

MINUSCULE COXETER DRESSIANS

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ABSTRACT. In this extended abstract, we study special tropical prevarieties which we call Coxeter Dressians. They arise from equations capturing a generalization of valuated symmetric basis exchange for Coxeter matroids. In particular, we study subdivisions of the associated Coxeter matroid polytopes. We show that the subdivisions induced by points of the Coxeter Dressian consist of cells which are strong Coxeter matroidal. This generalizes well-known results in type A to other Lie types. Finally, we implement explicit computations of Coxeter Dressians in **OSCAR**.

1. INTRODUCTION

Tropicalization is a process that famously translates algebro-geometric objects into combinatorial ones. Importantly, it is sensitive to chosen coordinates and embeddings of given varieties. Systems of polynomial equations are ubiquitous throughout the disciplines of algebra, geometry, and adjacent fields. In this project, we are interested in tropicalizing so-called *strong exchange equations*, which appear naturally in representation theory. The spaces they cut out are known as *generalized flag varieties*.

One such instance is the Grassmann–Plücker equations. It is well-known, that these equations cut out the Grassmannian, which parametrizes subspaces of a given vector space. Its tropical counterpart, the Dressian, parametrizes tropical linear spaces, and is obtained by explicitly tropicalizing the equations. The study of tropical linear spaces, going back to [Spe08], has been proven to be one of the most fruitful endeavors in tropical geometry, with numerous connections to combinatorics, algebra, and biology.

When one studies tropical linear spaces, or valuated matroids, in the background of the theory one always finds the root system A_n . An example is the Gel’fand–Serganova theorem, or Speyer’s characterization of valuated matroids [Spe08] of valuations inducing matroidal subdivision of the hypersimplex. The goal of this paper is to go beyond the type A_n case, and introduce valuated matroids as well as Dressians for different root systems.

Coxeter matroids are generalizations of matroids which arise as collections of cosets in W/P , where W is a reflection group of any Lie type, and P is a standard parabolic subgroup. We study Coxeter matroids which possess the *strong exchange property*, which captures the symmetric exchange property of usual matroids. We further restrict to a special subclass, those Coxeter matroids which are said to be *minuscule*. Minuscule strong exchange Coxeter matroids are the most faithful incarnation of usual matroids outside Lie type A .

For us, the relevant tropical prevarieties are defined by the equations which recently appeared in [CDF_S25]. We call the prevarieties of interest *Coxeter Dressians*. Let $\mathbb{T} := \mathbb{R} \cup \{\infty\}$ be the semifield of tropical numbers with min-notation. For minuscule W/P , the Coxeter Dressian is denoted $\text{Dr}(W, P) \subseteq \mathbb{T}^{W/P}$. For each W/P , these defining equations are obtained by tropicalizing polynomials which generate the ideal of the related generalized flag variety [CDF_S25]. When feasible, we compute and study concrete examples of these prevarieties. Every Coxeter matroid admits an encoding as a polytope. We study the tropical geometry associated with regular polyhedral subdivisions into cells which are again Coxeter matroidal. Our main theorem is as follows.

Theorem A (Theorem 3.3). *Let (W, P) be minuscule, and $\mu \in \text{Dr}(W, P)$ be a point in the corresponding Coxeter Dressian. Then μ induces a subdivision of its matroid polytope into strong Coxeter matroid polytopes.*

Theorem A holds for all Lie types admitting minuscule matroids. That is, for types A, B, C, D, E_6, E_7 . We show that, in previously unknown cases for types D and E_6 , the converse holds. We give a counter example to the converse for type B . For E_7 , we show that a proper subset of the strong exchange equations is sufficient to detect strong matroidal subdivisions.

Simplifying systems equations can be crucial for explicit computations. In the A_n case, it is well-known that the 3-term Grassmann–Plücker relations imply the other relations [DW92, Spe08]. A similar statement is known in the type D_n [Rin12]. We prove the type B_n analog.

