

KESTEN'S CRITERION FOR DISCRETE PROBABILITY MEASURE-PRESERVING GROUPOIDS

SOHAM CHAKRABORTY, MILAN DONVIL, FELIPE FLORES, AND MARIO KLISSE

ABSTRACT. Inspired by Kesten's criterion for the amenability of groups, we establish a characterization of the amenability of discrete probability measure-preserving groupoids in terms of the operator norms of symmetric invariant Markov operators.

1. INTRODUCTION

Amenability of groups, introduced by von Neumann in [vNeu29], is one of the central notions in modern group theory, probability theory, ergodic theory, and functional analysis. A group is called *amenable* if it possesses an invariant mean. This concept admits numerous equivalent characterizations; for instance, via the existence of *Følner sequences* (see [Føl55]), the *nuclearity* of the group's reduced group C^* -algebra (see [Lan73]), the existence of approximately invariant sequences of probability measures (see [Rei68]), or the existence of fixed points for continuous affine actions on compact convex subsets of locally convex topological vector spaces (see [Kai57]). For further characterizations, see [Pie84].

In his seminal work [Kes59a] (see also [Kes59b]), Kesten studied random walks on Cayley graphs of finitely generated groups and obtained yet another characterization of amenability, relating it to the decay of return probabilities to the identity. More precisely, if the support of a given random walk generates the group, then the probability of returning to the identity in $2n$ steps decays exponentially if and only if the group is non-amenable.

Kesten's criterion has been proven useful in several contexts, see e.g. [Kes59b; KV83; Oll05; ST10; BV25]. It can be formulated naturally in terms of the spectral radius of the Markov operator associated with the random walk, in a way that we explicitly cite now.

Theorem (Kesten's criterion [Kes59a]). *Let G be a countable discrete group, let μ be a symmetric probability measure on G whose support generates G , and let P_μ be the associated Markov operator on $\ell^2(G)$ defined by $(P_\mu \xi)(g) := \sum_{h \in G} \xi(gh) \mu(h)$ for $\xi \in \ell^2(G)$, $g \in G$. Then G is amenable if and only if the spectral radius of P_μ is equal to 1.*

Over time, amenability has found natural extensions in many other areas of mathematics. In particular, Zimmer introduced an analog for discrete group actions and countable measured equivalence relations in [Zim77a; Zim77b; Zim78] in the late seventies; soon after, Renault extended the notion to general measured groupoids [Ren80] (see also [AR00; CHI04]).

As in the case of groups, Renault's amenability for measured groupoids admits several equivalent formulations (see, e.g., [AR00]). In [Kai05], Kaimanovich introduced the *fiberwise Liouville property* for measured groupoids, formulated in terms of invariant Markov operators acting fiberwise with respect to a Haar system. A measured groupoid equipped with such an invariant Markov

Date: December 11, 2025.

2020 *Mathematics Subject Classification.* Primary: 37A20, Secondary: 37A15, 37A30.

operator is called *fiberwise Liouville* if almost all fiberwise actions admit no non-trivial bounded harmonic functions. A strengthened version of this notion, adapting the classical Choquet–Deny property to the discrete measured groupoid setting, was developed in [BCDKK24]. Kaimanovich showed that a measured groupoid which admits an invariant fiberwise Liouville Markov operator is amenable and conjectured that the reverse implication should hold as well, which was confirmed by Chu and Li in [CL18] (see also [BK21]). This yields a groupoid analog of the characterization of amenability of groups in terms of bounded harmonic functions, proven in one direction and conjectured by Furstenberg [Fur73] and proven in the other direction independently by Kaimanovich–Vershik [KV79; KV83] and Rosenblatt [Ros81].

Analogs of Kesten’s criterion have been formulated in a variety of contexts, including invariant random subgroups [AGV14], group extensions of topological Markov chains [Sta13], and quantum groups [Ban99]. Building on Kaimanovich’s framework of invariant Markov operators on groupoids, the goal of this article is to establish an analog of Kesten’s criterion in the setting of discrete probability measure-preserving measured groupoids. To this end, denote by P^π the Markov operator associated with a Borel field of probability measures π (see Subsection 3.1). We introduce the following definition.

Definition A (see Definition 3.5). *Let (\mathcal{G}, μ) be a discrete probability measure-preserving groupoid. We say that (\mathcal{G}, μ) satisfies Kesten’s criterion if, for every symmetric Borel field of probability measures π , the restriction of the Markov operator P^π to any invariant Borel subset $E \subseteq \mathcal{G}^{(0)}$ with $\mu(E) > 0$ has operator norm equal to 1.*

Whereas the original Kesten’s criterion stated above shows that for groups it suffices that a single Markov operator associated with a symmetric non-degenerate measure has norm one in order to deduce amenability, this is no longer true for general measured groupoids, as observed in [Kai01]. Instead, we obtain the following characterization, whose proof relies on Hayes’ approach in [Hay24] (see also [AFH24]).

Theorem B. *Let (\mathcal{G}, μ) be a discrete probability measure-preserving groupoid. Then (\mathcal{G}, μ) is amenable if and only if it satisfies Kesten’s criterion.*

Our theorem fits into the broader program of finding connections between probability theory and the study of measured equivalence relations and measured groupoids. In that same program, one finds the already mentioned works of Kaimanovich [Kai01; Kai05], Hayes [Hay24], and Abert, Fraczyk and Hayes [AFH24]. These works make explicit use of equivalence relations. Note, however, that the study of random unimodular random graphs, trees, or surfaces requires the study of (often covertly defined) particular measured equivalence relations (see [AL07; AB22; AFH25]). These objects arise naturally in the study of percolation theory [BLS15].

We also deduce the following interpretation of the norm of P^π associated with a symmetric Borel field of probability measures π , and hence of the amenability of the ambient groupoid. In particular, we provide a formula involving averages of return probabilities (see Remark 3.9), which recovers part of Kesten’s original inspiration.

Theorem C. *Let (\mathcal{G}, μ) be a discrete probability measure-preserving groupoid, and let π be a symmetric Borel field of probability measures on \mathcal{G} . Then the E -spectral radius of π ,*

$$\rho_E(\mathcal{G}, \mu) := \lim_{n \rightarrow \infty} (\mu(E)^{-1} \langle (P^\pi)^{2n} \chi_E, \chi_E \rangle)^{\frac{1}{2n}},$$

exists and satisfies $\rho_E(\mathcal{G}, \pi) \leq 1$ for every Borel subset $E \subseteq \mathcal{G}^{(0)}$ with $\mu(E) > 0$. Moreover, $\rho_{\mathcal{G}^{(0)}}(\mathcal{G}, \pi) = \|P^\pi\|$.

Structure of the article. The paper is organized as follows. Section 2 collects the necessary background on functional analysis, invariant means and almost invariant vectors for measure-preserving transformations, as well as on measured groupoids. In Section 3, we introduce and study invariant Markov operators on groupoids induced by symmetric Borel fields of probability measures and establish our main results: Theorem B and Theorem C. Finally, we have an appendix that contains examples illustrating that non-symmetric Borel fields of probability measures may give rise to unbounded Markov operators, hence justifying the assumption of symmetry.

2. PRELIMINARIES AND NOTATION

2.1. General notation. We denote by $\mathbb{N} := \{0, 1, 2, \dots\}$ the set of non-negative integers, by $\mathbb{N}_{\geq 1} := \{1, 2, \dots\}$ the set of positive integers, and $[n] := \{1, \dots, n\}$ for $n \in \mathbb{N}_{\geq 1}$. For a set S we write $\#S$ for the number of elements in S and χ_S for the characteristic function on S . The *symmetric difference* of two sets A and B is denoted by $A \triangle B := (A \setminus B) \cup (B \setminus A)$.

2.2. Functional-analytic preliminaries. For Banach spaces \mathcal{X} and \mathcal{Y} , we denote the Banach space of all bounded linear operators $T : \mathcal{X} \rightarrow \mathcal{Y}$, equipped with the operator norm, by $\mathbb{B}(\mathcal{X}, \mathcal{Y})$. In the case where $\mathcal{X} = \mathcal{Y}$, we abbreviate $\mathbb{B}(\mathcal{X}) := \mathbb{B}(\mathcal{X}, \mathcal{X})$.

