

HOMOLOGICAL MILNOR-WITT MODULES AND CHOW-WITT GROUPS OVER GENERAL BASES

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ABSTRACT. We introduce a general theory of Milnor–Witt cycle modules over a base scheme S , extending simultaneously Rost’s classical cycle modules and Feld’s Milnor–Witt theory over fields, and yielding a Borel–Moore type intersection theory with quadratic coefficients. We define *homological* Milnor–Witt modules and their Rost–Schmid complexes, establish proper pushforwards, smooth pullbacks, Gysin morphisms and duality, and introduce *pinnings* allowing a cohomological version and a duality equivalence between the two. As applications, we construct Chow–Witt groups for possibly singular S -schemes, satisfying localization, homotopy invariance and canonical duality. When S is regular of dimension at most 1, we prove generalized Bloch formulas identifying these groups with unramified Milnor–Witt cohomology, hence establishing their representability in motivic homotopy theory. For 1-dimensional arithmetic schemes, we compute homology groups of the Rost–Schmid complex associated with Milnor–Witt K-theory in the regular case, and deduce a comparison between Chow–Witt and Chow groups, as well as finiteness results in the general case, in analogy with the Bass–Tate conjecture. This work provides the cycle-theoretic foundation of an ongoing project on describing perverse homotopy t -structures via Milnor–Witt modules.

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Date: 11-12-2025.

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INTRODUCTION

Intersection theory is one of the historical pillars of algebraic geometry. Starting from Bézout’s theorem, the notion of intersection multiplicities has been a major focus of the early developments of the subject, evolving into the well-established theory of Chow groups developed during the second half of the twentieth century. More recently, motivated by the Gersten conjecture, Bloch’s formula and the horizon of mixed motives, Rost introduced the theory of *cycle modules* [Ros96] for schemes over a field. This provides a powerful framework for enriching Chow groups with coefficients coming from Milnor K -theory or more general cohomological functors, and opens the way to new homological methods in intersection theory, such as localization long exact sequences and spectral sequences.

Subsequent developments by Schmid, Barge-Morel and Fasel showed that Chow groups admit a *quadratic refinement*, now called the *Chow-Witt groups* [Fas08, FS09], in which degrees and intersection multiplicities are valued in the Grothendieck–Witt ring GW , and whose cohomological counterparts encode subtle quadratic and arithmetic information, allowing numerous applications to Euler classes, quadratic enumerative geometry, and real or hermitian phenomena. Over a perfect base field k , a quadratic analogue of Rost’s cycle modules has been defined by Feld in [Fel20].

The aim of this paper is to introduce a general theory of *Milnor-Witt cycle (ho)modules* over a base scheme, and to deduce from it a Borel-Moore type intersection theory extending the Chow-Witt groups to singular and arithmetic schemes. Our approach extends simultaneously Rost’s classical theory of cycle modules and Feld’s theory of Milnor-Witt cycle modules over a field, and produces a framework of an intersection theory with quadratic coefficients, without any base field. In particular, we give a cycle-theoretic definition of Chow-Witt groups for possibly singular or mixed characteristic schemes, together with expected functorialities.

More precisely, we introduce the notion of a *homological Milnor-Witt cycle module* (aka *MW-homomodule*, Section 1.2.2) over an excellent base scheme S with a dimension function. Although the definition appears notably different from Rost's and Feld's definitions, the axioms are indeed modeled on natural examples of Borel-Moore theories, such as Quillen's K' -theory (or G -theory) or Gille's homological Witt groups (Section 1.3.6). To any MW-homomodule we associate a Rost-Schmid type complex of quadratic cycles, whose homology satisfies the usual functorialities such as homotopy invariance, proper pushforwards, smooth pullbacks, and localization long exact sequences. In particular, MW-homomodules form a natural category of coefficients for intersection theory, equipped with a Leray-type spectral sequence (Section 2.1.2). We then construct, for essentially smoothable lci morphisms, Gysin morphisms and excess-intersection formulas, and show that the resulting Borel-Moore theory is (canonically) *SL-oriented*, in a sense slightly generalized from Panin-Walter [PW18]: it admits an *action* of Thom and Euler classes. Furthermore, those theories satisfy the homological form of cdh-descent, and more generally for hyperenvelope (Section 1.5.9, Section 1.5.10).

A new feature of our theory is the introduction of *pinnings* (Section 3.1.3), a datum induced by a dualizing complex, which allows us to define (cohomological) MW-modules (Section 3.3.2) whose axioms are closer to Rost's and Feld's theories. Natural examples of MW-modules include Milnor and Milnor-Witt K-theories, Witt groups and the powers of the fundamental ideal of the Witt ring (Section 3.3.6). We then establish a duality equivalence Section 3.3.4 between homological and cohomological versions, which is reminiscent of cohomological purity theorems. Assuming S is essentially of finite type over a field, this equivalence allows us to compare MW-modules with Feld's theory, and to show that Rost's cycle modules coincide with *oriented* MW-modules.

From these constructions we derive several concrete applications.

- (1) We define homological (resp. cohomological) *Chow-Witt groups* for any S -scheme essentially of finite type (resp. essentially lci) X , and any twist by a line bundle \mathcal{L}/X , extending the definition of Fasel-Srinivas [FS09] from regular schemes to possibly singular bases, and without assuming that 2 is invertible on S . These groups satisfy the property of a Borel-Moore homology relative to S as described above, including localization sequences, A^1 -homotopy invariance, and a canonical duality isomorphism

$$\widetilde{CH}_p(X/S, \mathcal{L}) \simeq \widetilde{CH}^{d-p}(X, \omega_{X/S} \otimes \mathcal{L}^\vee),$$

for $d = \dim(X/S)$ (see Section 4.2.6 for details). When X and S are regular, our theory recovers Fasel-Srinivas' definition after carefully choosing the gradings.

- (2) Based on the main result of [BHP23], we prove a *generalized Bloch formula* over a regular scheme S of dimension at most 1, identifying Chow-Witt groups with the zeroth cohomology of a complex of *unramified Milnor-Witt sheaves*; see Section 4.3.5 for more details. As a corollary, we deduce the representability of both cohomological and homological Chow-Witt groups in the motivic homotopy category over S ; see Section 4.3.8. This result was previously known only when S is the spectrum of a field.

(3) In general, one always has a comparison map (see 4.1.2.a):

$$F : \widetilde{\mathrm{CH}}_p(X/S, \mathcal{L}) \rightarrow \mathrm{CH}_p(X).$$

It fits into various long exact sequences derived from the Milnor conjecture on Milnor K-theory and quadratic forms; see Section 3.4.6. When $X = S$ is excellent of dimension 1 and $p = 0$, we give a criterion in Section 4.4.6 for F to be an isomorphism. This applies to many examples: arbitrary orders in a number ring, affine regular curves over finite fields, and some other examples built out of spectra of number rings: doubled points (non-separated) or normal crossings (singular).

In fact, the third result can be strengthened. The following result gives a complete computation of the *graded Chow-Witt groups*, namely the cohomology of the Rost-Schmid complex with coefficients in Milnor-Witt K-theory K_*^{MW} , over Dedekind domains of finite type over \mathbf{Z} :

Theorem A (Theorem 4.4.20). *Let R be a Dedekind domain of finite type over \mathbf{Z} , K its function field, and r the number of real places of K , L an invertible R -module. Put $X = \mathrm{Spec}(R)$, $\mathcal{L} = \mathrm{Spec}(\mathrm{Sym}(L^\vee))$. Then one gets:*

$$A_0(X, K_q^{MW}, \mathcal{L}) = \begin{cases} 0 & q > 0, \\ \mathrm{Cl}(R) & q = 0, \\ \mathrm{Cl}(R)/2 & q < 0, \end{cases} \quad A_1(X, K_q^{MW}, \mathcal{L}) = \begin{cases} \underline{W}(R, L) & q < 0, \\ \underline{GW}(R, L) & q = 0, \\ \underline{K}_1^{MW}(R, L) & q = 1, \\ \mathbf{Z}^r & q > 1, \end{cases}$$

Moreover, one has short exact sequences:

$$\begin{aligned} 0 \rightarrow \underline{I}(R, L) \rightarrow \underline{GW}(R, L) \rightarrow \mathbf{Z} \rightarrow 0 \\ 0 \rightarrow \underline{I}^2(R, L) \rightarrow \underline{K}_1^{MW}(R, L) \rightarrow R^\times \rightarrow 0. \end{aligned}$$

Here, $\mathrm{Cl}(R)$ (resp. $\underline{W}(R, L)$) refers to the class group of R (resp. the unramified symmetric Witt group of R). The underlined group refers to unramified invariants, and we refer the reader to Proposition 4.4.17 for the explicit computation of $\underline{I}(R, L)$, $\underline{I}^2(R, L)$, which can be expressed in terms of the units of R , and the unramified Milnor K -group of R in degree 2, two meaningful arithmetic invariants.

In fact, one also obtains a complete computation of the unramified (symmetric) Witt group $\underline{W}(R, L)$, under the above assumptions. Besides, we show that *purity* for the (symmetric) Witt group holds for R , namely the canonical map

$$W(R, L) \rightarrow \underline{W}(R, L)$$

from the L -twisted symmetric Witt theory to unramified Witt theory, is an isomorphism. This result gives a computation of $W(R, L)$ regardless of characteristics, and was previously known if 2 is invertible in R (see [BW02]). See Proposition 4.4.17 and Theorem 4.4.19 for details.

As an illuminating corollary of Theorem A, we deduce the following finiteness result which we further link with the Bass-Tate conjecture (see Section 4.5).

Theorem B (Section 4.5.12). *Let M be one of the MW-modules in $\{K_*^M, K_*^W, W, K_*^{MW}, GW, K_*\}$ over \mathbf{Z} . Then for any scheme X of finite type over \mathbf{Z} of dimension ≤ 1 , for all pair of integers p, q , and every line bundle \mathcal{L} over X , the group $A_p(X, \mathcal{M}_q, \mathcal{L})$ is a finitely generated abelian group.*

In other words, the homology groups of the Rost-Schmid complexes of a finite type \mathbf{Z} -scheme of dimension at most 1 with coefficients in all those MW-modules are finitely generated abelian groups. We may hope that Theorem B, together with the whole theory developed in this paper, notably the spectral sequence 2.1.2, will shed light on deep finiteness conjectures such as the Bass conjecture (see Conjecture 4.5.5).

Although our constructions are purely algebraic, they can be viewed as a first layer of cohomological theories represented in the motivic homotopy theory. Indeed, they provide the cycle-theoretic framework of a broader program — outlined in [DFJ22] — aimed at describing the heart of the Ayoub-Bondarko-Dégliše perverse homotopy t -structure in terms of Milnor–Witt homomodules. The present work should therefore be viewed as a self-contained but foundational step toward that long-term objective.

Outline of the paper. Section 1 develops the category of homological MW-modules and the associated Rost–Schmid complex together with the four basic maps and their compatibilities. Section 2 proves the homotopy invariance and constructs the Gysin morphisms, excess intersection and base-change formulas, as well as the SL-orientation. Section 3 introduces pinnings, defines cohomological MW-modules and establishes the duality equivalence. Section 4 contains the applications: Chow–Witt groups over arbitrary bases, unramified sheaves and Bloch formulas, explicit arithmetic computations, and finiteness theorems.

Future work. As already mentioned, a more general motivic interpretation will be developed in subsequent papers, towards the conjecture of Ayoub. In a forthcoming one ([FJ25]), the second and third-named authors will define and compute *quadratic lengths*, and construct pullback maps on Rost-Schmid complexes associated to flat Gorenstein morphisms. Moreover, the formalism of the present paper is used in [Fel22] to study birational invariants.

NOTATION AND CONVENTIONS

0.0.1. ALGEBRAIC GEOMETRY CONVENTIONS. We will work exclusively with excellent schemes in the sense of [GD67, §7.8.5].¹

We use the following conventions: a morphism $f : X \rightarrow S$ of schemes is:

- *essentially of finite type* if f is the projective limit of a cofiltered system $(f_i)_{i \in I}$ of morphisms of finite type with affine and étale transition maps;
- *smoothable* if it admits a factorization $X \xrightarrow{i} P \xrightarrow{p} S$ such that i is a closed immersion and p is smooth;

¹This is used to ensure that DVRs have no defect in the sense of [Dég25, Prop. 5.3.12]. We expect to remove this assumption in a subsequent paper.

- *slci* if it is smoothable and a local complete intersection (*i.e.* it admits a global factorization $f = p \circ i$, p smooth and i a regular closed immersion);
- *essentially slci* if it is a limit of slci morphisms with étale transition maps.

Given a morphism of schemes $f : Y \rightarrow X$, we let L_f be its cotangent complex, an object of $D_{\text{coh}}^b(Y)$, and when the latter is perfect (e.g. if f is essentially slci), we let ω_f be its determinant. When S is clear from the context, we simply write $\omega_X = \omega_f$.

In this paper, all schemes are assumed to be excellent noetherian and finite dimensional, and equipped with a dimension function δ . More precisely, for a scheme S , the expression “base scheme” will mean an excellent noetherian and finite dimensional scheme equipped with a fixed dimension function δ (sometimes made explicit in the text). Then an S -scheme will always be assumed essentially of finite type over S , and equipped with the dimension function induced by δ .

Our conventions on points, traits and valuations are spelled out in 0.0.2.

0.0.2. POINTS AND TRAITS. Let S be a base scheme. A *point* x of S is a map

$$x : \text{Spec}(E) \longrightarrow S$$

essentially of finite type with E a field; we also say that E is an S -field. A *trait* t of S is a map

$$t : \text{Spec}(\mathcal{O}) \longrightarrow S$$

essentially of finite type with \mathcal{O} a discrete valuation ring; we also say that \mathcal{O} is an S -DVR. A *singular trait* of S is a map $\text{Spec}(A) \rightarrow S$ essentially of finite type with A a local ring of dimension 1. We write $x \in X$ to denote a point of the underlying topological space of a scheme X .

Given an S -field E , an S -valuation on E is a valuation v on E whose valuation ring \mathcal{O}_v defines a trait

$$\text{Spec}(\mathcal{O}_v) \rightarrow S.$$

We denote by $\kappa(v)$ the residue field, by \mathfrak{m}_v the valuation ideal and by

$$\nu_v = \mathfrak{m}_v / \mathfrak{m}_v^2$$

its cotangent space at the closed point.

