

QUASI-ISOMETRIC RIGIDITY FOR A PRODUCT OF LATTICES

JOSIAH OH

ABSTRACT. We demonstrate quasi-isometric rigidity for the product of a non-uniform rank one lattice and a nilpotent lattice. Specifically, we show that any finitely-generated group quasi-isometric to such a product is, up to finite noise, an extension of a non-uniform rank one lattice by a nilpotent lattice. Furthermore, we show under extra conditions that this extension is nilcentral, a notion which generalizes central extensions to extensions by a nilpotent group.

1. INTRODUCTION

1.1. Background. One theme of geometric group theory is the rich relationship between the algebraic structure of groups and the large-scale geometry of spaces on which they act. By the “large-scale geometry” of a space we mean the metric structure that is preserved by quasi-isometries, maps which preserve distances up to a controlled error. Gromov proposed in a 1983 ICM address [Gro83] a broad research program of studying finitely generated groups as geometric objects and classifying them up to quasi-isometry. One aspect involves identifying instances of quasi-isometric rigidity, the phenomenon which occurs when algebraic properties of a finitely generated group are determined by its large-scale geometry. For example, a celebrated theorem by Gromov [Gro81] states that finitely generated groups of polynomial growth are virtually nilpotent (the converse is also true [Wol68]). Since growth rate is invariant under quasi-isometry, it follows that any finitely generated group which is quasi-isometric to a space of polynomial growth necessarily has a nilpotent subgroup of finite index. Thus we have an instance of when the quasi-isometry type of a group determines some of its algebraic structure.

One of the landmark results in research on quasi-isometric rigidity is the complete quasi-isometry classification of lattices in semisimple Lie groups. A large body of work in the 1980s and 1990s by several people over many papers culminated in a general theorem on the rigidity of the class of lattices among all finitely generated groups (see [Far97] for a detailed survey). Informally, this theorem states that any group quasi-isometric to a lattice in a semisimple Lie group is almost a lattice in that Lie group. More precisely,

Theorem 1 (Rigidity of lattices). *If Γ is a finitely generated group quasi-isometric to an irreducible lattice in a semisimple Lie group G , then there is a short exact sequence*

$$1 \longrightarrow F \longrightarrow \Gamma \longrightarrow \Lambda \longrightarrow 1$$

where Λ is a lattice in G , and F is a finite group.

Date: December 11, 2025.

2010 Mathematics Subject Classification. 20F65, 20F69, 20F18, 51F99.

Key words and phrases. quasi-isometry, quasi-isometric rigidity, nilpotent Lie group, negatively curved symmetric space, non-uniform lattice, neutered space, nilcentral extension.

The first author was partially supported by the NSF, under grant DMS-1547357.

One of the major breakthroughs leading to this general classification was the work of Schwartz [Sch95] on non-uniform lattices in rank one semisimple Lie groups. These Lie groups agree, up to index 2, with the isometry groups of the negatively curved symmetric spaces: real, complex, quaternionic hyperbolic space, and the Cayley hyperbolic plane.

Theorem 2 (Schwartz). *Let X be a negatively curved symmetric space other than the real hyperbolic plane \mathbb{H}^2 . If Γ is a finitely generated group quasi-isometric to a non-uniform lattice Λ in $\text{Isom}(X)$, then there exists a short exact sequence*

$$1 \longrightarrow F \longrightarrow \Gamma \longrightarrow \Lambda' \longrightarrow 1.$$

where $\Lambda' \leq \text{Isom}(X)$ is a non-uniform lattice commensurable to Λ , and F is a finite group.

In the case of $X = \mathbb{H}^2$, non-uniform lattices in $\text{Isom}(\mathbb{H}^2) = \text{PSL}(2, \mathbb{R})$ are virtually the fundamental groups of complete hyperbolic surfaces with finitely many punctures, and hence are virtually free. Any group quasi-isometric to a virtually free group is also virtually free, and every free group can be realized as a non-uniform lattice in $\text{PSL}(2, \mathbb{R})$, so quasi-isometric rigidity does hold in this case. However, the additional conclusion of commensurability fails. One reason for this failure is that commensurability preserves arithmeticity, and $\text{PSL}(2, \mathbb{R})$ contains both arithmetic and non-arithmetic non-uniform lattices. But since these lattices are virtually free, they are quasi-isometric to each other.

Following Schwartz, lattices such as Λ and Λ' shall from now on be called *non-uniform rank one lattices*. Then an informal summary might be: any group quasi-isometric to a non-uniform rank one lattice is almost a commensurable non-uniform rank one lattice.

In an extensive study on high-dimensional graph manifolds [FLS15], Frigerio–Lafont–Sisto prove, among many other things, a quasi-isometric rigidity result for products of the form $\pi_1(M) \times \mathbb{Z}^d$, where M is a complete non-compact finite-volume hyperbolic m -manifold, $m \geq 3$.

Theorem 3 (Frigerio–Lafont–Sisto). *If Γ is a finitely generated group quasi-isometric to $\pi_1(M) \times \mathbb{Z}^d$, then there exist short exact sequences*

$$1 \longrightarrow \mathbb{Z}^d \xrightarrow{j} \Gamma' \longrightarrow \Delta \longrightarrow 1$$

$$1 \longrightarrow F \longrightarrow \Delta \longrightarrow \pi_1(M') \longrightarrow 1$$

where $\Gamma' \leq \Gamma$ has finite index, M' is a finite-sheeted covering of M , Δ is a group, and F is a finite group. Moreover, $j(\mathbb{Z}^d)$ is contained in the center of Γ' . In other words, Γ is virtually a central extension by \mathbb{Z}^d of a finite extension of $\pi_1(M')$.

Observe that the case $d = 0$ is covered by Schwartz’ theorem, and indeed, the proof of this theorem applies many of the ideas and results from [Sch95].

1.2. Main results. Our main contribution is a generalization of Theorem 3 to products $\Lambda \times L$, where Λ is a non-uniform rank one lattice and L is a lattice in a simply connected nilpotent Lie group. From now on, a lattice such as L is called a *nilpotent lattice*. Our first theorem says that up to finite noise, any group quasi-isometric to $\Lambda \times L$ is an extension of a non-uniform rank one lattice commensurable to Λ by a nilpotent lattice quasi-isometric to L .

Theorem A. Let $X \neq \mathbb{H}^2$ be a negatively curved symmetric space. Let Λ be a non-uniform lattice in $\text{Isom}(X)$ and L be a nilpotent lattice. If Γ is a finitely generated group quasi-isometric to $\Lambda \times L$, then there exist short exact sequences

$$(1) \quad 1 \longrightarrow L' \longrightarrow \Gamma' \longrightarrow \Delta \longrightarrow 1,$$

$$1 \longrightarrow F \longrightarrow \Delta \longrightarrow \Lambda' \longrightarrow 1.$$

where $\Gamma' \leq \Gamma$ and $\Lambda' \leq \Lambda$ have finite index, L' is a nilpotent lattice quasi-isometric to L , Δ is a group, and F is a finite group.

The general outline of the proof is similar to that of the proof of Theorem 3. First, the quasi-isometry between Γ and $\Lambda \times L$ induces a quasi-action of Γ on $B \times N$, where $B \subset X$ is the neutered space associated to Λ , and N is the simply connected nilpotent Lie group in which L is a lattice. A theorem of Kapovich–Kleiner–Leeb [KKL98] guarantees that quasi-isometries $B \times N \rightarrow B \times N$ project to quasi-isometries $B \rightarrow B$, up to bounded error. Thus the quasi-action of Γ on $B \times N$ induces a quasi-action of Γ on B . A key result in [Sch95] is that quasi-isometries of the neutered space B have finite distance (with respect to the sup norm) from isometries of X . Thus we obtain a homomorphism $\theta : \Gamma \rightarrow \text{Isom}(X)$, and we show that the image $\text{im } \theta$ is a non-uniform lattice commensurable to Λ . The action $\theta : \Gamma \rightarrow \text{Isom}(X)$ came from a quasi-action on B , which itself was coarsely projected from a quasi-action on $B \times N$. Hence we are able to show that the kernel of θ is quasi-isometric to N . Then $\Gamma = \text{im } \theta / \ker \theta$, so we pass to finite-index subgroups as necessary to obtain the desired short exact sequences.

