

Anti-Ramsey Number of Stars in 3-uniform hypergraphs*

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Abstract

An edge-colored hypergraph is called a *rainbow hypergraph* if all the colors on its edges are distinct. Given two positive integers n, r and an r -uniform hypergraph \mathcal{G} , the anti-Ramsey number $ar_r(n, \mathcal{G})$ is defined to be the minimum number of colors t such that there exists a rainbow copy of \mathcal{G} in any exactly t -edge-coloring of the complete r -uniform hypergraph of order n . Let \mathcal{F}_k denote the 3-graph (k -star) consisting of k edges sharing exactly one vertex. Tang, Li and Yan [23] determined the value of $ar_3(n, \mathcal{F}_3)$ when $n \geq 20$. In this paper, we determine the anti-Ramsey number $ar_3(n, \mathcal{F}_{k+1})$, where $k \geq 3$ and $n > \frac{5}{2}k^3 + \frac{15}{2}k^2 + 26k - 3$.

Key words: anti-Ramsey number; k -star; rainbow hypergraph; matching

1 Introduction

For a set S and a positive integer k , we use $\binom{S}{k}$ to denote the collection of all possible subsets of k elements of S . A *hypergraph* $\mathcal{F} = (V(\mathcal{F}), E(\mathcal{F}))$ consists of a vertex set $V(\mathcal{F})$ and an edge set $E(\mathcal{F})$, where each edge in $E(\mathcal{F})$ is a non-empty subset of $V(\mathcal{F})$. The number of edges of \mathcal{F} is denoted by $e(\mathcal{F})$, that is, $e(\mathcal{F}) := |E(\mathcal{F})|$. If $|e| = r$ for any $e \in E(\mathcal{F})$, then \mathcal{F} is called an *r -uniform hypergraph* (or r -graph, for simplicity). For $u \in V(\mathcal{F})$, let $N_{\mathcal{F}}(u) := \{e \mid e \subseteq V(\mathcal{F}) \setminus \{u\} \text{ and } e \cup \{u\} \in E(\mathcal{F})\}$ be the neighborhood of u in \mathcal{F} . The *degree* of u in \mathcal{F} , denoted by $d_{\mathcal{F}}(u)$, is the size of $N_{\mathcal{F}}(u)$. For $X \subseteq V(\mathcal{F})$, we define $\mathcal{F} - X$ as the subhypergraph of \mathcal{F} obtained by removing all vertices in X and all edges intersecting with X in \mathcal{F} . Similarly, if $Y \subseteq E(\mathcal{F})$, we use $\mathcal{F} - Y$ to denote the hypergraph resulting from deleting all the edges in Y from \mathcal{F} . When $X = \{x\}$, $Y = \{e\}$, we respectively write $\mathcal{F} - X = \mathcal{F} - x$, $\mathcal{F} - Y = \mathcal{F} - e$. Specifically, to avoid confusion, for an edge $e \in E(\mathcal{F})$, we use $\mathcal{F} - V(e)$ to denote the subgraph of \mathcal{F} obtained by removing all vertices in e and all edges intersecting with e in \mathcal{F} . For a non-empty subset $X \subseteq V(\mathcal{F})$, let $\mathcal{F}[X]$ denote the subgraph *induced* by X . For two disjoint sets U and W , we use $U \times W$ to denote the collection of 2-sets that intersect U and W . That is, $U \times W = \{\{x, y\} \mid x \in U \text{ and } y \in W\}$.

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For two vertex-disjoint hypergraphs \mathcal{F}, \mathcal{H} , the *union* of \mathcal{F} and \mathcal{H} denoted by $\mathcal{F} \cup \mathcal{H}$ is the hypergraph with vertex set $V(\mathcal{F}) \cup V(\mathcal{H})$ and edge set $E(\mathcal{F}) \cup E(\mathcal{H})$. When there is no confusion, for $T \subseteq V(\mathcal{F})$, we also use $N_{\mathcal{F}}(T)$ to denote the $(k - |T|)$ -graph with vertex set $V(\mathcal{F}) - T$ and edge set $N_{\mathcal{F}}(T)$. A *matching* in a hypergraph H is a set of pairwise disjoint edges in H , and we use $\nu(H)$ to denote the maximum size of a matching in H .

A *t-edge-coloring* of a hypergraph is an assignment of t colors to its edges, and an *exactly t-edge-coloring* uses all t colors. An edge-colored graph is called *rainbow* if all edges have distinct colors. Let $[n] = \{1, \dots, n\}$. The complete r -graph with order n is denoted by K_n^r . Given a positive integer n and a hypergraph \mathcal{F} , the *anti-Ramsey number* $ar_r(n, \mathcal{F})$ is the minimum number of colors t such that each edge-coloring of K_n^r with exactly t colors contains a rainbow copy of \mathcal{F} . Given an edge-coloring C of \mathcal{F} , the colored hypergraph is F -*free* if \mathcal{F} has no rainbow subhypergraph which is isomorphic to F .

Given a hypergraph H and a family of hypergraphs \mathcal{H} , H is called \mathcal{H} -*free* if for any $F \in \mathcal{H}$, H does not contain F as a subhypergraph. The Turán number of a family of r -graphs \mathcal{F} , written $ex_r(n, \mathcal{F})$, is the largest possible number of edges in an \mathcal{F} -free r -graph on n vertices. When $\mathcal{F} = \{F\}$, we use $ex_r(n, F)$ instead of $ex_r(n, \{F\})$.

Anti-Ramsey numbers were introduced by Erdős, Simonovits and Sós [6] in 1973. They found that the anti-Ramsey numbers are closely related to the Turán numbers. For an r -graph F , there is a natural lower bound of $ar_r(n, F)$ in terms of Turán number as follows,

$$ar_r(n, F) \geq ex_r(n, \{F - e : e \in E(F)\}) + 2. \quad (1)$$

This trivial lower bound is easily obtained by coloring a rainbow Turán extremal r -graph for $\{F - e : e \in F\}$ in K_n^r , and the remaining edges with an additional color [15].

In 1973, Erdős, Simonovits, and Sós [6] proved that there exists an integer $n_0(p)$ such that for all $n > n_0(p)$, the anti-Ramsey number $ar(K_p, n)$ satisfies the equation $ar(K_p, n) = ex(n, K_{p-1}) + 2$. Later, Montellano-Ballesteros and Neumann-Lara [17], as well as Schiermeyer [20], independently extended this result to cover all values of n and p where $n > p \geq 3$. Jiang [14] and Montellano-Ballesteros [18] independently determined the anti-Ramsey numbers for stars in graphs. A variety of results regarding the anti-Ramsey numbers of 2-graphs have been achieved. These results cover various graph structures such as paths, cycles, and matchings. For a comprehensive overview, we recommend the survey paper [10]. Regarding 3-graphs, Guo, Lu, and Peng [13] established the exact value of the anti-Ramsey number for matchings. Gu, Li, and Shi [11] investigated the anti-Ramsey numbers of paths and cycles in hypergraphs. For other related results on the anti-Ramsey numbers of paths, cycles, and matchings in hypergraphs, interested readers are referred to [9, 15, 16, 18, 22, 23]. In this paper, we aim to investigate the anti-Ramsey numbers of stars of 3-graphs.