We let \mathcal{F}_S (resp. \mathcal{G}_S) be the full subcategory of the category of S -schemes essentially of finite type whose objects consist of S -fields (resp. S -fields and S -DVRs).

0.0.3. DIMENSION FUNCTIONS. A *dimensional scheme* is a pair (S, δ) with δ a dimension function. We will use the notation of [BD17, 1.1] (see there for a reminder on dimension functions). In particular, one denotes by $\delta(S) = \delta_+(S)$ (resp. $\delta_-(S)$) the maximum (resp. minimum) of the function δ . In a situation where a base scheme (S, δ) is given and X/S is a scheme essentially of finite type, we will always assume, unless explicitly stated otherwise, that the dimension function on X is induced by that of S (see [BD17, 1.1.7]). Moreover, we simply put $X_{(p)} = \{x \in X \mid \delta(x) = p\}$.

If, in addition, X/S is essentially of finite type, one defines the δ -codimension function of X/S as $\text{co } \delta_X = d_{X/S} - \delta_X$, where $d_{X/S} = \dim(f)$ is the relative dimension function ([BD17, 1.1.9]). Then we put: $X^{(p)} = \{x \in X \mid \text{co } \delta_X(x) = p\}$.

0.0.4. LINEAR ALGEBRA CONVENTIONS. We will fix a coefficient ring R , commutative with unit. We use the notation of [Mor12, Chap. 3] for the *Milnor-Witt K -theory ring* $\mathcal{K}_*^{MW}(E)$ of a field E and its elements (see also [Fel20, Dég25]). We also denote by $\text{GW}(E)$, $\text{W}(E)$ the Grothendieck-Witt ring and Witt ring of the field E , built out of **symmetric bilinear forms** rather than quadratic forms.

Further conventions:

- M MW-modules (cohomological Milnor-Witt cycle modules)
- \mathcal{M} MW-homomodules (homological Milnor-Witt cycle modules)
- \mathcal{L} or simply $*$ line bundle
- v, w virtual vector bundles
- f, g scheme morphisms; ϕ, ψ ring morphisms
- λ, λ' pinning (Section 3.1.3)

Acknowledgments. The authors warmly thank Håkon Kolderup for a crucial improvement of the proof of Section 4.4.6 (formerly Corollary 4.4.6), allowing us to pass from totally imaginary number fields to arbitrary ones.

They also thank Adrien Dubouloz, Jean Fasel, Marc Levine, Paul Arne Østvær and Heng Xie for conversations, exchanges and ideas that led to the present paper.

The work of the first two named authors are supported by the ANR HQDIAG project no ANR-21-CE40-0015. The second named author is supported by the ANR DAG-Arts. The third-named author is supported by the National Key Research and Development Program of China Grant Nr.2021YFA1001400, the National Natural Science Foundation of China Grant Nr.12471014, Nr. 12101455 and the Fundamental Research Funds for the Central Universities.

1. HOMOLOGICAL MILNOR-WITT CYCLE MODULES

1.1. Homological Milnor-Witt cycle premodules. The following definition is a homological version and generalization of [Fel20, Definition 3.1].

Definition 1.1.1. A *homological Milnor-Witt cycle premodule* \mathcal{M} , or simply *MW-prehomomodule*, over the base scheme S and with coefficients in the ring R is an R -module $\mathcal{M}_n(E)$ for any field E over S and any integer n along with the following data (D1'), ..., (D4') and the following rules (R1a'), ..., (R3e').

(D2'): Let $\phi : E \rightarrow F$ be a finite field extension over S and n an integer. There exists a morphism $\phi^* : \mathcal{M}_n(F) \rightarrow \mathcal{M}_n(E)$.

(D3'): Let E be a field over S . For any element x of $\mathcal{K}_n^{MW}(E)$, there is a morphism $\gamma_x : \mathcal{M}_n(E) \rightarrow \mathcal{M}_{m+n}(E)$ such that $\mathcal{M}_*(E)$ is a left \mathbf{Z} -graded module over the graded ring $\mathcal{K}_*^{MW}(E)$.

(D4'): Let \mathcal{O} be a 1-dimensional local domain essentially of finite type over S with fraction field F and residue field κ . There exists a *residue* map, morphism of abelian groups

$$(1.1.1.a) \quad \partial_{\mathcal{O}} : \mathcal{M}_n(F) \rightarrow \mathcal{M}_{n-1}(\kappa).$$

Let E be a field over S , n an integer and \mathcal{L} a 1-dimensional vector space over E , denote by

$$(1.1.1.b) \quad \mathcal{M}_n(E, \mathcal{L}) := \mathcal{M}_n(E) \otimes_{R[E^\times]} R[\mathcal{L}^\times]$$

where $R[\mathcal{L}^\times]$ is the free R -module generated by the nonzero elements of \mathcal{L} and where the action of $R[E^\times]$ on $\mathcal{M}_n(E)$ is given by $u \mapsto \langle u \rangle$ thanks to (D3').

(D1'): Let $\phi : E \rightarrow F$ be a field extension and n an integer. There exists a morphism of abelian groups $\phi_! : \mathcal{M}_n(E) \rightarrow \mathcal{M}_n(F, \omega_{F/E})$.

(R1a'): Let ϕ and ψ be two composable morphisms of fields. One has $(\psi \circ \phi)_! = \psi_! \circ \phi_!$, up to the obvious identification for twists.

(R1b'): Let ϕ and ψ be two composable finite morphisms of fields. One has: $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

(R1c'): Consider $\phi : E \rightarrow F$ and $\psi : E \rightarrow L$ with ϕ finite and ψ separable. Let A be the ring $F \otimes_E L$. For each $p \in \text{Spec } A$, let $\phi_p : L \rightarrow A/p$ and $\psi_p : F \rightarrow A/p$ be the morphisms induced by ϕ and ψ . One has

$$\psi_! \circ \phi^* = \sum_{p \in \text{Spec } A} (\phi_p)^* \circ (\psi_p)_!.$$

(R2'): Let $\phi : E \rightarrow F$ be a field extension, let x be in $K_n^{MW}(E)$ and y be in $K_{n'}^{MW}(F)$.

(R2a'): We have $\phi^* \circ \gamma_x = \gamma_{\phi^*(x)} \circ \phi^*$.

(R2b'): Suppose ϕ finite. We have $\phi_! \circ \gamma_{\phi^*(x)} = \gamma_x \circ \phi_!$.

(R2c'): Suppose ϕ finite. We have $\phi_! \circ \gamma_y \circ \phi^* = \gamma_{\phi_!(y)}$.

(R3a'): Let $\phi : E \rightarrow F$ be a field extension and w be a valuation on F which restricts to a non trivial valuation v on E . Assume that the extension $\mathcal{O}_w/\mathcal{O}_v$ of DVRs is unramified. Let $\bar{\phi} : \kappa(v) \rightarrow \kappa(w)$ be the induced morphism. Then one has: $\partial_w \circ \phi_! = \bar{\phi}_! \circ \partial_v$.

(R3b'): Let $E \rightarrow F$ be a finite extension of fields over k , let v be a valuation on E . For each extension w of v , we denote by $\phi_w : \kappa(v) \rightarrow \kappa(w)$ the morphism induced by ϕ . We have

$$\partial_v \circ \phi^* = \sum_w (\phi_w)^* \circ \partial_w.$$

(R3c'): Let $\phi : E \rightarrow F$ be a morphism of S -fields and let w be a valuation on F which restricts to the trivial valuation on E . Then $\partial_w \circ \phi_! = 0$.

(R3d'): Let ϕ and w be as in (R3c'), and let $\bar{\phi} : E \rightarrow \kappa(w)$ be the induced morphism. For any uniformizer π of v , we have: $\partial_w \circ \gamma_{[\pi]} \circ \phi_! = \bar{\phi}_!$.

(R3e'): Let E be an S -field, v be a valuation on E and u be a unit of v . Then

$$\begin{aligned} \partial_v \circ \gamma_{[u]} &= \gamma_{-[\bar{u}]} \circ \partial_v; \\ \partial_v \circ \gamma_\eta &= \gamma_\eta \circ \partial_v. \end{aligned}$$

Remark 1.1.2. Note that axiom (R3a') above is weaker than its counterpart in [Fel20, §2, Rule R3a] because we only consider the case without ramification. A stronger result with ramification will be proved later.

Remark 1.1.3. In the original definition of Milnor-Witt cycle modules over a perfect base field [Fel20, §2], there is the added rule

R4a: Let E be a field over the base, n an integer, \mathcal{L} a 1-dimensional vector space over E and let Θ be an endomorphism of \mathcal{L} . Denote by Δ the canonical map² from the group of automorphisms of \mathcal{L} to the group $K_0^{MW}(E)$. Then the map

$$\Theta^* : \mathcal{M}(E, \mathcal{L}) \rightarrow \mathcal{M}(E, \mathcal{L}).$$

induced by $(D2')$ and the map

$$\gamma_{\Delta(\Theta)} : \mathcal{M}(E, \mathcal{L}) \rightarrow \mathcal{M}(E, \mathcal{L}).$$

induced by $(D3')$ coincide.

Thanks to our convention in the definition of the twist by \mathcal{L} (which differs from [Fel20]), this rule is redundant according to our axioms (as it is now a consequence of the definition).

Remark 1.1.4. Finally, the twists defined above only depends on the class of \mathcal{L} modulo squares. Let us be more precise. First, for any line bundle \mathcal{L}' , there exists an equality: $\mathcal{M}_n(E, \mathcal{L}'^{\otimes 2}) = \mathcal{M}_n(E)$, according to Formula (1.1.1.b). One deduces that any isomorphism $\varphi : \mathcal{L} \rightarrow \mathcal{L}'^{\otimes 2}$ (also called an *orientation of \mathcal{L}*) induces a canonical isomorphism:

$$(1.1.4.a) \quad \varphi_* : \mathcal{M}_n(E, \mathcal{L}) \rightarrow \mathcal{M}_n(E, \mathcal{L}'^{\otimes 2}) = \mathcal{M}_n(E).$$

Remark 1.1.5. It is reasonable to expect the rule (R3b') to hold because the base scheme S is excellent. Otherwise, one would have to ask for the identity:

$$\partial_v \circ \phi^* = \sum_w (d_w)_\epsilon (\phi_w)^* \circ \partial_w,$$

where we used the notations of (R3b') and where d_w is the *defect* of w (see [Dég25, Prop. 5.3.12])

Remark 1.1.6. If \mathcal{M} is a MW-prehomodule and n an integer, then we can define the n -shifted MW-prehomodule associated to \mathcal{M} , denoted $\mathcal{M}\{n\}$, by the formula:

$$\mathcal{M}\{n\}_m(E) = \mathcal{M}_{m+n}(E)$$

for any integer m and any S -field E .

Example 1.1.7. The main source of MW-prehomodule comes from a twisted Borel-Moore homology theory³ H_* relative to the base S .

- (1) One can consider $H_* = K'_*$ the Quillen K' -theory or Thomason's G -theory. The associated MW-prehomodule \mathcal{K}_* is in fact oriented in the sense of Section 3.5.2, so that twists by line bundles are trivial.

²See [Fel20, §2] for more details.

³This can be defined by using the properties of bivariant theories in [DJK21, §2], and working over the fix base scheme S .

- (2) Let us assume that S admits a dualizing complex K_S . Then one can define the hermitian analog of K -theory using the fundamental work [CDH⁺23, CDH⁺25] as follows. Let $f : X \rightarrow S$ be a morphism essentially of finite type, and let $K_X := f^! K_S$ seen as a dualizing complex. Let \mathcal{L} be a line bundle over X and n be an integer. One considers the stable ∞ -category $D_{coh}^b(X)$ of derived coherent sheaves on X , and equipped it with the Poincaré structure with quadratic functor $\mathcal{Q} = \underline{\mathrm{Hom}}(-, \mathcal{L} \otimes K_X[n])$ ([CDH⁺23, Def. 1.2.8]). Then we denote by $\mathrm{GW}^{BM[n]}(X, \mathcal{L})$ the associated Grothendieck-Witt spectrum as defined in [CDH⁺25, Def. 4.2.1], and put $\mathrm{GW}_*^{BM[n]}(X, \mathcal{L}) = \pi_* \mathrm{GW}^{BM[n]}(X, \mathcal{L})$. Taking all n and \mathcal{L} together, one gets a twisted Borel-Moore homology theory. The pushforward and pullback functorialities follow from the functoriality of the construction of the associated Grothendieck-Witt spectrum, and the localization long exact sequence from [CDH⁺25, Cor. 4.2.2]. Note this theory is periodic with period 4 in n ([CDH⁺23, Cor. 3.5.16]). Therefore, we get an associated family of four MW-homomodules, $\mathcal{GW}^{[i]}$ for $0 \leq i \leq 3$, with the convention that for a point $x : \mathrm{Spec}(E) \rightarrow S$ and an integer $n \in \mathbf{Z}$, $\mathcal{GW}_{\{n\}}^{[i]}(E) = \mathrm{GW}_n^{BM, [i+n]}(x)$. Note we have used the notation $\{n\}$ to avoid the ambiguity in the indexing.
- (3) Same as above but with the Witt groups (i.e. take the bordification of the GW -hermitian K -theory GW^{BM} as in [CDH⁺25, §4.4]). The resulting MW-prehomomodule is denoted by \mathcal{W}_* . Note that by definition, this MW-prehomomodule is periodic: $\mathcal{W}_n(E) = \mathcal{W}_{n+1}(E)$.
- (4) More generally, let \mathbb{E} be a motivic spectrum over S . Then the associated Borel-Moore homology $H_* = \mathbb{E}_{\{*\}}^{BM}$ does induce a MW-prehomomodule. In fact, we will get a family indexed by integers $n \in \mathbf{Z}$ of such MW-prehomomodules, by considering $\mathbb{E}[n]$. We will generally denote by $\mathcal{H}_{n,*}$ this last MW-prehomomodule.

This example contains all the previous ones. We refer the reader to [DFJ22, §3].