Theorem 3 asserts that \mathbb{Z}^d can be made central in the group extension. In our more general setting, however, the extension is by a nilpotent group. So we define the notion of a nilcentral extension, analogous to that of a central extension, and find conditions which are sufficient to guarantee that (1) may be taken to be a nilcentral extension. Given a nilpotent group G with upper central series $1 = Z_0 \triangleleft Z_1 \triangleleft \cdots \triangleleft Z_n = G$, define

$$\Sigma(G) := \max_i \text{rank}(Z_{i+1}/Z_i).$$

Theorem B. Assume the hypotheses of Theorem A and let L' be the nilpotent lattice obtained from the conclusion of the theorem. If X is either quaternionic hyperbolic space or the Cayley hyperbolic plane, and $\dim \text{Isom}(X) > \Sigma(L')$, then the group extension (1) in the conclusion of Theorem A is virtually nilcentral.

Our proof of this theorem relies on a form of Margulis–Corlette–Gromov–Schoen superrigidity for Lie groups with Kazhdan’s property (T). The precise statement we apply is in [FH12].

2. PRELIMINARIES

Let (X, d_X) and (Y, d_Y) be metric spaces, and let $k \geq 1$ and $c \geq 0$ be real numbers. A map $f : X \rightarrow Y$ is a (k, c) -quasi-isometric embedding if for all $a, b \in X$,

$$\frac{1}{k}d_X(a, b) - c \leq d_Y(f(a), f(b)) \leq kd_X(a, b) + c.$$

A (k, c) -quasi-isometric embedding f is a (k, c) -quasi-isometry if there is a (k, c) -quasi-isometric embedding $g : Y \rightarrow X$ such that $d_X(x, (g \circ f)(x)) \leq c$ for all $x \in X$, and $d_Y(y, (f \circ g)(y)) \leq c$ for all $y \in Y$. Such a map g is a quasi-inverse of f . For maps

$h_1, h_2 : X \rightarrow Y$, let $d_Y(h_1, h_2)$ denote $\sup_{x \in X} d_Y(h_1(x), h_2(x))$. If $d_Y(h_1, h_2) < \infty$ then we say that h_1 has finite distance from h_2 . With this notation, g is a quasi-inverse of f if and only if $d_X(\text{id}_X, g \circ f) \leq c$ and $d_Y(\text{id}_Y, f \circ g) \leq c$. A (k, c) -quasi-isometric embedding is a (k', c') -quasi-isometry for some $k' \geq 1$ and $c' \geq 0$ if and only if it is *coarsely surjective*, that is, its image is r -dense for some $r \geq 0$. A map $f : X \rightarrow Y$ is a *quasi-isometry* between X and Y if it is a (k, c) -quasi-isometry for some $k \geq 1, c \geq 0$. Two metric spaces X and Y are *quasi-isometric* if there exists a quasi-isometry between them. Observe that a composition of quasi-isometries is again a quasi-isometry.

Let G be a group with a finite, symmetric generating set S . For $g \in G$, let $|g|_S$ denote the length of a shortest word representing g with letters in S . Then G is endowed with a *word metric* d_S defined by $d_S(g, h) = |g^{-1}h|_S$. Note that left multiplication of G on itself is an isometric action. If S' is another finite, symmetric generating set for G , then the identity map $(G, d_S) \rightarrow (G, d_{S'})$ is a quasi-isometry (in fact it is bi-Lipschitz). So the word metric on a finitely generated group is well-defined up to quasi-isometry. In other words, the quasi-isometry type of a finitely generated group is well-defined.

Recall the fundamental observation in geometric group theory which equates the quasi-isometry type of a group with the quasi-isometry type of a metric space on which it acts geometrically. Let X be a geodesic metric space which is *proper*, i.e., closed balls are compact. An isometric group action $G \curvearrowright X$ is *properly discontinuous* if for all compact $K \subseteq X$ the set $\{g \in G : g \cdot K \cap K \neq \emptyset\}$ is finite, and *cocompact* if $G \backslash X$ is compact. The action is *geometric* if it is properly discontinuous and cocompact.

Lemma 4 (Milnor-Schwarz). *If a group G acts geometrically on a proper geodesic metric space X , then G is finitely generated and quasi-isometric to X . A quasi-isometry $G \rightarrow X$ is given by $g \mapsto g \cdot x_0$, where $x_0 \in X$ is any basepoint.*

For example, if M is a compact Riemannian manifold with Riemannian universal cover \widetilde{M} , then $\pi_1(M)$ acts geometrically on \widetilde{M} and therefore $\pi_1(M)$ is quasi-isometric to \widetilde{M} .

A more general version of the Milnor-Schwarz lemma exists for quasi-actions, which we now define. Let (X, d) be a geodesic metric space, and let $\text{QI}(X)$ be the set of quasi-isometries $X \rightarrow X$ (in contrast to our use, $\text{QI}(X)$ is sometimes used to denote the *quasi-isometry group* of X whose elements are equivalence classes of quasi-isometries). For $k \geq 1$, a k -quasi-action of a group G on X is a map $h : G \rightarrow \text{QI}(X)$ such that

- (1) For all $g \in G$, $h(g)$ is a (k, k) -quasi-isometry with k -dense image,
- (2) $d(h(1), \text{id}_X) \leq k$;
- (3) For all $g_1, g_2 \in G$, $d(h(g_1g_2), h(g_1)h(g_2)) \leq k$,

where $h(g_1)h(g_2)$ means $h(g_1) \circ h(g_2)$. A k -quasi-action h is k' -*cobounded* if every G -orbit is k' -dense in X . A (*cobounded*) *quasi-action* is a map which is a (k' -cobounded) k -quasi-action for some $k \geq 1$ ($k' \geq 1$). Now we can state a stronger version of the Milnor-Schwarz lemma.

Lemma 5 ([FLS15] Lemma 1.4). *Let X be a geodesic metric space with basepoint x_0 , and let G be a group. Let $h : G \rightarrow \text{QI}(X)$ be a cobounded quasi-action of G on X , and suppose that for each $r > 0$, the set $\{g \in G : h(g)B(x_0, r) \cap B(x_0, r) \neq \emptyset\}$ is finite. Then G is finitely generated and quasi-isometric to X . A quasi-isometry $G \rightarrow X$ is given by $g \mapsto h(g)(x_0)$.*

Here are two more lemmas related to quasi-actions that will be useful later.

Lemma 6. *If a quasi-action has at least one dense orbit, then it is cobounded.*

Proof. Suppose $h : G \rightarrow \text{QI}(X)$ is a k -quasi-action such that the G -orbit of $x \in X$ is k' -dense in X . Let $y, p \in X$ be arbitrary, and take $g_1, g_2 \in G$ such that $d(y, h(g_1)(x)) \leq k'$ and $d(p, h(g_2)(x)) \leq k'$. Then

$$\begin{aligned} d(p, h(g_2g_1^{-1})(y)) &\leq d(p, h(g_2)(x)) + d(h(g_2)(x), h(g_2g_1^{-1})(y)) \\ &\leq k' + d(h(g_2)(x), h(g_2)h(g_1^{-1})(y)) + d(h(g_2)h(g_1^{-1})(y), h(g_2g_1^{-1})(y)) \\ &\leq k' + kd(x, h(g_1^{-1})(y)) + k + k \\ &\leq k' + 2k + k^2d(h(g_1)(x), h(g_1)h(g_1^{-1})(y)) + k^3 \\ &\leq k' + 2k + k^3 + k^2d(h(g_1)(x), h(1)(y)) + k^2d(h(1)(y), h(g_1)h(g_1^{-1})(y)) \\ &\leq k' + 2k + k^3 + k^2d(h(g_1)(x), y) + k^2d(y, h(1)(y)) + k^3 \\ &\leq k' + 2k + 3k^3 + k^2k'. \end{aligned}$$

So h is $(k' + 2k + 3k^3 + k^2k')$ -cobounded. \square

Lemma 7. *A quasi-isometry between a finitely generated group G and the fundamental group $\pi_1(M)$ of a geodesic metric space induces a cobounded quasi-action of G on the metric universal cover \widetilde{M} .*