The *dual space* $\mathcal{X}^* := \mathbb{B}(\mathcal{X}, \mathbb{C})$ of \mathcal{X} carries a natural locally convex topology weaker than the norm topology, called the *weak*-topology*, which is generated by the subbase of sets of the form

$$\{\psi \in \mathcal{X}^* \mid |\phi(v) - \psi(v)| < \varepsilon \text{ for every } v \in F\},$$

where $\phi \in \mathcal{X}^*$, $\varepsilon > 0$, and where $F \subseteq \mathcal{X}$ is finite. By the *Banach–Alaoglu theorem*, the closed unit ball $\{\phi \in \mathcal{X}^* \mid \|\phi\| \leq 1\}$ of \mathcal{X}^* is compact with respect to this topology.

We will also make use of the *spectral theorem* for bounded self-adjoint operators on Hilbert spaces. Let \mathcal{H} be a Hilbert space and $T \in \mathbb{B}(\mathcal{H})$ a self-adjoint operator. Then every pair of vectors $\xi, \eta \in \mathcal{H}$ admits a complex Borel measure $\mu_{\xi, \eta}$ on the spectrum $\sigma(T) \subseteq \mathbb{R}$ of T such that

$$\langle T\xi, \eta \rangle = \int_{\sigma(T)} t d\mu_{\xi, \eta}(t). \quad (2.1)$$

If $\|\xi\| = 1$, the measure $\mu_{\xi, \xi}$ is a probability measure.

Since T is assumed to be self-adjoint, the norm closure $C^*(1, T)$ of the span of all elements T^k with $k \in \mathbb{N}$ is a unital closed subalgebra of $\mathbb{B}(\mathcal{H})$, which is invariant under taking the adjoint in $\mathbb{B}(\mathcal{H})$. The *continuous functional calculus* asserts that there exists a unital isometric isomorphism $C(\sigma(T)) \rightarrow C^*(1, T)$, $f \mapsto f(T)$ of algebras which maps $\text{id}_{\sigma(T)} \in C(\sigma(T))$ to T . Here $C(\sigma(T))$ denotes the complex-valued continuous functions on $\sigma(T)$, equipped with the supremum norm. For $f \in C(\sigma(T))$ the identity in (2.1) then extends via

$$\langle f(T)\xi, \eta \rangle = \int_{\sigma(T)} f(t) d\mu_{\xi, \eta}(t) \quad \text{for all } \xi, \eta \in \mathcal{H}.$$

Further details on these constructions can be found in Conway's classical text [Con90].

2.3. Invariant means and almost invariant vectors. A *standard measure space* is a pair (Y, ν) where Y is a standard Borel space and ν is a σ -finite Borel measure on Y . If ν is a probability measure, then (Y, ν) is called a *standard probability space*.

Let Γ be a group, and let (Y, ν) be a σ -finite measure space equipped with a right action $\Gamma \curvearrowright (Y, \nu)$ by measure-preserving transformations. For $p \in [1, \infty]$ and $\gamma \in \Gamma$, we define a bounded linear operator $\alpha_p(\gamma) \in \mathbb{B}(L^p(Y, \nu))$ by

$$(\alpha_p(\gamma)f)(x) = f(x\gamma) \quad \text{for } f \in L^p(Y, \nu), x \in Y.$$

By abuse of notation, we will usually suppress the parameter p and simply write $\alpha := \alpha_p$.

A functional $m \in L^\infty(Y, \nu)^*$ is called a *mean* on Y if $m(1) = 1$ and $m(f) \geq 0$ whenever $f \geq 0$. Since the set of all means on Y is a weak*-closed subset of the unit ball of $L^\infty(Y, \nu)^*$, it is weak*-compact. A mean $m \in L^\infty(Y, \nu)^*$ is called Γ -*invariant* if $m(\alpha(\gamma)f) = m(f)$ for all $f \in L^\infty(Y, \nu)$, $\gamma \in \Gamma$.

In the next section, we will use several standard characterizations of the existence of almost invariant vectors and the connection with invariant means found in [Hay24, Theorem 2.6, Theorem 2.7, Theorem 2.8].

2.4. Measured groupoids. For a groupoid \mathcal{G} we denote its *unit space* by $\mathcal{G}^{(0)}$, and we write s and t for the *source* and *target* maps, respectively. The set of *composable pairs* of \mathcal{G} is denoted by $\mathcal{G}^{(2)} := \{(g, h) \in \mathcal{G} \times \mathcal{G} \mid s(g) = t(h)\}$, while g^{-1} denotes the inverse of an element $g \in \mathcal{G}$.

For any subset $E \subseteq \mathcal{G}^{(0)}$ we set

$$\mathcal{G}_E := s^{-1}(E), \quad \mathcal{G}^E := t^{-1}(E), \quad \mathcal{G}_E^E := \mathcal{G}_E \cap \mathcal{G}^E,$$

and for $x \in \mathcal{G}^{(0)}$ we abbreviate $\mathcal{G}_x := \mathcal{G}_{\{x\}}$, $\mathcal{G}^x := \mathcal{G}^{\{x\}}$, and $\mathcal{G}_x^x := \mathcal{G}_{\{x\}}^{\{x\}}$. Furthermore, for subsets $A, B \subseteq \mathcal{G}$ we define AB to be the set of products ab with $a \in A$ and $b \in B$ for which $s(a) = t(b)$, which may be empty.

A *discrete Borel groupoid* is a groupoid \mathcal{G} endowed with the structure of a standard Borel space such that the unit space $\mathcal{G}^{(0)}$ is a Borel subset of \mathcal{G} , the maps s, t , multiplication, and inversion are Borel measurable functions, and all source and target fibres are at most countable (i.e. s and t are countable-to-one). Given such a groupoid and a Borel probability measure μ on $\mathcal{G}^{(0)}$, we define measures μ_s and μ_t on \mathcal{G} by

$$\mu_s(A) := \int_{\mathcal{G}^{(0)}} \#(\mathcal{G}_x \cap A) d\mu(x), \quad \mu_t(A) := \int_{\mathcal{G}^{(0)}} \#(\mathcal{G}^x \cap A) d\mu(x),$$

for every Borel set $A \subseteq \mathcal{G}$. We say that the pair (\mathcal{G}, μ) is a *discrete probability measure-preserving (discrete p.m.p.) groupoid* if $\mu_s = \mu_t$; in this case we also say that \mathcal{G} *preserves* the measure μ .

Let (\mathcal{G}, μ) be a discrete p.m.p. groupoid. A Borel subset $E \subseteq \mathcal{G}^{(0)}$ is called *invariant* if $\mu(t(\mathcal{G}E) \Delta E) = 0$. The pair (\mathcal{G}, μ) is called *ergodic* if every invariant Borel set $E \subseteq \mathcal{G}^{(0)}$ satisfies $\mu(E) \in \{0, 1\}$.

A Borel subset $\gamma \subseteq \mathcal{G}$ is a (Borel) *bisection* if for every $x \in \mathcal{G}^{(0)}$ both sets $\gamma\{x\}$ and $\{x\}\gamma$ contain at most one element. The collection of all Borel bisections forms an inverse semigroup under the product $(\gamma, \gamma') \mapsto \gamma\gamma'$. The *full group* $[\mathcal{G}]$ of \mathcal{G} consists of all Borel bisections γ satisfying $\gamma\gamma^{-1} = \gamma^{-1}\gamma = \mathcal{G}^{(0)}$. For $\gamma \in [\mathcal{G}]$ and $x \in \mathcal{G}^{(0)}$ we identify the singleton sets γx and $x\gamma$ with the unique elements they contain. With this convention, for any Borel subset $E \subseteq \mathcal{G}^{(0)}$ with $\mu(E) > 0$ we obtain a unitary representation $\alpha_E : [\mathcal{G}] \longrightarrow \mathcal{U}(L^2(\mathcal{G}|_E, \mu_E))$ via

$$(\alpha_E(\gamma)\xi)(g) := \xi(g\gamma) \quad \text{for } \xi \in L^2(\mathcal{G}|_E, \mu_E), \gamma \in [\mathcal{G}], g \in \mathcal{G}|_E,$$

where the *restriction* $\mathcal{G}|_E$ of \mathcal{G} to E is the discrete p.m.p. groupoid obtained by equipping \mathcal{G}_E^E with the normalized measure $\mu_E := \mu(E)^{-1}\mu|_E$. This construction is a special case of the representation

introduced in Subsection 2.3. Indeed, one easily checks that $[\mathcal{G}]$ acts on $\mathcal{G}|_E$ by right multiplication in a measure-preserving way. When $E = \mathcal{G}^{(0)}$, we omit the index and simply write $\alpha = \alpha_{\mathcal{G}^{(0)}}$.

