

# Anti-Ramsey Number of Stars in 3-uniform hypergraphs\*

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## Abstract

An edge-colored hypergraph is called a *rainbow hypergraph* if all the colors on its edges are distinct. Given two positive integers  $n, r$  and an  $r$ -uniform hypergraph  $\mathcal{G}$ , the anti-Ramsey number  $ar_r(n, \mathcal{G})$  is defined to be the minimum number of colors  $t$  such that there exists a rainbow copy of  $\mathcal{G}$  in any exactly  $t$ -edge-coloring of the complete  $r$ -uniform hypergraph of order  $n$ . Let  $\mathcal{F}_k$  denote the 3-graph ( $k$ -star) consisting of  $k$  edges sharing exactly one vertex. Tang, Li and Yan [23] determined the value of  $ar_3(n, \mathcal{F}_3)$  when  $n \geq 20$ . In this paper, we determine the anti-Ramsey number  $ar_3(n, \mathcal{F}_{k+1})$ , where  $k \geq 3$  and  $n > \frac{5}{2}k^3 + \frac{15}{2}k^2 + 26k - 3$ .

*Key words:* anti-Ramsey number;  $k$ -star; rainbow hypergraph; matching

## 1 Introduction

For a set  $S$  and a positive integer  $k$ , we use  $\binom{S}{k}$  to denote the collection of all possible subsets of  $k$  elements of  $S$ . A *hypergraph*  $\mathcal{F} = (V(\mathcal{F}), E(\mathcal{F}))$  consists of a vertex set  $V(\mathcal{F})$  and an edge set  $E(\mathcal{F})$ , where each edge in  $E(\mathcal{F})$  is a non-empty subset of  $V(\mathcal{F})$ . The number of edges of  $\mathcal{F}$  is denoted by  $e(\mathcal{F})$ , that is,  $e(\mathcal{F}) := |E(\mathcal{F})|$ . If  $|e| = r$  for any  $e \in E(\mathcal{F})$ , then  $\mathcal{F}$  is called an  *$r$ -uniform hypergraph* (or  *$r$ -graph*, for simplicity). For  $u \in V(\mathcal{F})$ , let  $N_{\mathcal{F}}(u) := \{e \mid e \subseteq V(\mathcal{F}) \setminus \{u\} \text{ and } e \cup \{u\} \in E(\mathcal{F})\}$  be the neighborhood of  $u$  in  $\mathcal{F}$ . The *degree* of  $u$  in  $\mathcal{F}$ , denoted by  $d_{\mathcal{F}}(u)$ , is the size of  $N_{\mathcal{F}}(u)$ . For  $X \subseteq V(\mathcal{F})$ , we define  $\mathcal{F} - X$  as the subhypergraph of  $\mathcal{F}$  obtained by removing all vertices in  $X$  and all edges intersecting with  $X$  in  $\mathcal{F}$ . Similarly, if  $Y \subseteq E(\mathcal{F})$ , we use  $\mathcal{F} - Y$  to denote the hypergraph resulting from deleting all the edges in  $Y$  from  $\mathcal{F}$ . When  $X = \{x\}$ ,  $Y = \{e\}$ , we respectively write  $\mathcal{F} - X = \mathcal{F} - x$ ,  $\mathcal{F} - Y = \mathcal{F} - e$ . Specifically, to avoid confusion, for an edge  $e \in E(\mathcal{F})$ , we use  $\mathcal{F} - V(e)$  to denote the subgraph of  $\mathcal{F}$  obtained by removing all vertices in  $e$  and all edges intersecting with  $e$  in  $\mathcal{F}$ . For a non-empty subset  $X \subseteq V(\mathcal{F})$ , let  $\mathcal{F}[X]$  denote the subgraph *induced* by  $X$ . For two disjoint sets  $U$  and  $W$ , we use  $U \times W$  to denote the collection of 2-sets that intersect  $U$  and  $W$ . That is,  $U \times W = \{\{x, y\} \mid x \in U \text{ and } y \in W\}$ .

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For two vertex-disjoint hypergraphs  $\mathcal{F}, \mathcal{H}$ , the *union* of  $\mathcal{F}$  and  $\mathcal{H}$  denoted by  $\mathcal{F} \cup \mathcal{H}$  is the hypergraph with vertex set  $V(\mathcal{F}) \cup V(\mathcal{H})$  and edge set  $E(\mathcal{F}) \cup E(\mathcal{H})$ . When there is no confusion, for  $T \subseteq V(\mathcal{F})$ , we also use  $N_{\mathcal{F}}(T)$  to denote the  $(k - |T|)$ -graph with vertex set  $V(\mathcal{F}) - T$  and edge set  $N_{\mathcal{F}}(T)$ . A *matching* in a hypergraph  $H$  is a set of pairwise disjoint edges in  $H$ , and we use  $\nu(H)$  to denote the maximum size of a matching in  $H$ .

A *t-edge-coloring* of a hypergraph is an assignment of  $t$  colors to its edges, and an *exactly t-edge-coloring* uses all  $t$  colors. An edge-colored graph is called *rainbow* if all edges have distinct colors. Let  $[n] = \{1, \dots, n\}$ . The complete  $r$ -graph with order  $n$  is denoted by  $K_n^r$ . Given a positive integer  $n$  and a hypergraph  $\mathcal{F}$ , the *anti-Ramsey number*  $ar_r(n, \mathcal{F})$  is the minimum number of colors  $t$  such that each edge-coloring of  $K_n^r$  with exactly  $t$  colors contains a rainbow copy of  $\mathcal{F}$ . Given an edge-coloring  $C$  of  $\mathcal{F}$ , the colored hypergraph is *F-free* if  $\mathcal{F}$  has no rainbow subhypergraph which is isomorphic to  $F$ .

Given a hypergraph  $H$  and a family of hypergraphs  $\mathcal{H}$ ,  $H$  is called  *$\mathcal{H}$ -free* if for any  $F \in \mathcal{H}$ ,  $H$  does not contain  $F$  as a subhypergraph. The Turán number of a family of  $r$ -graphs  $\mathcal{F}$ , written  $ex_r(n, \mathcal{F})$ , is the largest possible number of edges in an  $\mathcal{F}$ -free  $r$ -graph on  $n$  vertices. When  $\mathcal{F} = \{F\}$ , we use  $ex_r(n, F)$  instead of  $ex_r(n, \{F\})$ .

Anti-Ramsey numbers were introduced by Erdős, Simonovits and Sós [6] in 1973. They found that the anti-Ramsey numbers are closely related to the Turán numbers. For an  $r$ -graph  $F$ , there is a natural lower bound of  $ar_r(n, F)$  in terms of Turán number as follow,

$$ar_r(n, F) \geq ex_r(n, \{F - e : e \in E(F)\}) + 2. \quad (1)$$

This trivial lower bound is easily obtained by coloring a rainbow Turán extremal  $r$ -graph for  $\{F - e : e \in F\}$  in  $K_n^r$ , and the remaining edges with an additional color [15].