Proof. Indeed, since $\pi_1(M)$ is quasi-isometric to \widetilde{M} , we get a quasi-isometry $\varphi : G \rightarrow \widetilde{M}$. Let $\psi : \widetilde{M} \rightarrow G$ be a quasi-inverse of φ . For each $g \in G$, define $h(g) : \widetilde{M} \rightarrow \widetilde{M}$ by

$$h(g)(x) = \varphi(g\psi(x)), \quad x \in \widetilde{M}.$$

Then $h(g)$ is the composition of three quasi-isometries with fixed constants. So $h(g)$ is a quasi-isometry with constants that are independent of g . Moreover, $h(1) = \varphi \circ \psi$ has finite distance from $\text{id}_{\widetilde{M}}$. It is also easily checked that $h(g_1g_2)$ has finite distance (bounded independently of g_1, g_2) from $h(g_1) \circ h(g_2)$. So h is a quasi-action, and since φ is coarsely surjective and the left multiplication action of G on itself is transitive, h is cobounded. \square

Finally, we recall some facts about non-uniform rank one lattices and their associated neutered spaces. Details may be found in [Sch95] and [DK18]. Let X be a negatively curved symmetric space with distance function d_X , and let Λ be a non-uniform lattice in $\text{Isom}(X)$. We call such a group a *non-uniform rank one lattice*. By Selberg's lemma [Sel60], Λ is virtually torsion-free. Since we are concerned with the quasi-isometry type of Λ , we may assume without loss of generality that Λ is torsion-free. Then X/Λ is a finite-volume non-compact manifold. By truncating the cusps of X/Λ we obtain a compact manifold with boundary Y , and the universal cover is a *neutered space* $B \subset X$. By definition, B is the complement in X of the union of an infinite equivariant family of open horoballs with pairwise disjoint closures. We refer to the boundaries of these horoballs as the *peripheral horospheres* (or just *horospheres*) of B . Since X/Λ has finite volume, it has finitely many cusps and Y has finitely many boundary components. It follows that there is an $R > 0$ such that the d_X -distance between every pair of distinct horospheres of B is at least R . A horosphere O is centered at a point p in the ideal boundary ∂X . We say that p is the *basepoint* of O and that O is *based* at p . Since X/Λ has finite volume, the basepoints of the horospheres of B are dense in ∂X . In general, two horospheres in X share a basepoint if and only if the Hausdorff distance between them is finite.

Since B is a subspace of X , it has two natural metrics: the restricted metric $d_X|_B$, and the induced length metric $d_B(x, y) := \inf_p \text{length}_X(p)$, where the infimum is taken over all paths in B between x and y . We will often use the following fact, sometimes without mention.

Lemma 8 ([DK18] Lemma 24.2). *The identity map $(B, d_B) \rightarrow (B, d_X|_B)$ is 1-Lipschitz and uniformly proper.*

Recall that a map $f : Z \rightarrow W$ between proper metric spaces is *uniformly proper* if f is coarse Lipschitz and there exists a function $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\text{diam}(f^{-1}(B(w, r))) \leq \zeta(r)$ for all $w \in W, r > 0$.

3. PROJECTING A QUASI-ACTION

Let X, Λ, Y , and B be as above. Let L be a lattice in a simply connected nilpotent Lie group N . Define M to be the product $Y \times (N/L)$. Let Γ be a finitely generated group and assume that Γ is quasi-isometric to $\Lambda \times L$. Then by Lemma 7, the quasi-isometry between Γ and $\pi_1(M) = \Lambda \times L$ induces a k -cobounded k -quasi-action h of Γ on $\widetilde{M} = B \times N$ for some $k \geq 1$. We denote the metrics on $B \times N, X$, and B by d, d_X , and d_B , respectively.

We first show that h induces a quasi-action of Γ on B . In order to do so, we use a theorem of Kapovich–Kleiner–Leeb [KKL98] which implies that quasi-isometries of the product $B \times N$ coarsely project to quasi-isometries of B . Their theorem is more general than we need, so we only state its application in our situation.

Proposition 9. [KKL98, Theorem B] *There exist $D > 0$ and $k' \geq 1$, both depending only on k , for which the following holds: For each $\gamma \in \Gamma$, there exists a (k', k') -quasi-isometry $\psi(\gamma) : B \rightarrow B$ with k' -dense image such that the diagram*

$$\begin{array}{ccc} B \times N & \xrightarrow{h(\gamma)} & B \times N \\ \pi \downarrow & & \downarrow \pi \\ B & \xrightarrow{\psi(\gamma)} & B \end{array}$$

commutes up to an error bounded by D , where $\pi : B \times N \rightarrow B$ is the natural projection.

In an effort to make calculations more readable, we denote the value of π at $(b, n) \in B \times N$ by $\pi[(b, n)]$ instead of $\pi((b, n))$.

Lemma 10. *The map $\psi : \Gamma \rightarrow \text{QI}(B)$ is a K -cobounded K -quasi-action for some $K \geq 1$.*

Proof. By definition, we already know that each $\psi(\gamma)$ is a (k', k') -quasi-isometry with k' -dense image. In the following calculations, 1 denotes the identity element of Γ and e denotes the identity element of N . For all $b \in B$,

$$\begin{aligned} d_B(\psi(1)(b), b) &\leq d_B(\psi(1)(b), \pi[h(1)(b, e)]) + d_B(\pi[h(1)(b, e)], \pi[(b, e)]) \\ &\leq D + d(h(1)(b, e), (b, e)) \\ &\leq D + k. \end{aligned}$$

Let $\gamma_1, \gamma_2 \in \Gamma$ and $b \in B$. Then

$$\begin{aligned} d_B(\psi(\gamma_1)\psi(\gamma_2)(b), \psi(\gamma_1\gamma_2)(b)) &\leq d_B(\psi(\gamma_1)\psi(\gamma_2)(b), \pi[h(\gamma_1)h(\gamma_2)(b, e)]) \\ &\quad + d_B(\pi[h(\gamma_1)h(\gamma_2)(b, e)], \pi[h(\gamma_1\gamma_2)(b, e)]) \\ &\quad + d_B(\pi[h(\gamma_1\gamma_2)(b, e)], \psi(\gamma_1\gamma_2)(b)). \end{aligned}$$

The first term on the right can be bounded as follows.

$$\begin{aligned} d_B(\psi(\gamma_1)\psi(\gamma_2)(b), \pi[h(\gamma_1)h(\gamma_2)(b, e)]) &\leq d_B(\psi(\gamma_1)\psi(\gamma_2)(b), \psi(\gamma_1)(\pi[h(\gamma_2)(b, e)])) \\ &\quad + d_B(\psi(\gamma_1)(\pi[h(\gamma_2)(b, e)]), \pi[h(\gamma_1)h(\gamma_2)(b, e)]) \\ &\leq k'd_B(\psi(\gamma_2)(b), \pi[h(\gamma_2)(b, e)]) + k' + D \\ &\leq k'D + k' + D. \end{aligned}$$

Hence,

$$\begin{aligned} d_B(\psi(\gamma_1)\psi(\gamma_2)(b), \psi(\gamma_1\gamma_2)(b)) &\leq (k'D + k' + D) + d_B(\pi[h(\gamma_1)h(\gamma_2)(b, e)], \pi[h(\gamma_1\gamma_2)(b, e)]) \\ &\quad + d_B(\pi[h(\gamma_1\gamma_2)(b, e)], \psi(\gamma_1\gamma_2)(b)) \\ &\leq k'D + k' + D + d(h(\gamma_1)h(\gamma_2)(b, e), h(\gamma_1\gamma_2)(b, e)) + D \\ &\leq k'D + k' + 2D + k. \end{aligned}$$

Set $K = k'D + k' + 2D + k$. So ψ is a K -quasi-action. Let $b_1, b_2 \in B$. Since h is k -cobounded, there exists $\gamma \in \Gamma$ such that $d_B(h(\gamma)(b_1, e), (b_2, e)) \leq k$. Then

$$\begin{aligned} d_B(\psi(\gamma)(b_1), b_2) &\leq d_B(\psi(\gamma)(b_1), \pi[h(\gamma)(b_1, e)]) + d_B(\pi[h(\gamma)(b_1, e)], \pi[(b_2, e)]) \\ &\leq D + d(h(\gamma)(b_1, e), (b_2, e)) \\ &\leq D + k. \end{aligned}$$

So by possibly increasing K , it follows that ψ is K -cobounded. □

Next we show that the quasi-action ψ induces a homomorphism $\Gamma \rightarrow \text{Isom}(X)$. It is proved in [Sch95] that the quasi-isometry $\psi(\gamma)$ of B coarsely extends to an isometry of X . More precisely, for each $\gamma \in \Gamma$ there exists $\beta > 0$ and $\theta(\gamma) \in \text{Isom}(X)$ such that

$$(2) \quad d_X(\psi(\gamma)(b), \theta(\gamma)(b)) \leq \beta$$

for all $b \in B$. It turns out the constant β depends only on the quasi-isometry constants of $\psi(\gamma)$. Hence β depends on K and not on γ . Before we show that θ is our desired homomorphism, we need the following lemma which states that isometries of X which behave similarly on B must be identical.

