

# Estimating order scale parameters of two scale mixture of exponential distributions

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## Abstract

Estimation of the ordered scale parameter of a two scale mixture of the exponential distribution is considered under Stein loss and symmetric loss. Under certain conditions, we prove that the inadmissibility equivariant estimator exhibits several improved estimators. Consequently, we propose various estimators that dominate the best affine equivariant estimators (BAEE). Also, we propose a class of estimators that dominates BAEE. We have proved that the boundary estimator of this class is a generalized Bayes estimator. The results are applied to the multivariate Lomax distribution and the Exponential Inverse Gaussian (E-IG) distribution. Consequently, we have obtained improved estimators for the ordered scale parameters of two multivariate Lomax distributions and the exponential inverse Gaussian distribution. For each case, we have conducted a simulation study to compare the risk performance of the improved estimators.

**Keywords:** Stein-type estimators; Best affine equivariant estimators; Generalized Bayes; Scale invariant loss function; Multivariate Lomax distribution; E-IG distribution; Relative risk improvement.

**Mathematics Subject Classification** 62C99 · 62F10 · 62H12

## 1 Introduction

The problem of estimating the ordered scale (location) parameter of two or more distributions has been extensively studied in the literature because of its applications in various areas such as medical research, reliability engineering, agricultural studies, economics, etc. Some examples where the ordering of the parameter arises naturally are as follows.

- (i) It is natural to assume that the average yield of a certain crop is higher when using fertilizer than when no fertilizer is used.
- (ii) It is expected that the waiting time in a ticket counter of a popular railway station is higher than that of an unpopular railway station.
- (iii) In reliability engineering, the lifetimes of machine components operate under normal operating conditions typically last longer than those tested operate in high-stress conditions, leading to the natural order restriction on the rate of failure.

For some more examples and a detailed review, we refer to [37] and [40]. We also refer to [30] and [39] for more applications of estimation under order restriction. In the literature, several authors have studied the estimation of location and scale parameters under ordered restrictions for various

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probability distributions. For some initial work in this direction, we refer to [4], [9], [2], [20], [21], [22] and reference therein. This manuscript will investigate component-wise estimation of the ordered scale parameter of a two-scale mixture of exponential distributions. Let  $X_1, X_2, \dots, X_n$  have the joint density

$$f(x_1, x_2, \dots, x_n; \mu, \sigma) = \int_0^\infty \frac{\tau^n}{\sigma \tau^{p_1}} e^{-\frac{\tau}{\sigma} \sum_{i=1}^n (x_i - \mu)} I_{(\mu, \infty)}(x_{(1)}) dH(\tau) \quad (1)$$

where  $x_{(1)} = \min\{x_1, x_2, \dots, x_n\}$ ,  $I_{(a,b)}(\cdot)$  is the indicator function and  $\mu \in \mathbb{R}$ ,  $\sigma > 0$  are unknown. The interpretation of the equation (1) is, given  $\tau > 0$ ,  $X_1, X_2, \dots, X_n$  are i.i.d. exponential random variables with location parameter  $\mu$  and scale parameter  $\sigma/\tau$  that is  $Exp(\mu, \sigma/\tau)$ . This model is named a scale mixture of exponential distributions. It was originally introduced by [24] in assessing the reliability of a parallel and series system in terms of their component reliabilities. The authors have argued that the distribution of the lifetimes of  $n$  independent components will not be independent when these are working in the same laboratory environment. The common laboratory environment may influence the lifetime of the components. To address this issue, [24] proposes a change of scale parameter  $\sigma$  to  $\sigma/\tau$ . Here  $\tau$  is an unknown quantity whose uncertainty is measured by the distribution function  $H(\cdot)$ . For improved estimation of scale parameter and function scale parameter of a scale mixture of exponential distribution, we refer to [34], [32], and [35]. If  $\tau$  is degenerate at 1 then we get  $X_1, \dots, X_n$  are i.i.d. exponential distribution  $Exp(\mu, \sigma)$ .

Several authors have investigated estimating the ordered location and scale parameter of two or more exponential distributions. Improved estimation of the ordered location parameter of a two-exponential distribution under a squared error loss function has been investigated by [28]. They have proved the inadmissibility of standard estimators. [27] investigated component-wise estimation of the ordered location parameter of two exponential distributions with known scale parameters. They have proved that unrestricted minimum risk estimators are inadmissible. [41] studied the component-wise estimation of ordered scale and location parameters of several exponential distributions under a quadratic loss function. By proposing improved estimators, they proved that the unrestricted minimum risk equivariant estimator is inadmissible. They also proved that restricted MLE is inadmissible. Smooth improved estimators of the ordered scale parameter of the two gamma distributions are investigated by [26]. [8] discussed the estimation of the linear function of ordered scale parameters of two gamma distributions under the entropy loss function. They have given a necessary and sufficient condition under which MLE dominates the crude unbiased estimator. [16] dealt with estimating the ordered scale parameters of two exponential distributions with a common location parameter  $\mu$ . The authors obtained the UMVUE and showed that the restricted MLE dominates the usual MLE. Additionally, the authors proved an inadmissibility result as an application of the inadmissibility result they obtained in classes of scale and affine equivariant estimators. [33] investigated the estimation of ordered scale parameters of two multivariate Lomax distributions with unknown locations under a quadratic loss function. By proposing various improved estimators, the author showed that the best equivariant estimators are not admissible. [29] studied the component-wise estimation of the ordered scale parameter of two exponential distributions under a general scale-invariant loss function. They have proved the inadmissibility of the best affine equivariant estimators and restricted MLE by proposing several dominating estimators. For some more recent works on estimation of ordered parameters of exponential distributions, we refer to [6], [18], [17], and [1].

In the literature attention has not been given to the estimation ordered scale parameter scale mixture of exponential distributions. In this work we study component wise estimation of ordered scale parameter two scale mixture exponential distribution. In this article, we consider model: for a given  $\tau > 0$ , let  $X_1, X_2, \dots, X_{p_1}$  and  $Y_1, Y_2, \dots, Y_{p_2}$  be a random samples taken from the exponential distribution  $Exp(\mu_1, \frac{\sigma_1}{\tau})$  and  $Exp(\mu_2, \frac{\sigma_2}{\tau})$  respectively where  $-\infty < \mu_1, \mu_2 < \infty$  and  $0 < \sigma_1 \leq \sigma_2$  are unknown parameters. We assume that the mixing parameter  $\tau$  have a distribution function  $H(\cdot)$ . The complete and sufficient statistic is  $(S_1, X, S_2, Y)$  (see [23]), where  $S_1 = \sum_{i=1}^{p_1} (X_i - X_{(1)})$ ,  $X \equiv X_{(1)} = \min(X_1, X_2, \dots, X_{p_1})$ ,  $S_2 = \sum_{i=1}^{p_2} (Y_i - Y_{(1)})$  and  $Y \equiv Y_{(1)} = \min(Y_1, Y_2, \dots, Y_{p_2})$ .

Given  $\tau > 0$ , the statistics  $S_1$ ,  $S_2$ ,  $X$  and  $Y$  are independent random variables distributed as,

$$\begin{aligned} S_1|\tau &\sim \text{Gamma}(p_1 - 1, \sigma_1/\tau), & X|\tau &\sim \mathcal{E}(\mu_1, \sigma_1/(p_1\tau)) \\ S_2|\tau &\sim \text{Gamma}(p_2 - 1, \sigma_2/\tau), & Y|\tau &\sim \mathcal{E}(\mu_2, \sigma_2/(p_2\tau)) \end{aligned} \quad (2)$$

We consider the estimation of  $\sigma_1$  and  $\sigma_2$  with respect to the loss function

$$\begin{aligned} \text{Symmetric loss: } L_1(\delta_i, \sigma_i) &= \frac{\delta_i}{\sigma_i} + \frac{\sigma_i}{\delta_i} - 2 \\ \text{Stein's loss: } L_2(\delta_i, \sigma_i) &= \frac{\delta_i}{\sigma_i} - \ln\left(\frac{\delta_i}{\sigma_i}\right) - 1 \end{aligned}$$

where  $\delta_i$  is an estimator of  $\sigma_i$ . Our aim is to find various estimators of  $\sigma_i$ ,  $i = 1, 2$  which dominants BAEF under the order restriction  $\sigma_1 \leq \sigma_2$ . The main contribution of this article is as follows.

- (i) We obtain BAEF of  $\sigma_i$ ,  $i = 1, 2$  with respect to the loss function  $L_1(\cdot)$  and  $L_2(\cdot)$ . Our goal is to propose estimators that dominate BAEF when  $\sigma_1 \leq \sigma_2$ . We have derived several estimators whose risk is uniformly smaller than BAEF. Further, we have obtained a class of improved estimators and also shown that the boundary estimator of this class is generalized Bayes.
- (ii) As an application, we study the estimation of ordered scale parameters of the multivariate Lomax distribution and the Exponential Inverse Gaussian distribution.
- (iii) To compare the risk performance of the proposed estimators, we have conducted a detailed simulation study. The relative risk improvement of the proposed estimators with respect to BAEF has been tabulated.

At first, we will derive the best affine equivariant estimator of  $\sigma_i$ . For this, we invoke the principle of invariance and consider the affine group of transformations  $G_{a_i, b_i} = \{g_{a_i, b_i}(x) = a_i x + b_i, j = 1, \dots, p_i\}$ ,  $i = 1, 2$ . After some simplification, the form of the affine equivariant estimator is obtained as  $cS_i$ , where  $c$  is a positive constant.

**Lemma 1.** (i) Under the  $L_1(\cdot)$  loss function, for  $i = 1, 2$ , the BAEF of  $\sigma_i$  is  $\delta_{1i} = c_i S_i$  with

$$c_i = \sqrt{\frac{E(\tau)}{(p_i - 1)(p_i - 2)E(1/\tau)}}.$$

(ii) Under the  $L_2(\cdot)$  loss function and for  $i = 1, 2$ , the BAEF of  $\sigma_i$  is  $\delta_{2i} = d_i S_i$  with  $d_i = \frac{1}{(p_i - 1)E(1/\tau)}$ .

**Remark 1.** The class of estimators  $\mathcal{D}_1$  includes the standard estimators of  $\sigma_i$ , (for  $i = 1, 2$ ), in particular the uniformly minimum variance estimator (UMVUE),  $\frac{E(1/\tau)}{p_i - 1} S_i$  and the maximum likelihood estimation (MLE),  $\frac{E(\tau)}{p_i - 1} S_i$ .

**Remark 2.** If  $\tau = 1$  with probability one, then the BAEF of  $\sigma_i$  under  $L_1(\cdot)$  and  $L_2(\cdot)$  is coincides with the BAEF of  $\sigma_i$  which was derived [29] in Example 2.1.

**Remark 3.** When  $\tau = 1$  with probability one, then the MLE of  $\sigma_1$  is coincide with the BAEF and UMVUE of  $\sigma_1$  under the entropy loss function which was previously obtained by [29] in Example 2.1 and Remark 2.2 respectively.

We will use the following lemma of to prove our results. For the sake of completeness we state it bellow.

**Lemma 2** ([5]). Let  $f_1(x)$  and  $f_2(x)$  be the densities supported on the domain set  $\Omega_1$  and  $\Omega_2$  respectively, where  $\Omega_1 \subset \Omega_2$  and the ration  $f_1(x)/f_2(x)$  is nondecreasing in  $x \in \Omega_1$ . If  $X$  is a random variable having density  $f_1(x)$  or  $f_2(x)$  and also  $a(x)$ ,  $x \in \Omega_1$ , is nondecreasing (nonincreasing) then  $E_{f_1} a(X) \geq (\leq) E_{f_2} a(X)$ . Moreover, if  $a(x)$  and the ration  $f_1(x)/a_2(x)$  are strictly monotone, then  $E_{f_1} a(X) > (<) E_{f_2} a(X)$ .

In this paper, estimation of the scale parameter is considered from two mixture model with an unknown scale parameter. In Section 2 of this paper, we used the [38] technique to derive a class of improved estimators of scale parameter  $\sigma_1$  under two scale invariant loss functions  $L_1(\cdot)$  and  $L_2(\cdot)$ . Also we have derive a class of estimators improving upon the BAEE by using the [19] IERD approach. In Section 3 the estimation of the parameter  $\sigma_2$  is considered. Here we have used the [19] IERD approach to find derive a class of improved estimators of  $\sigma_2$  under two scale invariant loss function  $L_1(\cdot)$  and  $L_2(\cdot)$  respectively. Moreover we have obtain a double shrinkage estimator of the scale parameter  $\sigma_2$  by using the technique of [15] under the loss function  $L_1(\cdot)$  and  $L_2(\cdot)$ . As an application, the results obtained in Section 2, 3, are used to derive the improved estimators for scale mixture of exponential distributions and in particular for multivariate Lomax distribution (see in Subsection 4.1) and Exponential-Inverse Gaussian (E-IG) distribution (see Subsection 4.3). These are all presented in Section 4.

## 2 Improved estimation for $\sigma_1$ when $\sigma_1 \leq \sigma_2$

This section will consider the improved estimation of  $\sigma_1$  when  $\sigma_1 \leq \sigma_2$ . To use the constraints  $\sigma_1 \leq \sigma_2$ , we utilize the second sample and consider a class of estimators of the form

$$\mathcal{D}_1 = \left\{ \delta_{\varphi_1} = \varphi_1(W) S_1; \quad W = \frac{S_2}{S_1}, \quad \varphi_1(\cdot) \text{ is a positive function} \right\} \quad (3)$$

We can observe that the BAEE lies in this class. In the next theorem, we find an improved estimator that dominates the  $\delta_{\varphi_1}$ . As an application of this theorem, we derive an estimator that dominates the BAEE  $\delta_{j1}$  for  $j = 1, 2$

**Theorem 1.** (i) Under symmetric loss function  $L_1(\cdot)$ , the risk of the estimator

$$\delta_{1S1}^1(X, S) = \min \{ \varphi_1(W), \varphi_{11}(W) \} S_1,$$

is nowhere larger than that of the estimator  $\delta_{\varphi_1}$  provided  $P(\varphi_{11}(W) < \varphi_1(W)) > 0$ , where

$$\varphi_{11}(W) = \frac{(1+W)\sqrt{E(\tau)}}{\sqrt{(p_1+p_2-2)(p_1+p_2-3)E(1/\tau)}}.$$

(ii) For the Stein loss function  $L_2(\cdot)$ , the risk of the estimator

$$\delta_{1S1}^2(X, S) = \min \{ \varphi_1(W), \varphi_{12}(W) \} S_1,$$

is nowhere larger than that of the estimator  $\delta_{\varphi_1}$  provided  $P(\varphi_{12}(W) < \varphi_1(W)) > 0$ , where

$$\varphi_{12}(W) = \frac{(1+W)}{(p_1+p_2-2)E(\frac{1}{\tau})}.$$

**Proof: (i)** We can easily seen that the risk function of the estimator of the form (3) depend on the unknown parameter through  $\eta = \frac{\sigma_1}{\sigma_2} \leq 1$  and it can be written as

$$R(\delta_{\varphi_1}; \eta) = E \left\{ E \left[ \left( \varphi_1(W) S_1 / \sigma_1 \right) + \frac{1}{(\varphi_1(W) S_1 / \sigma_1)} - 2 \middle| W \right] \right\}$$

The inner conditional risk is minimized at

$$\varphi_1(w; \eta) = \left( \frac{E(1/V_1 | W = w)}{E(V_1 | W = w)} \right)^{\frac{1}{2}}. \quad (4)$$

We can observe that the conditional distribution of  $V_1 = \frac{S_1}{\sigma_1}$ , given  $W = w > 0$  and  $T = \tau > 0$  is proportional to

$$v_1^{p_1+p_2-3} e^{-\tau v_1(1+w\eta)} \tau^{p_1+p_2-2}, \quad w > 0, \quad v_1 > 0,$$

and we have

$$\varphi_1(w; \eta) = \left( \frac{\int_0^\infty \int_0^\infty v_1^{p_1+p_2-4} e^{-\tau v_1(1+w\eta)} \tau^{p_1+p_2-2} dv_1 dH(\tau)}{\int_0^\infty \int_0^\infty v_1^{p_1+p_2-2} e^{-\tau v_1(1+w\eta)} \tau^{p_1+p_2-2} dv_1 dH(\tau)} \right)^{\frac{1}{2}}$$

Using the transformation  $z_1 = \tau v_1(1+w\eta)$ , we obtain equation (2) in the following form,

$$\begin{aligned} \varphi_1(w; \eta) &= (1+w\eta) \left( \frac{\int_0^\infty \tau \int_0^\infty z_1^{p_1+p_2-4} e^{-z_1} dz_1 dH(\tau)}{\int_0^\infty \frac{1}{\tau} \int_0^\infty z_1^{p_1+p_2-2} e^{-z_1} dz_1 dH(\tau)} \right)^{\frac{1}{2}} = (1+w\eta) \left( \frac{E(\tau) (E(1/\tau))^{-1}}{(p_1+p_2-2)(p_1+p_2-3)} \right)^{\frac{1}{2}} \\ &\leq (1+w) \left( \frac{E(\tau) (E(1/\tau))^{-1}}{(p_1+p_2-2)(p_1+p_2-3)} \right)^{\frac{1}{2}} = \varphi_{11}(w) \end{aligned}$$

Now using the convexity of  $R(\delta_{\varphi_1}; \eta)$  we get the result. Proof of (ii) similar to (i). So we omit it.