Theorem B (Theorem 4.9). *Let $\mu \in \mathbb{T}^{2^{[n]}}$, such that $\text{supp}(\mu)$ is a strong Coxeter matroid of type B_n . Then μ belongs to the Coxeter Dressian if and only if μ satisfies the 4-term strong exchange equations.*

As the strong exchange equations reflect the structure of the associated Coxeter matroid polytopes, it is natural to study Coxeter Dressians and valuations via direct computation. Using OSCAR [OSC25], we implement code to compute the polytopes and equations, test if a height function is contained in the Coxeter Dressian, and determine if a subdivision is Coxeter matroidal. We directly compute special cases of Coxeter Dressians in type B_n for $n \in \{3, 4\}$, and type D_n for $n \in \{5, 6\}$. Furthermore, we compare these prevarieties to the secondary fans of the relevant polytopes, and compute the f -vectors. Finally, we perform an exhaustive computation to verify that strong exchange holds for all E_7 minuscule Coxeter matroids. Our computations can be found at the following `github` repository:

https://github.com/danteluber/strong_coxeter_dressians

For this extended abstract, we assume some knowledge of regular subdivisions and secondary fans. The standard reference is [DLRS10]. For all aspects of tropical geometry, we point to [Jos21, MS15]. Our reference for (Coxeter) matroid theory is [BGW03].

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2. COXETER MATROIDS AND THEIR POLYTOPES

We defer to Chapter 5 of [BGW03] for the formal definitions concerning (classifications of) root systems, reflection groups, and general cryptomorphisms for Coxeter matroids. However, we remind the reader that *root system* are special collections of vectors corresponding to mirrors over hyperplanes in \mathbb{R}^n with deep connections to representation theory. Any root system Φ gives rise to a *Weyl group* W , acting on Euclidean space as a finite reflection group. Any choice of *simple roots* $\Phi^s = \{s_1, \dots, s_n\}$ then yields generators of W , i.e., $W = \langle s_1, \dots, s_n \rangle$. When we refer to a Weyl group, we implicitly assume the datum of its root system is known. Finally, every irreducible root system is uniquely classified by type A_n, B_n, C_n, \dots (etc). So when we say (for example)

$W = W(B_n)$, we mean that W is a reflection group associated with the type B_n root system, and that W is generated by n simple reflections within Φ^s .

Now, let $W = \langle s_1, \dots, s_n \mid s_i \in \Phi^s \rangle$, and consider $P = \langle s_{i_1}, \dots, s_{i_k} \rangle$. Let $v \in V \setminus \{0\}$ be such that $\text{stab}_W(v) = P$. That is, $\sigma \cdot v = v$ if and only if $\sigma \in P$. Then we identify every element of the orbit $W \cdot v$ with a unique coset in W/P . For any $\mathcal{M} \subseteq W/P$, we generate the polytope

$$\Delta(\mathcal{M}) = \text{conv}(\{\sigma \cdot v \mid \sigma P \in \mathcal{M}\}).$$

The vertices of $\Delta(W/P)$ are identified with the cosets, and every edge of $\Delta(W/P)$ is parallel to an element of Φ . More generally, $\mathcal{M} \subseteq W/P$ is a *Coxeter matroid* if every edge of $\Delta(\mathcal{M})$ is parallel to an element of Φ . Throughout this document, we may interchangeably think of an element $A \in \mathcal{M}$ as a coset or a vertex of the associated polytope. In line with usual matroid theory, when \mathcal{M} is a Coxeter matroid, its elements will be referred to as *bases*.

2.1. Minuscule Coxeter Matroids and the Strong Exchange Property. Let W be a Weyl group associated to the root system Φ , $P \leq W$ parabolic, and $\mathcal{M} \subseteq W/P$. Then \mathcal{M} possesses the *strong exchange property* if for all $A, B \in \mathcal{M}$, there exists a reflection $s \in \Phi$ such that the mirror defined by s separates A and B , and $sA, sB \in \mathcal{M}$. Observe that this definition mimics the symmetric exchange property for usual matroids. Any collection of cosets with the strong exchange property is a Coxeter matroid [BGW03, Theorem 6.1.2], but the converse fails [BGW03, 4.2.4]. Finally, the *Bruhat order* is a partial ordering on W which extends to W/P . When the resulting poset is a lattice, then P , is said to be *minuscule* [Pro84]. In that case, P is always maximal. That is, $P = P_r := \langle s_1, \dots, \hat{s}_r, \dots, s_n \rangle$.

Remark 2.1. Minuscule parabolic subgroups may also be defined in terms of fundamental weights coming from representation theory. The quotient \mathbb{G}/\mathbb{P} of a reductive group by a parabolic subgroup is naturally a projective variety. For minuscule representations, the ambient projective space of this variety naturally comes with a basis indexed by the cosets of the Weyl group. This allows one to concretely write equations in explicit coordinates that cut out \mathbb{G}/\mathbb{P} [Ses78].

