

CONNECTING ORBITS IN QUASIAFFINE SPHERICAL VARIETIES VIA B -ROOT SUBGROUPS

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ABSTRACT. Given a connected reductive algebraic group G with a Borel subgroup B and a quasiaffine spherical G -variety X , we prove that every G -orbit Y contained in the regular locus of X can be connected by a B -normalized additive one-parameter group action with any minimal G -orbit in X containing Y in its closure. As a consequence, we show that the regular locus of X is transitive for the subgroup in the automorphism group of X generated by G and all B -normalized additive one-parameter subgroups.

1. INTRODUCTION

Let X be an irreducible algebraic variety over an algebraically closed field \mathbb{K} of characteristic zero. Every nontrivial action on X of the group $\mathbb{G}_a = (\mathbb{K}, +)$ induces a one-parameter subgroup in the automorphism group $\text{Aut}(X)$, called a \mathbb{G}_a -subgroup. If in addition X is equipped with an action of an algebraic group F and R is a \mathbb{G}_a -subgroup normalized by F , then we say that R is an F -root subgroup.

The variety X is said to be *spherical* if it is normal and admits a regular action of a connected reductive algebraic group G such that a Borel subgroup $B \subset G$ has an open dense orbit in X . In this situation, X automatically contains an open G -orbit; moreover, there are only finitely many G -orbits in X . In the recent years, a study of B -root subgroups on spherical varieties was initiated in the papers [AA, AZ1, AZ2]. One motivation for studying B -root subgroups is related to the problem of describing the automorphism groups for complete spherical varieties. In this case, the connected component of the group $\text{Aut}(X)$ is algebraic, so a description of B -root subgroups on X enables one to describe the G -module decomposition of the Lie algebra of $\text{Aut}(X)$. Indeed, every B -root subgroup induces a B -semiinvariant vector field on X , which is a highest weight vector of a simple G -module in the Lie algebra of $\text{Aut}(X)$. The goal of this note is to prove the following theorem, which provides another motivation for studying B -root subgroups on spherical varieties.

Theorem 1. *Let X be a quasiaffine spherical G -variety and let Y be a G -orbit contained in the regular locus of X . Then for every minimal G -orbit $Y' \subset X$ containing Y in its closure there exists a B -root subgroup on X that connects Y with Y' .*

This theorem is a consequence of Theorem 4 below.

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It follows from Theorem 1 that for a quasiaffine spherical G -variety X every G -orbit in the regular locus of X can be connected with the open G -orbit via a suitable chain of B -root subgroups, which yields the following corollary.

Corollary 1. *Let X be a quasiaffine spherical G -variety and let \tilde{G} be the subgroup in $\text{Aut } X$ generated by the images of G and all B -root subgroups on X . Then the regular locus of X is a single \tilde{G} -orbit.*

In the case where $G = B = T$ and X is affine (so that X is an affine toric T -variety), Theorem 1 and Corollary 1 were known from [AKZ].

This note may serve as a complement and further development of one of the main results of the preprint [Sha] (Theorem 3), which states that the regular locus of an affine spherical G -variety X is a single orbit of the subgroup in $\text{Aut}(X)$ generated by G and all \mathbb{G}_a -subgroups. In fact, Corollary 1 gives an affirmative answer to [Sha, Question 1]. We remark that the strategy of our proof of Theorem 1 goes the same lines as that used in [Sha].

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2. PRELIMINARIES

Given an algebraic variety X , we denote by X^{reg} the regular locus of X , by $\text{Aut}(X)$ the group of automorphisms of X , and by $\mathbb{K}[X]$ the algebra of regular functions on X .