1.2. Homological Milnor-Witt cycle modules.

1.2.1. Throughout this section, \mathcal{M} denotes a MW-prehomomodule over S , X a scheme over S , n an integer and $*$ a line bundle (over X). We put

$$\mathcal{M}_n(x, *) = \mathcal{M}_n(\kappa(x), *).$$

Let x, y be two points in X . We define a map

$$\partial_y^x : \mathcal{M}_n(x, *) \rightarrow \mathcal{M}_{n-1}(y, *)$$

as follows. Let $Z = \overline{\{x\}}$. If $y \notin Z^{(1)}$, then put $\partial_y^x = 0$. If $y \in Z^{(1)}$, then the local ring of Z at y gives us a map

$$(1.2.1.a) \quad \partial_y^x : \mathcal{M}_n(x, *) \rightarrow \mathcal{M}_{n-1}(y, *),$$

according to (D4').

Definition 1.2.2. A homological Milnor-Witt cycle module \mathcal{M} , or simply MW-homomodule, over S is a homological Milnor-Witt cycle premodule \mathcal{M} which satisfies the following conditions (FD') and (C').

(FD'): FINITE SUPPORT OF DIVISORS. Let X be an irreducible scheme over S with generic point ξ_X , n an integer, $*$ a line bundle over X , and ρ an element of $\mathcal{M}_n(\xi_X, *)$. Then $\partial_x(\rho) = 0$ for all but finitely many $x \in X^{(1)}$.

(C'): CLOSEDNESS. Let X be integral and local of dimension 2, n an integer, and $*$ a line bundle over X . Then

$$0 = \sum_{x \in X_{(1)}} \partial_{x_0}^x \circ \partial_x^\xi : \mathcal{M}_n(\xi_X, *) \rightarrow \mathcal{M}_{n-2}(x_0, *)$$

where ξ_X is the generic point and x_0 the closed point of X .

It is important to note that both conditions do not depend on the particular choice of the dimension function δ on X , as the difference of two dimension functions is a Zariski locally constant function on S .

1.2.3. Of course (C') makes sense only under presence of (FD') which guarantees finiteness in the sum. More generally, note that if (FD') holds, then for any scheme X , any $x \in X$ and any $\rho \in \mathcal{M}_n(x, *)$ one has $\partial_y^x(\rho) = 0$ for all but finitely many $y \in X$.

1.2.4. If X is irreducible with generic point ξ_X and (FD') holds for the reduced scheme X_{red} , we put for any integer n and any line bundle $*$:

$$d = (\partial_x^{\xi_X})_{x \in X_{(1)}} : \mathcal{M}_n(\xi_X, *) \rightarrow \bigoplus_{x \in X_{(1)}} \mathcal{M}_n(x, *).$$

Definition 1.2.5. A morphism $a : \mathcal{M} \rightarrow \mathcal{M}'$ of MW-hommodules is a natural transformation which commutes with the data (D1'), ..., (D4').

In particular, MW-hommodules over a base scheme S form a category which is denoted by MW-Homod_S . One can easily check that MW-Homod_S is in fact an abelian category.⁴

1.2.6. In the following, let F be a field over S and $\mathbf{A}_F^1 = \text{Spec } F[t]$ be the affine line over $\text{Spec } F$ with function field $F(t)$. Moreover, we fix n an integer and $*$ a line bundle over S .

Proposition 1.2.7. Let \mathcal{M} be a MW-hommodule over S . With the previous notations, the following properties hold.

(H'): HOMOTOPY PROPERTY FOR \mathbf{A}^1 . We have a short exact sequence

$$0 \rightarrow \mathcal{M}_n(F, *) \xrightarrow{\text{res}} \mathcal{M}_n(F(t), *) \xrightarrow{d} \bigoplus_{x \in (\mathbf{A}_F^1)_{(1)}} \mathcal{M}_{n-1}(\kappa(x), *) \rightarrow 0$$

where the map d is defined in 1.2.4.

(RC'): RECIPROCITY FOR CURVES. Let X be a proper curve over F with generic point ξ_X . Then

$$\mathcal{M}_n(\xi_X, *) \xrightarrow{d} \bigoplus_{x \in X_{(1)}} \mathcal{M}_{n-1}(x, *) \xrightarrow{c} \mathcal{M}_{n-1}(F, *)$$

⁴It is actually a Grothendieck abelian symmetric monoidal category. However, as in the case of a base field ([Dég11], [Fel21a]), this result is quite involved and we leave it for a later work.

is a complex, that is $c \circ d = 0$ (where $c = \sum_x \text{cores}_{\kappa(x)/F}$).

Remark 1.2.8. Axiom (FD') enables one to write down the differential d of the soon-to-be-defined complex $C_*(X, \mathcal{M}, *)$, axiom (C') guarantees that $d \circ d = 0$, property (H') yields the homotopy invariance of the Chow groups $A_*(X, \mathcal{M}, *)$ and finally (RC') is needed to establish proper pushforward.

1.3. The associated cycle complex.

1.3.1. Let \mathcal{M} and \mathcal{M}' be two MW-hommodules over S , let X and Y be two schemes and let $U \subset X$ and $V \subset Y$ be subsets. Fix an integer n and a line bundle $*$. Given a morphism

$$\alpha : \bigoplus_{x \in U} \mathcal{M}_n(x, *) \rightarrow \bigoplus_{y \in V} \mathcal{M}'_n(y, *),$$

we write $\alpha_y^x : \mathcal{M}_n(x, *) \rightarrow \mathcal{M}'_n(y, *)$ for the components of α .

1.3.2. CHANGE OF COEFFICIENTS Let $\alpha : \mathcal{M} \rightarrow \mathcal{M}'$ be a morphism between two MW-hommodules, let X be a scheme over S and $U \subset X$ a subset. We put

$$\alpha_{\#} : \bigoplus_{x \in U} \mathcal{M}_n(x, *) \rightarrow \bigoplus_{x \in U} \mathcal{M}'_n(x, *)$$

where $(\alpha_{\#})_x^x = \alpha_{\kappa(x)}$ and $(\alpha_{\#})_y^x = 0$ for $x \neq y$.

1.3.3. Let \mathcal{M} be a MW-hommodule, let X be a scheme with a line bundle denoted by $*$ and p be an integer. Recall that $X_{(p)}$ is the set of (p) -dimensional points of X . Define

$$C_p(X, \mathcal{M}, *) = \bigoplus_{x \in X_{(p)}} \mathcal{M}_n(x, *)$$

and

$$d = d_X : C_p(X, \mathcal{M}, *) \rightarrow C_{p-1}(X, \mathcal{M}, *)$$

where $d_y^x = \partial_y^x$ as in 1.2.1. This definition makes sense by axiom (FD').

More precisely, for any integer q and any line bundle \mathcal{L} , we define:

$$(1.3.3.a) \quad C_p(X, \mathcal{M}_q, \mathcal{L}) = \bigoplus_{x \in X_{(p)}} \mathcal{M}_{q+p}(\kappa(x), \mathcal{L}_x).$$

and

$$d = d_X : C_p(X, \mathcal{M}_q, \mathcal{L}) \rightarrow C_{p-1}(X, \mathcal{M}_q, \mathcal{L})$$

where $d_y^x = \partial_y^x$ as in 1.2.1. This definition makes sense by axiom (FD').

We have

$$C_p(X, \mathcal{M}, *) = \bigoplus_{q \in \mathbb{Z}} C_p(X, \mathcal{M}_q, *).$$

Proposition 1.3.4. *With the previous notations, we have $d \circ d = 0$.*

Proof. Same as in [Ros96, §3.3]. Axiom (C') is needed. □

Definition 1.3.5. The complex $(C_p(X, \mathcal{M}, *), d)_{p \in \mathbb{Z}}$ (resp. $(C_p(X, \mathcal{M}_q, *), d)_{p \in \mathbb{Z}}$) of graded abelian groups (resp. abelian groups) is called the *cycle complex on X with coefficients in \mathcal{M} (resp. \mathcal{M}_q)*.

We define the p -th *Chow-Witt group with coefficients in \mathcal{M}* as the p -th homology group of the complex $C_*(X, \mathcal{M}, *)$ (resp. $C_*(X, \mathcal{M}_q, *)$). We denote it by $A_p(X, \mathcal{M}, *)$ (resp. $A_p(X, \mathcal{M}_q, *)$).

As a dimension function is Zariski locally constant, this complex only depends on δ up to a shift over each connected components of S . If we want to be precise, we use the notation $C_p^\delta(X, \mathcal{M}, *)$, $A_p^\delta(X, \mathcal{M}, *)$. Note that these groups of homological type (see below for the functoriality) are in fact trigraded. In the notation $A_p(X, \mathcal{M}_q, \mathcal{L})$, the integer p is the simplicial degree, q is the \mathbf{G}_m -degree,⁵ and the line bundle \mathcal{L} is a Thom-twist.⁶

Example 1.3.6. Consider one of the twisted Borel-Moore homology H_* of Section 1.1.7.

Then the MW-prehomodule \hat{H}_*^δ is in fact a MW-homodule. This can be shown by identifying the complex $E_{1,c}^{*,0}$ of the δ -niveau spectral sequence associated with H_* (see [BD17] for the definition and conventions) with the complex constructed above for the MW-prehomodule \hat{H}_*^δ . We refer the reader to [ADN20, §3.2.6] for this identification.

1.3.7. COHOMOLOGICAL CONVENTIONS. For $f : X \rightarrow S$ essentially lci and \mathcal{L} an invertible \mathcal{O}_X -module, we define:

$$(1.3.7.a) \quad C^p(X, \mathcal{M}_q, \mathcal{L}) = \bigoplus_{x \in X^{(p)}} \mathcal{M}_{q-p}(\kappa(x), \omega_{X,x} \otimes \mathcal{L}_x^\vee)$$

so that if X/S is of relative dimension d

$$(1.3.7.b) \quad C^p(X, \mathcal{M}_q, \mathcal{L}) = C_{d-p}(X, \mathcal{M}_{q-d}, \omega_X \otimes \mathcal{L}^\vee).$$

In this way, we have defined a cohomological complex $C^*(X, \mathcal{M}_q, *)$, called the *cocycle complex of X with coefficients in \mathcal{M}_q* . We deduce a cohomological complex of tri-graded abelian groups simply denoted by $C^*(X, \mathcal{M}, *)$. The corresponding cohomological groups in degree p are denoted by $A^p(X, \mathcal{M}_q, *)$, $A^p(X, \mathcal{M}, *)$ respectively, and called the *cohomological Chow-Witt groups of X with coefficients in \mathcal{M}* . If we want to be precise about the dimension function δ on S , we write A_δ^* .

One deduces a (tautological) *duality isomorphism*:

$$(1.3.7.c) \quad A^*(X, \mathcal{M}_*, \mathcal{L}) \simeq A_{d-*}(X, \mathcal{M}_{*-d}, \omega_X \otimes \mathcal{L}^\vee).$$

1.4. The four basic maps. The purpose of this section is to introduce the four basic operations on cycle complexes. If we count the operation d defined in the previous subsection, this will give five “basic maps” which are analogous to those of Rost (see [Ros96, §3]); they are the basic

⁵This can also be called the weight, but this terminology can cause confusion. In motivic homotopy, the \mathbf{G}_m -degree corresponds to the twist $\mathbb{1}(1)[1]$ and the weight is usually considered to be the twists $\mathbb{1}(1)$.

⁶In motivic terms, it is more precisely a twist by the Thom space of the virtual vector bundle $\mathcal{L} - \mathcal{O}_X$ of virtual rank 0.

foundations for the construction of more refined maps such as Gysin morphisms (see Section 2.2).

In the following, we fix \mathcal{M} a MW-homodule, q an integer and $*$ a line bundle over an S -scheme X .

1.4.1. PUSHFORWARD Let $f : Y \rightarrow X$ be a S -morphism of schemes. Define

$$f_* : C_p(Y, \mathcal{M}_q, *) \rightarrow C_p(X, \mathcal{M}_q, *)$$

as follows. If $x = f(y)$ and if $\kappa(y)$ is finite over $\kappa(x)$, then $(f_*)_x^y = \text{cores}_{\kappa(y)/\kappa(x)}$. Otherwise, $(f_*)_x^y = 0$.

With cohomological conventions, if f has a relative dimension s , we have

$$f_* : C^p(Y, \mathcal{M}_q, * \otimes \omega_f) \rightarrow C^{p+s}(X, \mathcal{M}_{q-s}, *).$$

1.4.2. PULLBACK Let $f : Y \rightarrow X$ be an *essentially smooth* morphism of schemes. Assume Y connected and denote by d the relative dimension of f . Define

$$f^! : C_p(X, \mathcal{M}_q, *) \rightarrow C_{p+d}(Y, \mathcal{M}_{q-d}, * \otimes \omega_f)$$

as follows. If $f(y) = x$, then $(f^!)_y^x = \text{res}_{\kappa(y)/\kappa(x)}$. Otherwise, $(f^!)_y^x = 0$. If Y is not connected, take the sum over each connected component.

With cohomological convention, we obtain:

$$f^! : C^p(X, \mathcal{M}_q, *) \rightarrow C^p(Y, \mathcal{M}_q, *).$$

Remark 1.4.3. The fact that the morphism f is (essentially) smooth implies that there are no multiplicities to consider. The case where the morphism f is flat will be handled later.

1.4.4. MULTIPLICATION WITH UNITS Let a_1, \dots, a_n be global units in \mathcal{O}_X^* . Define

$$[a_1, \dots, a_n] : C_p(X, \mathcal{M}_q, *) \rightarrow C_p(X, \mathcal{M}_{q+n}, *)$$

as follows. Let x be in $X_{(p)}$ and $\rho \in \mathcal{M}(\kappa(x), *)$. We consider $[a_1(x), \dots, a_n(x)]$ as an element of $K^{MW}(\kappa(x), *)$. If $x = y$, then put $[a_1, \dots, a_n]_y^x(\rho) = [a_1(x), \dots, a_n(x)] \cdot \rho$. Otherwise, put $[a_1, \dots, a_n]_y^x(\rho) = 0$.

1.4.5. MULTIPLICATION WITH η Define

$$\eta : C_p(X, \mathcal{M}_q, *) \rightarrow C_p(X, \mathcal{M}_{q-1}, *)$$

as follows. If $x = y$, then $\eta_y^x(\rho) = \gamma_\eta(\rho)$. Otherwise, $\eta_y^x(\rho) = 0$.