Let \mathcal{F}_k (k -star) denote the 3-graph consisting of k edges sharing exactly one vertex, called the core of the star. Let $f(n, k)$ denote the maximum number of edges in 3-graph without k -stars. $f(n, 2)$ was determined exactly by Erdős and Sós [21]. Duke and Erdős [4] established linear lower and upper bounds on $f(n, k)$ for fixed k (where the bounds scale linearly with n). These bounds were subsequently improved in [8], where the exact value of $f(n, 3)$ was determined for all integers $n \geq 54$. Further refinements were made in [1], yielding bounds that are nearly best possible. Finally, Chung and Frankl [2] derived the exact value of $f(n, k)$ for 3-graphs in the regime $n \geq \frac{5}{2}k^3$.

Theorem 1 (Erdős and Sós, [21]) *For all $n \geq 3$,*

$$f(n, 2) = \begin{cases} n, & \text{if } n \equiv 0 \pmod{4}, \\ n-1, & \text{if } n \equiv 1 \pmod{4}, \\ n-2, & \text{if } n \equiv 2, 3 \pmod{4}. \end{cases}$$

Theorem 2 (Chung and Frankl, [2]) *Suppose that $k \geq 3$ is odd and $n > k(k-1)(5k+2)/2$, then $f(n, k) = (n-2k)k(k-1) + 2\binom{k}{3}$. Moreover, a 3-graph \mathcal{F} has $f(n, k)$ edges and contains no k -star if and only if \mathcal{F} is isomorphic to \mathcal{F}_k^o .*

Theorem 3 (Chung and Frankl, [2]) *Suppose that $k \geq 4$ is even and $n > 2k^3 - 9k + 7$, then $f(n, k) = \frac{1}{2}nk(2k-3) - \frac{1}{2}(2k^3 - 9k + 6)$. Moreover, a 3-graph \mathcal{F} has $f(n, k)$ edges and contains no k -star if and only if \mathcal{F} is isomorphic to \mathcal{F}_k^e .*

Chung and Frankl [2] proposed the following extremal graph construction.

The construction of \mathcal{F}_k^o : Let k be odd and let S and R be two disjoint sets of $[n]$ of size k . Consider the 3-graph \mathcal{F}_k^o with vertex set $[n]$ and edge set

$$E(\mathcal{F}_k^o) = \{T \in \binom{V}{3} : |T \cap S| \geq 2 \text{ and } |T \cap R| = \emptyset\} \cup \{T \in \binom{V}{3} : |T \cap R| \geq 2 \text{ and } |T \cap S| = \emptyset\}.$$

The construction of \mathcal{F}_k^e : Let k be even. Let G_k be the 2-graph with $2k-1$ vertices, denoted by $x_1, \dots, x_{k-1}, y_1, \dots, y_{k-1}$ and z . The edge set of G_k consists of all the pairs (x_i, y_j) , except for (x_i, y_i) with $2i > k$ together with the pairs $(x_i, z), (y_i, z)$ with $2i > k$. It is easy to see that it has all degrees equal $k-1$ except for the degree of z , which is $k-2$. Let \mathcal{F}_k^e denote the 3-graph on n vertices such that each edge either intersects the vertex set $V(G_k)$ in an edge of G_k or contains two distinct edges of G_k , together with all the triples of the form $\{x_i, y_i, z\}$ for $1 \leq i \leq k/2$.

The anti-Ramsey number $ar_3(n, \mathcal{F}_2) = 2$ follows immediately from the definition. Tang, Li and Yan [23] determined the value of $ar_3(n, \mathcal{F}_3)$ when $n \geq 20$.

Theorem 4 (Tang, Li and Yan [23]) *For all $n \geq 20$,*

$$ar_3(n, \mathcal{F}_3) = \begin{cases} f(n, 2) + 2, & \text{if } n \equiv 0 \pmod{4}, \\ f(n, 2) + 2, & \text{if } n \equiv 1 \pmod{4}, \\ f(n, 2) + 3, & \text{if } n \equiv 2, 3 \pmod{4}. \end{cases}$$

In this paper, we determine the anti-Ramsey number of a star in 3-graphs for sufficiently large n .

Theorem 5 *For $k \geq 3$ and $n > \frac{5}{2}k^3 + \frac{15}{2}k^2 + 26k - 3$, $ar_3(n, \mathcal{F}_{k+1}) = f(n, k) + 2$.*

2 Technical Lemmas

Theorem 6 (Tutte, [24]) *A graph G has a perfect matching if and only if*

$$o(G - S) \leq |S| \quad \text{for any } S \subseteq V(G),$$

where $o(G - S)$ denotes the number of connected components of odd order in $G - S$.

A graph G is called a *factor-critical* graph if $G - v$ has a perfect matching for each $v \in V(G)$. Gallai [12] introduced the concept of factor-critical graphs. By definition and Theorem 6, one can see that the following statement holds.

Lemma 7 (Gallai, [12]) *A graph G of odd order is factor-critical if and only if*

$$o(G - S) \leq |S| \quad \text{for any non-empty set } S \subseteq V(G),$$

where $o(G - S)$ denotes the number of connected components of odd order in $G - S$.

Here we use a “weight function” method developed by Chung and Frankl [2] to characterize the structure of a 3-graph \mathcal{F} . Suppose that \mathcal{F} is a family of 3-element subsets of the n -set $V = V(\mathcal{F})$. Let P denote the set of all pairs of vertices in V . For each $\{u, v\}$ in P , the pair frequency is

$$z(u, v) = |\{w : \{u, v, w\} \in \mathcal{F}\}|.$$

We also have

$$\begin{aligned} A &:= \{\{u, v\} \in P : z(u, v) \geq 2k - 1\} \\ B &:= \{\{u, v\} \in P : 2k - 2 \geq z(u, v) \geq k\} \\ C &:= P - A - B \end{aligned}$$

The weight function $\omega : \mathcal{F} \times P \rightarrow R$ distributes weights to pairs within each triple in \mathcal{F} according to the pair frequency.

For a fixed triple $T \in \mathcal{F}$, the three pairs in T are denoted by $z(p_1) \geq z(p_2) \geq z(p_3)$. The weight function w is defined as follows:

- If $p_1, p_2, p_3 \in A \cup B$ or $p_1, p_2, p_3 \in B \cup C$, then $\omega(T, p_i) = \frac{1}{3}$.
- Suppose $p_1 \in A, p_3 \in C$. If $p_2 \in A \cup B$, then $\omega(T, p_1) = \omega(T, p_2) = \frac{1}{2}, \omega(T, p_3) = 0$. If $p_2 \in C$, then $\omega(T, p_1) = 1, \omega(T, p_2) = \omega(T, p_3) = 0$.
- For convenience we set also $\omega(T, p) = 0$ for $p \notin T$.

Obviously, we have

$$\sum_{1 \leq i \leq 3} \omega(T, p_i) = 1$$

and

$$\sum_{T \in \mathcal{F}} \sum_p \omega(T, p) = e(\mathcal{F})$$

Lemma 8 ([2, 25]) *For every vertex v in a 3-graph \mathcal{F} which does not contain a k -star, the following holds:*

$$W_v = \sum_{p \in N_{\mathcal{F}}(v)} \omega(\{v\} \cup p, p) \leq k(k - 1).$$

Moreover $W_v \leq k(k - 1) - \frac{2}{3}$ unless $N_{\mathcal{F}}(v)$ is the disjoint union of two complete 2-graphs on k vertices and every edge of $N_{\mathcal{F}}(v)$ is A-type. If k is even, then one has the stronger inequality,

$$W_v \leq k(k - 3/2)$$

Moreover, $W_v \leq k(k-3/2) - 1/2$ unless $N_{\mathcal{F}}(v) = K_{k-1} \cup C$, where C is a factor-critical graph of order $k+1$ with degree sequence $k-1, \dots, k-1, k-2$ or $N_{\mathcal{F}}(v)$ with maximum degree $k-1$ satisfies the following three conditions.