Lemma 11. *Let $\alpha, \alpha' \in \text{Isom}(X)$. If there is a constant c such that $d_X(\alpha(b), \alpha'(b)) \leq c$ for all $b \in B$, then $\alpha = \alpha'$.*

Proof. By considering $(\alpha')^{-1} \circ \alpha$, we may assume without loss of generality that $\alpha' = \text{Id}_X$. Let O be a peripheral horosphere of B and let $p \in \partial X$ be its basepoint. Let $\{b_n\}$ be a sequence of points in O converging to p . Since $d_X(\alpha(b_n), b_n) \leq c$ for all n , the sequence $\{\alpha(b_n)\}$ also converges to p . So the extension of α to ∂X must fix p . Since the basepoints of the horospheres of B are dense in ∂X and the extension of α to ∂X is a homeomorphism, α must fix ∂X pointwise. Only the identity map on X extends to the identity map on ∂X , so we must have $\alpha = \text{id}_X$. □

Lemma 12. *The map $\theta : \Gamma \rightarrow \text{Isom}(X)$ is a group homomorphism.*

Proof. Let $\gamma_1, \gamma_2 \in \Gamma$. We need to show that $\theta(\gamma_1\gamma_2) = \theta(\gamma_1)\theta(\gamma_2)$. By Lemma 11, it suffices to produce a constant c such that for all $b \in B$,

$$d_X(\theta(\gamma_1\gamma_2)(b), \theta(\gamma_1)\theta(\gamma_2)(b)) \leq c.$$

Since ψ is a K -quasi-action, we have that for all $b \in B$,

$$\begin{aligned} d_X(\theta(\gamma_1\gamma_2)(b), \theta(\gamma_1)\theta(\gamma_2)(b)) &\leq d_X(\theta(\gamma_1\gamma_2)(b), \psi(\gamma_1\gamma_2)(b)) + d_X(\psi(\gamma_1\gamma_2)(b), \psi(\gamma_1)\psi(\gamma_2)(b)) \\ &\quad + d_X(\psi(\gamma_1)\psi(\gamma_2)(b), \theta(\gamma_1)\psi(\gamma_2)(b)) \\ &\quad + d_X(\theta(\gamma_1)\psi(\gamma_2)(b), \theta(\gamma_1)\theta(\gamma_2)(b)) \\ &\leq \beta + K + \beta + \beta. \end{aligned}$$

So set $c = K + 3\beta$. □

Now that we have a homomorphism $\theta : \Gamma \rightarrow \text{Isom}(X)$, we can extract information about the algebraic structure of Γ by studying the image and kernel of θ .

4. THE IMAGE OF θ

Let Λ_Γ denote the image of θ . Our goal is to show that $\Lambda_\Gamma < \text{Isom}(X)$ is a non-uniform lattice that is commensurable with Λ . Our arguments in this section closely follow those in [FLS15]. From now on, unless otherwise stated, the Hausdorff distance $\text{dist}_H(\cdot, \cdot)$ between subsets of X will be considered with respect to d_X .

Recall that a quasi-isometry of the neutered space B coarsely permutes the peripheral horospheres [Sch95]. The inequality (2) says that the isometry $\theta(\gamma)$ is at finite distance (on B) from the quasi-isometry $\psi(\gamma)$. So it follows that $\theta(\gamma)$ also coarsely permutes the peripheral horospheres of B . We make this precise in the following lemma. Let Ω denote the set of peripheral horospheres of B .

Lemma 13. *There exists a $\beta' > 0$ such that the following holds. For each $\alpha \in \Lambda_\Gamma$ and horosphere $O \in \Omega$, there is a unique horosphere $O' \in \Omega$ such that $\text{dist}_H(\alpha(O), O') \leq \beta'$.*

Proof. Let α and O be given. Then $\alpha = \theta(\gamma)$ for some $\gamma \in \Gamma$. The quasi-isometry $\psi(\gamma)$ permutes the peripheral horospheres of B up to a constant error which depends only on the quasi-isometry constants of $\psi(\gamma)$. Since ψ is a K -quasi-action, this means there is a constant $C = C(K)$ and a unique $O' \in \Omega$ such that $\text{dist}_H(\psi(\gamma)(O), O') \leq C$. From (2) we also have $\text{dist}_H(\psi(\gamma)(O), \theta(\gamma)(O)) \leq \beta$. Set $\beta' = \beta + C$. Then β' is independent of α and O , and

$$\text{dist}_H(\alpha(O), O') \leq \text{dist}_H(\theta(\gamma)(O), \psi(\gamma)(O)) + \text{dist}_H(\psi(\gamma)(O), O') = \beta + C = \beta'.$$

□

Let $P \subset \partial X$ denote the set of basepoints of the peripheral horospheres of B . Since each element of Λ_Γ coarsely permutes these horospheres, we obtain an action of Λ_Γ on P .

Lemma 14. *The group Λ_Γ acts on the set P with finitely many orbits, and given $p \in P$, every element of Λ_Γ that fixes p under this action also preserves the peripheral horosphere in Ω based at p .*

Proof. Let $\alpha \in \Lambda_\Gamma$ and $p \in P$. Let $O \in \Omega$ be the horosphere based at p . By Lemma 13, there is a unique $O' \in \Omega$ such that $\text{dist}_H(\alpha(O), O') < \infty$. Let $p' \in P$ be the basepoint of O' . Since the Hausdorff distance between $\alpha(O)$ and O' is finite, the horosphere $\alpha(O)$ is also based at p' . So in the extension of α to ∂X , we in fact have $\alpha(p) = p'$. Thus $\alpha \cdot p = p'$ defines an action of Λ_Γ on P .

Next we show that this action has finitely many orbits. Fix a point $b \in B$. For any $r > 0$, $\overline{B_X(b, r)}$ is compact, and since there is a lower bound on the d_X -distance between horospheres of B , only finitely many of them may intersect $\overline{B_X(b, r)}$. Hence, we denote by $A \subset \Omega$ the finite subset of horospheres whose d_X -distance from b is at most $K + \beta + \beta'$, and let $P_0 \subset P$ be the finite set of basepoints of horospheres in A . We show that every orbit in P has a representative in P_0 . Fix $p \in P$. Let $O \in \Omega$ be the horosphere based at p , and let $x \in O$. Since the quasi-action of Γ on B is K -cobounded, there is a $\gamma \in \Gamma$ such that $d_X(b, \psi(\gamma)(x)) \leq d_B(b, \psi(\gamma)(x)) \leq K$. Let $\alpha = \theta(\gamma)$. Then $d_X(b, \alpha(x)) \leq K + \beta$ by (2). By Lemma 13, there is a horosphere $O' \in \Omega$ such that the Hausdorff distance between $\alpha(O)$ and O' is at most β' . In particular, there exists $x' \in O'$ such that $d_X(\alpha(x), x') \leq \beta'$. Then

$$d_X(b, x') \leq d_X(b, \alpha(x)) + d_X(\alpha(x), x') \leq K + \beta + \beta'.$$

So $O' \in A$ and its basepoint p' is in P_0 . By definition of the action of Λ_Γ on P , we have $\alpha \cdot p = p'$. Hence $p = \alpha^{-1} \cdot p'$, so p is in the orbit of a point in P_0 . This proves the first part of the statement.