2.2. Set-theoretic Identification of Cosets. In the types A_{n-1}, B_n, C_n , and D_n with minuscule parabolic P one can identify cosets in W/P with subsets of $[n]$. We may identify the Weyl group $W(A_{n-1})$ with the symmetric group S_n . The generating set is that of neighbor transpositions $s_i = (i, i+1)$, and all $P_r = \langle s_1, \dots, \hat{s}_r, \dots, s_n \rangle = \text{stab}_{S_n}([r])$ are minuscule. We then identify any coset wP_r with $w([r]) \in 2^{[n]}$ for $w \in S_n$. Moreover, under this identification, the polytope $\Delta(A_n/P_r)$ is identified with the (r, n) -hypersimplex $\Delta(r, n) = \text{conv}\{e_A \mid A \in \binom{[n]}{r}\}$.

The Weyl group of B_n and C_n is the hyperoctahedral group, i.e., the group of permutations of $[n] \sqcup -[n]$ commuting with the involution $i \mapsto -i$. As generators we have $s_i = (i, i+1)(-i, -(i+1))$ for $i = 1, \dots, n-1$, and $s_n = (n, -n)$. The only minuscule parabolic subgroup of B_n is $P_n = \langle s_1, \dots, s_{n-1} \rangle = \text{stab}_{W(B_n)}(-[n])$. We now identify cosets in $W(B_n)/P$ with subsets of $[n]$ via $wP \mapsto w(-[n]) \cap [n]$. Hence, the polytope $\Delta(B_n/P_n) = \text{conv}\{e_A - (-\frac{1}{2}, \dots, -\frac{1}{2}) \mid A \in 2^{[n]}\}$ is exactly the shifted unit n -cube.

2.3. Strong Exchange Equations. Let (W, P) be minuscule, and $\mathcal{M} \subseteq W/P$. Let $\mu_{\mathcal{M}} \in \{0, \infty\}^{W/P}$ such that $\mu_{\mathcal{M}}(A) = 0$ if $A \in \mathcal{M}$ and $\mu_{\mathcal{M}}(A) = \infty$ otherwise. Recently, in [CDFSS25] the authors introduced collections of tropical equations \mathcal{F} such that the minimum of $f \in \mathcal{F}$ is attained at least twice when evaluated at $\mu_{\mathcal{M}}$ if and only if \mathcal{M} is a strong Coxeter matroid. These equations are square-free, quadratic, and reflect the polyhedral structure of $\Delta(W/P)$, where the (degree-2)-monomials correspond to pairs of antipodes on a face. These equations are similar in spirit to those studied in [BB16, BB19], however the latter works only cover usual matroids, and consider different hyperfields to detect matroidal properties.

3. THE COXETER DRESSIAN

In this section, we will introduce valuated Coxeter matroids as well as the spaces parameterizing them. Recall that \mathbb{T} denotes the tropical min-semifield with operations denoted by \oplus and \odot .

Definition 3.1. Let W be a Weyl group and P a minuscule parabolic subgroup. Let \mathcal{F} be the collection of associated strong exchange equations (see Equations 1, 2, 3, 4, 5). The (*strong*) *Coxeter Dressian* of W/P is the tropical prevariety

$$\text{Dr}(W, P) = \bigcap_{f \in \mathcal{F}} V^{\text{trop}}(f) \subseteq \mathbb{T}^{W/P}.$$

We will call points $\mu \in \text{Dr}(W, P)$ *valuated Coxeter matroids*. Usually, to declutter notation, we will write the Coxeter Dressian, for example, as $\text{Dr}(A_n, P_r)$.

Definition 3.2. Let $E \subset \mathbb{R}^n$ be finite. We say that a set $\mathcal{F} \subseteq \mathbb{T}[x_e | e \in E]$ of tropical polynomials with trivial coefficients is *affinely invariant* if for any function $\mu : E \rightarrow \mathbb{T}$ and for all affine linear functions $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ we have

$$\mu \in V^{\text{trop}}(\mathcal{F}) \text{ if and only if } \mu + \varphi|_E \in V^{\text{trop}}(\mathcal{F}).$$

Theorem 3.3. Let $E \subset \mathbb{R}^n$ be finite, and $\mu : E \rightarrow \mathbb{T}$, with induced subdivision $S(\mu)$ of $\text{conv}(\text{supp}(\mu))$. Let $\mathcal{F} \subseteq \mathbb{T}[x_e | e \in E]$ be an affinely invariant set of tropical polynomials with trivial coefficients. If $\mu \in V^{\text{trop}}(\mathcal{F}) \subseteq \mathbb{T}^E$, then for any cell $C \in S(\mu)$, the function

$$\mu_C : E \rightarrow \mathbb{T}, \quad e \mapsto \begin{cases} 0 & \text{if } e \in C \\ \infty & \text{else} \end{cases}$$

satisfies all tropical equations in \mathcal{F} . In other words, $\mu_C \in V^{\text{trop}}(\mathcal{F})$.