- (a) There exists $S \subseteq V(N_{\mathcal{F}}(v))$ such that $N_{\mathcal{F}}(v) - S$ consists of isolated vertices and one factor-critical component denoted by F_0 with $2k-1-2|S| \geq k+1$ vertices and degree sequence $(k-1, \dots, k-1, k-2)$;
- (b) $N_{\mathcal{F}}(v) - V(F_0)$ is the edge disjoint union of $|S|$ stars, each with maximum degree $k-1$; and
- (c) every edge of $N_{\mathcal{F}}(v)$ is in A , and all edges connecting v to $V(N_{\mathcal{F}}(v))$ are in C .

Chung and Frankl [2] first determined the value of $f(n, k)$, though they did not characterize the corresponding extremal case: when $N_{\mathcal{F}}(v) = K_{k-1} \cup C$ (where C is a factor-critical graph of order $k+1$ with degree sequence $(k-1, \dots, k-1, k-2)$). Later, Zhu et al. [25] adopted a similar approach to fully characterize all extremal graphs (a method analogous to the one Chung and Frankl used in [2] when handling the function $\text{ex}_3(n, \mathcal{F}_k)$).

Write $c(n, k) := \text{ex}_3(n, \mathcal{F}_k) + 2$. Given an edge-coloring $c : E(K_n^3) \rightarrow [c(n, k)]$ such that c is surjective and the colored hypergraph denoted by H contains no rainbow \mathcal{F}_{k+1} . For $U \subseteq V(H)$, let $Z_c(U) := \{c(e) \mid e \in H, U \subseteq e\}$ and let $z_c(U) := |Z_c(U)|$. When $U = \{x\}$, we denote $Z_c(U)$ and $z_c(U)$ by $Z_c(x)$ and $z_c(x)$, respectively. If $z_c(\{u, v\}) \leq 3k$, the pair $\{u, v\}$ is *good* in H , saying *bad*, otherwise.

Lemma 9 *There exist $2k+6$ disjoint good pairs in H .*

Proof. Firstly, we show that the following claim.

Claim 1. For every u , there exist at least $n-k-1$ vertices $v \in V(H)$ such that $z_c(u, v) \leq 3k$.

By contradiction. Suppose the result does not hold. Then there exists $u \in V(H)$ and $S \in \binom{V(H)-u}{k+1}$ such that $z_c(u, v) \geq 3k+1$ for all $v \in S$. Write $S := \{v_1, \dots, v_{k+1}\}$. We choose a maximal rainbow star \mathcal{F}_r in H with center u such that $V(\mathcal{F}_r) \cap S = r$. Since H is a complete graph, we may choose e_1 such that $\{u, v_1\} \subseteq e_1$ and $|e_1 \cap S| = 1$. So we have $V(\mathcal{F}_r) \neq \emptyset$. Next we show that $r \geq k+1$, which will contradict the hypothesis. Otherwise, suppose that $r \leq k$. Then let $v \in S - V(\mathcal{F}_r)$. Since $z_c(u, v) \geq 3k+1$ and the number of edges containing $\{u, v\}$ and intersecting $V(\mathcal{F}_r) - u$ is at most $2r$, there exists an edge f containing $\{u, v\}$ and $f - \{u, v\} \notin V(\mathcal{F}_r)$ and $c(f) \notin \{c(e) \mid e \in \mathcal{F}_r\}$. So $\{f\} \cup E(\mathcal{F}_r)$ induces a rainbow \mathcal{F}_{r+1} , which contradicts the choice of \mathcal{F}_r . This completes the proof of claim 1.

Since $n-k-1 > 2(2k+6)$, by Claim 1, we may greedily choose a set of $2k+6$ vertex-disjoint good pairs in H . This completes the proof. \square

The following lemma is crucial for the proof of our main theorem.

Lemma 10 *Let $k \geq 5$ be an integer and let G be a graph with at most $2k-1$ vertices. If G contains at most one vertex of degree $k-2$ and the rest vertices has degree $k-1$, then for any $f \in E(G)$, $G - f$ is also factor-critical.*

Proof. Otherwise, suppose that the result does not hold. Then there exists $f \in E(G)$ such that $G - f$ is not factor-critical. Then by Lemma 7, there exists $S \subseteq V(G)$ such that $o(G - f - S) \geq |S| + 1$. Recall that $\delta(G) \geq k - 2$ and G contains at most one vertex of degree $k - 2$.

Claim 1. G is $(k - 2)$ -edge-connected.

Otherwise, suppose that there exists $M \subseteq E(G)$ such that $|M| \leq k - 3$ and $G - M$ is not connected. Let F_1 and F_2 be two connected components of $G - M$ such that $|V(F_1)| \leq |V(F_2)|$. Since $\delta(G) \geq k - 2$, then we have $2 \leq |V(F_1)| \leq k - 1$. So we have

$$\begin{aligned} |M| &\geq e_G(V(F_1), V(G) - V(F_1)) \\ &\geq |V(F_1)|(k - |V(F_1)|) - 1 \\ &\geq k - 2, \end{aligned}$$

which contradicts the hypothesis that $|M| \leq k - 3$. This completes the proof of Claim 1.

By Claim 1, we have $q \geq 2$. Let C_1, \dots, C_q denote these odd components of $G - S - f$ such that $|V(C_1)| \geq \dots \geq |V(C_q)|$.

Claim 2. $|V(C_1)| \geq k$.

Otherwise, suppose $|V(C_1)| \leq k - 1$. Then for $1 \leq i \leq q$, $e_G(V(C_i), V(G) - V(C_i)) \geq k - 2$ with equality if and only if the vertex of degree $k - 2$ in G belongs to C_i . Thus

$$\begin{aligned} (k - 1)|S| &\geq \sum_{x \in S} d_G(x) \\ &\geq \sum_{i=1}^q e_G(S, V(C_i)) \\ &\geq (k - 1)(|S| + 1) - 3 \\ &> (k - 1)|S| \quad (\text{since } k \geq 5), \end{aligned}$$

a contradiction. This completes the proof of Claim 2.

By Claim 2, we have $|V(G) - V(C_1)| \leq k - 1$.

Suppose that $\sum_{i=3}^q |V(C_i)| \geq 1$. For any $x \in V(\cup_{i=2}^q C_i)$, it follows that $d_{G-f}(x) \leq k - 3$. Recall that G contains exactly one vertex of degree $k - 2$. This implies that $\sum_{i=2}^q |V(C_i)| \leq 2$, which further yields $q = 3$ and $|V(C_2)| = 1$. Then for $x \in V(C_2 \cup C_3)$,

$$d_G(x) \leq |S| + 1 \leq 3,$$

which contradicts the fact that G contains exactly one vertex of degree $k - 2$.