Now let $p \in P$ be the basepoint of $O \in \Omega$ and suppose $\alpha \in \Lambda_\Gamma$ fixes p . Since $\alpha \cdot p = p$, the horosphere $\alpha(O)$ is also based at p . If $\alpha(O) \neq O$ then $\delta = \text{dist}_H(O, \alpha(O)) > 0$. Moreover, α must be a hyperbolic isometry whose axis has p as an endpoint. So $\text{dist}_H(O, \alpha^n(O)) = n\delta$. Then for large n , the Hausdorff distance between O and $\alpha^n(O)$ would exceed β' , contradicting Lemma 13. Thus $\alpha(O) = O$. \square

Let $P_0 = \{p_1, \dots, p_k\} \subset P$ be a finite set of representatives for the action of Λ_Γ on P , given by Lemma 14. By removing elements from P_0 as necessary, we may assume that no two elements in P_0 represent the same orbit. So the orbits of the points in P_0 form a partition of P . For each $i = 1, \dots, k$, let $O_i \in \Omega$ be the horosphere based at p_i , and let \widehat{O}_i be the horosphere contained in the horoball bounded by O_i such that $\text{dist}_H(\widehat{O}_i, O_i) = \beta'$. Let $\widehat{\Omega}$ be the set of horospheres obtained by translating $\widehat{O}_1, \dots, \widehat{O}_k$ by the elements of Λ_Γ . The second part of Lemma 14 guarantees that no two distinct horospheres of $\widehat{\Omega}$ share the same basepoint. Let \widehat{B} denote the complement in X of the union U of the open horoballs bounded by the horospheres in $\widehat{\Omega}$, and endow \widehat{B} with the path-metric induced by d_X . Note that U is Λ_Γ -invariant by construction, and so \widehat{B} is Λ_Γ -invariant. It follows that Λ_Γ acts isometrically on \widehat{B} . We observe in the next lemma that \widehat{B} is a thickening of B .

Lemma 15. *The following containments hold:*

$$B \subseteq \widehat{B} \subseteq N_{2\beta'}(B),$$

where $N_r(B) = \{x \in X : d_X(x, B) \leq r\}$ for $r > 0$. In particular, B and \widehat{B} are quasi-isometric.

Proof. Let $\widehat{O} \in \widehat{\Omega}$. Then \widehat{O} is based at some $p \in P$, and we let O be the horosphere in Ω that is based at p . By definition, $\widehat{O} = \alpha(\widehat{O}_i)$ for some $i \in \{1, \dots, k\}$ and $\alpha \in \Lambda_\Gamma$. Recall that \widehat{O}_i is

the horosphere contained in the horoball bounded by O_i such that $\text{dist}_H(\widehat{O}_i, O_i) = \beta'$. Since α is an isometry, $\alpha(\widehat{O}_i)$ must be the horosphere contained in the horoball bounded by $\alpha(O_i)$ such that $\text{dist}_H(\alpha(\widehat{O}_i), \alpha(O_i)) = \beta'$. By Lemma 13, $\text{dist}_H(\alpha(O_i), O) \leq \beta'$. Altogether, \widehat{O} must be contained in the horoball bounded by O , and

$$\text{dist}_H(O, \widehat{O}) \leq \text{dist}_H(O, \alpha(O_i)) + \text{dist}_H(\alpha(O_i), \alpha(\widehat{O}_i)) \leq 2\beta'.$$

The first part of the lemma follows, and so the inclusion $B \hookrightarrow \widehat{B}$ is a quasi-isometry. \square

Since B/Λ has finitely many boundary components which lift to the peripheral horospheres, there is an $R > 0$ such that the d_X -distance between distinct horospheres of B is at least R . Then $B \subseteq \widehat{B}$ implies that the d_X -distance between distinct elements of $\widehat{\Omega}$ is also at least R . We are now equipped to show that Λ_Γ is a non-uniform lattice.

Proposition 16. *The group Λ_Γ is a non-uniform lattice in $\text{Isom}(X)$, and admits \widehat{B} as its associated neutered space.*

Proof. We first show that Λ_Γ is a discrete subgroup. Recall that $P \subset \partial X$ is a dense subset. So pick horospheres $\widehat{O}_1, \dots, \widehat{O}_{m+1}$ in $\widehat{\Omega}$, where $m = \dim X$, with basepoints p_1, \dots, p_{m+1} such that $\{p_1, \dots, p_{m+1}\}$ is not contained in the ideal boundary of any hyperplane of X . Then the only isometry of X whose extension to ∂X fixes every p_i is the identity map. For each $i = 1, \dots, m+1$, pick some $x_i \in \widehat{O}_i$, and set

$$U = \{\alpha \in \text{Isom}(X) : d_X(\alpha(x_i), x_i) < R \text{ for } i = 1, \dots, m+1\}.$$

Then $U \subset \text{Isom}(X)$ is an open neighborhood of the identity, with respect to the compact-open topology. Let $\alpha \in \Lambda_\Gamma \cap U$. Since $\alpha \in \Lambda_\Gamma$, α permutes $\widehat{\Omega}$. Since $\alpha \in U$, $d_X(\alpha(\widehat{O}_i), \widehat{O}_i) < R$. Together, these imply that $\alpha(\widehat{O}_i) = \widehat{O}_i$, and hence $\alpha \cdot p_i = p_i$, for each $i = 1, \dots, m+1$. So $\alpha = \text{id}_X$ and $\Lambda_\Gamma \cap U = \{\text{id}_X\}$. Thus Λ_Γ is a discrete subgroup.

Next we show that X/Λ_Γ has finite volume. Recall that the quasi-action of Γ on B is K -cobounded. It follows from the definition of Λ_Γ and the containment $\widehat{B} \subseteq N_{2\beta'}(B)$ that the Λ_Γ -orbit of a (every) point in \widehat{B} is $(K + 3\beta')$ -dense in \widehat{B} . Hence the quotient orbifold $\widehat{B}/\Lambda_\Gamma$ is compact, and by Lemma 14, it has finitely many boundary components V_1, \dots, V_j . Suppose $\widehat{O}_i \in \widehat{\Omega}$ projects onto the boundary component V_i . Note that $\alpha(\widehat{O}_i) \cap \widehat{O}_i \neq \emptyset$ if and only if $\alpha(\widehat{O}_i) = \widehat{O}_i$ if and only if α is in the stabilizer Λ_{Γ_i} of the basepoint of \widehat{O}_i . The subset $V_i = \widehat{O}_i/\Lambda_{\Gamma_i}$ of the compact quotient $\widehat{B}/\Lambda_\Gamma$ is closed, and therefore, compact. Let $W_i \subset X$ be the horoball bounded by \widehat{O}_i . Then W_i/Λ_{Γ_i} has finite volume, and since $(\bigcup_{i=1}^j W_i) \cup \widehat{B}$ projects surjectively onto X/Λ_Γ , we conclude that X/Λ_Γ has finite volume. Thus Λ_Γ is a non-uniform lattice. \square

We are now in a position to use Schwartz' rigidity theorem for non-uniform rank one lattices.

Corollary 17. *The group Λ_Γ is commensurable with Λ .*

Proof. Recall that B and \widehat{B} are quasi-isometric by Lemma 15. Since Λ acts geometrically on B and Λ_Γ acts geometrically on \widehat{B} , the Milnor-Schwarz lemma implies that the two non-uniform lattices Λ and Λ_Γ are quasi-isometric. Then the corollary follows from Schwartz' theorem. \square

5. THE KERNEL OF θ

Next we analyze the kernel of θ , and our goal is to show that $\ker \theta$ is virtually a nilpotent lattice. To this end, we first show that via the quasi-action h of Γ on $B \times N$, each $\gamma \in \ker \theta$ coarsely preserves the fibers $\{b\} \times N \subset B \times N$.

Lemma 18. *There is a $\delta > 0$ such that for each $\gamma \in \ker \theta$ the following holds. For every fiber $F = \{b\} \times N \subset B \times N$ and every $x \in F$, we have $h(\gamma)(x) \in N_\delta(F)$*

Proof. Fix $\gamma \in \ker \theta$. Then $\theta(\gamma) = \text{id}_X$ and so $d_X(\psi(\gamma)(b), b) \leq \beta$ for all $b \in B$. By Lemma 8, there exists $\delta' > 0$ so that in B , every d_X -ball of radius β is contained in a d_B -ball of radius δ' . Then $d_B(\psi(\gamma)(b), b) \leq \delta'$ for all $b \in B$. Take a fiber $F = \{b_0\} \times N \subset B \times N$ and let $x \in F$. Then

$$d_B(\pi[h(\gamma)(x)], b_0) \leq d_B(\pi[h(\gamma)(x)], \psi(\gamma)(\pi[x])) + d_B(\psi(\gamma)(b_0), b_0) \leq D + \delta'.$$

Set $\delta = D + \delta'$ and note that this value does not depend on the choice of γ . Then observe that $d_B(\pi[h(\gamma)(x)], b_0) \leq \delta$ if and only if $h(\gamma)(x) \in N_\delta(F)$. \square

Fix a fiber $F_0 = \{b_0\} \times N \subset B \times N$. For $\gamma \in \ker \theta$ and $x \in F_0$, Lemma 18 guarantees that there exists a point $f(\gamma)(x)$ in F_0 such that $d(f(\gamma)(x), h(\gamma)(x)) \leq \delta$. So for each $\gamma \in \ker \theta$ we have a map $f(\gamma) : F_0 \rightarrow F_0$.