Suppose that X is an affine algebraic variety. A derivation ∂ of $\mathbb{K}[X]$ is called *locally nilpotent* (LND for short) if for every $f \in \mathbb{K}[X]$ there is $n \in \mathbb{Z}_{\geq 0}$ such that $\partial^n(f) = 0$. Every LND ∂ on $\mathbb{K}[X]$ defines a \mathbb{G}_a -action on $\mathbb{K}[X]$ via the formula $(t, f) \mapsto \exp(t\partial)f$ ($t \in \mathbb{K}$, $f \in \mathbb{K}[X]$) and hence induces a \mathbb{G}_a -subgroup in $\text{Aut}(X)$. By [Fre, § 1.5] this yields a bijection between LNDs on $\mathbb{K}[X]$ modulo proportionality and \mathbb{G}_a -subgroups on X . Moreover, if X is equipped with an action of an algebraic group F , then it is easy to see that ∂ is F -semiinvariant if and only if the corresponding \mathbb{G}_a -subgroup is F -normalized (and hence an F -root subgroup).

Below we shall need the following elementary facts.

Lemma 1. *Let ∂ be an LND on an algebra A over \mathbb{Q} without zero divisors. Let $a, b \in A$ such that $ab \in \text{Ker}(\partial)$, then either $a \in \text{Ker}(\partial)$ or $b \in \text{Ker}(\partial)$.*

Proof. Let n, m be the minimal numbers such that $\partial^{n+1}(a) = \partial^{m+1}(b) = 0$. Then by Leibnitz rule we get

$$\partial^{n+m}(ab) = \sum_{k=0}^{n+m} \binom{n+m}{k} \partial^k(a) \partial^{n+m-k}(b) = \binom{n+m}{n} \partial^n(a) \partial^m(b) = 0.$$

This implies the vanishing of either $\partial^n(a)$ or $\partial^m(b)$, which contradicts the assumption. \square

Corollary 2. *Let ∂ be an LND on an algebra A over \mathbb{Q} and let $f \in A$ be an invertible element. Then $f \in \text{Ker}(\partial)$.*

Lemma 2. *Let X be an irreducible affine variety, let $f \in \mathbb{K}[X]$, and consider the principal open subset $X_f = \{x \in X \mid f(x) \neq 0\}$. For every \mathbb{G}_a -subgroup R on X_f there exists a \mathbb{G}_a -subgroup \tilde{R} on X that preserves X_f and has on it the same orbits as R .*

Proof. Let ∂ be an LND on $\mathbb{K}[X_f]$ corresponding to R , and let $\{f_i\}$ be a finite generating set of the algebra $\mathbb{K}[X]$. Then $\partial(f_i) = h_i/f^{n_i}$ for some $h_i \in \mathbb{K}[X]$ and $n_i \in \mathbb{Z}_{\geq 0}$. Put $N = \max\{n_i\}$. By Corollary 2 we have $\partial(f) = 0$, so $\tilde{\partial} = f^N\partial$ is also an LND on $\mathbb{K}[X_f]$. Let \tilde{R} be the \mathbb{G}_a -subgroup on X_f corresponding to $\tilde{\partial}$. Clearly, $\tilde{\partial}(f_i) \in \mathbb{K}[X]$ for all i , so by the Leibnitz rule we have $\tilde{\partial}(\mathbb{K}[X]) \subset \mathbb{K}[X]$. It follows that $\tilde{\partial}$ is an LND on $\mathbb{K}[X]$, hence the action of \tilde{R} extends to the whole X .

Let us show that for every point $x_0 \in X_f$ the orbits $\tilde{R}x_0$ and Rx_0 coincide. Let $\mathfrak{m} \subset \mathbb{K}[X]$ be the maximal ideal of x_0 . Taking into account that $f = f(x_0) \bmod \mathfrak{m}$, for the rescaled parameterization $\tilde{t} = f(x_0)^N t$ we have

$$\begin{aligned} \exp(t\tilde{\partial}) \exp(-\tilde{t}\partial)\mathfrak{m} &= \sum_{i=0} \frac{t^i(f^N\partial)^i}{i!} (\exp(-\tilde{t}\partial)\mathfrak{m}) = \\ &= \sum_{i=0} \frac{(tf^N)^i \partial^i}{i!} \exp(-\tilde{t}\partial)\mathfrak{m} \subset \sum_{i=0} \frac{(tf^N(x_0))^i \partial^i}{i!} (\exp(-\tilde{t}\partial)\mathfrak{m}) + \mathfrak{m} \subset \mathfrak{m}. \end{aligned}$$