We proceed by considering the case where $\sum_{i=3}^q |V(C_i)| = 0$. This immediately implies that $q = 2$ and $|S| = 1$. By Claim 1, G is 2-connected. Hence $f \in E_G(V(C_1), V(C_2))$. If $|V(C_2)| \geq 3$, then there exist two vertices, say $x_1, x_2 \in V(C_2) - V(f)$ such that $d_G(x_i) \leq k - 2$, which contradicts the fact that G contains exactly one vertex of degree $k - 2$. We thus conclude that $|V(C_2)| = 1$. Then for $x \in V(C_2)$, $d_G(x) \leq 2$, which once again contradicts the condition that $\delta(G) \geq 3$. This completes the proof. \square

Lemma 11 Let G be a simple graph with $|V(G)| = 6$ and degree sequence $(3, 3, 3, 3, 2, 2)$. Then G has a Hamiltonian cycle.

Proof. Let $V(G) = \{w_1, w_2, w_3, w_4, u, v\}$, where $d_G(u) = d_G(v) = 2$ and $d_G(w_i) = 3$ for $i \in [4]$. Without loss of generality, assume that $uw_1, vw_2 \in E(G)$.

Firstly, we consider the case where $uv \in E(G)$. The subgraph $G_1 = G - \{u, v\}$ has a degree sequence of $(3, 3, 2, 2)$. Then G_1 has a path P of length three that connects w_1 and w_2 . The edge-set $(E(P) \cup \{uw_1, vw_2, uv\})$ forms a Hamiltonian cycle of G .

Next, we consider the case where $uv \notin E(G)$. If $G - \{u, v\}$ is a 4-cycle, denoted as $w_1w_2w_3w_4$, then by symmetry, we may assume that either $N_G(u) = \{w_1, w_3\}$ or $N_G(u) = \{w_1, w_4\}$. In either of these two cases, G has a Hamiltonian cycle. Otherwise, $G - \{u, v\}$ is a not 4-cycle, which implies that $N_G(u) \cap N_G(v) \neq \emptyset$. Note that $N_G(u) \neq N_G(v)$. So, we can assume that $w_3 \in N_G(u) \cap N_G(v)$. Without loss of generality, suppose that $N_G(u) = \{w_1, w_3\}$ and $N_G(v) = \{w_2, w_3\}$. It follows that the edge set $E(G) - \{w_4w_3, w_1w_2\}$ induces a Hamiltonian cycle of G . This completes the proof. \square

3 Proof of Theorem 5

By (1), we have $ar_3(n, \mathcal{F}_{k+1}) \geq f(n, k) + 2$. Therefore the lower bound is followed. Next we assume $k \geq 3$. Let $c(n, k) := f(n, k) + 2$.

For the upper bound, we prove it by contradiction. Suppose that the result does not hold. Then there exists an edge-coloring $c : E(K_n^3) \rightarrow [c(n, k)]$ such that c is surjective and the colored hypergraph denoted by \mathcal{G} contains no rainbow \mathcal{F}_{k+1} . By Lemma 9, we can denote a set of $2k+6$ disjoint good pairs in \mathcal{G} by $Q = \{\{u_1, v_1\}, \dots, \{u_{2k+6}, v_{2k+6}\}\}$, that is $z_c(u_i, v_i) \leq 3k$ for $1 \leq i \leq 2k+6$. Define a color set

$$C_Q = \bigcup_{p \in Q} \{c(T) \mid T \subseteq \mathcal{G}, p \subseteq T\}$$

and let $q = |C_Q|$. Then by Lemma 9, we have the following inequality

$$q \leq 6k^2 + 18k.$$

Let G be a rainbow subgraph of \mathcal{G} with $c(n, k) - q$ edges and vertex set $[n]$ such that

$$\{c(e) \mid e \in E(G)\} \cap C_Q = \emptyset.$$

Claim 1. G is \mathcal{F}_k -free.

Otherwise, suppose that G contains a copy of \mathcal{F}_k denoted by \mathcal{F} with center u . Since $|Q| \geq 2k+6$ and $|V(\mathcal{F})| = 2k+1$, there exists $\{u_i, v_i\} \in Q$ such that $\{u_i, v_i\} \cap V(\mathcal{F}) = \emptyset$. By the definition of G , $E(\mathcal{F}) \cup \{u, u_i, v_i\}$ induces a rainbow copy of \mathcal{F}_{k+1} , contradicting the hypothesis. This completes the proof of Claim 1.

Let $w : G \times P \rightarrow R$ be a weighted function defined as in Section 2. And we denote $W_v = \sum_{p \in N(v)} w(v \cup p, p)$. Next we discuss two cases.

Case 1. k is odd.

Then we have

$$e(G) = c(n, k) - q \geq (n - 2k)k(k - 1) + 2\binom{k}{3} + 2 - 6k^2 - 18k. \quad (2)$$

By Lemma 8, we have $W_v \leq k(k - 1)$ for all $v \in V(G)$. Let $W := \{v \in V(G) \mid W_v = k(k - 1)\}$. Then we have the following claim.

Claim 2. $W \neq \emptyset$.

Suppose to the contrary that $W_v < k(k - 1)$ for all $v \in V(G)$. By the definition of weight function, we have $W_v \leq k(k - 1) - 2/3$ for all $v \in V(G)$. So

$$e(G) = \sum_{v \in V(G)} W_v \leq nk(k - 1) - 2n/3. \quad (3)$$

Combining (2) and (3), we may infer that

$$n \leq \frac{5}{2}k^3 + \frac{15}{2}k^2 + 26k - 3,$$

a contradiction. This completes the proof of Claim 2.

By Claim 2, we may choose $x \in W$ such that $W_x = k(k - 1)$. Then by Lemma 8, $N_G(x)$ is the disjoint union of two complete 2-graphs on k vertices and every edge of $N_G(x)$ is in A -type. Denote the two complete subgraphs by R_1 and R_2 .

Claim 3. $|\{c(e) \mid x \in e \text{ and } e \subseteq \binom{[n]}{3}\}| \leq 2\binom{k}{2} + 1$.

Otherwise, suppose that $|\{c(e) \mid x \in e \text{ and } e \subseteq \binom{[n]}{3}\}| \geq 2\binom{k}{2} + 2$. Then one can see that

$$|\{c(p \cup \{x\}) \mid p \notin E(R_1 \cup R_2) \text{ and } p \in \binom{[n] - x}{2}\}| \geq 2. \quad (4)$$

We choose $T_1, T_2 \in \binom{[n] - \{x\}}{2} - (\binom{V(R_1)}{2} \cup \binom{V(R_2)}{2})$ such that $c(T_1 \cup \{x\}) \neq c(T_2 \cup \{x\})$, and for $i \in [2]$, $c(T_i \cup \{x\}) \notin \{c(p \cup \{x\}) \mid p \in E(R_1 \cup R_2)\}$. For $i \in [2]$, $F - V(T_i)$ has a matching M_i of size $k - 1$. Note that

$$|V(M_1 \cup M_2) \cup T_1 \cup T_2| \leq 2k + 4.$$

So by Lemma 9, we may choose $T_0 \in Q$ such that $x \notin T_0$ and $T_0 \cap V(M_1 \cup M_2 \cup \{T_1, T_2\}) = \emptyset$. It follows that either $\{p \cup \{x\} \mid p \in M_1 \cup \{T_1, T_0\}\}$ or $\{p \cup \{x\} \mid p \in M_2 \cup \{T_2, T_0\}\}$ induces a rainbow copy of \mathcal{F}_{k+1} , a contradiction. This completes the proof of Claim 3.