**Remark 4.** When  $\tau = 1$  with probability one, Theorem 1 reduces to the result which was previously derived by [29] in Example 2.3 and Example 2.4 under entropy and symmetric loss function respectively.

Now we aim to derive a class of improved estimators. For this purpose, we applied integral expression of risk difference (IERD) approach of [19]. In the following theorem we give sufficient conditions under which we obtain a class of improved estimators.

**Theorem 2.** (i) Suppose  $\varphi_1(u)$  be a function satisfies the following conditions:

- (a)  $\varphi_1(u)$  is non-decreasing in  $u$  and  $\lim_{u \rightarrow \infty} \varphi_1(u) = \left( \frac{E(\tau)}{(p_1-1)(p_1-2)E(1/\tau)} \right)^{1/2}$
- (b)  $\varphi_1(u) \geq \varphi_*^1(u)$ , where  $\varphi_*^1(u) = \left( \frac{E(\tau)B\left(\frac{u}{1+u}; p_1-2, p_2-1\right)}{E(1/\tau)(p_1+p_2-2)(p_1+p_2-3)B\left(\frac{u}{1+u}; p_1, p_2-1\right)} \right)^{1/2}$  with  $B(x; a, b)$  is the incomplete Beta function.

Then the estimator  $\delta_{\varphi_1}$  defined in (3) dominates  $\delta_{11}$  under the symmetric loss function  $L_1$ .

(ii) Let  $\varphi_1(u)$  satisfies the following conditions:

- (a)  $\varphi_1(u)$  is non-decreasing in  $u$  and  $\lim_{u \rightarrow \infty} \varphi_1(u) = \frac{1}{(p_1-1)E(1/\tau)}$
- (b)  $\varphi_1(u) \geq \varphi_*^2(u) = \frac{B\left(\frac{u}{1+u}; p_1-1, p_2-1\right)}{E(1/\tau)(p_1+p_2-2)B\left(\frac{u}{1+u}; p_1, p_2-1\right)}.$

Then the risk of  $\delta_{\varphi_1}$  is nowhere larger than that of  $\delta_{21}$  with respect to the Stein type loss function  $L_2(\cdot)$ .

**Proof:** We denote  $g_i$  and  $G_i$  is the density and cdf of the random variable  $V_i = \frac{\tau S_i}{\sigma_i}$  given  $\tau > 0$  respectively.

(i) The risk difference of the estimators  $\delta_{11}$  and  $\delta_{\varphi_1}$  is

$$\begin{aligned}\Delta_1 &= E \left[ \int_1^\infty \left( \varphi'_1 \left( \frac{1}{\eta} \frac{v_2}{v_1} t \right) \frac{1}{\eta} v_2 - \frac{\varphi'_1 \left( \frac{1}{\eta} \frac{v_2}{v_1} t \right) \frac{1}{\eta} \frac{v_2}{v_1}}{\varphi_1^2 \left( \frac{1}{\eta} \frac{v_2}{v_1} t \right) v_1} \right) dt \right] \\ &\geq E \left[ \int_1^\infty \left( v_2 - \frac{v_2}{v_1^2 \varphi_1^2 \left( t \frac{v_2}{v_1} \right)} \right) \varphi'_1 \left( \frac{1}{\eta} \frac{v_2}{v_1} t \right) dt \right] \\ &= \int_0^\infty \int_0^\infty \int_0^\infty \int_1^\infty \left( v_2 - \frac{v_2}{v_1^2 \varphi_1^2 \left( t \frac{v_2}{v_1} \right)} \right) \varphi'_1 \left( \frac{1}{\eta} \frac{v_2}{v_1} t \right) \tau g_1(\tau v_1) \tau g_2(\tau v_2) dt dv_1 dv_2 dH(\tau)\end{aligned}$$

Making the transformation  $u = t \frac{v_2}{v_1}$  and  $x = \frac{uv_1}{t}$ , the we get

$$\begin{aligned}\Delta_1 &\geq \int_0^\infty \int_0^\infty \int_0^\infty \int_0^{uv_1} \left( 1 - \frac{1}{v_1^2 \varphi_1^2(u)} \right) v_1 \varphi'_1 \left( \frac{u}{\eta} \right) \tau g_1(\tau v_1) \tau g_2(\tau x) dx dv_1 du dH(\tau) \\ &= \int_0^\infty \varphi'_1 \left( \frac{u}{\eta} \right) \int_0^\infty \int_0^\infty \left( 1 - \frac{1}{v_1^2 \varphi_1^2(u)} \right) v_1 \tau g_1(\tau v_1) \int_1^{\tau uv_1} g_2(x) dx dv_1 dH(\tau) du\end{aligned}$$

Now the risk difference  $\Delta_1 \geq 0$  if

$$\varphi_1(u) \geq \left( \frac{\int_0^\infty \int_0^\infty \frac{1}{v_1} \tau g_1(\tau v_1) G_2(\tau uv_1) dv_1 dH(\tau)}{\int_0^\infty \int_0^\infty v_1 \tau g_1(\tau v_1) G_2(\tau uv_1) dv_1 dH(\tau)} \right)^{1/2}$$

Put  $\tau v_1 = y$ , we have

$$\varphi_1(u) \geq \left( \frac{\int_0^\infty \tau \int_0^\infty \frac{1}{y} g_1(y) G_2(yu) dy dH(\tau)}{\int_0^\infty \frac{1}{\tau} \int_0^\infty y g_1(y) G_2(yu) dy dH(\tau)} \right)^{1/2} = \left( \frac{E(\tau) \int_0^\infty y^{p_1-3} e^{-y} \int_0^{yu} x^{p_2-2} e^{-x} dx dy}{E(1/\tau) \int_0^\infty y^{p_1-1} e^{-y} \int_0^{yu} x^{p_2-2} e^{-x} dx dy} \right)^{1/2}$$

Take  $z = \frac{x}{yu}$ , we have

$$\begin{aligned}\varphi_1(u) &\geq \left( \frac{E(\tau) \int_0^\infty y^{p_1+p_2-4} \int_0^1 z^{p_2-2} e^{-y(1+zu)} dz dy}{E(1/\tau) \int_0^\infty y^{p_1+p_2-2} \int_0^1 z^{p_2-2} e^{-y(1+zu)} dz dy} \right)^{1/2} \\ &= \left( \frac{E(\tau) B \left( \frac{u}{1+u}; p_1 - 2, p_2 - 1 \right)}{E(1/\tau) (p_1 + p_2 - 2)(p_1 + p_2 - 3) B \left( \frac{u}{1+u}; p_1, p_2 - 1 \right)} \right)^{1/2} = \varphi_*^1(u).\end{aligned}$$

(ii) The proof for the Stein type loss function  $L_2(\cdot)$  is similar to (i). So we omit it.

**Remark 5.** The boundary estimator  $\delta_{\varphi_*^1}$  and  $\delta_{\varphi_*^2}$  are [7]-type estimator for  $\sigma_1$  under the loss functions  $L_1(\cdot)$  and  $L_2(\cdot)$  respectively.

**Remark 6.** Now we prove that  $\delta_{\varphi_*^1}$  is a generalized Bayes estimator of  $\sigma_1$  with respect to  $L_1(\cdot)$ . We consider the prior distribution as

$$\pi(\sigma_1, \sigma_2, \mu_1, \mu_2) = \frac{1}{\sigma_1 \sigma_2} I_{\sigma_1 \leq \sigma_2}.$$

The corresponding posterior distribution, for given  $\tau > 0$ , is proportional to

$$\pi(\sigma_1, \sigma_2, \mu_1, \mu_2 | X, S_1, Y, S_2) \propto \frac{\tau^{p_1+p_2-2}}{\sigma_1^{p_1} \sigma_2^{p_2}} e^{-\frac{\tau S_1}{\sigma_1} - \frac{\tau S_2}{\sigma_2}} \frac{p_1 \tau}{\sigma_1} e^{-\frac{p_1 \tau}{\sigma_1} (X - \mu_1)} \frac{p_2 \tau}{\sigma_2} e^{-\frac{p_2 \tau}{\sigma_2} (Y - \mu_2)}, \quad (5)$$

where  $\mu_1 \leq x$ ,  $\mu_2 \leq y$ ,  $0 < \sigma_1 \leq \sigma_2$ . For the symmetric loss function  $L_1(\cdot)$ , the generalized Bayes estimator of  $\sigma_1$  is obtained as follows

$$\delta_{1B}^1 = \left( \frac{E(\sigma_1 | X, S_1, Y, S_2)}{E\left(\frac{1}{\sigma_1} | X, S_1, Y, S_2\right)} \right)^{1/2},$$

where the expectation are taken with respect to the posterior distribution specified in the equation (5). After some calculation, it is found that the generalized Bayes estimator coincides with the estimator  $\delta_{\varphi_+^1}$ .

**Remark 7.** By using a similar argument, we can prove that the generalized Bayes estimator of  $\sigma_1$  under the entropy loss function  $L_2(\cdot)$ , with respect to the same prior distribution  $\pi(\sigma_1, \sigma_2, \mu_1, \mu_2)$ , also coincides with the  $\delta_{\varphi_+^2}$ .

When  $\tau = 1$  with probability one, then the Theorem 2 reduces to the following result which was previously derived by [29] in Corollary 2.11 and Corollary 2.10 respectively.

**Theorem 3.** (i) Suppose  $\varphi_1(u)$  be a function satisfies the following conditions:

- (a)  $\varphi_1(u)$  is non-decreasing in  $u$  and  $\lim_{u \rightarrow \infty} \varphi_1(u) = \left( \frac{1}{(p_1-1)(p_1-2)} \right)^{1/2}$
- (b)  $\varphi_1(u) \geq \varphi_*^1(u) = \left( \frac{B\left(\frac{u}{1+u}; p_1-2, p_2-1\right)}{(p_1+p_2-2)(p_1+p_2-3)B\left(\frac{u}{1+u}; p_1, p_2-1\right)} \right)^{1/2}$ .

Then the estimator  $\delta_{\varphi_1}$  defined in (3) dominates  $\delta_{11}$  under the symmetric loss function  $L_1(\cdot)$ .

(ii) Let  $\varphi_1(u)$  satisfies the following conditions:

- (a)  $\varphi_1(u)$  is non-decreasing in  $u$  and  $\lim_{u \rightarrow \infty} \varphi_1(u) = \frac{1}{(p_1-1)}$
- (b)  $\varphi_1(u) \geq \varphi_*^2(u) = \frac{B\left(\frac{u}{1+u}; p_1-1, p_2-1\right)}{(p_1+p_2-2)B\left(\frac{u}{1+u}; p_1, p_2-1\right)}$ .

Then the risk of  $\delta_{\varphi_1}$  is nowhere larger than that of  $\delta_{21}$  with respect to the Stein type loss function  $L_2(\cdot)$ .

Now we define a bigger class of estimators based on the statistics  $(S_1, S_2, X)$  of the form

$$\mathcal{D}_2 = \left\{ \delta_{\varphi_2} = \varphi_2(W, W_1) S_1; \quad W = \frac{S_2}{S_1}, \quad W_1 = \frac{X}{S_1} \right\} \quad (6)$$

In the next theorem we will propose an estimator that dominates the estimator  $\delta_{\varphi_2}$ . As a consequences of this theorem we will obtain an estimator that dominates BAEE of  $\sigma_1$ .

**Theorem 4.** (i) Under the  $L_1(\cdot)$  loss function, the risk of the estimator

$$\delta_{1S_2}^1 = \begin{cases} \min \{ \varphi_2(W, W_1), \varphi_{21}(W, W_1) \} S_1, & W_1 > 0 \\ \varphi_2(W, W_1) S_1, & \text{otherwise} \end{cases} \quad (7)$$

is nowhere larger than that of the estimator  $\delta_{\varphi_2}$  provide  $P(\varphi_2(W, W_1) > \varphi_{21}(W, W_1)) > 0$ , where

$$\varphi_{21}(W, W_1) = \frac{(1 + W + p_1 W_1)}{\sqrt{(p_1 + p_2 - 1)(p_1 + p_2 - 2)}} \min \left\{ \left( \frac{E(\tau^{p_1+p_2+1})}{E(\tau^{p_1+p_2-1})} \right)^{1/2}, \left( \frac{E(\tau)}{E(\tau^{-1})} \right)^{1/2} \right\}.$$

(ii) For the loss function  $L_2(\cdot)$ , the risk of the estimator

$$\delta_{1S2}^2 = \begin{cases} \min \{ \varphi_2 W, W_1 \}, \varphi_{22}(W, W_1) \} S_1, & W_1 > 0 \\ \varphi_2(W, W_1) S_1, & \text{otherwise} \end{cases} \quad (8)$$

is nowhere larger than that of the estimator  $\delta_{\varphi_2}$  provide  $P(\varphi_2(W, W_1) > \varphi_2^2(W, W_1)) > 0$ , where

$$\varphi_{22}(W, W_1) = \frac{(1 + W + p_1 W_1)}{p_1 + p_2 - 1} \min \left\{ \frac{E(\tau^{p_1+p_2})}{E(\tau^{p_1+p_2-1})}, \frac{1}{E(\tau)} \right\}.$$

**Proof: (i)** Under the loss function  $L_1(\cdot)$ , the risk of the estimator  $\delta_{\varphi_2}(W, W_1)$  can be expressed as

$$R(\delta_{\varphi_2}, \mu_1, \sigma_1, \sigma_2) = E^{W, W_1} E \left[ \left( \varphi_2(W, W_1) V_1 + \frac{1}{\varphi_2(W, W_1) V_1} - 2 \right) \middle| W, W_1 \right]$$

For given  $\tau > 0$ , the conditional density of  $V_1$  given  $W = w, W_1 = w_1$  obtain as

$$f_{\eta, \rho}(v_1 | w, w_1) = \frac{v_1^{p_1+p_2-2} e^{-\tau v_1(1+w\eta+p_1 w_1)} e^{p_1 \tau \rho} \tau^{p_1+p_2-1}}{\int_0^\infty \int_{\frac{\rho}{w_1}}^\infty v_1^{p_1+p_2-2} e^{-\tau v_1(1+w\eta+p_1 w_1)} e^{p_1 \tau \rho} \tau^{p_1+p_2-1} dv_1 dH(\tau)}, \quad \max\{0, \frac{\rho}{w_1}\} \leq v_1 < \infty,$$

where  $\eta = \frac{\sigma_1}{\sigma_2} \leq 1, \rho = \frac{\mu_1}{\sigma_1} \in \mathbb{R}$ . It can be easily seen that the conditional risk function

$$R_1(\delta_{\varphi_2}, \eta, \rho) = E \left[ \left( \varphi_2(w, w_1) V_1 + \frac{1}{\varphi_2(w, w_1) V_1} - 2 \right) \middle| W = w, W_1 = w_1 \right]$$

is minimized at

$$\varphi_2(w, w_1; \eta, \rho) = \left( \frac{E[1/V_1 | W = w, W_1 = w_1]}{E[V_1 | W = w, W_1 = w_1]} \right)^{1/2}$$

Now we will consider two cases. First case, we consider  $\mu_1 > 0, w_1 > 0$ . In this case we have

$$\varphi_2(w, w_1; \eta, \rho) = \left( \frac{\int_0^\infty \int_{\frac{\rho}{w_1}}^\infty v_1^{p_1+p_2-3} e^{-\tau v_1(1+w\eta+p_1 w_1)} e^{p_1 \tau \rho} \tau^{p_1+p_2-1} dv_1 dH(\tau)}{\int_0^\infty \int_{\frac{\rho}{w_1}}^\infty v_1^{p_1+p_2-1} e^{-\tau v_1(1+w\eta+p_1 w_1)} e^{p_1 \tau \rho} \tau^{p_1+p_2-1} dv_1 dH(\tau)} \right)^{1/2}.$$

Now by taking the transformation  $z_1 = \tau v_1(1 + w\eta + p_1 w_1)$ , we obtain

$$\varphi_2(w, w_1; \eta, \rho) = (1 + w\eta + p_1 w_1) \left( \frac{\int_0^\infty \tau e^{p_1 \tau \rho} \int_\xi^\infty z_1^{p_1+p_2-3} e^{-z_1} dz_1 dH(\tau)}{\int_0^\infty 1/\tau e^{p_1 \tau \rho} \int_\xi^\infty z_1^{p_1+p_2-1} e^{-z_1} dz_1 dH(\tau)} \right)^{1/2}.$$