1.4.6. BOUNDARY MAPS Let X be a scheme, let $i : Z \rightarrow X$ be a closed immersion and let $j : U = X \setminus Z \rightarrow X$ be the inclusion of the open complement. We will refer to (Z, i, X, j, U) as a boundary triple and define

$$\partial = \partial_Z^U : C_p(U, \mathcal{M}_q, *) \rightarrow C_{p-1}(Z, \mathcal{M}_q, *)$$

by taking ∂_y^x to be as the definition in 1.2.1 with respect to X . The map ∂_Z^U is called the boundary map associated to the boundary triple, or just the boundary map for the closed immersion $i : Z \rightarrow X$.

1.4.7. GENERALIZED CORRESPONDENCES We will use the notation

$$\alpha : [X, *] \bullet \rightarrow [Y, *]$$

or simply

$$\alpha : X \bullet \rightarrow Y$$

to denote maps of complexes which are sums of composites of the five basic maps f_* , $g^!$, $[a]$, η , ∂ for schemes over k . Unlike Rost in [Ros96, §3], we look at these morphisms up to quasi-isomorphisms so that a morphism $\alpha : X \bullet \rightarrow Y$ may be a weak inverse of a well-defined morphism of complexes.

1.5. Compatibilities. In this section we establish the basic compatibilities for the maps considered in the last section. Fix \mathcal{M} a MW-homodule.

Proposition 1.5.1. (1) *Let $f : X \rightarrow Y$ and $f' : Y \rightarrow Z$ be two morphisms of schemes. Then*

$$(f' \circ f)_* = f'_* \circ f_*.$$

(2) *Let $g : Y \rightarrow X$ and $g' : Z \rightarrow Y$ be two essentially smooth morphisms. Then:*

$$(g \circ g')^! = g^! \circ g'^!.$$

(3) *Consider a pullback diagram of schemes*

$$\begin{array}{ccc} U & \xrightarrow{g'} & Z \\ f' \downarrow & & \downarrow f \\ Y & \xrightarrow{g} & X \end{array}$$

with g and g' essentially smooth. Then

$$g^! \circ f_* = f'_* \circ g'^!.$$

Proof. (1) This is clear from the definition and by (R1b').

(2) The claim is trivial by (R1a') (again, there are no multiplicities).

(3) This reduces to the rule (R1c') (see [Ros96, Proposition 4.1]). □

Proposition 1.5.2. *Let $f : Y \rightarrow X$ be a morphism of schemes. If a is a unit on X , then*

$$f_* \circ [\tilde{f}^!(a)] = [a] \circ f_*$$

where $\tilde{f}^! : \mathcal{O}_X^ \rightarrow \mathcal{O}_Y^*$ is induced by f .*

Proof. This comes from (R2b'). □

Proposition 1.5.3. *Let a be a unit on a scheme X .*

(1) *Let $g : Y \rightarrow X$ be an essentially smooth morphism. One has*

$$g^! \circ [a] = [\tilde{g}^!(a)] \circ g^!.$$

(2) *Let (Z, i, X, j, U) be a boundary triple. One has*

$$\partial_Z^U \circ [\tilde{j}^!(a)] = -[\tilde{i}^!(a)] \circ \partial_Z^U.$$

Moreover,

$$\partial_Z^U \circ \eta = \eta \circ \partial_Z^U.$$

Proof. The first result comes from (R2a'), the second from (R2b') and (R3e'). \square

Proposition 1.5.4. *Let $h : X \rightarrow X'$ be a morphism of schemes. Let $Z' \hookrightarrow X'$ be a closed immersion. Consider the induced diagram given by $U' = X' \setminus Z'$ and pullback:*

$$\begin{array}{ccccc} Z & \hookrightarrow & X & \longleftarrow & U \\ \downarrow f & & \downarrow h & & \downarrow g \\ Z' & \hookrightarrow & X' & \longleftarrow & U' \end{array}$$

(1) *If h is proper, then*

$$f_* \circ \partial_Z^U = \partial_{Z'}^{U'} \circ g_*.$$

(2) *If h is essentially smooth, then*

$$f^! \circ \partial_Z^U = \partial_{Z'}^{U'} \circ g^!.$$

Proof. This will follow from Proposition 1.5.6. \square

Lemma 1.5.5. *Let $g : Y \rightarrow X$ be a smooth morphism of schemes of finite type over a field of constant fiber dimension 1, let $\sigma : X \rightarrow Y$ be a section of g and let $t \in \mathcal{O}_Y$ be a global parameter defining the subscheme $\sigma(X)$. Moreover, let $\tilde{g} : U \rightarrow X$ be the restriction of g where $U = Y \setminus \sigma(X)$ and let ∂ be the boundary map associated to σ . Then*

$$\partial \circ [t] \circ \tilde{g}^! = (\text{Id}_X)_* \text{ and } \partial \circ \tilde{g}^! = 0,$$

Proof. Same as [Fel20, Proposition 6.5]. \square

Proposition 1.5.6. (1) *Let $f : X \rightarrow Y$ be a proper morphism of schemes. Then*

$$d_Y \circ f_* = f_* \circ d_X.$$

(2) *Let $g : Y \rightarrow X$ be an essentially smooth morphism. Then*

$$g^! \circ d_X = d_Y \circ g^!.$$

(3) *Let a be a unit on X . Then*

$$d_X \circ [a] = -[a] \circ d_X.$$

Moreover,

$$d_X \circ \eta = \eta \circ d_X.$$

(4) *Let (Z, i, X, j, U) be a boundary triple. Then*

$$d_Z \circ \partial_Z^U = -\partial_Z^U \circ d_U.$$

Proof. Same as [Ros96, Proposition 4.6]. The first assertion comes from Section 1.5.1.1 and (R3b'), the second from (R3c'), Section 1.5.1.3, Section 1.5.6.1 and (R3a') (note that the proof is actually much easier in our case since there are no multiplicities to consider). The third assertion follows from the definitions and Proposition 1.5.3.2, the fourth from the fact that $d \circ d = 0$. \square

1.5.7. According to the previous section (see Proposition 1.5.6), the morphisms f_* for f proper, g^* for g essentially smooth, multiplication by $[a_1, \dots, a_n]$ or η , ∂_Y^U (anti)commute with the differentials, and thus define maps on the homology groups.

1.5.8. Let (Z, i, X, j, U) be a boundary triple. We can split the complex $C_*(X, \mathcal{M}_q, *)$ as

$$C_*(X, \mathcal{M}_q, *) = C_*(Z, \mathcal{M}_q, *) \oplus C_*(U, \mathcal{M}_q, *)$$

so that one get the associated *localization long exact sequence*:

$$(1.5.8.a) \quad \dots \xrightarrow{\partial} A_p(Z, \mathcal{M}_q, *) \xrightarrow{i_*} A_p(X, \mathcal{M}_q, *) \xrightarrow{j^!} A_p(U, \mathcal{M}_q, *) \xrightarrow{\partial} A_{p-1}(Z, \mathcal{M}_q, *) \xrightarrow{i_*} \dots$$

One deduces that Chow-Witt groups with coefficients satisfy the following form of cdh-descent:

Proposition 1.5.9. *Consider, as above, an MW-homodule \mathcal{M} over S . Then any cdh-distinguished square⁷*

$$\begin{array}{ccc} T & \xrightarrow{k} & Y \\ q \downarrow & \Delta & \downarrow p \\ Z & \xrightarrow{i} & X \end{array}$$

is sent to a homotopy cartesian square by the covariant functor $C_(-, \mathcal{M}, *)$. In particular, one gets a long exact sequence:*

$$\dots \xrightarrow{\partial} A_p(T, \mathcal{M}_q, *) \xrightarrow{q_* - k_*} A_p(Z, \mathcal{M}_q, *) \oplus A_p(Y, \mathcal{M}_q, *) \xrightarrow{i_* + p_*} A_p(X, \mathcal{M}_q, *) \xrightarrow{\partial} \dots$$

Further, Gillet introduced the notion of homological descent for *hyperenvelopes*. We will refer to [Gei14] for this notion. As a corollary of the above proposition and of Theorem 1 of *op. cit.*, one deduces the following result.

Corollary 1.5.10. *The Rost-Schmid complex $C_*(-, \mathcal{M}, *)$ with coefficients in \mathcal{M} satisfies homological descent for hyperenvelopes of S -schemes essentially of finite type.*

2. REFINED FUNCTORIALITY, DUALITY AND ORIENTATION

Here we prove further properties of Chow groups with coefficients in a fixed MW-homodule \mathcal{M} over a base scheme S . We use the convention of Section 1 for schemes.

2.1. Homotopy invariance. As in [Fel20, Section 9], we define a coniveau spectral sequence that will help us reduce the homotopy invariance property to the known case (H').

We start with a lemma, establishing a property usually called *continuity*.

Lemma 2.1.1. *Let $(X_i)_{i \in I}$ be a pro-object of S -schemes with essentially étale transition maps and which admits a projective limit X in the category of S -schemes (essentially of finite type).*

Then the canonical map of bigraded abelian groups

$$A_p(X, \mathcal{M}, *) \rightarrow \varprojlim_{i \in I} A_p(X_i, \mathcal{M}, *)$$

⁷Recall: Δ is cartesian, i is a closed immersion, p is proper and an isomorphism over $X - Z$.

is an isomorphism.

In fact, the same property is true for the cycle complex with coefficients in \mathcal{M} and we can use the fact that filtered colimits are exact.

2.1.2. A RELATIVE NIVEAU SPECTRAL SEQUENCE. Let X be an S -scheme and $\pi : V \rightarrow X$ an essentially smooth morphism of schemes.

According to our conventions, we let δ be the dimension function on X . Recall from [BD17, Def. 3.1.1] that a δ -flag on X is a sequence $(Z_p)_{p \in \mathbb{Z}}$ of reduced closed subschemes of X such that, for all p , we have $Z_p \subset Z_{p+1}$ and $\delta(Z_p) \leq p$; there is a unique morphism of δ -flags $(Z_p) \rightarrow (Z'_p)$ whenever we have termwise inclusions $Z_p \subset Z'_p$. The category of δ -flags $\text{Flag}_\delta(X)$ on X is non-empty and filtered.

Let $\mathfrak{Z} = (Z_p)_{p \in \mathbb{Z}}$ be a δ -flag of X . Let $\pi^* Z^p = V \times_X Z^p$. For $p, q \in \mathbb{Z}$, define

$$\begin{aligned} D_{p,q}^{\mathfrak{Z}} &= A_{p+q}(\pi^* Z_p, \mathcal{M}, *) \\ E_{p,q}^{1,\mathfrak{Z}} &= A_{p+q}(\pi^* Z_p - \pi^* Z_{p-1}, \mathcal{M}, *) \end{aligned}$$

which are functorial on $\mathfrak{Z} \in \text{Flag}_\delta(X)$. By Proposition 1.5.4, we have a functorial long exact sequence

$$\cdots \longrightarrow D_{p-1,q+1}^{\mathfrak{Z}} \xrightarrow{i_{p,*}} D_{p,q}^{\mathfrak{Z}} \xrightarrow{j_p^!} E_{p,q}^{1,\mathfrak{Z}} \xrightarrow{\partial_p} D_{p-1,q}^{\mathfrak{Z}} \longrightarrow \cdots$$

so that $(D_{p,q}^{\mathfrak{Z}}, E_{p,q}^{1,\mathfrak{Z}})_{p,q \in \mathbb{Z}}$ is an exact couple where $j_p^!$ and $i_{p,*}$ are induced by the canonical immersions. Since the dimension of X is bounded, this exact couple defines a convergent homological spectral sequence

$$(2.1.2.a) \quad E_{p,q}^1 = A_{p+q}(\pi^* Z_p - \pi^* Z_{p-1}, \mathcal{M}, *) \Rightarrow A_{p+q}(\pi^* Z_\infty, \mathcal{M}, *)$$

where $Z_\infty = \bigcup_{p \in \mathbb{Z}} Z_p$ (see for example [McC01, Chapter 3]).

For $p, q \in \mathbb{Z}$, denote by

$$\begin{aligned} D_{p,q}^{1,\pi} &= \text{colim}_{\mathfrak{Z} \in \text{Flag}_\delta(X)} D_{p,q}^{\mathfrak{Z}}, \\ E_{p,q}^{1,\pi} &= \text{colim}_{\mathfrak{Z} \in \text{Flag}_\delta(X)} E_{p,q}^{1,\mathfrak{Z}}. \end{aligned}$$

Since filtered colimits are exact in the category of abelian groups, taking colimit over $\mathfrak{Z} \in \text{Flag}_\delta(X)$ of the spectral sequence (2.1.2.a) yields a convergent spectral sequence

$$(2.1.2.b) \quad E_{p,q}^{1,\pi} = \text{colim}_{\mathfrak{Z} \in \text{Flag}_\delta(X)} A_{p+q}(\pi^* Z_p - \pi^* Z_{p-1}, \mathcal{M}, *) \Rightarrow A_{p+q}(V, \mathcal{M}, *).$$

We need to compute the spectral sequence (2.1.2.b). This is done in the following theorem.

Proposition 2.1.3. *For $p, q \in \mathbb{Z}$, we have a canonical isomorphism*

$$E_{p,q}^{1,\pi} \simeq \bigoplus_{x \in X_{(p)}} A_q(V_x, \mathcal{M}, *).$$

Proof. Denote by \mathcal{I}_p the set of pairs (Z, Z') where Z is a reduced closed subscheme of X of δ -dimension p and $Z' \subset Z$ is a closed subset containing the singular locus of Z . Notice that any such pair (Z, Z') can be (functorially) extended into a δ -flag of X . Moreover, for any x in X , consider $\overline{\{x\}}$ the reduced closure of x in X and $\mathfrak{F}(x)$ be the set of closed subschemes Z' of

$\overline{\{x\}}$ containing its singular locus. Then we get isomorphisms, where the last one follows from Section 2.1.1, and this concludes the proof:

$$\begin{aligned}
 E_{p,q}^{1,\pi} &\simeq \lim_{\mathfrak{z} \in \text{Flag}_\delta(X)} A_q(V \times_X (Z_p - Z_{p-1}), \mathcal{M}, *) \\
 &\simeq \lim_{(Z, Z') \in \mathcal{I}_p} A_q(V \times_X (Z - Z'), \mathcal{M}, *) \\
 &\simeq \bigoplus_{x \in X_{(p)}} \lim_{Z' \in \mathfrak{F}(x)} A_q(V \times_X (\overline{\{x\}} - Z'), \mathcal{M}, *) \\
 &\simeq \bigoplus_{x \in X_{(p)}} A_q(V_x, \mathcal{M}, *).
 \end{aligned}$$

□

Theorem 2.1.4 (Homotopy Invariance). *Let X be a scheme, V a vector bundle of rank n over X , $\pi : V \rightarrow X$ the canonical projection. Then, for every $q \in \mathbf{Z}$, the canonical morphism*

$$\pi^! : A_q(X, \mathcal{M}, *) \rightarrow A_{q+n}(V, \mathcal{M}, *)$$

is an isomorphism.