In 1973, Erdős, Simonovits, and Sós [6] proved that there exists an integer  $n_0(p)$  such that for all  $n > n_0(p)$ , the anti-Ramsey number  $ar(K_p, n)$  satisfies the equation  $ar(K_p, n) = ex(n, K_{p-1}) + 2$ . Later, Montellano-Ballesteros and Neumann-Lara [17], as well as Schiermeyer [20], independently extended this result to cover all values of  $n$  and  $p$  where  $n > p \geq 3$ . Jiang [14] and Montellano-Ballesteros [18] independently determined the anti-Ramsey numbers for stars in graphs. A variety of results regarding the anti-Ramsey numbers of 2-graphs have been achieved. These results cover various graph structures such as paths, cycles, and matchings. For a comprehensive overview, we recommend the survey paper [10]. Regarding 3-graphs, Guo, Lu, and Peng [13] established the exact value of the anti-Ramsey number for matchings. Gu, Li, and Shi [11] investigated the anti-Ramsey numbers of paths and cycles in hypergraphs. For other related results on the anti-Ramsey numbers of paths, cycles, and matchings in hypergraphs, interested readers are referred to [9, 15, 16, 18, 22, 23]. In this paper, we aim to investigate the anti-Ramsey numbers of stars of 3-graphs.

Let  $\mathcal{F}_k$  ( $k$ -star) denote the 3-graph consisting of  $k$  edges sharing exactly one vertex, called the core of the star. Let  $f(n, k)$  denote the maximum number of edges in 3-graph without  $k$ -stars.  $f(n, 2)$  was determined exactly by Erdős and Sós [21]. Duke and Erdős [4] established linear lower and upper bounds on  $f(n, k)$  for fixed  $k$  (where the bounds scale linearly with  $n$ ). These bounds were subsequently improved in [8], where the exact value of  $f(n, 3)$  was determined for all integers  $n \geq 54$ . Further refinements were made in [1], yielding bounds that are nearly best possible. Finally, Chung and Frankl [2] derived the exact value of  $f(n, k)$  for 3-graphs in the regime  $n \geq \frac{5}{2}k^3$ .

**Theorem 1 (Erdős and Sós, [21])** For all  $n \geq 3$ ,

$$f(n, 2) = \begin{cases} n, & \text{if } n \equiv 0 \pmod{4}, \\ n - 1, & \text{if } n \equiv 1 \pmod{4}, \\ n - 2, & \text{if } n \equiv 2, 3 \pmod{4}. \end{cases}$$

**Theorem 2 (Chung and Frankl, [2])** Suppose that  $k \geq 3$  is odd and  $n > k(k-1)(5k+2)/2$ , then  $f(n, k) = (n-2k)k(k-1) + 2\binom{k}{3}$ . Moreover, a 3-graph  $\mathcal{F}$  has  $f(n, k)$  edges and contains no  $k$ -star if and only if  $\mathcal{F}$  is isomorphic to  $\mathcal{F}_k^o$ .

**Theorem 3 (Chung and Frankl, [2])** Suppose that  $k \geq 4$  is even and  $n > 2k^3 - 9k + 7$ , then  $f(n, k) = \frac{1}{2}nk(2k-3) - \frac{1}{2}(2k^3 - 9k + 6)$ . Moreover, a 3-graph  $\mathcal{F}$  has  $f(n, k)$  edges and contains no  $k$ -star if and only if  $\mathcal{F}$  is isomorphic to  $\mathcal{F}_k^e$ .

Chung and Frankl [2] proposed the following extremal graph construction.

*The construction of  $\mathcal{F}_k^o$ :* Let  $k$  be odd and let  $S$  and  $R$  be two disjoint sets of  $[n]$  of size  $k$ . Consider the 3-graph  $\mathcal{F}_k^o$  with vertex set  $[n]$  and edge set

$$E(\mathcal{F}_k^o) = \{T \in \binom{V}{3} : |T \cap S| \geq 2 \text{ and } |T \cap R| = \emptyset\} \cup \{T \in \binom{V}{3} : |T \cap R| \geq 2 \text{ and } |T \cap S| = \emptyset\}.$$

*The construction of  $\mathcal{F}_k^e$ :* Let  $k$  be even. Let  $G_k$  be the 2-graph with  $2k-1$  vertices, denoted by  $x_1, \dots, x_{k-1}, y_1, \dots, y_{k-1}$  and  $z$ . The edge set of  $G_k$  consists of all the pairs  $(x_i, y_j)$ , except for  $(x_i, y_i)$  with  $2i > k$  together with the pairs  $(x_i, z), (y_i, z)$  with  $2i > k$ . It is easy to see that it has all degrees equal  $k-1$  except for the degree of  $z$ , which is  $k-2$ . Let  $\mathcal{F}_k^e$  denote the 3-graph on  $n$  vertices such that each edge either intersects the vertex set  $V(G_k)$  in an edge of  $G_k$  or contains two distinct edges of  $G_k$ , together with all the triples of the form  $\{x_i, y_i, z\}$  for  $1 \leq i \leq k/2$ .

The anti-Ramsey number  $ar_3(n, \mathcal{F}_2) = 2$  follows immediately from the definition. Tang, Li and Yan [23] determined the value of  $ar_3(n, \mathcal{F}_3)$  when  $n \geq 20$ .

**Theorem 4 (Tang, Li and Yan [23])** For all  $n \geq 20$ ,

$$ar_3(n, \mathcal{F}_3) = \begin{cases} f(n, 2) + 2, & \text{if } n \equiv 0 \pmod{4}, \\ f(n, 2) + 2, & \text{if } n \equiv 1 \pmod{4}, \\ f(n, 2) + 3, & \text{if } n \equiv 2, 3 \pmod{4}. \end{cases}$$

In this paper, we determine the anti-Ramsey number of a star in 3-graphs for sufficiently large  $n$ .

**Theorem 5** For  $k \geq 3$  and  $n > \frac{5}{2}k^3 + \frac{15}{2}k^2 + 26k - 3$ ,  $ar_3(n, \mathcal{F}_{k+1}) = f(n, k) + 2$ .

## 2 Technical Lemmas

**Theorem 6 (Tutte, [24])** A graph  $G$  has a perfect matching if and only if

$$o(G - S) \leq |S| \text{ for any } S \subseteq V(G),$$

where  $o(G - S)$  denotes the number of connected components of odd order in  $G - S$ .

A graph  $G$  is called a *factor-critical* graph if  $G - v$  has a perfect matching for each  $v \in V(G)$ . Gallai [12] introduced the concept of factor-critical graphs. By definition and Theorem 6, one can see that the following statement holds.

**Lemma 7 (Gallai, [12])** *A graph  $G$  of odd order is factor-critical if and only if*

$$o(G - S) \leq |S| \quad \text{for any non-empty set } S \subseteq V(G),$$

where  $o(G - S)$  denotes the number of connected components of odd order in  $G - S$ .

Here we use a “weight function” method developed by Chung and Frankl [2] to characterize the structure of a 3-graph  $\mathcal{F}$ . Suppose that  $\mathcal{F}$  is a family of 3-element subsets of the  $n$ -set  $V = V(\mathcal{F})$ . Let  $P$  denote the set of all pairs of vertices in  $V$ . For each  $\{u, v\}$  in  $P$ , the pair frequency is

$$z(u, v) = |\{w : \{u, v, w\} \in \mathcal{F}\}|.$$

We also have

$$\begin{aligned} A &:= \{\{u, v\} \in P : z(u, v) \geq 2k - 1\} \\ B &:= \{\{u, v\} \in P : 2k - 2 \geq z(u, v) \geq k\} \\ C &:= P - A - B \end{aligned}$$

The weight function  $\omega : \mathcal{F} \times P \rightarrow \mathbb{R}$  distributes weights to pairs within each triple in  $\mathcal{F}$  according to the pair frequency.