For given  $\tau > 0$ , we can easily seen that

$$\frac{\int_\xi^\infty z_1^{p_1+p_2-3} e^{-z_1} dz_1}{\int_\xi^\infty z_1^{p_1+p_2-1} e^{-z_1} dz_1} = E_\xi(Z_1^{-2}),$$

where  $Z_1$  has density  $g(z_1, \xi) \propto z_1^{p_1+p_2-1} e^{-z_1} I_{(\xi, \infty)}(z_1)$  with  $\xi = \frac{\tau \rho}{w_1}(1 + w\eta + p_1 w_1) > 0$ . For  $\xi > 0, \frac{g(z_1, \xi)}{g(z_1, 0)}$  is non-decreasing then we have  $E_\xi Z_1^{-2} \leq E_0 Z_1^{-2} = \frac{1}{(p_1+p_2-1)(p_1+p_2-2)}$  and hence we get  $\int_\xi^\infty z_1^{p_1+p_2-3} e^{-z_1} dz_1 \leq \frac{1}{(p_1+p_2-1)(p_1+p_2-2)} \int_\xi^\infty z_1^{p_1+p_2-1} e^{-z_1} dz_1$ . Now we obtain that

$$\varphi_2(w, w_1; \eta, \rho) \leq \frac{(1 + w\eta + p_1 w_1)}{\sqrt{(p_1 + p_2 - 1)(p_1 + p_2 - 2)}} \left( \frac{\int_0^\infty \tau e^{p_1 \tau \rho} \int_\xi^\infty z_1^{p_1+p_2-1} e^{-z_1} dz_1 dH(\tau)}{\int_0^\infty \frac{1}{\tau} e^{p_1 \tau \rho} \int_\xi^\infty z_1^{p_1+p_2-1} e^{-z_1} dz_1 dH(\tau)} \right)^{1/2}$$



Again we take the transformation  $z_1 = \tau x$  then we have

$$\begin{aligned}\varphi_2(w, w_1; \eta, \rho) &\leq \frac{(1 + w\eta + p_1 w_1)}{\sqrt{(p_1 + p_2 - 1)(p_1 + p_2 - 2)}} \left( \frac{\int_0^\infty \int_{\xi_1}^\infty x^{p_1+p_2-1} \tau^{p_1+p_2+1} e^{p_1 \tau \rho} e^{-\tau x} dx dH(\tau)}{\int_0^\infty \int_{\xi_1}^\infty x^{p_1+p_2-1} \tau^{p_1+p_2-1} e^{p_1 \tau \rho} e^{-\tau x} dx dH(\tau)} \right)^{1/2} \\ &= \frac{(1 + w\eta + p_1 w_1)}{\sqrt{(p_1 + p_2 - 1)(p_1 + p_2 - 2)}} \left( \frac{\int_0^\infty \int_{\xi_1}^\infty x^{p_1+p_2-1} \tau^{p_1+p_2+1} e^{-(x-p_1 \rho) \tau} dx dH(\tau)}{\int_0^\infty \int_{\xi_1}^\infty x^{p_1+p_2-1} \tau^{p_1+p_2-1} e^{-(x-p_1 \rho) \tau} dx dH(\tau)} \right)^{1/2}\end{aligned}\quad (9)$$

where,  $\xi_1 = \frac{\rho}{w_1}(1 + w\eta + p_1 w_1) > 0$ . Now for  $k = x - p_1 \rho > 0$ , we obtain

$$\frac{\int_0^\infty \tau^{p_1+p_2+1} e^{-k\tau} dH(\tau)}{\int_0^\infty \tau^{p_1+p_2-1} e^{-k\tau} dH(\tau)} = \int_0^\infty \tau^2 f_k(\tau) dH(\tau)$$

where  $f_k(\tau) \propto \tau^{p_1+p_2-1} e^{-k\tau}$ . But,  $\frac{f_k(\tau)}{f_0(\tau)}$  is decreasing in  $\tau$ , so by using Lemma (2) we have,

$$\int_0^\infty \tau^2 f_k(\tau) dH(\tau) \leq \int_0^\infty \tau^2 f_0(\tau) dH(\tau) = \frac{E(\tau^{p_1+p_2+1})}{E(\tau^{p_1+p_2-1})} \quad (10)$$

From (9) and (10) with  $\eta \leq 1$  we get

$$\varphi_2(w, w_1; \eta, \rho) \leq \frac{(1 + w + p_1 w_1)}{\sqrt{(p_1 + p_2 - 1)(p_1 + p_2 - 2)}} \left( \frac{E(\tau^{p_1+p_2+1})}{E(\tau^{p_1+p_2-1})} \right)^{1/2} \quad (11)$$

Now we consider the case  $\mu_1 \leq 0, w_1 > 0$ . In this case we have,

$$\varphi_2(w, w_1; \eta, \rho) = \left( \frac{\int_0^\infty \int_0^\infty v_1^{p_1+p_2-3} e^{-\tau v_1(1+w\eta+p_1 w_1)} e^{\rho p_1 \tau} \tau^{p_1+p_2-1} dv_1 dH(\tau)}{\int_0^\infty \int_0^\infty v_1^{p_1+p_2-1} e^{-\tau v_1(1+w\eta+p_1 w_1)} e^{\rho p_1 \tau} \tau^{p_1+p_2-1} dv_1 dH(\tau)} \right)^{1/2}$$

After using the transformation  $z_1 = \tau v_1(1 + w\eta + p_1 w_1)$  we have,

$$\begin{aligned}\varphi_2(w, w_1; \eta, \rho) &= (1 + w\eta + p_1 w_1) \left( \frac{\int_0^\infty \tau e^{p_1 \rho \tau} \int_0^\infty z^{p_1+p_2-3} e^{-z} dz dH(\tau)}{\int_0^\infty 1/\tau e^{p_1 \tau \rho} \int_0^\infty z^{p_1+p_2-1} e^{-z} dz dH(\tau)} \right)^{1/2} \\ &= \frac{(1 + w\eta + p_1 w_1)}{\sqrt{(p_1 + p_2 - 1)(p_1 + p_2 - 2)}} \left( \frac{\int_0^\infty \tau e^{p_1 \rho \tau} dH(\tau)}{\int_0^\infty 1/\tau e^{p_1 \tau \rho} dH(\tau)} \right)^{1/2} \\ &= \frac{(1 + w\eta + p_1 w_1)}{\sqrt{(p_1 + p_2 - 1)(p_1 + p_2 - 2)}} \left( \int_0^\infty \tau^2 f_\rho(\tau) dH(\tau) \right)^{1/2}\end{aligned}\quad (12)$$

where  $f_\rho(\tau) = \frac{\frac{1}{\tau} e^{n\tau\rho}}{\int_0^\infty \frac{1}{\tau} e^{\tau n\rho} dH(\tau)}$ . Set  $f_0(\tau) = \frac{\tau^{-1}}{\int_0^\infty \tau^{-1} dH(\tau)}$ . Now  $\frac{f_\rho(\tau)}{f_0(\tau)}$  is decreasing in  $\tau$ , then by Lemma (2) we have

$$\int_0^\infty \tau^2 f_\rho(\tau) dH(\tau) \leq \int_0^\infty \tau^2 f_0(\tau) dH(\tau) = \frac{E(\tau)}{E(\tau^{-1})}. \quad (13)$$

Hence from equation (12) and (13) we obtain

$$\varphi_2(w, w_1; \eta, \rho) \leq \frac{(1 + w + p_1 w_1)}{\sqrt{(p_1 + p_2 - 1)(p_1 + p_2 - 2)}} \left( \frac{E(\tau)}{E(\tau^{-1})} \right)^{1/2}. \quad (14)$$

Hence from the equations (14) and (11) we get,

$$\varphi_2(w, w_1; \eta, \rho) \leq \frac{(1 + w + p_1 w_1)}{\sqrt{(p_1 + p_2 - 1)(p_1 + p_2 - 2)}} \min \left\{ \left( \frac{E(\tau^{p_1 + p_2 + 1})}{E(\tau^{p_1 + p_2 - 1})} \right)^{1/2}, \left( \frac{E(\tau)}{E(\tau^{-1})} \right)^{1/2} \right\} = \varphi_{12}(w, w_1).$$

Now using the convexity of  $R_1(\delta_{\varphi_2}, \eta, \rho)$  for  $P(\varphi_{12}(W, W_1) < \varphi_2(W, W_1)) > 0$  we get the result. Proof of (ii) is similar to (i), so we omit it.

Now the Theorem 4 can be extended by using the information contained in the statistics  $Y$ , substituting  $Y$  for  $X$  within the class  $\mathcal{D}_2$  in (6). We get the following theorem

**Theorem 5.** (i) Under the  $L_1(\cdot)$  loss function, the risk of the estimator

$$\delta_{1S3}^1 = \begin{cases} \min \{ \varphi_3(W, W_2), \varphi_{31}(W, W_2) \} S_1, & W_2 > 0 \\ \varphi_3(W, W_2) S_1, & \text{otherwise} \end{cases} \quad (15)$$

is nowhere larger than that of the estimator  $\delta_{\varphi_3}$  provide  $P(\varphi_3(W, W_2) > \varphi_{31}(W, W_2)) > 0$  where  $W_2 = \frac{Y}{S_1}$  and

$$\varphi_{31}(W, W_2) = \frac{(1 + W + p_2 W_2)}{\sqrt{(p_1 + p_2 - 1)(p_1 + p_2 - 2)}} \min \left\{ \left( \frac{E(\tau^{p_1 + p_2 + 1})}{E(\tau^{p_1 + p_2 - 1})} \right)^{1/2}, \left( \frac{E(\tau)}{E(\tau^{-1})} \right)^{1/2} \right\}.$$

(ii) For the loss function  $L_2(\cdot)$ , the risk of the estimator

$$\delta_{1S3}^2 = \begin{cases} \min \{ \varphi_3(W, W_2), \varphi_{32}(W, W_2) \} S_1, & W_2 > 0 \\ \varphi_3(W, W_2) S_1, & \text{otherwise} \end{cases} \quad (16)$$

is nowhere larger than that of the estimator  $\delta_{\varphi_3}$  provide  $P(\varphi_3(W, W_2) > \varphi_{32}(W, W_2)) > 0$ , where

$$\varphi_{32}(W, W_2) = \frac{(1 + W + p_2 W_2)}{p_1 + p_2 - 1} \min \left\{ \frac{E(\tau^{p_1 + p_2})}{E(\tau^{p_1 + p_2 - 1})}, \frac{1}{E(\tau^{-1})} \right\}.$$

**Remark 8.** When  $\tau = 1$  with probability one, then the Theorem 4 and Theorem 5 reduces to the following result which was previously obtained by [29] in Example 2.3 and Example 2.4. under entropy and symmetric loss functions respectively.

### 3 Improved estimation for $\sigma_2$ when $\sigma_1 \leq \sigma_2$

In this section, we will derive estimators of the parameter  $\sigma_2$  which will improve upon the BAEE under the restriction  $\sigma_1 \leq \sigma_2$ . We consider estimators of the form

$$\mathcal{C}_1 = \left\{ \delta_{\psi_1} = \psi_1(U) S_2; U = \frac{S_1}{S_2}, \psi_1(\cdot) \text{ is a positive function} \right\} \quad (17)$$

and derive [38]-type estimators for  $\sigma_2$ , which gives an improvement upon  $\delta_{j2}$  for  $j = 1, 2$  as we proved in the next theorem.

**Theorem 6.** (i) Under the loss function  $L_1(\cdot)$ , the risk of the estimator

$$\delta_{2S1}^1(X, S) = \max \{ \psi_1(U), \psi_{11}(U) \} S_2,$$

is nowhere larger than that of the estimator  $\delta_{\psi_1}$  provided  $P(\psi_{11}(U) > \psi_1(U)) > 0$ , where

$$\psi_{11}(U) = (1 + U) \left( \frac{E(\tau)}{(p_1 + p_2 - 2)(p_1 + p_2 - 3)E(1/\tau)} \right)^{1/2}.$$

(ii) For the loss function  $L_2(\cdot)$ , the risk of the estimator

$$\delta_{2S1}^2(X, S) = \max \{ \psi_1(U), \psi_{12}(U) \} S_2,$$

is nowhere larger than that of the estimator  $\delta_{\psi_1}$  provided  $P(\psi_{12}(U) > \psi_1(U)) > 0$ , where  $\psi_{12}(U) = \frac{(1+U)}{(p_1+p_2-2)E(\frac{1}{\tau})}$ .

**Proof.** Proof of this theorem is similar to the Theorem 1.

**Remark 9.** If  $\tau = 1$  with probability one, then the Theorem 6 reduces to the following result which was previously obtained by [29] in Example 3.3 and Example 3.4 under entropy and symmetric loss functions respectively.

As in the previous section, we derive an improved estimator for  $\sigma_2$  within the class of estimators  $\mathcal{C}_1$  using the IERD method of [19]. We have the following theorem as follows

**Theorem 7.** (i) Assume that the function  $\psi_1(z)$  satisfies the following conditions:

$$\begin{aligned} (a) \quad & \psi_1(z) \text{ is non-decreasing in } z \text{ and } \lim_{z \rightarrow 0} \psi_1(z) = \left( \frac{E(\tau)}{(p_2-1)(p_2-2)E(1/\tau)} \right)^{1/2} \\ (b) \quad & \psi_1(z) \leq \psi_*^1(z) = \left( \frac{E(\tau) \left[ \frac{B(p_1-1, p_2-2)}{\Gamma(p_1-1)} - B\left(\frac{z}{1+z}; p_1-1, p_2-2\right) \right]}{E(1/\tau)(p_1+p_2-2)(p_1+p_2-3) \left[ \frac{B(p_1-1, p_2)}{\Gamma(p_1-1)} - B\left(\frac{z}{1+z}; p_1-1, p_2\right) \right]} \right)^{1/2}. \end{aligned}$$

Then the risk of the estimator  $\delta_{\psi_1}$  defined in (17) dominates  $\delta_{12}$  under the loss function  $L_1(\cdot)$ .

(ii) Let  $\psi_1(z)$  satisfies the following conditions:

$$\begin{aligned} (a) \quad & \psi_1(z) \text{ is non-decreasing in } z \text{ and } \lim_{z \rightarrow 0} \psi_1(z) = \frac{1}{(p_2-1)E(1/\tau)} \\ (b) \quad & \psi_1(z) \leq \psi_*^2(z) = \frac{\frac{B(p_1-1, p_2-1)}{\Gamma(p_1-1)} - B\left(\frac{z}{1+z}; p_1-1, p_2-1\right)}{E(1/\tau)(p_1+p_2-2) \left[ \frac{B(p_1-1, p_2)}{\Gamma(p_1-1)} - B\left(\frac{z}{1+z}; p_1, p_2-1\right) \right]}. \end{aligned}$$

Then the risk of the estimator  $\delta_{\psi_1}$  defined in (17) is nowhere larger than that of  $\delta_{22}$  with respect to the  $L_2(\cdot)$  loss function.

**Proof.** Proof of this theorem is similar to the Theorem 2.

**Remark 10.** The boundary estimator  $\delta_{\psi_*^1}$  and  $\delta_{\psi_*^2}$  are [7]-type estimator for  $\sigma_2$  under the loss functions  $L_1(\cdot)$  and  $L_2(\cdot)$  respectively.

**Remark 11.** Now we prove that  $\delta_{\psi_*^1}$  is a generalized Bayes estimator of  $\sigma_2$  under the loss function  $L_1(\cdot)$ . We consider the prior distribution as

$$\pi(\sigma_1, \sigma_2, \mu_1, \mu_2) = \frac{1}{\sigma_1 \sigma_2} I_{\sigma_1 \leq \sigma_2}.$$

The corresponding posterior distribution, for given  $\tau > 0$ , is proportional to

$$\pi(\sigma_1, \sigma_2, \mu_1, \mu_2 | X, S_1, Y, S_2) \propto \frac{\tau^{p_1+p_2-2}}{\sigma_1^{p_1} \sigma_2^{p_2}} e^{-\frac{\tau S_1}{\sigma_1} - \frac{\tau S_2}{\sigma_2}} \frac{p_1 \tau}{\sigma_1} e^{-\frac{p_1 \tau}{\sigma_1} (X - \mu_1)} \frac{p_2 \tau}{\sigma_2} e^{-\frac{p_2 \tau}{\sigma_2} (Y - \mu_2)}, \quad (18)$$

where  $\mu_1 \leq x$ ,  $\mu_2 \leq y$ ,  $0 < \sigma_1 \leq \sigma_2$ . For the symmetric loss function  $L_1(\cdot)$ , the generalized Bayes estimator of  $\sigma_2$  is obtained as follows

$$\delta_{2B}^1 = \left( \frac{E(\sigma_1 | X, S_1, Y, S_2)}{E\left(\frac{1}{\sigma_1} | X, S_1, Y, S_2\right)} \right)^{1/2},$$

where the expectation are taken with respect to the posterior distribution specified in the equation (18). After some calculation, it is found that the generalized Bayes estimator coincides with the estimator  $\delta_{\psi_*^1}$ .

