I|\}$, see Figure 6.*

Suppose $I = [(x_1, y_1), \dots, (x_{|I|}, y_{|I|})]$ is a jump sequence. Throughout this paper, for $i \in \{1, \dots, |I|\}$ we will let $I_i^x = x_i$, $I_i^y = y_i$ and $I_i = (x_i, y_i)$. Further, for $i \in \{-|I|, \dots, -1\}$ we will let $I_i^x = x_{|I|+i+1}$, $I_i^y = y_{|I|+i+1}$ and $I_i = (x_{|I|+i+1}, y_{|I|+i+1})$, that is, $I_{-1} = I_{|I|} = (x_{|I|}, y_{|I|})$.



■ **Figure 6** An illustration of notation and definitions regarding paths, jumps, jump sequences and chain extensions (Definitions 12, 13, and 14). $I = [(p_2, p_5), (p_3, p_6), (p_5, p_8), (p_7, p_{10}), (p_9, p_{11})]$ is a jump sequence. To illustrate notation, $I_2 = (p_3, p_6)$, $I_{-1} = (p_9, p_{11})$, $I_2^x = p_3$ and $I_{-1}^y = p_{11}$. Further, I is a maximum length positive chain extension of (p_1, p_3) . That is $I = T^+(p_1, p_3)$. I does not describe a single path, but the odd and the even path as described in Observation 15 are drawn in teal and magenta respectively.

► **Definition 13** (Chain extension). *Let G be a graph with some path subgraph $P = (p_1, \dots, p_n)$. Let $u, v \in V(P)$. Without loss of generality, we say that $u \in V(P[: v])$. We say a jump sequence $I = [(x_1, y_1), \dots, (x_{|I|}, y_{|I|})]$ is a positive chain extension of (u, v) (with respect to P), if $(x_1, y_1) = (x^+(u, v), y^+(u, v))$ and $(x_i, y_i) = (x^+(u, y_{i-1}), y^+(u, y_{i-1}))$ for every $i \in \{2, \dots, |I|\}$. See Figure 6. Likewise, I is a negative chain extension of (u, v) if $(x_1, y_1) = (x^-(u, v), y^-(u, v))$ and $(x_i, y_i) = (x^-(v, y_{i-1}), y^-(v, y_{i-1}))$.*

When considering a specific graph G and a subpath $P \subseteq G$, for a pair of vertices $u, v \in V(P)$, we let $T^+(u, v)$ denote the maximum length positive chain extension from (u, v) and $T^-(u, v)$ denote the maximum length negative chain extension from (u, v) . We note that $T^+(u, v)$ and $T^-(u, v)$ do not necessarily exist but if they do they are unique. Note that as $T^+(u, v)$ is maximal, there is no positive jump out of $(u, T^+(u, v)^y_{-1})$. Likewise, there is no negative jump out of $(T^-(u, v)^y_{-1}, v)$.

► **Definition 14** (Paths associated with a jump sequence). *For a jump sequence I we say the path $Z^P(x_1, y_1) \cup \bigcup_{2 \leq i \leq |I|} (P[y_{i-1} : x_i] \cup Z^P(x_i, y_i))$ is associated with I , where, for every $i \in \{1, \dots, |I|\}$, $Z^P(x_i, y_i)$ is that arbitrary jump fixed between x_i and y_i .*

We highlight that there is not necessarily a path associated with a given jump sequence. Moreover, depending on which arbitrary jumps have been chosen, the resulting vertices may or may not form a valid path. Nonetheless, whenever it is relevant, we will justify the existence of such a path and explain why it suffices to consider an arbitrary path.

► **Observation 15.** *Let G be a graph containing some path $P \subseteq G$. Suppose that for $u, v \in V(P)$, T is a chain extension of (u, v) , with respect to P . There exist paths associated with the jump sequences $[(T_i^x, T_i^y) : i \bmod 2 = 1, 1 \leq i \leq |T|]$ and $[(T_i^x, T_i^y) : i \bmod 2 =$*

$0, 1 \leq i \leq |T|$, which we call the odd and even path of T , respectively. Further, if $|T| \geq 2$ the odd and the even path of T are disjoint.

Proof. We will prove this observation by induction on the length of $|T|$. We will assume that T is a positive chain extension. The case where T is a negative chain extension will follow symmetrically. For the notation, we direct the reader to the above definitions and notation regarding jump sequences. Suppose $|T| = 2$. The paths $Z^P(T_1^x, T_1^y)$ and $Z^P(T_2^x, T_2^y)$ are disjoint else there exists some jump from T_1^x to T_2^y , thus contradicting that the jump between T_1^x and T_1^y was a maximum jump.

We now assume that for some $\delta \geq 2$, every chain extension of (u, v) with length δ results in an odd path and an even path which are disjoint. Suppose $|T| = \delta + 1$. If for some $i \in \{1, \dots, |T| - 1\}$, the path $Z^P(T_{-1}^x, T_{-1}^y)$ intersects the path $Z^P(T_i^x, T_i^y)$ at some vertex which is not T_{-1}^x then there is some jump from T_i^x to T_{-1}^y , hence the jump between T_i^x and T_i^y were not maximum.

As $T - (T_{-1}^x, T_{-1}^y)$ is a chain extension of (u, v) with length δ , by assumption the paths associated with the jump sequences $[(T_i^x, T_i^y) : i \bmod 2 = 1, 1 \leq i \leq |T| - 1]$ and $[(T_i^x, T_i^y) : i \bmod 2 = 0, 1 \leq i \leq |T| - 1]$ are disjoint. We note that by definition T_{-1}^x and T_{-1}^y are the endpoints of the maximum jump from (u, T_{-2}^y) but not of a jump from (u, T_{-3}^y) (or (u, v) if $|T| = 3$), else the jump between (T_{-3}^x, T_{-3}^y) (or (u, v) if $|T| = 3$) was not maximum. It follows that, $T_{-1}^x \in V(P[T_{-3}^y : T_{-2}^y]) \setminus \{T_{-2}^y\}$ (or $V(P[v : T_{-2}^y]) \setminus \{T_{-2}^y\}$ if $|T| = 3$). Suppose δ is even.

It follows that the path $P[T_{-3}^y : T_{-1}^x] + Z^P(T_{-1}^x, T_{-1}^y) - (T_{-3}^y)$ is disjoint from both the odd and the even path of $T - (T_{-1}^x, T_{-1}^y)$. That is, if δ is even, then the odd path of $T - (T_{-1}^x, T_{-1}^y)$ extended via $P[T_{-3}^y : T_{-1}^x] + Z^P(T_{-1}^x, T_{-1}^y)$ is disjoint from the even path of $T - (T_{-1}^x, T_{-1}^y)$, that is the odd and the even paths of T are disjoint. That is, if δ is even, then our observation holds in the inductive case.

Suppose now that δ is odd, then the even path of $T - (T_{-1}^x, T_{-1}^y)$ extended via $P[T_{-3}^y : T_{-1}^x] + Z^P(T_{-1}^x, T_{-1}^y)$ is disjoint from the even path of $T - (T_{-1}^x, T_{-1}^y)$, that is the odd and the even paths of T are disjoint. Thus concluding the inductive case and so also the proof of this observation. \blacktriangleleft

4 Subdivided \mathbb{H}_1 Graphs: Structural Results

In this section we develop those structural results concerning subdivided \mathbb{H}_1 graphs. As outlined in the previous section, jumps will play a central role in the proofs. The next two lemmas form the foundation of our reasoning: the first shows that we can either identify a desired subgraph or reason about the endpoints of jumps, while the second guarantees the existence of such jumps in the graph.

► **Lemma 16.** *For any $m, k \geq 1$, let G be a graph with minimum degree at least 3. Suppose G contains vertices a, b and an induced path P with length at least $3m^2 + 7k + 2$, such that:*

- $a \in V(P[k+1 : -(k+1)])$
- $\text{dist}_P(a, b) \geq 2k$
- *there is a jump from a to b with respect to P , see Definition 9*

Then G contains a protected fan on $m+k+2$ vertices or some $\mathbb{H}_m^{1,k,k,k}$ subgraph.

Proof. Let G be a graph with those properties described in the Lemma statement and let $P = (p_1, \dots, p_\ell)$ be that induced path of length at least $3m^2 + 7k + 2$. Given P has length at least $3m^2 + 7k + 2$, after possibly relabelling a and b , we claim that at least one of the

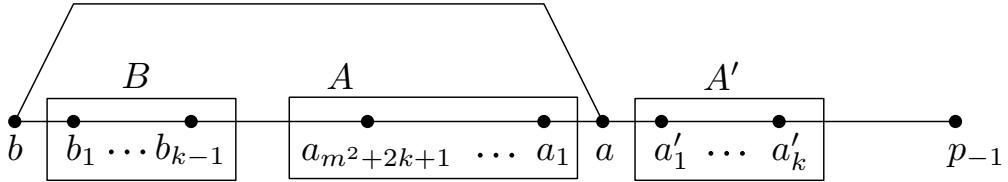


Figure 7 An illustration Lemma 16, in particular of the case where $a \in V(P[b :])$ and $\text{dist}_P(a, b) \geq m^2 + 3k + 1$. The paths A , A' and B are highlighted alongside the vertices a_1, \dots, a_{m^2+3k+1} , a'_1, \dots, a'_k and b_1, \dots, b_{k-1} . We note that the vertices a_1, \dots and b_1, \dots each have a pair of labels, i.e. $a_1 = b_{\text{dist}_P(a,b)-1}$ and $b_1 = a_{\text{dist}_P(a,b)-1}$.

following must hold.

$$\begin{aligned}\text{dist}_P(a, b) &\geq m^2 + 3k + 1 \\ \text{dist}_P(p_1, a) &\geq m^2 + 2k + 1 \text{ and } a \in V(P[:b]) \text{ or} \\ \text{dist}_P(a, p_{-1}) &\geq m^2 + 2k + 1 \text{ and } a \in V(P[b:])\end{aligned}$$

Suppose towards a contradiction that none of the above inequalities hold. We will consider the case where $a \in V(P[:b])$, the case where $a \in V(P[b:])$ will follow symmetrically. Given that $\text{dist}_P(p_1, a) \leq m^2 + 2k$ and $\text{dist}_P(a, b) \leq m^2 + 3k$ it follows that $\text{dist}_P(p_1, b) \leq 2m^2 + 5k$ and so $\text{dist}_P(b, p_{-1}) \geq m^2 + 2k + 1$. That is exchanging a and b we find that $a \in V(P[b:])$ and $\text{dist}_P(a, p_{-1}) \geq m^2 + 2k + 1$, a contradiction. We may therefore assume that at least one of the above inequalities hold.

We will now define disjoint paths A , A' and B . The path A will have length $m^2 + 2k$, the path A' will have length $k - 1$ and the path B will have length $k - 2$. Further, the paths $(a) + A$, $(a) + A'$ and $(b) + B$ will each be a subpath of P . Let $(a, \dots, b_{k-1}, \dots, b_1, b) = P[a : b]$ be the subpath of P between a and b and $B = (b_1, \dots, b_{k-1})$. If $\text{dist}_P(a, b) \geq m^2 + 3k + 1$, then let $(a, a_1, \dots, a_{m^2+3k+1}, \dots, b) = P[a : b]$ be the subpath of P between a and b , note that for $i \in \{1, \dots, \text{dist}_P(a, b) - 1\}$, $a_i = b_{\text{dist}_P(a,b)-i}$. In addition, we let $(a, a'_1, \dots, a'_k, \dots, p_{-1}) = P[a :]$, if $b \in V(P[:a])$, and $(p_1, \dots, a'_k, \dots, a'_1, a) = P[:]a$ otherwise. See Figure 7.

If $a \in V(P[:b])$ and $\text{dist}_P(p_1, a) \geq m^2 + 2k + 1$, then let $(p_1, \dots, a_{m^2+2k+1}, \dots, a_1, a) = P[:]a$ and $(a, a'_1, \dots, a'_k, \dots, b) = P[a : b]$. Else, $a \in V(P[b:])$ and $\text{dist}_P(a, p_{-1}) \geq m^2 + 2k + 1$. Let $(a, a_1, \dots, a_{m^2+2k+1}, \dots, p_{-1}) = P[a :]$ and $(a, a'_1, \dots, a'_k, \dots, b) = P[a : b]$. Let $A = (a_1, \dots, a_{m^2+2k+1})$ and $A' = (a'_1, \dots, a'_k)$. We note that the paths A , A' and B have the properties outlined.

Let Z denote that path from a to b in $G - (V(P) \setminus \{a, b\})$. We will first consider the case where Z has length 2, that is without loss of generality there is a vertex $x \in V(G)$ such that $Z = (a, x, b)$.

▷ **Claim 16.1.** Either G contains a $\mathbb{H}_m^{1,k,k,k}$ subgraph or the vertices $\{x, a_{m^2-m+1}, \dots, a_{m^2+k+1}\}$ form a protected fan on $m + k + 2$ vertices.

Proof. To simplify arguments, we will also denote a by a_0 . Recall that the paths A , A' and B are disjoint subpaths of P and have length $m^2 + 2k$, $k - 1$ and $k - 2$ respectively. We also make the following pair of observations.

Observation 1: if there is some $i \in \{m, \dots, m^2 + k + 1\}$ such that a_i is adjacent to some $y_i \in V(G \setminus P)$ with $y_i \neq x$ and a_{i-m} is adjacent to x , then G contains $\mathbb{H}_m^{1,k,k,k}$ with degree three vertices a_{i-m} and a_i ; the path (a_i, y_i) of length 1; and paths of length at least k via $(x, b) + B$, $A[i-m : 1] + (a) + A'$ and $A[i : i+k]$.

Observation 2: if there is some $i \in \{m+1, \dots, m^2+k+1\}$ such that a_i is adjacent to some $y_i \in V(G \setminus P)$ with $y_i \neq x$ and a_{i-m+1} is adjacent to x , then G contains $\mathbb{H}_m^{1,k,k,k}$ with degree three vertices x, a_i ; the path (a_i, y_i) of length 1; and paths of length at least k via $(x) + B, (x) + A'$ and $A[i : i+k]$.

We will now apply these observations to show that, if G does not contain $\mathbb{H}_m^{1,k,k,k}$ as a subgraph, then for each $\delta \in \{1, \dots, m\}$, the vertices $a_{\delta m-(\delta-1)}, \dots, a_{\delta m}$ each have exactly one neighbour outside P , namely x . Our proof will follow by induction on δ . We highlight that, by definition, G has minimum degree at least 3 meaning every vertex in A is either adjacent to x or has some neighbour not in $V(P) \cup \{x\}$. As $a_0 = a$ and a is adjacent to x , it follows by Observation 1, that if G does not contain $\mathbb{H}_m^{1,k,k,k}$ as a subgraph then $N(a_m) \setminus V(P) = \{x\}$. That is our claim holds in the base case. Suppose now $N(\{a_{\delta m-(\delta-1)}, \dots, a_{\delta m}\}) \setminus V(P) = \{x\}$ for some $\delta \in \{1, \dots, m-1\}$. For every $i \in \{(\delta+1)m-(\delta-1), \dots, (\delta+1)m\}$, by assumption $N(a_{i-m}) \setminus V(P) = \{x\}$. That is by Observation 1, if G does not contain $\mathbb{H}_m^{1,k,k,k}$ as a subgraph then $N(a_i) \setminus V(P) = \{x\}$. Likewise, for $i = (\delta+1)m - ((\delta+1)-1)$, by assumption $N(a_{i-m+1}) \setminus V(P) = \{x\}$. That is by Observation 2, if G does not contain $\mathbb{H}_m^{1,k,k,k}$ as a subgraph then $N(a_i) \setminus V(P) = \{x\}$. That is, the vertices $a_{(\delta+1)m-((\delta+1)-1)}, \dots, a_{(\delta+1)m}$ each have exactly one neighbour outside P , namely x , thus concluding our inductive step.

Hence, if G does not contain $\mathbb{H}_m^{1,k,k,k}$ as a subgraph, then the vertices $a_{m^2-m+1}, \dots, a_{m^2}$ each have exactly one neighbour outside P , namely x . We will now also show that, if G does not contain $\mathbb{H}_m^{1,k,k,k}$ as a subgraph, then for every $i \in \{m^2+1, \dots, m^2+k+1\}$, $N(a_i) \setminus V(P) = \{x\}$, that is the vertices $\{x, a_{m^2-m+1}, \dots, a_{m^2+k+1}\}$ form a protected fan on $m+k+2$ vertices. Assume towards a contradiction, that there is some minimum $i \in \{m^2+1, \dots, m^2+k+1\}$ such that $N(a_i) \setminus V(P) \neq \{x\}$. As i is minimal and $N(a_{m^2-m+1}, \dots, a_{m^2}) \setminus V(P) = \{x\}$, it follows that $N(a_{i-m}) \setminus V(P) = \{x\}$. By Observation 1 either $N(a_i) \setminus V(P) = \{x\}$ or G contains $\mathbb{H}_m^{1,k,k,k}$ as a subgraph. By assumption $N(a_i) \setminus V(P) \neq \{x\}$ and G does not contain $\mathbb{H}_m^{1,k,k,k}$ as a subgraph, that is we have a contradiction and find that the vertices $\{x, a_{m^2-m+1}, \dots, a_{m^2+k+1}\}$ form a protected fan on $m+k+2$ vertices. \triangleleft

Claim 16.1 implies that if Z has length 2, then our Lemma holds. We will now show that this can be also used to solve the more general case, i.e., where Z has length greater than 2. In particular, we will show that either G contains $\mathbb{H}_m^{1,k,k,k}$ as a subgraph or there is some pair a', b' such that $a' \in V(P[k+1 : -(k+1)])$, $\text{dist}_P(a', b') \geq 2k$ and there is a path of length 2 from a' to b' which is internally disjoint from P . That is, by Claim 16.1 either G contains a $\mathbb{H}_m^{1,k,k,k}$ subgraph or a protected fan on $m+k+2$ vertices.

\triangleright Claim 16.2. If G does not contain $\mathbb{H}_m^{1,k,k,k}$ as a subgraph, then there is some $z \in V(Z) \setminus \{a, b\}$ such that z is adjacent to a_m and $a_{m \lceil \frac{2k+m}{m} \rceil}$.

Proof. We will first show that a_m is adjacent to some vertex $z \in V(Z) \setminus \{a, b\}$. As G has minimum degree at least 3, if a_m is not adjacent to some vertex $z \in V(Z) \setminus \{a, b\}$ then it must have some neighbour $y \in V(G) \setminus (V(P) \cup V(Z))$. However, now G contains $\mathbb{H}_m^{1,k,k,k}$ as a subgraph with degree 3 vertices a and a_m ; a path of length 1 via (a_m, y) ; and paths of length at least k via $(a) + Z + B, (a) + A', A[m : m+k]$, a contradiction. That is a_m is adjacent to some vertex in $z \in V(Z) \setminus \{a, b\}$.

We will now show that for every $\delta \in \{1, \dots, \lceil \frac{2k+m}{m} \rceil\}$, $a_{\delta m}$ is adjacent to z . Suppose there is some minimum $\delta \in \{1, \dots, \lceil \frac{2k+m}{m} \rceil\}$, such that $a_{\delta m}$ is not adjacent to z . As G has minimum degree at least 3, $a_{\delta m}$ has some neighbour $y \in V(G \setminus P)$, such that $y \neq z$. As δ is minimal and by definition a_m is adjacent to z , it follows that $a_{\delta m-m}$ is adjacent to z . We highlight that for either $Z' = Z[a : z]$ or $Z' = Z[z : b]$, $y \notin V(Z')$. Now we observe that G contains $\mathbb{H}_m^{1,k,k,k}$ as a subgraph with degree 3 vertices $a_{\delta m-m}$ and $a_{\delta m}$; a path of length 1 via

$(a_{\delta m}, y)$; and paths of length at least k via $A[\delta m - m : 1] + (a) + A'$, $(a_{\delta m - m}) + Z' + (b) + B$ and $A[\delta m : \delta m + k]$, a contradiction.

It follows that $a_{\delta m}$ is adjacent to z for every $\delta \in \{1, \dots, \lceil \frac{2k+m}{m} \rceil\}$, that is $a_{m \lceil \frac{2k+m}{m} \rceil}$ is adjacent to z concluding the proof of this claim. \triangleleft

Let $a' = a_m$ and $b' = a_{m \lceil \frac{2k+m}{m} \rceil}$. By definition, $a' \in V(P[k+1 : -(k+1)])$ and $\text{dist}_P(a', b') \geq 2k$. Further, from Claim 16.2, there is a path of length 2 from a' to b' which is internally disjoint from P . That is, by Claim 16.1 either G contains a $\mathbb{H}_m^{1,k,k,k}$ subgraph or a protected fan on $m+k+2$ vertices which concludes the proof of this lemma. \blacktriangleleft

We also need the following lemma.

► **Lemma 17.** *For any $m, k \geq 1$, let G be a graph with minimum degree at least 3. Suppose G contains a vertex a and an induced path P with length at least $3m^2 + 7k + 2$, such that:*

- $a \in V(P[k+1 : -(k+1)])$
- *there is some path $Z = (a, z_1, \dots, z_{2k-1})$ of length at least $2k-1$ in $G \setminus (P - a)$*

Then G contains a protected fan on $m+k+2$ vertices or some $\mathbb{H}_m^{1,k,k,k}$ subgraph.

Proof. Let G be a graph with those properties described in the Lemma statement and let $P = (p_1, \dots, p_\ell)$ be that induced path of length at least $3m^2 + 7k + 2$. Without loss of generality, $\text{dist}_P(a, p_{-1}) \geq \text{dist}_P(p_1, a)$ and so $\text{dist}_P(a, p_{-1}) \geq \frac{3m^2+7k+2}{2} - 1$. Let $a_0 = a$, $A' = (a'_1, \dots, a'_k, \dots, p_1) = P[a : p_1] - \{a\}$ and $A = (a_0, \dots, p_{-1}) = P[a : p_{-1}]$.

We now claim that either G contains some $\mathbb{H}_m^{1,k,k,k}$ subgraph or for every $\delta \in \{1, \dots, \lceil \frac{2k}{m} \rceil\}$, the vertex $a_{\delta m}$ is adjacent to some vertex in $Z - a$. Suppose there is some minimum $\delta \in \{1, \dots, \lceil \frac{2k}{m} \rceil\}$ such that $a_{\delta m}$ is not adjacent to some vertex in $Z - a$. As G has minimum degree at least 3, $a_{\delta m}$ has some neighbour $y \notin V(P) \cup Z$. Further, as δ is minimal and $a_0 = a$, there is some $i \in \{1, \dots, 2k-1\}$ such that $a_{\delta m - m}$ is adjacent to z_i . If $i \geq k$, then let $Z' = Z[z_i : z_1]$ else let $Z' = Z[z_i : z_{2k-1}]$, note in each case Z' has length at least $k-1$. Now G contains $\mathbb{H}_m^{1,k,k,k}$ as a subgraph with degree 3 vertices $a_{\delta m - m}$ and $a_{\delta m}$; a path of length 1 via $(a_{\delta m}, y)$; and paths of length at least k via $A[\delta m - m :] + (a) + A'$, $(a) + Z'$ and $A[\delta m : \delta m + k]$. That is, either G contains some $\mathbb{H}_m^{1,k,k,k}$ subgraph or for every $\delta \in \{1, \dots, \lceil \frac{2k}{m} \rceil\}$, the vertex $a_{\delta m}$ is adjacent to some vertex in $Z - a$.