Proof. Consider a cell C of $S(\mu)$. Let $\tilde{\mu} = \mu + \varphi$ for some affine linear function φ such that $\tilde{\mu}|_C = 0$ and $\tilde{\mu}|_{S \setminus C} > 0$. Since \mathcal{F} is affinely invariant, $\tilde{\mu} \in V^{\text{trop}}(\mathcal{F})$. Since $\mu, \tilde{\mu}$ only differ by an affine linear function, their induced subdivisions agree. Let $f \in \mathcal{F}$, then $f = \bigoplus_\alpha x^{\odot \alpha}$ for finitely many $\alpha \in \mathbb{N}^E$. Since this minimum is attained at least twice, when plugging in $\tilde{\mu}$, there are two cases:

- If the evaluation $f(\tilde{\mu}) = 0$, then there are at least two monomials of f supported on C . In that case, surely $f(\mu_C) = 0$, and the minimum is attained at these monomials. Thus, $\mu_C \in V^{\text{trop}}(\mathcal{F})$.
- If $f(\tilde{\mu}) > 0$, then all monomials of f contain a variable in $E \setminus C$. It follows, that $f(\mu_C) = \infty$, and hence trivially μ_C tropically satisfies f . \square

Theorem 3.4. For all minuscule types $A_n, B_n, C_n, D_n, E_6, E_7$, the set of strong exchange equations are affinely invariant. Thus, by Theorem 3.3, all height functions $V(\Delta(W/P)) \rightarrow \mathbb{T}$ satisfying the respective equations induce strong matroidal subdivisions.

Proof. We prove affine invariance case by case in Section 4. More specifically, we deal with cases where valuations are not already well studied. We do so via Propositions 4.1 (type (B_n, P_n)), 4.4 (type (D_n, P_1)), 4.12 (type (E_7, P_1)). Proofs for the other cases can be done in a similar fashion, but are omitted here. From affine invariance, the indicator vectors of the cells satisfy respective strong exchange equations, hence by [CDF25, Theorem 1.1], the cells are strong matroid polytopes. \square

4. THE MINUSCULE TYPES

In this section we will explicitly look at tropical strong exchange equations individually for each minuscule Lie type. These equations were systematically analyzed in [CDF25] over the Boolean semifield.

4.1. Type A_n . As mentioned in the introduction, valuations of type A Coxeter matroids have been extensively studied even outside the minuscule case [Spe08, BEZ21, JLLO23]. For the equations corresponding to usual matroids, we defer to [MS15, Chapter 4]. The type A strong exchange

equations are exactly the Grassmann–Plücker equations. They are affinely invariant, but we will omit a proof in this abstract.

4.2. Type B_n . Recall that in type B_n , the Weyl group is the hyperoctahedral group, and the only minuscule parabolic subgroup is $P_n = \langle s_1, \dots, s_{n-1} \rangle = \text{stab}_{B_n}(-[n])$. We identify cosets $wP_n \in B_n/P_n$ with subsets of $[n]$ via $wP_n \mapsto w(-[n]) \cap [n]$. Fixing (B_n, P_n) , the *strong exchange equations* are

$$(1) \quad f_{I,J}^B := \begin{cases} \bigoplus_{i \in I \Delta J} x_{I \Delta i} \odot x_{J \Delta i} & \text{if } |I \Delta J| \equiv 0 \pmod{2} \\ x_I \odot x_J \oplus \bigoplus_{i \in I \Delta J} x_{I \Delta i} \odot x_{J \Delta i} & \text{if } |I \Delta J| \equiv 1 \pmod{2}, \end{cases}$$

for all $I, J \subseteq [n]$ with $|I \Delta J| \geq 3$. They are tropicalizations of quadratic algebraic equations cutting out the type B_n minuscule flag variety.