3. MAIN RESULTS

In this section we introduce invariant Markov operators on groupoids and establish our main results, Theorem B and Theorem C. Throughout, the pair (\mathcal{G}, μ) will always denote a discrete p.m.p. groupoid.

3.1. Invariant Markov operators on groupoids. Let (\mathcal{G}, μ) be a discrete p.m.p. groupoid and let $\pi : \mathcal{G} \rightarrow \mathbb{C}$ be a Borel function. We call π *symmetric* if $\pi(g^{-1}) = \overline{\pi(g)}$ for μ_t -almost every $g \in \mathcal{G}$. Moreover, π is called a (Borel) *field of probability measures* if $\pi(g) \geq 0$ for μ_t -almost every $g \in \mathcal{G}$ and $\sum_{g \in \mathcal{G}^x} \pi(g) = 1$ for μ -almost every $x \in \mathcal{G}^{(0)}$.

Following Hahn [Hah78], the *I-norm* of a Borel function $\pi : \mathcal{G} \rightarrow \mathbb{C}$ is defined by

$$\|\pi\|_I = \max \left\{ \text{ess sup}_{x \in \mathcal{G}^{(0)}} \sum_{g \in \mathcal{G}_x} |\pi(g)|, \text{ess sup}_{x \in \mathcal{G}^{(0)}} \sum_{g \in \mathcal{G}^x} |\pi(g)| \right\}.$$

If π is symmetric, then $\|\pi\|_I = \text{ess sup}_{x \in \mathcal{G}^{(0)}} \sum_{g \in \mathcal{G}^x} |\pi(g)|$, so in particular every symmetric field of probability measures has *I*-norm equal to 1.

Whenever π has finite *I*-norm, it induces for each $p \in [1, \infty]$ a bounded operator $P_p^\pi \in \mathbb{B}(L^p(\mathcal{G}, \mu_t))$, as established in the following lemma. As before, we usually omit the index p and write $P^\pi = P_\pi^\pi$. For $p = 2$, the operator P^π will be referred to as the *invariant Markov operator* associated with π . Note that Kaimanovich focuses his work on the case $p = \infty$ [Kai05].

Lemma 3.1. *Let (\mathcal{G}, μ) be a discrete p.m.p. groupoid, let $p \in [1, \infty]$, and let $\pi : \mathcal{G} \rightarrow \mathbb{C}$ be a Borel function with $\|\pi\|_I < \infty$. Then the map $P^\pi : L^p(\mathcal{G}, \mu_t) \rightarrow L^p(\mathcal{G}, \mu_t)$ given by $P^\pi(\xi)(g) := \sum_{h \in \mathcal{G}^{s(g)}} \xi(gh) \pi(h)$ for every $\xi \in L^p(\mathcal{G}, \mu_t)$ and μ_t -almost every $g \in \mathcal{G}$, is well-defined and defines an element of $\mathbb{B}(L^p(\mathcal{G}, \mu_t))$ with operator norm at most $\|\pi\|_I$.*

Proof. Let $p \in [1, \infty]$ and let $\pi : \mathcal{G} \rightarrow \mathbb{C}$ be a Borel function with $\|\pi\|_I < \infty$. We distinguish three cases.

- *Case 1:* Consider the case $p = 1$ and let $\xi \in L^1(\mathcal{G}, \mu_t)$. Then, $\sum_{h \in \mathcal{G}^{r(g)}} |\xi(h)| |\pi(g^{-1}h)| \leq \|\pi\|_I \|\xi|_{\mathcal{G}^{r(g)}}\|_1$ for μ_t -almost every $g \in \mathcal{G}$. Thus the sum defining $P^\pi(\xi)$ converges absolutely, and, arguing as in case 3, we get that $\|P^\pi\| \leq \|\pi\|_I$.
- *Case 2:* For $p = \infty$ and $\xi \in L^\infty(\mathcal{G}, \mu_t)$ one has that

$$\sum_{h \in \mathcal{G}^{s(g)}} |\xi(gh)| |\pi(h)| \leq \|\xi\|_\infty \sum_{h \in \mathcal{G}^{s(g)}} |\pi(h)| \leq \|\pi\|_I \|\xi\|_\infty$$

for μ_t -almost every $g \in \mathcal{G}$. As above, P^π is well defined and $\|P^\pi\| \leq \|\pi\|_I$.

- *Case 3:* Now assume that $1 < p < \infty$ and let $q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. With Hölder's inequality it follows that for $\xi \in L^p(\mathcal{G}, \mu_t)$,

$$\begin{aligned} \sum_{h \in \mathcal{G}^{s(g)}} |\xi(gh)| |\pi(h)| &\leq \left(\sum_{h \in \mathcal{G}^{s(g)}} |\xi(gh)|^p |\pi(h)| \right)^{\frac{1}{p}} \left(\sum_{h \in \mathcal{G}^{s(g)}} |\pi(h)| \right)^{\frac{1}{q}} \\ &\leq \|\pi\|_I^{\frac{1}{q}} \left(\sum_{h \in \mathcal{G}^{s(g)}} |\xi(gh)|^p |\pi(h)| \right)^{\frac{1}{p}} \end{aligned} \tag{3.1}$$

for μ_t -almost every $g \in \mathcal{G}$. By $\sum_{h \in \mathcal{G}^{r(g)}} |\xi(h)|^p |\pi(g^{-1}h)| \leq \|\pi\|_I \|\xi|_{\mathcal{G}^{s(g)}}\|_p^p$ the sum on the right-hand side of (3.1) converges absolutely. This in particular implies that $P^\pi(\xi)$ is well-defined with

$$\begin{aligned} \|P^\pi(\xi)\|_p^p &= \int_{\mathcal{G}} \left| \sum_{h \in \mathcal{G}^{s(g)}} \xi(gh) \pi(h) \right|^p d\mu_t(g) \\ &= \int_{\mathcal{G}^{(0)}} \left(\sum_{g \in \mathcal{G}^x} \left| \sum_{h \in \mathcal{G}^{s(g)}} \xi(gh) \pi(h) \right|^p \right) d\mu(x) \\ &\leq \|\pi\|_I^p \|\xi\|_p^p. \end{aligned}$$

This finishes the proof. \square

From the lemma it in particular follows that every symmetric Borel field of probability measures gives rise to a Markov operator of norm at most 1.

Given Borel functions π_1 and π_2 on a discrete p.m.p. groupoid (\mathcal{G}, μ) with $\|\pi_1\|_I, \|\pi_2\|_I < \infty$, their *convolution product* is defined by

$$(\pi_1 * \pi_2)(g) := \sum_{h \in \mathcal{G}^{r(g)}} \pi_1(h) \pi_2(h^{-1}g), \quad \text{for } g \in \mathcal{G}.$$

This produces again a Borel function with finite I -norm. The convolution is associative with $P^{\pi_1 * \pi_2} = P^{\pi_1} P^{\pi_2}$ and $\chi_{\mathcal{G}^{(0)}} * \pi = \pi = \pi * \chi_{\mathcal{G}^{(0)}}$.

Furthermore, Borel functions on groupoids admit a natural *involution*: for a Borel function $\pi : \mathcal{G} \rightarrow \mathbb{C}$ the function $\pi^* : \mathcal{G} \rightarrow \mathbb{C}, g \mapsto \overline{\pi(g^{-1})}$ is again Borel.

3.2. Amenable groupoids. As mentioned earlier, Renault's notion of amenability for (discrete) measured groupoids admits several equivalent characterizations (see, e.g., [AR00]). For our purposes, the following formulation will be the most convenient.