**Remark 12.** In the case when  $\tau = 1$  with probability one, then the Theorem 7 reduces to the following result which was previously derived by [29] in Corollary 3.8 and Corollary 3.9 for entropy and symmetric loss functions respectively.

**Remark 13.** By using a similar argument, we can prove that the generalized Bayes estimator of  $\sigma_2$  under the loss function  $L_2(\cdot)$ , with respect to the same prior distribution  $\pi(\sigma_1, \sigma_2, \mu_1, \mu_2)$ , also coincides with the  $\delta_{\psi_*^2}$ .

Now we derive an improved estimators for  $\sigma_2$ , from a class of estimators where we use the statistic  $S_2$  and  $Y$ . The class of estimators is considered as

$$\mathcal{C}_2 = \left\{ \delta_{\psi_2} = \psi_2(U_1) S_2; U_1 = \frac{Y}{S_2}, \psi_2(\cdot) \text{ is a positive function} \right\} \quad (19)$$

**Theorem 8.** (i) Under the  $L_1(\cdot)$  loss function, the risk of the estimator

$$\delta_{\psi_{21}}^1 = \begin{cases} \min \{ \psi_2(U_1), \psi_{21}(U_1) \} S_2, & U_1 > 0 \\ \psi_2(U_1) S_2, & \text{otherwise} \end{cases} \quad (20)$$

is nowhere larger than that of the estimator  $\delta_{\psi_2}$  provide  $P(\psi_2(U_1) > \psi_{21}(U_1)) > 0$ , where

$$\psi_{21}(U_1) = \frac{(1 + p_2 U_1)}{\sqrt{p_2(p_2 - 1)}} \min \left\{ \left( \frac{E(\tau^{p_2+2})}{E(\tau^{p_2})} \right)^{1/2}, \left( \frac{E(\tau)}{E(\tau^{-1})} \right)^{1/2} \right\}.$$

(ii) Under the loss function  $L_2(\cdot)$ , the risk of the estimator

$$\delta_{\psi_{22}}^2 = \begin{cases} \min \{ \psi_2(U_1), \psi_{22}(U_1) \} S_2, & U_1 > 0, \\ \psi_2(U_1) S_2, & \text{otherwise} \end{cases}$$

is nowhere larger than that of  $\delta_{\psi_2}$ , provided  $P(\psi_2(U_1) > \psi_{22}(U_1)) > 0$ , where  $\psi_{22}(U_1) = \frac{(1+p_2 U_1)}{p_2 E(\frac{1}{\tau})}$ .

**Proof.**

(i) Under the loss function  $L_1(\cdot)$ , the risk of the estimator  $\delta_{\psi_2}(U_1)$  can be expressed as

$$R(\delta_{\psi_2}; \mu_2, \sigma_2) = E^{U_1} E \left[ \left( \psi_2(U_1) V_2 + \frac{1}{\psi_2(U_1) V_2} - 2 \right) \middle| U_1 \right]$$

For a given  $\tau > 0$ , the conditional density of  $V_2$  given  $U_1 = u_1$  is obtain as

$$f_{\rho_2}(v_2|u_1) = \frac{p_2}{\Gamma(p_2 - 1)} v_2^{p_2-1} e^{-\tau v_2(1+p_2 u_1)} e^{p_2 \rho_2 \tau} \tau^{p_2}, \quad v_2 > \max \left\{ 0, \frac{\rho_2}{u_1} \right\}, \quad u_1 \in \mathbb{R}$$

where  $\rho_2 = \frac{\mu_2}{\sigma_2} \in \mathbb{R}$ . It can be easily seen that the conditional risk function

$$R_2(\delta_{\psi_2}, \rho_2) = E \left[ \left( \psi_2(U_1) V_2 + \frac{1}{\psi_2(U_1) V_2} - 2 \right) \middle| U_1 = u_1 \right]$$

is minimized at

$$\psi_2(u_1; \rho_2) = \left( \frac{E \left[ 1/V_2 \mid U_1 = u_1 \right]}{E \left[ V_2 \mid U_1 = u_1 \right]} \right)^{1/2}$$

Now we will consider two cases. First case, we consider  $\mu_2 > 0$ ,  $u_2 > 0$ . In this case we have

$$\psi_2(u_1; \rho_2) = \left( \frac{\int_0^\infty \int_{\frac{\rho_2}{u_1}}^\infty v_2^{p_2-2} e^{-\tau v_2(1+p_2 u_1)} e^{p_2 \rho_2 \tau} \tau^{p_2} dv_2 dH(\tau)}{\int_0^\infty \int_{\frac{\rho_2}{u_1}}^\infty v_2^{p_2} e^{-\tau v_2(1+p_2 u_1)} e^{p_2 \rho_2 \tau} \tau^{p_2} dv_2 dH(\tau)} \right)^{1/2} \quad (21)$$

Using the transformation  $z_4 = \tau v_2(1 + p_2 u_1)$ , we obtain the equation (21) as follows,

$$\psi_2(u_1; \rho_2) = (1 + p_2 u_1) \left( \frac{\int_0^\infty \tau e^{p_2 \rho_2 \tau} \int_{\xi_*}^\infty z_4^{p_2-2} e^{-z_4} dz_4 dH(\tau)}{\int_0^\infty \frac{1}{\tau} e^{p_2 \rho_2 \tau} \int_{\xi_*}^\infty z_4^{p_2} e^{-z_4} dz_4 dH(\tau)} \right)^{1/2}$$

Now for a given  $\tau > 0$ , we can easily seen that

$$\frac{\int_{\xi_*}^\infty z_4^{p_2-2} e^{-z_4} dz_4}{\int_{\xi_*}^\infty z_4^{p_2} e^{-z_4} dz_4} = E_{\xi_*}(Z_4^{-2}),$$

where  $Z_4$  has density  $g_2(z_4, \xi_*) \propto z_4^{p_2} e^{-z_4} I_{(\xi_*, \infty)}(z_4)$  with  $\xi_* = \frac{\tau \rho_2}{u_1}(1 + p_2 u_1)$ . For  $\xi_* > 0$ ,  $\frac{g_2(z_4, \xi_*)}{g_2(z_4, 0)}$  is non-decreasing then we have  $E_{\xi_*}(Z_4^{-2}) \leq E_0(Z_4^{-2}) = \frac{1}{p_2(p_2-1)}$  and hence  $\int_{\xi_*}^\infty z_4^{p_2-2} e^{-z_4} dz_4 \leq \frac{1}{p_2(p_2-1)} \int_{\xi_*}^\infty z_4^{p_2} e^{-z_4} dz_4$ . Finally we have

$$\psi_2(u_1; \rho_2) \leq \frac{(1 + p_2 u_1)}{\sqrt{p_2(p_2 - 1)}} \left( \frac{\int_0^\infty \tau e^{p_2 \rho_2 \tau} \int_{\xi_*}^\infty z_4^{p_2} e^{-z_4} dz_4 dH(\tau)}{\int_0^\infty \frac{1}{\tau} e^{p_2 \rho_2 \tau} \int_{\xi_*}^\infty z_4^{p_2} e^{-z_4} dz_4 dH(\tau)} \right)^{1/2}$$

Again we take a transformation  $z_4 = \tau x_4$  then we have

$$\psi_2(u_1; \rho_2) \leq \frac{(1 + p_2 u_1)}{\sqrt{p_2(p_2 - 1)}} \left( \frac{\int_0^\infty \int_{\xi_{**}}^\infty x_4^{p_2} \tau^{p_2+2} e^{-(x-p_2\rho)\tau} dx_4 dH(\tau)}{\int_0^\infty \int_{\xi_{**}}^\infty x_4^{p_2} \tau^{p_2} e^{-(x_4-p_2\rho)\tau} dx_4 dH(\tau)} \right)^{1/2} \quad (22)$$

where,  $\xi_{**} = \frac{\rho_2}{u_1}(1 + p_2 u_1)$ . Now for  $l = x_4 - p_2 \rho > 0$ , we obtain

$$\frac{\int_0^\infty \tau^{p_2+2} e^{-l\tau} dH(\tau)}{\int_0^\infty \tau^{p_2} e^{-l\tau} dH(\tau)} = \int_0^\infty \tau^2 f_l(\tau) dH(\tau)$$

where  $f_l(\tau) \propto \tau^{p_2} e^{-l\tau}$ . But,  $\frac{f_l(\tau)}{f_0(\tau)}$  is decreasing in  $\tau$ , so by using Lemma (2) we have,

$$\int_0^\infty \tau^2 f_l(\tau) dH(\tau) \leq \int_0^\infty \tau^2 f_0(\tau) dH(\tau) = \frac{E(\tau^{p_2+2})}{E(\tau^{p_2})} \quad (23)$$

From (22) and (23) we get

$$\psi_2(u_1; \rho_2) \leq \frac{(1 + p_2 u_1)}{\sqrt{p_2(p_2 - 1)}} \left( \frac{E(\tau^{p_2+2})}{E(\tau^{p_2})} \right)^{1/2} \quad (24)$$

Now we consider the second case where,  $\mu_2 \leq 0$  and  $u_1 > 0$ . In this case we have,

$$\psi_2(u_1; \rho_2) = \left( \frac{\int_0^\infty \int_0^\infty v_2^{p_2-2} e^{-\tau v_2(1+p_2 u_1)} e^{p_2 \rho_2 \tau} \tau^{p_2} dv_2 dH(\tau)}{\int_0^\infty \int_0^\infty v_2^{p_2} e^{-\tau v_2(1+p_2 u_1)} e^{p_2 \rho_2 \tau} \tau^{p_2} dv_2 dH(\tau)} \right)^{1/2}$$

Using the transformation  $z_4 = \tau v_2(1 + p_2 u_1)$ , we have

$$\begin{aligned} \psi_2(u_1; \rho_2) &= (1 + p_2 u_1) \left( \frac{\int_0^\infty \tau e^{p_2 \rho_2 \tau} \int_0^\infty z_4^{p_2-2} e^{-z_4} dz_4 dH(\tau)}{\int_0^\infty \frac{1}{\tau} e^{p_2 \rho_2 \tau} \int_0^\infty z_4^{p_2} e^{-z_4} dz_4 dH(\tau)} \right)^{1/2} \\ &= \frac{(1 + p_2 u_1)}{\sqrt{p_2(p_2 - 1)}} \left( \frac{\int_0^\infty \tau e^{p_2 \rho_2 \tau} dH(\tau)}{\int_0^\infty \frac{1}{\tau} e^{p_2 \rho_2 \tau} dH(\tau)} \right)^{1/2} \\ &= \frac{(1 + p_2 u_1)}{\sqrt{p_2(p_2 - 1)}} \left( \int_0^\infty \tau^2 f_{\rho_2}(\tau) dH(\tau) \right)^{1/2} \end{aligned} \quad (25)$$

where  $f_{\rho_2}(\tau) = \frac{\frac{1}{\tau} e^{p_2 \tau \rho}}{\int_0^\infty \frac{1}{\tau} e^{p_2 \tau \rho} dH(\tau)}$ . Set  $f_0(\tau) = \frac{\tau^{-1}}{\int_0^\infty \tau^{-1} dH(\tau)}$ . Now  $\frac{f_{\rho_2}(\tau)}{f_0(\tau)}$  is non-increasing in  $\tau$ , then by Lemma 2 we have

$$\int_0^\infty \tau^2 f_{\rho_2}(\tau) dH(\tau) \leq \int_0^\infty \tau^2 f_0(\tau) dH(\tau) = \frac{E(\tau)}{E(1/\tau)} \quad (26)$$

Thus from the equations (25) and (26) we obtain

$$\psi_2(u_1; \rho_2) \leq \frac{(1 + p_2 u_1)}{\sqrt{p_2(p_2 - 1)}} \left( \frac{E(\tau)}{E(1/\tau)} \right)^{1/2} \quad (27)$$

Hence for any  $\mu_2$  and  $u_1 > 0$ , we have

$$\psi_2(u_1; \rho_2) \leq \frac{(1 + p_2 u_1)}{\sqrt{p_2(p_2 - 1)}} \min \left\{ \left( \frac{E(\tau^{p_2+2})}{E(\tau^{p_2})} \right)^{1/2}, \left( \frac{E(\tau)}{E(\tau^{-1})} \right)^{1/2} \right\} = \psi_{21}(u_1).$$

Now using the convexity of  $R_2(\delta_{\psi_2}, \rho_2)$  for  $P(\psi_{21}(U_1) < \psi_2(U_1)) > 0$  we get the result. Proof of (ii) is similar to (i), so we omit it.

In Theorems 6 and 8, we derived improved estimator for  $\sigma_2$  by using the statistics  $S_2$  along with either the statistics  $S_1$  or  $Y$ . Now we want to use simultaneously the information that the insights these statistics offer, then double shrinkage estimators are applied, as it demonstrated in Theorem 9 (see [15]).

**Theorem 9.** *To estimating the parameter  $\sigma_2$ , under the symmetric loss function  $L_1(\cdot)$  with the constraint  $\sigma_1 \leq \sigma_2$ , we consider the estimator*

$$\delta_{\psi_{11} + \psi_{21} - c_2} = (\psi_{11}(U) + \psi_{21}(U_1) - c_2) S_2$$

where  $\psi_{11}$  is defined in Theorem 6, and  $\psi_{21}$  is defined in Theorem 8 and  $c_2 = \sqrt{\frac{E(\tau)}{(p_2-1)(p_2-2)E(1/\tau)}}$ . then the estimator  $\delta_{\psi_{11} + \psi_{21} - c_2}$  has smaller or equal risk function than that of the estimator  $\delta_{\psi_{11}} = \max \{ \psi_{11}(U), c_2 \} S_2$ ,  $\delta_{\psi_{21}} = \min \{ \psi_{21}(U_1), c_2 \} S_2$  and so of the estimator  $\delta_{12} = c_2 S_2 \forall 0 < \frac{\sigma_1}{\sigma_2} \leq 1$ ,  $\mu_2 \in \mathbb{R}$  and  $\tau > 0$ .

**Proof.** Considering the case where  $\psi_{11}(U) > c_2$ , and  $\psi_{21}(U_1) < c_2$ , we establish that  $\forall \theta = (\mu_1, \mu_2, \sigma_1, \sigma_2)$ ,

$$\begin{aligned} R(\delta_{\psi_{11}+\psi_{21}-c_2}; \theta) - R(\delta_{\psi_{21}}; \theta) &= E \left[ \psi_{21}(U_1)V_2 - c_2V_2 + \frac{1}{(\psi_{11}(U) + \psi_{21}(U_1) - c_2)V_2} + \frac{1}{\psi_{11}(U)V_2} \right] \\ &\leq E \left[ \psi_{11}(U)V_2 + \frac{1}{\psi_{11}(U)V_2} - 2 \right] + E \left[ c_2V_2 + \frac{1}{c_2V_2} - 2 \right] \leq 0 \end{aligned}$$

because of Theorem 6. In a similar way, we use the result in Theorem 8,  $R(\delta_{\psi_{11}+\psi_{21}-c_2}; \theta) \leq R(\delta_{\psi_{21}}; \theta) \leq 0$ . Hence the theorem is proved.

**Remark 14.** A similar result can be derived for the Stein-type loss function  $L_2(\cdot)$  by applying the same line of reasoning as described above.

## 4 Application

In this section, we apply the previously derived results from Section 2 and 3 to two specific distribution of the mixing parameter  $\tau$ , each leading to a special and widely used model. First, we assume that the mixing parameter  $\tau$  follows a Gamma distribution  $\Gamma(b, 1)$ . Then the joint density of  $X_1, X_2, \dots, X_{p_1}$  in (1) is

$$f_1(x_1, x_2, \dots, x_{p_1}; \mu_1, \sigma_1) = \frac{\Gamma(p_1 + b)}{\Gamma(b)\sigma_1^{p_1}} \frac{1}{\left(1 + \frac{1}{\sigma_1} \sum_{i=1}^{p_1} (x_i - \mu_1)\right)^{p_1+b}} I_{(\mu_1, \infty)}(x_{(1)}) \quad (28)$$

Under this assumption, the model (1) reduces to a multivariate Lomax distribution. The Lomax distribution, also known as the Pareto Type II distribution, has the probability density function. This distribution has been extensively applied across various domains, including socioeconomics, reliability analysis, and life testing, biological and medical sciences, modeling business failure data etc. There are some important contributions in this directions are [25], [11], [24], [14], [12]. Secondly, when the mixing parameter  $\tau$  follows an Inverse Gaussian distribution  $IG(m, n)$ , then (1) becomes

$$\begin{aligned} f_2(x_1, x_2, \dots, x_{p_1}; \mu_1, \sigma_1) &= \frac{\exp \left\{ \frac{n}{m} - n \sqrt{\frac{1}{m^2} + \frac{2}{\sigma_1 n} \sum_{i=1}^{p_1} (x_i - \mu_1)} \right\}}{\sigma_1^{p_1} \left( \frac{1}{m^2} + \frac{2}{\sigma_1 n} \sum_{i=1}^{p_1} (x_i - \mu_1) \right)^{p_1/2}} \times \\ &\quad \sum_{i=1}^{p_1-1} \frac{(p_1 - 1 + i)!}{i!(p_1 - 1 - i)!} \left[ 2n \sqrt{\left( \frac{1}{m^2} + \frac{2}{\sigma_1 n} \sum_{i=1}^{p_1} (x_i - \mu_1) \right)} \right]^{-i} I_{(0, \infty)}(x_{(1)}) \end{aligned} \quad (29)$$

With this assumption, the model (1) reduces to an exponential-Inverse Gaussian (E-IG) distribution. This distribution was used by [31] in studying quantile estimation within a mixture of exponential distribution having unknown location and scale parameters. The E-IG distribution has been applied in various domains, including reliability, actuarial science and survival analysis. There are some work in this direction, we refer to [3], [13], [10].