Proposition 4.1. *The type B_n strong exchange equations 1 are affinely invariant.*

Proof. Let $\mu : 2^{[n]} \rightarrow \mathbb{T}$ be arbitrary, and $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ be affine linear. We consider the variables as vertices of the n -cube $Q_n = [-\frac{1}{2}, \frac{1}{2}]^n$. We fix $I, J \subseteq [n]$ with $|I \Delta J| \geq 3$. For each monomial $x_{I \Delta i} \odot x_{J \Delta i}$ in $f_{I,J}^B$, evaluating at $\mu + \varphi|_{V(Q_n)}$ yields $\mu_{I \setminus i} + \mu_{J \cup i} + (\varphi(e_I) + \varphi(e_J) - \varphi(0))$, the last term being independent of i . The same holds for $x_I \odot x_J$, in the case $|I \Delta J| \equiv 1 \pmod{2}$. Thus μ attains the minimum at least twice if and only if $\mu + \varphi|_{V(\Delta(r,n))}$ does. \square

It is now natural to ask, whether type B_n behaves as nicely as type A_n , and the converse of Theorem 3.4 also holds. This turns out to be wrong already for $n = 3$, as the following example shows.

Example 4.2. We get a single strong exchange equation in type B_3 , which is supported on the entire 3-cube and given by

$$f_{\emptyset,[3]}^B = x_\emptyset \odot x_{123} \oplus x_1 \odot x_{23} \oplus x_2 \odot x_{13} \oplus x_3 \odot x_{12}.$$

Some functions on the 3-cube induce strong matroidal subdivisions, but are not in the Coxeter Dressian. One such example is:

$$(\mu_\emptyset, \mu_1, \mu_2, \mu_3, \mu_{12}, \mu_{13}, \mu_{23}, \mu_{123}) = (0, 0, 0, -1, 2, 1, 1, 0).$$

There are 5 maximal cells in the induced subdivision, each of which are tetrahedra. Notice that none of these cells contain any antipodes. We visualize the subdivision in Figure 1. However, by direct computation, we observe that there *is* an element of $\text{Dr}(B_3, P_3)$ that induces this subdivision.

Lemma 4.3. *Let $\mu \in \mathbb{R}^{2^{[n]}}$ satisfy the all strong exchange equations supported on the 3-dimensional faces of the 4-cube. Then μ is already a strong valuated Coxeter matroid.*

4.3. Type C_n . The only minuscule parabolic of type C is P_1 . Cosets $wP_1 \in W(C_n)/P_1$ are identified with $[n] \sqcup -[n]$ via $wP_1 \mapsto w(1)$. Hence, the polytope associated to (C_n, P_1) is given by the *cross polytope*

$$\diamond_n = \text{conv}(\pm e_i : i \in [n]).$$

All pairs $(\pm e_i, \pm e_j)$ for $i \neq j$ form edges on \diamond_n . On the other hand, $2e_i$ is an element of the type C_n root system. In other words, all possible subpolytopes of \diamond_n are strong matroids, and all subdivisions of \diamond_n are strong matroidal. In particular, there are no strong exchange type C_n equations, and the Coxeter Dressian $\text{Dr}(C_n, P_1)$ is exactly the secondary fan of \diamond_n .

4.4. Type D_n . Recall that in type D_n , the Weyl group is the intersection of the alternating group with the hyperoctahedral group. There are three minuscule parabolic subgroups P_1, P_{n-1} , and P_n . Note that the last two are isomorphic as subgroups of D_n , and they were studied in [Rin12].

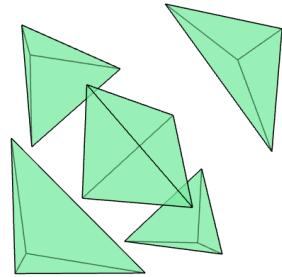


FIGURE 1. A strong exchange triangulation of the 3-cube

4.4.1. *Minuscule Parabolic P_1 in type D .* Note that $W(D_n)/P_1 \cong W(C_n)/P_1$, hence the polytope that arises in this situation is again the cross-polytope \diamondsuit_n of Section 4.3. However, $2e_i$ is not an element of the D_n root system, so the segment connecting $e_i, -e_i$ is not an admissible edge of a type D_n matroidal subdivision of \diamondsuit_n . It follows in a straightforward way that any subdivision of \diamondsuit_n that is not a triangulation is strong matroidal. The singular strong exchange equation defining $\text{Dr}(D_n, P_1)$ is of the form:

$$(2) \quad f_{\diamondsuit_n}^D = \bigoplus_{i=1}^n x_i \odot x_{-i}.$$

Proposition 4.4. *The type D_n strong exchange equation 2 is affinely invariant.*