Definition 3.2 ([AR00, Definition 3.2.8]). Let (\mathcal{G}, μ) be a discrete measured groupoid. We say that (\mathcal{G}, μ) is *amenable* if there exists a bounded linear map $\Phi : L^\infty(\mathcal{G}, \mu_t) \rightarrow L^\infty(\mathcal{G}^{(0)}, \mu)$ of norm 1 such that $\Phi(f) = f$ for all $f \in L^\infty(\mathcal{G}^{(0)}, \mu)$, and $\Phi(\alpha(\gamma)(f)) = \alpha(\gamma)(\Phi(f))$ for all $f \in L^\infty(\mathcal{G}, \mu_t)$ and all $\gamma \in [\mathcal{G}]$. The map Φ is called a *global invariant mean*.

The proof of the “if” direction of Theorem B requires the following lemma.

Lemma 3.3. *Let (\mathcal{G}, μ) be a discrete p.m.p. groupoid. Then (\mathcal{G}, μ) is amenable if and only if there exists a $[\mathcal{G}]$ -invariant mean $m \in L^\infty(\mathcal{G}, \mu_t)^*$ such that $m(f) = \int_{\mathcal{G}^{(0)}} f d\mu$ for all $f \in L^\infty(\mathcal{G}^{(0)}, \mu)$.*

Proof. For the “only if” direction, assume that there exists a global invariant mean $\Phi : L^\infty(\mathcal{G}, \mu_t) \rightarrow L^\infty(\mathcal{G}^{(0)}, \mu)$ and define $m \in L^\infty(\mathcal{G}, \mu_t)^*$ by $m(f) := \int_{\mathcal{G}^{(0)}} \Phi(f) d\mu$ for $f \in L^\infty(\mathcal{G}, \mu_t)$. Then m is a $[\mathcal{G}]$ -invariant mean, and $m(f) = \int_{\mathcal{G}^{(0)}} f d\mu$ for all $f \in L^\infty(\mathcal{G}^{(0)}, \mu)$.

For the “if” direction, let $m \in L^\infty(\mathcal{G}, \mu_t)^*$ be a $[\mathcal{G}]$ -invariant mean satisfying $m(f) = \int_{\mathcal{G}^{(0)}} f d\mu$ for $f \in L^\infty(\mathcal{G}^{(0)}, \mu)$. Fix $f \in L^\infty(\mathcal{G}, \mu_t)$. For every $k \in L^\infty(\mathcal{G}^{(0)}, \mu)$ we have

$$|m(kf)| \leq m(|fk|) \leq \|f\|_\infty m(|k|) = \|f\|_\infty \|k\|_1.$$

Since μ is a probability measure, $L^\infty(\mathcal{G}^{(0)}, \mu)$ is dense in $L^1(\mathcal{G}^{(0)}, \mu)$. Hence there exists a unique functional $L_f \in L^1(\mathcal{G}^{(0)}, \mu)^* \cong L^\infty(\mathcal{G}^{(0)}, \mu)$ with $\|L_f\| \leq \|f\|_\infty$ such that $L_f(k) = m(fk)$ for all

$k \in L^\infty(\mathcal{G}^{(0)}, \mu)$. Let $\Phi(f) \in L^\infty(\mathcal{G}^{(0)}, \mu)$ be the function corresponding to L_f so that $m(fk) = \int_{\mathcal{G}^{(0)}} \Phi(f)k d\mu$ for all $k \in L^1(\mathcal{G}^{(0)}, \mu)$.

Positivity of m shows that Φ is positive and has norm 1. Moreover, since $m(f) = \int_{\mathcal{G}^{(0)}} f d\mu$ for all $f \in L^\infty(\mathcal{G}^{(0)}, \mu)$, we have $\Phi(f) = f$ on $L^\infty(\mathcal{G}^{(0)}, \mu)$. Finally,

$$\int_{\mathcal{G}^{(0)}} \Phi(f) d\mu = m(f) = m(\alpha(\gamma)f) = \int_{\mathcal{G}^{(0)}} \Phi(\alpha(\gamma)f) d\mu$$

for all $f \in L^\infty(\mathcal{G}^{(0)}, \mu)$ and all $\gamma \in [\mathcal{G}]$, proving that Φ is a global invariant mean. \square

Lemma 3.3 shows that to prove amenability of a discrete p.m.p. groupoid (\mathcal{G}, μ) it suffices to construct a $[\mathcal{G}]$ -invariant mean on $L^\infty(\mathcal{G}, \mu_t)$. In the context of Theorem B we achieve this by producing Γ -invariant means for arbitrary countable subgroups $\Gamma \leq [\mathcal{G}]$, and then applying a compactness argument based on the weak*-topology. This strategy is heavily inspired by Hayes' approach in [Hay24].

The following proposition states a classical characterization of amenability which we will use in the “only if” direction of the proof of Theorem B, often referred to as the *weak Godement condition*; see, for example, [Ana11, Definition 7.1].

Proposition 3.4. *A discrete measured groupoid (\mathcal{G}, μ) is amenable if and only if there exists a sequence $(\xi_n)_{n \in \mathbb{N}}$ of Borel functions on \mathcal{G} such that:*

- (i) $\sum_{g \in \mathcal{G}_x} |\xi_n(g)|^2 = 1$ for μ -almost every $x \in \mathcal{G}^{(0)}$ and every $n \in \mathbb{N}$;
- (ii) $F_n \rightarrow 1$ in the weak*-topology on $L^\infty(\mathcal{G}, \mu_t) \cong L^1(\mathcal{G}, \mu_t)^*$, where $F_n \in L^\infty(\mathcal{G}, \mu_t)$ is given by $F_n(h) := \sum_{g \in \mathcal{G}_{t(h)}} \xi_n(gh) \overline{\xi_n(g)}$ for $h \in \mathcal{G}$.

3.3. Proof of Kesten's criterion for amenability. We now work with Markov operators on restrictions of groupoids. For a discrete p.m.p. groupoid (\mathcal{G}, μ) , an invariant Borel subset $E \subseteq \mathcal{G}^{(0)}$ with $\mu(E) > 0$, and a Borel function $\pi: \mathcal{G} \rightarrow \mathbb{C}$ with $\|\pi\|_I < \infty$, we denote by $P_E^\pi \in \mathbb{B}(L^2(\mathcal{G}|_E, (\mu_E)_t))$ the Markov operator associated with $(\mathcal{G}|_E, \mu_E)$ and the restriction $\pi|_{\mathcal{G}|_E}$.

Definition A can then be formulated as follows.

Definition 3.5. Let (\mathcal{G}, μ) be a discrete p.m.p. groupoid. We say that (\mathcal{G}, μ) satisfies *Kesten's criterion* if, for every invariant Borel subset $E \subseteq \mathcal{G}^{(0)}$ with $\mu(E) > 0$ and every symmetric Borel field of probability measures $\pi: \mathcal{G} \rightarrow \mathbb{C}$, one has $\|P_E^\pi\| = 1$.

In the next lemma, a countable subgroup $\Gamma \leq [\mathcal{G}]$ is said to *cover* \mathcal{G} if the union $\bigcup_{\gamma \in \Gamma} \gamma$ is a co-null set in \mathcal{G} .

Lemma 3.6. *Let (\mathcal{G}, μ) be a discrete p.m.p. groupoid such that $\|P_E^\pi\| = 1$ for every invariant Borel subset $E \subseteq \mathcal{G}^{(0)}$ with $\mu(E) > 0$ and every symmetric field $\pi: \mathcal{G} \rightarrow \mathbb{C}$ of probability measures. Then for any invariant Borel subsets $E_1, \dots, E_k \subseteq \mathcal{G}^{(0)}$ of positive measure and any countable subgroup $\Gamma \leq [\mathcal{G}]$ covering \mathcal{G} , there exists a Γ -invariant mean $m \in L^\infty(\mathcal{G}, \mu)^*$ satisfying $m(\chi_{E_i}) = \mu(E_i)$ for $1 \leq i \leq k$.*

Proof. Let \mathcal{F} denote the σ -algebra generated by E_1, \dots, E_k , and let $\nu \in \text{Prob}(\Gamma)$ be a symmetric probability measure whose support generates Γ . There exists a finite partition $(A_j)_{1 \leq j \leq l}$ of $\mathcal{G}^{(0)}$ into invariant Borel subsets of positive measure such that $\mathcal{F} = \{\bigcup_{j \in D} A_j \mid D \subseteq [l]\}$.