### 4.1 Multivariate Lomax distribution

In this subsection, we will derive the improved estimator of the parameters  $\sigma_1$  and  $\sigma_2$  for multivariate Lomax distribution under the loss functions  $L_1(\cdot)$  and  $L_2(\cdot)$ . In this case, we have  $E^\tau(\tau^n) = \frac{\Gamma(n+b)}{\Gamma(b)}$

for any  $n \in \mathbb{Z}$ . So that the BAAE of  $\sigma_i$  under the loss function  $L_1(\cdot)$  is  $\delta_{1i} = c_i S_i$ , where

$$c_i = \left( \frac{b(b-1)}{(p_i-1)(p_i-2)} \right)^{1/2}$$

for  $i = 1, 2$  and for the loss function  $L_2(\cdot)$  we have the BAAE of  $\sigma_i$  is  $\delta_{2i} = d_i S_i$  with

$$d_i = \frac{b-1}{p_i-1}.$$

Estimation of  $\sigma_1$  and  $\sigma_2$  under the square error loss function for multivariate Lomax distribution has been studied by [33], which is a special case of this work. Here we will derive the better estimator than  $\delta_{1i}$  and  $\delta_{2i}$  of  $\sigma_i$  for multivariate Lomax distribution under the loss function  $L_1(\cdot)$  and  $L_2(\cdot)$  for  $i = 1, 2$  respectively.

Now as an application of Theorem 1, 4 and 5 the [38]-type improved estimators of  $\sigma_1$  are obtained for multivariate Lomax distribution as follows which is better than that of  $\delta_{11}$  and  $\delta_{12}$  under the loss function  $L_1(\cdot)$  and  $L_2(\cdot)$  respectively.

**Theorem 10.** (i) Under  $L_1(\cdot)$  loss function, we have  $\varphi_{11}(W) = (1+W) \left( \frac{b(b-1)}{(p_1+p_2-2)(p_1+p_2-3)} \right)^{\frac{1}{2}}$ ,  $\varphi_{21}(W, W_1) = (1+W+p_1 W_1) \left( \frac{b(b-1)}{(p_1+p_2-1)(p_1+p_2-2)} \right)^{1/2}$ ,  $\varphi_{31}(W, W_2) = \frac{(1+W+p_2 W_2)(b(b-1))^{1/2}}{((p_1+p_2-1)(p_1+p_2-2))^{1/2}}$ . The improved estimator of  $\sigma_1$  are obtained as follows

$$\delta_{11}^1(X, S) = \min \{ \varphi_{11}(W), c_1 \} S_1$$

$$\delta_{12}^1 = \begin{cases} \min \{ \varphi_{21}(W, W_1), c_1 \} S_1, & W_1 > 0 \\ c_1 S_1, & \text{otherwise} \end{cases}$$

$$\delta_{13}^1 = \begin{cases} \min \{ \varphi_{31}(W, W_2), c_1 \} S_1, & W_2 > 0 \\ c_1 S_1, & \text{otherwise} \end{cases}$$

(ii) Under the Stein loss function  $L_2(\cdot)$  we have  $\varphi_{12}(W) = (1+W) \frac{b-1}{(p_1+p_2-2)}$ ,  $\varphi_{22}(W, W_1) = (1+W+p_1 W_1) \frac{b-1}{p_1+p_2-1}$ ,  $\varphi_{32}(W, W_2) = (1+W+p_2 W_2) \frac{b-1}{p_1+p_2-1}$ . The improved estimators are obtained as follows

$$\delta_{11}^2(X, S) = \min \{ \varphi_{12}(W), d_1 \} S_1$$

$$\delta_{12}^2 = \begin{cases} \min \{ \varphi_{22}(W, W_1), d_1 \} S_1, & W_1 > 0 \\ d_1 S_1, & \text{otherwise} \end{cases}$$

$$\delta_{13}^2 = \begin{cases} \min \{ \varphi_{32}(W, W_2), d_1 \} S_1, & W_2 > 0 \\ d_1 S_1, & \text{otherwise} \end{cases}$$

If we use the both information  $X$  and  $Y$ , analogous result to Theorem 4 can be derived, as described in the following theorem.

**Theorem 11.** (i) Under the loss function  $L_1(\cdot)$  we have  $\varphi_{41}(W, W_1, W_2) = (1+W+p_1 W_1+p_2 W_2) \left( \frac{b(b-1)}{(p_1+p_2)(p_1+p_2-1)} \right)^{1/2}$ . The improved estimator of  $\sigma_1$  is obtained as

$$\delta_{14}^1 = \begin{cases} \min \{ \varphi_{41}(W, W_1, W_2), c_1 \} S_1, & W_1 > 0, W_2 > 0 \\ c_1 S_1, & \text{otherwise} \end{cases}$$



(ii) For the Stein type loss function  $L_2(\cdot)$ , we have  $\varphi_{42}(W, W_1, W_2) = (1 + W + p_1 W_1 + p_2 W_2) \frac{b-1}{p_1+p_2}$ . We get the improved estimators as

$$\delta_{14}^2 = \begin{cases} \min \{ \varphi_{42}(W, W_1, W_2), d_1 \} S_1, & W_1 > 0, W_2 > 0 \\ d_1 S_1, & \text{otherwise} \end{cases}$$

As in the Theorem 2, the IERD method of [19] is applied for an estimator of the form (3). In that case we have the following theorem

**Theorem 12.** (i) Under the loss function  $L_1(\cdot)$ , the risk of the estimator  $\delta_{\varphi_1}$  given in (3) is nowhere greater than that of  $\delta_{11}$  provided the function  $\varphi_1(w)$  satisfies

$$(a) \varphi_1(w) \text{ is non-decreasing in } w \text{ and } \lim_{w \rightarrow \infty} \varphi_1(w) = \sqrt{\frac{b(b-1)}{(p_1-1)(p_1-2)}}$$

$$(b) \varphi_1(w) \geq \varphi_*^1(w), \varphi_*^1(w) \text{ is defined as } \varphi_*^1(w) = \left( \frac{b(b-1)B\left(\frac{w}{1+w}; p_1-2, p_2-1\right)}{(p_1+p_2-2)(p_1+p_2-3)B\left(\frac{w}{1+w}; p_1, p_2-1\right)} \right)^{1/2}$$

(ii) Suppose the following conditions are hold true.

$$(a) \varphi_1(w) \text{ is non-decreasing in } w \text{ and } \lim_{w \rightarrow \infty} \varphi_1(w) = \frac{b-1}{p_1-1}$$

$$(b) \varphi_1(w) \geq \varphi_*^2(w), \varphi_*^2(w) \text{ is defined as } \varphi_*^2(w) = \left( \frac{(b-1)B\left(\frac{w}{1+w}; p_1-1, p_2-1\right)}{(p_1+p_2-2)B\left(\frac{w}{1+w}; p_1, p_2-1\right)} \right)^{1/2}$$

Then the risk of the estimator  $\delta_{\varphi}$  given in (3) is nowhere greater than that of  $\delta_{12}$  under the loss function  $L_2(\cdot)$ .

We have derive the improved estimator for  $\sigma_2$  under the loss function  $L_1(\cdot)$  and  $L_2(\cdot)$  respectively.

**Theorem 13.** (i) For  $L_1(\cdot)$  loss function, we have  $\psi_{11}(U) = (1 + U) \left( \frac{b(b-1)}{(p_1+p_2-2)(p_1+p_2-3)} \right)^{1/2}$ ,  $\psi_{21}(U_1) = (1 + p_2 U_1) \sqrt{\frac{b(b-1)}{p_2(p_2-1)}}$ . The improve estimator of  $\sigma_2$  are obtained as follows

$$\delta_{21}^1(X, S) = \max \{ \psi_{11}(U), c_2 \} S_2$$

$$\delta_{22}^1 = \begin{cases} \min \{ \psi_{21}(U_1), c_2 \} S_2, & U_1 > 0 \\ c_2 S_2, & \text{otherwise} \end{cases}$$

(ii) Under  $L_2(\cdot)$  loss function, we have  $\psi_{12}(U) = (1 + U) \frac{b-1}{(p_1+p_2-1)}$ ,  $\psi_{22}(U_1) = (1 + p_2 U_1) \frac{(b-1)}{p_2}$ . We get the improve estimators of  $\sigma_2$  as

$$\delta_{21}^2(X, S) = \max \{ \psi_{12}(U), d_2 \} S_2$$

$$\delta_{22}^2 = \begin{cases} \min \{ \psi_{22}(U_1), d_2 \} S_2, & U_1 > 0 \\ d_2 S_2, & \text{otherwise} \end{cases}$$

As in the previous theorem, now we present an improved estimator for  $\sigma_2$  within the class of estimators (17) using the IERD method of [19]. In that case, we have the theorem as follows.

**Theorem 14.** (i) Under the loss function  $L_1(\cdot)$ , the risk of the estimator  $\delta_{\psi_1}$  given in (17) is nowhere larger than that of  $\delta_{12}$  provided the function  $\psi_1(u)$  satisfies the following conditions

(a)  $\psi_1(u)$  is non-decreasing in  $u$  and  $\lim_{u \rightarrow 0} \psi_1(u) = \left( \frac{b(b-1)}{(p_2-1)(p_2-2)} \right)^{1/2}$

(b)  $\psi_1(u) \leq \psi_*^1(u) = \left( \frac{b(b-1) \left[ \frac{B(p_1-1, p_2-2)}{\Gamma(p_1-1)} - B\left(\frac{u}{1+u}; p_1-1, p_2-2\right) \right]}{(p_1+p_2-2)(p_1+p_2-3) \left[ \frac{B(p_1-1, p_2)}{\Gamma(p_1-1)} - B\left(\frac{u}{1+u}; p_1-1, p_2\right) \right]} \right)^{1/2}$

(ii) Let us assume that the function  $\psi_1(u)$  satisfies the following conditions

(a)  $\psi_1(u)$  is non-decreasing in  $u$  and  $\lim_{u \rightarrow 0} \psi_1(u) = \frac{b-1}{p_2-1}$

(b)  $\psi_1(u) \leq \psi_*^2(u) = \frac{(b-1) \left[ \frac{B(p_1-1, p_2-1)}{\Gamma(p_1-1)} - B\left(\frac{u}{1+u}; p_1-1, p_2-1\right) \right]}{(p_1+p_2-2) \left[ \frac{B(p_1-1, p_2)}{\Gamma(p_1-1)} - B\left(\frac{u}{1+u}; p_1, p_2-1\right) \right]}$

Then the risk of the estimator  $\delta_{\psi_1}$  given in (17) is nowhere larger than that of  $\delta_{22}$  under the loss function  $L_2(\cdot)$ .

Now we will give result as an application of the Theorem 9

**Theorem 15.** (i) Under the loss function  $L_1(\cdot)$ , we consider the estimator

$$\delta_{D1} = (\psi_{11}(U) + \psi_{21}(U_1) - c_2) S_2$$

where  $\psi_{11}$  and  $\psi_{21}$  are defined in Theorem 13, and  $c_2 = \left( \frac{b(b-1)}{(p_2-1)(p_2-2)} \right)^{1/2}$ . Then the estimator  $\delta_{D1}$  has smaller or equal risk function than that of the estimator  $\delta_{\psi_{11}}$ ,  $\delta_{\psi_{21}}$  and so of the estimator  $\delta_{12} = c_2 S_2$ .

(ii) The estimator

$$\delta_{D2} = (\psi_{21}(U) + \psi_{22}(U_1) - d_2) S_2$$

has the smaller risk than that of the estimator  $\delta_{\psi_{21}}$ ,  $\delta_{\psi_{22}}$  and so of the estimator  $\delta_{22} = d_2 S_2$  under the loss function  $L_2(\cdot)$ .

## 4.2 Simulation study

In this Subsection, we will do simulation study to compare the risk performance of the estimators proposed in the Section 4.1. To carry out the simulation study, we have generated 50,000 random samples from two populations following the distributions  $Exp\left(\mu_1, \frac{\sigma_1}{\tau}\right)$  and  $Exp\left(\mu_2, \frac{\sigma_2}{\tau}\right)$  for various values of  $(\mu_1, \mu_2)$  and  $(\sigma_1, \sigma_2)$ . For multivariate Lomax distribution, we generate samples of  $\tau$  from  $\Gamma(b, 1)$  distribution for  $b = 3, 5, 7$ . Observed that the risk of the estimators are depends on the parameters  $\sigma_1$  and  $\sigma_2$  through  $\eta = \sigma_1/\sigma_2$ . The performance of the improved estimators is calculated using the relative risk improvement (RRI) with respect to the BAE under the Stein loss  $L_2(\cdot)$ . The RRI of an estimator  $\delta$  is with the respect to  $\delta_1$  is defined by

$$\text{RRI}(\delta) = \frac{\text{Risk}(\delta_1) - \text{Risk}(\delta)}{\text{Risk}(\delta_1)} \times 100\%$$

It is known that in a neighborhood of  $(\mu_1, \mu_2) = (0, 0)$ , [38]-type estimators gives the best performance. Thus we have computed the RRI around the neighborhood of  $(\mu_1, \mu_2) = (0, 0)$ . Further in the table where the risk improvement is given as 0.000% this implies that it is less than 0.001%. In Table ML1-ML4, we have presented RRI values of  $\delta_{12}^2$ ,  $\delta_{13}^2$  and  $\delta_{14}^2$  with respect to the BAE  $\delta_{21}$ . From the tabulated RRI values we have observed that the RRI of  $\delta_{12}^2$ ,  $\delta_{13}^2$  and  $\delta_{14}^2$  are decreasing function of  $b$  for  $b > 2$ . Also the RRI of  $\delta_{12}^2$ ,  $\delta_{13}^2$  and  $\delta_{14}^2$  are increasing function of  $\eta$  for  $0 < \eta \leq 1$ .

In Table ML5 we have tabulated the RRI  $\delta_{11}^2$  and  $\delta_{\varphi_*^2}$ . It can be easily seen that the risk of these estimators are independent of the parameter  $\mu_1$  and  $\mu_2$ . Thus we have compute the RRI of  $\delta_{11}^2$  and

$\delta_{\varphi_*^2}$  only at  $(\mu_1, \mu_2) = (0, 0)$ . Here the RRI of  $\delta_{11}^2$  and  $\delta_{\varphi_*^2}$  are decreasing function of  $b$ . RRI of  $\delta_{11}^2$  is increasing in  $\eta$  but RRI of the estimator  $\delta_{\varphi_*^2}$  is not monotone it is increasing when  $\eta < 0.7$  (approximately) and decreasing when  $\eta > 0.7$  (approximately). Overall, Table ML5 shows that  $\delta_{\varphi_*^2}$  performs better than  $\delta_{11}^2$  when  $0 < \eta < 0.5$  (approximately) and when  $\eta > 0.5$  (approximately) then  $\delta_{11}^2$  performed better than  $\delta_{\varphi_*^2}$ .

Now we discuss the numerical performance of the estimators for the parameter  $\sigma_2$  under the loss function  $L_2(\cdot)$ . In Table ML6, we present the RRI of  $\delta_{21}^2$  and  $\delta_{\psi_*^2}$  over  $\delta_{22}$  for estimating  $\sigma_2$ . The RRI of  $\delta_{21}$  is increasing function of  $\eta$ , whereas the RRI of  $\delta_{\psi_*^2}$  is not monotonic in  $\eta$ ; it is increasing when  $\eta < 0.7$  (approximately) and then it decreases. In Table ML7, the RRI of  $\delta_{D2}$  is an increasing function of  $\eta$  and a decreasing function of  $b$ . Moreover, the double shrinkage estimator  $\delta_{D2}$  gives higher RRI values when the  $(\mu_1, \mu_2) = (0, 0)$ . Among the estimators  $\delta_{21}^2$ ,  $\delta_{\psi_*^2}$  and  $\delta_{D2}$  for estimating  $\sigma_2$ , we can conclude that  $\delta_{D2}$  performs better than  $\delta_{21}^2$  and  $\delta_{\psi_*^2}$  when  $(\mu_1, \mu_2) = (0, 0)$ . In all other cases,  $\delta_{\psi_*^2}$  preferable than  $\delta_{D2}$  and  $\delta_{21}^2$  for estimating  $\sigma_2$ .