Proposition 19. *The map $\gamma \mapsto f(\gamma)$ is a cobounded quasi-action of $\ker \theta$ on F_0 .*

Proof. First we show that $f(\gamma)$ is a quasi-isometry. Fix $\gamma \in \ker \theta$, and let $x, y \in F_0$. Then

$$\begin{aligned} d(f(\gamma)(x), f(\gamma)(y)) &\leq d(h(\gamma)(x), h(\gamma)(y)) + 2\delta \\ &\leq kd(x, y) + k + 2\delta \end{aligned}$$

and

$$\begin{aligned} d(f(\gamma)(x), f(\gamma)(y)) &\geq d(h(\gamma)(x), h(\gamma)(y)) - 2\delta \\ &\geq \frac{1}{k}d(x, y) - k - 2\delta \end{aligned}$$

Let $z \in F_0$. Since $h(\gamma)$ has k -dense image, there is some $y = (b, n) \in B \times N$ for which $d(z, h(\gamma)(y)) \leq k$. By Lemma 18, $d_B(\pi[h(\gamma)(y)], b) \leq \delta$. So it turns out that b cannot be too far from b_0 . Indeed,

$$\begin{aligned} d_B(b_0, b) &\leq d_B(b_0, \pi[h(\gamma)(y)]) + d_B(\pi[h(\gamma)(y)], b) \\ &\leq d(z, h(\gamma)(x)) + \delta \\ &\leq k + \delta. \end{aligned}$$

Set $x = (b_0, n)$. Then $d(x, y) = d_B(b_0, b) \leq k + \delta$, and so

$$\begin{aligned} d(z, f(\gamma)(x)) &\leq d(z, h(\gamma)(y)) + d(h(\gamma)(y), h(\gamma)(x)) + d(h(\gamma)(x), f(\gamma)(x)) \\ &\leq k + (kd(x, y) + k) + \delta \\ &\leq k + k(k + \delta) + k + \delta. \end{aligned}$$

It follows that $f(\gamma)$ is a quasi-isometry.

For all $x \in F_0$,

$$d(f(1)(x), x) \leq d(f(1)(x), h(1)(x)) + d(h(1)(x), x) \leq \delta + k.$$

Let $\gamma_1, \gamma_2 \in \Gamma$ and $x \in F_0$. Then

$$\begin{aligned} d(f(\gamma_1\gamma_2)(x), f(\gamma_1)f(\gamma_2)(x)) &\leq d(f(\gamma_1\gamma_2)(x), h(\gamma_1\gamma_2)(x)) + d(h(\gamma_1\gamma_2)(x), f(\gamma_1)f(\gamma_2)(x)) \\ &\leq \delta + d(h(\gamma_1\gamma_2)(x), h(\gamma_1)h(\gamma_2)(x)) \\ &\quad + d(h(\gamma_1)h(\gamma_2)(x), f(\gamma_1)f(\gamma_2)(x)) \\ &\leq \delta + k + d(h(\gamma_1)h(\gamma_2)(x), h(\gamma_1)f(\gamma_2)(x)) \\ &\quad + d(h(\gamma_1)f(\gamma_2)(x), f(\gamma_1)f(\gamma_2)(x)) \\ &\leq \delta + k + [kd(h(\gamma_2)(x), f(\gamma_2)(x)) + k] + \delta \\ &\leq \delta + k + k\delta + k + \delta. \end{aligned}$$

It follows that f is a quasi-action, and it remains to show that f is cobounded.

Recall that it suffices to show that the orbit of $x_0 = (b_0, e) \in F_0$ is r -dense in F_0 for some $r > 0$. First we claim that if $\gamma \in \Gamma$ is such that $h(\gamma)(x_0) \in N_k(F_0)$, then $\theta(\gamma)$ moves b_0 by a distance which is bounded independently of γ . Indeed, if $h(\gamma)(x_0) \in N_k(F_0)$ then

$$\begin{aligned} d_X(\theta(\gamma)(b_0), b_0) &\leq d_X(\theta(\gamma)(b_0), \psi(\gamma)(b_0)) + d_B(\psi(\gamma)(\pi[x_0]), \pi[h(\gamma)(x_0)]) + d_B(\pi[h(\gamma)(x_0)], \pi[x_0]) \\ &\leq \beta + D + k. \end{aligned}$$

By an Arzelà-Ascoli theorem, the isometries of X which move b_0 by a distance at most $\beta + D + k$ lie in a compact subset C of the compact-open topology on $\text{Isom}(X)$. Since $\Lambda_\Gamma \subset \text{Isom}(X)$ is discrete, the intersection $A = \Lambda_\Gamma \cap C$ is finite. In summary, $h(\gamma)(x_0) \in N_k(F_0)$ implies $\theta(\gamma) \in A$. For each $a \in A$, pick $\gamma_a \in \theta^{-1}(\{a\})$ and set $M = \max_{a \in A} d(x_0, h(\gamma_a^{-1})(x_0))$.

Now, let $y \in F_0$ be arbitrary. Since the quasi-action h of Γ on $B \times N$ is k -cobounded, there is a $\gamma \in \Gamma$ such that $d(y, h(\gamma)(x_0)) \leq k$. Then $h(\gamma)(x_0) \in N_k(F_0)$, so $\theta(\gamma)$ is equal to some $a \in A$. Then $\gamma\gamma_a^{-1} \in \ker \theta$ and

$$\begin{aligned} d(y, h(\gamma\gamma_a^{-1})(x_0)) &\leq d(y, h(\gamma)h(\gamma_a^{-1})(x_0)) + k \\ &\leq d(y, h(\gamma)(x_0)) + d(h(\gamma)(x_0), h(\gamma)h(\gamma_a^{-1})(x_0)) + k \\ &\leq kd(x_0, h(\gamma_a^{-1})(x_0)) + 3k \\ &\leq kM + 3k. \end{aligned}$$

Hence, $d(y, f(\gamma\gamma_a^{-1})(x_0)) \leq kM + 3k + \delta$, and so the $(\ker \theta)$ -orbit of x_0 is $(kM + 3k + \delta)$ -dense in F_0 . □

Proposition 20. *The group $\ker \theta$ is finitely generated and quasi-isometric to N .*

Proof. Set $x_0 = (b_0, e) \in F_0$. Since f is a cobounded quasi-action of $\ker \theta$ on F_0 , and F_0 is isometric to N , the proposition follows from the stronger version of the Milnor-Schwarz lemma once we show that for $r > 0$, the set $\{\gamma \in \ker \theta : f(\gamma)B_{F_0}(x_0, r) \cap B_{F_0}(x_0, r) \neq \emptyset\}$ is finite. By definition, $d(f(\gamma)(x), h(\gamma)(x)) \leq \delta$ for $\gamma \in \ker \theta$ and $x \in F_0$. So it suffices to show that for $r > 0$, the set $\{\gamma \in \Gamma : h(\gamma)B(x_0, r) \cap B(x_0, r) \neq \emptyset\}$ is finite, where $B(x_0, r) = B_{B \times N}(x_0, r)$. Recall that h is defined by $h(\gamma)(x) = \varphi(\gamma\varphi^{-1}(x))$ for $\gamma \in \Gamma$ and $x \in B \times N$, where $\varphi : \Gamma \rightarrow B \times N$ is a quasi-isometry and φ^{-1} is a quasi-inverse. Since φ is a quasi-isometry, the set $\{\gamma' \in \Gamma : \varphi(\gamma') \in B(x_0, r)\}$ is finite. Since φ^{-1} is a quasi-isometry, $\varphi^{-1}(B(x_0, r))$ is also finite. It follows that the set $\{\gamma \in \Gamma : \varphi(\gamma\varphi^{-1}(B(x_0, r))) \cap B(x_0, r) \neq \emptyset\}$ must be finite, as desired. \square

6. QUASI-ISOMETRIC RIGIDITY AND NILCENTRAL EXTENSIONS

We are now ready to prove our main result which is quasi-isometric rigidity for products of the form $\Lambda \times L$, where Λ is a non-uniform rank one lattice and L is a nilpotent lattice.