Define $\pi: \mathcal{G} \rightarrow [0, \infty)$ by $\pi(g) := \nu(\{\gamma \in \Gamma \mid g \in \gamma\})$. A simple computation shows that π is a symmetric Borel field of probability measures. Similarly,

$$\begin{aligned} (P_{A_j}^\pi \xi)(g) &= \sum_{h \in \mathcal{G}^{s(g)}} \xi(gh) \nu(\{\gamma \in \Gamma \mid h \in \gamma\}) \\ &= \sum_{\gamma \in \Gamma} \xi(g\gamma) \nu(\gamma) \\ &= (\alpha_{A_j}(\nu)(\xi))(g). \end{aligned}$$

for all $1 \leq j \leq l$, $\xi \in L^2(\mathcal{G}|_{A_j}, (\mu_{A_j})_t)$, $g \in \mathcal{G}|_{A_j}$ where $\alpha_{A_j}(\nu) := \sum_{\gamma \in \Gamma} \nu(\gamma) \alpha_{A_j}(\gamma)$, so that $P_{A_j}^\pi = \alpha_{A_j}(\nu)$. By assumption, $\|\alpha_{A_j}(\nu)\| = \|P_{A_j}^\pi\| = 1$, so [Hay24, Theorem 2.6] and [Hay24, Theorem 2.7] yield a Γ -invariant mean $m_j \in L^\infty(\mathcal{G}|_{A_j}, (\mu_{A_j})_t)^*$.

Define $m \in L^\infty(\mathcal{G}, \mu_t)^*$ by $m(f) := \sum_{j=1}^l \mu(A_j) m_j(f|_{\mathcal{G}|_{A_j}})$ for $f \in L^\infty(\mathcal{G}, \mu_t)$. Since the A_j are invariant and disjoint, m is Γ -invariant and satisfies $m(\chi_{A_j}) = \mu(A_j)$ for all $1 \leq j \leq l$, hence $m(\chi_{E_i}) = \mu(E_i)$ for all i . The result then follows from Theorem [Hay24, Theorem 2.8]. \square

Proof of Theorem B. For the “if” direction assume that the discrete p.m.p. groupoid (\mathcal{G}, μ) satisfies Kesten’s criterion. For a countable subgroup $\Gamma \leq [\mathcal{G}]$ and a finite tuple $\mathbf{E} = (E_1, \dots, E_k)$ of invariant Borel subsets of positive measure, let $\mathcal{M}_{\Gamma, \mathbf{E}}$ denote the set of Γ -invariant means $m \in L^\infty(\mathcal{G}, \mu_t)^*$ with $m(\chi_{E_i}) = \mu(E_i)$ for $1 \leq i \leq k$. By Lemma 3.6, each $\mathcal{M}_{\Gamma, \mathbf{E}}$ is non-empty and weak*-compact.

By the finite intersection property the intersection $\bigcap_{\Gamma, \mathbf{E}} \mathcal{M}_{\Gamma, \mathbf{E}}$ is non-empty. Hence there exists a $[\mathcal{G}]$ -invariant mean $m \in L^\infty(\mathcal{G}, \mu_t)^*$ satisfying $m(\chi_E) = \mu(E)$ for every invariant Borel subset $E \subseteq \mathcal{G}^{(0)}$. By [Hay24, Theorem 2.8], $m(f) = \int_{\mathcal{G}^{(0)}} f d\mu$ for all $f \in L^\infty(\mathcal{G}^{(0)}, \mu)$, and Lemma 3.3 then implies that (\mathcal{G}, μ) is amenable.

For the “only if” direction assume that (\mathcal{G}, μ) is amenable. Let $\pi: \mathcal{G} \rightarrow [0, \infty)$ be a symmetric Borel field of probability measures and let $E \subseteq \mathcal{G}^{(0)}$ be an invariant Borel subset with $\mu(E) > 0$. Lemma 3.1 shows that $\|P_E^\pi\| \leq 1$.

To show the reverse inequality, let $(\xi_n)_{n \in \mathbb{N}}$ be sequence of Borel functions as in Proposition 3.4, and for each n let $\xi_n^E \in L^2(\mathcal{G}|_E, (\mu_E)_t)$ be the restriction of ξ_n to $\mathcal{G}|_E$. Then,

$$\begin{aligned} \langle P_E^\pi \xi_n^E, \xi_n^E \rangle &= \frac{1}{\mu(E)} \int_E \sum_{g \in \mathcal{G}_x} (P_E^\pi \xi_n^E)(g) \overline{\xi_n^E(g)} d\mu(x) \\ &= \frac{1}{\mu(E)} \int_E \sum_{g \in \mathcal{G}_x} \sum_{h \in \mathcal{G}^x} \xi_n(gh) \overline{\xi_n(g)} \pi(h) d\mu(x) \\ &= \frac{1}{\mu(E)} \int_E \sum_{h \in \mathcal{G}^x} \pi(h) \sum_{g \in \mathcal{G}_x} \xi_n(gh) \overline{\xi_n(g)} d\mu(x) \\ &\rightarrow \frac{1}{\mu(E)} \int_E \sum_{h \in \mathcal{G}^x} \pi(h) d\mu(x) \\ &= 1. \end{aligned}$$

Hence $\|P_E^\pi\| = 1$, completing the proof. \square

Remark 3.7. For groups, it suffices that one Markov operator associated with a symmetric non-degenerate measure has norm 1 in order to conclude amenability. As shown by the examples in [Kai01], this is no longer true at the level of general groupoids. However, for a measurable bundle

of groups such a reduction still works. Indeed, if a Markov operator P has norm 1, there exists a sequence $(\xi_n)_{n \in \mathbb{N}} \subseteq L^2(\mathcal{G}, \mu_t)$ of positive unit vectors such that $\langle P\xi_n, \xi_n \rangle \rightarrow 1$. A computation analogous to that in the proof of the “only if” direction of Theorem B shows that the ξ_n are asymptotically invariant. Moreover, the ξ_n may be chosen so that the sum over the target fibres equals 1. Since in a bundle of groups the source and target fibres coincide, Proposition 3.4 then implies that the bundle is amenable.

3.4. A formula for the spectral radius. In [Kes59a; Kes59b], Kesten characterized the amenability of finitely generated groups in terms of the return probabilities of random walks on their Cayley graphs. Motivated by this perspective, we introduce the following notion.

Definition 3.8. Let (\mathcal{G}, μ) be a discrete p.m.p. groupoid. Given a Borel subset $E \subseteq \mathcal{G}^{(0)}$ with $\mu(E) > 0$ and a symmetric Borel field of probability measures $\pi : \mathcal{G} \rightarrow [0, \infty)$, we define

$$\rho_E(\mathcal{G}, \pi) := \lim_{n \rightarrow \infty} \mu(E)^{-\frac{1}{2n}} \langle (P^\pi)^{2n} \chi_E, \chi_E \rangle^{\frac{1}{2n}} \quad (3.2)$$

and call it the *E-spectral radius of π* . When $E = \mathcal{G}^{(0)}$, we write simply $\rho(\mathcal{G}, \pi) := \rho_{\mathcal{G}^{(0)}}(\mathcal{G}, \pi)$ and refer to this as the *spectral radius of π* .

Recall that Theorem C asserts that the limit in (3.2) always exists, satisfies $\rho_E(\mathcal{G}, \pi) \leq 1$, and that $\rho(\mathcal{G}, \pi) = \|P^\pi\|$. Before proving this, we record two observations.