Let \mathcal{G}' be a rainbow edge-colored subgraph of $\mathcal{G} - \{x\}$ obtained by deleting the triples colored by the colors from $\{c(e) \mid x \in e \text{ and } e \in \binom{[n]}{3}\}$. By Claim 3, $e(\mathcal{G}') \geq f(n - 1, k) + 1$. Thus \mathcal{G}' has a rainbow copy of \mathcal{F}_k with core y . Choosing $y' \in [n] - V(\mathcal{F}_k) - \{x\}$, in view of the preceding analysis, we know that $c(\{x, y, y'\}) \notin \{c(e) \mid e \in \mathcal{F}_k\}$. It follows that $\{x, y, y'\} \cup \mathcal{F}_k$ a rainbow copy of \mathcal{F}_{k+1} , a contradiction. This completes the proof.

Case 2. $k \geq 4$ is even.

Then we have

$$e(G) \geq c(n, k) - q \geq \frac{1}{2}nk(2k - 3) - \frac{1}{2}(2k^3 - 9k + 6) - 6k^2 - 18k. \quad (5)$$

By Lemma 8, $W_v \leq k(k - 3/2)$ for all $v \in V(G)$. By Theorem 4, we may assume that $k \geq 4$.

Claim 4. There exists $v \in V(G)$ such that $W_v = k(k - 3/2)$.

Otherwise, we may assume that $W_v \leq k(k - 3/2) - 1/2$ for all $v \in V(G)$. Thus we get

$$e(G) \leq nk(k - 3/2) - n/2. \quad (6)$$

Combining (5) and (6), we may infer that $n \leq 2k^3 + 12k^2 + 27k + 6$, a contradiction. This completes the proof of Claim 4.

We choose $x \in V(G)$ with $W_x = k(k - 3/2)$. Suppose that following inequality holds

$$|\{c(e) \mid x \in e \text{ and } e \subseteq \binom{[n]}{3}\}| \leq k(k - \frac{3}{2}) + 1, \quad (7)$$

Let G' be a rainbow subgraph of \mathcal{G} avoiding the colors appearing in $\{c(e) \mid x \in e \text{ and } e \subseteq \binom{[n]}{3}\}$. By (7), we have $e(G') > f(n - 1, k)$. Hence G' contains a copy of \mathcal{F}_k denoted by \mathcal{M} . Let y be the core of \mathcal{M} . We select a vertex $x' \in [n] - V(\mathcal{M}) - \{x\}$. Now $E(\mathcal{M}) \cup \{\{x', x, y\}\}$ induces a rainbow copy of \mathcal{F}_{k+1} , a contradiction. This contradiction concludes our proof. So next it is sufficient for us to show that (7) holds.

Let $F := N_G(x)$ denote the 2-graph with edge set $N_G(x)$ and vertex set $V(N_G(x))$. By Lemma 8, F consists of two vertex-disjoint factor-critical graphs or satisfies (a), (b), and (c).

We first prove the following claim, which provides a method for constructing a rainbow \mathcal{F}_{k+1} in \mathcal{G} as described below.

Claim 5. Let $x_0 \in V(F)$ with $d_F(x_0) = k - 1$, and let $\{y_0, w\} \subseteq [n] - (N_F(x_0) \cup \{x, x_0\})$ be a 2-element set. If $c(\{x_0, w, x\}) \neq c(\{x_0, y_0, x\})$, then \mathcal{G} has a rainbow \mathcal{F}_{k+1} with core x_0 .

Let $N_F(x_0) = \{y_2, \dots, y_k\}$. Our next step is to construct a rainbow copy of \mathcal{F}_{k+1} with the core x_0 . According to Lemma 8, for $2 \leq i \leq k$, $x_0 y_i$ is A-type. Consequently, we know that $d_G(\{x_0, y_i\}) \geq 2k - 1$. For $2 \leq i \leq k$, we can pick a set $\{y_{i,1}, \dots, y_{i,2k-1}\} \subseteq N_G(\{x_0, y_i\})$. Let T_1 be a 2-element subset chosen from the set $[n] - V(F) - \{x, w, y_0\} - (\bigcup_{i=2}^k \bigcup_{j=1}^{2k-1} \{y_{i,j}\})$. Since $c(\{x_0, w, x\}) \neq c(\{x_0, y_0, x\})$, we have either $c(T_1 \cup \{x_0\}) \neq c(\{x_0, y_0, x\})$ or $c(T_1 \cup \{x_0\}) \neq c(\{x_0, w, x\})$. Without loss of generality, assume that $c(T_1 \cup \{x_0\}) \neq c(\{x_0, y_0, x\})$. Since G is a rainbow subgraph and $k \geq 4$, we can re-order the vertices y_2, \dots, y_k in such a way that

$$c(T_1 \cup \{x_0\}) \notin \{c(\{x_0, y_{k-1}, y_{k-1,j}\}) \mid j \in [2k - 1]\}$$

and

$$\{c(\{x_0, y_0, x\}), c(T_1 \cup \{x_0\})\} \cap \{c(\{x_0, y_k, y_{k,j}\}) \mid j \in [2k - 1]\} = \emptyset.$$

Next, we aim to extend the sets $T_1 \cup \{x_0\}$, $\{x_0, y_0, x\}$ to form a rainbow copy of \mathcal{F}_{k+1} with the core x_0 . Since $k \geq 4$, we can find a vertex $y'_{2,1} \in N_G(\{x_0, y_2\}) - \{x, y_3, \dots, y_k\}$, such that $\{\{T_1 \cup \{x_0\}, \{x_0, y_0, x\}, \{x_0, y'_{2,1}, y_2\}\}\}$ induces a rainbow copy \mathcal{F}_3 . Now, assume that for some $i < k-1$, we have already constructed a collection $\{\{T_1 \cup \{x_0\}, \{x_0, y_0, x\}, \{x_0, y'_{2,1}, y_2\}, \dots, \{x_0, y'_{i,1}, y_i\}\}$ that induce a rainbow copy of \mathcal{F}_{i+1} , where $\{y'_{2,1}, \dots, y'_{i,1}\} \cap N_F(x_0) = \emptyset$. Define $W_i := \{\bigcup_{j=2}^i \{y_j, y'_{j,1}\}\} \cup \{y_0, x_0, x\} \cup V(T_1)$. Let's consider the step for $i+1$. Since $d_G(\{y_{i+1}, x_0\}) \geq 2k - 1$ and $T_1 \cap N_G(\{y_{i+1}, x_0\}) = \emptyset$, there exists $y'_{i+1,1} \in N_G(\{y_{i+1}, x_0\}) - (W_i \cup N_F(x_0))$

such that $\{\{T_1 \cup \{x_0\}, \{x_0, y_0, x\}, \{x_0, y'_{2,1}, y_2\}, \dots, \{x_0, y'_{i+1,1}, y_{i+1}\}\}$ induces a rainbow copy of \mathcal{F}_{i+2} . Continuing the process until $i = k - 1$, we can obtain a rainbow copy of \mathcal{F}_k denoted by \mathcal{M} . Recall that $d_G(\{y_k, x_0\}) \geq 2k - 1$ and $T_1 \cap N_G(\{y_k, x_0\}) = \emptyset$. As a result, there exists a vertex $y'_{k,1}$ in the set $N_G(\{y_k, x_0\}) - W_{k-1}$. Moreover, by the choice of y_k , $c(\{x_0, y_k, y'_{k,1}\}) \notin \{c(\{x_0, y_0, x\}), c(T_1 \cup \{x_0\})\}$. Hence $E(\mathcal{M}) \cup \{x_0, y_k, y'_{k,1}\}$ induces a rainbow copy of \mathcal{F}_{k+1} , a contradiction. This completes the proof of Claim 5.