For a fixed triple  $T \in \mathcal{F}$ , the three pairs in  $T$  are denoted by  $z(p_1) \geq z(p_2) \geq z(p_3)$ . The *weight function*  $w$  is defined as follows:

- If  $p_1, p_2, p_3 \in A \cup B$  or  $p_1, p_2, p_3 \in B \cup C$ , then  $\omega(T, p_i) = \frac{1}{3}$ .
- Suppose  $p_1 \in A, p_3 \in C$ . If  $p_2 \in A \cup B$ , then  $\omega(T, p_1) = \omega(T, p_2) = \frac{1}{2}, \omega(T, p_3) = 0$ . If  $p_2 \in C$ , then  $\omega(T, p_1) = 1, \omega(T, p_2) = \omega(T, p_3) = 0$ .
- For convenience we set also  $\omega(T, p) = 0$  for  $p \notin T$ .

Obviously, we have

$$\sum_{1 \leq i \leq 3} \omega(T, p_i) = 1$$

and

$$\sum_{T \in \mathcal{F}} \sum_p \omega(T, p) = e(\mathcal{F})$$

**Lemma 8 ([2, 25])** *For every vertex  $v$  in a 3-graph  $\mathcal{F}$  which does not contain a  $k$ -star, the following holds:*

$$W_v = \sum_{p \in N_{\mathcal{F}}(v)} \omega(\{v\} \cup p, p) \leq k(k - 1).$$

Moreover  $W_v \leq k(k - 1) - \frac{2}{3}$  unless  $N_{\mathcal{F}}(v)$  is the disjoint of two complete 2-graphs on  $k$  vertices and every edge of  $N_{\mathcal{F}}(v)$  is  $A$ -type. If  $k$  is even, then one has the stronger inequality,

$$W_v \leq k(k - 3/2)$$

Moreover,  $W_v \leq k(k-3/2) - 1/2$  unless  $N_{\mathcal{F}}(v) = K_{k-1} \cup C$ , where  $C$  is a factor-critical graph of order  $k+1$  with degree sequence  $k-1, \dots, k-1, k-2$  or  $N_{\mathcal{F}}(v)$  with maximum degree  $k-1$  satisfies the following three conditions.

- (a) There exists  $S \subseteq V(N_{\mathcal{F}}(v))$  such that  $N_{\mathcal{F}}(v) - S$  consists of isolated vertices and one factor-critical component denoted by  $F_0$  with  $2k-1-2|S| \geq k+1$  vertices and degree sequence  $(k-1, \dots, k-1, k-2)$ ;
- (b)  $N_{\mathcal{F}}(v) - V(F_0)$  is the edge disjoint union of  $|S|$  stars, each with maximum degree  $k-1$ ; and
- (c) every edge of  $N_{\mathcal{F}}(v)$  is in  $A$ , and all edges connecting  $v$  to  $V(N_{\mathcal{F}}(v))$  are in  $C$ .

Chung and Frankl [2] first determined the value of  $f(n, k)$ , though they did not characterize the corresponding extremal case: when  $N_{\mathcal{F}}(v) = K_{k-1} \cup C$  (where  $C$  is a factor-critical graph of order  $k+1$  with degree sequence  $(k-1, \dots, k-1, k-2)$ ). Later, Zhu et al. [25] adopted a similar approach to fully characterize all extremal graphs (a method analogous to the one Chung and Frankl used in [2] when handling the function  $\text{ex}_3(n, \mathcal{F}_k)$ ).

Write  $c(n, k) := \text{ex}_3(n, \mathcal{F}_k) + 2$ . Given an edge-coloring  $c : E(K_n^3) \rightarrow [c(n, k)]$  such that  $c$  is surjective and the colored hypergraph denoted by  $H$  contains no rainbow  $\mathcal{F}_{k+1}$ . For  $U \subseteq V(H)$ , let  $Z_c(U) := \{c(e) \mid e \in H, U \subseteq e\}$  and let  $z_c(U) := |Z_c(U)|$ . When  $U = \{x\}$ , we denote  $Z_c(U)$  and  $z_c(U)$  by  $Z_c(x)$  and  $z_c(x)$ , respectively. If  $z_c(\{u, v\}) \leq 3k$ , the pair  $\{u, v\}$  is *good* in  $H$ , saying *bad*, otherwise.

**Lemma 9** *There exist  $2k+6$  disjoint good pairs in  $H$ .*

**Proof.** Firstly, we show that the following claim.

**Claim 1.** For every  $u$ , there exist at least  $n-k-1$  vertices  $v \in V(H)$  such that  $z_c(u, v) \leq 3k$ .

By contradiction. Suppose the result does not hold. Then there exists  $u \in V(H)$  and  $S \in \binom{V(H)-u}{k+1}$  such that  $z_c(u, v) \geq 3k+1$  for all  $v \in S$ . Write  $S := \{v_1, \dots, v_{k+1}\}$ . We choose a maximal rainbow star  $\mathcal{F}_r$  in  $H$  with center  $u$  such that  $V(\mathcal{F}_r) \cap S = r$ . Since  $H$  is a complete graph, we may choose  $e_1$  such that  $\{u, v_1\} \subseteq e_1$  and  $|e_1 \cap S| = 1$ . So we have  $V(\mathcal{F}_r) \neq \emptyset$ . Next we show that  $r \geq k+1$ , which will contradict the hypothesis. Otherwise, suppose that  $r \leq k$ . Then let  $v \in S - V(\mathcal{F}_r)$ . Since  $z_c(u, v) \geq 3k+1$  and the number of edges containing  $\{u, v\}$  and intersecting  $V(\mathcal{F}_r) - u$  is at most  $2r$ , there exists an edge  $f$  containing  $\{u, v\}$  and  $f - \{u, v\} \not\subseteq V(\mathcal{F}_r)$  and  $c(f) \notin \{c(e) \mid e \in \mathcal{F}_r\}$ . So  $\{f\} \cup E(\mathcal{F}_r)$  induces a rainbow  $\mathcal{F}_{r+1}$ , which contradicts the choice of  $\mathcal{F}_r$ . This completes the proof of claim 1.

Since  $n-k-1 > 2(2k+6)$ , by Claim 1, we may greedily choose a set of  $2k+6$  vertex-disjoint good pairs in  $H$ . This completes the proof.  $\square$

The following lemma is crucial for the proof of our main theorem.

**Lemma 10** *Let  $k \geq 5$  be an integer and let  $G$  be a graph with at most  $2k-1$  vertices. If  $G$  contains at most one vertex of degree  $k-2$  and the rest vertices has degree  $k-1$ , then for any  $f \in E(G)$ ,  $G - f$  is also factor-critical.*

**Proof.** Otherwise, suppose that the result does not hold. Then there exists  $f \in E(G)$  such that  $G - f$  is not factor-critical. Then by Lemma 7, there exists  $S \subseteq V(G)$  such that  $o(G - f - S) \geq |S| + 1$ . Recall that  $\delta(G) \geq k - 2$  and  $G$  contains at most one vertex of degree  $k - 2$ .

**Claim 1.**  $G$  is  $(k - 2)$ -edge-connected.

Otherwise, suppose that there exists  $M \subseteq E(G)$  such that  $|M| \leq k - 3$  and  $G - M$  is not connected. Let  $F_1$  and  $F_2$  be two connected components of  $G - M$  such that  $|V(F_1)| \leq |V(F_2)|$ . Since  $\delta(G) \geq k - 2$ , then we have  $2 \leq |V(F_1)| \leq k - 1$ . So we have

$$\begin{aligned} |M| &\geq e_G(V(F_1), V(G) - V(F_1)) \\ &\geq |V(F_1)|(k - |V(F_1)|) - 1 \\ &\geq k - 2, \end{aligned}$$

which contradicts the hypothesis that  $|M| \leq k - 3$ . This completes the proof of Claim 1.