Theorem 21. *Let $X \neq \mathbb{H}^2$ be a negatively curved symmetric space. Let Λ be a non-uniform lattice in $\text{Isom}(X)$ and L be a nilpotent lattice. If Γ is a finitely generated group quasi-isometric to $\Lambda \times L$, then there exist short exact sequences*

$$(1) \quad 1 \longrightarrow L' \longrightarrow \Gamma' \longrightarrow \Delta \longrightarrow 1,$$

$$1 \longrightarrow F \longrightarrow \Delta \longrightarrow \Lambda' \longrightarrow 1.$$

where $\Gamma' \leq \Gamma$ and $\Lambda' \leq \Lambda$ have finite index, L' is a nilpotent lattice quasi-isometric to L , Δ is a group, and F is a finite group.

Proof. By Proposition 20, $\ker \theta$ is quasi-isometric to N , where N is the simply connected nilpotent Lie group in which L is a lattice. In particular $\ker \theta$ has polynomial growth. Then by Gromov's polynomial growth theorem, $\ker \theta$ is virtually nilpotent. So $\ker \theta$ has a finite-index, hence finitely generated, nilpotent subgroup K . Such groups are virtually torsion-free, so take a finite-index torsion-free subgroup $K' \leq K$. Since $\ker \theta$ is finitely generated, it has only finitely many subgroups with the same index as K' . Let L' denote their intersection. Then L' is a finite-index characteristic subgroup of $\ker \theta$. Moreover, L' is finitely generated, nilpotent, and torsion-free. A theorem of Malcev [Mal51] then says that L' embeds as a lattice in a simply connected nilpotent Lie group N' , known as the (real) Malcev completion of L' (see also [Rag72, Theorem 2.18]). Since L' is characteristic in $\ker \theta$, it is normal in Γ . By construction, Γ/L' is an extension of $\Lambda_\Gamma = \Gamma/\ker \theta$ by the finite group $F = \ker \theta/L'$. By Corollary 17, Λ_Γ is virtually a finite-index subgroup $\Lambda' \leq \Lambda$ which is also a non-uniform lattice in $\text{Isom}(X)$. Let $\Gamma' = \theta^{-1}(\Lambda')$ and $\Delta = \Gamma'/L'$. Then we obtain the desired short exact sequences.

Since lattices in simply connected nilpotent Lie groups are uniform, $L, N, \ker \theta$, and L' are all quasi-isometric to each other. The quotient Γ'/L' is a finite extension of a finite-index subgroup of Λ , so Γ'/L' and Λ are virtually isomorphic, hence quasi-isometric, to each other. \square

Remark 22. One may wonder whether the fact that N and N' are quasi-isometric determines any algebraic relation between them or between L and L' . However, it is an open problem whether quasi-isometric simply connected nilpotent Lie groups are necessarily isomorphic.

Remark 23. It is natural to wonder whether the short exact sequence (1) splits, perhaps after passing to further finite-index subgroups. Indeed, a conclusion such as this would strengthen even further the connection between the algebraic structure of Γ and that of $\Lambda \times L$. However, it was shown in [FLS15] that the sequence does not in general virtually split when Λ is the fundamental group of a complete finite-volume real hyperbolic m -manifold, $m \geq 3$, and $N = \mathbb{Z}^d$.

In the setting of Theorem 3 where $L' = \mathbb{Z}^d$, Γ' may be chosen so that the group extension (1) is a central extension. In our more general setting where L' is not necessarily abelian, it does not make sense to ask whether (1) is a central extension. Despite this, we are still able to derive some extra structure for the group extension (1), analogous to centrality, when we focus our attention to the case when X is either quaternionic hyperbolic space or the Cayley hyperbolic plane. Our argument works for these specific symmetric spaces because we rely on a superrigidity theorem that applies when $\text{Isom}(X)$ has Kazhdan's property (T). Indeed, for those symmetric spaces, $\text{Isom}(X)$ is either $\text{Sp}(n, 1)$ or $F_{4(-20)}$, both of which are known to have property (T) by Kostant [Kos69] (see also [BdlHV08]). On the other hand, if X is real or complex hyperbolic space, then $\text{Isom}(X)$ is $\text{SO}(n, 1)$ or $\text{SU}(n, 1)$ and does not have property (T).

To state the superrigidity theorem, we first give a definition. Let V be a finite-dimensional vector space over \mathbb{R} or \mathbb{C} . Given a lattice Λ in a locally compact group G and a representation $\pi : \Lambda \rightarrow \text{GL}(V)$, we say that π *almost extends to a continuous representation of G* if there exist representations $\pi_1 : G \rightarrow \text{GL}(V)$ and $\pi_2 : \Lambda \rightarrow \text{GL}(V)$ such that: π_2 has bounded image, the images of π_1 and π_2 commute, and $\pi(\gamma) = \pi_1(\gamma)\pi_2(\gamma)$ for all $\gamma \in G$.

Theorem 24. [FH12, Theorem 3.7] *Let G be a semisimple Lie group with no compact factors and Kazhdan's property (T). Let Λ be a lattice in G and $\pi : \Lambda \rightarrow \text{GL}(V)$ a finite-dimensional representation. Then π almost extends to a continuous representation of G .*

We now give a precise definition of the extra structure we alluded to above. Let G be a group with a nilpotent normal subgroup N . The upper central series of N is

$$1 = Z_0 \triangleleft Z_1 \triangleleft \cdots \triangleleft Z_n = N$$

where $Z_1 = Z(N)$ and $Z_{i+1}/Z_i = Z(N/Z_i)$. Recall that the Z_i are characteristic subgroups of N , so in particular they are normal in G . We say that N is a *nilcentral* subgroup of G if $Z(N/Z_i) \subseteq Z(G/Z_i)$ for each $i = 0, \dots, n$. Observe that N is nilcentral in G if and only if $Z(N) \subseteq Z(G)$ and $N/Z(N)$ is nilcentral in $G/Z(N)$. An abelian subgroup is nilcentral if and only if it is central, and a nilpotent group N is nilcentral in $N \times G$ for every group G . We now give an example of a group with a normal subgroup which is nilpotent but not nilcentral.

Example 25. Let N be the discrete Heisenberg group with group presentation

$$N = \langle a, b, c \mid [a, c], [b, c], [a, b]c^{-1} \rangle.$$

Let $H = \mathbb{Z}_2 = \{\pm 1\}$ act on N via the action $\varphi : H \rightarrow \text{Aut}(N)$ defined by

$$\varphi_{-1}(a) = a, \quad \varphi_{-1}(b) = b, \quad \varphi_{-1}(c) = c^{-1}.$$

Let $G = N \rtimes_{\varphi} H$. Then N is a nilpotent normal subgroup of G but it is not nilcentral because $Z(N) \not\subseteq Z(G)$. Indeed,

$$(c, 1)(a, -1) = (ca, -1) \quad \text{and} \quad (a, -1)(c, 1) = (ac^{-1}, -1),$$

but $(ca, -1) = (ac^{-1}, -1) = (c^{-1}a, -1)$ if and only if $c = c^{-1}$, which is not the case. Thus $c \in Z(N)$, but $(c, 1) \notin Z(G)$.

Let

$$(3) \quad 1 \longrightarrow N \xrightarrow{j} G \longrightarrow H \longrightarrow 1$$

be a short exact sequence of groups, where N is nilpotent. We say that (3) is a *nilcentral* extension if $j(N)$ is nilcentral in G . Note that an abelian extension is nilcentral if and only if it is central. Our theorem at the end of the section provides a family of non-trivial examples of nilpotent extensions that are nilcentral. On the other hand, if $\varphi : H \rightarrow \text{Aut}(N)$ is as in Example 25, then

$$1 \longrightarrow N \longrightarrow N \rtimes_{\varphi} H \longrightarrow H \longrightarrow 1$$

is a nilpotent extension which is not nilcentral.

We will see that in the situation of Theorem A under some additional conditions, passing to a finite-index subgroup of Γ' does yield a nilcentral extension. To this end, we make the following definition. We say that (3) is a *virtually nilcentral* extension if G has a finite-index subgroup G' containing $j(N)$ such that $j(N)$ is nilcentral in G' . Recall that this means $Z(N/Z_i) \subseteq Z(G'/Z_i)$ for $i = 0, \dots, n$, where N is identified with $j(N) \subseteq G$. We now give a more tractable characterization of virtually nilcentral extensions.