Next by Lemma 8, we proceed by discussing two subcases.

Subcase 2.1. $F = R_1 \cup R_2$, where $R_1 \cong K_{k-1}$ and R_2 is a factor-critical graph of order $k + 1$ with degree sequence $(k - 1, \dots, k - 1, k - 2)$.

Claim 6. $\{c(g \cup \{x\}) \mid g \in \binom{[n] - \{x\} - V(R_1 \cup R_2)}{2}\}$ contains exactly one color denoted by c_0 which belongs to the color set C_Q .

We will use a proof by contradiction. Firstly, suppose that there exists $g \in \binom{[n] - \{x\} - V(R_1 \cup R_2)}{2}$ such that $c(g \cup \{x\}) \in \{c(p \cup \{x\}) \mid p \in E(R_1 \cup R_2)\}$. Write $c(g \cup \{x\}) = c(g' \cup \{x\})$, where $g' \in E(R_1 \cup R_2)$. Since R_1 and R_2 are factor-critical graphs, $F - g'$ has a matching M_1 of size $k - 1$. Let $h \in Q$ such that $x \notin h$ and $h \cap V(M_1 \cup \{g\}) = \emptyset$. It follows that $\{p \cup \{x\} \mid p \in M_1 \cup \{g, h\}\}$ induces a rainbow copy of \mathcal{F}_{k+1} , a contradiction. Subsequently, suppose that there are two distinct edges $g_1, g_2 \in \binom{[n] - \{x\} - V(R_1) - V(R_2)}{2}$ such that $c(g_1 \cup \{x\}) \neq c(g_2 \cup \{x\})$ and neither of them belongs to $\{c(p \cup \{x\}) \mid p \in E(R_1 \cup R_2)\}$. Let M_2 be a matching of size $k - 1$ in $R_1 \cup R_2$. Since $|Q| \geq 2k + 6$, we may choose $h \in Q$ such that $x \notin h$ and $h \cap V(M_2 \cup \{g_1, g_2\}) = \emptyset$. It follows that either $\{p \cup \{x\} \mid p \in M_2 \cup \{g_1, h\}\}$ or $\{p \cup \{x\} \mid p \in M_2 \cup \{g_2, h\}\}$ induces a rainbow copy of \mathcal{F}_{k+1} , a contradiction. This completes the proof of Claim 6.

Claim 7. For any $f \in ([n] - V(R_1) - \{x\}) \times V(R_1)$, we have $c(f \cup \{x\}) \in \{c_0\} \cup \{c(p \cup \{x\}) \mid p \in E(F)\}$; and for any $g \in ([n] - V(R_1 \cup R_2) - \{x\}) \times V(R_2)$, we have $c(g \cup \{x\}) = c_0$.

We will use a proof by contradiction. Firstly, suppose that there exists $f_1 \in ([n] - V(R_1) - \{x\}) \times V(R_1)$ such that $c(f_1 \cup \{x\}) \notin \{c_0\} \cup \{c(p \cup \{x\}) \mid p \in E(R_1 \cup R_2)\}$. Since R_1 and R_2 are factor-critical graphs, $F - V(f_1)$ contains a matching M_3 of size $k - 1$. Then we can select $h \in \binom{[n] - V(R_1 \cup R_2) - \{x\}}{2}$ such that $h \cap V(f_1) = \emptyset$. By Claim 6, we have $c(h \cup \{x\}) = c_0$. Thus $\{p \cup \{x\} \mid p \in M_3 \cup \{f_1, h\}\}$ induces a rainbow copy of \mathcal{F}_{k+1} , which is a contradiction. Next consider that there exists $g_1 \in ([n] - V(R_1 \cup R_2) - \{x\}) \times V(R_2)$ such that $c(g_1 \cup \{x\}) \neq c_0$. If $c(g_1 \cup \{x\}) \notin \{c(p \cup \{x\}) \mid p \in E(R_1 \cup R_2)\}$, let M_4 be a matching of size $k - 1$ of $F - V(g_1)$ and let h' be a 2-subset of $[n] - (g_1 \cup V(F) \cup \{x\}) \cup V(M)$, then $\{p \cup \{x\} \mid p \in M_4 \cup \{g_1, h'\}\}$ induces a rainbow copy of \mathcal{F}_{k+1} a contradiction again. So we may assume that $c(g_1 \cup \{x\}) \in \{c(p \cup \{x\}) \mid p \in E(R_1 \cup R_2)\}$. Select an edge $g_2 \in E(R_1 \cup R_2)$ such that $c(g_2 \cup \{x\}) = c(g_1 \cup \{x\})$. Then, pick an element $g_0 \in \binom{[n] - V(F) - V(g_1 \cup g_2) - \{x\}}{2}$. By Claim 6, we have $c(g_0 \cup \{x\}) = c_0$. If $F - V(g_1) - g_2$ has a matching M of size $k - 1$, then $\{e \cup \{x\} \mid e \in M \cup \{g_0, g_1\}\}$ induces a rainbow copy of \mathcal{F}_{k+1} , which is a contradiction. Consequently, we can assume that

$$\nu(F - V(g_1) - g_2) < k - 1. \quad (8)$$

If $k \geq 6$, by Lemma 10, $\nu(F - V(g_1) - g_2) = k - 1$, which contradicts inequality (8). So we can conclude $k = 4$. Write $V(R_2) := \{y_0, y_1, \dots, y_4\}$ such that $d_{R_2}(y_0) = 2$, and $y_0y_1, y_0y_4 \in$

$E(R_2)$. If $g_1 \cap \{y_1, y_4\} = \emptyset$ or $g_1 \cap g_2 \neq \emptyset$, then $F - V(g_1) - g_2$ has a matching of size 3, which again contradicts inequality (8). Moreover, by symmetry, we may assume that $y_1 \in g_1 \cap V(R_2)$ and $g_1 \cap g_2 = \emptyset$. Recall that for every edge $e \in E(R_2)$, e is type A in G , which means $d_G(\{y_1, y\}) \geq 7$ for $y \in \{y_0, y_2, y_3\}$. Note that $\{y_2, y_3, y_0\} \cap (g_1 \cup \{x\}) = \emptyset$. Let $w \in [n] - V(F) - \{x\}$. By Claim 7, we have $c(\{x, w, y_1\}) = c_0 \neq c(g_1 \cup \{x\})$. With the same discussion as Claim 6, $\{x\} \cup g_1$ can be extended into a rainbow copy with core y_1 denoted by \mathcal{M} of \mathcal{F}_4 , where $\{c(e) \mid e \in E(\mathcal{F}_4)\} \subseteq \{c(e) \mid e \in E(\mathcal{M})\}$. By Lemma 9, we may choose $h \in Q$ such that $h \cap V(\mathcal{M}) = \emptyset$. Then $\{h \cup \{y_1\}\} \cup E(\mathcal{M})$ induces a rainbow copy of \mathcal{F}_5 in \mathcal{G} , a contradiction. This completes the proof of Claim 7.