Since both ideals are maximal, the last inclusion is an equality, which implies $\exp(t\tilde{\partial})x_0 = \exp(\tilde{t}\partial)x_0$ for all $t \in \mathbb{K}$. \square

Every irreducible quasiaffine spherical G -variety X naturally embeds in the affine spherical variety $\tilde{X} = \text{Spec } \mathbb{K}[X]$ with closed G -stable boundary $\tilde{X} \setminus X$ of codimension at least two. In view of this, in what follows we shall work with affine spherical varieties X together with a fixed closed G -stable boundary subset ∂X keeping in mind the possibility to extend the results to the open quasiaffine subvariety $X \setminus \partial X$.

Lemma 3. *Let X be an irreducible affine G -variety and let $\partial X \subset X$ be a closed G -stable subvariety. Let Y be a G -orbit not contained in ∂X and let $D_Y \subset Y$ be a proper closed B -stable subvariety. Then there is a B -semiinvariant function $f \in \mathbb{K}[X]$ vanishing on $\partial X \cup D_Y$ and not vanishing identically on Y .*

Proof. Let \bar{Y} be the closure of Y in X and let $\pi: \mathbb{K}[X] \rightarrow \mathbb{K}[\bar{Y}]$ be the natural surjective map given by restricting functions. Let $I_{\partial X} \subset \mathbb{K}[X]$ be the ideal defining ∂X . Similarly, let $I_{D_Y} \subset \mathbb{K}[\bar{Y}]$ be the ideal defining D_Y . Since Y is not contained in the union of ∂X and D_Y , there is a function $F \in \mathbb{K}[X]$ that vanishes on $\partial X \cup D_Y$ and not on Y , so that $F \in I_{\partial X}$ and $\pi(F) \in I_{D_Y} \setminus \{0\}$. By the Lie–Kolchin theorem we may assume that $\pi(F)$ is B -semiinvariant. Since G is reductive, there is a G -module decomposition $\langle GF \rangle = \text{Ker } \pi|_{\langle GF \rangle} \oplus V$ for some G -submodule $V \subset \langle GF \rangle$. Consider the expression $F = f_0 + f$ with respect to this decomposition. Clearly, π maps V isomorphically to its image $\pi(V) = \pi(\langle GF \rangle)$, so f is also B -semiinvariant and hence has all the desired properties. \square

3. LOCAL STRUCTURE THEOREMS

Now let us recall some versions of the Local Structure Theorem. We will use the form discovered by F. Knop ([Kno]) in slightly less general formulation since we work only with affine varieties.

Let X be a normal irreducible affine G -variety.

Consider a nonzero B -semiinvariant function $f \in k[X]$ and let D be its divisor of zeros. Let P be the stabilizer in G of the line spanned by f . Then P is a parabolic subgroup in G containing B . Consider also one more parabolic subgroup $P(X) \supset B$ defined as the stabilizer of a generic B -orbit in X .

Consider the following P -equivariant map:

$$\psi: X \setminus D \rightarrow \mathfrak{g}^*, x \mapsto l_x, \text{ where } l_x(\xi) = \frac{\xi f}{f}(x).$$

Let us recall the version of the local structure theorem obtained by Knop.

Theorem 2 ([Kno], Thm. 2.3, Prop. 2.4). *For some $x_0 \in X \setminus D$ put $l := \psi(x_0)$, $L := G_l$, and $Z := \psi^{-1}(l)$. Then the following assertions hold.*

- (a) *The image of ψ is a single P -orbit isomorphic to P_u .*
- (b) *L is a Levi subgroup of P and there is an isomorphism*

$$P *_L Z \rightarrow X \setminus D \simeq P_u \times Z.$$

- (c) *Assume that $P = P(X)$. Then the kernel L_0 of the action of L on Z contains the derived subgroup of L .*

From the conjugacy of maximal tori in G it follows that x_0 can be chosen so that $L \supset T$. In the situation of (c), we see that the torus $A = L/L_0 = P/(L_0 P_u)$ acts effectively on Z .