Let $b = a_{m \lceil \frac{2k}{m} \rceil}$. From above, if G does not contain some $\mathbb{H}_m^{1,k,k,k}$ subgraph, then b is adjacent to some vertex in $Z - a$. Note that this implies that there is some path from a to b which is internally disjoint from P . By definition $a \in V(P[k+1 : -(k+1)])$ and $\text{dist}_P(a, b) \geq 2k$. Note that this implies that there is some path from a to b which is internally disjoint from P . It then follows from Lemma 16 that either G contains a protected fan on $m+k+2$ vertices or some $\mathbb{H}_m^{1,k,k,k}$ subgraph. \blacktriangleleft

Using the above lemma we can now prove the following lemma.

► **Lemma 18.** *For every $m, k \geq 1$, let G be a graph with minimum degree at least 3. Suppose G contains some induced path P with length at least $3m^2 + 7k + 2$ and vertices $u, v \in V(P[k+1 : -(k+1)])$ such that $\text{dist}_P(u, v) \geq 3$. Then at least one of the following properties must hold:*

- i) G contains a protected fan on $m+k+2$ vertices,
- ii) G contains some $\mathbb{H}_m^{1,k,k,k}$ subgraph,
- iii) G contains some L-type subgraph with treedepth bound $(4k-3)(\text{dist}_P(u, v)-1) - 1$ and length bound $\text{dist}_P(u, v) - 2$, see Definition 6, or
- iv) there is some jump out of (u, v) with respect to P , see Definition 10.

Proof. We direct the reader to Section 3 for definitions and notation regarding jumps and L -type subgraphs.

Suppose there exists some graph G and vertices $u, v \in V(G)$ as described in the Lemma statement. We highlight that if iv) does not hold, that is there is no jump out of (u, v) , then there is some connected component C in $G - \{u, v\}$ such that $V(P[u : v]) \subseteq V(C) \cup \{u, v\}$ and $V(C) \cap V(P \setminus V(P[u : v])) = \emptyset$. Suppose $\text{td}(C) \geq (4k - 3)(\text{dist}_P(u, v) - 1)$, then by Theorem 3, C contains a path of length at least $(4k - 3)(\text{dist}_P(u, v) - 1)$. Given that $|V(P) \cap V(C)| = \text{dist}_P(u, v) - 1$, it follows that $C - V(P)$ contains a path of length at least $(4k - 3)$. Let $Q = (q_1, \dots, q_{4k-2})$, denote this path. As C is a connected component, it contains some path from $V(P[u : v])$ to $V(Q)$. Let $p \in V(P[u : v]) \setminus \{u, v\}$ and $q_i \in V(Q)$ be vertices such that there is a path from p to q_i in C which is internally disjoint from $P[u : v]$ and Q . Without loss of generality we may assume that this path from p to q_i consists of a single edge. Let $Q' = (q_i, \dots, q_{i+2k-2})$, if $i \leq 2k$, and $Q' = (q_i, \dots, q_{i-(2k-2)})$ otherwise. As $p \in V(P[k + 1 : -(k + 1)])$ and there is some path $(p) + Q'$ of length at least $2k - 1$ in $G \setminus (P - p)$, from Lemma 17, we find that either G contains a protected fan on $m + k + 2$ vertices or some $\mathbb{H}_m^{1,k,k,k}$ subgraph. That is either i) or ii) is satisfied.

It follows that if i), ii) and iv) do not hold, then $\text{td}(C) \leq (4k - 3)(\text{dist}_P(u, v) - 1) - 1$ and $N(V(C)) \setminus V(C) = \{u, v\}$. We now claim that there is some L -type subgraph $C' \subseteq C$ with a witness pair $\{\hat{u}, \hat{v}\}$. Let u' be that distinct vertex in $N(u) \cap V(P[u : v])$ and v' be that distinct vertex in $N(v) \cap V(P[u : v])$. As G has minimum degree at least 3 and C is a connected component in $G - \{u, v\}$, it follows that $|N(u') \cap V(C)|, |N(v') \cap V(C)| \geq 2$. If $|N(u) \cap V(C)| \geq 2$, then let $\hat{u} = u$, else, let $\hat{u} = u'$. Likewise, if $|N(v) \cap V(C)| \geq 2$, then let $\hat{v} = v$, else, let $\hat{u} = v'$. If $\hat{u} \neq u$, then $N(u) \cap V(C) = \{u'\}$, symmetrically, if $\hat{v} \neq v$, then $N(v) \cap V(C) = \{v'\}$. It follows that $C - \{\hat{u}, \hat{v}\}$ consists of possibly multiple disjoint connected components in the graph $G - \{\hat{u}, \hat{v}\}$ each with treedepth at most that of C . By definition $|N(u') \cap (V(C) \setminus \{\hat{u}, \hat{v}\})|, |N(v') \cap (V(C) \setminus \{\hat{u}, \hat{v}\})| \geq 2$, that is, $C - \{\hat{u}, \hat{v}\}$ is an L -type subgraph with a witness set $\{\hat{u}, \hat{v}\}$, thus concluding the proof of this lemma. \blacktriangleleft

We now have those necessary components to prove our main structural theorem regarding subdivided \mathbb{H}_1 graphs.

► **Theorem 19.** *For any $k, m \geq 1$, there is some function $c(k, m)$, such that every graph G with treedepth at least $c(k, m)$ and minimum degree at least 3 contains either some protected fan on $m + k + 2$ vertices, an L -type subgraph with treedepth at most $(4k - 3)(8k^2 - 6k + 2m + 8) - 1$ and length bound $m + k$ or some $\mathbb{H}_m^{1,k,k,k}$ subgraph.*

Proof. Let G be a graph with minimum degree at least 3. We note that $K_{4k+m+1,4k+m+1}$ contains $\mathbb{H}_m^{1,k,k,k}$ as a subgraph, that is if G contains $K_{4k+m+1,4k+m+1}$ then our theorem holds. That is we assume that G is $K_{4k+m+1,4k+m+1}$ -subgraph-free. By Theorem 4, there is some function $c(k, m)$ such that if $\text{td}(G) \geq c(k, m)$ then G contains some induced path P with length at least $8k^2 - 6k + 3m^2 + 8$. As $8k^2 - 6k + 3m^2 + 8 > 3m^2 + 7k + 2$, the length of P is sufficient for the application of Lemmas 16, 17 and 18. We direct the reader to Section 3 for definitions and notation regarding L -type subgraphs, jumps and chain extensions. In particular, recall that for every pair $a, b \in V(P)$ we have fixed some arbitrary jump between a and b , if such a jump exists. We let $Z^P(a, b)$ denote this arbitrary jump.

Let q be the middle vertex of P and $r \neq r'$ be the pair of vertices such that $\text{dist}_P(q, r) = \text{dist}_P(q, r') = \lceil \frac{k+m+2}{2} \rceil$. Note that $\text{dist}_P(r, r') \geq k + m + 2$. Whenever we refer to a L -type subgraph, it is understood to be with respect to treedepth bound $(4k - 3)(8k^2 - 11k + m + 5) - 1$ and length bound $m + k$ unless stated otherwise.

▷ **Claim 19.1.** Either G contains some L -type subgraph, a protected fan on $m + k + 2$ vertices, some $\mathbb{H}_m^{1,k,k,k}$ subgraph or there is some chain extension of r, r' with size at least $2k$.

Proof. As $\text{dist}_P(q, r) = \text{dist}_P(q, r') \leq \frac{m+k+4}{2}$, $r, r' \in V(P[k+1 : -(k+1)])$ and $\text{dist}_P(r, r') \leq m + k + 4$. By Lemma 18 either G contains some protected fan on $m + k + 2$ vertices, some $\mathbb{H}_m^{1,k,k,k}$ subgraph, an L -type subgraph with treedepth bound at most $(4k - 3)(m + k + 4 - 1) - 1 \leq (4k - 3)(8k^2 - 6k + 2m + 8) - 1$ and length bound at least $\text{dist}_P(r, r') - 2 \geq m + k$, or there is some jump out of (r, r') . In the first three cases, this implies that our claim holds.

That is, we assume that there is some jump out of (r, r') . Without loss of generality we may assume this is a positive jump. Let $T^+(r, r')$ be the maximum positive chain extension of (r, r') . We note that, by definition, for $i \in \{1, \dots, |T^+(r, r')|\}$, there is some path from $T^+(r, r')_i^x$ to $T^+(r, r')_i^y$ which is both internally vertex disjoint and edge disjoint from P . That is by Lemma 16, if $\text{dist}_P(T^+(r, r')_i^x, T^+(r, r')_i^y) \geq 2k$ then G either contains some protected fan on $m + k + 2$ vertices or some $\mathbb{H}_m^{1,k,k,k}$ subgraph and our claim holds.

Assume now that $\text{dist}_P(T^+(r, r')_i^x, T^+(r, r')_i^y) \leq 2k - 1$, for all $i \in \{1, \dots, |T^+(r, r')|\}$. By definition $T^+(r, r')_i^x \in V(P[r, r']) \setminus \{r, r'\}$ and so $\text{dist}_P(q, T^+(r, r')_1^x) \leq \frac{m+k+4}{2} - 1$, it therefore follows that $\text{dist}_P(q, T^+(r, r')_i^y) \leq i(2k - 1) + \frac{m+k+4}{2} - 1$ for every $i \in \{1, \dots, |T^+(r, r')|\}$. That is either $|T^+(r, r')| \geq 2k$, in which case our claim holds, or $\text{dist}_P(q, T^+(r, r')_{-1}^y) \leq (2k - 1)^2 + \frac{m+k+4}{2} - 1$. We highlight that $\text{dist}_P(r, T^+(r, r')_{-1}^y) \geq \text{dist}_P(r, r') \geq m + k + 2$ and $\text{dist}_P(r, T^+(r, r')_{-1}^y) \leq (2k - 1)^2 + m + k + 4 - 1$. That is by Lemma 18 either G contains some protected fan on $m + k + 2$ vertices, some $\mathbb{H}_m^{1,k,k,k}$ subgraph, an L -type subgraph with treedepth bound $(4k - 3)((2k - 1)^2 + m + k + 4 - 1 - 1) - 1 \leq (4k - 3)(8k^2 - 6k + 2m + 8) - 1$, for $k \geq 2$, or there is some jump out of $(r, T^+(r, r')_{-1}^y)$. In each of these cases, except for the final, our claim holds. That is we now assume that there is some jump out of $(r, T^+(r, r')_{-1}^y)$. By maximality of $T^+(r, T^+(r, r')_{-1}^y)$, this is a negative jump.

Let $T^-(r, T^+(r, r')_{-1}^y)$ be the maximum negative chain extension of $(r, T^+(r, r')_{-1}^y)$. Note that by definition, $T^-(r, T^+(r, r')_{-1}^y)_1^y \in V(P[: r]) \setminus \{r\}$, that is if $T^-(r, T^+(r, r')_{-1}^y)_1^x \in V(P[r' :])$ then $\text{dist}_P(T^-(r, T^+(r, r')_{-1}^y)_1^x, T^-(r, T^+(r, r')_{-1}^y)_1^y) \geq 2k$. From Lemma 16, this implies that either G contains a protected fan on $m + k + 2$ vertices or some $\mathbb{H}_m^{1,k,k,k}$ subgraph, that is we assume that $T^-(r, T^+(r, r')_{-1}^y)_1^x \in V(P[r : r']) \setminus \{r, r'\}$. That is $T^-(r, T^+(r, r')_{-1}^y) = T^-(r, r')$.

Now applying the same arguments as for $T^+(r, r')$, we find that either $|T^-(r, r')| \geq 2k$, G contains a protected fan on $m + k + 2$ vertices, some $\mathbb{H}_m^{1,k,k,k}$ subgraph or $\text{dist}_P(q, T^-(r, r')_{-1}^y) \leq (2k - 1)^2 + \frac{m+k+4}{2} - 1$. Further,

$$m + k + 2 \leq \text{dist}_P(T^-(r, r')_{-1}^y, T^+(r, r')_{-1}^y) \leq 2((2k - 1)^2 + m + k + 4 - 1) = 8k^2 - 6k + 2m + 8.$$

By Lemma 18, we either find an L -type subgraph, or we obtain that there is some jump out of $(T^-(r, r')_{-1}^y, T^+(r, r')_{-1}^y)$. Suppose this jump has endpoints (x, y) . By maximality of $T^-(r, r')$ and $T^+(r, r')$, it is not the case that both $x, y \in V(P[: r'])$ or that both $x, y \in V(P[r :])$, that is we may assume that one of $x, y \in V(P[r :])$ and the other is in $V(P[: r'])$. Note this implies that $\text{dist}_P(x, y) \geq 2k$. Applying Lemma 16, we find that G contains either a protected fan on $m + k + 2$ vertices or some $\mathbb{H}_m^{1,k,k,k}$ subgraph. This concludes the proof of this claim. \triangleleft

From Claim 19.1, either G contains some L -type subgraph, a protected fan on $m + k + 2$ vertices, some $\mathbb{H}_m^{1,k,k,k}$ subgraph or there is some chain extension T of r, r' with size at least $2k$. We will now use this chain extension to show that either G contains a protected fan on $m + k + 2$ vertices or some $\mathbb{H}_m^{1,k,k,k}$ subgraph.

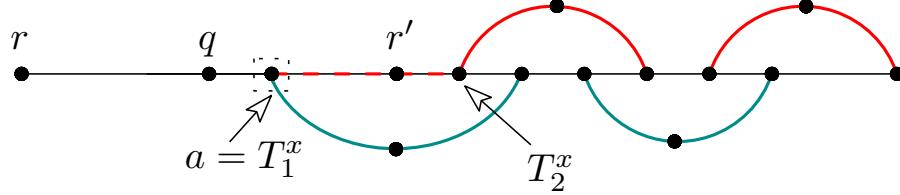


Figure 8 Illustration of the chain extension T from Claim 19.1, with $T = T^+(r, r')$. The odd path of T is shown in solid green, and the even path in solid red. The path Z_1 corresponds to the odd path of T and Z_2 corresponds to the even path of T extended to a . In the figure, Z_2 consists of both the solid and dashed red paths.

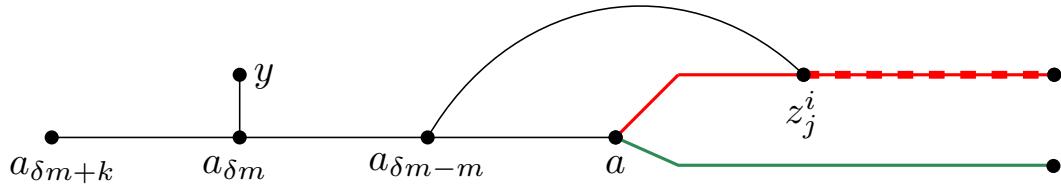


Figure 9 The path Z_1 is shown in solid red and the path Z_2 is shown in solid green. $a_{\delta m-m}$ is adjacent to some z_j^i , in the case depicted $i = 1$ and $j < k$. The path Z' is shown in dashed red and the path $Z'' = Z_2$, that is, it is seen in solid green.

By Observation 15, there exists a pair of disjoint paths described by T . As P is an induced path, to recall notation regarding jump sequences, for every $i \in \{1, \dots, |T|\}$, $Z^P(T_i)$ must have length at least 2. That is the odd and even path of T each have length at least $2 \cdot \frac{2k}{2} = 2k$. We let $a = T_1^x$. Let $Z_1 = (a, z_1^1, \dots, z_{2k-1}^1, \dots)$ denote the odd path of T and $Z_2 = (a, z_1^2, \dots, z_{2k-1}^2, \dots)$ denote the even path of T extended to a via the path $P[a : T_2^x]$. If $T = T^+(r, r')$, we let $A = (a, a_1, \dots, p_1)$ be that subpath of P between a and p_1 . If $T = T^-(r, r')$, let (a, a_1, \dots, p_{-1}) be that subpath of P between a and p_{-1} . Hence, $V(A) \cap Z_1 \cap Z_2 = \{a\}$.

Let $a_0 = a$. We claim that either G contains some $\mathbb{H}_m^{1,k,k,k}$ subgraph or for every $\delta \in \{1, \dots, \lceil \frac{2k}{m} \rceil\}$, the vertex $a_{\delta m}$ has some neighbour in $(Z_1 \cup Z_2) - \{a\}$. Suppose there is some minimum $\delta \in \{1, \dots, \lceil \frac{2k}{m} \rceil\}$ such that $a_{\delta m}$ is not adjacent to some vertex in $(Z_1 \cup Z_2) - \{a\}$. As G has minimum degree at least 3, $a_{\delta m}$ has some neighbour $y \notin V(P) \cup V(Z_1) \cup V(Z_2)$. Further, as δ is minimal and $a_0 = a$, there is some $i \in \{1, 2\}$ and $j \in \{1, \dots, 2k-1\}$ such that $a_{\delta m-m}$ is adjacent to z_j^i . If $j \geq k$, then let $Z' = Z_i[1 : j]$ else let $Z' = Z_i[j : 2k-1]$, note in each case Z' has length at least $k-1$. Let $Z'' = Z'_1$, if $i = 2$, and $Z'' = Z'_2$ otherwise. See Figure 9. Now G contains $\mathbb{H}_m^{1,k,k,k}$ as a subgraph with degree 3 vertices $a_{\delta m-m}$ and $a_{\delta m}$; a path of length 1 via $(a_{\delta m}, y)$; and paths of length at least k via $A[\delta m-m : a] + Z''$, $(a_{\delta m-m}) + Z'$ and $A[\delta m : \delta m+k]$. That is, either G contains some $\mathbb{H}_m^{1,k,k,k}$ subgraph or for every $\delta \in \{1, \dots, \lceil \frac{2k}{m} \rceil\}$, the vertex $a_{\delta m}$ is adjacent to some vertex in $Z'_1 \cup Z'_2$.

Let $b = a_{m \lceil \frac{2k}{m} \rceil}$. From above, if G does not contain some $\mathbb{H}_m^{1,k,k,k}$ subgraph, then b is adjacent to some vertex z in $Z'_1 \cup Z'_2$. By definition, there is some $i \in \{1, \dots, |T|\}$ such that $z \in V(Z^P(T_i))$. It follows that there is some path $Z^P(b, T_i^y)$. As $\text{dist}_P(b, T_i^y) \geq 2k$, it follows from Lemma 16 that either G contains a protected fan on $m+k+2$ vertices or some $\mathbb{H}_m^{1,k,k,k}$ subgraph. This concludes the proof of this theorem. \blacktriangleleft

5 Subdivided Stars (\mathbb{H}_0): Structural Results

This section is dedicated to subdivided star graphs. That is, we will prove the following theorem:

► **Theorem 20.** *For any $k \geq 1$, let $c = 16(2k - 1)(k - 1)$. If G is proper bridgeless, contains some degree 4 vertex, has treedepth at least $8(7k^3 + 15k^2 - \frac{4k}{9} + 3)^2 + 6$ and no T-type subgraph, with treedepth bound c , then G contains $S_{1,k,k,k}$ as a subgraph.*

Although the arguments to prove this result have several similarities with those used for subdivided \mathbb{H}_0 graphs, there are also crucial differences. In particular, unlike Theorem 25, here we require not only that the long paths intersect in a single vertex, but also that this vertex has degree at least 4.

To show Theorem 20 we begin with the following structural result from the literature.

► **Theorem 21 ([14]).** *Let $q, r \geq 1$. The subclass of connected $S_{1,1,q,r}$ -subgraph-free graphs that are not subcubic and are quasi-bridgeless has treedepth at most $2(q + r + 3)^2 + 6$.*

As G is proper bridgeless, has maximum degree at least 4 and treedepth at least $8(\ell + 3)^2 + 6$, for $\ell = 7k^3 + 15k^2 - \frac{4k}{9}$. By Theorem 21, G contains some $S_{1,1,\ell,\ell}$ as a subgraph. Let x be the centre of this $S_{1,1,\ell,\ell}$. Note that x is the middle vertex of some path P of length 2ℓ .

The central idea of the proof of Theorem 20 will be to show that there exists some vertex x with degree at least 4 and three paths each with length at least $2k$ sharing the single common vertex x . The following observation will then be used to find some $S_{1,k,k,k}$.

► **Observation 22.** *For every $k \geq 1$, if G contains some $S_{2k,2k,2k}$ subgraph with centre x and $\deg(x) \geq 4$, then G contains $S_{1,k,k,k}$ as a subgraph.*

Proof. Let G be a graph that contains a subgraph isomorphic to $S_{2k,2k,2k}$. Let x denote the centre of this subdivided star, and for each $i \in \{1, 2, 3\}$ and $j \in \{1, \dots, 2k\}$, let f_j^i denote the j th vertex along the i th branch.

If x has degree at least 4, then x has some neighbour $y \notin \{f_1^1, f_1^2, f_1^3\}$. Suppose $y \neq f_j^i$ for any $i \in \{1, 2, 3\}$ and $j \in \{2, \dots, k\}$. Then G contains $S_{1,k,k,k}$ with centre x , one branch of length 1 by the edge xy and three branches of length k by the paths (f_1^1, \dots, f_k^1) , (f_1^2, \dots, f_k^2) and (f_1^3, \dots, f_k^3) , respectively.

Therefore, we may assume that $y = f_j^i$ for some $i \in \{1, 2, 3\}$ and $j \in \{2, \dots, k\}$. Without loss of generality, let $i = 1$. In this case, G again contains $S_{1,k,k,k}$ as a subgraph, with centre x , one branch of length 1 via the edge xf_1^1 , and three branches of length k given by the paths $(f_j^1, \dots, f_{j+k}^1)$, (f_1^2, \dots, f_k^2) and (f_1^3, \dots, f_k^3) , respectively. ◀

The following lemma is crucial to construct those required paths. We note that this lemma is analogous to Lemma 18, however, now that these long paths must intersect at a vertex of degree at least 4, several additional conditions are now necessary.

► **Lemma 23.** *For every $k \geq 1$, let G be some $S_{1,k,k,k}$ -subgraph-free graph. Suppose that G contains some path P with length at least $4k - 2$ as a subgraph.*

Suppose there exist $u, v \in V(P[2k + 1 : -(2k + 1)])$ and some $x \in V(P[u : v]) \setminus \{v\}$ with $|N(x) \setminus V(P)| \geq 2$, such that:

- i) *there is no jump from x to some $y \in V(P) \setminus V(P[u : v])$, with respect to P (see Definition 9);*
- ii) *there exist internally disjoint paths D_v^1, D_v^2 from x to v ;*
- iii) *either $x = u$, or there exist internally disjoint paths D_u^1, D_u^2 from x to u ;*
- iv) *if $x \neq u$, then the paths D_v^1 , D_v^2 and D_u^1 are internally disjoint.*

Then either G contains some $S_{1,k,k,k}$ as a subgraph; G contains some T -type subgraph with treedepth bound $16(2k-1)(k-1)$; or there exists some jump out of (u, v) .

Proof. Let G be a graph with a path $P = (p_1, \dots, p_\ell) \subseteq G$, for some $\ell \geq 4k+3$ and some pair of vertices u, v as described above.