By Claim 1, we have  $q \geq 2$ . Let  $C_1, \dots, C_q$  denote these odd components of  $G - S - f$  such that  $|V(C_1)| \geq \dots \geq |V(C_q)|$ .

**Claim 2.**  $|V(C_1)| \geq k$ .

Otherwise, suppose  $|V(C_1)| \leq k - 1$ . Then for  $1 \leq i \leq q$ ,  $e_G(V(C_i), V(G) - V(C_i)) \geq k - 2$  with equality if and only if the vertex of degree  $k - 2$  in  $G$  belongs to  $C_i$ . Thus

$$\begin{aligned} (k - 1)|S| &\geq \sum_{x \in S} d_G(x) \\ &\geq \sum_{i=1}^q e_G(S, V(C_i)) \\ &\geq (k - 1)(|S| + 1) - 3 \\ &> (k - 1)|S| \quad (\text{since } k \geq 5), \end{aligned}$$

a contradiction. This completes the proof of Claim 2.

By Claim 2, we have  $|V(G) - V(C_1)| \leq k - 1$ .

Suppose that  $\sum_{i=3}^q |V(C_i)| \geq 1$ . For any  $x \in V(\cup_{i=2}^q C_i)$ , it follows that  $d_{G-f}(x) \leq k - 3$ . Recall that  $G$  contains exactly one vertex of degree  $k - 2$ . This implies that  $\sum_{i=2}^q |V(C_i)| \leq 2$ , which further yields  $q = 3$  and  $|V(C_2)| = 1$ . Then for  $x \in V(C_2 \cup C_3)$ ,

$$d_G(x) \leq |S| + 1 \leq 3,$$

which contradicts the fact that  $G$  contains exactly one vertex of degree  $k - 2$ .

We proceed by considering the case where  $\sum_{i=3}^q |V(C_i)| = 0$ . This immediately implies that  $q = 2$  and  $|S| = 1$ . By Claim 1,  $G$  is 2-connected. Hence  $f \in E_G(V(C_1), V(C_2))$ . If  $|V(C_2)| \geq 3$ , then there exist two vertices, say  $x_1, x_2 \in V(C_2) - V(f)$  such that  $d_G(x_i) \leq k - 2$ , which contradicts the fact that  $G$  contains exactly one vertex of degree  $k - 2$ . We thus conclude that  $|V(C_2)| = 1$ . Then for  $x \in V(C_2)$ ,  $d_G(x) \leq 2$ , which once again contradicts the condition that  $\delta(G) \geq 3$ . This completes the proof.  $\square$

**Lemma 11** *Let  $G$  be a simple graph with  $|V(G)| = 6$  and degree sequence  $(3, 3, 3, 3, 2, 2)$ . Then  $G$  has a Hamiltonian cycle.*

**Proof.** Let  $V(G) = \{w_1, w_2, w_3, w_4, u, v\}$ , where  $d_G(u) = d_G(v) = 2$  and  $d_G(w_i) = 3$  for  $i \in [4]$ . Without loss of generality, assume that  $uw_1, vw_2 \in E(G)$ .

Firstly, we consider the case where  $uv \in E(G)$ . The subgraph  $G_1 = G - \{u, v\}$  has a degree sequence of  $(3, 3, 2, 2)$ . Then  $G_1$  has a path  $P$  of length three that connects  $w_1$  and  $w_2$ . The edge-set  $(E(P) \cup \{uw_1, vw_2, uv\})$  forms a Hamiltonian cycle of  $G$ .

Next, we consider the case where  $uv \notin E(G)$ . If  $G - \{u, v\}$  is a 4-cycle, denoted as  $w_1w_2w_3w_4$ , then by symmetry, we may assume that either  $N_G(u) = \{w_1, w_3\}$  or  $N_G(u) = \{w_1, w_4\}$ . In either of these two cases,  $G$  has a Hamiltonian cycle. Otherwise,  $G - \{u, v\}$  is a not 4-cycle, which implies that  $N_G(u) \cap N_G(v) \neq \emptyset$ . Note that  $N_G(u) \neq N_G(v)$ . So, we can assume that  $w_3 \in N_G(u) \cap N_G(v)$ . Without loss of generality, suppose that  $N_G(u) = \{w_1, w_3\}$  and  $N_G(v) = \{w_2, w_3\}$ . It follows that the edge set  $E(G) - \{w_4w_3, w_1w_2\}$  induces a Hamiltonian cycle of  $G$ . This completes the proof.  $\square$

### 3 Proof of Theorem 5

By (1), we have  $ar_3(n, \mathcal{F}_{k+1}) \geq f(n, k) + 2$ . Therefore the lower bound is followed. Next we assume  $k \geq 3$ . Let  $c(n, k) := f(n, k) + 2$ .

For the upper bound, we prove it by contradiction. Suppose that the result does not hold. Then there exists an edge-coloring  $c : E(K_n^3) \rightarrow [c(n, k)]$  such that  $c$  is surjective and the colored hypergraph denoted by  $\mathcal{G}$  contains no rainbow  $\mathcal{F}_{k+1}$ . By Lemma 9, we can denote a set of  $2k + 6$  disjoint good pairs in  $\mathcal{G}$  by  $Q = \{\{u_1, v_1\}, \dots, \{u_{2k+6}, v_{2k+6}\}\}$ , that is  $z_c(u_i, v_i) \leq 3k$  for  $1 \leq i \leq 2k + 6$ . Define a color set

$$C_Q = \cup_{p \in Q} \{c(T) \mid T \subseteq \mathcal{G}, p \subseteq T\}$$

and let  $q = |C_Q|$ . Then by Lemma 9, we have the following inequality

$$q \leq 6k^2 + 18k.$$

Let  $G$  be a rainbow subgraph of  $\mathcal{G}$  with  $c(n, k) - q$  edges and vertex set  $[n]$  such that

$$\{c(e) \mid e \in E(G)\} \cap C_Q = \emptyset.$$

**Claim 1.**  $G$  is  $\mathcal{F}_k$ -free.

Otherwise, suppose that  $G$  contains a copy of  $\mathcal{F}_k$  denoted by  $\mathcal{F}$  with center  $u$ . Since  $|Q| \geq 2k + 6$  and  $|V(\mathcal{F})| = 2k + 1$ , there exists  $\{u_i, v_i\} \in Q$  such that  $\{u_i, v_i\} \cap V(\mathcal{F}) = \emptyset$ . By the definition of  $G$ ,  $E(\mathcal{F}) \cup \{u, u_i, v_i\}$  induces a rainbow copy of  $\mathcal{F}_{k+1}$ , contradicting the hypothesis. This completes the proof of Claim 1.

Let  $w : G \times P \rightarrow R$  be a weighted function defined as in Section 2. And we denote  $W_v = \sum_{p \in N(v)} w(v \cup p, p)$ . Next we discuss two cases.

**Case 1.**  $k$  is odd.

Then we have

$$e(G) = c(n, k) - q \geq (n - 2k)k(k - 1) + 2 \binom{k}{3} + 2 - 6k^2 - 18k. \quad (2)$$

By Lemma 8, we have  $W_v \leq k(k - 1)$  for all  $v \in V(G)$ . Let  $W := \{v \in V(G) \mid W_v = k(k - 1)\}$ . Then we have the following claim.