To get a relative version of the local structure theorem (see Brion, Luna, Vust [BLV]), let Y be a G -orbit in X , denote by \bar{Y} its closure in X , let $D_Y \subset Y$ be a proper closed B -stable subvariety such that $P(Y) = P(Y \setminus D_Y)$, and let ∂X be a closed G -stable subvariety in X containing $\bar{Y} \setminus Y$. By Lemma 3 there is a B -semiinvariant function $f \in \mathbb{K}[X]$ vanishing on $\partial X \cup D_Y$ and not vanishing identically on Y . Let D be the divisor of zeros of f , put $X_f = X \setminus D$ and let P be the stabilizer in G of the line spanned by f . We now apply Theorem 2 to f and a point $x_0 \in Y \setminus D$; consider the resulting Levi subgroup $L \subset P$ and the closed L -stable subvariety $Z = \psi^{-1}(\psi(x_0)) \subset X_f$.

Theorem 3. *The following assertions hold.*

- (a) *The variety $X_f \simeq P *_L Z \simeq P_u \times Z$ is affine.*
- (b) *The subsets $Y \setminus D$ and $Z_Y = Z \cap Y$ are closed in X_f and $Y \setminus D \simeq P_u \times A \times C$ where $A = L/L_0$ is a torus and C is a closed subvariety in Z on which L_0 acts trivially.*

Corollary 3. *Suppose that Y is spherical. Then $Y \setminus D \simeq P_u \times A$ and $Z_Y \simeq A$.*

Corollary 4. *Suppose that X is spherical. Then Z is spherical as an L -variety and Z_Y is the unique closed L -orbit in Z .*

Proof. Since X is spherical, it contains an open B -orbit Bz_0 . The subset $X_f \simeq P_u \times Z$ being open in X , the point z_0 can be taken from Z . Then the open set Bz_0 is isomorphic to $P_u \times (Z \cap Bz_0)$. Using the semidirect product decomposition $B = P_u \rtimes B_L$ (where $B_L = B \cap L$) we get $Z \cap Bz_0 \simeq B_L z_0$, which is dense in Z . This implies both claims. \square

Proposition 1. *Let Y be a spherical G -orbit in X^{reg} , let $z \in Z_Y$ be a point, and let N_z be the normal space to Y at z . Assume that either Z has a unique closed L_0 -orbit or $\mathbb{K}[N_z]^{L_0} = \mathbb{K}$. Then $Z \simeq L *__{L_0} N_z$.*

Proof. By construction, Z_Y is a closed subset of the affine variety Z . The sphericity of Y implies that Z_Y is isomorphic to A , which yields the required result by the Luna slice theorem; see [PV, Theorem 6.7, 6.8]. \square

4. MAIN RESULTS

Proposition 2. *Retain the assumptions of Proposition 1. Then there exists a B -root subgroup on X_f that moves Y .*

Proof. Corollary 3 yields $Z_Y \simeq A$. By Proposition 1 we have $Z \simeq L *_{L_0} N_z$. Since $X_f \simeq P *_L Z \simeq P_u \times Z$, the variety X_f is isomorphic to the homogeneous vector bundle $\mathcal{N} = P *_{L_0} N_z$ over P/L_0 . Since $P/L_0 \simeq P_u \times A$ is affine, it follows that the space of global sections $H^0(P/L_0, \mathcal{N})$ is nonzero, hence by the Lie–Kolchin theorem it contains a nonzero B -semiinvariant section s . Then s defines a B -root subgroup on \mathcal{N} by the formula $(t; v_y) \mapsto v_y + ts_y$ where $v_y \in \mathcal{N}_y$ for $y \in P/L_0 \subset Y$, s_y is the value of s at y , and $t \in \mathbb{K}$. By Lemma 2 we can extend the constructed action to the whole X fixing ∂X . \square

Remark 1. Since the variety P/L_0 is affine, the vector bundle \mathcal{N} is generated by global sections, that is, for any vector $v \in N_y$ one can find a section $s \in H^0(P/L_0, \mathcal{N})$ such that $s_y = v$. This shows that by a \mathbb{G}_a -action that is not necessary B -normalized one can connect the spherical orbit Y (satisfying $\mathbb{K}[N_z]^{L_0} = \mathbb{K}$) with any G -orbit that contains Y in its closure.