▷ **Claim 23.1.** If $u \neq x$, then there exist paths $\hat{D}_v^1, \hat{D}_v^2, \hat{D}_u^1, \hat{D}_u^2$, such that \hat{D}_v^1, \hat{D}_v^2 are internally disjoint x - v -paths; \hat{D}_u^1, \hat{D}_u^2 are internally disjoint x - u -paths; $\hat{D}_v^1, \hat{D}_v^2, \hat{D}_u^1$ are internally disjoint; and $\hat{D}_u^1, \hat{D}_u^2, \hat{D}_v^1$ are internally disjoint.

Proof. Suppose D_u^2 is internally disjoint from either D_v^1 or D_v^2 . If D_u^2 is internally disjoint from D_v^1 , then let $\hat{D}_v^1 = D_v^1$ and $\hat{D}_v^2 = D_v^2$. Otherwise, let $\hat{D}_v^1 = D_v^2$ and $\hat{D}_v^2 = D_v^1$. We note that now the paths $\hat{D}_v^1, \hat{D}_v^2, \hat{D}_u^1 = D_u^1, \hat{D}_u^2 = D_u^2$ have the desired disjointness properties.

We now assume that D_u^2 intersects both the path D_v^1 and D_v^2 . Let d be that final vertex in D_u^2 , such that $d \in V(D_v^1) \cup V(D_v^2)$, that is the subpath of D_u^2 from d to u contains no vertex from $V(D_v^1) \cup V(D_v^2)$ except for d . We can now define a path from x to u which intersects only one of D_v^1 and D_v^2 , if $d \in V(D_v^2)$ this is via $D_v^2[:d] + D_u^2[d:]$. Otherwise, this is via $D_v^1[:d] + D_u^2[d:]$. Let $\hat{D}_v^1 = D_v^1, \hat{D}_v^2 = D_v^2, \hat{D}_u^1 = D_u^1$ and $\hat{D}_u^2 = D_u^2[:d] + D_u^2[d:]$ in the first case and $\hat{D}_v^1 = D_v^2, \hat{D}_v^2 = D_v^1, \hat{D}_u^1 = D_u^1$ and $\hat{D}_u^2 = D_v^1[:d] + D_u^2[d:]$ in the second. We find that the resulting $\hat{D}_v^1, \hat{D}_v^2, \hat{D}_u^1, \hat{D}_u^2$ have the desired disjointness properties. \triangleleft

By Claim 23.1, if $u \neq x$, then we can define paths $\hat{D}_v^1, \hat{D}_v^2, \hat{D}_u^1, \hat{D}_u^2$, with the properties outlined above. Let $D_v^1 = \hat{D}_v^1, D_v^2 = \hat{D}_v^2, D_u^1 = \hat{D}_u^1, D_u^2 = \hat{D}_u^2$.

Let C_1 be that component of $G - \{u, v\}$ containing the path $D_v^1 - \{u, v\}$. If C_1 contains some vertex $y \in V(P) \setminus V(P[u:v])$, then there is some path from x to y in $G[V(C_1) \cup \{x\}]$, without loss of generality y is the single vertex from $V(P) \setminus V(P[u:v])$ on this path. From i), there is no jump from x to y . It follows that this path from x to y must contain some vertex from $P[u:v] - \{u, v\}$, let $x' \in V(P[u:v]) \setminus \{u, v\}$ be that vertex which is closest to y . By definition there is a jump between x' and y and this is a jump out of the interval (u, v) . It follows, if C_1 contains some vertex $y \in V(P) \setminus V(P[u:v])$, the lemma holds.

Hence, in the following we assume that C_1 contains no vertices from $V(P) \setminus V(P[u:v])$. Further, if D_v^1 has length at least $2k+1$, then the paths $P[:x], D_v^1 - \{v\}$ and $D_v^2 + P[v:]$ form a $S_{2k,2k,2k}$ subgraph with centre x . By Observation 22, it follows that G contains some $S_{1,k,k,k}$, hence our Lemma holds. Symmetrically, the same follows for D_v^2, D_u^1 and D_u^2 , that is, each of D_v^1, D_v^2, D_u^1 and D_u^2 have length at most $2k$.

We first consider the case where $u = x$, without loss of generality we may assume that $x \in P[:v]$. Suppose $\text{td}(C_1) \geq 8(2k-1)(k-1)$, then by Theorem 3, C_1 contains a path of length at least $8(2k-1)(k-1)$. Given that $|(V(D_v^1) \cup V(D_v^2)) \cap V(C_1)| \leq 2(2k-1)$, it follows that $C_1 - (V(D_v^1) \cup V(D_v^2))$ contains a path of length at least $4(k-1)$. Let $Q = (q_1, \dots, q_{4k-3})$, denote this path. As C_1 is a connected component, it contains some path from $V(D_v^1)$ to $V(Q)$. Let $d \in V(D_v^1) \cup V(D_v^2) \setminus \{u, v\}$ and $q_i \in V(Q)$ be vertices such that there is a path from d to q_i in C_1 which is internally disjoint from D_v^1, D_v^2 and Q . Without loss of generality we may assume that $d \in V(D_v^1)$ and this path from d to q_i consists of a single edge.

Let $Q' = (q_i, \dots, q_{i+2k-1})$, if $i \leq 2(k-1)$, and $Q' = (q_i, \dots, q_{i-(2k-1)})$ otherwise. The paths $P[:x], D_v^2 \cup P[v:]$ and $D_v^1[:d] \cup Q'$ form a $S_{2k,2k,2k}$ subgraph with centre x , from Observation 22 G contains $S_{1,k,k,k}$ as a subgraph and so our lemma holds.

That is, we assume $\text{td}(G[C_1]) < 8(2k-1)(k-1)$. If $(V(D_v^1) \cup V(D_v^2)) \setminus \{x, v\} \subseteq C_1$ then as the paths D_v^1, D_v^2 are internally disjoint, $|N(x) \cap (V(C_1) \cup \{v\})|, |N(v) \cap (V(C_1) \cup \{x\})| \geq 2$. It follows that $G[C_1]$ is a T -type subgraph with a witness set $\{x, v\} = \{u, v\}$ and so our lemma holds. Else, let C_2 be that component of $G - \{x, v\}$ containing the path $\hat{D}_v^2 - \{x, v\}$.

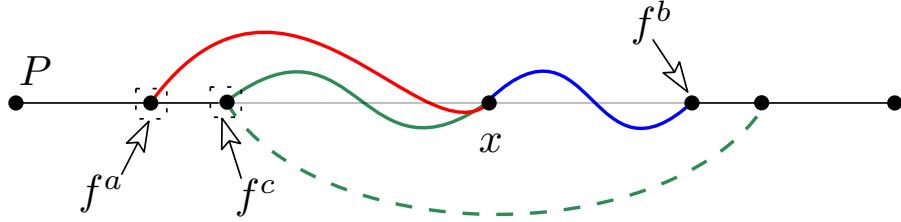


Figure 10 An illustration of Theorem 20, as described in the proof sketch. The active frontier path is drawn in solid red, the inert frontier path is drawn in solid blue and the candidate path is drawn in solid green. In the case depicted, in the inductive step the candidate path is extended via that jump drawn in dashed green. The previous inert frontier becomes the new candidate, the previous active frontier becomes the new inert frontier and the previous candidate becomes the new active frontier.

By symmetry $\text{td}(G[C_2]) < 8(2k - 1)(k - 1)$. Given the paths D_v^1 and D_v^2 are internally disjoint, $|N(v) \cap (V(C_1) \cup V(C_2) \cup \{x, v\})| \geq 2$. It follows that $G[V(C_1) \cup V(C_2)]$ is a T -type subgraph with a witness set $\{x, v\} = \{u, v\}$. The case where $u = x$ is now complete.

We now assume $u \neq x$. Suppose $\text{td}(G[C_1]) \geq 16(2k - 1)(k - 1)$, then by Theorem 3, $G[C_1]$ contains a path of length at least $16(2k - 1)(k - 1)$. Given $(V(D_v^1) \cup V(D_v^2) \cup V(D_u^1) \cup V(D_u^2)) \leq 4(2k - 1)$, it follows that $C_1 - (V(D_v^1) \cup V(D_v^2) \cup V(D_u^1) \cup V(D_u^2))$ contains some path of length at least $4(k - 1)$.

Let $Q = (q_1, \dots, q_{4k-3})$, denote this path. Let $d \in (V(D_v^1) \cup V(D_v^2) \cup V(D_u^1) \cup V(D_u^2)) \setminus \{u, v\}$ and $q_i \in V(Q)$ be a pair such that there is a path from d to q_i in C_1 which is internally disjoint from $D_v^1, D_v^2, D_u^1, D_u^2$ and Q . Without loss of generality $d \in V(D_v^1)$. We also assume that this path consists of a single edge, else we replace this edge by the respective path in the below reasoning. Let $Q' = (q_i, \dots, q_{i+2k-1})$, if $i \leq 2(k - 1)$, and $Q' = (q_i, \dots, q_{i-(2k-1)})$ otherwise. The paths $D_u^1 \cup P[: u], D_v^2 \cup P[v:]$ and $D_v^1[: d] \cup Q'$ form a $S_{2k, 2k, 2k}$ subgraph with centre x , from Observation 22, G contains $S_{1,k,k,k}$ as a subgraph and so our lemma holds. That is, we assume that $\text{td}(G[C_1]) < 16(2k - 1)(k - 1)$. We note as $x \in C_1$, it follows that $(V(D_v^1) \cup V(D_v^2) \cup V(D_u^1) \cup V(D_u^2)) \setminus \{u, v\} \subseteq C_1$. As the paths D_v^1, D_v^2 are internally disjoint $|N(v) \cap (C_1 \cup \{u\})| \geq 2$ and as the paths D_u^1, D_u^2 are internally disjoint $|N(u) \cap (C_1 \cup \{v\})| \geq 2$. That is $G[C_1]$ is a T -type subgraph with a witness set $\{x, v\} = \{u, v\}$ and concluding the proof of this lemma. \blacktriangleleft

We now sketch the proof of Theorem 20. Recall that our goal is to find three paths of length at least $2k$ with a single common vertex x . These paths will be constructed inductively. In every step of the induction, we either find some $S_{1,k,k,k}$ subgraph, or a pair of vertices $b^-, b^+ \in V(P)$ such that b^-, b^+ and x and P meet the conditions to apply Lemma 23. By Lemma 23, we either find some $S_{1,k,k,k}$, some T -type subgraph (with the desired treedepth bound) or some jump out of (b^-, b^+) . In the first case, the theorem follows, in the second we get a contradiction to the assumption that G contains no such T -type subgraph. That is we assume that such a jump exists. These jumps are used to construct three paths each of length at least $2k$ sharing the single common vertex x , that is our theorem follows.

The paths are built inductively. At the end of each inductive step we obtain three paths which share the single common vertex x , together with a pair of vertices b^-, b^+ which satisfy the conditions of Lemma 23. Moreover, in every block of three steps each of the paths grows in length by at least one, as ensured by the addition of a new jump. It follows that after $6k$ steps each of these paths have length at least $2k$, as desired.

To ensure that these paths have the desired properties, we label one of these paths as the active frontier, one of these as the inert frontier and the other as the candidate. Let f^a be the final vertex of the active frontier, f^b be the final vertex of the inert frontier, and f^c be the final vertex of the candidate path. We find that $f^c \in P[f^a : x]$ and either, $f^a \in P[: x]$ and $f^b \in P[: x]$, or, $f^a \in P[x :]$ and $f^b \in P[x :]$. See Figure 10.

During the inductive step, the inert frontier remains unchanged, the candidate path is extended by at least one edge (via the addition of a new jump), and the active frontier may either stay the same or increase in length. In the next step, the previous inert frontier becomes the new candidate, and one of the remaining two paths (either the former active frontier or the former candidate) takes the role of active frontier. If the previous active frontier does not increase in one step, it becomes the inert frontier in the step after, that is in the following step it is extended. This rotation ensures that after every three steps each of the three paths has grown by at least one.

To maintain disjointness, we introduce auxiliary properties ensuring that the extended paths intersect only at x and that we can identify a pair of vertices b^- , b^+ such that, together with x , they satisfy the conditions of Lemma 23. Concretely, we find the candidate and active frontiers can be extended to one of b^- , b^+ , while a subpath of the active frontier path, together with the inert frontier, extends to the other. These extensions of the active, inert and candidate paths remain internally disjoint, thus meeting the conditions for Lemma 23.

We now present the proof of Theorem 20 in full detail.

Theorem 20 (restated). *For any $k \geq 1$, let $c = 16(2k - 1)(k - 1)$. If G is proper bridgeless, contains some degree 4 vertex, has treedepth at least $8(7k^3 + 15k^2 - \frac{4k}{9} + 3)^2 + 6$ and no T -type subgraph, with treedepth bound c , then G contains $S_{1,k,k}$ as a subgraph.*

Proof. For $k \geq 1$, let $c = 16(2k - 1)(k - 1)$. We direct the reader to Section 3 for definitions and notation regarding T -type subgraphs, jumps and chain extensions. In this proof, whenever we refer to a T -type subgraph, it is understood to be with respect to the treedepth bound c unless explicitly stated otherwise. Let G be a graph with maximum degree at least 4, treedepth at least $8(7k^3 + 15k^2 - \frac{4k}{9} + 3)^2 + 6$ and no T -type subgraph.

Let $\ell = 7k^3 + 15k^2 - \frac{4k}{9}$. As G has maximum degree at least 4, treedepth at least $2(2\ell+3)^2+6$ and is proper bridgeless, it follows from Theorem 21, that G contains some $S_{1,1,\ell,\ell}$ subgraph. Let x denote the vertex at the centre of this $S_{1,1,\ell,\ell}$ and let $A^+ = (a_1^+, \dots, a_\ell^+)$ and $A^- = (a_1^-, \dots, a_\ell^-)$ be those paths such that $(x) + A^+$ and $(x) + A^-$ correspond to that pair of paths of length ℓ in $S_{1,1,\ell,\ell}$. Let $P = (a_\ell^-, \dots, a_1^-, x, a_1^+, \dots, a_\ell^+)$. Note we will consider all jumps with respect to P . As noted in Section 3, for every pair $a, b \in V(P)$ we will fix some arbitrary jump between a and b , if such a jump exists, and denote this by $Z^P(a, b)$.

In the following we will show that G contains a $S_{2k,2k,2k}$ subgraph with centre x . Observation 22 then implies that G contains $S_{1,k,k,k}$ as a subgraph and thus the theorem follows. To show the existence of a $S_{2k,2k,2k}$ subgraph, we will either describe some $S_{2k,2k,2k}$ subgraph with centre x , or inductively define pairs $(b_1^-, b_1^+), \dots, (b_{6k}^-, b_{6k}^+)$ together with the jump sequences $I_1^i \subseteq \dots \subseteq I_{6k}^i$ for $i \in \{1, 2, 3\}$. We will then show that the jump sequences $I_{6k}^1, I_{6k}^2, I_{6k}^3$ each have a corresponding path and that these paths can be extended such that they have a single common vertex x . We will use these paths to construct a $S_{2k,2k,2k}$ subgraph.

Intuitively, for $\delta \in \{1, \dots, 6k - 1\}$, in the inductive step from δ to $\delta + 1$ we will add at least one jump to at least one of I_δ^1 , I_δ^2 or I_δ^3 . We further show that after 3 steps each of I_δ^1 , I_δ^2 and I_δ^3 has increased in length by at least 1 so after at most $6k$ steps we find $S_{2k,2k,2k}$.

The vertices $b_{\delta-1}^-, b_{\delta-1}^+, b_\delta^-, b_\delta^+, b_{\delta+1}^-, b_{\delta+1}^+$ will be such that $P[b_{\delta-1}^- : b_{\delta-1}^+] \subseteq P[b_\delta^- : b_\delta^+] \subseteq P[b_{\delta+1}^- : b_{\delta+1}^+]$. If a jump between (a, b) is added to one of I_δ^1 , I_δ^2 or I_δ^3 , then

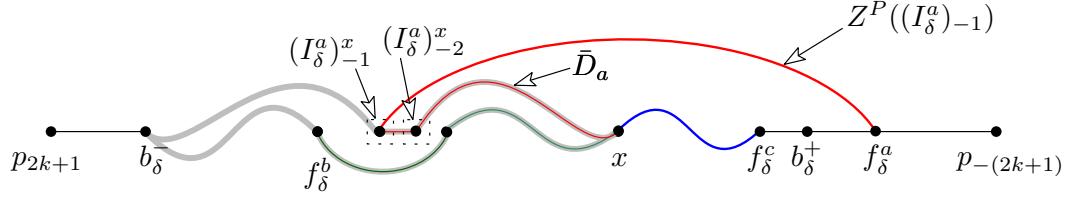


Figure 11 This diagram illustrates Properties 1-3. The paths D_a , D_b and D_c are shown in red, green and blue respectively. The paths D_a^{ext} and D_b^{ext} are both shown in gray. Note that D_b^{ext} is that gray path containing D_b as a subpath and D_a^{ext} is that gray path containing \bar{D}_a as a subpath.

$$a, b \notin V(P[b_{\delta-1}^- : b_{\delta-1}^+]) \setminus \{b_{\delta-1}^-, b_{\delta-1}^+\}.$$

Further, either every jump added to I_δ^1 , I_δ^2 and I_δ^3 will have an endpoint in $V(P[b_\delta^+ : b_{\delta+1}^+]) \setminus \{b_{\delta+1}^+\}$, in which case we say that b_δ^+ is *active*, or every jump added to I_δ^1 , I_δ^2 and I_δ^3 will have an endpoint in $V(P[b_\delta^- : b_{\delta+1}^-]) \setminus \{b_{\delta+1}^-\}$, in which case we say that b_δ^- is *active*.

To formalise these ideas and allow us to do this, the pairs of vertices and jump sequences will satisfy the following properties.

P1. $\text{dist}_P(x, b_1^+) = \text{dist}_P(x, b_1^-) \leq 4k^2 - 1$

P2. Exactly one of b_δ^+ and b_δ^- is active. If b_δ^+ is active we have the following properties, else the properties hold after interchanging b_δ^+ , $b_{\delta+1}^+$ with b_δ^- , $b_{\delta+1}^-$, and vice versa.

a) $b_{\delta+1}^-$ is active,

b) $b_{\delta+1}^- = b_\delta^-$,

c) there is no negative jump out of (b_δ^-, b_δ^+) , and

d) $\text{dist}_P(b_\delta^+, b_{\delta+1}^+) \leq 4k^2 - \frac{k(4\delta-14)}{3} + \frac{\delta(\delta-7)+11}{9} - 1$.

These sub-properties will imply that if b_δ^+ is active then for every jump with endpoints (a, b) added to I_δ^1 , I_δ^2 or I_δ^3 , we have $a \in V(P[b_\delta^+ : b_{\delta+1}^+]) \setminus \{b_{\delta+1}^+\}$.

P3. One of I_δ^1 , I_δ^2 and I_δ^3 is labelled the positive frontier, another the negative frontier and another the candidate. Again suppose b_δ^+ is active, otherwise, in the following, we replace b_δ^+ with b_δ^- and the positive with the negative frontier accordingly.

As b_δ^+ is active, we let the positive frontier be the active frontier and the negative frontier be the inert frontier. Let $a \neq b \neq c \in \{1, 2, 3\}$ be such that, I_δ^a is the active frontier, I_δ^b is the inert frontier and I_δ^c is the candidate. Using the notation introduced regarding jump sequences, for $i \in \{1, 2, 3\}$, if $|I_\delta^i| \geq 1$ we let $s_\delta^i = (I_\delta^i)_1^x$ and $f_\delta^i = (I_\delta^i)_{-1}^y$.

That is, s_δ^i and f_δ^i correspond to the *starting* and *final* vertex of the path described by I_δ^i .

If $I_\delta^i = []$, then let $s_\delta^i = f_\delta^i = x$. The following properties will hold.

a) $|I_\delta^a| \geq \lfloor \frac{\delta+2}{3} \rfloor$, $|I_\delta^b| \geq \lfloor \frac{\delta+1}{3} \rfloor$ and $|I_\delta^c| \geq \lfloor \frac{\delta}{3} \rfloor$.

b) $f_\delta^a = y^+(b_\delta^-, b_\delta^+)$ with $(I_\delta^a)_{-1}^x \in V(P[b_\delta^- : b_\delta^+])$ and $b_\delta^+ \in V(P[f_\delta^c : f_\delta^a]) \setminus \{f_\delta^a\}$.

c) Those paths described by I_δ^a , I_δ^b , I_δ^c can be extended to x via $P[x : s_\delta^a]$, $P[x : s_\delta^b]$ and $P[x : s_\delta^c]$ respectively. Let D_a , D_b and D_c denote these extended paths.

d) Let \bar{D}_a denote the subpath of D_a from x to $(I_\delta^a)_{-2}^y$. If $|I_\delta^a| = 1$, then let $\bar{D}_a = (x)$.

The path \bar{D}_a can be extended to b_δ^- , we denote the resulting path by D_a^{ext} . The path D_b can also be extended to b_δ^- , we denote the resulting path by D_b^{ext} .

e) The paths $P[: b_\delta^-]$, $P[b_\delta^+ :]$, $P[f_\delta^c : f_\delta^a]$, D_a^{ext} , D_b^{ext} and D_c are pairwise internally disjoint.

f) The path $Z^P((I_\delta^a)_{-1})$ is disjoint from both D_b^{ext} and D_c .

In our inductive step the *inert* frontier will remain the same, that is $I_{\delta+1}^b = I_\delta^b$. The *candidate* is the next list of pairs which will necessarily increase in length, that is $|I_{\delta+1}^c| \geq |I_\delta^c| + 1$, the *active* frontier may remain the same.

An illustration of these properties can be seen in Figure 11. If b_{6k}^+ is active, from Properties P2a) and b), for every $\delta \in \{2, \dots, 6k - 1\}$, if δ is odd, then $b_\delta^+ = b_\delta^+$. Further, from Property P2d), if δ is even then $\text{dist}_P(b_\delta^+, b_{\delta+1}^+) \leq 4k^2 - \frac{k(4\delta-14)}{3} + \frac{\delta(\delta-7)+11}{9} - 1$. Note similar reasoning follows for b_{6k}^- where we replace odd by even and visa versa. The case where b_{6k}^- is active now follows symmetrically.

Combining this with Property P1, it follows that

$$\begin{aligned} \text{dist}_P(x, b_{6k}^+), \text{dist}_P(x, b_{6k}^-) &\leq 4k^2 - 1 + \sum_{\delta=1}^{3k} \left(4k^2 - \frac{k(4\delta-14)}{3} + \frac{\delta(\delta-7)+11}{9} - 1 \right) \\ &= 7k^3 + 13k^2 - \frac{4k}{9} - 1. \end{aligned}$$

Likewise, for every $\delta \in \{1, \dots, 6k\}$, we obtain that $\text{dist}_P(x, b_\delta^+), \text{dist}_P(x, b_\delta^-) \leq 7k^3 + 13k^2 - \frac{4k}{9} - 1$. As $\text{dist}_P(x, a_\ell^+) = \text{dist}_P(x, a_\ell^-) = \ell$ and $\ell = 7k^3 + 15k^2 - \frac{4k}{9}$, we get that $b_\delta^+, b_\delta^- \in V(P[2k+1 : -(2k+1)])$ for $\delta \in \{1, \dots, 6k\}$.

Base case: Define b_1^+ and b_1^- with properties P1 and P2a–d.