Remark 3.9. (i) Let (\mathcal{G}, μ) be a discrete p.m.p. groupoid, $E \subseteq \mathcal{G}^{(0)}$ a Borel subset with $\mu(E) > 0$, and $\pi : \mathcal{G} \rightarrow [0, \infty)$ a symmetric Borel field of probability measures. For $n \in \mathbb{N}_{\geq 1}$ denote by π^{*n} the n -fold convolution power of π . Then

$$((P^\pi)^n \chi_E)(g) = \begin{cases} \pi^{*n}(g^{-1}), & t(g) \in E, \\ 0, & \text{otherwise,} \end{cases}$$

for every $g \in \mathcal{G}$. Consequently,

$$\langle (P^\pi)^n \chi_E, \chi_E \rangle = \int_E \pi^{*n}(x) d\mu(x)$$

and hence

$$\rho_E(\mathcal{G}, \pi) = \lim_{n \rightarrow \infty} \mu(E)^{-\frac{1}{2n}} \left(\int_E \pi^{*(2n)}(x) d\mu(x) \right)^{\frac{1}{2n}}.$$

Following the framework of Kaimanovich in [Kai05], the quantity $\mu(E)^{-1} \int_E \pi^{*(2n)}(x) d\mu(x)$ may be interpreted as the normalized μ -average of the probability of returning to the set E after $2n$ steps.

(ii) For a discrete p.m.p. groupoid (\mathcal{G}, μ) the same computation used in the previous remark shows that $P^\pi(\chi_{\mathcal{G}^{(0)}}) = \overline{\pi^*}$ for every Borel function $\pi : \mathcal{G} \rightarrow \mathbb{C}$ with $\|\pi\|_I < \infty$. Hence, appealing to Lemma 3.1,

$$\|\pi\|_2 = \|P^\pi(\chi_{\mathcal{G}^{(0)}})\|_2 \leq \|P^\pi\| \leq \|\pi\|_I.$$

Therefore, if an operator $T \in \mathbb{B}(L^2(\mathcal{G}, \mu_t))$ can be written as a norm limit $T = \lim_{i \rightarrow \infty} P^{\pi_i}$ for suitable Borel functions $\pi_i : \mathcal{G} \rightarrow \mathbb{C}$ with finite I -norms, then T itself has to be of the form $T = P^\eta$, for some Borel function $\eta \in L^2(\mathcal{G}, \mu_t)$. In this case, η can be retrieved as $\eta = \overline{T(\chi_{\mathcal{G}^{(0)}})^*}$ and it follows that $T(\chi_{\mathcal{G}^{(0)}})$ is non-zero whenever $T \neq 0$.

We now proceed to the proof of Theorem C.

Proof of Theorem C. Since π is symmetric, P^π is self-adjoint. Indeed, for $\xi, \eta \in L^2(\mathcal{G}, \mu_t)$,

$$\begin{aligned} \langle P^\pi \xi, \eta \rangle &= \int_{\mathcal{G}^{(0)}} \sum_{g \in \mathcal{G}_x} (P^\pi \xi)(g) \overline{\eta(g)} d\mu(x) = \int_{\mathcal{G}^{(0)}} \sum_{g \in \mathcal{G}_x} \sum_{h \in \mathcal{G}^x} \xi(gh) \overline{\eta(g)} \pi(h) d\mu(x) \\ &= \int_{\mathcal{G}^{(0)}} \sum_{h \in \mathcal{G}^x} \pi(h) \sum_{g \in \mathcal{G}_x} \xi(gh) \overline{\eta(g)} d\mu(x) = \int_{\mathcal{G}^{(0)}} \sum_{h \in \mathcal{G}_x} \pi(h^{-1}) \sum_{g \in \mathcal{G}_x} \xi(gh^{-1}) \overline{\eta(g)} d\mu(x) \\ &= \int_{\mathcal{G}^{(0)}} \sum_{h \in \mathcal{G}_x} \pi(h) \sum_{g \in \mathcal{G}_{t(h)}} \xi(g) \overline{\eta(gh)} d\mu(x) = \int_{\mathcal{G}^{(0)}} \sum_{g \in \mathcal{G}_x} \xi(g) \sum_{h \in \mathcal{G}^x} \overline{\eta(gh)} \pi(h) d\mu(x) \\ &= \int_{\mathcal{G}^{(0)}} \sum_{g \in \mathcal{G}_x} \xi(g) \overline{(P^\pi \eta)(g)} d\mu(x) = \langle \xi, P^\pi \eta \rangle. \end{aligned}$$

Fix a Borel subset $E \subseteq \mathcal{G}^{(0)}$ with $\mu(E) > 0$ and define the unit vector $\xi_E := \mu(E)^{-1/2} \chi_E \in L^2(\mathcal{G}, \mu_t)$. By the spectral theorem for self-adjoint operators (together with the discussion in Subsection 2.2), there exists a probability measure ν_E on the spectrum $\sigma(P^\pi) \subseteq [-1, 1]$ such that

$$\langle f(P^\pi) \xi_E, \xi_E \rangle = \int_{\sigma(P^\pi)} f(t) d\nu_E(t) \quad (3.3)$$

for every continuous function $f \in C(\sigma(P^\pi))$. In particular, the limit in (3.2) exists and equals the L^∞ -norm of the function $t \mapsto |t|$ on $\sigma(P^\pi)$.

Now assume that $E = \mathcal{G}^{(0)}$. Since P^π is self-adjoint, there exists $\lambda \in \sigma(P^\pi)$ with $\|P^\pi\| = |\lambda|$. We claim that $\lambda \in \text{supp}(\nu_E)$. Indeed, if we suppose otherwise, then some open neighborhood U of λ satisfies $\nu_E(U) = 0$. Choose a non-negative $f \in C(\sigma(P^\pi))$ with $f(\lambda) = 1$ and $f \equiv 0$ on $U^c \cap \sigma(P^\pi)$. By (3.3),

$$\langle f(P^\pi) \xi_E, \xi_E \rangle = \int_{U \cap \sigma(P^\pi)} f(t) d\nu_E(t) = 0.$$

However, $\sqrt{f(P^\pi)}$ is a non-zero positive operator in the smallest norm closed, self-adjoint subalgebra of $\mathbb{B}(L^2(\mathcal{G}, \mu_t))$ containing P^π , and hence can be approximated in norm by operators of the form P^{π_i} , where the π_i are Borel functions of finite I -norm. By Remark 3.9 (ii),

$$\langle f(P^\pi) \xi_E, \xi_E \rangle = \|\sqrt{f(P^\pi)} \xi_E\|_2^2 \neq 0,$$

a contradiction. Thus $\lambda \in \text{supp}(\nu_E)$.

Combining this with the previous paragraph yields $\rho(\mathcal{G}, \pi) = |\lambda| = \|P^\pi\|$, as desired. \square

4. APPENDIX: UNBOUNDED MARKOV OPERATORS

The purpose of this appendix is to illustrate how Borel fields of probability measures with infinite I -norm can give rise to unbounded Markov operators. We have two examples. The first one occurs in a discrete probability measure-preserving non-ergodic groupoid, whereas the second one occurs in a measure-preserving discrete ergodic groupoid.

4.1. First example. We begin by recalling that any countable Borel equivalence relation gives rise to a discrete p.m.p. groupoid. More precisely, let (X, μ) be a standard probability space, and let $\mathcal{R} \subseteq X \times X$ be a countable Borel equivalence relation. Then \mathcal{R} becomes a discrete measured groupoid with unit space $\mathcal{R}^{(0)} := \text{Diag}(X) \cong X$ and structure maps

$$s(x, y) := (y, y), \quad t(x, y) := (x, x), \quad (x, y)(y, z) := (x, z), \quad (x, y)^{-1} := (y, x)$$

for all $(x, y), (y, z) \in \mathcal{R}$. We say that \mathcal{R} is *probability measure-preserving* (p.m.p.) if the resulting groupoid is p.m.p.

If X is countable and $\mathcal{R} = X \times X$ is the full equivalence relation, then a Borel system of probability measures reduces to a function $\pi: \mathcal{R} \rightarrow [0, \infty)$ such that $\sum_{x \in X} \pi(x, y) = 1$ for every $y \in X$. The associated Markov operator satisfies

$$P^\pi(\xi)(x, y) = \sum_{z \in X} \xi((x, y)(y, z))\pi(y, z) = \sum_{z \in X} \xi(x, z)\pi(y, z),$$

for every $\xi \in L^2(\mathcal{R}, \mu_t)$; hence P^π identifies with the (possibly infinite) matrix $[\pi(y, x)]_{x, y \in X}$.