**Claim 2.**  $W \neq \emptyset$ .

Suppose to the contrary that  $W_v < k(k - 1)$  for all  $v \in V(G)$ . By the definition of weight function, we have  $W_v \leq k(k - 1) - 2/3$  for all  $v \in V(G)$ . So

$$e(G) = \sum_{v \in V(G)} W_v \leq nk(k - 1) - 2n/3. \quad (3)$$

Combining (2) and (3), we may infer that

$$n \leq \frac{5}{2}k^3 + \frac{15}{2}k^2 + 26k - 3,$$

a contradiction. This completes the proof of Claim 2.

By Claim 2, we may choose  $x \in W$  such that  $W_x = k(k - 1)$ . Then by Lemma 8,  $N_G(x)$  is the disjoint union of two complete 2-graphs on  $k$  vertices and every edge of  $N_G(x)$  is in  $A$ -type. Denote the two complete subgraphs by  $R_1$  and  $R_2$ .

**Claim 3.**  $|\{c(e) \mid x \in e \text{ and } e \subseteq \binom{[n]}{3}\}| \leq 2\binom{k}{2} + 1$ .

Otherwise, suppose that  $|\{c(e) \mid x \in e \text{ and } e \subseteq \binom{[n]}{3}\}| \geq 2\binom{k}{2} + 2$ . Then one can see that

$$|\{c(p \cup \{x\}) \mid p \notin E(R_1 \cup R_2) \text{ and } p \in \binom{[n] - x}{2}\}| \geq 2. \quad (4)$$

We choose  $T_1, T_2 \in \binom{[n] - \{x\}}{2} - ((V(R_1)) \cup (V(R_2)))$  such that  $c(T_1 \cup \{x\}) \neq c(T_2 \cup \{x\})$ , and for  $i \in [2]$ ,  $c(T_i \cup \{x\}) \notin \{c(p \cup \{x\}) \mid p \in E(R_1 \cup R_2)\}$ . For  $i \in [2]$ ,  $F - V(T_i)$  has a matching  $M_i$  of size  $k - 1$ . Note that

$$|V(M_1 \cup M_2) \cup T_1 \cup T_2| \leq 2k + 4.$$

So by Lemma 9, we may choose  $T_0 \in Q$  such that  $x \notin T_0$  and  $T_0 \cap V(M_1 \cup M_2 \cup \{T_1, T_2\}) = \emptyset$ . It follows that either  $\{p \cup \{x\} \mid p \in M_1 \cup \{T_1, T_0\}\}$  or  $\{p \cup \{x\} \mid p \in M_2 \cup \{T_2, T_0\}\}$  induces a rainbow copy of  $\mathcal{F}_{k+1}$ , a contradiction. This completes the proof of Claim 3.

Let  $\mathcal{G}'$  be a rainbow edge-colored subgraph of  $\mathcal{G} - \{x\}$  obtained by deleting the triples colored by the colors from  $\{c(e) \mid x \in e \text{ and } e \in \binom{[n]}{3}\}$ . By Claim 3,  $e(\mathcal{G}') \geq f(n - 1, k) + 1$ . Thus  $\mathcal{G}'$  has a rainbow copy of  $\mathcal{F}_k$  with core  $y$ . Choosing  $y' \in [n] - V(\mathcal{F}_k) - \{x\}$ , in view of the preceding analysis, we know that  $c(\{x, y, y'\}) \notin \{c(e) \mid e \in \mathcal{F}_k\}$ . It follows that  $\{x, y, y'\} \cup \mathcal{F}_k$  is a rainbow copy of  $\mathcal{F}_{k+1}$ , a contradiction. This completes the proof.

**Case 2.**  $k \geq 4$  is even.

Then we have

$$e(G) \geq c(n, k) - q \geq \frac{1}{2}nk(2k - 3) - \frac{1}{2}(2k^3 - 9k + 6) - 6k^2 - 18k. \quad (5)$$



By Lemma 8,  $W_v \leq k(k - 3/2)$  for all  $v \in V(G)$ . By Theorem 4, we may assume that  $k \geq 4$ .

**Claim 4.** There exists  $v \in V(G)$  such that  $W_v = k(k - 3/2)$ .

Otherwise, we may assume that  $W_v \leq k(k - 3/2) - 1/2$  for all  $v \in V(G)$ . Thus we get

$$e(G) \leq nk(k - 3/2) - n/2. \quad (6)$$

Combining (5) and (6), we may infer that  $n \leq 2k^3 + 12k^2 + 27k + 6$ , a contradiction. This completes the proof of Claim 4.

We choose  $x \in V(G)$  with  $W_x = k(k - 3/2)$ . Suppose that following inequality holds

$$|\{c(e) \mid x \in e \text{ and } e \subseteq \binom{[n]}{3}\}| \leq k(k - \frac{3}{2}) + 1, \quad (7)$$

Let  $G'$  be a rainbow subgraph of  $\mathcal{G}$  avoiding the colors appearing in  $\{c(e) \mid x \in e \text{ and } e \subseteq \binom{[n]}{3}\}$ . By (7), we have  $e(G') > f(n - 1, k)$ . Hence  $G'$  contains a copy of  $\mathcal{F}_k$  denoted by  $\mathcal{M}$ . Let  $y$  be the core of  $\mathcal{M}$ . We select a vertex  $x' \in [n] - V(\mathcal{M}) - \{x\}$ . Now  $E(\mathcal{M}) \cup \{\{x', x, y\}\}$  induces a rainbow copy of  $\mathcal{F}_{k+1}$ , a contradiction. This contradiction concludes our proof. So next it is sufficient for us to show that (7) holds.

Let  $F := N_G(x)$  denote the 2-graph with edge set  $N_G(x)$  and vertex set  $V(N_G(x))$ . By Lemma 8,  $F$  consists of two vertex-disjoint factor-critical graphs or satisfies (a), (b), and (c).

We first prove the following claim, which provides a method for constructing a rainbow  $\mathcal{F}_{k+1}$  in  $\mathcal{G}$  as described below.

**Claim 5.** Let  $x_0 \in V(F)$  with  $d_F(x_0) = k - 1$ , and let  $\{y_0, w\} \subseteq [n] - (N_F(x_0) \cup \{x, x_0\})$  be a 2-element set. If  $c(\{x_0, w, x\}) \neq c(\{x_0, y_0, x\})$ , then  $\mathcal{G}$  has a rainbow  $\mathcal{F}_{k+1}$  with core  $x_0$ .

Let  $N_F(x_0) = \{y_2, \dots, y_k\}$ . Our next step is to construct a rainbow copy of  $\mathcal{F}_{k+1}$  with the core  $x_0$ . According to Lemma 8, for  $2 \leq i \leq k$ ,  $x_0 y_i$  is  $A$ -type. Consequently, we know that  $d_G(\{x_0, y_i\}) \geq 2k - 1$ . For  $2 \leq i \leq k$ , we can pick a set  $\{y_{i,1}, \dots, y_{i,2k-1}\} \subseteq N_G(\{x_0, y_i\})$ . Let  $T_1$  be a 2-element subset chosen from the set  $[n] - V(F) - \{x, w, y_0\} - (\bigcup_{i=2}^k \bigcup_{j=1}^{2k-1} \{y_{i,j}\})$ . Since  $c(\{x_0, w, x\}) \neq c(\{x_0, y_0, x\})$ , we have either  $c(T_1 \cup \{x_0\}) \neq c(\{x_0, y_0, x\})$  or  $c(T_1 \cup \{x_0\}) \neq c(\{x_0, w, x\})$ . Without loss of generality, assume that  $c(T_1 \cup \{x_0\}) \neq c(\{x_0, y_0, x\})$ . Since  $G$  is a rainbow subgraph and  $k \geq 4$ , we can re-order the vertices  $y_2, \dots, y_k$  in such a way that

$$c(T_1 \cup \{x_0\}) \notin \{c(\{x_0, y_{k-1}, y_{k-1,j}\}) \mid j \in [2k - 1]\}$$

and

$$\{c(\{x_0, y_0, x\}), c(T_1 \cup \{x_0\})\} \cap \{c(\{x_0, y_k, y_{k,j}\}) \mid j \in [2k - 1]\} = \emptyset.$$