Let $i \in \{1, \dots, \ell\}$ be the largest index, such that there is some jump from x to a_i^+ . That is, the largest i such that there is a path from x to a_i^+ which is both edge-disjoint and internally vertex-disjoint from P . If such an i exists let $y_0^+ = a_i^+$, if no such i exists let $y_0^+ = x$. Likewise, let $i' \in \{1, \dots, \ell\}$ be the largest index such that there is some jump from x to $a_{i'}^-$. Let $y_0^- = a_{i'}^-$, if such an i' exists, and $y_0^- = x$ otherwise.

If $\text{dist}_P(x, y_0^+) > 2k$, then the paths $P[:x]$, $P[x : a_{2k}^+]$ and $Z^P(x, a_i^+) + P[a_i^+ :]$ each have length at least $2k$ and share the single common vertex x . That is G contains a $S_{2k,2k,2k}$ subgraph with centre x and the theorem follows. Hence, we may assume that $\text{dist}_P(x, y_0^+) \leq 2k$ and symmetrically $\text{dist}_P(x, y_0^-) \leq 2k$.

▷ **Claim 20.1.** If $y_0^- = y_0^+ = x$, then either $\{x\}$ is the witness set for some T -type subgraph or G contains some $S_{2k,2k,2k}$ with centre x .

Proof. As x has degree at least 4, there exist at least two distinct vertices $q, q' \in N(x) \setminus P$. Let C and C' be those components of $G - x$ containing q and q' , respectively. Note that possibly $C = C'$. Since $y_0^- = y_0^+ = x$, every path from either q or q' to P must contain x and hence, $(V(C) \cup V(C')) \cap V(P) = \emptyset$. Suppose $\text{td}(C) \geq 4k + 1$. From Theorem 3, C contains some path Q of length at least $4k + 1$. As C is connected and contains q , either $q \in V(Q)$ or there is some shortest path from q to Q in C . Note this path can be extended via a subpath of Q with length at least $2k - 2$. By definition x and q are adjacent, that is we obtain a path Q' of length $2k$ from x in $G[V(C) \cup \{x\}]$. As $(V(C) \cup V(C')) \cap V(P) = \emptyset$, G contains $S_{2k,2k,2k}$ with centre x and paths of length at least $2k$ via Q' , $P[:x]$ and $P[x :]$. That is, our claim holds. Symmetrically, the same holds for C' . Hence, we may assume that $\text{td}(C), \text{td}(C') \leq 4k$. Note now $C \cup C'$ is a T -type subgraph with witness set $\{x\}$. Thus concluding the proof of this claim. ◇

Hence, by Claim 20.1, if $y_0^- = y_0^+ = x$ then our theorem follows. Thus, we may assume that $y_0^+ \neq x$ or $y_0^- \neq x$. Without loss of generality, suppose $y_0^+ \neq x$, else we exchange the roles of A^+ and A^- .

We further claim that y_0^-, y_0^+, x meet the conditions for Lemma 23. To see this, note first that, as $\text{dist}_P(x, y_0^-), \text{dist}_P(x, y_0^+) \leq 2k$, it follows that $y_0^-, y_0^+ \in V(P[2k+1 : -(2k+1)])$. Condition i) follows as y_0^-, y_0^+ were maximal. By definition of y_0^+ there exists a pair of internally disjoint paths from x to y_0^+ via $P[x : y_0^+]$ and $Z^P(x, y_0^+)$, respectively, showing condition ii). If $x \neq y_0^-$, then there also exist internally disjoint paths from x to y_0^- via

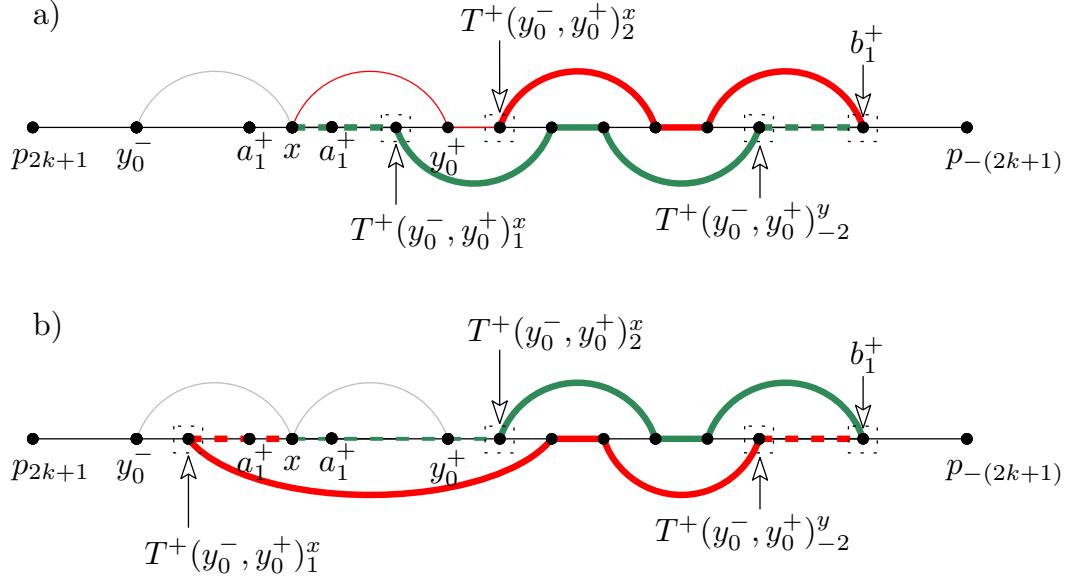


Figure 12 An illustration of the cases of Claim 20.2. In Figure a) $T^+(y_0^-, y_0^+)_1^x \in V(P[x : y_0^+])$ and in Figure b) $T^+(y_0^-, y_0^+)_1^x \in V(P[y_0^- : x])$. The odd path of T is drawn in solid red and the even path of T is drawn in solid green. These paths can also be extended to (x, b_1^+) via the dashed lines. The odd and the even path of $T^+(y_0^-, y_0^+)$ are emphasised using a thicker line. Note in Case a) the even path of $T^+(y_0^-, y_0^+)$ is a subpath of the odd path of T , that is, it is drawn in red. In Case b) $T = T^+(y_0^-, y_0^+)$ and so the even path of $T^+(y_0^-, y_0^+)$ is also the even path of T , that is, it is drawn in red.

$P[y_0^- : x]$ and $Z^P(y_0^-, x)$, respectively, showing condition iii) holds. Given the paths $P[x : y_0^+]$, $Z^P(y_0^-, x)$ and $P[y_0^- : x]$ are internally disjoint, condition iv) also follows. That is, by applying Lemma 23 we find that there must exist some jump out of (y_0^-, y_0^+) . Recalling the notation introduced regarding maximum jumps, we find that at least one of $(x^+(y_0^-, y_0^+), y^+(y_0^-, y_0^+))$ or $(x^-(y_0^-, y_0^+), y^-(y_0^-, y_0^+))$ must exist. Notice in both cases this corresponds to either a positive or negative chain extension of (y_0^-, y_0^+) with length 1.

Suppose that $(x^+(y_0^-, y_0^+), y^+(y_0^-, y_0^+))$ exists. The case where $(x^-(y_0^-, y_0^+), y^-(y_0^-, y_0^+))$ is the only existing pair will follow symmetrically. Recall that $T^+(y_0^-, y_0^+)$ is the maximum positive chain extension of (y_0^-, y_0^+) . As $(x^+(y_0^-, y_0^+), y^+(y_0^-, y_0^+)) \in T^+(y_0^-, y_0^+)$, necessarily $|T^+(y_0^-, y_0^+)| \geq 1$. Let $b_1^- = y_0^-$ and $b_1^+ = T^+(y_0^-, y_0^+)_{-1}^y$. Recall that $T^+(y_0^-, y_0^+)_{-1}^y$ corresponds to the second endpoint of the final jump of $T^+(y_0^-, y_0^+)$. Again we direct the reader to Section 3 for details of definitions and notation regarding jump sequences. We say b_1^- is active. By maximality of $T^+(y_0^-, y_0^+)$ there is no positive jump out of (b_1^-, b_1^+) , that is Property P2c) holds in the base case.

Another useful idea to introduce is that of extending a path via P . Let $Q = (q_1, \dots, q_r)$ be a path, and let $s, t \in P$. If $q_1, q_r \in P$, we call the path $P[s : q_1] + Q + P[q_r : t]$ Q extended to (s, t) via P . Note we also require that the paths $P[s : q_1] - q_1$, Q and $P[q_r : t] - q_r$ are disjoint. Additionally, may extend the empty path to (s, t) via P , which will be the path $P[s : t]$.

▷ **Claim 20.2.** There is some jump sequence T , such that either $T = T^+(y_0^-, y_0^+)$ or $T = (x, y_0^+) + T^+(y_0^-, y_0^+)$. We note that in the first case T is the maximum positive chain extension of (y_0^-, y_0^+) and in the second case T is the maximum positive chain extension of (a_1^-, a_1^+) . Note that a_1^- and a_1^+ are both adjacent to x in P . Further, the odd path of T (see

Observation 15 for odd and even paths) extended to (x, b_1^+) via P and the even path of T extended to (x, b_1^+) via P are internally disjoint. In addition, these paths are disjoint from $P[: y_0^-]$ and either the path $P[y_0^- : x]$ or $Z^P(y_0^-, x)$.

Proof. We first note that both the odd path and the even path of $T^+(y_0^-, y_0^+)_1^x \in V(P[x : y_0^+])$, see Figure 12a). Recall that by the notation introduced regarding jump sequences, $T^+(y_0^-, y_0^+)_1^x$ is the first vertex of the odd path. Let $T = (x, y_0^+) + T^+(y_0^-, y_0^+)$. Note T is a positive chain extension of (a_1^-, a_1^+) . Further, $T_1^x = x$ and $T_2^x = T^+(y_0^-, y_0^+)_1^x$, that is the odd path of $T^+(y_0^-, y_0^+)$ is a subpath of the even path of T , see Figure 12a). If $|T|$ is odd, then the odd path of T is a (x, b_1^+) path. Further, this path is disjoint from the paths $P[x : T_1^y] - \{x, T_1^y\}$ and $P[T_{-1}^x : b_1^+] - \{T_{-1}^x, b_1^+\}$. By Observation 15, the odd and even path of T are disjoint, that is the odd path is internally disjoint from the even path of T extended to (x, b_1^+) via P , that is our claim holds. Similarly, if $|T|$ is even, then the even path of T extended via $P[x : T_2^x]$ and the odd path of T extended via $P[T_{-2}^y : b_1^+]$ are internally disjoint from each other and the path $P[: x]$. That is our claim holds.

If $T^+(y_0^-, y_0^+)_1^x \in V(P[y_0^- : x])$ let $T = T^+(y_0^-, y_0^+)$, see Figure 12b). By definition, T is a positive chain extension of (y_0^-, y_0^+) . Extending the odd and the even path of T to (x, b_1^+) via P we again find that these paths are internally disjoint. Note, if $|T| \geq 2$, then the odd path is extended by $P[x : T_1^x]$ and the even path is extended by $P[x : T_2^x]$, further, depending on if $|T|$ is odd or even one of these paths will also be extended via the path $P[T_{-2}^y : b_1^+]$. If $|T| = 1$, then the even path of T is empty. The odd and the even path of T extended to (x, b_1^+) via P correspond to the paths $P[x : T_1^x] + Z^P(T_1^x, b_1^+)$ and $P[x : b_1^+]$, respectively. Note that both of them are disjoint from the path $Z^P(y_0^-, x)$, thus proving our claim. \triangleleft

\triangleright **Claim 20.3.** If $\text{dist}_P(x, b_1^+) \geq 4k^2$, then G contains some $S_{2k, 2k, 2k}$ subgraph with centre x .

Proof. We first claim, if $\text{dist}_P(x, b_1^+) \geq 4k^2$, then at least one of the following must hold,

- $|T^+(y_0^-, y_0^+)| \geq 4k - 2$,
- $\text{dist}_P(T^+(y_0^-, y_0^+)_i^y, T^+(y_0^-, y_0^+)^y_{i+1}) \geq 2k - \lceil \frac{i}{2} \rceil$ for some $i \in \{1, \dots, |T^+(y_0^-, y_0^+)| - 1\}$, or
- $\text{dist}_P(y_0^+, T^+(y_0^-, y_0^+)_1^y) \geq 2k$.

Towards a contradiction, suppose $|T^+(y_0^-, y_0^+)| \leq 4k - 3$, $\text{dist}_P(y_0^+, T^+(y_0^-, y_0^+)_1^y) \leq 2k - 1$ and $\text{dist}_P(T^+(y_0^-, y_0^+)_i^y, T^+(y_0^-, y_0^+)^y_{i+1}) \leq 2k - \lceil \frac{i}{2} \rceil - 1$ for every $i \in \{1, \dots, |T^+(y_0^-, y_0^+)| - 1\}$. Recall that $\text{dist}_P(x, y_0^+) \leq 2k$ and $b_1^+ = T^+(y_0^-, y_0^+)^y_{-1}$. Hence, it follows that for every $i \in \{1, \dots, |T^+(y_0^-, y_0^+)|\}$, we get that

$$\text{dist}_P(x, T^+(y_0^-, y_0^+)_i^y) \leq 2k + \sum_{j=1}^i \left(2k - \left\lceil \frac{j-1}{2} \right\rceil - 1 \right),$$

that is, as $|T^+(y_0^-, y_0^+)| \leq 4k - 3$,

$$\begin{aligned} \text{dist}_P(x, b_1^+) &\leq 2k + (4k - 3)(2k - 1) - \sum_{j=1}^{4k-3} \left\lceil \frac{j-1}{2} \right\rceil \\ &= 2k + (4k - 3)(2k - 1) - (2k - 2)(2k - 1) = 4k^2 - 2k + 1, \end{aligned}$$

a contradiction.

That is we assume that $|T^+(y_0^-, y_0^+)| \geq 4k - 2$, $\text{dist}_P(T^+(y_0^-, y_0^+)_i^y, T^+(y_0^-, y_0^+)^y_{i+1}) \geq 2k - \lceil \frac{i}{2} \rceil$ for some $i \in \{1, \dots, |T^+(y_0^-, y_0^+)| - 1\}$, or $\text{dist}_P(y_0^+, T^+(y_0^-, y_0^+)_1^y) \geq 2k$.

From Claim 20.2, for every $i \in \{1, \dots, |T^+(y_0^-, y_0^+)| - 1\}$ there exist paths from x to y_0^- , $T^+(y_0^-, y_0^+)_i^y$ and $T^+(y_0^-, y_0^+)_i^y$ which share a single common vertex which is x . Further, by definition these paths have lengths at least 0, $\lceil \frac{i}{2} \rceil + 1$ and $\lceil \frac{i+1}{2} \rceil + 1$ respectively.

Suppose that $|T^+(y_0^-, y_0^+)| \geq 4k - 2$. We extend that path from x to y_0^- via $P[: y_0^-]$. This path alongside those paths from x to $T^+(y_0^-, y_0^+)_i^y$ and $T^+(y_0^-, y_0^+)_i^y$ describe three paths of length at least $2k$ with a single common vertex x . That is G contains some $S_{2k,2k,2k}$ subgraph with centre x .

Suppose now that $\text{dist}_P(T^+(y_0^-, y_0^+)_i^y, T^+(y_0^-, y_0^+)_i^y) \geq 2k - \lceil \frac{i}{2} \rceil$. We extend that path from x to y_0^- via the path $P[: y_0^-]$, that path from x to $T^+(y_0^-, y_0^+)_i^y$ via the path $P[T^+(y_0^-, y_0^+)_i^y : T^+(y_0^-, y_0^+)_i^y] - T^+(y_0^-, y_0^+)_i^y$ and that path from x to $T^+(y_0^-, y_0^+)_i^y$ via the path $P[T^+(y_0^-, y_0^+)_i^y :]$. We note that these three extended paths have length at least $2k$ and have a single common vertex x . That is G contains some $S_{2k,2k,2k}$ subgraph with centre x .

This leaves only the case where $\text{dist}_P(y_0^+, T^+(y_0^-, y_0^+)_1^y) \geq 2k$. Claim 20.2 also implies that there exist paths from x to y_0^- , y_0^+ and $T^+(y_0^-, y_0^+)_1^y$ which share a single common vertex which is x . If $\text{dist}_P(y_0^+, T^+(y_0^-, y_0^+)_1^y) \geq 2k$, then extending the paths via $P[: y_0^-]$, $P[y_0^+ : T^+(y_0^-, y_0^+)_1^y] - T^+(y_0^-, y_0^+)_1^y$ and $P[T^+(y_0^-, y_0^+)_1^y :]$, respectively, we obtain three paths of length at least $2k$ with a single common vertex x . That is G contains some $S_{2k,2k,2k}$ subgraph with centre x . \triangleleft

It follows then from Claim 20.3, that Property P1 holds.

We now also claim that the vertices b_1^- and b_1^+ meet the conditions for Lemma 23 to be applied. To see this, we first note that from Claim 20.3 we get $b_1^-, b_1^+ \in V(P[2k+1 : -(2k+1)])$. Further, by the maximality of y_0^- and y_0^+ , condition i) of Lemma 23 holds for b_1^-, b_1^+ . By Claim 20.2, there exists a pair of internally disjoint paths from x to b_1^+ (via T) and another pair of internally disjoint paths from x to y_0^- (via $P[x : y_0^-]$ or $Z^P(x, y_0^-)$), implying condition ii) and iii) of Lemma 23 also hold. At least one of these x to y_0^- paths is disjoint from both x to b_1^+ paths, that is, applying Lemma 23, either $y^+(b_1^-, b_1^+)$ or $y^-(b_1^-, b_1^+)$ must exist. As $b_1^- = y_0^-$ and $b_1^+ = T^+(y_0^-, y_0^+)_1^y$, it follows that $y^+(b_1^-, b_1^+)$ does not exist, else $T^+(y_0^-, y_0^+)_1^y$ was not maximum. That is $y^-(b_1^-, b_1^+)$ must exist. We let $(c_1^x, c_1^y) = (x^-(b_1^-, b_1^+), y^-(b_1^-, b_1^+))$. By definition, $c_1^x \in V(P[y_0^- :]) \setminus \{y_0^-\}$ and $c_1^y \in V(P[: b_1^-]) \setminus \{b_1^-\}$, that is intuitively $(x^-(b_1^-, b_1^+), y^-(b_1^-, b_1^+))$ cross b_1^- . We say that b_1^- is active. We highlight that Properties P2a-d are satisfied.

We will now consider the following specific case to simplify the remainder of the base case.

\triangleright **Claim 20.4.** Suppose that $T^-(y_0^-, y_0^+) = T^-(y_0^-, T^+(y_0^-, y_0^+)_1^y)$ and $T^+(y_0^-, y_0^+) = T^+(T^-(y_0^-, y_0^+)_1^y, y_0^+)$ and the paths $Z^P(T^-(y_0^-, y_0^+)_1^y)$, $Z^P(T^+(y_0^-, y_0^+)_1^y)$ are disjoint. Then either G contains some $S_{2k,2k,2k}$ subgraph with centre x or a T -type subgraph.

Proof. An illustration of the following is shown in Figure 13. We note that as $T^+(y_0^-, y_0^+) = T^+(T^-(y_0^-, y_0^+)_1^y, y_0^+)$ and these are maximal, there does not exist a positive jump out of the interval $(T^-(y_0^-, y_0^+)_1^y, T^+(y_0^-, y_0^+)_1^y)$. Further, as $T^-(y_0^-, y_0^+) = T^-(y_0^-, T^+(y_0^-, y_0^+)_1^y)$ there does not exist a negative jump out of the interval $(T^-(y_0^-, y_0^+)_1^y, T^+(y_0^-, y_0^+)_1^y)$.

We now claim that the vertices $(T^-(y_0^-, y_0^+)_1^y, T^+(y_0^-, y_0^+)_1^y)$ meet the criteria to apply Lemma 23. As there is no jump out of the interval $(T^-(y_0^-, y_0^+)_1^y, T^+(y_0^-, y_0^+)_1^y)$, it follows that either G contains some $S_{2k,2k,2k}$ subgraph with centre x or a T -type subgraph.

By Claim 20.3, we get that either G contains some $S_{2k,2k,2k}$ subgraph with centre x or $\text{dist}_P(x, T^+(y_0^-, y_0^+)_1^y) < 4k^2$. Symmetrically, the same holds for $T^-(y_0^-, y_0^+)_1^y$, that is,

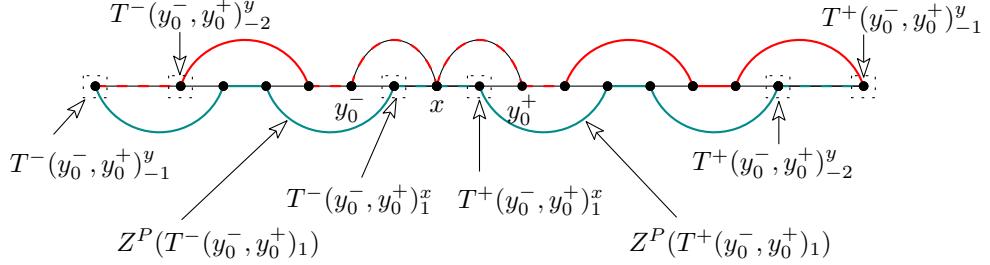


Figure 13 Illustration of Claim 20.4. Here $T^-(y_0^-, y_0^+) = T^-(y_0^-, T^+(y_0^-, y_0^+)_1^y)$, $T^+(y_0^-, y_0^+) = T^+(y_0^-, T^+(y_0^-, y_0^+)_1^y)$ and the paths $Z^P(T^-(y_0^-, y_0^+)_1)$, $Z^P(T^+(y_0^-, y_0^+)_1)$ are disjoint. The solid green paths correspond to the odd path of $T^-(y_0^-, y_0^+)$ and the odd path of $T^+(y_0^-, y_0^+)$. The dashed green paths show these extended to $(x, T^-(y_0^-, y_0^+)_1^y)$ and $(x, T^+(y_0^-, y_0^+)_1^y)$ paths. The solid red paths correspond to the even path of $T^-(y_0^-, y_0^+)$ and the even path of $T^+(y_0^-, y_0^+)$. The dashed red paths show these extended to $(x, T^-(y_0^-, y_0^+)_2^y)$ and $(x, T^+(y_0^-, y_0^+)_2^y)$ paths.

$T^-(y_0^-, y_0^+)_1^y, T^+(y_0^-, y_0^+)_1^y \in V(P[2k+1 : -(2k+1)])$. Condition i) holds by maximality of y_0^- and y_0^+ .

We note that both the odd and even paths of $T^+(y_0^-, y_0^+)$ and $T^-(y_0^-, y_0^+)$ are internally disjoint from the paths $Z^P(x, y_0^+)$ and $Z^P(x, y_0^-)$ else y_0^- or y_0^+ weren't maximum. It follows that we can extend the odd path of $T^+(y_0^-, y_0^+)$ via the path $P[x : T^+(y_0^-, y_0^+)_1^x]$ and the even path of $T^+(y_0^-, y_0^+)$ via the path $Z^P(x, y_0^+) + P[y_0^+ : T^+(y_0^-, y_0^+)_2^x]$. We note that these are internally disjoint $(x, T^+(y_0^-, y_0^+)_1^y)$ paths. Note, symmetrically, we can define internally disjoint $(x, T^-(y_0^-, y_0^+)_1^y)$ paths. It follows that condition ii) and iii) hold.