When we are interested in B -root subgroups on a spherical variety, the situation becomes more rigid.

Theorem 4. *Let X be an affine spherical G -variety and let $\partial X \subset X$ be a closed G -stable subvariety. Let Y be a G -orbit contained in $X^{\text{reg}} \setminus \partial X$. Then for every minimal G -orbit $Y' \neq Y$ with the property $\overline{Y'} \supset Y$ there exists a B -root subgroup on X that preserves the boundary ∂X and connects Y with Y' .*

Proof. Since B has an open orbit in X , it follows that the group $B_{L_0} = L_0 \cap B$ has an open orbit in N_z . This implies that N_z is a spherical L_0 -module and hence has a finite number of B_{L_0} -orbits. Let U_L be the unipotent radical of B_L and also of B_{L_0} . Then the action of B_{L_0} on $N_z^{U_L}$, which reduces to an action of the torus $T_{L_0} = B_{L_0}/U_L$, has a finite number of orbits. The latter implies that the T_{L_0} -weights of $N_z^{U_L}$ (which are the highest weights of N_z as an L_0 -module) are multiplicity free and linearly independent.

Since there are a finite number of L_0 -orbits on N_z , they are all stable under homotheties, so the nonzero orbits are in bijection with L_0 -orbits in the projective space $\mathbb{P}(N_z)$. In particular, the minimal nonzero L_0 -orbits in N_z are in bijection with the closed L_0 -orbits in $\mathbb{P}(N_z)$. By the Borel fixed point theorem, the latter are in bijection with B_{L_0} -fixed points in $\mathbb{P}(N_z)$, which correspond to highest-weight vectors in N_z regarded as an L_0 -module. Given a highest-weight vector $v \in N_z$, consider the corresponding B -semiinvariant section s with $s_z = v$. The fiberwise translation by s , which defines a B -root subgroup on X_f takes the zero section (which corresponds to Y) to a dense subset in $B *_{B_{L_0}} \mathbb{K}v \subset \mathcal{N}$, which corresponds to the minimal G -orbit that properly contains Y in its closure. \square

The next theorem provides a partial converse to Theorem 4.

Theorem 5. *Let X be a quasiaffine spherical G -variety and let R be a B -root subgroup on X . Let $Y \subset X^{\text{reg}}$ be a G -orbit such that all B -stable prime divisors in X not containing Y are R -stable. Suppose that R connects Y with a G -orbit Y' such that $Y \subset \overline{Y'}$. Then Y' is minimal with the property $Y \subset \overline{Y'}$ and $RY = Y \cup Y'$.*

Proof. It follows from the hypothesis that the vanishing set of the function f from Theorem 3 is R -stable, so the action of R can be restricted to X_f . Since R commutes with U , it permutes P_u -orbits, so there exists an R -equivariant quotient map $\pi_Z: X_f \rightarrow Z$ as well as the restriction of the action of R to $\mathbb{K}[X_f]^{P_u} = \mathbb{K}[Z]$. In particular, a G -orbit $Y' \supset Y$ can be connected with Y via R if and only if $Y' \cap Z$ can be connected with Z_Y via the action of R on Z . The action of R on Z still commutes with U_L . This implies that R preserves the U_L -fixed point set $Z^{U_L} \simeq L *_{L_0} N_z^{U_L} \simeq T *_{T_0} N_z^{U_L}$. Since $Z_Y \subset Z^{U_L}$, the set $Y' \cap Z$ can be connected with Z_Y by R only if $Y' \cap Z \subset Z^{U_L}$. Observe that Z^{U_L} is a toric T -variety and the restriction of R to Z^{U_L} is a T -root subgroup. It is well known that in this case any pair of T -orbits $O_1, O_2 \subset Z^{U_L}$ connected by R satisfy $\dim O_1 - \dim O_2 = \pm 1$. Then every G -orbit Y' satisfying $Y \subset \overline{Y'}$ and connected with Y via R satisfies $\dim(Y' \cap Z) = \dim Z_Y + 1$, hence Y' is minimal among all G -orbits properly containing Y in their closure. \square

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