By Claims 6 and 7, it is sufficient for us to show that for any $l \in \binom{V(R_2)}{2} \setminus E(R_2)$, $c(l \cup \{x\}) = c_0$. Suppose to the contrary that there exists $l_0 = \{x_0, y_0\} \in \binom{V(R_2)}{2} \setminus E(R_2)$ such that $c(\{x_0, y_0, x\}) \neq c_0$. Note that R_2 is a factor-critical graph of order $k+1$ with degree sequence $(k-1, \dots, k-1, k-2)$. Without loss of generality, suppose $d_{R_2}(x_0) = k-1$. By Claim 7, for $w \in [n] - V(F) - \{x\}$, $c(\{x_0, w, x\}) = c_0 \neq c(\{x_0, y_0, x\})$. By Claim 5, we can find a rainbow copy of \mathcal{F}_{k+1} with the core x_0 , which contradicts the hypothesis. Thus we can derive that the inequality (7) holds.

Subcase 2.2. F satisfies (a), (b) and (c).

Let U_k denote the set of vertices that serve as the centers of stars fulfilling condition (b) in Lemma 8. Let $V_k := V(F) \setminus (V(F_0) \cup U_k)$. Let F_0 be the factor-critical component that satisfies condition (a) in Lemma 8. Since $\nu(F) = k-1$ and the number of vertices in F_0 is at least $k+1$ (i.e., $|F_0| \geq k+1$), we can infer that the cardinality of the set U_k is at most $\frac{k}{2}-1$, that is, $|U_k| \leq \frac{k}{2}-1$.

Claim 8. The set $\{c(g \cup \{x\}) \mid g \in \binom{[n]-\{x\}-V(F)}{2}\}$ contains exactly one color denoted by c_1 , which belongs to the color set C_Q .

We will prove this statement by contradiction. First, suppose that there exists $g \in \binom{[n]-\{x\}-V(F)}{2}$ such that

$$c(g \cup \{x\}) \in \{c(f \cup \{x\}) \mid f \in E(F)\}.$$

Our goal is to construct a rainbow copy \mathcal{F}_{k+1} , which will lead to a contradiction. Without loss of generality, assume that $c(g \cup \{x\}) = c(g' \cup \{x\})$, where $g' \in E(F)$. Since F_0 is factor-critical and $U_k \leq \frac{k}{2}-1$, $F-g'$ has a matching M of size $k-1$. By Lemma 9, we can then find $h \in Q$ such that $x \notin h$ and $h \cap V(M \cup \{g\}) = \emptyset$. In this situation, the collection $\{p \cup \{x\} \mid p \in M \cup \{g, h\}\}$ forms a rainbow copy of \mathcal{F}_{k+1} , which contradicts our initial assumption. Next, consider the case when there are two distinct edges $g_1, g_2 \in \binom{[n]-\{x\}-V(F)}{2}$ such that $c(g_1 \cup \{x\}) \neq c(g_2 \cup \{x\})$ and neither $c(g_1 \cup \{x\})$ nor $c(g_2 \cup \{x\})$ belongs to the set $\{c(p \cup \{x\}) \mid p \in E(F)\}$. We select a matching M' of size $k-1$ in F . Subsequently, we can find an element $h \in Q$ such that $x \notin h$ and $h \cap V(M' \cup \{g_1, g_2\}) = \emptyset$. As a consequence, either the set $\{p \cup \{x\} \mid p \in M' \cup \{g_1, h\}\}$ or $\{p \cup \{x\} \mid p \in M' \cup \{g_2, h\}\}$ forms a rainbow copy of \mathcal{F}_{k+1} , a contradiction. This completes the proof of Claim 8.

Let $\mathcal{C} := \{c(p \cup \{x\}) \mid p \in E(F)\}$ and let $\mathcal{C}' := \{c_1\} \cup \{c(p \cup \{x\}) \mid p \in E(F)\}$.

Claim 9. For any $f \in \binom{[n] \setminus (U_k \cup \{x\})}{2}$ satisfying $|f \cap V(F_0)| = 1$, we have $c(f \cup \{x\}) = c_1$.

Suppose that there exists $g_1 \in \binom{[n]-U_k-\{x\}}{2}$ such that $|g_1 \cap V(F_0)| = 1$ and $c(g_1 \cup \{x\}) \neq c_1$. We choose $h \in \binom{[n]-V(F)-\{x\}}{2}$ such that $h \cap V(g_1) = \emptyset$. By Claim 8, we have $c(h \cup \{x\}) = c_1$. If $c(g_1 \cup \{x\}) \notin \mathcal{C}$, we proceed as follows. Let M_3 be a matching M_3 of size $k-1$ in $F - V(g_1)$. Then, the collection $\{p \cup \{x\} \mid p \in M_3 \cup \{g_1, h\}\}$ induces a rainbow copy of \mathcal{F}_{k+1} , a contradiction. Hence we may infer that

$$c(g_1 \cup \{x\}) \in \mathcal{C}. \quad (9)$$

Let $g_2 \in E(F)$ such that $c(g_2 \cup \{x\}) = c(g_1 \cup \{x\})$. If $F - V(g_1) - g_2$ contains a matching M_4 of size $k-1$, $\{p \cup \{x\} \mid p \in M_4 \cup \{g_1, h\}\}$ induces rainbow copy of \mathcal{F}_{k+1} , a contradiction. Thus we have

$$\nu(F - V(g_1) - g_2) < k-1. \quad (10)$$

Then by Lemma 10, we may infer that $k=4$. Let $g_1 \cap V(F_0) = \{y_0\}$. Consider $d_F(y_0) = 3$. By the proof of Claim 5, $\{x\} \cup g_1$ may be extended into a rainbow copy denoted by \mathcal{M} of \mathcal{F}_4 such that for any $e \in E(\mathcal{M})$, $c(e) \in \{c(f) \mid f \in E(G)\}$. Now we select $h \in Q$ such that $h \cap V(\mathcal{M}) = \emptyset$. It follows that $(\{y_0\} \cup h) \cup E(\mathcal{M})$ induces a rainbow copy of \mathcal{F}_5 , a contradiction. Next we may assume $d_F(y_0) = 2$. Recall that $g_2 \in E(F)$ such that $c(g_2 \cup \{x\}) = c(g_1 \cup \{x\})$. By Lemma 11, $F - V(g_1)$ contains a Hamilton cycle. So $F - V(g_1) - g_2$ contains a matching M_5 of size 3, which contradicts the inequality (10). This completes the proof of Claim 9.

Claim 10. For any $f \in \binom{[n] \setminus (U_k \cup \{x\} \cup V(F_0))}{2}$ satisfying $|f \cap V_k| = 1$, we have $c(f \cup \{x\}) = c_1$.

We proceed by contradiction. Suppose to the contrary that there exists $f_1 \in \binom{[n] \setminus (U_k \cup \{x\} \cup V(F_0))}{2}$ such that $c(f_1 \cup \{x\}) \neq c_1$. We choose $h \in \binom{[n]-V(F)-\{x\}}{2}$ such that $h \cap V(f_1) = \emptyset$. It follows that $c(h \cup \{x\}) = c_1$ by Claim 8. If $c(f_1 \cup \{x\}) \in \mathcal{C}$, let $f_2 \in E(F)$ such that $c(f_1 \cup \{x\}) = c(f_2 \cup \{x\})$; otherwise let $f_2 = \emptyset$. Since F_0 is a factor-critical graph, we know that $F - V(f_1) - f_2$ contains a matching M_2 of size $k-1$. Thus $\{p \cup \{x\} \mid p \in M_2 \cup \{f_1, h\}\}$ induce a rainbow copy of \mathcal{F}_{k+1} , a contradiction again. This completes the proof of Claim 10.

Claim 11. For any $T \in \binom{V_k}{2}$, $c(T \cup \{x\}) \in \mathcal{C}'$.