We now claim that the extended odd path of $T^-(y_0^-, y_0^+)$ is internally disjoint from both extended paths of $T^+(y_0^-, y_0^+)$. Suppose these paths are not internally disjoint. It follows that there exists some odd $i \in \{1, \dots, |T^-(y_0^-, y_0^+)|\}$ and $j \in \{2, \dots, |T^+(y_0^-, y_0^+)|\}$ such that the paths $Z^P(T^-(y_0^-, y_0^+)_i)$ and $Z^P(T^+(y_0^-, y_0^+)_j)$ intersect. It follows that there exists a jump between $T^-(y_0^-, y_0^+)_i^x$ and $T^+(y_0^-, y_0^+)_j^y$ hence $T^+(y_0^-, y_0^+)_1^y$ was not maximum. It follows that the extended odd path of $T^-(y_0^-, y_0^+)$ is internally disjoint from both extended paths of $T^+(y_0^-, y_0^+)$ and so condition iv) holds. That is applying Lemma 23, we find that either G contains some $S_{2k,2k,2k}$ subgraph with centre x or a T -type subgraph, thus concluding our proof. \triangleleft

Suppose that $T^+(y_0^-, y_0^+)_1^x \in V(P[x : y_0^+]) \setminus \{y_0^+\}$, $c_1^x \in V(P[y_0^- : x]) \setminus \{y_0^-\}$ and the paths $Z^P(T^+(y_0^-, y_0^+)_1)$ and $Z^P(c_1^x, c_1^y)$ are disjoint. We note that this implies that the maximum negative chain extension of (b_1^-, b_1^+) is also the maximum negative chain extension of (y_0^-, y_0^+) .

In this case we will *reverse* P , i.e. let $P = (a_\ell^+, \dots, a_1^+, x, a_1^-, \dots, a_\ell^-)$ and redefine vertices $y_0^-, y_0^+, b_1^-, b_1^+, c_1^x$ and c_1^y based on this new path.

As the maximum negative chain extension of (y_0^-, y_0^+) is also the maximum negative chain extension of $(y_0^-, T^+(y_0^-, y_0^+)_1^y)$, before reversing P , and b_1^+ becomes what was previously $T^-(y_0^-, y_0^+)_1^y$, when P is reversed. It follows that if, $T^+(y_0^-, y_0^+)_1^x \in V(P[x : y_0^+]) \setminus \{y_0^+\}$, $c_1^x \in V(P[y_0^- : x]) \setminus \{y_0^-\}$ and the paths $Z^P(T^+(y_0^-, y_0^+)_1)$ and $Z^P(c_1^x, c_1^y)$ are disjoint, then the same must hold where P is reversed. More formally, that is $T^-(y_0^-, T^+(y_0^-, y_0^+)_1^y) \in V(P[y_0^- : x]) \setminus \{y_0^-\}$ and $T^+(T^-(y_0^-, y_0^+)_1^y, y_0^+) \in V(P[x : y_0^+]) \setminus \{y_0^+\}$.

As $Z^P(T^-(y_0^-, T^+(y_0^-, y_0^+)_1^y))$, $Z^P(T^+(T^-(y_0^-, y_0^+)_1^y, y_0^+))$ are disjoint, it then follows by Claim 20.4 that either G contains some $S_{2k,2k,2k}$ subgraph with centre x or a T -type subgraph. In both of these cases our Theorem holds.

Base case: Define the jump sequences I_1^1, I_1^2, I_1^3 with properties **P3a–f**.

Recall that b_1^- is active. Note we will define the jump sequences I_1^1, I_1^2, I_1^3 such that I_1^1 is the positive frontier, I_1^2 is the candidate and I_1^3 is the inert frontier.

▷ **Claim 20.5.** We can define jump sequences I_1^1, I_1^2, I_1^3 with the following properties.

B1. $(I_1^1)_-^y = c_1^y, (I_1^1)_-^x \in V(P[b_1^- : b_1^+]) \setminus \{b_1^-, b_1^+\}$ and $b_1^- \in V(P[f_1^1 : f_1^2]) \setminus \{f_1^1\}$.

B2. The paths described by I_1^1, I_1^2, I_1^3 can be extended to x via $P[x : s_1^1], P[x : s_1^2]$ and $P[x : s_1^3]$ respectively. Let D_1, D_2 and D_3 denote these paths. Recall that for $i \in \{1, 2, 3\}$, if $I_1^i = []$, then $s_1^i = x$ and D_i consists of the single vertex x . Else, $s_1^i = (I_1^i)_1^x$.

B3. Let \bar{D}_1 be the subpath of D_1 from x to $(I_\delta^a)_-^y$. If $|I_\delta^a| = 1$, then let $\bar{D}_1 = (x)$. The paths \bar{D}_1 and D_3 can both be extended to b_δ^- . We denote these extended paths by D_1^{ext} and D_3^{ext} respectively.

B4. The paths $P[: b_1^-], P[b_1^+ :], P[c_1^y : y_0^+], D_1^{ext}, D_3^{ext}$ and D_2 are pairwise internally disjoint.

B5. The path $Z^P((I_1^1)_-^1)$ is disjoint from both D_3^{ext} and D_2 .

Proof. We first note that, if the path $Z^P(c_1^x, c_1^y)$ intersects either $Z^P(y_0^-, x)$ or $Z^P(x, y_0^+)$ then the path $Z^P(x, c_1^y)$ exists. This is a contradiction as it implies that y_0^- was not maximal.

We first consider the case where $c_1^x \in V(P[y_0^- : y_0^+]) \setminus \{y_0^+\}$. In this case, the path $Z^P(c_1^x, c_1^y)$ is internally disjoint from the even path of $T^+(y_0^-, y_0^+)$ else $T^+(y_0^-, y_0^+)_1^y$ was not maximal. Further, if $Z^P(c_1^x, c_1^y)$ is not disjoint from the odd path of $T^+(y_0^-, y_0^+)$ it may only intersect with vertices in $Z^P(T^+(y_0^-, y_0^+)_1)$, else again $T^+(y_0^-, y_0^+)_1^y$ was not maximal.

We now further consider the following subcases based on the position of c_1^x and $T^+(y_0^-, y_0^+)_1^x$. That is, the cases of: $c_1^x, T^+(y_0^-, y_0^+)_1^x \in V(P[x : y_0^+])$; $c_1^x, T^+(y_0^-, y_0^+)_1^x \in V(P[y_0^- : x])$; or one of c_1^x and $T^+(y_0^-, y_0^+)_1^x$ is in $V(P[y_0^- : x])$ and the other in $V(P[x : y_0^+])$. In the final of these cases, we will also differentiate between where $V(Z^P(c_1^x, c_1^y)) \cap V(Z^P(T^+(y_0^-, y_0^+)_1)) = \emptyset$ and where $V(Z^P(c_1^x, c_1^y)) \cap V(Z^P(T^+(y_0^-, y_0^+)_1)) \neq \emptyset$. Each of these cases are illustrated in Figure 14 with Table 1 giving the specific jump sequences I_1^1, I_1^2, I_1^3 , paths D_1, D_2, D_3 and chain extension \hat{T} corresponding to each case.

■ **Subcase 1.** If $c_1^x, T^+(y_0^-, y_0^+)_1^x \in V(P[x : y_0^+])$, let $I_1^1 = [(c_1^x, c_1^y)], I_1^2 = []$ and $I_1^3 = [(x, y_0^+)]$. Note, $V(P[f_1^1 : f_1^2]) \setminus \{f_1^1\} = V(P[c_1^y : x]) \setminus \{c_1^y\}$ so by definition, $b_1^- \in V(P[f_1^1 : f_1^2]) \setminus \{f_1^1\}$ and so Properties B1 and B2 hold by definition. Let D_1, D_2 and D_3 be those paths as defined in the claim statement.

Let \hat{T} denote the jump sequence $(x, y_0^+) + T^+(y_0^-, y_0^+)$. We note that \hat{T} is the maximum positive chain extension of (a_1^-, a_1^+) . Further, $\hat{T}_2^x = T^+(y_0^-, y_0^+)_1^x$ and $\hat{T}_{-1}^y = b_1^+$. We extend the even path of \hat{T} by $P[x : \hat{T}_2^x]$. Observe that the odd path of \hat{T} and this extended even path of \hat{T} both begin with the vertex x .

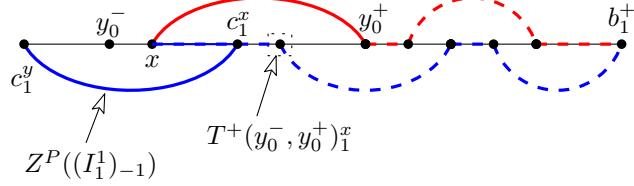
We now show that these paths can be further extended such that they both end with the vertex \hat{T}_{-1}^y , recall that $b_1^+ = \hat{T}_{-1}^y$. If $|\hat{T}|$ is odd then we obtain these (x, b_1^+) paths by extending the even path of \hat{T} by $P[\hat{T}_{-2}^y : \hat{T}_{-1}^y]$, else if $|\hat{T}|$ is even, we extend the odd path of \hat{T} by $P[\hat{T}_{-2}^y : \hat{T}_{-1}^y]$. We denote the extended odd path by D_3^{ext} and the extended even path by D_1^{ext} . We note that D_1^{ext} and D_3^{ext} are internally disjoint.

We note that $D_3 = Z^P(x, y_0^+)$ is a subpath of D_3^{ext} and $\bar{D}_1 = (x)$ is a subpath of D_1^{ext} . That is, Property B3 of our claim also holds.

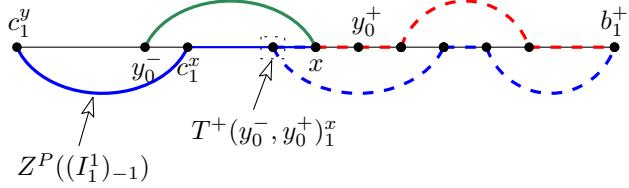
As $\hat{T} = T^+(a_1^-, a_1^+)$ and $\hat{T}_{-1}^y = b_1^+$, it follows that D_1^{ext}, D_3^{ext} and D_2 are internally disjoint from each other, $P[: x]$ and $P[b_1^+ :]$. Hence, Property B4 also holds. Finally, as $Z^P(c_1^x, c_1^y)$ is internally disjoint from the even path of $T^+(y_0^-, y_0^+)$, Property B5 also holds.

■ **Subcase 2.** If $c_1^x, T^+(y_0^-, y_0^+)_1^x \in V(P[y_0^- : x])$, let $I_1^1 = [(c_1^x, c_1^y)], I_1^2 = [(x, y_0^-)]$ and $I_1^3 = []$. Again, by definition, Properties B1 and B2 of our claim hold.

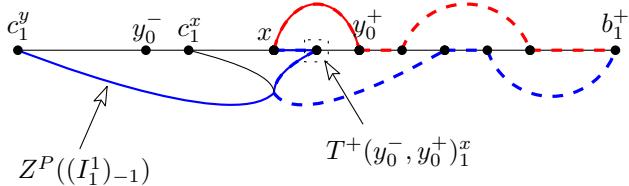
Case 1)



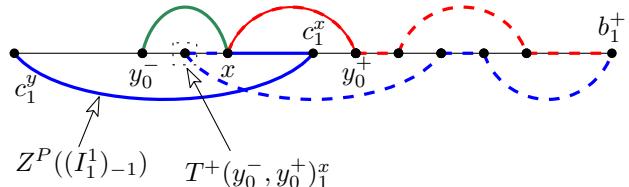
Case 2)



Case 3a)



Case 3b)



Case: $c_1^x \notin V(P[y_0^-, y_0^+]) \setminus \{y_0^+\}$

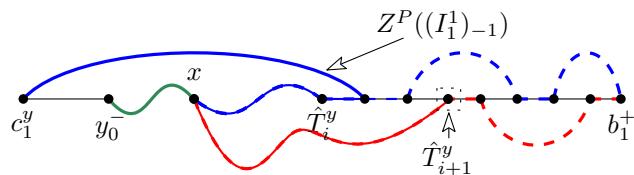


Figure 14 An illustration of the cases of Claim 20.5. The path D_2 is drawn in green, D_1 is drawn in solid blue and D_3 is drawn in solid red. The path D_1^{ext} is drawn in dashed blue and D_3^{ext} is drawn in dashed red. Note where a subpath is contained in both D_1 and D_1^{ext} (likewise for D_3 and D_3^{ext}) a dashed line is drawn on top of the solid line. See also Table 1 for a summary of the jump sequences I_1^1, I_1^2, I_1^3 , paths D_1, D_2, D_3 and chain extension \hat{T} corresponding to each case.

Let $\hat{T} = T^+(y_0^-, y_0^+)$. If $|\hat{T}| = 1$, then we let $D_1^{ext} = P[x : \hat{T}_1^y]$ and $D_3^{ext} = Z^P(\hat{T}_1)$. Otherwise, if $|\hat{T}| \geq 2$, then we extend the even path of \hat{T} by $P[x : \hat{T}_2^x]$ and the odd path of \hat{T} by $P[x : \hat{T}_1^x]$. In addition, if $|\hat{T}|$ is odd, then we extend the even path of \hat{T} by $P[\hat{T}_{-2}^y : \hat{T}_{-1}^y]$, else we extend the odd path of \hat{T} by $P[\hat{T}_{-2}^y : \hat{T}_{-1}^y]$. We denote the extended even path by D_3^{ext} and the extended odd path by D_1^{ext} . We note that D_1^{ext} and D_3^{ext} are internally disjoint (x, b_1^+) paths.

As $D_3 = (x)$ is a subpath of D_3^{ext} and $\bar{D}_1 = (x)$ is a subpath of D_1^{ext} , Property B3 of our claim also holds. Further, by definition of these paths, Property B4 also holds. Now as $Z^P(c_1^x, c_1^y)$ is internally disjoint from the even path of $T^+(y_0^-, y_0^+)$, Property B5 holds.

- **Subcase 3a.** Suppose one of c_1^x and $T^+(y_0^-, y_0^+)_1^x$ is in $V(P[y_0^- : x])$ and the other in $V(P[x : y_0^+])$ and $V(Z^P(c_1^x, c_1^y)) \cap V(Z^P(T^+(y_0^-, y_0^+)_1)) \neq \emptyset$. Let $I_1^1 = [(T^+(y_0^-, y_0^+)_1^x, c_1^y)]$, $I_1^2 = []$ and $I_1^3 = [(x, y_0^+)]$. By definition, Properties B1 and B2 hold.

Recall that by definition c_1^x minimises $\text{dist}_P(c_1^x, c_1^y)$ and $T^+(y_0^-, y_0^+)_1^x$ minimises $\text{dist}_P(T^+(y_0^-, y_0^+)_1^x, T^+(y_0^-, y_0^+)_1^y)$. It follows that $c_1^x \in V(P[y_0^- : x])$ and $T^+(y_0^-, y_0^+)_1^x \in V(P[x : y_0^+])$.

Let $\hat{T} = (x, y_0^+) + T^+(y_0^-, y_0^+)$. The even path of \hat{T} can be extended via the path $P[x : \hat{T}_2^x]$. In addition, if $|\hat{T}|$ is odd, then we extend the even path of \hat{T} by $P[\hat{T}_{-2}^y : \hat{T}_{-1}^y]$, else we extend the odd path of \hat{T} by $P[\hat{T}_{-2}^y : \hat{T}_{-1}^y]$. That is, we obtain a pair of internally disjoint (x, b_1^+) paths. We will denote the extended odd path by D_3^{ext} and that extended even path by D_1^{ext} . Given D_3 is a subpath of D_3^{ext} and $\bar{D}_1 = (x)$ is a subpath of D_1^{ext} , Property B3 of our claim also holds.

Note that $Z^P(T^+(y_0^-, y_0^+)_1^x, c_1^y)$ is internally disjoint from both $Z^P(x, y_0^-)$ and $Z^P(x, y_0^+)$, else y_0^- was not maximal. Further, as $Z^P(T^+(y_0^-, y_0^+)_1^x, c_1^y)$ is internally disjoint from the even path of $T^+(y_0^-, y_0^+)$ it is also internally disjoint from D_3^{ext} . That is properties B4 and B5 also hold.

- **Subcase 3b.** Again suppose one of c_1^x and $T^+(y_0^-, y_0^+)_1^x$ is in $V(P[y_0^- : x])$ and the other in $V(P[x : y_0^+])$. We now consider the case where $V(Z^P(c_1^x, c_1^y)) \cap V(Z^P(T^+(y_0^-, y_0^+)_1)) = \emptyset$. Suppose first that $T^+(y_0^-, y_0^+)_1^x \in V(P[x : y_0^+]) \setminus \{y_0^+\}$ and $c_1^x \in V(P[y_0^- : x]) \setminus \{y_0^-\}$. We recall that as $V(Z^P(c_1^x, c_1^y)) \cap V(Z^P(T^+(y_0^-, y_0^+)_1)) = \emptyset$, $T^+(y_0^-, y_0^+)_1^x \in V(P[x : y_0^+]) \setminus \{y_0^+\}$ and $c_1^x \in V(P[y_0^- : x]) \setminus \{y_0^-\}$, we must have chosen to reverse P . That following our observation resulting from Claim 20.4, $T^-(y_0^-, T^+(y_0^-, y_0^+)_{-1})_1^x \in V(P[y_0^- : x])$, $T^+(T^-(y_0^-, y_0^+)_{-1}, y_0^+)_1^x \in V(P[x : y_0^+])$ and so either G contains some $S_{2k, 2k, 2k}$ subgraph with centre x or a T -type subgraph. In both cases this implies that our Theorem holds, that is we assume that $T^+(y_0^-, y_0^+)_1^x \in V(P[y_0^- : x]) \setminus \{y_0^-\}$ and $c_1^x \in V(P[x : y_0^+]) \setminus \{y_0^+\}$. Let $\hat{T} = (x, y_0^+) + T^+(y_0^-, y_0^+)$, $I_1^1 = [(c_1^x, c_1^y)]$ and $I_1^2 = [(x, y_0^-)]$, $I_1^3 = [\hat{T}_2]$.

We extend the even path of \hat{T} by $P[x : \hat{T}_2^x]$. If $|\hat{T}|$ is odd then we extend the even path of \hat{T} by $P[\hat{T}_{-2}^y : \hat{T}_{-1}^y]$, else we extend the odd path of \hat{T} by $P[\hat{T}_{-2}^y : \hat{T}_{-1}^y]$. We denote the extended odd path by D_3^{ext} and the extended even path by D_1^{ext} . We note that D_1^{ext} and D_3^{ext} are internally disjoint (x, b_1^+) paths. Given D_3 is a subpath of D_3^{ext} and $\bar{D}_1 = (x)$ is a subpath of D_1^{ext} , Property B3 of our claim also holds. Now, Property B4 holds by definition.

Finally, Property 5 holds as $Z^P(c_1^x, c_1^y)$ is internally disjoint from the even path of $T^+(y_0^-, y_0^+)$ and so also D_1^{ext} .

In each of the subcases above, we defined I_1^1, I_1^2, I_1^3 satisfying the properties B1–B5. Hence, if $c_1^x \in V(P[y_0^- : y_0^+]) \setminus \{y_0^+\}$, then our claim holds.

We may now assume that $c_1^x \notin V(P[y_0^- : y_0^+]) \setminus \{y_0^+\}$, that is $c_1^x \in P[\hat{T}_i^y : \hat{T}_{i+1}^y] \setminus \{\hat{T}_{i+1}^y\}$ for some $i \in \{1, \dots, |\hat{T}| - 1\}$. Let T be that chain extension resulting from Claim 20.2. Recall

	Jump sequences	Paths D_1, D_2, D_3	\hat{T}
1.	$I_1^1 = [(c_1^x, c_1^y)]$ $I_1^2 = []$ $I_1^3 = [(x, y_0^+)]$	$D_1 = P[x : c_1^x] + Z^P(c_1^x, c_1^y)$ $D_2 = (x)$ $D_3 = Z^P(x, y_0^+)$	$\hat{T} = (x, y_0^+) + T^+(y_0^-, y_0^+)$
2.	$I_1^1 = [(c_1^x, c_1^y)]$ $I_1^2 = [(x, y_0^-)]$ $I_1^3 = []$	$D_1 = P[x : c_1^x] + Z^P(c_1^x, c_1^y)$ $D_2 = Z^P(x, y_0^-)$ $D_3 = (x)$	$\hat{T} = T^+(y_0^-, y_0^+)$
3a.	$I_1^1 = [(T^+(y_0^-, y_0^+), c_1^y)]$ $I_1^2 = []$ $I_1^3 = [(x, y_0^+)]$	$D_1 = P[x : c_1^x] + Z^P(T^+(y_0^-, y_0^+), c_1^y)$ $D_2 = (x)$ $D_3 = Z^P(x, y_0^+)$	$\hat{T} = (x, y_0^+) + T^+(y_0^-, y_0^+)$
3b.	$I_1^1 = [(c_1^x, c_1^y)]$ $I_1^2 = []$ $I_1^3 = [(x, y_0^+)]$	$D_1 = P[x : c_1^x] + Z^P(c_1^x, c_1^y)$ $D_2 = (x)$ $D_3 = Z^P(x, y_0^+)$	$\hat{T} = (x, y_0^+) + T^+(y_0^-, y_0^+)$

■ **Table 1** A summary of Subcases 1, 2, 3a, and 3b of Claim 20.5. For each case we give the jump sequences I_1^1, I_1^2, I_1^3 , the paths D_1, D_2, D_3 and the chain extension \hat{T} . A visual depiction of these cases can be seen in Figure 14.

that the odd and the even path of T can be extended to a pair of internally disjoint (x, b_1^+) paths, via the path $P[x : T_1^x]$ and the paths $P[x : T_2^x], P[T_{-2}^y : T_{-1}^y]$, if $|T| \geq 2$, and $P[x : T_1^y]$ if $|T| = 1$. We will denote these paths by D_1^{ext} and D_3^{ext} but not yet fix which of D_1^{ext} and D_3^{ext} refers to the odd path and which refers to the even path.

From Claim 20.2, either the path $P[y_0^- : x]$ is internally disjoint from both D_1^{ext} and D_3^{ext} or the path $Z^P(y_0^-, x)$ is internally disjoint from both D_1^{ext} and D_3^{ext} . If $P[y_0^- : x]$ is internally disjoint from both D_1^{ext} and D_3^{ext} , then let $I_1^2 = []$, else, let $I_1^2 = [(y_0^-, x)]$.

If i is odd, let D_1^{ext} be that odd extended path and D_3^{ext} be that even extended path. Let $I_1^3 = [\hat{T}_j : 1 \leq j \leq i, j \bmod 2 = 1]$ and $I_1^1 = [\hat{T}_j : 1 \leq j \leq i, j \bmod 2 = 0] + (c_1^x, c_1^y)$. If i is even, let D_1^{ext} be that even extended path and D_3^{ext} be that odd extended path, $I_1^3 = [\hat{T}_j : 1 \leq j \leq i, j \bmod 2 = 0]$ and $I_1^1 = [\hat{T}_j : 1 \leq j \leq i, j \bmod 2 = 1] + (c_1^x, c_1^y)$. By definition Properties B1 and B2 hold. Further, the subpath of D_1 to $(I_\delta^a)_{-2}^y$ is also a subpath of D_1^{ext} and D_3 is a subpath of D_3^{ext} , that is, Property B3 holds. As the paths $D_1^{ext}, D_3^{ext}, D_2$ are internally disjoint from each other and by definition also the paths $P[: b_1^-], P[b_1^+ :], P[c_1^y : y_0^+]$, Property B4 also holds.