Proposition 26. *The short exact sequence (3) is a virtually nilcentral extension if and only if for each $i = 0, \dots, n$, G/Z_i has a finite-index subgroup K_i containing N/Z_i , such that $Z(N/Z_i) \subseteq Z(K_i)$.*

Proof. The forward direction is done by taking $K_i = G'/Z_i$ for each i . For the reverse direction, recall that $K_i = G_i/Z_i$ for some $G_i \leq G$. So $N \leq G_i$ and $[G : G_i] = [G/Z_i : K_i] < \infty$. Let $G' = \bigcap G_i$. Since each G_i has finite index, so does G' . For each i , $N/Z_i \subset G'/Z_i \subset K_i$, and since $Z(N/Z_i) \subset Z(K_i)$, we have $Z(N/Z_i) \subset Z(G'/Z_i)$. \square

Now suppose we are in the situation of Theorem A and we have the following two short exact sequences.

$$(4) \quad 1 \longrightarrow N \longrightarrow \Gamma \longrightarrow \Delta \longrightarrow 1$$

$$(5) \quad 1 \longrightarrow F \longrightarrow \Delta \longrightarrow \Lambda \longrightarrow 1$$

where N is a nilpotent lattice (we are now using N to denote the lattice, not the ambient Lie group), F is finite, and Λ is a non-uniform lattice in $\text{Isom}(X)$, where $X \neq \mathbb{H}^2$ is a negatively curved symmetric space. Since N is finitely generated, nilpotent, and torsion-free, $Z(N/Z_i) = Z_{i+1}/Z_i$ is a finitely generated free abelian group. That is, $Z(N/Z_i) = \mathbb{Z}^{d_i}$ for some d_i , and so $\Sigma(N) = \max_i d_i$. We now focus on the situation when X is quaternionic hyperbolic space or the Cayley hyperbolic plane, and $\text{Isom}(X)$ has sufficiently large dimension.

Theorem 27. *Suppose X is either quaternionic hyperbolic space or the Cayley hyperbolic plane. If $\dim \text{Isom}(X) > \Sigma(N)$, then (4) is a virtually nilcentral extension.*

Proof. Fix $i \in \{0, \dots, n\}$. By Proposition 26 it suffices to show that Γ/Z_i has a finite-index subgroup K_i containing N/Z_i in which $Z(N/Z_i)$ is central. Since N/Z_i is normal in Γ/Z_i and $Z(N/Z_i)$ is characteristic in N/Z_i , we have $Z(N/Z_i) \triangleleft \Gamma/Z_i$. So Γ/Z_i acts by conjugation on $Z(N/Z_i)$, and we call this action ρ_1 . If $\rho_1 : \Gamma/Z_i \rightarrow \text{Aut}(Z(N/Z_i))$ has finite image, then we may take $K_i = \ker \rho_1$ to finish. Since $N/Z_i \leq \ker \rho_1$, we may project ρ_1 to an action ρ_2 of $\Delta = (\Gamma/Z_i)/(N/Z_i)$ on $Z(N/Z_i)$. Then it suffices to show that $\rho_2 : \Delta \rightarrow \text{Aut}(Z(N/Z_i))$ has finite image because $\text{im } \rho_2 = \text{im } \rho_1$.

Recall that $Z(N/Z_i) = \mathbb{Z}^d$ for some $d \leq r$. So $\text{Aut}(Z(N/Z_i)) = \text{GL}(d, \mathbb{Z})$, and the latter group naturally embeds into $\text{GL}(d, \mathbb{R})$. Composing ρ_2 with this inclusion yields a homomorphism $\rho_3 : \Delta \rightarrow \text{GL}(d, \mathbb{R})$, and it suffices to show that ρ_3 has bounded image. From (5), Δ is finitely generated because F is finite and the lattice Λ is finitely generated. Thus by Selberg's lemma, $\rho_3(\Delta)$ has a finite-index subgroup which is torsion-free. This subgroup has pre-image $\Delta' \leq \Delta$ which also has finite index. Let $\rho_4 : \Delta' \rightarrow \text{GL}(d, \mathbb{R})$ be the restriction $\rho_3|_{\Delta'}$, and note that since $[\Delta : \Delta'] < \infty$, it suffices to show that ρ_4 has bounded image. Since $\rho_4(\Delta')$ is torsion-free and F is finite, we have that $\Delta' \cap F \subset \ker \rho_4$. So ρ_4 projects to a homomorphism $\rho_5 : \Delta' / (\Delta' \cap F) \rightarrow \text{GL}(d, \mathbb{R})$. Since $\text{im } \rho_5 = \text{im } \rho_4$, it suffices to show that ρ_5 has bounded image.

Now, $\Delta' / (\Delta' \cap F)$ is isomorphic to $(\Delta' F)/F$, and since Δ' has finite index in Δ , $(\Delta' F)/F$ has finite index in $\Delta/F = \Lambda$. Thus, $\Delta' / (\Delta' \cap F)$ is a lattice in $\text{Isom}(X)$. The hypothesis on X implies $\text{Isom}(X)$ has property (T). Hence, we may apply Theorem 24 to ρ_5 to obtain representations $\pi_1 : \text{Isom}(X) \rightarrow \text{GL}(d, \mathbb{R})$ and $\pi_2 : \Delta' / (\Delta' \cap F) \rightarrow \text{GL}(d, \mathbb{R})$, where π_2 has bounded image, such that $\rho_5(\gamma) = \pi_1(\gamma)\pi_2(\gamma)$ for $\gamma \in \Delta' / (\Delta' \cap F)$. If $\dim \text{Isom}(X) > \Sigma(N) \geq d$, so that the simple group $\text{Isom}(X)$ has larger dimension than $\text{GL}(d, \mathbb{R})$, then π_1 must be trivial, in which case $\rho_5 = \pi_2$ has bounded image. \square

Remark 28. $\dim \text{Sp}(n, 1) = 2n^2 + 5n + 3$ and $\dim \text{F}_{4(-20)} = 52$.

REFERENCES

- [BdlHV08] Bachir Bekka, Pierre de la Harpe, and Alain Valette. *Kazhdan's Property (T)*. Cambridge University Press, 2008.
- [DK18] Cornelia Druțu and Michael Kapovich. *Geometric Group Theory*. American Mathematical Society, 2018.
- [Far97] Benson Farb. The quasi-isometry classification of lattices in semisimple lie groups. *Mathematical Research Letters*, 4:705–717, 1997.
- [FH12] David Fisher and Theron Hitchman. Strengthening Kazhdan's property (T) by Bochner methods. *Geometriae Dedicata*, 160(1):333–364, 2012.
- [FLS15] Roberto Frigerio, Jean-François Lafont, and Alessandro Sisto. Rigidity of high dimensional graph manifolds. *Astérisque*, 2015.
- [Gro81] Mikhael Gromov. Groups of polynomial growth and expanding maps (with an appendix by Jacques Tits). *Inst. Hautes Études Sci. Publ. Math.*, 53:53–73, 1981.
- [Gro83] Mikhael Gromov. Infinite groups as geometric objects. In *Proceedings of the ICM Warsaw*, 1983.
- [KKL98] Michael Kapovich, Bruce Kleiner, and Bernhard Leeb. Quasi-isometries and the de Rham decomposition. *Topology*, 37(6):1193–1211, 1998.
- [Kos69] Bertram Kostant. On the existence and irreducibility of certain series of representations. *Bull. Amer. Math. Soc.*, 75:627–642, 1969.
- [Mal51] Anatoly Ivanovich Malcev. On a class of homogeneous spaces. *Amer. Math. Soc. Translation*, 1951.
- [Rag72] Madabusi Santanam Raghunathan. *Discrete subgroups of Lie groups*, volume 3. Springer, 1972.

- [Sch95] Richard Evan Schwartz. The quasi-isometry classification of rank one lattices. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 82(1):133–168, 1995.
- [Sel60] Atle Selberg. On discontinuous groups in higher-dimensional symmetric spaces. “*Contributions to Function Theory*, 1960.
- [Wol68] Joseph Albert Wolf. Growth of finitely generated solvable groups and curvature of Riemannian manifolds. *Journal of differential Geometry*, 2(4):421–446, 1968.