Proof. From a noetherian induction and the localization sequence (1.5.8.a), we can reduce to the case where $V = \mathbf{A}_X^n$ is the affine trivial vector bundle of rank n . Moreover, we may assume $n = 1$ by induction. The spectral sequence (2.1.2.b) gives the spectral sequence

$$E_{p,q}^{1,\pi} \Rightarrow A_{p+q}^\delta(V, \mathcal{M}, *)$$

where $E_{p,q}^{1,\pi}$ is (abusively) defined as previously, but twisted accordingly. By Theorem 2.1.3, the page $E_{p,q}^{1,\pi}$ is isomorphic to $\bigoplus_{x \in X_{(p)}} A_q(V_x, \mathcal{M}, *)$. According to the property (H'), this last expression is isomorphic (via the map π) to $\bigoplus_{x \in X_{(p)}} A_q(\text{Spec } \kappa(x), \mathcal{M}, *)$. Using again 2.1.3, this group is isomorphic to $E_{p,q}^{1, \text{Id}_X}$, which converge to $A_{p+q}(X, \mathcal{M}, *)$. By Proposition 1.5.4, the map π induces a morphism of exact couples

$$(D_{p,q}^{1, \text{Id}_X}, E_{p,q}^{1, \text{Id}_X}) \rightarrow (D_{p,q+n}^{1,\pi}, E_{p,q+n}^{1,\pi})$$

hence we have compatible isomorphisms on the pages which induce the pullback

$$\pi^! : A_q(X, \mathcal{M}, *) \rightarrow A_{q+n}(V, \mathcal{M}, *).$$

□

2.2. Gysin morphisms for slci maps.

2.2.1. The classical approach of Fulton-Verdier ([Ful98, §6]) defines Gysin morphisms at the level of Chow groups associated with slci morphisms, which is a cornerstone of the intersection theory in Chow groups; such a construction has been extended to Chow groups with coefficients in cycle modules ([Ros96, §11]), and more recently to twisted Borel-Moore theories that are representable in the motivic stable homotopy category ([DJK21]).⁸ We use the same strategy here

⁸In a forthcoming work, we will extend Gysin morphisms to the level of cycles via ∞ -categorical techniques.

to define Gysin morphisms for Chow-Witt groups with coefficients in MW-hommodules, while carefully taking into account the twists as in [Fel20, §10].

2.2.2. For the remainder on the section, we fix \mathcal{M} a MW-hommodule. Let $i : Z \rightarrow X$ be a closed immersion of (possibly singular) S -schemes. Let t be a parameter of \mathbf{A}^1 and let

$$q : X \times_S (\mathbf{A}^1 \setminus \{0\}) \rightarrow X$$

be the canonical projection. Denote by $D = D_Z X$ the deformation space such that

$$D = U \sqcup N_Z X$$

where $U = X \times_S (\mathbf{A}^1 \setminus \{0\})$ (see [Ros96, §10] for more details).

Consider the morphism

$$J(X, Z) = J_{Z/X} : A^*(X, \mathcal{M}_q, *) \rightarrow A^*(N_Z X, \mathcal{M}_q, *)$$

defined by the composition:

$$\begin{array}{ccc} A_p(X, \mathcal{M}_q, *) & \xrightarrow{q^!} & A_{p+1}(U, \mathcal{M}_{q-1}, *) \\ J_{Z/X}^! \downarrow & & \downarrow [t] \\ A_p(N_Z X, \mathcal{M}_q, *) & \xleftarrow{\partial} & A_{p+1}(U, \mathcal{M}_q, *) \end{array}$$

where ∂ is the boundary map of 1.4.6, associated with the closed immersion $N_Z X \rightarrow D_Z X$. This defines a morphism (also denoted by $J(X, Z)$ or $J_{Z/X}$) by passing to homology.

Assume moreover that $i : Z \rightarrow X$ is regular of codimension n , the map $\pi : N_Z X \rightarrow Z$ is a vector bundle over X of dimension n . By homotopy invariance (Theorem 2.1.4), we have an isomorphism

$$\pi^! : A_{p-n}(Z, \mathcal{M}_q, \omega_i \otimes *) \rightarrow A_p(N_Z X, \mathcal{M}_q, *)$$

where we have used the canonical isomorphism $\pi^{-1}(\omega_{Z/X}) = \omega_{N_Z X/Z}$ of line bundles. Denote by $r_{Z/X} = (\pi^!)^{-1}$ its inverse.

Definition 2.2.3. With the previous notations, we define the map

$$i^! : A_*(X, \mathcal{M}_q, *) \rightarrow A_{*-n}(Z, \mathcal{M}_q, \omega_i \otimes *)$$

by putting $i^! = r_{Z/X} \circ J_{Z/X}$ and call it the *Gysin morphism of i* .

2.2.4. Let X and Y be two S -schemes and let $f : Y \rightarrow X$ be an slci morphism. Consider a factorization $Y \xrightarrow{i} P \xrightarrow{p} X$ of f into a regular closed immersion followed by a smooth morphism. As in [Fel20, Section 10], we see that the composition $i^! p^!$ does not depend on the chosen factorization.

Definition 2.2.5. Keeping the previous notations and let d be the relative dimension of f . We define the Gysin morphism associated to f as the morphism (where we have made twists explicit)

$$f^! = i^! \circ p^! : A_*(X, \mathcal{M}, \mathcal{L}) \rightarrow A_{*+d}(Y, \mathcal{M}, \omega_f \otimes f^{-1} \mathcal{L}).$$

Proposition 2.2.6. Consider slci morphisms $Z \xrightarrow{g} Y \xrightarrow{f} X$ such that $g \circ f$ is smoothable, thus slci. Then

$$g^! \circ f^! = (f \circ g)^!.$$

Proof. As in [Fel20, Prop. 10.12]. □

2.2.7. The drawback of working with slci (recall: smoothable lci) morphisms is that they are not necessarily stable under composition. If one wants an appropriate site \mathcal{S}_S of S -schemes and such that every morphism is slci, one can consider the following ones:

- (1) the site of smoothable separate (e.g. quasi-projective in the sense of Hartshorne) S -schemes with slci morphisms;
- (2) the site of separated (or affine) smooth S -schemes.

In any of those cases, morphisms in the underlying site admit a canonical factorization into a regular closed immersion followed by a smooth morphism, using the S -graph construction.

Proposition 2.2.8 (Base change for slci morphisms). *Consider a cartesian square of S -schemes*

$$\begin{array}{ccc} Y' & \xrightarrow{g} & X' \\ q \downarrow & & \downarrow p \\ Y & \xrightarrow{f} & X \end{array}$$

with p proper and f slci. Assume that the square is transversal in the sense that the morphism g is slci and the canonical map $p^! \omega_f \rightarrow \omega_g$ is an isomorphism. We have (up to the canonical isomorphism induced by the previous isomorphism):

$$f^! \circ p_* = q_* \circ g^!$$

which is compatible with horizontal and vertical compositions of transversal squares.

Proof. As in [Fel20, Prop. 10.13]. □

Applying the above proposition to the case $p = i$ is a closed immersion, one gets:

Corollary 2.2.9. *The localization long exact sequence (1.5.8.a) associated with a closed immersion i is functorial with respect to pullbacks along slci morphism transversal to i .*

2.3. Excess intersection. Following classical lines, it is possible to refine Section 2.2.8, by introducing *Euler classes*. As we want a general formula, without relying on an underlying product structure, we introduce an operation which plays the role of the multiplication by an Euler class.

Definition 2.3.1. Let $E \rightarrow X$ be a vector bundle of rank n , with zero section $s : X \rightarrow E$. One defines, for any integer q and any line bundle $*$ over X , the *Euler class multiplication map* as the following composite map:

$$\epsilon(E) := s^! s_* : A_*(X, \mathcal{M}_q, *) \rightarrow A_{*-n}(X, \mathcal{M}_{q+n}, \det(-E) \otimes *).$$

One easily deduce from the properties of Gysin morphisms obtained in the previous section the following properties, analogous to that of the Euler classes in Chow-Witt theory over a field ([Fas08]):

- (1) *Invariance.* Given an isomorphism $\phi : E \xrightarrow{\sim} F$, one gets: $(\det(\phi)^\vee)_*(\epsilon(F)) = \epsilon(E)$.

- (2) *Base change.* For $p : Y \rightarrow X$ slci, $p^! \circ \epsilon(E) = \epsilon(p^{-1}E) \circ p^!$.
- (3) *Additivity.* $\epsilon(E \oplus F) = \epsilon(E) \circ \epsilon(F)$
- (4) *Triviality.* If $E = \mathbf{A}^1 \oplus E'$, $\epsilon(E) = 0$.

One also deduces the following formula, as expected.

Proposition 2.3.2 (Excess intersection). *Consider a cartesian square as in Section 2.2.8 but we only assume that p is proper, and both f, g are slci.*

In that case, there exists an excess intersection bundle ξ over Y' , well-defined up to isomorphism, of rank $e = \dim(g) - \dim(f)$ and such that $p^! \omega_g = \det(\xi) \otimes \omega_f$. We have, up to the canonical isomorphism induced by the previous isomorphism:

$$f^! \circ p_* = q_* \circ \epsilon(\xi) \circ g^!$$

Proof. One just reduces to the case where p is a regular closed immersion, comes back to the definition of Gysin morphisms of regular closed immersions via deformation to the normal cone, use the functoriality properties of the latter as described in the proof of [Dég04, Th. 2.1], and finally use Section 2.2.9 and Section 2.2.8. \square

2.4. From homology to cohomology; supports and SL-orientation. Recall that we have associated to an MW-homodule \mathcal{M} a cocycle complex, and the associated cohomological Chow-Witt groups with coefficients (Section 1.3.7). The conventions are chosen so that for any smoothable map $f : Y \rightarrow X$ between essentially slci S -schemes, the Gysin map of Section 2.2.5 induces the following pullback map:

$$(2.4.0.a) \quad f^* : A^n(X, \mathcal{M}_q, \mathcal{L}) \rightarrow A^n(Y, \mathcal{M}_q, f^* \mathcal{L}).$$

In particular, cohomological Chow-Witt groups form a cohomology theory over any of the site introduced in Section 2.2.7. One can even consider a suitable notion of cohomology with supports, generalizing [BCD⁺20, Chap. 2, Def. 2.8]

Definition 2.4.1. Let (X, Z) be a closed S -pair such that X is an essentially slci S -schemes, Z an arbitrary closed subscheme of X . Let \mathcal{L} be a line bundle over X . One defines the *cycle complex of (X, \mathcal{L}) with support in Z and coefficients in \mathcal{M}* as:

$$C_Z^p(X, \mathcal{M}_q, \mathcal{L}) = \bigoplus_{x \in (X^{(p)} \cap Z)} \mathcal{M}_{q-p}(\kappa(x), \omega_f \otimes \mathcal{L}^\vee)$$

seen as a subcomplex of $C^*(X, \mathcal{M}_q, \mathcal{L})$. We let $A_Z^*(X, \mathcal{M}_q, \mathcal{L})$ be the associated cohomology groups, called *Chow-Witt group with support in Z* and coefficients in \mathcal{M} .

It follows from this definition that the duality isomorphism (1.3.7.b) can be extended to a version with supports:

$$(2.4.1.a) \quad A_Z^p(X, \mathcal{M}_q, \mathcal{L}) \simeq A_{d-p}(Z, \mathcal{M}_{q-d}, \omega_X|_Z \otimes \mathcal{L}^\vee|_Z)$$

By construction, we have the usual long exact sequence of cohomology with support, which corresponds by using duality to the localization long exact sequence (1.5.8.a). Finally, cohomology with support allows us to express the fact that Chow-Witt groups with coefficients are

canonically SL-oriented, in the sense of Panin and Walter. As in the previous section, we give a formulation which is valid without a product structure.

Proposition 2.4.2. *Consider the above notation. Let X be an essentially slci S -scheme, and $\pi : E \rightarrow X$ be a vector bundle of rank r over X . Then there exists a canonical Thom isomorphism:*

$$\tau_E : A^{p-r}(X, \mathcal{M}_{q-r}, \mathcal{L} \otimes \det(-E)) \simeq A_X^p(E, \mathcal{M}_q, \pi^* \mathcal{L})$$

which satisfies the properties of the Thom classes of an SL-oriented theory (see e.g. [PW18, Def. 3.1, Th. 3.2]).

Indeed, τ_E is given by the following composite of duality isomorphisms, where d is the dimension of X :

$$A_X^p(E, \mathcal{M}_q, \pi^* \mathcal{L}) \simeq A_{d+r-p}(X, \mathcal{M}_{q-d-r}, \omega_E|_X \otimes \mathcal{L}^\vee) \simeq A_{p-r}(X, \mathcal{M}_{q-r}, \mathcal{L} \otimes \det(-E)).$$

Moreover, one further deduces from Section 1.1.4 that cohomological Chow-Witt groups with coefficients are SL^c -oriented: an orientation $\varphi : \det(E) \xrightarrow{\sim} \mathcal{L}^{\otimes 2}$ of E/X induces a canonical Thom isomorphism:

$$\tau_\varphi : A_X^p(E, \mathcal{M}_q, \pi^* \mathcal{L}) \simeq A^{p-r}(X, \mathcal{M}_{q-r}, \mathcal{L} \otimes \det(-E)) \simeq A^{p-r}(X, \mathcal{M}_{q-r}, \mathcal{L}).$$

One can therefore apply all the results of SL^c and Sp -oriented theories. This includes a symplectic projective bundle theorem, an action of Borel classes, (see [PW18], formulated without ring structure as above). With further work, one can even get a projective bundle theorem as in [Fas20, Th. 4.2].