Next, we aim to extend the sets  $T_1 \cup \{x_0\}$ ,  $\{x_0, y_0, x\}$  to form a rainbow copy of  $\mathcal{F}_{k+1}$  with the core  $x_0$ . Since  $k \geq 4$ , we can find a vertex  $y'_{2,1} \in N_G(\{x_0, y_2\}) - \{x, y_3, \dots, y_k\}$ , such that  $\{T_1 \cup \{x_0\}, \{x_0, y_0, x\}, \{x_0, y'_{2,1}, y_2\}\}$  induces a rainbow copy  $\mathcal{F}_3$ . Now, assume that for some  $i < k - 1$ , we have already constructed a collection  $\{T_1 \cup \{x_0\}, \{x_0, y_0, x\}, \{x_0, y'_{2,1}, y_2\}, \dots, \{x_0, y'_{i,1}, y_i\}\}$  that induce a rainbow copy of  $\mathcal{F}_{i+1}$ , where  $\{y'_{2,1}, \dots, y'_{i,1}\} \cap N_F(x_0) = \emptyset$ . Define  $W_i := \{\bigcup_{j=2}^i \{y_j, y'_{j,1}\}\} \cup \{y_0, x_0, x\} \cup V(T_1)$ . Let's consider the step for  $i + 1$ . Since  $d_G(\{y_{i+1}, x_0\}) \geq 2k - 1$  and  $T_1 \cap N_G(\{y_{i+1}, x_0\}) = \emptyset$ , there exists  $y'_{i+1,1} \in N_G(\{y_{i+1}, x_0\}) - (W_i \cup N_F(x_0))$

such that  $\{\{T_1 \cup \{x_0\}, \{x_0, y_0, x\}, \{x_0, y'_{2,1}, y_2\}, \dots, \{x_0, y'_{i+1,1}, y_{i+1}\}\}$  induces a rainbow copy of  $\mathcal{F}_{i+2}$ . Continuing the process until  $i = k - 1$ , we can obtain a rainbow copy of  $\mathcal{F}_k$  denoted by  $\mathcal{M}$ . Recall that  $d_G(\{y_k, x_0\}) \geq 2k - 1$  and  $T_1 \cap N_G(\{y_k, x_0\}) = \emptyset$ . As a result, there exists a vertex  $y'_{k,1}$  in the set  $N_G(\{y_k, x_0\}) - W_{k-1}$ . Moreover, by the choice of  $y_k$ ,  $c(\{x_0, y_k, y'_{k,1}\}) \notin \{c(\{x_0, y_0, x\}), c(T_1 \cup \{x_0\})\}$ . Hence  $E(\mathcal{M}) \cup \{x_0, y_k, y'_{k,1}\}$  induces a rainbow copy of  $\mathcal{F}_{k+1}$ , a contradiction. This completes the proof of Claim 5.

Next by Lemma 8, we proceed by discussing two subcases.

**Subcase 2.1.**  $F = R_1 \cup R_2$ , where  $R_1 \cong K_{k-1}$  and  $R_2$  is a factor-critical graph of order  $k + 1$  with degree sequence  $(k - 1, \dots, k - 1, k - 2)$ .

**Claim 6.**  $\{c(g \cup \{x\}) \mid g \in \binom{[n] - \{x\} - V(R_1 \cup R_2)}{2}\}$  contains exactly one color denoted by  $c_0$  which belongs to the color set  $C_Q$ .

We will use a proof by contradiction. Firstly, suppose that there exists  $g \in \binom{[n] - \{x\} - V(R_1 \cup R_2)}{2}$  such that  $c(g \cup \{x\}) \in \{c(p \cup \{x\}) \mid p \in E(R_1 \cup R_2)\}$ . Write  $c(g \cup \{x\}) = c(g' \cup \{x\})$ , where  $g' \in E(R_1 \cup R_2)$ . Since  $R_1$  and  $R_2$  are factor-critical graphs,  $F - g'$  has a matching  $M_1$  of size  $k - 1$ . Let  $h \in Q$  such that  $x \notin h$  and  $h \cap V(M_1 \cup \{g\}) = \emptyset$ . It follows that  $\{p \cup \{x\} \mid p \in M_1 \cup \{g, h\}\}$  induces a rainbow copy of  $\mathcal{F}_{k+1}$ , a contradiction. Subsequently, suppose that there are two distinct edges  $g_1, g_2 \in \binom{[n] - \{x\} - V(R_1) - V(R_2)}{2}$  such that  $c(g_1 \cup \{x\}) \neq c(g_2 \cup \{x\})$  and neither of them belongs to  $\{c(p \cup \{x\}) \mid p \in E(R_1 \cup R_2)\}$ . Let  $M_2$  be a matching of size  $k - 1$  in  $R_1 \cup R_2$ . Since  $|Q| \geq 2k + 6$ , we may choose  $h \in Q$  such that  $x \notin h$  and  $h \cap V(M_2 \cup \{g_1, g_2\}) = \emptyset$ . It follows that either  $\{p \cup \{x\} \mid p \in M_2 \cup \{g_1, h\}\}$  or  $\{p \cup \{x\} \mid p \in M_2 \cup \{g_2, h\}\}$  induces a rainbow copy of  $\mathcal{F}_{k+1}$ , a contradiction. This completes the proof of Claim 6.

**Claim 7.** For any  $f \in ([n] - V(R_1) - \{x\}) \times V(R_1)$ , we have  $c(f \cup \{x\}) \in \{c_0\} \cup \{c(p \cup \{x\}) \mid p \in E(F)\}$ ; and for any  $g \in ([n] - V(R_1 \cup R_2) - \{x\}) \times V(R_2)$ , we have  $c(g \cup \{x\}) = c_0$ .

We will use a proof by contradiction. Firstly, suppose that there exists  $f_1 \in ([n] - V(R_1) - \{x\}) \times V(R_1)$  such that  $c(f_1 \cup \{x\}) \notin \{c_0\} \cup \{c(p \cup \{x\}) \mid p \in E(R_1 \cup R_2)\}$ . Since  $R_1$  and  $R_2$  are factor-critical graphs,  $F - V(f_1)$  contains a matching  $M_3$  of size  $k - 1$ . Then we can select  $h \in \binom{[n] - V(R_1 \cup R_2) - \{x\}}{2}$  such that  $h \cap V(f_1) = \emptyset$ . By Claim 6, we have  $c(h \cup \{x\}) = c_0$ . Thus  $\{p \cup \{x\} \mid p \in M_3 \cup \{f_1, h\}\}$  induces a rainbow copy of  $\mathcal{F}_{k+1}$ , which is a contradiction. Next consider that there exists  $g_1 \in ([n] - V(R_1 \cup R_2) - \{x\}) \times V(R_2)$  such that  $c(g_1 \cup \{x\}) \neq c_0$ . If  $c(g_1 \cup \{x\}) \notin \{c(p \cup \{x\}) \mid p \in E(R_1 \cup R_2)\}$ , let  $M_4$  be a matching of size  $k - 1$  of  $F - V(g_1)$  and let  $h'$  be a 2-subset of  $[n] - (g_1 \cup V(F) \cup \{x\}) \cup V(M)$ , then  $\{p \cup \{x\} \mid p \in M_4 \cup \{g_1, h'\}\}$  induces a rainbow copy of  $\mathcal{F}_{k+1}$  a contradiction again. So we may assume that  $c(g_1 \cup \{x\}) \in \{c(p \cup \{x\}) \mid p \in E(R_1 \cup R_2)\}$ . Select an edge  $g_2 \in E(R_1 \cup R_2)$  such that  $c(g_2 \cup \{x\}) = c(g_1 \cup \{x\})$ . Then, pick an element  $g_0 \in \binom{[n] - V(F) - V(g_1 \cup g_2) - \{x\}}{2}$ . By Claim 6, we have  $c(g_0 \cup \{x\}) = c_0$ . If  $F - V(g_1) - g_2$  has a matching  $M$  of size  $k - 1$ , then  $\{e \cup \{x\} \mid e \in M \cup \{g_0, g_1\}\}$  induces a rainbow copy of  $\mathcal{F}_{k+1}$ , which is a contradiction. Consequently, we can assume that