Suppose to the contrary that there exists $T_1 \in \binom{V_k}{2}$ such that $c(T_1 \cup \{x\}) \notin \mathcal{C}'$. Since $d_F(w) = 3$ for $w \in U_k$ and $k \geq 4$, $F - V(T_1)$ has a matching M of size $k-1$. We may choose $T_2 \in \binom{[n]-\{x\} \cup V(F) \cup T_1}{2}$. It follows that $c(T_2 \cup \{x\}) = c_1$ by Claim 8. So $\{p \cup \{x\} \mid p \in M \cup \{T_2, T_1\}\}$ induces a rainbow copy of \mathcal{F}_{k+1} , a contradiction. This completes the proof of Claim 11.

Let $z \in V(F_0)$ such that $d_{F_0}(z) = k-2$.

Claim 12. For any $T \in \left((U_k \times V(F_0 - z)) \cup \binom{V(F_0)}{2} \right) \setminus E(F)$, $c(T \cup \{x\}) \in \mathcal{C}'$.

Suppose to the contrary that there exists $T \in \left((U_k \times V(F_0 - z)) \cup \binom{V(F_0)}{2} \right) \setminus E(F)$ such that $c(T \cup \{x\}) \notin \mathcal{C}'$. Write $T = \{y_1, y_2\}$. Without loss of generality, let y_1 be a vertex in F_0 with degree $k-1$. We choose $w \in [n] - (V(F) \cup \{x\})$. By Claim 9, we have $c(\{x, w, y_1\}) = c_1$. So we have $c(\{x, w, y_1\}) \neq c(\{x, y_1, y_2\})$. Recall that $\{x, y_1\}, \{y_1, y_2\} \notin E(F)$. Then by Claim

5, we can find a rainbow copy of \mathcal{F}_{k+1} in \mathcal{G} with core y_1 , a contradiction. This completes the proof of Claim 12.

Claim 13. For any $T \in \binom{[n] - (V(F_0) - z) \cup \{x\})}{2} \setminus E(F)$, $c(T \cup \{x\}) \in \mathcal{C}'$.

Suppose to the contrary that there exists $T \in \binom{[n] - (V(F_0) \cup \{x\})}{2} \setminus E(F)$ such that $c(T \cup \{x\}) \notin \mathcal{C}'$. Write $T = \{x_0, y_0\}$. By Claims 8, 9, 10 and 11, we have $T \cap U_k \neq \emptyset$. Without loss generality, suppose that $x_0 \in U_k$. Then $d_F(x_0) = k-1$. Let $w \in V(F_0) - N_F(x_0)$ such that $d_{F_0}(w) = k-1$. By Claim 12, we have $c(\{w, x, x_0\}) \in \{c(p \cup \{x\}) \mid p \in E(F)\} \cup \{c_1\}$. Thus it follows that $c(\{w, x, x_0\}) \neq c(\{x, x_0, y_0\})$. By Claim 5, we can find a rainbow copy of \mathcal{F}_{k+1} with core x_0 , a contradiction. This completes the proof of Claim 13.

By Claims 8, 9, 10, 11, 12 and 13, for any $e \in \binom{[n] - x}{2}$, $c(e \cup \{x\}) \in \mathcal{C}'$. Thus we can derive that the inequality (7) holds. This complete the proof of Theorem 5. \square

References

- [1] F. Chung, Unavoidable stars in 3-graphs, *J. Combin. Theory Ser. A*, **35** (1983), 252–262.
- [2] F. Chung and P. Frankl, The maximum number of edges in a 3-graph not containing a given star, *Graphs Combin.*, **3** (1987), 111–126.
- [3] G. Dirac, Some theorems on abstract graphs, *Proc. London Math. Soc.*, **2** (1952), 68–81.
- [4] R. Duke and P. Erdős, Systems offinite sets having a common intersection, In: *Proceedings, 8th S-E Conf. Combinatorics, Graph Theory and Computing*, 1977, pp. 247-252.
- [5] P. Erdős and R. Rado, Intersection theorems for systems of sets, *J. London Math. Soc.*, **35** (1960), 85–90.
- [6] P. Erdős and M. Simonovits, V. Sós, Anti-Ramsey theorems, Infinite and finite Sets, *Colloq. Math. Soc. János Bolyai*, **10** (1975), 633–643.
- [7] P. Erdős and V. Sós, Remarks on the connection of graph theory, finite geometry and block designs, *Accad. Naz. Lincei, Rome*, **2** (1976), 223–233.
- [8] P. Frankl, An extremal set theoretical characterization of some Steiner systems, *Combinatorica*, **3** (1983), 193–199.
- [9] P. Frankl and A. Kupavskii, Two problems of P. Erdős on matchings in set families—in the footsteps of Erdős and Kleitman, *J. Combin. Theory Ser. B*, **138** (2019), 286–313.
- [10] S. Fujita, C. Magnant and K. Ozeki, Rainbow generalizations of Ramsey theory: A survey, *Graphs Combin.*, **26** (2010), 1–30.
- [11] R. Gu, J. Li and Y. Shi, Anti-Ramsey numbers of paths and cycles in hypergraphs, *SIAM J. Discrete Math.*, **34**(1) (2020), 271–307.
- [12] T. Gallai, Kritische Graphen für die perfekte Matchung, *MTA Math. Kutató Int. Közl.*, **8** (1963), 373–395.

- [13] M. Guo, H. Lu and X. Peng, Anti-Ramsey number of matchings in 3-uniform hypergraphs, *SIAM J. Discrete Math.*, **37**(3) (2023), 1970–1987.
- [14] T. Jiang, Edge-colorings with no large polychromatic stars, *Graphs Combin.*, **18** (2002), 303–308.
- [15] T. Li, Y. Tang, G. Wang and G. Yan, Anti-Ramsey numbers of loose paths and cycles in uniform hypergraphs, *Graphs Combin.*, **41** (2025), Paper No. 76.
- [16] X. Liu and J. Song Hypergraph anti-Ramsey theorems, *J. Graph Theory*, **108** (2025), 808—816.
- [17] J. Montellano-Ballesteros and V. Neumann-Lara, An anti-Ramsey theorem, *Combinatorica*, **22** (2002), 445–449.
- [18] J. Montellano-Ballesteros, On totally multicolored stars, *J. Graph Theory*, **51**(2006), 225–243.
- [19] L. Ōzkahya and M. Young, Anti-Ramsey number of matchings in hypergraphs, *Discrete Math.*, **313** (2013), 2359–2364.
- [20] I. Schiermeyer, Rainbow numbers for matchings and complete graphs, *Discrete Math.*, **286** (2004), 157–162.
- [21] V. Sós, Remarks on the connection of graph theory, finite geometry and block designs, *Accad. Naz. Lincei, Rome*, **2** (1976), 223–233.
- [22] Y. Tang and T. Li, Anti-Ramsey numbers of cycles of length three in uniform hypergraphs, *Acta Math. Appl. Sin. Engl. Ser.*, **41** (2025), 797–805.
- [23] Y. Tang, T. Li and G. Yan, Anti-Ramsey number of disjoint union of star-like hypergraphs, *Discrete Math.*, **347** (2024), 11374.
- [24] W. Tutte, The factorization of linear graphs, *J. London Math. Soc.*, **22** (1947), 107–111.
- [25] X. Zhu, Y. Chen, D. Gerbner, E. Györi and H. Karim, The maximum number of triangles in F_k -free graphs, *Eur. J. Comb.*, **114** (2023), 103793.