The same observations can be found for the loss function  $L_1(\cdot)$ , and hence we omit the discussion for the sake of simplicity.

Table ML1: RRI of improved estimators of  $\sigma_1$  under  $L_2(\cdot)$  over  $\delta_{21}$  when  $(\mu_1, \mu_2) = (-0.5, -0.3)$

$(p_1, p_2)$	$\eta$	$b = 3$			$b = 5$			$b = 7$		
		$\delta_{12}^2$	$\delta_{13}^2$	$\delta_{14}^2$	$\delta_{12}^2$	$\delta_{13}^2$	$\delta_{14}^2$	$\delta_{12}^2$	$\delta_{13}^2$	$\delta_{14}^2$
(5,5)	0.1	0.000	0.132	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.3	0.019	0.651	0.000	0.000	0.077	0.000	0.000	0.000	0.000
	0.5	0.055	1.004	0.003	0.037	0.146	0.000	0.000	0.002	0.000
	0.7	0.124	1.209	0.007	0.030	0.196	0.000	0.000	0.008	0.000
	0.9	0.415	1.307	0.018	0.046	0.222	0.000	0.015	0.011	0.000
(5,8)	0.1	0.000	0.027	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.3	0.025	0.229	0.000	0.000	0.009	0.000	0.000	0.000	0.000
	0.5	0.061	0.397	0.000	0.045	0.027	0.000	0.000	0.002	0.000
	0.7	0.228	0.487	0.000	0.041	0.042	0.000	0.000	0.002	0.000
	0.9	0.514	0.528	0.003	0.068	0.047	0.000	0.000	0.002	0.000
(15,10)	0.1	0.000	0.022	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.3	0.000	0.133	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.5	0.010	0.216	0.000	0.000	0.002	0.000	0.000	0.000	0.000
	0.7	0.028	0.270	0.000	0.000	0.003	0.000	0.000	0.000	0.000
	0.9	0.083	0.299	0.000	0.000	0.003	0.000	0.000	0.000	0.000
(15,21)	0.1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.3	0.000	0.015	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.5	0.015	0.034	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.7	0.042	0.050	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.9	0.123	0.058	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table ML2: RRI of improved estimators of  $\sigma_1$  under  $L_2(\cdot)$  over  $\delta_{21}$  when  $(\mu_1, \mu_2) = (0, 0)$ 

$(p_1, p_2)$	$\eta$	$b = 3$			$b = 5$			$b = 7$		
		$\delta_{12}^2$	$\delta_{13}^2$	$\delta_{14}^2$	$\delta_{12}^2$	$\delta_{13}^2$	$\delta_{14}^2$	$\delta_{12}^2$	$\delta_{13}^2$	$\delta_{14}^2$
(5,5)	0.1	6.201	4.130	5.374	2.488	1.299	1.851	1.395	0.574	0.903
	0.3	16.604	14.444	16.737	12.089	9.706	11.761	9.943	7.432	9.323
	0.5	21.010	20.042	22.286	18.212	16.824	19.063	16.635	14.999	17.223
	0.7	22.554	22.370	24.514	21.029	20.642	22.753	20.046	19.474	21.597
	0.9	22.684	22.722	24.931	21.806	21.869	23.952	21.168	21.172	23.186
(5,8)	0.1	5.821	4.461	5.277	1.586	1.007	1.301	0.675	0.357	0.501
	0.3	18.478	16.795	18.317	12.030	10.260	11.626	9.400	7.578	8.866
	0.5	24.330	23.598	25.006	20.102	19.043	20.457	18.303	17.029	18.520
	0.7	26.264	26.177	27.536	23.857	23.618	24.895	23.117	22.743	24.062
	0.9	26.125	26.166	27.693	24.639	24.712	26.049	24.550	24.599	25.884
(15,10)	0.1	3.511	2.845	3.330	0.538	0.311	0.438	0.108	0.055	0.079
	0.3	12.958	11.814	13.011	7.634	6.467	7.417	5.108	4.071	4.820
	0.5	18.313	17.689	18.989	14.808	13.911	15.212	12.639	11.623	12.849
	0.7	20.727	20.562	21.902	19.126	18.820	20.133	18.081	17.689	18.961
	0.9	21.426	21.430	22.886	20.933	20.954	22.320	20.658	20.681	22.011
(15,21)	0.1	4.173	3.737	4.013	0.375	0.266	0.321	0.073	0.058	0.068
	0.3	16.837	16.076	16.714	8.762	8.036	8.563	5.393	4.769	5.190
	0.5	24.350	23.990	24.646	18.957	18.362	19.054	15.915	15.228	15.900
	0.7	27.291	27.280	27.988	24.972	24.840	25.510	23.813	23.565	24.233
	0.9	27.337	27.388	28.278	26.582	26.650	27.432	26.590	26.632	27.385

Table ML3: RRI of improved estimators of  $\sigma_1$  under  $L_2(\cdot)$  over  $\delta_{21}$  when  $(\mu_1, \mu_2) = (0.2, 0.5)$ 

$(p_1, p_2)$	$\eta$	$b = 3$			$b = 5$			$b = 7$		
		$\delta_{12}^2$	$\delta_{13}^2$	$\delta_{14}^2$	$\delta_{12}^2$	$\delta_{13}^2$	$\delta_{14}^2$	$\delta_{12}^2$	$\delta_{13}^2$	$\delta_{14}^2$
(5,5)	0.1	1.594	0.354	0.330	0.0089	0.0000	0.0000	0.000	0.000	0.000
	0.3	9.077	3.727	3.553	1.7531	0.1742	0.1321	0.212	0.000	0.000
	0.5	15.690	8.247	7.956	5.6334	1.0682	0.8109	1.529	0.086	0.048
	0.7	20.396	12.781	12.330	10.4379	2.7107	2.1788	4.269	0.347	0.206
	0.9	23.474	16.791	16.265	15.0131	4.9811	4.0250	7.899	0.978	0.616
(5,8)	0.1	2.575	0.688	0.688	0.0665	0.0000	0.0000	0.000	0.000	0.000
	0.3	12.724	4.553	4.411	3.2661	0.1417	0.1120	0.751	0.000	0.000
	0.5	20.824	9.922	9.584	9.7035	1.0646	0.8375	4.095	0.090	0.060
	0.7	25.874	15.234	14.696	16.4680	2.9339	2.4684	9.540	0.423	0.291
	0.9	28.599	19.912	19.243	22.0138	5.5910	4.7432	15.443	1.157	0.845
(15,10)	0.1	0.837	0.514	0.379	0.0000	0.0000	0.0000	0.000	0.000	0.000
	0.3	4.840	2.905	2.091	0.1547	0.0197	0.0000	0.000	0.000	0.000
	0.5	9.790	6.365	4.685	1.3615	0.2858	0.0721	0.075	0.000	0.000
	0.7	14.329	10.154	7.734	3.6888	1.1976	0.3757	0.516	0.050	0.000
	0.9	18.041	13.771	10.813	6.8062	2.6978	1.1038	1.575	0.236	0.028
(15,21)	0.1	1.849	0.544	0.439	0.0000	0.0000	0.0000	0.000	0.000	0.000
	0.3	10.270	3.983	3.315	1.2532	0.0087	0.0000	0.074	0.000	0.000
	0.5	18.835	8.858	7.476	5.9416	0.3289	0.1511	1.353	0.000	0.000
	0.7	25.142	14.104	12.129	12.7033	1.5524	0.8446	4.866	0.059	0.004
	0.9	29.063	19.130	16.767	19.6184	3.6281	2.2394	10.365	0.304	0.111

Table ML4: RRI of improved estimators of  $\sigma_1$  under  $L_2(\cdot)$  over  $\delta_{21}$  when  $(\mu_1, \mu_2) = (0.5, 1)$ 

$(p_1, p_2)$	$\eta$	$b = 3$			$b = 5$			$b = 7$		
		$\delta_{12}^2$	$\delta_{13}^2$	$\delta_{14}^2$	$\delta_{12}^2$	$\delta_{13}^2$	$\delta_{14}^2$	$\delta_{12}^2$	$\delta_{13}^2$	$\delta_{14}^2$
(5,5)	0.1	0.426	0.069	0.026	0.000	0.000	0.000	0.000	0.000	0.000
	0.3	4.155	1.454	1.076	0.189	0.000	0.000	0.000	0.000	0.000
	0.5	8.803	3.911	3.151	1.143	0.142	0.059	0.085	0.000	0.000
	0.7	13.163	6.805	5.658	2.813	0.513	0.269	0.348	0.010	0.000
	0.9	16.928	9.747	8.249	5.020	1.190	0.673	0.997	0.081	0.021
(5,8)	0.1	1.213	0.282	0.242	0.000	0.000	0.000	0.000	0.000	0.000
(15,10)	0.3	7.493	2.039	1.749	0.634	0.000	0.000	0.039	0.000	0.000
	0.5	14.430	4.873	4.251	3.013	0.115	0.059	0.489	0.000	0.000
	0.7	20.269	8.303	7.274	6.722	0.493	0.295	1.746	0.006	0.000
	0.9	24.689	11.791	10.436	11.026	1.234	0.780	3.941	0.086	0.033
	0.1	0.335	0.217	0.121	0.000	0.000	0.000	0.000	0.000	0.000
(15,21)	0.3	1.945	1.349	0.835	0.000	0.000	0.000	0.000	0.000	0.000
	0.5	4.359	3.086	1.821	0.059	0.015	0.000	0.000	0.000	0.000
	0.7	7.198	5.232	3.123	0.324	0.103	0.010	0.000	0.000	0.000
	0.9	10.085	7.616	4.632	0.960	0.340	0.040	0.018	0.000	0.000
	0.1	0.848	0.140	0.082	0.000	0.000	0.000	0.000	0.000	0.000
	0.3	5.601	1.794	1.316	0.075	0.000	0.000	0.000	0.000	0.000
	0.5	11.790	4.245	3.140	1.083	0.007	0.000	0.031	0.000	0.000
	0.7	17.878	7.296	5.424	3.484	0.098	0.015	0.321	0.000	0.000
	0.9	23.063	10.626	8.066	6.941	0.401	0.120	1.232	0.000	0.000

Table ML5: RRI values of  $\delta_{11}^2$  and  $\delta_{\phi_*^2}$  of  $\sigma_1$  under  $L_2(\cdot)$ 

$(p_1, p_2)$	$\eta$	$b = 3$		$b = 5$		$b = 7$	
		$\delta_{11}^2$	$\delta_{\phi_*^2}$	$\delta_{11}^2$	$\delta_{\phi_*^2}$	$\delta_{11}^2$	$\delta_{\phi_*^2}$
(5,5)	0.1	4.599	10.570	1.711	7.384	0.908	6.133
	0.3	13.895	17.416	9.671	15.787	7.698	14.931
	0.5	18.277	17.207	15.538	16.203	14.023	15.509
	0.7	19.812	15.114	18.404	13.851	17.462	12.879
	0.9	19.783	12.483	19.130	10.552	18.589	9.087
(5,8)	0.1	4.849	11.486	1.205	6.947	0.477	5.494
	0.3	16.792	21.422	10.538	18.475	7.988	17.567
	0.5	22.719	21.758	18.510	19.858	16.665	19.514
	0.7	24.633	19.026	22.351	16.792	21.600	16.266
	0.9	24.228	15.243	22.983	11.920	22.958	10.760
(15,10)	0.1	2.942	5.488	0.382	1.979	0.078	1.029
	0.3	11.668	14.917	6.596	11.596	4.284	9.971
	0.5	16.915	18.069	13.422	16.631	11.327	15.927
	0.7	19.255	18.262	17.694	17.532	16.689	17.176
	0.9	19.792	17.132	19.400	16.165	19.170	15.517
(15,21)	0.1	3.870	6.430	0.314	1.599	0.061	0.644
	0.3	16.168	19.918	8.210	14.091	4.953	11.521
	0.5	23.664	24.917	18.232	22.277	15.218	21.193
	0.7	26.522	24.960	24.257	23.734	23.095	23.444
	0.9	26.351	22.534	25.719	20.815	25.749	20.012

Table ML6: RRI of improved estimators of  $\sigma_2$  under  $L_2(\cdot)$  over  $\delta_{22}$ 

$(p_1, p_2)$	$\eta$	$b = 3$		$b = 5$		$b = 7$	
		$\delta_{21}^2$	$\delta_{\psi_*^2}$	$\delta_{21}^2$	$\delta_{\psi_*^2}$	$\delta_{21}^2$	$\delta_{\psi_*^2}$
(5,5)	0.1	0.945	8.226	0.507	8.075	0.336	8.169
	0.3	6.764	21.075	5.657	22.857	5.315	24.027
	0.5	12.924	24.087	13.154	25.687	13.425	26.708
	0.7	17.546	20.372	19.624	19.528	20.896	19.100
	0.9	20.203	11.894	23.876	6.790	26.072	3.858
(5,8)	0.1	0.397	4.467	0.116	4.115	0.054	4.029
	0.3	4.067	14.341	2.911	15.182	2.439	15.616
	0.5	8.816	19.149	8.274	20.364	7.929	20.676
	0.7	12.836	19.738	13.666	19.851	13.983	19.062
	0.9	15.612	17.117	17.802	14.711	18.817	11.919
(15,10)	0.1	0.508	2.247	0.114	1.316	0.025	0.892
	0.3	5.660	12.861	4.100	13.232	3.048	13.041
	0.5	11.867	19.625	12.022	23.145	11.312	24.602
	0.7	15.991	20.208	19.143	24.702	20.098	26.399
	0.9	17.086	15.269	22.600	17.661	25.090	17.424
(15,21)	0.1	0.293	1.012	0.037	0.413	0.005	0.214
	0.3	3.708	7.811	2.207	6.970	1.398	6.410
	0.5	8.723	14.543	7.774	15.459	6.824	15.914
	0.7	13.056	18.768	14.027	20.645	14.005	21.798
	0.9	15.732	20.344	18.542	21.150	19.689	21.701

Table ML7: RRI of  $\delta_{D2}$  for  $\sigma_2$  under  $L_2(\cdot)$  over  $\delta_{22}$ 

$(p_1, p_2)$	$\eta$	$(\mu_1, \mu_2) = (-0.5, -0.3)$			$(\mu_1, \mu_2) = (0, 0)$			$(\mu_1, \mu_2) = (0.2, 0.5)$			$(\mu_1, \mu_2) = (0.5, 1)$		
		$b = 3$	$b = 5$	$b = 7$	$b = 3$	$b = 5$	$b = 7$	$b = 3$	$b = 5$	$b = 7$	$b = 3$	$b = 5$	$b = 7$
(5,5)	0.1	0.413	0.084	0.020	8.495	7.570	6.747	2.528	0.544	0.336	1.679	0.517	0.336
	0.3	0.809	0.129	0.035	14.592	12.941	11.924	8.347	5.694	5.315	7.497	5.667	5.315
	0.5	1.208	0.191	0.054	21.231	20.982	20.595	14.507	13.191	13.425	13.657	13.164	13.425
	0.7	1.503	0.245	0.071	26.417	28.196	28.885	19.129	19.661	20.896	18.280	19.634	20.896
	0.9	1.685	0.279	0.082	29.663	33.299	35.039	21.786	23.913	26.072	20.936	23.886	26.072
(5,8)	0.1	0.105	0.026	0.000	5.677	5.088	4.726	0.996	0.126	0.054	0.729	0.116	0.054
	0.3	0.194	0.026	0.000	9.500	7.996	7.203	4.666	2.921	2.439	4.398	2.911	2.439
	0.5	0.313	0.042	0.003	14.532	13.675	13.019	9.415	8.284	7.929	9.148	8.274	7.929
	0.7	0.425	0.059	0.005	18.898	19.537	19.617	13.435	13.676	13.983	13.167	13.666	13.983
	0.9	0.506	0.070	0.008	22.049	24.237	25.135	16.211	17.812	18.817	15.944	17.802	18.817
(15,10)	0.1	0.076	0.001	0.000	4.752	4.208	3.902	0.720	0.118	0.025	0.563	0.114	0.025
	0.3	0.168	0.002	0.000	10.100	8.331	7.017	5.872	4.104	3.048	5.714	4.100	3.048
	0.5	0.263	0.008	0.001	16.675	16.685	15.701	12.079	12.025	11.312	11.922	12.022	11.312
	0.7	0.334	0.017	0.003	21.251	24.469	25.259	16.203	19.146	20.098	16.045	19.143	20.098
	0.9	0.356	0.023	0.005	22.828	28.713	31.244	17.298	22.603	25.090	17.140	22.600	25.090
(15,21)	0.1	0.014	0.000	0.000	2.512	2.238	2.038	0.388	0.037	0.005	0.366	0.037	0.005
	0.3	0.027	0.000	0.000	6.000	4.455	3.461	3.803	2.207	1.398	3.781	2.207	1.398
	0.5	0.048	0.001	0.000	11.160	10.192	9.054	8.818	7.774	6.824	8.796	7.774	6.824
	0.7	0.063	0.003	0.000	15.683	16.731	16.575	13.151	14.027	14.005	13.129	14.027	14.005
	0.9	0.073	0.004	0.000	18.568	21.603	22.730	15.827	18.542	19.689	15.804	18.542	19.689

### 4.3 Exponential-Inverse Gaussian (E-IG) distribution

In this subsection, we will derive the improved estimator of the parameter for Exponential-Inverse Gaussian distribution under the loss function  $L_1(\cdot)$  and  $L_2(\cdot)$ . In this case, we have  $E(\tau) = m$ ,  $E(\tau^{-1}) = \frac{1}{m} + \frac{1}{n}$  (see [36]). We get the BAEE of  $\sigma_i$  under the loss function  $L_1(\cdot)$  is as  $\delta_{1i} = c_i S_i$  with

$$c_i = \left( \frac{m}{(p_i - 1)(p_i - 2) \left( \frac{1}{m} + \frac{1}{n} \right)} \right)^{1/2}$$

for  $i = 1, 2$  and for the loss function  $L_2(\cdot)$  we have the BAEE of  $\sigma_i$  is  $\delta_{2i} = d_i S_i$  with

$$d_i = \frac{1}{(p_i - 1) \left( \frac{1}{m} + \frac{1}{n} \right)}.$$

Estimation of a quantile in a mixture model of exponential distributions is considered by [34]. In particular, improved estimators for a quantile of an Exponential-Inverse Gaussian distribution and the multivariate Lomax distribution with unknown location and scale parameters are derived. Now we will applied the result Theorem 1, 4 and 5 for Exponential-Inverse Gaussian distribution to find the improved estimator of the parameter  $\sigma_1$  under two scale invariant loss function  $L_1(\cdot)$  and  $L_2(\cdot)$ .