Proof. Consider f^{E_7} , and $\mu \in \mathbb{T}^{V(\diamondsuit_n)}$. For $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ affine linear, we have that $x_i \odot x_{-i}$ evaluated at $\mu + \varphi|_{V(\diamondsuit_n)}$ yields $\mu_i + \mu_{-i} + 2\varphi(0)$. Therefore, μ attains the minimum at least twice if and only if $\mu + \varphi$ does. \square

Proposition 4.5. *A height function μ induces a type D strong matroidal subdivision of \diamondsuit_n if and only if $\mu \in \text{Dr}(D_n, P_1)$.*

Proof. One direction follows immediately from Proposition 4.4. For the other, suppose the μ induced subdivision is strong matroidal, and let $\Delta(\mathcal{M})$ be a maximal cell. Then $\Delta(\mathcal{M}) = C_v$,

$$C_v = \text{conv}(e_i : \langle v, e_i \rangle) + \mu_i \leq \langle v, e_k \rangle + \mu_k \text{ where } i, k \in [n] \cup [-n].$$

Now, observe that for C_v to be full dimensional, C_v must use at least $n+1$ vertices of \diamondsuit_n . As a consequence, we must have some $i \in [n]$ where $e_i, e_{-i} \in C_v$. Furthermore, since $\Delta(\mathcal{M}) = C_v$ is a strong matroid, we have at least one j such that $e_j, e_{-j} \in C_v$. This leads to the following collection of relations:

$$\langle v, e_{\pm i} \rangle + \mu_{\pm i} = \langle v, e_{\pm j} \rangle + \mu_{\pm j} \leq \langle v, e_{\pm k} \rangle + \mu_{\pm k} \text{ for any } k \in [n].$$

From these, one can deduce $\mu_i + \mu_{-i} = \mu_j + \mu_{-j} \leq \mu_k + \mu_{-k}$ for any k . \square

Proposition 4.6. *For $n \in \{5, 6\}$, the Coxeter Dressian $\text{Dr}(D_n, P_1)$ is isomorphic to a proper subfan of the secondary fan of \diamondsuit_n . That is, the coarsest polyhedral structure of the former agrees with the fan structure of the latter, the Coxeter Dressian.*

Proof. We verify this computationally. See the related notebook on the [github](#) repository. \square

It follows from Proposition 4.6, and the fact that any subdivision of \diamondsuit_n which isn't a triangulation is strong matroidal, that $\text{Dr}(D_n, P_1)$ is isomorphic to the subfan of the secondary fan consisting of codimension-1 cones.

4.4.2. *Minuscule Parabolic P_n .* We identify cosets $A \in W(D_n)/P_n$ with even subsets of $[n]$ via $A \mapsto A(-[n]) \cap [n]$. In analogy to the B_n case, the corresponding polytope is the *demicube*, the subpolytope of the cube with even vertices. We note that the parabolics P_{n-1} and P_n are isomorphic as subgroups of $W(D_n)$, hence we only consider the latter case. The strong exchange equations are given by

$$(3) \quad f_{I,J}^D := \bigoplus_{i \in I \triangle J} x_{I \triangle i} \odot x_{J \triangle i}$$

for $I, J \subseteq [n]$ with $|I \triangle J| \geq 4$ and I, J of odd cardinality. Affine invariance follows exactly like in 4.1. There is a simple construction relating valuated strong Coxeter matroids of type B_n to those of type D_{n+1} , with minuscule parabolic P_{n+1} .

Proposition 4.7. *The embedding $\mathbb{T}^{2^{[n]}} \hookrightarrow \mathbb{T}^{2^{[n+1]}}$ given by*

$$2^{[n]} \longrightarrow 2^{[n+1]} \quad A \mapsto \overline{A} = \begin{cases} A & \text{if } |A| \text{ even} \\ A \cup \{n+1\} & \text{if } |A| \text{ odd.} \end{cases}$$

induces a linear isomorphism $\text{Dr}(B_n, P_n) \cong \text{Dr}(D_{n+1}, P_{n+1})$.

Proof. We first show that type B_n strong exchange equations are in bijection to type D_{n+1} strong exchange equations via $f_{I,J}^B \mapsto f_{I',J'}^D$. Here $I' = I \cup \{n+1\}$ if $|I|$ even, and $I' = I$ if $|I|$ odd, similarly, J' . Bijectivity follows from the fact that I', J' are always of odd cardinality, and $|I \Delta J| \geq 3$ if and only if $|I' \Delta J'| \geq 4$.