Note, the path $Z^P(c_1^x, c_1^y)$ is internally disjoint from the even path of T , if i is odd, and the odd path of T otherwise, else, $T^+(y_0^-, y_0^+)_{i+1}^y$ was not maximal. $Z^P(c_1^x, c_1^y)$ is also disjoint from the path D_2 , else $T^+(y_0^-, y_0^+)_1^y$ was not maximal, that is Property B5 holds thus concluding the proof of this claim. \triangleleft

Let I_1^1, I_1^2, I_1^3 be those jump sequences resulting from Claim 20.5. We let I_1^1 be the active frontier, I_1^2 be the candidate and I_1^3 be the inert frontier. We note that Property B1 implies that $|I_1^1| \geq 1$ and so Property P3a holds in the base case. Further, the Properties B1-B5 of Claim 20.5 correspond to Properties P3b-f. That is the vertices b_1^-, b_1^+ and jump sequences I_1^1, I_1^2, I_1^3 satisfy Properties P1, P2a-d and P3a-f, thus concluding the base case.

Inductive case

We now suppose that for some $1 \leq \delta \leq 6k - 1$ there exist the pairs $(b_1^-, b_1^+), \dots, (b_\delta^+, b_\delta^-)$ and jump sequences $I_1^i \subseteq \dots \subseteq I_\delta^i$ for $i \in \{1, 2, 3\}$ satisfying Properties P1-P3. Let $a \neq b \neq c \in \{1, 2, 3\}$ be such that I_δ^a is the active frontier, I_δ^b is the inert frontier and I_δ^c

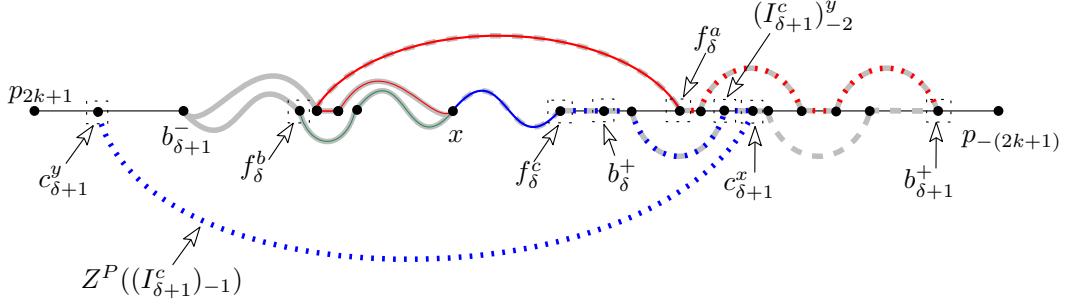


Figure 15 Here the paths D_a , D_b , D_c are shown in solid red, green and blue respectively. The paths D_a^{ext} and D_b^{ext} from x to $b_{\delta+1}^-$ are shown in solid gray. In the case depicted $I_{\delta+1}^c$ is the active frontier, $I_{\delta+1}^b$ is the candidate and $I_{\delta+1}^a$ is the inert frontier. The path D_a^{ind} consists of both the solid and dotted red paths, the path D_c^{ind} consist of the solid and dotted blue paths and $D_b^{ind} = D_b$. The path \hat{D}_a^{ext} is that dashed gray path containing D_a^{ind} as a subpath and the path \hat{D}_c^{ext} is that dashed gray path containing that subpath of D_c^{ind} from x to $(I_{\delta+1}^c)^y_{-2}$.

the candidate. We consider the case where b_δ^+ is active, the case where b_δ^- is active follows symmetrically. From Properties P3a and P3c we have paths D_a , D_b and D_c with length at least $\lfloor \frac{\delta+2}{3} \rfloor$, $\lfloor \frac{\delta+1}{3} \rfloor$ and $\lfloor \frac{\delta}{3} \rfloor$ respectively. Further, combining properties P3d-f, it follows that the paths D_a , D_b , D_c and $P[f_\delta^c : f_\delta^a]$ are internally disjoint. We also assume by Property P3d that there exist the paths D_a^{ext} and D_b^{ext} as described.

We will now define vertices $b_{\delta+1}^+$, $b_{\delta+1}^-$ and construct jump sequences $I_{\delta+1}^1$, $I_{\delta+1}^2$, $I_{\delta+1}^3$ that will again satisfy the properties. Further, $I_{\delta+1}^b$ will be the candidate and one of $I_{\delta+1}^a$, $I_{\delta+1}^c$ will be the active frontier, the other will be the inert frontier.

Note that Property P1 holds by the base case. We say $b_{\delta+1}^-$ is active and $b_{\delta+1}^- = b_\delta^-$, as b_δ^+ is active, Properties P2a and P2b hold by definition. If $T^+(b_\delta^-, f_\delta^a)$ exists, let $b_{\delta+1}^+ = T^+(b_\delta^-, f_\delta^a)^y_{-1}$, else let $b_{\delta+1}^+ = f_\delta^a$. Either by definition or maximality of $T^+(b_\delta^-, f_\delta^a)$, it follows that $y^+(b_{\delta+1}^-, b_{\delta+1}^+)$ does not exist. That is, Property P2c holds for $\delta + 1$. It remains to show that Property P2d is also satisfied.

We will now construct a pair of internally disjoint paths from x to $b_{\delta+1}^+$ which we will call, \hat{D}_a^{ext} and \hat{D}_c^{ext} . D_a and D_c will form subpaths of \hat{D}_a^{ext} and \hat{D}_c^{ext} , respectively. Further, the paths \hat{D}_a^{ext} , \hat{D}_c^{ext} , D_b^{ext} , $P[: f_\delta^b]$ and $P[b_{\delta+1}^+ :]$ are pairwise internally disjoint.

We first consider the case where $T^+(b_\delta^-, f_\delta^a)$ exists. By Property P3b, $f_\delta^a = y^+(b_\delta^-, b_\delta^+)$ and so $T^+(b_\delta^-, f_\delta^a)_1^x \in V(P[b_\delta^+ : f_\delta^a]) \setminus \{f_\delta^a\}$. Further, both the odd and the even path of $T^+(b_\delta^-, f_\delta^a)$ are disjoint from the paths D_a^{ext} , D_b^{ext} , D_a , D_b and D_c else f_δ^a , i.e. $y^+(b_\delta^-, b_\delta^+)$, was not maximal. This implies that the path D_a can be extended via $P[f_\delta^a : T^+(b_\delta^-, f_\delta^a)_2^x]$ and the even path of $T^+(b_\delta^-, f_\delta^a)$ and D_c can be extended via $P[f_\delta^c : T^+(b_\delta^-, f_\delta^a)_1^x]$ and the odd path of $T^+(b_\delta^-, f_\delta^a)$, to obtain a pair of internally disjoint paths from x to $b_{\delta+1}^+$ which are also disjoint from D_b^{ext} , $P[: f_\delta^b]$ and $P[b_{\delta+1}^+ :] - \{b_{\delta+1}^+\}$. We denote these paths by \hat{D}_a^{ext} and \hat{D}_c^{ext} respectively. See Figure 15 for an illustration.

If $T^+(b_\delta^-, f_\delta^a)$ does not exist, then we let $\hat{D}_a^{ext} = D_a$ and $\hat{D}_c^{ext} = D_c + P[f_\delta^c : f_\delta^a]$. \triangleright

Claim 20.6. If $\text{dist}_P(b_\delta^+, b_{\delta+1}^+) \geq 4k^2 - \frac{k(4\delta-14)}{3} + \frac{\delta(\delta-7)+11}{9}$, then G contains some $S_{2k,2k,2k}$ subgraph with centre x .

Proof. Recall that there exist paths D_a , D_b and D_c . We let ℓ_a denote the length of the path D_a and ℓ_c denote the length of the path D_c .

For every $i \in \{1, \dots, |T^+(b_\delta^-, f_\delta^a)|\}$, there exist paths from x to f_δ^b , $T^+(b_\delta^-, f_\delta^a)_i^y$ and $T^+(b_\delta^-, f_\delta^a)_{i+1}^y$ which share the single common vertex x . That path to f_δ^b corresponds to

D_b . If i is even, then that path to $T^+(b_\delta^-, f_\delta^a)_i^y$ is a subpath of \hat{D}_a^{ext} and that path to $T^+(b_\delta^-, f_\delta^a)_{i+1}^y$ is a subpath of \hat{D}_c^{ext} . It follows that this path from x to $T^+(b_\delta^-, f_\delta^a)_i^y$ has length at least $\ell_a + \frac{i}{2}$ and that path from x to $T^+(b_\delta^-, f_\delta^a)_{i+1}^y$ has length at least $\ell_c + \frac{i}{2} + 1$. Similarly, if i is odd, then that path to $T^+(b_\delta^-, f_\delta^a)_i^y$ is a subpath of \hat{D}_c^{ext} and that path to $T^+(b_\delta^-, f_\delta^a)_{i+1}^y$ is a subpath of \hat{D}_a^{ext} . These paths have length at least $\ell_c + \frac{i+1}{2}$ and $\ell_a + \frac{i+1}{2}$ respectively.

Suppose $\text{dist}_P(b_{\delta+1}^+, b_\delta^+) \geq 4k^2 - \frac{k(4\delta-14)}{3} + \frac{\delta(\delta-7)+11}{9}$. We now claim that at least one of the following length or distance inequalities must hold,

$$\begin{aligned} |T^+(b_\delta^-, f_\delta^a)_{-1}^y| &> 2(2k - \ell_c) - 1 \\ |T^+(b_\delta^-, f_\delta^a)_{-1}^y| &> 2(2k - \ell_a) \\ \text{dist}_P(f_\delta^c, f_\delta^a) &> 2k - \ell_c \\ \text{dist}_P(f_\delta^a, T^+(b_\delta^-, f_\delta^a)_1^y) &> 2k - \ell_a \\ \text{dist}_P(T^+(b_\delta^-, f_\delta^a)_i^y, T^+(b_\delta^-, f_\delta^a)_{i+1}^y) &> 2k - \left(\ell_c + \frac{i+1}{2}\right) \text{ for some} \\ &\quad \mathbf{odd} \ i \in \{1, \dots, |T^+(b_\delta^-, f_\delta^a)_{-1}^y| - 1\}, \text{ or} \\ \text{dist}_P(T^+(b_\delta^-, f_\delta^a)_i^y, T^+(b_\delta^-, f_\delta^a)_{i+1}^y) &> 2k - \left(\ell_a + \frac{i}{2}\right) \text{ for some} \\ &\quad \mathbf{even} \ i \in \{1, \dots, |T^+(b_\delta^-, f_\delta^a)_{-1}^y| - 1\}. \end{aligned}$$

Assume towards a contradiction that none of the above length or distance inequalities holds. This implies that for every $i \in \{1, \dots, |T^+(b_\delta^-, f_\delta^a)_{-1}^y|\}$,

$$\text{dist}(b_\delta^+, T^+(b_\delta^-, f_\delta^a)_i^y) \leq \begin{cases} \sum_{j=0}^{\frac{i-1}{2}} (2k - (\ell_c + j)) + \sum_{j=0}^{\frac{i-1}{2}} (2k - (\ell_a + j)), & \text{if } i \text{ is odd.} \\ \text{dist}(b_\delta^+, T^+(b_\delta^-, f_\delta^a)_{i-1}^y) + 2k - 1 - (\ell_a + \frac{i}{2}) = \\ \sum_{j=0}^{\frac{i-2}{2}} (2k - (\ell_c + j)) + \sum_{j=0}^{\frac{i-2}{2}} (2k - (\ell_a + j)) + 2k - 1 - (\ell_a + \frac{i}{2}), & \text{if } i \text{ is even.} \end{cases}$$

Further, as $|T^+(b_\delta^-, f_\delta^a)_{-1}^y| \leq 2(2k - \ell_c) - 1$, $|T^+(b_\delta^-, f_\delta^a)_{-1}^y| \leq 2(2k - \ell_a)$ and $b_{\delta+1}^+ = T^+(b_\delta^-, f_\delta^a)_{-1}^y$, it follows that,

$$\begin{aligned} \text{dist}(b_\delta^+, b_{\delta+1}^+) &\leq \sum_{j=0}^{\frac{(2(2k-\ell_c)-2)}{2}} (2k - (\ell_c + j)) + \sum_{j=0}^{\frac{(2(2k-\ell_c)-2)}{2}} (2k - (\ell_a + j)), \text{ and,} \\ \text{dist}(b_\delta^+, b_{\delta+1}^+) &\leq \sum_{j=0}^{\frac{2(2k-\ell_a)-2}{2}} (2k - (\ell_c + j)) \\ &\quad + \sum_{j=0}^{\frac{2(2k-\ell_a)-2}{2}} (2k - (\ell_a + j)) + 2k - 1 - \left(\ell_a + \frac{2(2k-\ell_a)}{2}\right) \\ &= \sum_{j=0}^{\frac{2(2k-\ell_a)-2}{2}} (2k - (\ell_c + j)) + \sum_{j=0}^{\frac{2(2k-\ell_a)-2}{2}} (2k - (\ell_a + j)) - 1. \end{aligned}$$

By Property P3a and P3b, $\ell_a \geq \lfloor \frac{\delta+2}{3} \rfloor \geq \frac{\delta-1}{3}$ and $\ell_c \geq \lfloor \frac{\delta}{3} \rfloor \geq \frac{\delta-3}{3}$. It follows that

$\text{dist}(b_\delta^+, b_{\delta+1}^+) \leq 4k^2 - \frac{k(4\delta-14)}{3} + \frac{\delta(\delta-7)+11}{9} - 1$, a contradiction. That is we may assume that at least one of the length or distance conditions are met.

We will now show that in each of these cases G contains some $S_{2k,2k,2k}$ subgraph with centre x .

If $|T^+(b_\delta^-, f_\delta^a)_{-1}^y| \geq 2(2k - \ell_c)$ then there exist paths from x to f_δ^b , $T^+(b_\delta^-, f_\delta^a)_{2(2k-\ell_c)-1}^y$ and $T^+(b_\delta^-, f_\delta^a)_{2(2k-\ell_c)}^y$ which share the single common vertex x . Further, that path from x to $T^+(b_\delta^-, f_\delta^a)_{2(2k-\ell_c)-1}^y$ has length at least $\ell_c + \frac{2(2k-\ell_c)-1+1}{2} = 2k$, that is, extending that path from x to f_δ^b by $P[: f_\delta^b]$ and that path from x to $T^+(b_\delta^-, f_\delta^a)_{2(2k-\ell_c)}^y$ by $P[T^+(b_\delta^-, f_\delta^a)_{2(2k-\ell_c)}^y :]$, we obtain three paths of length at least $2k$ with a single common vertex x . That is G contains some $S_{2k,2k,2k}$ subgraph with centre x .

Similarly, if $|T^+(b_\delta^-, f_\delta^a)_{-1}^y| \geq 2(2k - \ell_a) + 1$ then there exist paths from x to f_δ^b , $T^+(b_\delta^-, f_\delta^a)_{2(2k-\ell_a)}^y$ and $T^+(b_\delta^-, f_\delta^a)_{2(2k-\ell_a)+1}^y$ which share the single common vertex x . As that path from x to $T^+(b_\delta^-, f_\delta^a)_{2(2k-\ell_a)}^y$ has length at least $\ell_a + \frac{2(2k-\ell_a)}{2} = 2k$, extending that path from x to f_δ^b by $P[: f_\delta^b]$ and that path from x to $T^+(b_\delta^-, f_\delta^a)_{2(2k-\ell_a)+1}^y$ by $P[T^+(b_\delta^-, f_\delta^a)_{2(2k-\ell_a)+1}^y :]$, we obtain three paths of length at least $2k$ with a single common vertex x . That is G contains some $S_{2k,2k,2k}$ subgraph with centre x .

If $\text{dist}_P(f_\delta^c, f_\delta^a) \geq 2k + 1 - \ell_c$, then extending those paths from x to f_δ^a , f_δ^b and f_δ^c via the paths $P[f_\delta^a :]$, $P[: f_\delta^b]$ and $P[f_\delta^c : f_\delta^a] - \{f_\delta^a\}$, we again obtain three paths of length at least $2k$ with a single common vertex x . That is G contains some $S_{2k,2k,2k}$ subgraph with centre x . If $\text{dist}_P(f_\delta^a, T^+(b_\delta^-, f_\delta^a)_1^y) \geq 2k + 1 - \ell_a$, then similarly extending those paths from x to f_δ^b , f_δ^a and $T^+(b_\delta^-, f_\delta^a)_1^y$ via the paths $P[: f_\delta^b]$, $P[f_\delta^a : T^+(b_\delta^-, f_\delta^a)_1^y] - \{T^+(b_\delta^-, f_\delta^a)_1^y\}$ and $P[T^+(b_\delta^-, f_\delta^a)_1^y :]$, we obtain three paths of length at least $2k$ with a single common vertex x . That is G contains some $S_{2k,2k,2k}$ subgraph with centre x .

Finally, suppose for some $i \in \{1, \dots, |T^+(b_\delta^-, f_\delta^a)_{-1}^y| - 1\}$, we either have that i is odd and $\text{dist}_P(T^+(b_\delta^-, f_\delta^a)_i^y, T^+(b_\delta^-, f_\delta^a)_{i+1}^y) \geq 2k + 1 - (\ell_c + \frac{i+1}{2})$ or we have that i is even and $\text{dist}_P(T^+(b_\delta^-, f_\delta^a)_i^y, T^+(b_\delta^-, f_\delta^a)_{i+1}^y) \geq 2k + 1 - (\ell_a + \frac{i}{2})$. We can once again extend those paths from x to f_δ^b , $T^+(b_\delta^-, f_\delta^a)_i^y$ and $T^+(b_\delta^-, f_\delta^a)_{i+1}^y$ via the paths $P[: f_\delta^b]$, $P[T^+(b_\delta^-, f_\delta^a)_i^y : T^+(b_\delta^-, f_\delta^a)_{i+1}^y] - \{T^+(b_\delta^-, f_\delta^a)_{i+1}^y\}$ and $P[T^+(b_\delta^-, f_\delta^a)_{i+1}^y :]$. Each of these paths have length at least $2k$ with a single common vertex x . That is G contains some $S_{2k,2k,2k}$ subgraph with centre x . \square

It now follows from Claim 20.6, that Properties P2a–d hold for $b_{\delta+1}^-$ and $b_{\delta+1}^+$. We now claim that the vertices $b_{\delta+1}^-$, $b_{\delta+1}^+$ and x meet the conditions for the application of Lemma 23.

Note that, Claim 20.6 also implies $b_{\delta+1}^-, b_{\delta+1}^+ \in V(P[2k+1 : -(2k+1)])$. By maximality of the vertices y_0^+ and y_0^- , condition i) is satisfied. As $b_\delta^- = b_{\delta+1}^-$, by P3d there exist paths D_a^{ext} and D_b^{ext} from x to $b_{\delta+1}^-$ and by P3e these are internally disjoint, that is condition ii) is satisfied. There also exist internally disjoint paths \hat{D}_a^{ext} and \hat{D}_c^{ext} from x to $b_{\delta+1}^+$. Further, the paths D_a^{ext} , D_b^{ext} and \hat{D}_c^{ext} are also internally disjoint, that is conditions iii) and iv) are satisfied. Applying Lemma 23 either $y^+(b_{\delta+1}^-, b_{\delta+1}^+)$ or $y^-(b_{\delta+1}^-, b_{\delta+1}^+)$ must exist. By the maximality of $b_{\delta+1}^+$, it follows that $y^+(b_{\delta+1}^-, b_{\delta+1}^+)$ cannot exist and so $y^-(b_{\delta+1}^-, b_{\delta+1}^+)$ must exist. Let $(c_{\delta+1}^x, c_{\delta+1}^y) = (x^-(b_{\delta+1}^-, b_{\delta+1}^+), y^-(b_{\delta+1}^-, b_{\delta+1}^+))$.

By P2c, $y^-(b_\delta^-, b_\delta^+)$ does not exist and $b_{\delta+1}^- = b_\delta^-$. It follows that $Z^P(c_{\delta+1}^x, c_{\delta+1}^y)$ is internally disjoint from the paths D_a , D_b and D_c and $c_{\delta+1}^x \in V(P[b_\delta^+ : b_{\delta+1}^+]) \setminus \{b_{\delta+1}^+\}$. Further, either $c_{\delta+1}^x \in V(P[b_\delta^+ : f_\delta^a]) \setminus \{f_\delta^a\}$, $c_{\delta+1}^x \in V(P[f_\delta^a : T^+(b_\delta^-, f_\delta^a)_1^y]) \setminus \{T^+(b_\delta^-, f_\delta^a)_1^y\}$ or $c_{\delta+1}^x \in V(P[T^+(b_\delta^-, f_\delta^a)_i^y : T^+(b_\delta^-, f_\delta^a)_{i+1}^y]) \setminus T^+(b_\delta^-, f_\delta^a)_{i+1}^y$ for some $i \in \{1, \dots, |T^+(b_\delta^-, f_\delta^a)| - 1\}$.

Suppose that $c_{\delta+1}^x \in V(P[b_\delta^+ : f_\delta^a]) \setminus \{f_\delta^a\}$. It follows that the path $Z^P(c_{\delta+1}^x, c_{\delta+1}^y)$ is internally disjoint from \hat{D}_a^{ext} and $\hat{D}_b^{ext} \setminus Z^P(T^+(b_\delta^-, f_\delta^a)_1^y)$, else either $T^+(b_\delta^-, f_\delta^a)_1^y$ was not maximal or c_δ^x was not minimal with respect to distance from c_δ^y . We let $I_{\delta+1}^a = I_\delta^a$, $I_{\delta+1}^b = I_\delta^b$

and $I_{\delta+1}^c = I_\delta^c + (c_{\delta+1}^x, c_{\delta+1}^y)$. We say $I_{\delta+1}^a$ is the inert frontier, $I_{\delta+1}^c$ is the active frontier and $I_{\delta+1}^b$ is the candidate.

Suppose instead $c_{\delta+1}^x \in V(P[f_\delta^a : T^+(b_\delta^-, f_\delta^a)_2^y]) \setminus \{T^+(b_\delta^-, f_\delta^a)_1^y\}$. Similarly the path $Z^P(c_{\delta+1}^x, c_{\delta+1}^y)$ is internally disjoint from $\hat{D}_a^{ext} \setminus Z^P(T^+(b_\delta^-, f_\delta^a)_2)$ and \hat{D}_c^{ext} , else either $T^+(b_\delta^-, f_\delta^a)_2^y$ was not maximal or c_δ^x was not minimal with respect to distance from c_δ^y . We let $I_{\delta+1}^a = I_\delta^a + (c_{\delta+1}^x, c_{\delta+1}^y)$, $I_{\delta+1}^b = I_\delta^b$ and $I_{\delta+1}^c = I_\delta^c + (T^+(b_\delta^-, f_\delta^a)_1^x, T^+(b_\delta^-, f_\delta^a)_1^y)$. We say $I_{\delta+1}^a$ is the active frontier, $I_{\delta+1}^c$ is the inert frontier and $I_{\delta+1}^b$ is the candidate.