For what follows, fix a p.m.p. equivalence relation \mathcal{R}_0 on a standard probability space (Y, ν) . For each $n \in \mathbb{N}_{\geq 1}$ let \mathcal{S}_n denote the full equivalence relation on $([n], \nu_n)$, where ν_n is the normalized counting measure. Recall also that if (\mathcal{G}_1, μ_1) and (\mathcal{G}_2, μ_2) are discrete p.m.p. groupoids, then their product $(\mathcal{G}_1 \times \mathcal{G}_2, \mu_1 \times \mu_2)$ is again a discrete p.m.p. groupoid with the obvious operations.

Consider now the equivalence relation $\mathcal{R} := \bigsqcup_{n \in \mathbb{N}} (\mathcal{R}_0 \times \mathcal{S}_n)$, defined on the space $X := \bigsqcup_{n \in \mathbb{N}} (Y \times [n])$, and equipped with the measure $\mu := \sum_{n \in \mathbb{N}} (\nu \times \nu_n)$. With the obvious Borel structure, \mathcal{R} becomes a discrete p.m.p. groupoid.

Our construction of the Borel field π of probability measures is based on direct sums of finite-dimensional operators (i.e., matrices) whose sequence of norms diverges. The following lemma provides the relevant building blocks.

Lemma 4.1. *Let $n \in \mathbb{N}$ be an integer with $n > 1$ and let $\delta \in (0, \frac{1}{2})$. Then there exists a matrix $A_\delta := [A_\delta(i, j)]_{1 \leq i, j \leq n} \in \mathbb{M}_n(\mathbb{C})$ with strictly positive entries and column sums $\sum_{i=1}^n A_\delta(i, j) = 1$ for $1 \leq j \leq n$ such that the corresponding operator norm satisfies $\|A_\delta\| > \sqrt{n} - \delta$.*

Proof. For $\epsilon \in (0, 1)$, let $B(\epsilon)$ be the matrix whose n rows are identical and equal to

$$x_\epsilon := (1 - \epsilon, \frac{\epsilon}{n-1}, \dots, \frac{\epsilon}{n-1}).$$

Writing $\mathbf{1}$ for the row vector of all ones, we have $B(\epsilon) = \mathbf{1}^T x_\epsilon$. Thus,

$$\|B(\epsilon)\| \geq \|\mathbf{1}\|_2 \|x_\epsilon\|_2 = \sqrt{n} \sqrt{(1 - \epsilon)^2 + \frac{\epsilon^2}{n-1}}.$$

The function $F(\epsilon) := (1 - \epsilon)^2 + \frac{\epsilon^2}{n-1}$ is continuous on $[0, 1]$ and attains its maximum at $\epsilon = 0$. Hence, there exists a $\epsilon_0 > 0$ such that

$$\sqrt{F(\epsilon_0)} > 1 - \frac{\delta}{\sqrt{n}}.$$

Let $A_\delta := B(\epsilon_0)$, so that $\|A_\delta\| > \sqrt{n} - \delta$. □

Fix $\delta \in (0, \frac{1}{2})$ and let $\pi'_0: \mathcal{R}_0 \rightarrow [0, \infty)$ be a Borel field of probability measures. For $n \geq 1$, define a Borel field $\pi_n: \mathcal{S}_n \rightarrow [0, \infty)$ of probability measures by setting $\pi_n(i, j) := A_\delta(i, j)$ for $1 \leq i, j \leq n$, where A_δ is as in Lemma 4.1. These fields combine to a Borel field of probability

measures $\pi : \mathcal{R} \rightarrow [0, 1]$ defined to be equal to $\pi'_0 \times \pi_n$ on each component $\mathcal{R}_0 \times \mathcal{S}_n$. The associated Markov operator decomposes as

$$P^\pi = \bigoplus_{n \in \mathbb{N}} (P^{\pi'_0} \otimes P^{\pi_n}) \in \bigoplus_{n \in \mathbb{N}} \mathbb{B}(\ell^2(\mathcal{R}_0) \otimes \ell^2(\mathcal{S}_n)).$$

Since

$$\sup_{n \in \mathbb{N}} \|P^{\pi'_0} \otimes P^{\pi_n}\| = \|P^{\pi'_0}\| \left(\sup_{n \in \mathbb{N}} \|P^{\pi_n}\| \right) > \|P^{\pi'_0}\| \left(\sup_{n \in \mathbb{N}} (\sqrt{n} - \delta) \right) = \infty,$$

the operator P^π is unbounded.

The construction above enjoys several noteworthy features. First, if (Y, ν) is non-atomic, then so is the resulting unit space (X, μ) . Second, if \mathcal{R}_0 is amenable, then \mathcal{R} is amenable as well. Finally, the field π is non-degenerate in the sense of [BCDKK24, Definition 2.1].

4.2. Second example. For our second example, consider the full equivalence relation $\mathcal{R} := \mathbb{N} \times \mathbb{N}$ on \mathbb{N} , equipped with the counting measure μ . The pair (\mathcal{R}, μ) forms a *discrete measure-preserving* groupoid, meaning that the induced measures μ_s and μ_t introduced in Subsection 2.4 coincide; indeed, both agree with the counting measure on \mathbb{N}^2 . Moreover, (\mathcal{R}, μ) is ergodic. Note, however, that (\mathcal{R}, μ) is not a discrete p.m.p. groupoid.

For $k \in \mathbb{N}$ define

$$I_k := \mathbb{N} \cap \left[\frac{k(k+1)}{2}, \frac{(k+1)(k+2)}{2} \right),$$

so that $\mathbb{N} = \bigsqcup_{k \in \mathbb{N}} I_k$ is a partition of the natural numbers. Define a map $\pi : \mathcal{R} \rightarrow [0, \infty)$ by

$$\pi(m, n) := \begin{cases} 1, & \text{if } n \in I_m, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } (m, n) \in \mathcal{R}.$$

Then $\sum_{m \in \mathbb{N}} \pi(m, n) = 1$ for each $n \in \mathbb{N}$, so π is a (non-symmetric) Borel field of probability measures.

For each $k \in \mathbb{N}$ define $\xi_k \in L^2(\mathcal{R}, \mu_t)$ by

$$\xi_k(m, n) := \begin{cases} \pi(k, n), & \text{if } m = k, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } (m, n) \in \mathcal{R}.$$

A direct computation shows that

$$P^\pi(\xi_k)(m, n) = \sum_{a \in \mathbb{N}} \xi_k(m, a) \pi(n, a) = \begin{cases} \sum_{a \in \mathbb{N}} \pi(k, a) \pi(n, a), & \text{if } m = k, \\ 0, & \text{otherwise,} \end{cases}$$

so that in particular,

$$P^\pi(\xi_k)(k, n) = \sum_{a \in \mathbb{N}} \pi(k, a) \pi(n, a) = \#(I_k \cap I_n) = \begin{cases} \#I_k, & \text{if } n = k, \\ 0, & \text{otherwise.} \end{cases}$$

Hence,

$$\|P^\pi(\xi_k)\|_2^2 = \sum_{m, n \in \mathbb{N}} |P^\pi(\xi_k)(m, n)|^2 = \sum_{n \in \mathbb{N}} |P^\pi(\xi_k)(k, n)|^2 = (\#I_k)^2 = (k+1)^2,$$

while

$$\|\xi_k\|_2^2 = \sum_{m, n \in \mathbb{N}} |\xi_k(m, n)|^2 = \sum_{n \in \mathbb{N}} |\pi(k, n)|^2 = \sum_{n \in I_k} 1 = \#I_k = (k+1).$$

Consequently,

$$\sup_{k \in \mathbb{N}} \frac{\|P^\pi(\xi_k)\|_2}{\|\xi_k\|_2} = \sup_{k \in \mathbb{N}} \sqrt{k+1} = \infty,$$

and therefore the Markov operator P^π is not a bounded operator on $L^2(\mathcal{R}, \mu_t)$.