**Theorem 16.** (i) Under the loss function  $L_1(\cdot)$ , we have  $\varphi_{11}(W) = (1+W) \left( \frac{m}{(p_1+p_2-2)(p_1+p_2-3) \left( \frac{1}{m} + \frac{1}{n} \right)} \right)^{\frac{1}{2}}$ ,  $\varphi_{21}(W, W_1) = \frac{(1+W+p_1 W_1) m^{1/2}}{\left( (p_1+p_2-1)(p_1+p_2-2) \left( \frac{1}{m} + \frac{1}{n} \right) \right)^{1/2}}$ . The improved estimator of  $\sigma_1$  are obtained as follows

$$\delta_{11}^1(X, S) = \min \{ \varphi_{11}(W), c_1 \} S_1$$

$$\delta_{12}^1 = \begin{cases} \min \{ \varphi_{21}(W, W_1), c_1 \} S_1, & W_1 > 0 \\ c_1 S_1, & \text{otherwise} \end{cases}$$

$$\delta_{13}^1 = \begin{cases} \min \{ \varphi_{31}(W, W_2), c_1 \} S_1, & W_2 > 0 \\ c_1 S_1, & \text{otherwise} \end{cases} \quad (30)$$

(ii) For the loss function  $L_2(\cdot)$ , we have  $\varphi_{21}(W) = \frac{(1+W)}{(p_1+p_2-2) \left( \frac{1}{m} + \frac{1}{n} \right)}$ ,  $\varphi_{22}(W, W_1) = \frac{(1+W+p_1 W_1)}{(p_1+p_2-1) \left( \frac{1}{m} + \frac{1}{n} \right)}$ ,  $\varphi_{32}(W, W_1) = \frac{(1+W+p_2 W_2)}{(p_1+p_2-1) \left( \frac{1}{m} + \frac{1}{n} \right)}$ . We have the improved estimator of  $\sigma_1$  are as follows

$$\delta_{11}^2(X, S) = \min \{ \varphi_{21}(W), d_1 \} S_1$$

$$\delta_{12}^2 = \begin{cases} \min \{ \varphi_{22}(W, W_1), d_1 \} S_1, & W_1 > 0 \\ d_1 S_1, & \text{otherwise} \end{cases} \quad (31)$$

$$\delta_{13}^2 = \begin{cases} \min \{ \varphi_{32}(W, W_2), d_1 \} S_1, & W_2 > 0 \\ d_1 S_1, & \text{otherwise} \end{cases} \quad (32)$$

If we use the both information  $X$  and  $Y$ , analogous result to Theorem 4 can be derived, as described in the following theorem.

**Theorem 17.** (i) Under the loss function  $L_1(\cdot)$  we have  $\varphi_{41}(W, W_1, W_2) = (1 + W + p_1 W_1 + p_2 W_2) \left( \frac{m}{(1/m+1/n)(p_1+p_2)(p_1+p_2-1)} \right)^{1/2}$ . The improved estimator of  $\sigma_1$  is obtained as

$$\delta_{14}^1 = \begin{cases} \min \{ \varphi_{41}(W, W_1, W_2), d_1 \} S_1, & W_1 > 0, W_2 > 0 \\ d_1 S_1, & \text{otherwise} \end{cases}$$

(ii) For the Stein type loss function  $L_2(\cdot)$ , we have  $\varphi_{42}(W, W_1, W_2) = (1+W+p_1 W_1+p_2 W_2) \frac{mn}{(m+n)(p_1+p_2)}$ . We get the improved estimators as

$$\delta_{14}^2 = \begin{cases} \min \{ \varphi_{42}(W, W_1, W_2), d_1 \} S_1, & W_1 > 0, W_2 > 0 \\ d_1 S_1, & \text{otherwise} \end{cases}$$

As in the previous section, now we have obtain an improved estimator for  $\sigma_1$  within the class of estimators (3) using the IERD method of [19]. In that case, we have the following theorem as.

**Theorem 18.** (i) Under the loss function  $L_1(\cdot)$ , the risk of the estimator  $\delta_{\varphi_1}$  defined in (3) is nowhere larger than that of  $\delta_{11}$  provided the function  $\varphi_1(w)$  satisfies

$$(a) \varphi_1(w) \text{ is non-decreasing in } w \text{ and } \lim_{w \rightarrow \infty} \varphi_1(w) = \left( \frac{m}{(p_1-1)(p_1-2)(1/m+1/n)} \right)^{1/2}$$

$$(b) \varphi_1(w) \geq \varphi_1^*(w) = \left( \frac{m B\left(\frac{w}{1+w}; p_1-2, p_2-1\right)}{(1/m+1/n)(p_1+p_2-2)(p_1+p_2-3)B\left(\frac{w}{1+w}; p_1, p_2-1\right)} \right)^{1/2}.$$

(ii) Let us assume that the function  $\varphi_1(w)$  satisfies the following conditions:

$$(a) \varphi_1(w) \text{ is non-decreasing in } w \text{ and } \lim_{w \rightarrow \infty} \varphi_1(w) = \frac{1}{(p_1-1)(1/m+1/n)}$$

$$(b) \varphi_1(w) \geq \varphi_1^*(w) = \left( \frac{B\left(\frac{w}{1+w}; p_1-1, p_2-1\right)}{(1/m+1/n)(p_1+p_2-2)B\left(\frac{w}{1+w}; p_1, p_2-1\right)} \right)^{1/2}$$

Then the risk of the estimator  $\delta_{\varphi_1}$  defined in (3) is nowhere larger than that of  $\delta_{21}$ .

Now we have obtained the improve estimator of  $\sigma_2$  for  $L_1(\cdot)$  and  $L_2(\cdot)$  loss functions respectively.

**Theorem 19.** (i) For the loss  $L_1(\cdot)$ , we have  $\psi_{11}(U) = (1 + U) \left( \frac{m}{(p_1+p_2-2)(p_1+p_2-3)(1/m+1/n)} \right)^{1/2}$ ,  $\psi_{21}(U_1) = \frac{(1+p_2 U_1)m^{1/2}}{(p_2(p_2-1)(1/m+1/n))^{1/2}}$ . The improve estimator of  $\sigma_2$  are obtained as follows

$$\delta_{21}^1(X, S) = \max \{ \psi_{11}(U), c_2 \} S_2$$

$$\delta_{22}^1 = \begin{cases} \min \{ \psi_{21}(U_1), c_2 \} S_2, & U_1 > 0 \\ c_2 S_2, & \text{otherwise} \end{cases}$$

(ii) Under  $L_2(\cdot)$  loss function, we have  $\psi_{12}(U) = \frac{(1+U)}{(p_1+p_2-2)(1/m+1/n)}$ ,  $\psi_{22}(U_1) = \frac{(1+p_2 U_1)}{p_2(1/m+1/n)}$ . We get the improve estimator of  $\sigma_2$  as

$$\delta_{21}^2(X, S) = \max \{ \psi_{12}(U), d_2 \} S_2,$$

$$\delta_{22}^2 = \begin{cases} \min \{ \psi_{22}(U_1), d_2 \} S_2, & U_1 > 0 \\ d_2 S_2, & \text{otherwise} \end{cases}$$



We now obtain improved estimator of  $\sigma_2$  with in the class of estimator (17) using the [19].

**Theorem 20.** (i) Under the loss function  $L_1(\cdot)$ , the risk of the estimator  $\delta_{\psi_1}$  is nowhere larger than that of  $\delta_{12}$  provided the function  $\psi_1(u)$  satisfies the following conditions

$$(a) \psi_1(u) \text{ is non-decreasing in } u \text{ and } \lim_{u \rightarrow 0} \psi_1(u) = \left( \frac{m}{(p_2-1)(p_2-2)(1/m+1/n)} \right)^{1/2}$$

$$(b) \psi_1(u) \leq \psi_*^1(u) = \left( \frac{m \left[ \frac{B(p_1-1, p_2-2)}{\Gamma(p_1-1)} - B\left(\frac{u}{1+u}; p_1-1, p_2-2\right) \right]}{(1/m+1/n)(p_1+p_2-2)(p_1+p_2-3) \left[ \frac{B(p_1-1, p_2)}{\Gamma(p_1-1)} - B\left(\frac{u}{1+u}; p_1-1, p_2\right) \right]} \right)^{1/2}.$$

(ii) Let us assume that the function  $\psi_1(u)$  satisfies the following conditions

$$(a) \psi_1(u) \text{ is non-decreasing in } u \text{ and } \lim_{u \rightarrow 0} \psi_1(u) = \frac{1}{(p_2-1)(1/m+1/n)}$$

$$(b) \psi_1(u) \leq \psi_*^2(u) = \frac{\frac{B(p_1-1, p_2-1)}{\Gamma(p_1-1)} - B\left(\frac{u}{1+u}; p_1-1, p_2-1\right)}{(1/m+1/n)(p_1+p_2-2) \left[ \frac{B(p_1-1, p_2)}{\Gamma(p_1-1)} - B\left(\frac{u}{1+u}; p_1, p_2-1\right) \right]}.$$

Then the risk of the estimator  $\delta_{\psi_1}$  defined in (17) is nowhere larger than that of  $\delta_{22}$ .

As an application of the Theorem 9, we have obtain the following results

**Theorem 21.** (i) Under the loss function  $L_1(\cdot)$ , we consider the estimator

$$\delta_{D1} = (\psi_{11}(U) + \psi_{21}(U_1) - c_2) S_2$$

where  $\psi_{11}$  and  $\psi_{21}$  are defined in Theorem 13, and  $c_2 = \left( \frac{m}{(p_2-1)(p_2-2)(1/m+1/n)} \right)^{1/2}$ . Then the estimator  $\delta_{D1}$  has smaller or equal risk function than that of the estimator  $\delta_{\psi_{11}}$ ,  $\delta_{\psi_{21}}$  and so of the estimator  $\delta_{12} = c_2 S_2$ .

(ii) The estimator

$$\delta_{D2} = (\psi_{21}(U) + \psi_{22}(U_1) - d_2) S_2$$

has the smaller risk than that of the estimator  $\delta_{\psi_{21}}$ ,  $\delta_{\psi_{22}}$  and so of the estimator  $\delta_{22} = d_2 S_2$  under the loss function  $L_2(\cdot)$ .

#### 4.4 Simulation study

In this Subsection, we have discuss the numerical performance of the proposed estimators derived in Sections 4.3. For this purpose, we generate the mixing distribution from an inverse Gaussian distribution  $IG(m, n)$  with parameter values  $(m, n) = (3, 5), (4, 6), (8, 10)$ . The performance of the improved estimators is calculated using the relative risk improvement (RRI) with respect to the BAAE. The RRI is computed in the same manner as defined in the Subsection 4.2. In this numerical study, we consider the discussion for risk performance of  $\sigma_1$  and  $\sigma_2$  for E-IG distribution under the loss function  $L_2(\cdot)$ . In Table G1-G5, we have presented the RRI values of  $\delta_{12}^2$ ,  $\delta_{13}^2$ , and  $\delta_{14}^2$  with the respect to the BAAE  $\delta_{21}$ . From the tabulated RRI values we have observed that the RRI of  $\delta_{12}^2$ ,  $\delta_{13}^2$  and  $\delta_{14}^2$  increasing function of  $\eta$  for  $0 < \eta \leq 1$ . Also the RRI of  $\delta_{12}^2$ ,  $\delta_{13}^2$  and  $\delta_{14}^2$  decrease as  $(m, n)$  increases.

In Table G5, we have tabulated the RRI of  $\delta_{11}^2$  and  $\delta_{\varphi_*^2}$ . It can be easily seen that the risk of these estimators are independent of the parameter  $\mu_1$  and  $\mu_2$ . Thus we have computed the RRI of  $\delta_{11}^2$  and  $\delta_{\varphi_*^2}$  only at  $(\mu_1, \mu_2) = (0, 0)$ . Here the  $\delta_{11}^2$  and  $\delta_{\varphi_*^2}$  are decreasing function of  $(m, n)$ . RRI of  $\delta_{11}^2$  is an increasing function  $\eta$  but RRI of  $\delta_{\varphi_*^2}$  is not monotone function of  $\eta$ , it is increasing when  $\eta < 0.6$  (approximately) and decreases for  $\eta > 0.6$  (approximately). Overall, Table G5 shows that  $\delta_{\varphi_*^2}$  performs better than  $\delta_{11}^2$  when  $0 < \eta < 0.6$  (approximately) and when  $\eta > 0.6$  (approximately) then  $\delta_{11}^2$  performs better than  $\delta_{\varphi_*^2}$ .

Among the estimators in Table G1-G4, we observed that the best performance of the estimator varies with the values of  $(\mu_1, \mu_2)$ . Specifically:

- (i) When  $(\mu_1, \mu_2)$  away from  $(0,0)$  in negative direction (see Table G1) the RRI of  $\delta_{13}^2$  is greater than that of  $\delta_{12}^2$  and  $\delta_{14}^2$ .
- (ii)  $\delta_{12}^2$  performs better than the other two estimators when  $(\mu_1, \mu_2) = (0,0)$  (see Table G2).
- (iii) When  $(\mu_1, \mu_2)$  move away from  $(0,0)$  in positive direction (see Table G3, G4) the RRI of  $\delta_{12}^2$  is greater than that of  $\delta_{13}^2$  and  $\delta_{14}^2$ .

Now we discuss the risk performance of the estimators for the parameter  $\sigma_2$  under the loss function  $L_2(\cdot)$ . In Table G6, we present the RRI of  $\delta_{21}^2$  and  $\delta_{\psi_*^2}$  over  $\delta_{22}$  for estimating  $\sigma_2$ . The RRI of  $\delta_{21}^2$  is increasing function of  $\eta$  whereas the RRI of  $\delta_{\psi_*^2}$  is not monotonic in  $\eta$ ; it increases for  $0 < \eta < 0.7$  (approximately) and decreases for  $\eta > 0.7$  (approximately). The RRI of both estimators  $\delta_{21}^2$  and  $\delta_{\psi_*^2}$  are increasing function of  $(m, n)$ . In Table G6, the estimator  $\delta_{\psi_*^2}$  performs better than  $\delta_{21}^2$ . In Table G7, the RRI of  $\delta_{D2}$  is an increasing function of  $\eta$ . Furthermore, it is decreasing function of  $(m, n)$  when  $(\mu_1, \mu_2) < (0,0)$  and increasing when  $(\mu_1, \mu_2) \geq (0,0)$ . Among the estimators  $\delta_{21}^2$ ,  $\delta_{\psi_*^2}$  and  $\delta_{D2}$  for estimating  $\sigma_2$ , we can observe that  $\delta_{D2}$  performs better than  $\delta_{21}^2$  and  $\delta_{\psi_*^2}$  when  $(\mu_1, \mu_2) = (0,0)$ . In all other cases,  $\delta_{\psi_*^2}$  is preferable than  $\delta_{D2}$  and  $\delta_{21}^2$  for estimating  $\sigma_2$ .