Remark 2.4.3. Over an essentially slci S -scheme X , the previous Thom isomorphisms can be used to define the Euler class multiplication map of Section 2.3.1. Indeed, with the above notations, $\mathcal{L} = \det(E)$, $s : X \rightarrow E$ the zero section, one deduces from definitions that the following composite is the Euler class multiplication map:

$$\begin{aligned} A_*(X, \mathcal{M}, *) &\stackrel{(1)}{\simeq} A^*(X, \mathcal{M}, \det(E) \otimes *^\vee) \xrightarrow{\tau_E} A_X^*(E, \mathcal{M}, *^\vee) \\ &\xrightarrow{s_*} A^*(E, \mathcal{M}, *^\vee) \stackrel{s^*}{\simeq} A^*(X, \mathcal{M}, *^\vee) \stackrel{(1)}{\simeq} A_*(X, \mathcal{M}, \det(-E) \otimes *), \end{aligned}$$

where the map s_* forgets the support, the maps (1) follow from duality (1.3.7.b).

3. COHOMOLOGICAL MW-MODULES

The notion of *Milnor-Witt cycle modules* is introduced by the second-named author in [Fel20] over a perfect field which, after slight changes, can be generalized to more general base schemes (see [BHP23] for the case of a regular base scheme). In the spirit of the general formalism of *bivariant theories* (see [Dég18]), they should be considered as *cohomological* Milnor-Witt cycle modules, that we simply call MW-modules in contrast with homological Milnor-Witt cycle modules, *i.e* MW-homodules, that were introduced in Section 1. Our main result on MW-modules will be a duality theorem relating these objects to their homological counterparts, which hold for possibly singular schemes, see Theorems 3.5.9 and 3.3.4 below.

3.1. Pinning and dualizing complexes.

Definition 3.1.1. We define the *fibred category of virtual schemes* $\mathcal{K}^!$ as the category whose objects are pairs (X, v) where X is a scheme and v a virtual vector bundle over X , and with morphisms $(Y, w) \rightarrow (X, v)$ the pairs (f, ϕ) where $f : Y \rightarrow X$ is an slci morphism of schemes and $\phi : w \rightarrow \tau_f + f^{-1}(v)$ an isomorphism of virtual vector bundles over Y .

The composition of two twisted morphisms

$$(Z, w) \xrightarrow{(g, \psi)} (Y, v) \xrightarrow{(f, \phi)} (X, u)$$

is given by the composition of morphisms of schemes $fg = f \circ g$ and by the following composite isomorphism:

$$w \xrightarrow{\psi} \tau_g + g^{-1}v \xrightarrow{Id + g^{-1}(\phi)} \tau_g + g^{-1}\tau_f + g^{-1}f^{-1}(u) \simeq \tau_{fg} + (fg)^{-1}(u)$$

where the last isomorphism is induced by the canonical isomorphism of virtual vector bundles (coming from the distinguished triangles of cotangent complexes).

Note that we will denote a morphism (f, ϕ) of one of the above two types simply by f when ϕ is clear.

Similar to the pseudo-functor \mathcal{K} , we have a pseudo-functor from the category of S -fields (see Section 0.0.2)

$$(3.1.1.a) \quad \begin{aligned} \mathcal{L} : (\mathcal{F}_S)^{op} &\rightarrow \text{Spaces} \\ x &\mapsto \mathcal{L}(x) \end{aligned}$$

where $\mathcal{L}(x)$ is the space of line bundles on x . However, instead of the usual pull-back functoriality of line bundles, we consider the exceptional functor between line bundles: if $f : \text{Spec } F \rightarrow \text{Spec } E$ is a morphism of S -fields, we define

$$(3.1.1.b) \quad \begin{aligned} f^! : \mathcal{L}(E) &\rightarrow \mathcal{L}(F) \\ v &\mapsto f^*v \otimes \omega_{F/E}. \end{aligned}$$

With this new functoriality we obtain another pseudo-functor

$$(3.1.1.c) \quad \begin{aligned} \mathcal{L}^! : (\mathcal{F}_S)^{op} &\rightarrow \text{Spaces} \\ x &\mapsto \mathcal{L}(x). \end{aligned}$$

3.1.2. A morphism in the category \mathcal{G}_S (see Section 0.0.2) is in particular a morphism between regular schemes, and is therefore a perfect morphism (see [Sta25, tag 0685]). For a morphism $f : Y \rightarrow X$ in \mathcal{G}_S , we define

$$(3.1.2.a) \quad \begin{aligned} f^! : \mathcal{L}(X) &\rightarrow \mathcal{L}(Y) \\ v &\mapsto f^*v \otimes \omega_f. \end{aligned}$$

We obtain an extension of the previous pseudo-functor

$$(3.1.2.b) \quad \begin{aligned} \mathcal{L}^! : (\mathcal{G}_S)^{op} &\rightarrow \text{Spaces} \\ X &\mapsto \mathcal{L}(X). \end{aligned}$$

Definition 3.1.3. A **pinning** (or *épinglage*) on a scheme S is a Cartesian section of $\mathcal{L}^!$. In other words, a pinning on S is a natural transformation of pseudo-functors

$$(3.1.3.a) \quad \lambda : * \rightarrow \mathcal{L}^!$$

where $*$ denotes the constant functor $(\mathcal{G}_S)^{op} \rightarrow \text{Spaces}$ sending every object in \mathcal{G}_S to the one-point space.

3.1.4. Concretely, a pinning λ is the following data:⁹

- (1) for an S -field E , a line bundle λ_E on E ;
- (2) for a morphism of S -fields $\phi : F \rightarrow E$, an isomorphism $\phi^* \lambda_E \otimes \omega_{F/E} \simeq \lambda_F$;
- (3) for an S -DVR A , a line bundle λ_A on A , together with isomorphisms $j^* \lambda_A \simeq \lambda_\eta$ and $i^* \lambda_A \otimes (\mathfrak{m}/\mathfrak{m}^2) \simeq \lambda_s$, where \mathfrak{m} is the maximal ideal of A and $i : s \rightarrow A$ and $j : \eta \rightarrow A$ are the inclusions of the closed point and the generic point;
- (4) for a morphism $\phi : A \rightarrow E$ where A is an S -DVR and E is an S -field, an isomorphism $\phi^* \lambda_E \otimes \omega_{A/E} \simeq \lambda_A$;
- (5) for an extension $\phi : B \rightarrow A$ of S -DVRs, an isomorphism $\phi^* \lambda_A \otimes \omega_{B/A} \simeq \lambda_B$.

There are natural compatibilities with the isomorphisms above, which we do not write down.

Example 3.1.5. Let \mathcal{K} be a (coherent) dualizing complex on S (see [Har66, Chapter V, Proposition 2.1]). For every object $f : X \rightarrow S$ in \mathcal{G}_S , the object $f^! \mathcal{K}$ is a dualizing complex on S , and since X is a regular scheme, there is a line bundle $K_{(X)}$ on X and an integer $\mu_K(X)$ such that

$$(3.1.5.a) \quad f^! \mathcal{K} = K_{(X)}[-\mu_K(X)].$$

Then there is a pinning λ on S such that $\lambda_X = K_{(X)}$. Note also that when X is a point of S , the map $X \mapsto -\mu_K(X)$ defines a dimension function on S (see [Har66, Chapter V, §7]).

In particular, every regular scheme (or even Gorenstein scheme) has a canonical pinning since the structure sheaf is a dualizing complex. For example, if S is regular, then the pinning on S above is such that for every S -field $\text{Spec } E$ we have

$$(3.1.5.b) \quad \lambda_E = \omega_{E/S}.$$

Remark 3.1.6. Example 3.1.5 is indeed the prototype of a pinning, which incarnates the data of “local orientations” at each point of a scheme. In Theorem 3.3.4 below we will establish a duality of Milnor-Witt cycle modules for schemes with pinnings: such an idea of relating dualizing complexes to duality already appeared in [JX25] in the special case of real étale sheaves. See also [ILO14, Exp. XVII] for a construction of similar spirit in the context of étale sheaves.

Also note that since any scheme essentially of finite type over a regular scheme (or even a Gorenstein scheme) has a dualizing complex, the existence of a pinning is a relatively weak condition and does not really impose any restriction on the singularities of the scheme.

3.1.7. One can define several operations on pinnings, reflecting the theory of dualizing complexes. Let λ be a pinning on a scheme S .

⁹We abuse notation writing A for an object $\text{Spec}(A)$ of \mathcal{G}_S .

Given a morphism $f : T \rightarrow S$ essentially of finite type, we define a pinning $f^*(\lambda)$ on T by the restriction of λ to \mathcal{G}_T . Given \mathcal{L} a line bundle on S , we define another pinning $\mathcal{L} \otimes \lambda$ on S such that for every object $f : X \rightarrow S$ in \mathcal{G}_S , $(\mathcal{L} \otimes \lambda)_X = \lambda_X \otimes f^*\mathcal{L}$. Finally, when f is essentially slci, one can define a twisted pullback by the formula: $f^!\lambda = \omega_f \otimes f^*(\lambda)$.

3.2. Cohomological Milnor-Witt cycle premodules. The following definition is a generalization of [Fel20, Definition 3.1] and [BHP23]. We use Notation Section 0.0.2. The base scheme S can be arbitrary.

Definition 3.2.1. A (cohomological) Milnor-Witt cycle premodule M over S , or simply *MW-premodule*, over the base scheme S and with coefficients in R is the data of an R -module $M_n(E)$ for any S -field E and any integer n , along with the following data (D1), ..., (D4) and the following rules (R1a), ..., (R3e).

(D1): For a morphism of S -fields $\phi : E \rightarrow F$, we have a map $\phi_* : M_n(E) \rightarrow M_n(F)$.

(D3): For an S -field E and an element $x \in \mathcal{K}_m^{MW}(E)$, a map $\gamma_x : M_n(E) \rightarrow M_{n+m}(E)$, making $M(E)$ a left \mathbf{Z} -graded module over the \mathbf{Z} -graded ring $\mathcal{K}_*^{MW}(E)$ (i.e. we have $\gamma_x \circ \gamma_y = \gamma_{x \cdot y}$ and $\gamma_1 = \text{Id}$).

Exactly as in the construction in (1.1.1.b), the axiom (D3) allows us to define, for every S -field E and every 1-dimensional E -vector space \mathcal{L} , a graded R -module

$$(3.2.1.a) \quad M_n(E, \mathcal{L}) := M_n(E) \otimes_{R[E^\times]} R[\mathcal{L}^\times].$$

(D2): For a finite morphism of S -fields $\phi : E \rightarrow F$, we have a *trace* map

$$\phi^! : M_n(F, \omega_{F/E}) \rightarrow M_n(E).$$

(D4): For an S -field E and an S -valuation v on E , we have a *residue* map:

$$\partial_v : M_n(E) \rightarrow M_{n-1}(\kappa(v), \nu_v).$$

(R1a): For any composable morphisms of fields, one has $(\varphi \circ \psi)_* = \varphi_* \circ \psi_*$.

(R1b): For any composable finite morphisms of fields, one has $(\varphi \circ \psi)^! = \psi^! \circ \varphi^!$, up to the obvious identification for twists.

(R1c): Let $\phi : E \rightarrow F$ and $\psi : E \rightarrow L$ be morphisms of fields with ϕ finite and ψ separable. Let A be the artinian ring $F \otimes_E L$. For each prime $p \in \text{Spec } A$, let $\phi_p : L \rightarrow A/p$ and $\psi_p : F \rightarrow A/p$ be the morphisms induced by ϕ and ψ . One has:

$$\psi_* \circ \phi^! = \sum_{p \in \text{Spec } A} (\phi_p)^! \circ (\psi_p)_*.$$

(R2): Let $\phi : E \rightarrow F$ be a morphism of S -fields. Let $x \in \mathcal{K}_m^{MW}(E)$ and $y \in \mathcal{K}_l^{MW}(F)$. Then:

(R2a): we have $\phi_* \circ \gamma_x = \gamma_{\phi_*(x)} \circ \phi_*$.

(R2b): If ϕ is finite, then: $\phi^! \circ \gamma_{\phi_*(x)} = \gamma_x \circ \phi^!$.

(R2c): If ϕ is finite, then: $\phi^! \circ \gamma_y \circ \phi_* = \gamma_{\phi^!(y)}$.

(R3a): Let $\phi : E \rightarrow F$ be a morphism of S -fields. Let w be an S -valuation on F which restricts to a non-trivial valuation v on E of ramification index 1, and denote by $\bar{\phi} : \kappa(v) \rightarrow \kappa(w)$ the induced morphism of S -fields. Then $\partial_w \circ \phi_* = \bar{\phi}_* \circ \partial_v$.

- (R3b):** Let $\phi : E \rightarrow F$ be a finite morphism of S -fields, and let v be an S -valuation on E . For each extension w of v , denote by $\phi_w : \kappa(v) \rightarrow \kappa(w)$ the induced morphism of S -fields. Then: $\partial_v \circ \phi^! = \sum_w (\phi_w)^! \circ \partial_w$.
- (R3c):** Let $\phi : E \rightarrow F$ be a morphism of S -fields, and let w be an S -valuation on F which restricts to the trivial valuation on E . Then: $\partial_w \circ \phi_* = 0$.
- (R3d):** Let ϕ and w be as in (R3c) and denote by $\bar{\phi} : E \rightarrow \kappa(w)$ be the induced morphism of S -fields. Then for any uniformizer π of w , we have: $\partial_w \circ \gamma_{[\pi]} \circ \phi_* = \bar{\phi}_*$.
- (R3e):** Let $\text{Spec } E$ be an S -field, v be an S -valuation on E and u be a unit of v . Then

$$\begin{aligned}\partial_v \circ \gamma_{[u]} &= \gamma_{-[\bar{u}]} \circ \partial_v; \\ \partial_v \circ \gamma_\eta &= \gamma_\eta \circ \partial_v.\end{aligned}$$

- Remark 3.2.2.* (1) Our axioms, while slightly different from the ones in [Fel20] and [BHP23], are indeed equivalent to the latter, see Section 3.3.5 below.¹⁰
- (2) Note that we distinguish the notion of S -fields from the one of S -points: the latter refers to points on (the underlying topological space of) S , which are particular cases of S -fields.