$$\nu(F - V(g_1) - g_2) < k - 1. \quad (8)$$

If  $k \geq 6$ , by Lemma 10,  $\nu(F - V(g_1) - g_2) = k - 1$ , which contradicts inequality (8). So we can conclude  $k = 4$ . Write  $V(R_2) := \{y_0, y_1, \dots, y_4\}$  such that  $d_{R_2}(y_0) = 2$ , and  $y_0 y_1, y_0 y_4 \in$

$E(R_2)$ . If  $g_1 \cap \{y_1, y_4\} = \emptyset$  or  $g_1 \cap g_2 \neq \emptyset$ , then  $F - V(g_1) - g_2$  has a matching of size 3, which again contradicts inequality (8). Moreover, by symmetry, we may assume that  $y_1 \in g_1 \cap V(R_2)$  and  $g_1 \cap g_2 = \emptyset$ . Recall that for every edge  $e \in E(R_2)$ ,  $e$  is type  $A$  in  $G$ , which means  $d_G(\{y_1, y\}) \geq 7$  for  $y \in \{y_0, y_2, y_3\}$ . Note that  $\{y_2, y_3, y_0\} \cap (g_1 \cup \{x\}) = \emptyset$ . Let  $w \in [n] - V(F) - \{x\}$ . By Claim 7, we have  $c(\{x, w, y_1\}) = c_0 \neq c(g_1 \cup \{x\})$ . With the same discussion as Claim 6,  $\{x\} \cup g_1$  can be extended into a rainbow copy with core  $y_1$  denoted by  $\mathcal{M}$  of  $\mathcal{F}_4$ , where  $\{c(e) \mid e \in E(F_4)\} \subseteq \{c(e) \mid e \in E(\mathcal{M})\}$ . By Lemma 9, we may choose  $h \in Q$  such that  $h \cap V(\mathcal{M}) = \emptyset$ . Then  $\{h \cup \{y_1\}\} \cup E(\mathcal{M})$  induces a rainbow copy of  $\mathcal{F}_5$  in  $\mathcal{G}$ , a contradiction. This completes the proof of Claim 7.

By Claims 6 and 7, it is sufficient for us to show that for any  $l \in \binom{V(R_2)}{2} \setminus E(R_2)$ ,  $c(l \cup \{x\}) = c_0$ . Suppose to the contrary that there exists  $l_0 = \{x_0, y_0\} \in \binom{V(R_2)}{2} \setminus E(R_2)$  such that  $c(\{x_0, y_0, x\}) \neq c_0$ . Note that  $R_2$  is a factor-critical graph of order  $k+1$  with degree sequence  $(k-1, \dots, k-1, k-2)$ . Without loss of generality, suppose  $d_{R_2}(x_0) = k-1$ . By Claim 7, for  $w \in [n] - V(F) - \{x\}$ ,  $c(\{x_0, w, x\}) = c_0 \neq c(\{x_0, y_0, x\})$ . By Claim 5, we can find a rainbow copy of  $\mathcal{F}_{k+1}$  with the core  $x_0$ , which contradicts the hypothesis. Thus we can derive that the inequality (7) holds.

**Subcase 2.2.**  $F$  satisfies (a), (b) and (c).

Let  $U_k$  denote the set of vertices that serve as the centers of stars fulfilling condition (b) in Lemma 8. Let  $V_k := V(F) \setminus (V(F_0) \cup U_k)$ . Let  $F_0$  be the factor-critical component that satisfies condition (a) in Lemma 8. Since  $\nu(F) = k-1$  and the number of vertices in  $F_0$  is at least  $k+1$  (i.e.,  $|F_0| \geq k+1$ ), we can infer that the cardinality of the set  $U_k$  is at most  $\frac{k}{2} - 1$ , that is,  $|U_k| \leq \frac{k}{2} - 1$ .

**Claim 8.** The set  $\{c(g \cup \{x\}) \mid g \in \binom{[n] - \{x\} - V(F)}{2}\}$  contains exactly one color denoted by  $c_1$ , which belongs to the color set  $C_Q$ .

We will prove this statement by contradiction. First, suppose that there exists  $g \in \binom{[n] - \{x\} - V(F)}{2}$  such that

$$c(g \cup \{x\}) \in \{c(f \cup \{x\}) \mid f \in E(F)\}.$$

Our goal is to construct a rainbow copy  $\mathcal{F}_{k+1}$ , which will lead to a contradiction. Without loss of generality, assume that  $c(g \cup \{x\}) = c(g' \cup \{x\})$ , where  $g' \in E(F)$ . Since  $F_0$  is factor-critical and  $U_k \leq \frac{k}{2} - 1$ ,  $F - g'$  has a matching  $M$  of size  $k-1$ . By Lemma 9, we can then find  $h \in Q$  such that  $x \notin h$  and  $h \cap V(M \cup \{g\}) = \emptyset$ . In this situation, the collection  $\{p \cup \{x\} \mid p \in M \cup \{g, h\}\}$  forms a rainbow copy of  $\mathcal{F}_{k+1}$ , which contradicts our initial assumption. Next, consider the case when there are two distinct edges  $g_1, g_2 \in \binom{[n] - \{x\} - V(F)}{2}$  such that  $c(g_1 \cup \{x\}) \neq c(g_2 \cup \{x\})$  and neither  $c(g_1 \cup \{x\})$  nor  $c(g_2 \cup \{x\})$  belongs to the set  $\{c(p \cup \{x\}) \mid p \in E(F)\}$ . We select a matching  $M'$  of size  $k-1$  in  $F$ . Subsequently, we can find an element  $h \in Q$  such that  $x \notin h$  and  $h \cap V(M' \cup \{g_1, g_2\}) = \emptyset$ . As a consequence, either the set  $\{p \cup \{x\} \mid p \in M' \cup \{g_1, h\}\}$  or  $\{p \cup \{x\} \mid p \in M' \cup \{g_2, h\}\}$  forms a rainbow copy of  $\mathcal{F}_{k+1}$ , a contradiction. This completes the proof of Claim 8.

Let  $\mathcal{C} := \{c(p \cup \{x\}) \mid p \in E(F)\}$  and let  $\mathcal{C}' := \{c_1\} \cup \{c(p \cup \{x\}) \mid p \in E(F)\}$ .

**Claim 9.** For any  $f \in \binom{[n] \setminus (U_k \cup \{x\})}{2}$  satisfying  $|f \cap V(F_0)| = 1$ , we have  $c(f \cup \{x\}) = c_1$ .