The same observations can be found for the loss function  $L_1(\cdot)$ , and hence we omit the discussion for the sake of simplicity.

Table G1: RRI of improved estimators of  $\sigma_1$  under  $L_2(\cdot)$  over  $\delta_{21}$  when  $(\mu_1, \mu_2) = (-0.5, -0.3)$

$(m, n)$		(3,5)			(4,6)			(8,10)		
$(p_1, p_2)$	$\eta$	$\delta_{12}^2$	$\delta_{13}^2$	$\delta_{14}^2$	$\delta_{12}^2$	$\delta_{13}^2$	$\delta_{14}^2$	$\delta_{12}^2$	$\delta_{13}^2$	$\delta_{14}^2$
(5,5)	0.1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.3	0.000	0.243	0.000	0.000	0.165	0.000	0.000	0.068	0.000
	0.5	0.102	0.993	0.008	0.057	0.702	0.000	0.022	0.254	0.000
	0.7	0.435	1.922	0.043	0.248	1.381	0.014	0.045	0.512	0.000
	0.9	1.169	2.879	0.106	0.670	2.097	0.039	0.153	0.783	0.012
(5,8)	0.1	0.000	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.3	0.000	0.100	0.000	0.000	0.047	0.000	0.000	0.012	0.000
	0.5	0.080	0.448	0.000	0.030	0.273	0.000	0.005	0.071	0.000
	0.7	0.439	0.963	0.024	0.271	0.624	0.010	0.031	0.169	0.000
	0.9	1.232	1.571	0.080	0.735	1.029	0.043	0.137	0.274	0.000
(15,10)	0.1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.3	0.000	0.004	0.000	0.000	0.001	0.000	0.000	0.000	0.000
	0.5	0.001	0.067	0.000	0.000	0.030	0.000	0.000	0.003	0.000
	0.7	0.006	0.263	0.000	0.007	0.142	0.000	0.000	0.011	0.000
	0.9	0.019	0.560	0.000	0.011	0.297	0.000	0.000	0.035	0.000
(15,21)	0.1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.3	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.5	0.000	0.006	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.7	0.004	0.029	0.000	0.005	0.007	0.000	0.000	0.000	0.000
	0.9	0.022	0.074	0.000	0.011	0.031	0.000	0.000	0.000	0.000

Table G2: RRI of improved estimators of  $\sigma_1$  under  $L_2(\cdot)$  over  $\delta_{21}$  when  $(\mu_1, \mu_2) = (0, 0)$ 

$(m, n)$		(3,5)			(4,6)			(8,10)		
$(p_1, p_2)$	$\eta$	$\delta_{12}^2$	$\delta_{13}^2$	$\delta_{14}^2$	$\delta_{12}^2$	$\delta_{13}^2$	$\delta_{14}^2$	$\delta_{12}^2$	$\delta_{13}^2$	$\delta_{14}^2$
(5,5)	0.1	0.097	0.028	0.005	0.093	0.026	0.004	0.083	0.023	0.002
	0.3	1.670	1.043	0.220	1.575	0.978	0.172	1.433	0.897	0.110
	0.5	4.055	3.363	0.935	3.826	3.163	0.758	3.503	2.914	0.510
	0.7	5.808	5.433	2.007	5.469	5.108	1.662	5.011	4.700	1.172
	0.9	6.632	6.466	3.221	6.235	6.069	2.721	5.707	5.580	1.977
(5,8)	0.1	0.012	0.002	0.000	0.011	0.002	0.000	0.010	0.002	0.000
	0.3	1.068	0.739	0.139	1.006	0.696	0.109	0.913	0.635	0.068
	0.5	4.028	3.434	1.009	3.807	3.236	0.835	3.483	2.966	0.584
	0.7	6.815	6.474	2.625	6.429	6.106	2.225	5.893	5.614	1.632
	0.9	8.252	8.122	4.563	7.774	7.644	3.959	7.120	7.022	3.040
(15,10)	0.1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.3	0.075	0.050	0.000	0.070	0.047	0.000	0.061	0.041	0.000
	0.5	0.601	0.484	0.009	0.558	0.451	0.005	0.489	0.398	0.001
	0.7	1.543	1.426	0.062	1.439	1.337	0.036	1.267	1.187	0.011
	0.9	2.228	2.151	0.195	2.080	2.012	0.119	1.827	1.783	0.043
(15,21)	0.1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.3	0.010	0.006	0.000	0.010	0.006	0.000	0.008	0.005	0.000
	0.5	0.358	0.315	0.013	0.331	0.290	0.008	0.287	0.252	0.002
	0.7	1.725	1.637	0.140	1.612	1.532	0.097	1.412	1.348	0.047
	0.9	3.304	3.269	0.556	3.090	3.060	0.412	2.719	2.702	0.225

Table G3: RRI of improved estimators of  $\sigma_1$  under  $L_2(\cdot)$  over  $\delta_{21}$  when  $(\mu_1, \mu_2) = (0.2, 0.5)$ 

$(m, n)$		(3,5)			(4,6)			(8,10)		
$(p_1, p_2)$	$\eta$	$\delta_{12}^2$	$\delta_{13}^2$	$\delta_{14}^2$	$\delta_{12}^2$	$\delta_{13}^2$	$\delta_{14}^2$	$\delta_{12}^2$	$\delta_{13}^2$	$\delta_{14}^2$
(5,5)	0.1	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.3	0.483	0.041	0.000	0.330	0.019	0.000	0.083	0.000	0.000
	0.5	2.630	0.560	0.015	2.031	0.340	0.005	0.847	0.066	0.000
	0.7	5.935	2.182	0.113	4.942	1.479	0.043	2.625	0.420	0.001
	0.9	9.297	4.753	0.393	8.164	3.479	0.166	5.050	1.252	0.012
(5,8)	0.1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.3	0.439	0.017	0.000	0.333	0.006	0.000	0.145	0.000	0.000
	0.5	2.951	0.493	0.025	2.445	0.314	0.009	1.310	0.061	0.000
	0.7	7.138	2.113	0.236	6.304	1.435	0.106	4.092	0.427	0.008
	0.9	11.290	5.012	0.742	10.458	3.648	0.391	7.730	1.327	0.050
(15,10)	0.1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.3	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.5	0.098	0.024	0.000	0.055	0.010	0.000	0.009	0.000	0.000
	0.7	0.693	0.229	0.000	0.453	0.126	0.000	0.115	0.022	0.000
	0.9	2.241	1.002	0.000	1.605	0.612	0.000	0.513	0.130	0.000
(15,21)	0.1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.3	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.5	0.118	0.002	0.000	0.082	0.000	0.000	0.022	0.000	0.000
	0.7	1.155	0.136	0.000	0.870	0.072	0.000	0.351	0.003	0.000
	0.9	4.213	0.874	0.006	3.476	0.514	0.000	1.748	0.092	0.000

Table G4: RRI of improved estimators of  $\sigma_1$  under  $L_2(\cdot)$  over  $\delta_{21}$  when  $(\mu_1, \mu_2) = (0.5, 1)$ 

$(m, n)$		(3,5)			(4,6)			(8,10)		
$(p_1, p_2)$	$\eta$	$\delta_{12}^2$	$\delta_{13}^2$	$\delta_{14}^2$	$\delta_{12}^2$	$\delta_{13}^2$	$\delta_{14}^2$	$\delta_{12}^2$	$\delta_{13}^2$	$\delta_{14}^2$
(5,5)	0.1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.3	0.060	0.001	0.000	0.027	0.000	0.000	0.000	0.000	0.000
	0.5	0.750	0.097	0.000	0.469	0.046	0.000	0.082	0.000	0.000
	0.7	2.445	0.573	0.005	1.649	0.306	0.000	0.471	0.036	0.000
	0.9	4.910	1.639	0.038	3.597	0.993	0.009	1.264	0.176	0.000
(5,8)	0.1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.3	0.127	0.000	0.000	0.073	0.000	0.000	0.011	0.000	0.000
	0.5	1.255	0.091	0.000	0.878	0.039	0.000	0.297	0.002	0.000
	0.7	4.070	0.580	0.021	3.080	0.317	0.007	1.238	0.042	0.000
	0.9	7.906	1.720	0.121	6.368	1.062	0.043	3.057	0.201	0.002
(15,10)	0.1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.3	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.5	0.005	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.7	0.086	0.032	0.000	0.039	0.011	0.000	0.000	0.000	0.000
	0.9	0.413	0.182	0.000	0.220	0.087	0.000	0.027	0.005	0.000
(15,21)	0.1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.3	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.5	0.016	0.000	0.000	0.005	0.000	0.000	0.000	0.000	0.000
	0.7	0.319	0.006	0.000	0.186	0.000	0.000	0.032	0.000	0.000
	0.9	1.649	0.134	0.000	1.076	0.057	0.000	0.269	0.000	0.000

Table G5: RRI values of  $\delta_{11}^2$  and  $\delta_{\phi_*^2}$  of  $\sigma_1$  under  $L_2(\cdot)$ 

$(m, n)$		(3,5)		(4,6)		(8,10)	
$(p_1, p_2)$	$\eta$	$\delta_{11}^2$	$\delta_{\phi_*^2}$	$\delta_{11}^2$	$\delta_{\phi_*^2}$	$\delta_{11}^2$	$\delta_{\phi_*^2}$
(5,5)	0.1	0.054	1.304	0.052	1.228	0.046	1.125
	0.3	1.209	4.437	1.137	4.166	1.039	3.821
	0.5	3.217	4.616	3.026	4.311	2.778	3.950
	0.7	4.854	3.163	4.560	2.915	4.193	2.667
	0.9	5.651	1.028	5.296	0.879	4.865	0.797
(5,8)	0.1	0.006	0.947	0.006	0.894	0.005	0.817
	0.3	0.815	5.185	0.768	4.885	0.698	4.471
	0.5	3.457	6.308	3.260	5.923	2.980	5.419
	0.7	6.173	4.741	5.822	4.416	5.343	4.034
	0.9	7.671	1.757	7.217	1.569	6.623	1.424
(15,10)	0.1	0.000	0.011	0.000	0.011	0.000	0.009
	0.3	0.058	0.690	0.054	0.643	0.046	0.567
	0.5	0.512	1.647	0.475	1.534	0.417	1.349
	0.7	1.362	1.649	1.273	1.527	1.124	1.325
	0.9	2.024	0.703	1.892	0.634	1.671	0.513
(15,21)	0.1	0.000	0.001	0.000	0.001	0.000	0.001
	0.3	0.008	0.456	0.007	0.425	0.006	0.373
	0.5	0.320	2.096	0.295	1.955	0.256	1.722
	0.7	1.627	2.837	1.522	2.640	1.334	2.316
	0.9	3.211	1.554	3.005	1.433	2.651	1.224

Table G6: RRI of improved estimators of  $\sigma_2$  under  $L_2(\cdot)$  over  $\delta_{22}$

$(m, n)$		(3,5)		(4,6)		(8,10)	
$(p_1, p_2)$	$\eta$	$\delta_{21}^2$	$\delta_{\psi_*^2}$	$\delta_{21}^2$	$\delta_{\psi_*^2}$	$\delta_{21}^2$	$\delta_{\psi_*^2}$
(5,5)	0.1	2.050	10.377	2.292	10.642	2.726	11.113
	0.3	12.451	25.715	12.926	25.882	13.775	26.209
	0.5	21.428	29.647	21.781	29.855	22.435	30.276
	0.7	27.048	26.216	27.230	26.693	27.627	27.588
	0.9	29.401	17.703	29.478	18.643	29.725	20.330
(5,8)	0.1	1.020	5.968	1.201	6.198	1.546	6.639
	0.3	8.523	18.297	8.994	18.545	9.862	19.064
	0.5	16.189	24.404	16.628	24.670	17.485	25.239
	0.7	21.829	25.709	22.160	26.123	22.868	26.945
	0.9	25.161	23.428	25.419	24.125	26.029	25.411
(15,10)	0.1	1.111	3.762	1.337	4.124	1.833	4.845
	0.3	11.309	19.323	11.976	19.751	13.267	20.626
	0.5	21.588	28.613	22.100	28.777	23.131	29.238
	0.7	27.706	30.255	27.940	30.368	28.538	30.749
	0.9	29.194	25.554	29.272	25.918	29.633	26.693
(15,21)	0.1	0.556	1.737	0.702	1.982	1.046	2.484
	0.3	7.820	12.486	8.430	12.970	9.605	13.915
	0.5	16.712	21.803	17.310	22.134	18.444	22.788
	0.7	23.525	27.100	23.951	27.275	24.796	27.639
	0.9	27.427	28.581	27.712	28.779	28.329	29.124

Table G7: RRI of improved estimators of  $\delta_{D2}$  for  $\sigma_2$  under  $L_2(\cdot)$  over  $\delta_{22}$

$(m, n)$		(3, 5)	(4, 6)	(8, 10)	(3, 5)	(4, 6)	(8, 10)	(3, 5)	(4, 6)	(8, 10)	(3, 5)	(4, 6)	(8, 10)
$(p_1, p_2)$	$\eta$	$(\mu_1, \mu_2) = (-0.5, -0.3)$			$(\mu_1, \mu_2) = (0, 0)$			$(\mu_1, \mu_2) = (0.2, 0.5)$			$(\mu_1, \mu_2) = (0.5, 1)$		
(5,5)	0.1	0.637	0.436	0.102	8.474	8.697	9.067	2.429	2.479	2.742	2.073	2.296	2.726
	0.3	1.499	1.016	0.245	19.315	19.780	20.574	12.830	13.114	13.791	12.473	12.931	13.775
	0.5	2.244	1.489	0.364	28.962	29.286	29.854	21.807	21.968	22.450	21.451	21.785	22.435
	0.7	2.729	1.776	0.434	35.316	35.439	35.697	27.427	27.418	27.643	27.071	27.235	27.627
	0.9	2.955	1.902	0.463	38.405	38.386	38.432	29.780	29.665	29.741	29.423	29.482	29.725
(5,8)	0.1	0.169	0.098	0.013	5.538	5.694	5.962	1.041	1.206	1.546	1.020	1.201	1.546
	0.3	0.397	0.235	0.038	13.276	13.725	14.519	8.544	8.999	9.862	8.523	8.994	9.862
	0.5	0.627	0.364	0.061	21.324	21.730	22.493	16.209	16.633	17.485	16.189	16.628	17.485
	0.7	0.784	0.450	0.076	27.409	27.686	28.264	21.849	22.165	22.868	21.829	22.160	22.868
	0.9	0.874	0.499	0.084	31.200	31.376	31.814	25.182	25.424	26.029	25.161	25.419	26.029
(15,10)	0.1	0.062	0.032	0.006	4.806	5.011	5.449	1.112	1.337	1.833	1.111	1.337	1.833
	0.3	0.259	0.120	0.028	15.245	15.898	17.141	11.310	11.976	13.267	11.309	11.976	13.267
	0.5	0.436	0.197	0.045	25.957	26.444	27.405	21.589	22.100	23.131	21.588	22.100	23.131
	0.7	0.533	0.238	0.057	32.580	32.762	33.246	27.707	27.940	28.538	27.706	27.940	28.538
	0.9	0.546	0.239	0.061	34.590	34.581	34.774	29.195	29.272	29.633	29.194	29.272	29.633
(15,21)	0.1	0.002	0.002	0.000	2.468	2.598	2.897	0.556	0.702	1.046	0.556	0.702	1.046
	0.3	0.030	0.016	0.000	9.820	10.417	11.551	7.820	8.430	9.605	7.820	8.430	9.605
	0.5	0.052	0.028	0.000	18.884	19.466	20.550	16.712	17.310	18.444	16.712	17.310	18.444
	0.7	0.068	0.037	0.000	25.911	26.309	27.085	23.525	23.951	24.796	23.525	23.951	24.796
	0.9	0.077	0.043	0.000	30.039	30.281	30.805	27.427	27.712	28.329	27.427	27.712	28.329

## 5 Concluding remarks

In this work, we are tried to investigate the problem of estimating the scale parameters  $\sigma_1$  and  $\sigma_2$  from two scale mixture of exponential distribution under two scale invariant loss function  $L_1(\cdot)$  and  $L_2(\cdot)$ . We begin by deriving results for the scenario in which the mixing parameter  $\tau$  is unknown. As an application of derived theory, we then obtain two application based on multivariate Lomax distribution and E-IG distribution. We consider a class of scale invariant estimators and proposed sufficient conditions under which we obtain estimators that dominate the BAEE. Furthermore, we derive several [38]-type estimators that utilize all the information contained in the samples from the two populations and show that they dominate the BAEE. It has been observed that that a generalized Bayes estimator coincides with the [7]-type estimator. To utilize the additional information available from both samples, we construct a double-shrinkage estimator for  $\sigma_2$  as presented in Theorem 9. A numerical study has been conducted to compare the risk performance of these estimators.

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