Example 3.2.3. According to [Fel20, Th. 4.13], \mathcal{K}_*^{MW} defines a MW-cycle module over a perfect field k . More generally, over any base scheme S , \mathcal{K}_*^{MW} does satisfy the axioms of a MW-premodule. We refer the reader to [Dég25, §5] for the construction of the functoriality and the proof of the relations in this generality.

3.3. Milnor-Witt cycle modules.

3.3.1. As in the homological case, we would like to promote *Milnor-Witt cycle premodules* into *Milnor-Witt cycle modules* (see Definition 1.2.2). The main obstruction for doing so in the cohomological setting lies in the difference between the axioms for residue maps: indeed, the axiom (D4) only requires residues for (discrete) valuation rings, while the axiom (D4') requires residues for all 1-dimensional local domains. In some sense, this difficulty is similar to the problem of defining direct images in bivariant theory (see [DJK21]). We deal with this problem in the particular case of cycle premodules by using the notion of pinning introduced in Section 3.1.3 and Notation Section 3.1.4.

Let us fix (S, λ) a base scheme (excellent according to our conventions) with a pinning and M a MW-premodule over S . We introduce the following operations on M :

- (1) If A is an S -DVR with fraction field F , maximal ideal \mathfrak{m} and residue field κ , we define the following a map using (D4)

(3.3.1.a)

$$M_n(F, \lambda_F) \simeq M_n(F) \otimes_{R[A^\times]} R[\lambda_A^\times] \xrightarrow{(D4)} M_{n-1}(\kappa, \mathfrak{m}/\mathfrak{m}^2) \otimes_{R[A^\times]} R[\lambda_A^\times] \simeq M_{n-1}(\kappa, \lambda_\kappa).$$

¹⁰Beware that in axiom (D4), [Fel20] uses the normal bundle N_v while we are using here the conormal sheaf ν_v . This explains the difference of signs in the first formula (R3e).

(2) For a finite morphism of S -fields $\phi : E \rightarrow F$, we define the following map using (D2)

$$(3.3.1.b) \quad \phi^! : M_n(F, \lambda_F) \simeq M_n(F, f^* \lambda_E \otimes \omega_{F/E}) \xrightarrow{(D2)} M_n(E, \lambda_E).$$

(3) Let \mathcal{O} be a 1-dimensional local domain which is essentially of finite type over S , with fraction field F and residue field κ . Let A be the integral closure of \mathcal{O} (in F). Since S is noetherian and \mathcal{O} is essentially of finite type over S , we have that \mathcal{O} is noetherian. By Krull-Akizuki theorem ([Sta25, Tag 09IG]), we know that A is a Dedekind domain with finitely many closed points denoted by z_1, \dots, z_n and that each closed point z_i corresponds to a discrete valuation on F whose residue field $\kappa(z_i)$ is finite over κ . Denote by d_i the defect of the field extension $\kappa(z_i)/\kappa$: we have $d_i = 1$ in our setting since the base scheme S is universally Japanese (see [Dég25, Prop. 5.3.12] for more detail). Denote by $\phi_i : \kappa \rightarrow \kappa(z_i)$ the induced finite morphism of S -fields. We then define a map

$$(3.3.1.c) \quad \partial_{\mathcal{O}} : M_n(F, \lambda_F) \xrightarrow[(3.3.1.a)]{(\partial_{z_i})_{i=1}^n} \bigoplus_{i=1}^n M_{n-1}(\kappa(z_i), \lambda_{\kappa(z_i)}) \xrightarrow[(3.3.1.b)]{\sum_{i=1}^n (d_i) \epsilon \phi_i^!} M_{n-1}(\kappa, \lambda_{\kappa}).$$

(4) Let X be an S -scheme essentially of finite type, and let x and y be two points on X . We define a map

$$(3.3.1.d) \quad \partial_y^x : M_*(x, \lambda_x) \rightarrow M_{*-1}(y, \lambda_y)$$

as follows: let Z be the reduced Zariski closure of x in X , and if $y \in Z^{(1)}$, then the map (3.3.1.d) is defined as the map $\partial_{\mathcal{O}_{Z,y}}$ in (3.3.1.c) associated to the 1-dimensional local domain $\mathcal{O}_{Z,y}$; otherwise we put $\partial_y^x = 0$.

Definition 3.3.2. Using the map (3.3.1.d) defined above, we will say that the MW-premodule M is a *Milnor-Witt cycle module* over S , or simply MW-module, if the following conditions hold:

- (FD): FINITE SUPPORT OF DIVISORS. Let X be an S -scheme. Let ξ be a point on X and let ρ be an element of $M(\xi)$. Then $\partial_x^\xi(\rho) = 0$ for all but finitely many x .
- (C): CLOSEDNESS. Let X be a local, integral S -scheme of dimension 2, with generic point ξ and closed point x_0 . Then

$$0 = \sum_{x \in X^{(1)}} \partial_{x_0}^x \circ \partial_x^\xi : M_*(\xi, \lambda_\xi) \rightarrow M_{*-2}(x_0, \lambda_{x_0}).$$

The notion of a MW-module in Section 3.3.2 depends a priori on the choice of a pinning. The following result shows that it actually does not.

Proposition 3.3.3. *Let λ and λ' be two pinnings on a base scheme S , then they define the same Milnor-Witt cycle modules.*

It follows that MW-cycle modules are well defined over any scheme with a pinning, and does not depend on the choice of a pinning. For a scheme S with a pinning, we denote by MW-Homod_S (resp. MW-Mod_S) the category of MW-cycle homodules (resp. modules) over S .

Proof. It is straightforward to check that the two pinningings λ and $\mathcal{L} \otimes \lambda$ (see Section 3.1.7 for the definition of the latter) define the same MW-cycle modules.

If X is an integral local scheme and λ, λ' are two pinningings on X , then there exists a line \mathcal{L} bundle on X such that $\lambda' \simeq \mathcal{L} \otimes \lambda$. Indeed, every line bundle over a local scheme is trivial, and a trivialization is uniquely determined by a global invertible section. Let ξ be the generic point of X , and the isomorphism $\lambda_\xi \simeq \lambda'_\xi$ determines an invertible section s at ξ . Then the set of points of X at which s is invertible is stable under immediate specializations (use the property that a pinning has sections over all discrete valuation rings and argue as in 3.3.1) and contains ξ , and therefore agrees with the whole X since X is integral. It follows that s defines a global invertible section on X , which induces an isomorphism $\lambda' \simeq \mathcal{L} \otimes \lambda$.

Now we are ready to prove the proposition: for the axiom (FD), we may assume that X is a 1-dimensional local domain, and the result follows the fact that there is a line bundle \mathcal{L} on X such that $f^*(\lambda') \simeq \mathcal{L} \otimes f^*(\lambda)$, where $f^*(\lambda)$ is defined as in 3.1.7; the axiom (C) follows from a similar argument. \square

We now prove a general duality theorem for schemes with a pinning:

Theorem 3.3.4 (Duality). *Let (S, λ) be a base scheme with a pinning. Given a MW-(pre)module M over S , for every S -field $\text{Spec } E$, we let*

$$(3.3.4.a) \quad \mathcal{D}_\lambda(M)_n(E) = M_n(E, \lambda_E).$$

Reciprocally given a MW-cycle (pre)homomodule \mathcal{M} over S , we let

$$(3.3.4.b) \quad D_\lambda(\mathcal{M})_n(E) = \mathcal{M}_n(E, \lambda_E^\vee).$$

Then these two constructions establish two functors that are equivalences of categories inverse to each other

$$(3.3.4.c) \quad \mathcal{D}_\lambda : \text{MW-Mod}_S \simeq \text{MW-Homod}_S : D_\lambda.$$

Proof. It is not hard to see that the maps (3.3.4.a) and (3.3.4.b) define functors \mathcal{D}_λ and D_λ that are inverse to each other. Keeping the previous notations, it remains to check that $D_\lambda(\mathcal{M})$ lives in MW-Mod_S and $\mathcal{D}_\lambda(M)$ lives in MW-Homod_S . We now define the functorialities (D1') through (D4').

(D1'): For a morphism of S -fields $\phi : E \rightarrow F$, we define the map

$$(3.3.4.d) \quad \phi! : \mathcal{D}_\lambda(M)_n(E) = M_n(E, \lambda_E) \xrightarrow{(D1)} M_n(F, \phi^* \lambda_E) \simeq M_n(F, \lambda_F \otimes \omega_{F/E}) = \mathcal{D}_\lambda(M)_n(F, \omega_{F/E})$$

thanks to (D1).

(D2'): For a finite morphism of S -fields $\phi : E \rightarrow F$, we have a map

$$(3.3.4.e) \quad \phi^* : \mathcal{D}_\lambda(M)_n(F) = M_n(F, \lambda_F) \xrightarrow{(3.3.1.b)} M_n(E, \lambda_E) = \mathcal{D}_\lambda(M)_n(E).$$

(D3'): Completely analogous to (D3).

(D4'): If \mathcal{O} is a 1-dimensional local domain which is essentially of finite type over S , with fraction field F and residue field κ , then we have a map

$$(3.3.4.f) \quad \partial_{\mathcal{O}} : \mathcal{D}_{\lambda}(M)_n(F) = M_n(F, \lambda_F) \xrightarrow{(3.3.1.c)} M_{n-1}(\kappa, \lambda_{\kappa}) = \mathcal{D}_{\lambda}(M)_{n-1}(\kappa).$$

For the rest of the axioms it suffices to proceed line by line and check that the corresponding versions match with each other, which makes use of the compatibilities of the isomorphisms stated in 3.1.4. The proof for $\mathcal{D}_{\lambda}(\mathcal{M})$ is similar, and we leave the details to the reader. \square

Remark 3.3.5. Assume that the base scheme S is the spectrum of a perfect field $\mathrm{Spec}(k)$ (resp. the spectrum of an excellent DVR) equipped with its canonical pinning. Then the category of MW-module (in the sense of Section 3.3.2) is equivalent to the category of Milnor-Witt cycle module in the sense of [Fel20, §2] (resp. in the sense of [BHP23]). Indeed, the definitions are almost identical. The main point is that under the axiom (D3), twists by line bundles can be defined using the twist *à la Morel* in (3.2.1.a) and are consequences of the other axioms (see also Section 1.1.3).

Example 3.3.6. Let (S, λ) be a base scheme with a pinning. Then the MW-premodule K_*^{MW} over S constructed in Section 3.2.3 is actually a MW-module.

Indeed, this follows from the proof of [BHP23, Th. 1.4], though an argument is missing point in *loc. cit.* First, the axiom (FD) of Section 3.3.2 is obvious (see e.g. [Dég25, Lem. 3.1.2]). The problem in the proof of *loc. cit.* happens with axiom (C). In fact, the reduction of Step 2 in *loc. cit.*, to the case of a local *complete* 2-dimensional domain A uses the existence of flat pullbacks on the Rost-Schmid complex. Indeed, to prove axiom (C), one reduces to a 2-dimension local excellent scheme $X_0 = \mathrm{Spec}(A_0)$. Then the completion $\widehat{A} = \widehat{A_0}$ is flat over A of relative dimension 0.¹¹ One then needs to use the existence of the flat pullback along $f : X = \mathrm{Spec}(A) \rightarrow X_0 = \mathrm{Spec}(A_0)$, as in Step 3 of the proof of [EKM08, Prop. 49.30], to reduce to the case of the local complete scheme X , using that A and A_0 have the same residue field. Such a pullback is defined in [FJ25], and the above proof shows that K_*^{MW} satisfies both axioms of Section 3.3.2.

One then applies Theorem 3.3.4 to deduce a MW-cycle homodule $\mathcal{D}_{\lambda}(K_*^{MW})$ over S , with the formula:

$$(3.3.6.a) \quad \mathcal{D}_{\lambda}(K_*^{MW})_n(E) = K_n^{MW}(E, \lambda_E) = \mathcal{K}_*^{MW}(E) \otimes_{\mathbf{Z}[E^{\times}]} \mathbf{Z}[\lambda_E^{\times}].$$

3.3.7. Let (S, λ) be a base scheme with a pinning (recall that, according to our conventions, S is equipped with a dimension function δ). According to Section 3.3.4, one can transport all constructions obtained earlier for MW-homodules to the case of a MW-module M . Explicitly, give an S -scheme X essentially of finite type (resp. essentially slci) and a line bundle \mathcal{L} over X , one defines the homological (resp. cohomological) complex of cycles of X with coefficients in

¹¹Flatness is classical (see [Sta25, Tag 00MB] and relative dimension 0 follows as A_0 and A have the same residue field.

M and twists \mathcal{L} as follows:

$$(3.3.7.a) \quad C_p(X, M_q, \mathcal{L}) = C_p(X, \mathcal{D}_\lambda(M)_q, \mathcal{L}) = \bigoplus_{x \in X^{(p)}} M_{q+p}(x, \lambda_x \otimes \mathcal{L}_x)$$

$$(3.3.7.b) \quad C^p(X, M_q, \mathcal{L}) = C^p(X, \mathcal{D}_\lambda(M)_q, \mathcal{L}) = \bigoplus_{x \in X^{(p)}} M_{q-p}(x, \omega_{X,x} \otimes \lambda_x \otimes \mathcal{L}_x^\vee).$$

so that if X/S is of relative dimension d

$$(3.3.7.c) \quad C^p(X, M_q, \mathcal{L}) \simeq C_{d-p}(X, M_{q-d}, \omega_X \otimes \mathcal{L}^\vee).$$

See respectively (1.3.3.a), (1.3.7.a), (1.3.7.b), and use Section 1.1.4.

One deduces a twisted Borel-Moore homology theory $A_*(X, M, *)$ and respectively a cohomology theory $A^*(X, M, *)$ satisfying the expected axioms, as proved in the previous sections. Finally, it is possible to define a cohomology with support, Thom isomorphism, and deduce that these theories are SL-oriented. See Section 2.4.