Suppose that there exists  $g_1 \in \binom{[n]-U_k-\{x\}}{2}$  such that  $|g_1 \cap V(F_0)| = 1$  and  $c(g_1 \cup \{x\}) \neq c_1$ . We choose  $h \in \binom{[n]-V(F)-\{x\}}{2}$  such that  $h \cap V(g_1) = \emptyset$ . By Claim 8, we have  $c(h \cup \{x\}) = c_1$ . If  $c(g_1 \cup \{x\}) \notin \mathcal{C}$ , we proceed as follows. Let  $M_3$  be a matching  $M_3$  of size  $k-1$  in  $F - V(g_1)$ . Then, the collection  $\{p \cup \{x\} \mid p \in M_3 \cup \{g_1, h\}\}$  induces a rainbow copy of  $\mathcal{F}_{k+1}$ , a contradiction. Hence we may infer that

$$c(g_1 \cup \{x\}) \in \mathcal{C}. \quad (9)$$

Let  $g_2 \in E(F)$  such that  $c(g_2 \cup \{x\}) = c(g_1 \cup \{x\})$ . If  $F - V(g_1) - g_2$  contains a matching  $M_4$  of size  $k-1$ ,  $\{p \cup \{x\} \mid p \in M_4 \cup \{g_1, h\}\}$  induces rainbow copy of  $\mathcal{F}_{k+1}$ , a contradiction. Thus we have

$$\nu(F - V(g_1) - g_2) < k-1. \quad (10)$$

Then by Lemma 10, we may infer that  $k = 4$ . Let  $g_1 \cap V(F_0) = \{y_0\}$ . Consider  $d_F(y_0) = 3$ . By the proof of Claim 5,  $\{x\} \cup g_1$  may be extended into a rainbow copy denoted by  $\mathcal{M}$  of  $\mathcal{F}_4$  such that for any  $e \in E(\mathcal{M})$ ,  $c(e) \in \{c(f) \mid f \in E(G)\}$ . Now we select  $h \in Q$  such that  $h \cap V(\mathcal{M}) = \emptyset$ . It follows that  $(\{y_0\} \cup h) \cup E(\mathcal{M})$  induces a rainbow copy of  $\mathcal{F}_5$ , a contradiction. Next we may assume  $d_F(y_0) = 2$ . Recall that  $g_2 \in E(F)$  such that  $c(g_2 \cup \{x\}) = c(g_1 \cup \{x\})$ . By Lemma 11,  $F - V(g_1)$  contains a Hamilton cycle. So  $F - V(g_1) - g_2$  contains a matching  $M_5$  of size 3, which contradicts the inequality (10). This completes the proof of Claim 9.

**Claim 10.** For any  $f \in \binom{[n] \setminus (U_k \cup \{x\} \cup V(F_0))}{2}$  satisfying  $|f \cap V_k| = 1$ , we have  $c(f \cup \{x\}) = c_1$ .

We proceed by contradiction. Suppose to the contrary that there exists  $f_1 \in \binom{[n] \setminus (U_k \cup \{x\} \cup V(F_0))}{2}$  such that  $c(f_1 \cup \{x\}) \neq c_1$ . We choose  $h \in \binom{[n]-V(F)-\{x\}}{2}$  such that  $h \cap V(f_1) = \emptyset$ . It follows that  $c(h \cup \{x\}) = c_1$  by Claim 8. If  $c(f_1 \cup \{x\}) \in \mathcal{C}$ , let  $f_2 \in E(F)$  such that  $c(f_1 \cup \{x\}) = c(f_2 \cup \{x\})$ ; otherwise let  $f_2 = \emptyset$ . Since  $F_0$  is a factor-critical graph, we know that  $F - V(f_1) - f_2$  contains a matching  $M_2$  of size  $k-1$ . Thus  $\{p \cup \{x\} \mid p \in M_2 \cup \{f_1, h\}\}$  induce a rainbow copy of  $\mathcal{F}_{k+1}$ , a contradiction again. This completes the proof of Claim 10.

**Claim 11.** For any  $T \in \binom{V_k}{2}$ ,  $c(T \cup \{x\}) \in \mathcal{C}'$ .

Suppose to the contrary that there exists  $T_1 \in \binom{V_k}{2}$  such that  $c(T_1 \cup \{x\}) \notin \mathcal{C}'$ . Since  $d_F(w) = 3$  for  $w \in U_k$  and  $k \geq 4$ ,  $F - V(T_1)$  has a matching  $M$  of size  $k-1$ . We may choose  $T_2 \in \binom{[n]-\{x\} \cup V(F) \cup T_1}{2}$ . It follows that  $c(T_2 \cup \{x\}) = c_1$  by Claim 8. So  $\{p \cup \{x\} \mid p \in M \cup \{T_2, T_1\}\}$  induces a rainbow copy of  $\mathcal{F}_{k+1}$ , a contradiction. This completes the proof of Claim 11.

Let  $z \in V(F_0)$  such that  $d_{F_0}(z) = k-2$ .

**Claim 12.** For any  $T \in \left( (U_k \times V(F_0 - z)) \cup \binom{V(F_0)}{2} \right) \setminus E(F)$ ,  $c(T \cup \{x\}) \in \mathcal{C}'$ .

Suppose to the contrary that there exists  $T \in \left( (U_k \times V(F_0 - z)) \cup \binom{V(F_0)}{2} \right) \setminus E(F)$  such that  $c(T \cup \{x\}) \notin \mathcal{C}'$ . Write  $T = \{y_1, y_2\}$ . Without loss of generality, let  $y_1$  be a vertex in  $F_0$  with degree  $k-1$ . We choose  $w \in [n] - (V(F) \cup \{x\})$ . By Claim 9, we have  $c(\{x, w, y_1\}) = c_1$ . So we have  $c(\{x, w, y_1\}) \neq c(\{x, y_1, y_2\})$ . Recall that  $\{x, y_1\}, \{y_1, y_2\} \notin E(F)$ . Then by Claim

5, we can find a rainbow copy of  $\mathcal{F}_{k+1}$  in  $\mathcal{G}$  with core  $y_1$ , a contradiction. This completes the proof of Claim 12.

**Claim 13.** For any  $T \in \left([n] - (V(F_0 - z) \cup \{x\})\right) \setminus E(F)$ ,  $c(T \cup \{x\}) \in \mathcal{C}'$ .

Suppose to the contrary that there exists  $T \in \left([n] - (V(F_0) \cup \{x\})\right) \setminus E(F)$  such that  $c(T \cup \{x\}) \notin \mathcal{C}'$ . Write  $T = \{x_0, y_0\}$ . By Claims 8, 9, 10 and 11, we have  $T \cap U_k \neq \emptyset$ . Without loss generality, suppose that  $x_0 \in U_k$ . Then  $d_F(x_0) = k - 1$ . Let  $w \in V(F_0) - N_F(x_0)$  such that  $d_{F_0}(w) = k - 1$ . By Claim 12, we have  $c(\{w, x, x_0\}) \in \{c(p \cup \{x\}) \mid p \in E(F)\} \cup \{c_1\}$ . Thus it follows that  $c(\{w, x, x_0\}) \neq c(\{x, x_0, y_0\})$ . By Claim 5, we can find a rainbow copy of  $\mathcal{F}_{k+1}$  with core  $x_0$ , a contradiction. This completes the proof of Claim 13.

By Claims 8,9,10,11,12 and 13, for any  $e \in \left([n] - x\right)$ ,  $c(e \cup \{x\}) \in \mathcal{C}'$ . Thus we can derive that the inequality (7) holds. This complete the proof of Theorem 5.  $\square$

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