Finally, suppose $c_{\delta+1}^x \in V(P[T^+(b_\delta^-, f_\delta^a)_i^y : T^+(b_\delta^-, f_\delta^a)_{i+1}^y]) \setminus T^+(b_\delta^-, f_\delta^a)_{i+1}^y$ for some $i \in \{1, \dots, |T^+(b_\delta^-, f_\delta^a)|\}$. If i is odd, then the path $Z^P(c_{\delta+1}^x, c_{\delta+1}^y)$ is internally disjoint from \hat{D}_a^{ext} and $\hat{D}_c^{ext} \setminus Z^P(T^+(b_\delta^-, f_\delta^a)_{i+1})$. We let $I_{\delta+1}^a = I_\delta^a + [T^+(b_\delta^-, f_\delta^a)_j : 1 \leq j \leq i, j \bmod 2 = 0]$, $I_{\delta+1}^b = I_\delta^b$ and $I_{\delta+1}^c = I_\delta^c + [T^+(b_\delta^-, f_\delta^a)_j : 1 \leq j \leq i, j \bmod 2 = 1] + (c_{\delta+1}^x, c_{\delta+1}^y)$. We say $I_{\delta+1}^a$ is the inert frontier, $I_{\delta+1}^c$ is the active frontier and $I_{\delta+1}^b$ is the candidate.

If i is even, then the path $Z^P(c_{\delta+1}^x, c_{\delta+1}^y)$ is internally disjoint from $\hat{D}_a^{ext} \setminus Z^P(T^+(b_\delta^-, f_\delta^a)_{i+1})$ and \hat{D}_c^{ext} . We let $I_{\delta+1}^a = I_\delta^a + [T^+(b_\delta^-, f_\delta^a)_j : 1 \leq j \leq i, j \bmod 2 = 0] + (c_{\delta+1}^x, c_{\delta+1}^y)$, $I_{\delta+1}^b = I_\delta^b$ and $I_{\delta+1}^c = I_\delta^c + [T^+(b_\delta^-, f_\delta^a)_j : 1 \leq j \leq i, j \bmod 2 = 1]$. We say $I_{\delta+1}^a$ is the active frontier, $I_{\delta+1}^c$ is the inert frontier and $I_{\delta+1}^b$ is the candidate.

We note given $|I_{\delta+1}^c| = |I_\delta^c| + 1$ Property P3a holds for $\delta + 1$. Further, Property P3b holds by definition. Let us denote those paths described by $I_{\delta+1}^a$, $I_{\delta+1}^b$, $I_{\delta+1}^c$ extended to x by $P[x : s_{\delta+1}^a]$, $P[x : s_{\delta+1}^b]$ and $P[x : s_{\delta+1}^c]$, respectively, by D_a^{ind} , D_b^{ind} , D_c^{ind} . Note the paths $D_a^{ind} \setminus (c_{\delta+1}^x, c_{\delta+1}^y)$ and $D_c^{ind} \setminus (c_{\delta+1}^x, c_{\delta+1}^y)$ form subpaths of \hat{D}_a^{ext} and \hat{D}_c^{ext} . That is properties P3c-f also hold and concluding inductive case.

We now assume that we have the jump sequences I_{6k}^1 , I_{6k}^2 and I_{6k}^3 . By properties P3a-f, there exist three paths of length at least $2k$ from x , each sharing only the common vertex x . That is, G contains a $S_{2k, 2k, 2k}$ with centre x . \blacktriangleleft

6 The Two Algorithms

We now apply Theorems 20 and 19 in order to show that COLOURING can be solved in polynomial time for $S_{1,k,k,k}$ -subgraph-free graphs and for $\mathbb{H}_m^{1,k,k,k}$ -subgraph-free graphs. In both cases we need another structural result that allows us to assume that the graphs in the input do not contain protected fans, T -type and L -type subgraphs. We first consider $\mathbb{H}_m^{1,k,k,k}$ -subgraph-free graphs.

► **Lemma 24.** *Let (G, r) be an instance of COLOURING and let $c, m, k \geq 1$ be constants. We can in polynomial time obtain a graph G' such that:*

- G' has minimum degree at least 3;
- G' contains no protected fan of order $m + k + 2$;
- G' contains no L -type subgraph, with respect to the bound c and length $m + k$;
- G' can be coloured with r colours, if and only if G can;
- if G is $\mathbb{H}_m^{1,k,k,k}$ -subgraph-free, then so is G' .

Proof. Towards this we will define three operations. The first we call degree 2 vertex removal, the second we call the fan-contraction operation and the third is the L removal operation. We will first describe each of these operations, showing that the resulting graph can be coloured using r colours if, and only if, the original graph could. We then also claim that if the original graph was $\mathbb{H}_m^{1,k,k,k}$ -subgraph-free then so is the resulting graph. Whenever we refer to a L -type subgraph, it is always understood to be with respect to this pair of constants c and $m + k$.

Degree 2 vertex removal:

If $|V(G)| \geq 2$, $r \geq 3$ and G contains some degree 2 vertex, v , then (G, r) is a yes-instance of COLOURING if, and only if, $(G - v, r)$ is a yes-instance. That is we first exhaustively remove degree 2 vertices. We highlight that the family of $\mathbb{H}_m^{1,k,k,k}$ -subgraph-free graphs is closed under vertex deletion.

Fan-contraction operation:

Suppose some $Q \subseteq V(G)$ is a protected fan with centre vertex z and end vertices x, y . We note if $r \geq 4$, then G can be coloured with r colours if and only if $G - (Q \setminus \{x, y, z\})$ can. We let $G^F(Q) = G - (Q \setminus \{x, y, z\})$, and we call this a *type 1 fan-contraction*. Assume now $r = 3$. If $m + k + 2$ is odd then x, y, z must each take different colours in every 3-colouring of G . Let $G^F(Q)$ be the graph obtained from G by deleting the vertices $Q \setminus \{x, y, z\}$ and adding the edge xy . That is, the vertices $\{x, y, z\}$ induce a triangle in $G^F(Q)$. We call this a *type 2 fan-contraction*. If $m + k + 2$ is even, then x and y must take the same colour. We let $G^F(Q)$ be the graph obtained from G by replacing the vertices $Q \setminus \{z\}$ by a new vertex w such that $N(w) = N(x) \cup N(y)$. We call this a *type 3 fan-contraction*. We note this operation takes polynomial time to apply and $|V(G^F(Q))| \leq |V(G)| - 1$.

▷ **Claim 24.1.** Let $G^F(Q)$ be that graph obtained from G by applying the fan-contraction operation to some protected fan on vertices $Q \subseteq V(G)$, with $|Q| = m + k + 2$. If $G^F(Q)$ contains some $\mathbb{H}_m^{1,k,k,k}$ subgraph then so must G .

Proof. Let $z, x, y \in Q$ denote the centre and end vertices of this fan, respectively. We let $R = (r_1, \dots, r_{m+k+1})$ denote that path of length $m + k$ between x and y in $G[Q] - z$, we highlight that $r_1 = x$ and $r_{m+k+1} = y$.

Suppose $G^F(Q)$ contains some $\mathbb{H}_m^{1,k,k,k}$ subgraph F . Let f, f' denote the degree 3 vertices of F , let $f + M + f'$ denote that path of length m between f and f' . Let f_1^1 be that isolated vertex in $F - (V(M) \cup \{f, f'\})$ and F_2, F_3, F_4 denote those paths of length $k - 1$. For each $i \in \{2, 3, 4\}$ and $j \in \{1, \dots, k\}$, let f_j^i denote the j th vertex of F_i . Without loss of generality f_1^1 and f_1^2 are adjacent to f and f_1^3 and f_1^4 are adjacent to f' . See Figure 2.

If $G^F(Q)$ has been obtained by a type 1 fan-contraction, then $F \subseteq G$ and so G contains $\mathbb{H}_m^{1,k,k,k}$ as a subgraph. Likewise, if $G^F(Q)$ has been obtained via a type 2 operation but the edge $xy \notin E(F)$ or if $G^F(Q)$ has been obtained via a type 3 operation but that vertex $w \in V(G^F(Q)) \setminus V(G)$ is not in F , then $F \subseteq G$. That is in each case we find $\mathbb{H}_m^{1,k,k,k}$ as a subgraph. We are now left with two cases: $G^F(Q)$ has been obtained via a type 2 operation and F contains the edge xy or $G^F(Q)$ has been obtained via a type 3 operation F contains that new vertex $w \notin V(G)$.

Suppose $G^F(Q)$ has been obtained via a type 2 fan-contraction. If both $x, y \in V(F) \setminus M$, then without loss of generality either $x = f$ and $y \in \{f_1^1, f_1^2\}$; or $x = f'$ and $y \in \{f_1^3, f_1^4\}$; or $x = f_j^i$ and $y = f_{j+1}^i$, for some $i \in \{2, 3, 4\}$, $j \in \{1, \dots, k - 1\}$. Let F' be that graph obtained by replacing the edge $xy \in E(F)$ by the path (r_1, \dots, r_{m+k+1}) . Note, F' contains $\mathbb{H}_m^{1,k,k,k}$ as a subgraph and $F' \subseteq G$, that is G contains $\mathbb{H}_m^{1,k,k,k}$ as a subgraph.

We now assume that at least one of x or $y \in M$. Recall that the case where $(x, y) \notin E(F)$ was covered previously, that is we assume that x and y appear consecutively in the path $(f) + M + (f')$. Without loss of generality, $\text{dist}_F(x, f') \leq \text{dist}_F(y, f')$ and $\text{dist}_F(x, f') = i$, for some $i \in \{0, \dots, m - 1\}$. Let F' be that graph obtained from F by replacing the edge $xy \in E(F)$ by the path (r_1, \dots, r_{m+k+1}) and adding the edges zr_{m-i+1} and zr_{m+k-i} . We note that $\text{dist}_F(f', r_{m-i+1}) = m$, $\text{dist}_F(f, r_{m+k-i}) = m$ and $F' \subseteq G$. We now claim that F' contains $\mathbb{H}_m^{1,k,k,k}$ as a subgraph. If $z \notin V(M[1 : i - 1]) \cup F_3 \cup F_4 \cup \{f'\}$, then F' contains $\mathbb{H}_m^{1,k,k,k}$ as a subgraph with degree 3 vertices f', r_{m-i+1} , a path of length 1 via the edge

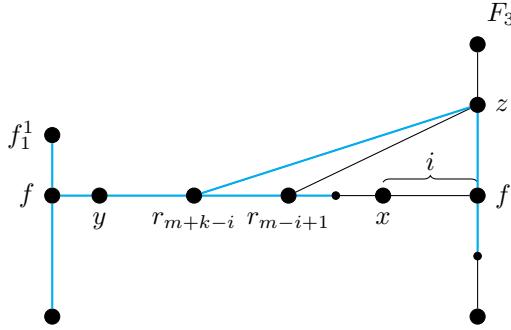


Figure 16 An illustration of the graph F' from Claim 24.1, where $G^F(Q)$ has been obtained via a type 2 fan-contraction and $z \in V(F_3)$. That $\mathbb{H}_m^{1,k,k,k}$ subgraph of F' is highlighted in blue.

zr_{m-i+1} and paths of length k via $R[m-i+1 : m+k-i+1]$, $f' + F_3$ and $f' + F_4$. If $z \in V(M[1 : i-1]) \cup F_3 \cup F_4 \cup \{f'\}$, then we note there exists some path Z of length at least $k-1$ from z in $F[V(M[1 : i-1]) \cup F_3 \cup F_4 \cup \{f'\}]$. It follows that F' contains $\mathbb{H}_m^{1,k,k,k}$ as a subgraph with degree 3 vertices f and r_{m+k-i} , a path of length 1 via the edge ff_1^1 and paths of length k via $f + F_2$, $f' + R[m+k-(i+1) : m-i]$, and $f' + Z$. See Figure 16 for an illustration where $z \in V(F_3)$. That is, if $G^F(Q)$ has been obtained via a type 2 fan-contraction then our claim holds.

The case where $G^F(Q)$ has been obtained via a type 3 fan-contraction follows similarly. Recall that we may assume that F contains some vertex $w \notin V(G)$, else F is also a subgraph of G . Further, $N(w) = N(x) \cup N(y)$, meaning if either $N(w) \cap V(F) \subseteq N(x)$ or $N(w) \cap V(F) \subseteq N(y)$, then F is again a subgraph of G . That is G contains $\mathbb{H}_m^{1,k,k,k}$ as a subgraph. We therefore assume that there is some pair r', r'' such that $r' \in N(x) \cap V(F)$ and $r'' \in N(y) \cap V(F)$. If $w \notin V(M) \cup \{f, f'\}$, then let F' be that graph obtained by replacing the path $(r', w, r'') \subseteq F$ by the path $(r', r_1, \dots, r_{m+k+1}, r'')$. Note, F' contains $\mathbb{H}_m^{1,k,k,k}$ as a subgraph and $F' \subseteq G$, that is G contains $\mathbb{H}_m^{1,k,k,k}$ as a subgraph.

If $w \in V(M) \cup \{f, f'\}$, then $\text{dist}_F(w, f') = i$ for some $i \in \{0, \dots, m\}$. Let F' be that graph obtained from F by replacing the vertex w with the path (r_1, \dots, r_{m+k+1}) and adding the edges zr_{m-i+1} and $zr_{m+k+1-i}$. Note that $\text{dist}(f', r_{m-i+1}) = m$, $\text{dist}(f, r_{m+k+1-i}) = m$ and $F' \subseteq G$. If $z \notin V(M[1 : i-1]) \cup F_3 \cup F_4 \cup \{f'\}$, then F' contains $\mathbb{H}_m^{1,k,k,k}$ as a subgraph with degree 3 vertices f' and r_{m-i+1} , a path of length 1 via the edge zr_{m-i+1} and paths of length k via $R[m-i+1 : m+k-i+1]$, F_3 , and F_4 . If $z \in V(M[1 : i-1]) \cup F_3 \cup F_4 \cup \{f'\}$, then we note there exists some path Z of length at least $k-1$ from z in $F[V(M[1 : i-1]) \cup F_3 \cup F_4 \cup \{f'\}]$. It follows that F' contains $\mathbb{H}_m^{1,k,k,k}$ as a subgraph with degree 3 vertices f and $r_{m+k+1-i}$, a path of length 1 via the edge ff_1^1 and paths of length k via $f + F_2$, $R[m+k+1-i : m+1-i]$, and $r_{m+k+1-i} + Z$. \triangleleft

L removal operation:

Suppose there is some $C \subseteq V(G)$ such that $G[C]$ is a minimal L -type subgraph. By Lemma 8, $G[C \cup S]$ has treedepth at most $3k+2$. That is, by Courcelle's theorem [4] we can decide, in polynomial time, if $G[C \cup S]$ can be coloured using r colours. If not, then (G, r) is a no-instance of COLOURING. Assume now $G[C \cup S]$ can be coloured using r colours.

Let $S = \{u, v\}$. We branch on the two non-isomorphic colourings of u and v . For each branch, we consider the PRECOLOURING EXTENSION problem on the graph $G[C \cup \{u, v\}]$, with precoloured vertices u and v . Since $\text{td}(G[C \cup u, v]) \leq 3k+2$, it follows from [11], who showed that even the more general problem LIST COLOURING is polynomial-time solvable

on graphs of bounded treewidth, that each branch can be processed in polynomial time.

If there is both a proper colouring where u, v are coloured the same and where they are different, then G can be coloured with r colours if and only if, $G - C$ can be coloured. Let $G^T(C \cup \{u, v\}) = G - C$. This is again a type 1 L -removal operation.

If there is only a proper colouring of $G[C \cup \{u, v\}]$ where u, v are coloured differently, then let $G^T(C \cup \{u, v\})$ be the graph obtained from G by removing the vertices of C and adding the edge uv . We call this a type 2 operation. The remaining case is where u and v must take the same colour in every colouring of $G[C \cup S]$. Let $G^T(C \cup S)$ be the graph obtained from G by removing the vertices of $C \cup \{u, v\}$ and adding a new vertex w such that $N(w) = N(u) \cup N(v)$. We call this a type 3 operation. Note, G can be coloured using r colours if, and only if, $G^T(C \cup S)$ can be coloured using r colours.

\triangleright **Claim 24.2.** Let $G^T(C \cup S)$ be that graph obtained from G by applying the L removal operation to minimal L -type subgraph $G[C]$ with a witness set $S = \{u, v\}$. If $G^T(C \cup S)$ contains some $\mathbb{H}_m^{1,k,k,k}$ subgraph then so must G .

Proof. Note that $G^T(C \cup S)$ has been obtained via either a type 1, 2, 3 operation. If it was a type 1 operation, then $G^T(C \cup S)$ is a subgraph of G , that is our claim holds by definition. We therefore assume that $G^T(C \cup S)$ has been obtained via either an operation of type 2 or 3.

By definition, for some $\ell \geq m + k + 1$, there is some induced path $R = (r_1, \dots, r_\ell)$ where $u = r_1$ and $v = r_\ell$ in $G[C \cup S]$. Further, u and v have at least 2 neighbours in $C \cup S$. As G has minimum degree at least 3 and $N(C) \setminus C = \{u, v\}$, for every $i \in \{1, \dots, \ell\}$, r_i has some neighbour in $C \setminus V(R)$ which we will denote by z_i . We highlight that $\{z_1, \dots, z_\ell\} \cap V(G^T(C \cup S)) = \emptyset$.

Suppose $G^T(C \cup S)$ contains some $\mathbb{H}_m^{1,k,k,k}$ subgraph F . Let f, f' denote the degree 3 vertices of F , let $f + M + f'$ denote that path of length m between f and f' . Let f_1^1 be that isolated vertex in $F - (V(M) \cup \{f, f'\})$ and F_2, F_3, F_4 denote those paths of length $k - 1$. For each $i \in \{2, 3, 4\}$ and $j \in \{1, \dots, k\}$, let f_j^i denote the j th vertex of F_i . Without loss of generality f_1^1 and f_1^2 are adjacent to f and f_1^3 and f_1^4 are adjacent to f' . See Figure 2.

If $G^T(C \cup S)$ has been obtained via a type 2 operation but the edge $uv \notin E(F)$, then $F \subseteq G$. Likewise, if $G^T(C \cup S)$ has been obtained via a type 3 operation but that vertex $w \in V(G^T(C \cup S)) \setminus V(G)$ is not in F , then $F \subseteq G$.

If $G^T(C \cup S)$ has been obtained via a type 2 operation, let F' be that graph obtained from F by replacing the edge uv with the path R . If $G^T(C \cup S)$ has been obtained via a type 3 operation, let F' be that graph obtained from F by replacing the vertex w with the path R . Note in each case $F' \subseteq G$. If $u, v, w \in V(F) \setminus M$, then F' and so also G contains $\mathbb{H}_m^{1,k,k,k}$ as a subgraph.

Suppose now that at least one of u, v or $w \in M$. If $G^T(C \cup S)$ has been obtained via a type 2 operation, without loss of generality $\text{dist}_F(u, f') \leq \text{dist}_F(v, f')$ and $\text{dist}_F(u, f') = i$ for some $i \in \{0, \dots, m - 1\}$. If the operation had type 3, then $\text{dist}_F(w, f') = i$ for some $i \in \{0, \dots, m\}$. Let F' be that graph obtained from F by replacing the edge $uv \in E(F)$ or vertex x by the path (r_1, \dots, r_{m+k+1}) and adding the edge $z_{m-i+1}r_{m-i+1}$. We note that $\text{dist}_R(f', r_{m-i+1}) = m$. Now F' contains $\mathbb{H}_m^{1,k,k,k}$ as a subgraph with degree 3 vertices f', r_{m-i+1} , a path of length 1 via the edge $z_{m-i+1}r_{m-i+1}$ and paths of length k via $R[m - i + 1 : m + k - i + 1], f' + F_3$ and $f' + F_4$. \triangleleft

We have now shown that, after applying one of these operations, we obtain a resulting graph which can be coloured using r colours if, and only if, the original graph could. Further, if the original graph was $\mathbb{H}_m^{1,k,k,k}$ -subgraph-free then so is the resulting graph. We will now

show how these operations can be applied to obtain that graph G' described in the theorem statement.

We first, in polynomial time, exhaustively apply the degree 2 removal operation to obtain a graph with minimum degree at least 3. Further, in polynomial time, for every $Q \subseteq V(G)$ of size $m+k+2$ we can decide if $G[Q]$ is a protected fan in G . If $G[Q]$ is a protected fan we can construct a graph $G^F(Q)$, such that $|V(G^F(Q))| < |V(G)|$, $(G^F(Q), r)$ is a yes-instance of COLOURING if, and only if, (G, r) is. Further, if G is $\mathbb{H}_m^{1,k,k,k}$ -subgraph-free, then so is $G^F(Q)$.

Likewise, in polynomial time, for every $S \subseteq V(G)$, such that $|S| \leq 2$, we can decide if S is the witness set for some minimal L -type subgraph. If so we can construct a graph $G^T(C \cup S)$ such that $|G^T(C \cup S)| < |V(G)|$, $(G^T(C \cup S), r)$ is a yes-instance of COLOURING if and only if (G, r) is and if G is $\mathbb{H}_m^{1,k,k,k}$ -subgraph-free, then so is $G^T(C \cup S)$.

We will first apply the degree 2 removal operation. We will then alternate between exhaustively applying the fan contraction operation and applying the L removal operation. After each application of either the fan contraction operation or the L removal operation we again remove any degree 2 vertices. Doing this exhaustively, we obtain a graph G' such that G' contains no protected fan of order $m+k+2$, G' contains L -type subgraph, G' can be coloured with r colours, if and only if G can, and if G is $\mathbb{H}_m^{1,k,k,k}$ -subgraph-free, then so is G' . \blacktriangleleft

Given an instance (G, r) of COLOURING, we now apply Lemma 24 and obtain a modified instance (G', r) . By Theorem 19, the treedepth of G' is bounded by a constant and since COLOURING is polynomial-time solvable for graphs of bounded treewidth [15] and therefore also graphs of bounded treedepth we obtain the following.

► **Theorem 25.** COLOURING is solvable in polynomial time for $\mathbb{H}_m^{1,k,k,k}$ -subgraph-free graphs, for all $m, k \geq 1$.

Proof. Let (G, r) be an instance of COLOURING, where G is a $\mathbb{H}_m^{1,k,k,k}$ -subgraph-free graph. We apply Lemma 24 and obtain in polynomial time an instance (G', r) of COLOURING such that G' has minimum degree at least 3; G' contains no protected fan of order $m+k+2$; and G' contains no L -type subgraph, with treedepth bound $(4k-3)(8k^2-6k+2m+8)-1$ and length bound $m+k$. Further, G' can be coloured with r colours, if and only if G can and as G is $\mathbb{H}_m^{1,k,k,k}$ -subgraph-free, then so is G' .

By Theorem 19, the treedepth of G' is bounded by a constant. As COLOURING is polynomial-time solvable for graphs of bounded treewidth [15] and therefore also graphs of bounded treedepth the theorem follows. \blacktriangleleft

We continue with the application of Theorem 20 to solve COLOURING on $S_{1,k,k,k}$ -subgraph-free graphs. In order to apply Theorem 20 to obtain a polynomial-time algorithm for COLOURING, we first show a last structural result.