Let $\mu \in \text{Dr}(B_n, P_n)$, and $\bar{\mu} \in \mathbb{T}^{2^{[n+1]}}$ its corresponding image. We now show that μ satisfies $f_{I,J}^B$ if and only if $\bar{\mu}$ satisfies $f_{I',J'}^D$. If both I, J are of odd cardinality, then the respective equations are the same, and there is nothing to show. If I is odd, and J is even, then $I' = I$, and $J' = J \cup \{n+1\}$. In that case, $I' \Delta J' = I \Delta J \cup \{n+1\}$ is of odd cardinality. Hence, for all $i \in I \Delta J$, we have $I \Delta \{i\}$ is even, and $\overline{J \Delta i} = J' \Delta i$. Finally, for $i = \{n+1\}$, we have $I \Delta i = I$, and $J = J' \Delta i$, finishing the proof for the case $|I|$ even, $|J|$ odd. Lastly, let I, J both be even. Then $I \Delta J = I' \Delta J'$, and for all $i \in I \Delta J$ we have $\overline{I \Delta i} = I' \Delta i$, finishing the proof. \square

We note that the above is already true algebraically, as the same morphism over any field maps the minuscule type B_n flag variety isomorphically onto the even component of the spinor variety. There is also an equivalent map $\text{Dr}(B_n, P_n) \cong \text{Dr}(D_{n+1}, P_n)$, mapping subsets of $[n]$ to odd subsets of $[n+1]$. As the even and odd components of the spinor variety are isomorphic, this is not surprising. The following can be deduced immediately from Proposition 4.7 and [Rin12, Theorem 4.6].

Corollary 4.8. *For $n \leq 4$, the Coxeter Dressian $\text{Dr}(B_n, P_n)$ is the tropicalization of the minuscule type B_n flag variety. For $n \geq 6$, it is strictly larger than the tropicalization. In particular, not every valuated Coxeter matroid of type B_n is realizable for $n \geq 6$.*

Theorem 4.9. *Let $\mu \in \mathbb{T}^{2^{[n]}}$, such that $\text{supp}(\mu)$ is a strong Coxeter matroid of type B_n . Then the following statements are equivalent:*

- (a) μ is in the Coxeter Dressian,
- (b) μ satisfies the four term strong exchange equations $f_{I,J}^B$ for $|I \Delta J| \leq 4$.

Moreover, if all entries of μ are finite, in (b) it suffices to only consider the case $|I \Delta J| \leq 3$.

Proof. The statement (a) \Rightarrow (b) holds by definition. For the other direction, we need to show that μ satisfies all strong exchange equations $f_{I,J}^B$. Consider the image $\bar{\mu}$ from Proposition 4.7. By [Rin12, Theorem 5.1], $\bar{\mu}$ satisfies the type D_{n+1} strong exchange equations if and only if it satisfies the respective four-term strong exchange equations. Hence $\bar{\mu} \in \text{Dr}(D_{n+1}, P_{n+1})$, and we conclude $\mu \in \text{Dr}(B_n, P_n)$.

Let additionally all entries of μ be finite. Restricting μ to the various 4-cubes and applying Lemma 4.3 we obtain that μ satisfies all equations $f_{I,J}^B$ with $|I \Delta J| = 4$ if it does so for all $|I \Delta J| = 3$. \square

Example 4.10. The point $\mu \in \mathbb{T}^{2^{[4]}}$ given by $\mu_\emptyset = \mu_{1234} = \mu_{12} = 0$, $\mu_{34} = 1$, and $\mu_A = \infty$ for all other $A \in 2^{[4]}$ does not satisfy $f_{1,234}^B$. Hence, it is not in the Coxeter Dressian. However, it satisfies all equations supported on proper 3-faces of the 4-cube.