ACKNOWLEDGMENTS

S.C. and M.D. are supported by the ERC advanced grant 101141693. F.F. thankfully acknowledges support from the Simons Foundation Dissertation Fellowship SFI-MPS-SDF-00015100. He also thanks Professor Nicolás Matte Bon for the interesting discussions about Kesten's criterion. The authors would like to thank James Harbour for initial discussions and Ben Hayes for the discussions surrounding the cospectral radius on equivalence relations. Moreover, the authors also wish to thank Tey Berendschot for proposing the problem and the conferences "YMC*A" at SDU and "Orbit equivalence and topological and measurable dynamics" at CIRM for hosting part of this research.

REFERENCES

- [AB22] M. Abért and I. Biringer. "Unimodular measures on the space of all Riemannian manifolds". In: *Geom. Topol.* 26.5 (2022), pp. 2295–2404.
- [AFH24] M. Abert, M. Fraczyk, and B. Hayes. "Co-spectral radius for countable equivalence relations". In: *Ergodic Theory Dynam. Systems* 44.12 (2024), pp. 3385–3427.
- [AFH25] M. Abért, M. Fraczyk, and B. Hayes. "Growth dichotomy for unimodular random rooted trees". In: *Ann. Probab.* 53.5 (2025), pp. 1627–1644.
- [AGV14] M. Abért, Y. Glasner, and B. Virág. "Kesten's theorem for invariant random subgroups". In: *Duke Math. J.* 163.3 (2014), pp. 465–488.
- [AL07] D. Aldous and R. Lyons. "Processes on unimodular random networks". In: *Electron. J. Probab.* 12.54 (2007), pp. 1454–1508.
- [Ana11] C. Anantharaman-Delaroche. "Old and new about treeability and the Haagerup property for measured groupoids". In: hal-00596887 (2011).
- [AR00] C. Anantharaman-Delaroche and J. Renault. *Amenable groupoids*. Monographies de L'Enseignement Mathématique 36. L'Enseignement Mathématique, Geneva, 2000, p. 196.
- [Ban99] T. Banica. "Representations of compact quantum groups and subfactors". In: *J. Reine Angew. Math.* 509 (1999), pp. 167–198.
- [BLS15] I. Benjamini, R. Lyons, and O. Schramm. "Unimodular random trees". In: *Ergodic Theory Dynam. Systems* 35.2 (2015), pp. 359–373.
- [BCDKK24] T. Berendschot, S. Chakraborty, M. Donvil, S.-J. Kim, and M. Klisse. "The Choquet-Deny Property for Groupoids". 2024. arXiv: 2406.05004 [math.FA].
- [BV25] T. Berendschot and S. Vaes. "Measure equivalence embeddings of free groups and free group factors". In: *Ann. Scient. Éc. Norm. Sup.* (2025), pp. 389–418.
- [BK21] T. Bühl and V. A. Kaimanovich. "Amenability of groupoids and asymptotic invariance of convolution powers". In: *Topology, geometry, and dynamics—V. A. Rokhlin-Memorial*. Contemp. Math. 772. Amer. Math. Soc., [Providence], RI, 2021, pp. 69–92.

- [CL18] C. Chu and X. Li. “Amenability, Reiter’s condition and Liouville property”. In: *J. Funct. Anal.* 274.12 (2018), pp. 3291–3324.
- [Con90] J. B. Conway. *A course in functional analysis*. Second. Graduate Texts in Mathematics 96. Springer-Verlag, New York, 1990.
- [CHI04] K. Corlette, L. Hernández Lamoneda, and A. Iozzi. “A vanishing theorem for the tangential de Rham cohomology of a foliation with amenable fundamental groupoid”. In: *Geom. Dedicata* 103 (2004), pp. 205–223.
- [Føl55] E. Følner. “On groups with full Banach mean value”. In: *Math. Scand.* 3 (1955), pp. 243–254.
- [Fur73] H. Furstenberg. “Boundary theory and stochastic processes on homogeneous spaces”. In: *Harmonic analysis on homogeneous spaces (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972)*. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 1973, pp. 193–229.
- [Hah78] P. Hahn. “The regular representations of measure groupoids”. In: *Trans. Amer. Math. Soc.* 242 (1978), pp. 35–72.
- [Hay24] B. Hayes. “Coamenability and cospectral radius for orbit equivalence relations”. 2024. arXiv: 2410.16480 [math.DS].
- [Kai57] V. A. Kaimanovich. “Amenable semigroups”. In: *Illinois J. Math.* 1 (1957), pp. 509–544.
- [Kai01] V. A. Kaimanovich. “Equivalence relations with amenable leaves need not be amenable”. In: *Topology, ergodic theory, real algebraic geometry: Rokhlin’s Memorial*. Amer. Math. Soc. Transl. Ser. 2 202. Amer. Math. Soc., Providence, RI, 2001, pp. 151–166.
- [Kai05] V. A. Kaimanovich. “Amenability and the Liouville property”. In: *Israel J. Math.* 149 (2005), pp. 45–85.
- [KV79] V. A. Kaimanovich and A. M. Vershik. “Random walks on groups: boundary, entropy, uniform distribution”. In: *Dokl. Akad. Nauk SSSR* 249.1 (1979), pp. 15–18.
- [KV83] V. A. Kaimanovich and A. M. Vershik. “Random walks on discrete groups: boundary and entropy”. In: *Ann. Probab.* 11.3 (1983), pp. 457–490.
- [Kes59a] H. Kesten. “Symmetric random walks on groups”. In: *Trans. Amer. Math. Soc.* 92 (1959), pp. 336–354.
- [Kes59b] H. Kesten. “Full Banach mean values on countable groups”. In: *Math. Scand.* 7 (1959), pp. 146–156.
- [Lan73] C. Lance. “On nuclear C^* -algebras”. In: *J. Functional Analysis* 12 (1973), pp. 157–176.
- [Oll05] Y. Ollivier. *A January 2005 invitation to random groups*. Ensaios Matemáticos [Mathematical Surveys] 10. Sociedade Brasileira de Matemática, Rio de Janeiro, 2005.
- [Pie84] J. Pier. *Amenable locally compact groups*. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1984.
- [Rei68] H. Reiter. *Classical harmonic analysis and locally compact groups*. Clarendon Press, Oxford, 1968.
- [Ren80] J. Renault. *A groupoid approach to C^* -algebras*. Lecture Notes in Mathematics 793. Springer, Berlin, 1980.
- [Ros81] J. Rosenblatt. “Ergodic and mixing random walks on locally compact groups”. In: *Math. Ann.* 257 (1981), pp. 31–42.

- [ST10] Y. Shalom and T. Tao. “A finitary version of Gromov’s polynomial growth theorem”. In: *Geom. Funct. Anal.* (2010), pp. 1502–1547.
- [Sta13] M. Stadlbauer. “An extension of Kesten’s criterion for amenability to topological Markov chains”. In: *Adv. Math.* 235 (2013), pp. 450–468.
- [vNeu29] J. von Neumann. “Zur allgemeinen Theorie des Masses”. In: *Fundamenta Mathematicae* 13.1 (1929), pp. 73–116.
- [Zim77a] R. J. Zimmer. “Hyperfinite factors and amenable ergodic actions”. In: *Invent. Math.* 41.1 (1977), pp. 23–31.
- [Zim77b] R. J. Zimmer. “On the von Neumann algebra of an ergodic group action”. In: *Proc. Amer. Math. Soc.* 66.2 (1977), pp. 289–293.
- [Zim78] R. J. Zimmer. “Amenable ergodic group actions and an application to Poisson boundaries of random walks”. In: *J. Functional Analysis* 27.3 (1978), pp. 350–372.

DÉPARTEMENT DE MATHÉMATIQUES ET APPLICATIONS, ÉCOLE NORMALE SUPÉRIEURE, 45 RUE D’ULM, 75005 PARIS, FRANCE

Email address: soham.chakraborty@ens.psl.eu

Email address: milan.donvil@ens.psl.eu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, KERCHOV HALL 114. 141 CABELL DR, CHARLOTTEVILLE, VIRGINIA, UNITED STATES

Email address: hmy3tf@virginia.edu

DEPARTMENT OF MATHEMATICS, CHRISTIAN-ALBRECHTS UNIVERSITY KIEL,, HEINRICH-HECHT-PLATZ 6, KIEL, GERMANY

Email address: klisse@math.uni-kiel.de