We prove two more lemmas.

► **Lemma 26.** Let G be a graph and $c, k \geq 1$. Suppose there is some $C \subseteq V(G)$ such that $G[C]$ is a minimal T -type subgraph with respect to c and with some witness set S of size 2. Let G' be the graph obtained from G by removing the vertices of C and adding an edge between the two vertices of S . If G is $S_{1,k,k,k}$ -subgraph-free then so is the graph G' .

Proof. We will prove the contrapositive. Let $S = \{u, v\}$. Suppose there is some $F \subseteq G'$ such that F is isomorphic to $S_{1,k,k,k}$, then we claim G must also contain some subgraph isomorphic to $S_{1,k,k,k}$. Let f denote the degree 4 vertex of F . Let f_1^1 be that vertex lying

on the branch of length 1 and for each $i \in \{2, 3, 4\}$ and $j \in \{1, \dots, k\}$, let f_j^i denote the j th vertex along the i th branch, see Figure 17.

By definition of a T -type subgraph, we have that $|N(u) \cap (C \cup \{u, v\})|, |N(v) \cap (C \cup \{u, v\})| \geq 2$. Let u', u'' be two neighbours of u in $C \cup \{u, v\}$ and let v', v'' be two neighbours of v in $C \cup \{u, v\}$. Note, if u and v are not connected in $G[C \cup \{u, v\}]$ then there is some $C' \subsetneq C$ such that $v, v' \notin C'$ and $G[C']$ is a T -type subgraph with witness set $\{u\}$. By assumption $G[C]$ is minimal, that is without loss of generality there is a path from u to v via u'', v'' and vertices in C . We denote this path by $P^C = (u, u'', \dots, v'', v)$.

If $(u, v) \notin E(F)$ then F is a subgraph of G , that is, G contains a subgraph isomorphic to $S_{1,k,k,k}$. Let F' be the graph obtained from F by replacing the edge uv by the path P^C . Given the vertices of $P^C - \{u, v\}$ do not appear in G' , F' is (possibly the supergraph of) some $S_{1,k,k,k}$ subgraph in G . \blacktriangleleft

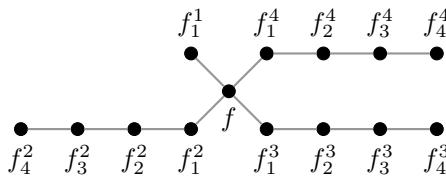


Figure 17 An illustration of $S_{1,4,4,4}$ with the corresponding vertex names.

► **Lemma 27.** *Let G be a graph and $c, k \geq 1$. Suppose there is some $C \subseteq V(G)$ such that $G[C]$ is a minimal T -type subgraph with respect to c and with some witness set S of size 2. Let G' be the graph obtained from G by removing the vertices of $C \cup S$ and adding a new vertex w such that $N(w) = N(S)$. If G is $S_{1,k,k,k}$ -subgraph-free then so is the graph G' .*

Proof. We will prove the contrapositive. Let $S = \{u, v\}$. Suppose there is some $F \subseteq G'$ such that F is isomorphic to $S_{1,k,k,k}$, then we claim G must also contain some subgraph isomorphic to $S_{1,k,k,k}$. Let f denote the degree 4 vertex of F . Let f_1^1 be that vertex lying on the branch of length 1 and for each $i \in \{2, 3, 4\}$ and $j \in \{1, \dots, k\}$, let f_j^i denote the j th vertex along the i th branch, see Figure 17.

By definition of a T -type subgraph, $|N(u) \cap (C \cup \{u, v\})|, |N(v) \cap (C \cup \{u, v\})| \geq 2$ holds. Let u', u'' be two neighbours of u in $C \cup \{u, v\}$ and let v', v'' be two neighbours of v in $C \cup \{u, v\}$. Note, if u and v are not connected in $G[C \cup \{u, v\}]$ then there is some $C' \subsetneq C$ such that $v, v' \notin C'$ and $G[C']$ is a T -type-subgraph with witness set $\{u\}$. By assumption $G[C]$ is minimal, that is, without loss of generality, there is a path from u to v via u'', v'' and vertices in C . We denote this path by $P^C = (u, u'', \dots, v'', v)$.

If $w \notin V(F)$, then F is a subgraph of G and so G contains a subgraph isomorphic to $S_{1,k,k,k}$. If $w \neq f$, then either $w = f_1^1$ or $w = f_j^i$ for some $i \in \{2, 3, 4\}$ and $j \in \{1, \dots, k\}$. Recall there exists the path $P^C \subseteq G$ with $V(P^C) \cap V(G') = \emptyset$. Let F' be the graph obtained by replacing the vertex w with the path P^C . Now F' is (possibly the supergraph of) some $S_{1,k,k,k}$ subgraph in G . Suppose now $w = f$.

It follows that either u or v is adjacent to at least 2 of f_1^2, f_1^3, f_1^4 in G . Without loss of generality, say v is adjacent to f_1^3 and f_1^4 . If v is also adjacent to f_1^2 in G , then replacing the vertex f_1^1 by v'' in F we find a $S_{1,k,k,k}$ subgraph in G . From this, we assume that v is not adjacent to f_1^2 and so u must be adjacent to f_1^2 . Let \hat{P}^C be a shortest path from u to v in $G[V(C) \cup \{u, v\}]$. Note \hat{P}^C cannot contain both v' and v'' else there is a shorter path containing only one of them. Without loss of generality, we may assume that \hat{P}^C does not

contain v' . Let F' be the graph obtained by replacing the edge wf_1^2 by the path \hat{P}^C and the edge wf_1^1 by the edge vv' . Once again F' is (possibly the supergraph of) some $S_{1,k,k,k}$ subgraph in G . \blacktriangleleft

Using the previous two lemmas we can now show the following lemma.

► **Lemma 28.** *Let (G, r) be an instance of COLOURING and let $c, k \geq 1$ be constants. We can in polynomial time obtain a graph G' such that:*

- G' contains no T -type subgraph (with respect to c);
- G' can be coloured with r colours, if and only if G can;
- if G is $S_{1,k,k,k}$ -subgraph-free, then so is G' .

Proof. Each of our T -type subgraphs will be with respect to the constant c . We now define a T removal operation as follows. Suppose G contains some set $C \subseteq V(G)$ such that $G[C]$ is a minimal T -type subgraph with witness set S . By Lemma 8, $\text{td}(G[C \cup S]) \leq 3k + 2$. We branch on the (at most) two non-isomorphic colourings of S . For each branch, we consider the precolouring extension problem on the graph $G[C \cup S]$, with the set S of precoloured vertices. Since $\text{td}(G[C \cup S]) \leq 3k + 2$, we can use the result of [11] again to process each branch in polynomial time.

If there is no proper r -colouring of $G[C \cup S]$, then (G, r) is a no-instance of COLOURING and so we return no. Assume now $G[C \cup S]$ can be coloured using r colours. If $|S| = 1$, then G can be coloured using r colours, if, and only if, both $G[C \cup S]$ and $G - C$ can be coloured using r . Let $G^T(C \cup S) = G - C$. As $|N(S) \cap C| \geq 2$, we get that $|V(G^T(C \cup S))| < |V(G)|$. In addition, if G is $S_{1,k,k,k}$ -subgraph-free then so is $G^T(C \cup S)$.

That is, we now assume that $|S| = 2$. Let $S = \{u, v\}$. If there is both a proper colouring where u, v are coloured the same and where they are different, then G can be coloured with r colours if and only if, $G - C$ can be coloured. Let $G^T(C \cup \{u, v\}) = G - C$. As the class of $S_{1,k,k,k}$ -subgraph-free graphs is closed under vertex deletion, if G is $S_{1,k,k,k}$ -subgraph-free, then so is $G^T(C \cup \{u, v\})$. If there is only a proper colouring of $G[C \cup \{u, v\}]$ where u, v are coloured differently, then let $G^T(C \cup \{u, v\})$ be the graph obtained from G by removing the vertices of C and adding the edge uv . It follows from Lemma 26, that if G is $S_{1,k,k,k}$ -subgraph-free then so is $G^T(C \cup S)$. As $|N(S) \cap C| \geq 2$, we get that $|V(G^T(C \cup S))| < |V(G)|$. The remaining case is where u and v must take the same colour in every colouring of $G[C \cup S]$. Let $G^T(C \cup S)$ be the graph obtained from G by removing the vertices of C and adding a new vertex w such that $N(w) = N(u) \cup N(v)$. Now, G can be coloured using r colours if, and only if, $G^T(C \cup S)$ can be coloured using r colours. As $|N(S) \cap C| \geq 2$, $|V(G^T(C \cup S))| < |V(G)|$. Further, from Lemma 27, if G is $S_{1,k,k,k}$ -subgraph-free then so is $G^T(C \cup S)$.

In polynomial time, for every $S \subseteq V(G)$, such that $|S| \leq 2$, we can decide if S is the witness set for some minimal T -type subgraph. If so we can construct a graph $G^T(C \cup S)$ such that $|G^T(C \cup S)| < |V(G)|$ and $(G^T(C \cup S), r)$ is a yes-instance of COLOURING if, and only if, (G, r) is. Applying this operation exhaustively, we obtain a graph G' such that G' contains no T -type subgraph, G' can be coloured with r colours, if and only if G can and if G is $S_{1,k,k,k}$ -subgraph-free, then so is G' . \blacktriangleleft

Given an instance (G, r) of COLOURING, we now apply Lemma 28 and obtain a modified instance (G', r) . We may assume by [14] that G' contains no bridges and further by [1] that G' contains some degree 4 vertex. By Theorem 20, the treedepth of G' is bounded by a constant and since COLOURING is polynomial-time solvable for graphs of bounded treewidth [15] and therefore also graphs of bounded treedepth we obtain the following.

► **Theorem 29.** COLOURING is solvable in polynomial time for $S_{1,k,k,k}$ -subgraph-free graphs, for all $k \geq 1$.

Proof. Let (G, r) be an instance of COLOURING, where G is a $S_{1,k,k,k}$ -subgraph-free graph. We apply Lemma 28 and obtain in polynomial time an instance (G', r) of COLOURING such that G' contains no T -type subgraph with treedepth bound $16(2k - 1)(k - 1)$, and G' is $S_{1,k,k,k}$ -subgraph-free. Further, G' can be coloured using r colours if, and only if, G can be coloured using r colours. In [14], it was shown that the graph obtained from G' by deleting all bridges can be coloured using r if, and only if G' can be coloured using r colours. That is, we assume that G' contains no bridges. If G' has maximum degree at most 3, by Brooks' Theorem [1], we solve COLOURING in polynomial time. That is we assume that G' contains some degree 4 vertex. Hence, from Theorem 20, as G' is $S_{1,k,k,k}$ -subgraph-free, $\text{td}(G) < 8(7k^3 + 15k^2 - \frac{4k}{9} + 3)^2 + 6$. As COLOURING is polynomial-time solvable for graphs of bounded treewidth [15] and therefore also for graphs of bounded treedepth the theorem follows. ◀

7 Wider Applicability of Our Techniques

Our main structural results, Theorems 19 and 20, can be applied to other problems, as we explain below.

Set of Conditions I. Let Π be a problem that is polynomial-time solvable for graphs of bounded treedepth. If, for every $k, m \geq 1$, there exist constants c, ℓ such that we are able to preprocess the input graph in polynomial time to a polynomial number of smaller graphs of minimum degree at least 3 that contains neither a large protected fan nor an L -type subgraph with length bound ℓ and treedepth bound c , then Π is polynomial-time solvable on $\mathbb{H}_m^{1,k,k,k}$ -subgraph-free graphs for all $k, m \geq 1$.

Set of Conditions II. Let Π be a problem that is polynomial-time solvable for both subcubic graphs and graphs of bounded treedepth. If, for every $k \geq 1$, there exists a constant c such that we are able to preprocess the input graph in polynomial time to a polynomial number of smaller graphs that each contain some vertex of degree at least 4 but neither contain a proper bridge nor a T -type subgraph with treedepth bound c , then Π is polynomial-time solvable for $S_{1,k,k,k}$ -subgraph-free graphs for all $k \geq 1$.

We note that if a problem satisfies the above two sets of conditions then the algorithm of [14] can be applied directly to obtain an algorithm for $S_{1,1,r,r}$ -subgraph-free graphs. Conversely, the above two sets of conditions are not satisfied by every problem considered in [14]: MATCHING CUT is NP-complete for bipartite graphs in which one partition class has maximum degree 2 [21] and thus on $(\mathbb{H}_1, \mathbb{H}_3, \mathbb{H}_5, \dots)$ -subgraph-free graphs. It is still open whether the polynomial-time result for $S_{1,1,r,r}$ -subgraph-free graphs for MATCHING CUT from [14] can be generalized to $S_{1,k,k,k}$ -subgraph-free graphs. This is in contrast to a related problem: STABLE CUT, which we recall is the problem of deciding if a connected graph G has an independent set I such that $G - I$ is disconnected. Below we apply our techniques to show the same results for STABLE CUT as we did for COLOURING:

► **Theorem 30.** STABLE CUT is solvable in polynomial time for $\mathbb{H}_m^{1,k,k,k}$ -subgraph-free graphs for all $k, m \geq 1$, and for $S_{1,k,k,k}$ -subgraph-free graphs for all $k \geq 1$.

Proof. Suppose first that G contains some cutset S of size at most 2. Note that we can find such a cutset in polynomial time by iterating over all possible sets. If either S has size 1 or $G[S]$ consists of a pair of non-adjacent vertices, then G contains a stable cut. Suppose

now $G[S]$ consists of two adjacent vertices. As S is a clique, it was shown by Le, Mosca and Müller [17] that G has a stable cutset if and only if there is some connected component $G[C]$ of $G - S$ such that $G[C \cup S]$ has a stable cutset. We therefore consider each such connected component of $G - S$ in turn.

That is, we now assume that G does not contain any cutset with size at most 2. Note this implies that G has minimum degree at least 3 as the neighbourhood of any vertex with degree at most 2 is a cutset with size at most 2. Further, we may assume G does not contain a bridge, a T -type subgraph or an L -type subgraph (for any treedepth and length bounds), since again this would give a cutset of size at most 2.

Suppose G is $S_{1,k,k,k}$ -subgraph-free. As STABLE CUT is polynomial-time solvable for graphs with maximum degree at most 3 [3], we assume that G contains some vertex with degree at least 4. As G is bridgeless, contains no T -type subgraph and is $S_{1,k,k,k}$ -subgraph-free, by Theorem 20, G has treedepth less than $2(7k^3 + 15k^2 - \frac{4k}{9} + 3)^2 + 6$. As STABLE CUT is readily seen to be definable in monadic second-order logic, it is polynomial-time solvable on graphs of bounded treewidth due to Courcelle's Theorem [4], and therefore also for graphs of bounded treedepth. Hence, we can decide if G contains a stable cut in polynomial time.

Suppose now G is $\mathbb{H}_m^{1,k,k,k}$ -subgraph-free. Recall that G has minimum degree at least 3 and no L -type subgraph. Suppose G contains some protected fan with vertex set F , centre z and ends x and y . For every vertex $v \in F \setminus \{x, y, z\}$, $N(v) \subseteq F$ and so v is not contained in any minimal stable cut of G . It follows that v can be deleted from G . We note that the class of $\mathbb{H}_m^{1,k,k,k}$ -subgraph-free graphs is closed under vertex deletion. We exhaustively delete such vertices to obtain a graph G' such that G' contains no protected fan (of size at least 3). It now follows from Theorem 19 that G' has treedepth bounded by some constant. Hence, we can apply Courcelle's Theorem [4] again to decide in polynomial time if G' contains a stable cut. This concludes the proof of the theorem. \blacktriangleleft

8 Conclusions

In contrast to H -free graphs [16] and H -minor-free graphs (see e.g. [13]), for which COLOURING is fully classified, even formulating a dichotomy for H -subgraph-free graphs is still challenging. However, our new dichotomies for COLOURING and STABLE CUT, being the first of their kind, open up the way for further progress for many graph problems. In this section we will summarize our findings and explore some new directions.

We proved that COLOURING on H -subgraph-free graphs is polynomial-time solvable on $\mathbb{H}_m^{1,k,k,k}$ -subgraph-free graphs for all $k, m \geq 1$ and on $S_{1,k,k,k}$ -subgraph-free graphs for all $k \geq 1$. Combining these results with known NP-completeness results yields a complete complexity classification of COLOURING on H -subgraph-free graphs whenever H is a subdivided \mathbb{H}_0 or a subdivided \mathbb{H}_1 . As mentioned in Section 1, combining our new results with known results leaves open the cases where

- (i) H is a tree of maximum degree 4 with exactly one vertex of degree 4 and at least one vertex of degree 3; or
- (ii) H is a subcubic tree with at least three vertices of degree 3.

Some graphs H of type (i) and (ii) are covered by existing NP-completeness results for smaller graphs H' (e.g. when H contains an $S_{2,2,2,2}$). Nevertheless, there still exist exactly four open cases of graphs H on eight vertices, three of type (i) and one of type (ii); see Figure 18. Our structural analysis does not apply to these four open cases and new ideas are needed. In fact, even the case when H is the 7-vertex graph obtained from \mathbb{H}_1 by adding a pendant vertex to one of the degree-3 vertices is not covered by our new technique. Indeed, there exists

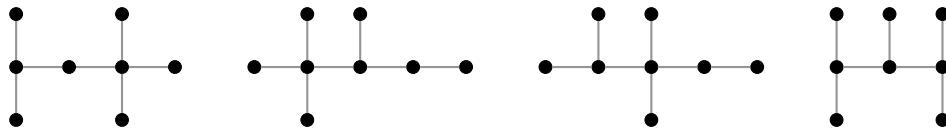


Figure 18 The four open cases of graphs H on eight vertices.

a family of H -subgraph-free graphs demonstrating that a modified Theorem 19, where \mathbb{H}_1 is replaced by H does not hold. Likewise, a modified of Theorem 20 with $S_{1,k,k,k}$ replaced by H also does not hold. This case, which is of type (i), is shown to be polynomial-time solvable in [9] by a rather involved tailor-made algorithm. Studying the proof technique in more detail would be a natural starting point.

Finally, as future work we propose to investigate in a systematic way which other graph problems satisfy the sets of conditions specified in Section 7. In particular, we ask whether MATCHING CUT is polynomial-time solvable for $S_{1,k,k,k}$ -subgraph-free graphs for every $k \geq 1$.

References

- 1 R. L. Brooks. On colouring the nodes of a network. *Mathematical Proceedings of the Cambridge Philosophical Society*, 37(2):194–197, 1941.
- 2 Maria Chudnovsky, Sepehr Hajebi, and Sophie Spirkl. List- k -coloring H -free graphs for all $k \geq 4$. *Combinatorica*, 44:1063–1068, 2024.
- 3 Vasek Chvátal. Recognizing decomposable graphs. *J. Graph Theory*, 8(1):51–53, 1984.
- 4 Bruno Courcelle. The monadic second-order logic of graphs. I. Recognizable sets of finite graphs. *Information and Computation*, 85:12–75, 1990.
- 5 Thomas Emden-Weinert, Stefan Hougardy, and Bernd Kreuter. Uniquely colourable graphs and the hardness of colouring graphs of large girth. *Combinatorics, Probability and Computing*, 7:375–386, 1998.
- 6 F. Galvin, I. Rival, and B. Sands. A Ramsey-type theorem for traceable graphs. *Journal of Combinatorial Theory, Series B*, 33(1):7–16, 1982.
- 7 Michael R. Garey, David S. Johnson, and Larry J. Stockmeyer. Some simplified NP-complete graph problems. *Theoretical Computer Science*, pages 237–267, 1976.
- 8 Petr A. Golovach, Matthew Johnson, Daniël Paulusma, and Jian Song. A survey on the computational complexity of coloring graphs with forbidden subgraphs. *Journal of Graph Theory*, 84:331–363, 2017.
- 9 Petr A. Golovach, Daniël Paulusma, and Bernard Ries. Coloring graphs characterized by a forbidden subgraph. *Discrete Applied Mathematics*, 180:101–110, 2015.
- 10 Sepehr Hajebi, Yanjia Li, and Sophie Spirkl. Complexity dichotomy for List-5-Coloring with a forbidden induced subgraph. *SIAM Journal on Discrete Mathematics*, 36:2004–2027, 2022.
- 11 Klaus Jansen and Petra Scheffler. Generalized coloring for tree-like graphs. *Discrete Applied Mathematics*, 75:135–155, 1997.
- 12 Justyna Jaworska, Bartłomiej Kielak, Tomáš Masařík, and Jana Masaříková. Constricting the computational complexity gap of the 4-Coloring problem in (P_t, C_3) -free graphs. *CoRR*, abs/2509.02423, 2025.
- 13 Matthew Johnson, Barnaby Martin, Jelle J. Oostveen, Sukanya Pandey, Daniël Paulusma, Siani Smith, and Erik Jan van Leeuwen. Complexity framework for forbidden subgraphs I: the framework. *Algorithmica*, 87:429–464, 2025.
- 14 Matthew Johnson, Barnaby Martin, Sukanya Pandey, Daniël Paulusma, Siani Smith, and Erik Jan van Leeuwen. Complexity framework for forbidden subgraphs III: When problems are tractable on subcubic graphs. *Proc. MFCS 2023, LIPIcs*, 272:57:1–57:15, 2023.
- 15 Daniel Kobler and Udi Rotics. Edge dominating set and colorings on graphs with fixed clique-width. *Discrete Applied Mathematics*, 126:197–221, 2003.

- 16 Daniel Král', Jan Kratochvíl, Zsolt Tuza, and Gerhard J. Woeginger. Complexity of coloring graphs without forbidden induced subgraphs. *Proc. WG 2001, LNCS*, 2204:254–262, 2001.
- 17 Van Bang Le, Raffaele Mosca, and Haiko Müller. On stable cutsets in claw-free graphs and planar graphs. *J. Discrete Algorithms*, 6(2):256–276, 2008.
- 18 László Lovász. Coverings and coloring of hypergraphs. *Congr. Numer.*, VIII:3–12, 1973.
- 19 Vadim V. Lozin and Marcin Kamiński. Coloring edges and vertices of graphs without short or long cycles. *Contributions to Discrete Mathematics*, 2, 2007.
- 20 Vadim V. Lozin, Barnaby Martin, Sukanya Pandey, Daniël Paulusma, Mark H. Siggers, Siani Smith, and Erik Jan van Leeuwen. Complexity framework for forbidden subgraphs II: edge subdivision and the "H"-graphs. *Proc. ISAAC 2024, LIPIcs*, 322:47:1–47:18, 2024.
- 21 Augustine M. Moshi. Matching cutsets in graphs. *Journal of Graph Theory*, 13:527–536, 1989.
- 22 Jaroslav Nešetřil and Patrice Ossona de Mendez. *Sparsity - Graphs, Structures, and Algorithms*, volume 28 of *Algorithms and combinatorics*. Springer, 2012.
- 23 Neil Robertson and Paul D. Seymour. Graph minors. III. planar tree-width. *Journal of Combinatorial Theory, Series B*, 36:49–64, 1984.
- 24 Susan S. Wang. Structure and coloring of graphs with only small odd cycles. *SIAM Journal on Discrete Mathematics*, 22:1040–1072, 2008.