4.5. Type E_6 . In type E_6 , the two minuscule parabolic subgroups are P_1 and P_6 . Since $P_1 \cong P_6 \cong W(D_5)$, we have $W(E_6)/P_1 \cong W(E_6)/P_6$ as sets with W -action. We thus consider both cases simultaneously. The polytope is commonly referred to as $2_{21} = \Delta(E_6/P_1)$ [Cox40]. The face structure of 2_{21} is studied in [Cox88]. We recall the relevant facts. The polytope 2_{21} is 6 dimensional and has 27 vertices, each at most edge-distance 2 from each other. Each distance 2 pair of vertices defines a \diamondsuit_5 cross polytope in the 5-skeleton of 2_{21} , of which there are 27. All other facets are simplices. For each distance-2 pair (A, B) , we get a strong exchange equation

$$(4) \quad f_{A,B}^{E_6} = x_A \odot x_B \oplus \bigoplus_{i=1}^4 x_{C_i} \odot x_{C'_i},$$

where the pairs (A, B) and (C_i, C'_i) for $i \in [4]$ are antipodes of the corresponding cross polytope. All strong exchange equations for minuscule E_6 arise in this way. By Proposition 4.4, the respective equations are affinely invariant.

Proposition 4.11. *A height function μ induces a strong matroidal subdivision of 2_{21} if and only if $\mu \in \text{Dr}(E_6, P_1)$.*

Proof. One direction follows from affine invariance. For the other, let f_\diamond correspond to a D_5 cross polytope \diamond in the 5-skeleton of 2_{21} , and consider the restriction of the subdivision induced by μ to \diamond . If the minimum on f_\diamond were unique, Proposition 4.6 tells us that there is a maximal cell of the restriction with peerless antipodes (A, B) . Since all possible peers for (A, B) in 2_{21} correspond to entries of f_\diamond , (A, B) form peerless antipodes of some maximal cell in the global subdivision. The result follows from the contrapositive. \square

4.6. **Type E_7 .** In type E_7 we only have a single minuscule parabolic P_1 , and the polytope is commonly known as $3_{21} = \Delta(E_7/P_1)$ [Cox88]. The polytope is 7 dimensional, with 56 vertices with maximum edge distance 3 between pairs. Each vertex has 27 neighbors. Each distance 2 pair induces a \diamond_6 facet, and distance 3 pairs are of the form $(v, -v)$. Since each element A has a unique distance 3 counterpart B , we just write $B = -A$. The strong exchange equations come in two forms. we have an equation for each \diamond_6 facet, one supported on all pairs of antipodes of 3_{21}

$$(5) \quad \begin{aligned} f_{A,B}^{E_7} &= x_A \odot x_B \bigoplus_{i=1}^5 x_{C_i} \odot x_{C'_i} \quad \text{when } d(A, B) = 2, \\ f^{E_7} &= \bigoplus_{A \in W/P} x_A \odot x_{-A}. \end{aligned}$$

Proposition 4.12. *The type E_7 strong exchange equations 5 are affinely invariant.*

Proof. Since the equations $f_{A,B}^{E_7}$ are the exact equations 2 on cross polytopes, the corresponding statement follows from Proposition 4.4. Consider f^{E_7} , and $\mu \in \mathbb{T}^{V(3_{21})}$. For $\varphi : \mathbb{R}^8 \rightarrow \mathbb{R}$ affine linear, we have that $x_A \odot x_{-A}$ evaluated at $\mu + \varphi|_{V(3_{21})}$ yields $\mu_A + \mu_{-A} + 2\varphi(0)$. Therefore, μ attains the minimum at least twice if and only if $\mu + \varphi$ does. \square

Over the Krasner hyperfield, the strong exchange equations (5) are equivalent to the symmetric exchange axiom of Borovik, Gelfand, and White [BGW98], which they conjectured to be equivalent to the Coxeter matroid axioms (e.g. the maximality property) in all minuscule types [BGW03, Section 6.16]. By means of an exhaustive search, we prove that this conjecture holds in E_7 .

Theorem 4.13. *All Coxeter matroids of type (E_7, P_7) satisfy the strong exchange condition.*

In combination with known results in the other types we conclude that in all types except B_n , all minuscule Coxeter matroids satisfy the strong exchange axiom. Moreover, Theorem 4.13 implies that the relations on the proper faces of 3_{21} are already sufficient to guarantee a subdivision consists of strong Coxeter matroids.

Theorem 4.14. *A height function μ induces a strong matroidal subdivision of 3_{21} if and only if it is contained in $\bigcap_{(A,B)} V^{\text{Trop}}(f_{A,B}^{E_7})$, where (A, B) runs over all pairs of elements of $W(E_7)/P_7$ with $d(A, B) = 2$.*

However, it is still open whether all μ satisfying the equivalent conditions of Theorem 4.14 are in the Dressian.

Conjecture 1. The support of $\text{Dr}(E_7, P_7)$ is equal to the support of $\bigcap V^{\text{Trop}}(f_{A,B}^{E_7})$.

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