3.4. Quotient and localization by universal Milnor-Witt elements.

3.4.1. Let us denote by $K_*^{MW}(\mathbf{Z})$ the \mathbf{Z} -graded ring obtained using Morel's generators and relations [Mor12, Def. 3.1] applied to the ring \mathbf{Z} instead of a field. Explicitly, one gets the following \mathbf{Z} -graded ring:

$$K_*^{MW}(\mathbf{Z}) = \mathbf{Z}[\rho, \eta] / (2\rho + \eta\rho^2, \eta\rho - \rho\eta, 2\eta + \eta^2\rho)$$

where $\rho = [-1]$ with Morel's notation, has degree $+1$ and η has degree -1 . It is an easy exercise to compute this ring, and one deduces:

- $K_0^{MW}(\mathbf{Z}) = \mathbf{Z}[\epsilon] / (\epsilon^2 - 1)$, as a ring, where $\epsilon = 1 + \eta\rho$. Note that $K_0^{MW}(\mathbf{Z}) \simeq \text{GW}(\mathbf{Z})$.
- for all $n > 0$, $K_n^{MW}(\mathbf{Z}) = \mathbf{Z} \cdot \rho^n$, and $K_{-n}^{MW}(\mathbf{Z}) = \mathbf{Z} \cdot \eta^n \simeq W(\mathbf{Z})$.

Let us consider a MW-(pre)homomodule \mathcal{M} and a MW-(pre)module M over the base scheme S , equipped with a pinning in the case of MW-modules. For any field, E , one gets a ring map: $K_*^{MW}(\mathbf{Z}) \rightarrow K_*^{MW}(E)$. Thus, according to (D3') and (D3), any element $e \in K_n^{MW}(\mathbf{Z})$ of degree n acts on \mathcal{M} and M , by maps $\gamma_e : \mathcal{M} \rightarrow \mathcal{M}\{n\}$ and $\gamma_e : M \rightarrow M\{n\}$, which are in fact morphisms of MW-(pre)homomodule and MW-(pre)module respectively (recall the notation of Section 1.1.6). (This can be deduced from the axioms and the fact that the symbol e is globally defined.)

Definition 3.4.2. Consider the setting of 3.4.1. We say that \mathcal{M} or M is *e-local* (resp. *e-trivial*) if γ_e is an isomorphism (resp. the zero map). If \mathcal{C} is the category of MW-(pre)homomodules, or that of MW-(pre)modules, one denotes by $\mathcal{C}[e^{-1}]$ (resp. $\mathcal{C}^{e=0}$) the full subcategory of \mathcal{C} made by the *e-local* (resp. *e-trivial*) objects.

The following result is straightforward.

Proposition 3.4.3. *Consider the setting of Definition 3.4.2. Then the fully faithful functor $j_e : \mathcal{C}[e^{-1}] \rightarrow \mathcal{C}$ (resp. $i_e : \mathcal{C}^{e=0} \rightarrow \mathcal{C}$) admits a left adjoint L'_e (resp. Q'_e) defined by the formula:*

$$L'_e(\mathcal{M}) = \mathcal{M}[e^{-1}] := \operatorname{colim} (\mathcal{M} \xrightarrow{\gamma_e} \mathcal{M}\{n\} \xrightarrow{\gamma_e} \dots)$$

$$\text{resp. } Q'_e(\mathcal{M}) = \mathcal{M}/e := \operatorname{coKer}(\mathcal{M} \xrightarrow{\gamma_e} \mathcal{M}\{n\}),$$

and similarly for the MW-module M . We will put: $L_e = j_e L'_e$ (resp. $Q_e = i_e Q'_e$) called the e -localization (e -quotient) functor.

Finally, $\mathcal{C}[e^{-1}]$ is a thick¹² abelian subcategory of \mathcal{C} , while $\mathcal{C}^{e=0}$ is only an abelian subcategory of \mathcal{C} stable under direct factors.

Example 3.4.4. Let S be a scheme which admits a pinning, and consider the notation of Section 3.3.6.

- (1) According to the presentation of each rings, one gets for any (S) -field E a canonical isomorphism: $K_*^{MW}(E)/\eta = K_*^M(E)$. In particular, one deduces the η -quotient:

$$K_*^{MW}/\eta = K_*^M.$$

Note that data (D^*) on the η -trivial MW-module K_*^M are the well-known operations on Milnor K-theory, and the theory does not depend on twists (see next subsection).

- (2) According to [Mor12, Cor. 3.11], $K_*^{MW}(E)[\eta^{-1}] = W(E)[\eta, \eta^{-1}]$, the right hand-side being the Witt ring in all degree (η being seen as a formal invertible variable of degree -1). One deduces that there exists a MW-module, simply denoted by W , which in degree n and for an S -field E associates the abelian group $W(E)$, and such that:

$$K_*^{MW}[\eta^{-1}] = W.$$

The notation indicates that the MW-module W is periodic: $W_n = W_{n+1}$. Finally, data (D^*) on the η -local MW-module W are given by the classical operations on Witt rings of fields.

- (3) Consider the element $h = 2 + \eta\rho$ in the ring $K_0^{MW}(\mathbf{Z})$ (the avatar of the hyperbolic form). Another result of Morel (see [Dég25, 2.3.5] for the statement and references) states that for any integer $n \in \mathbf{Z}$ and any (S) -field E , there exists a canonical isomorphism of abelian groups: $K_n^{MW}(E)/(h) = I^n(E)$ where $I(E) = \operatorname{Ker}(\operatorname{rk} : W(E) \rightarrow \mathbf{Z}/2)$ is the fundamental ideal of $W(E)$.¹³ In particular, one can define a sub-MW-module $I^* \subset W$ over S , and there exists a canonical isomorphism:

$$(3.4.4.a) \quad K_*^{MW}/h = I^*.$$

Based on the notation of the previous example, and on the, now proved, second Milnor conjecture, one deduces the following fundamental exact sequences:

¹²i.e. closed under subobjects, quotients and extensions

¹³According to the usual convention, one puts: $I^n(E) = W(E)$ for non-positive n .

Theorem 3.4.5. *Let S be a scheme S which admits a pinning. Let $2K_*^M$ (resp. $K_*^M/2$) be the image (resp. cokernel) of the multiplication by 2 map: $\gamma_2 : K_*^M \rightarrow K_*^M$.¹⁴ Then there exists exact sequences of MW-modules over S :*

$$(3.4.5.a) \quad 0 \rightarrow I^*\{1\} \xrightarrow{\nu} I^* \rightarrow K_*^M/2 \rightarrow 0$$

$$(3.4.5.b) \quad 0 \rightarrow I^*\{1\} \rightarrow K_*^{MW} \xrightarrow{F} K_*^M \rightarrow 0$$

$$(3.4.5.c) \quad 0 \rightarrow 2K_*^M \xrightarrow{H} K_*^{MW} \xrightarrow{P} I^* \rightarrow 0$$

where ν is the canonical inclusion, F and P are the canonical projection maps, respectively called the forgetful and Pfister map.

Moreover, $F \circ H : 2K_*^M \rightarrow K_*^M$ is the canonical injection, so that the three short exact sequences become split after inverting 2. Last, the map P induces an isomorphism of MW-modules over S :

$$(3.4.5.d) \quad W[1/2] \rightarrow I^*[1/2].$$

Proof. The first exact sequence follows directly from the *Milnor isomorphism*: for any field k and any 1-dimensional k -vector space L we have

$$(3.4.5.e) \quad K_n^M(k)/2 \simeq I^n(k, L)/I^{n+1}(k, L)$$

see [OVV07, Theorem 4.1], [Mor04, Théorème 2.1], and [Kat82] for the case of characteristic 2 - see also [Fas08, Lemme E.1.3] to deal with the twist by L .

Then the two remaining exact sequences follow from this isomorphism and from the following cartesian square of abelian groups:

$$(3.4.5.f) \quad \begin{array}{ccc} K_n^{MW}(E) & \xrightarrow{F} & K_n^M(E) \\ \downarrow & & \downarrow \\ K_n^{MW}(E)/h & & K_n^M(E)/2 \\ \downarrow \wr (3.4.4.a) & & (2) \downarrow \wr \\ I^n(E) & \longrightarrow & I^n(E)/I^{n+1}(E) \end{array}$$

(A curved arrow labeled P points from the left column to the right column.)

The remaining of the statement is now obvious. □

To state the main corollary of the previous theorem we will use the notation from Section 3.3.7, and anticipate the next section by using that of Section 3.5.11. One then deduces the following important computation tool.

Corollary 3.4.6. *Let (S, δ, λ) be a scheme equipped with a dimension function and a pinning. Let X be an S -scheme essentially of finite type and \mathcal{L} a line bundle over S . Then one gets long*

¹⁴Note that $K_*^M/2$ is also the 2-quotient of K_*^M in the sense of the above proposition.

exact sequences:

$$(3.4.6.a) \quad \dots \rightarrow A_p(X, I^{q+1}, \mathcal{L}) \xrightarrow{\nu} A_p(X, I^q, \mathcal{L}) \rightarrow A_p(X, K_q^M/2) \rightarrow A_{p-1}(X, I^{q-1}, \mathcal{L}) \rightarrow \dots$$

$$(3.4.6.b) \quad \dots \rightarrow A_p(X, I^{q+1}, \mathcal{L}) \rightarrow A_p(X, K_q^{MW}, \mathcal{L}) \xrightarrow{F} A_p(X, K_q^M) \rightarrow A_{p-1}(X, I^{q+1}, \mathcal{L}) \rightarrow \dots$$

$$(3.4.6.c) \quad \dots \rightarrow A_p(X, 2K_q^M) \xrightarrow{H} A_p(X, K_q^{MW}, \mathcal{L}) \xrightarrow{P} A_p(X, I^q, \mathcal{L}) \rightarrow A_{p-1}(X, 2K_q^M) \rightarrow \dots$$

which are functorial with respect to smoothable slci pullbacks, proper pushforwards, multiplication by Euler and Thom classes. Moreover, both exact sequence become split after inverting 2: up to identifying $2K_*^M$ and K_*^M , one has the relation $F \circ H = 2 \cdot \text{Id}$.

Remark 3.4.7. Absolute Chow-Witt homology. We indicate to the reader another possible notation for the complexes and their homology defined in Section 3.3.7, when it comes to the MW-module $M = K_*^{MW}$, or another variant obtained by localization or quotient as above (see Section 3.3.6). We fix a scheme X . To simplify the notation, we consider a pair $\hat{\lambda} = (\lambda, \delta)$ made by a pinning λ and a dimension function δ on S — such a pair could be called a *dimensional pinning*, and corresponds to a section of the Picard category of graded line bundles [Del87, 4.14]. Then one can put:

$$C_p(X, M_q, \hat{\lambda}) = C_p(X, \mathcal{D}_\lambda(M)_q, \mathcal{L}) = \bigoplus_{x \in X_{(\delta=p)}} M_{q+p}(x, \lambda_x \otimes \mathcal{L}_x),$$

where we have made the dimension function explicit. Let $A_p(X, M_q, \hat{\lambda})$ be its homology.

One can always twist these groups by a line bundle \mathcal{L} over X : $C_p(X, M_q, \hat{\lambda} \otimes \mathcal{L})$, where $\hat{\lambda} \otimes \mathcal{L} = (\mathcal{L} \otimes \lambda, \delta)$ with the notation of Section 3.1.7. One can also describe the functoriality of these complexes or their homology as follows, for $\pi : X' \rightarrow X$ proper and $f : Y \rightarrow X$ essentially slci:

$$\begin{aligned} \pi_* : A_p(X', M_q, \pi^* \hat{\lambda}) &\rightarrow A_p(X, M_q, \hat{\lambda}), \\ f^! : A_p(X, M_*, \hat{\lambda}) &\rightarrow A_p(Y, M_*, f^! \hat{\lambda}), \end{aligned}$$

where: $\pi^* \hat{\lambda} = (\pi^* \lambda, \delta|_{X'})$, $f^! \hat{\lambda} = (f^! \lambda, d(f) + \delta|_Y)$, with the notation of Section 3.1.7, and $d(f)$ is the Zariski local function on Y given by the dimension of the fibers.

3.5. Oriented Milnor-Witt cycle modules. The particular case of Section 3.4.2 of η -trivial objects is fundamental. The following lemma is obvious, as the symbols $\langle u \rangle \in K_0^{MW}(E)$ becomes equal to 1 in $K_0^M(E)$.

Lemma 3.5.1. *We have that a MW-premodule M (resp. MW-prehomomodule \mathcal{M}) is η -trivial if and only if for any field E , the action of $K_*^{MW}(E)$ on $M(E)$ (resp. $\mathcal{M}(E)$) factors through $K_*^M(E)$.*

If these conditions hold, for any line bundle \mathcal{L} over E , one gets a canonical identification:

$$(3.5.1.a) \quad M(E, \mathcal{L}) = M(E) \text{ resp. } \mathcal{M}(E, \mathcal{L}) = \mathcal{M}(E).$$

Definition 3.5.2. If the equivalent conditions of the first sentence of the previous lemma holds, we will say that M (resp. \mathcal{M}) is *orientable*. We also use the terminology *Milnor cycle pre(ho)module*, of simply M -module (resp. M -homomodule).

3.5.3. Thanks to Section 3.5.1, the axioms defining a M -premodule M are simpler than those defining a MW-premodule. They are in fact a simplification (but essentially equivalent) of Rost's axioms for cycle modules. First, the lemma shows that the data (D3) is equivalent of \mathbf{Z} -graded $K_*^M(E)$ -module structure on $M(E)$. Second, given the canonical trivialization of twists one can rewrite some of the functorialities of M as follows:

(d2): For a finite morphism of S -fields $\phi : E \rightarrow F$, we have a map of degree 0

$$(3.5.3.a) \quad \phi^! : M(F) \rightarrow M(E);$$

(d4): For an S -field $\text{Spec } E$ and an S -valuation v on E , we have a map of degree -1

$$(3.5.3.b) \quad \partial_v : M(E) \rightarrow M(\kappa(v)).$$

One can check that all the relations (R1a) though (R3e) continue to hold with the new functorialities 3.5.3.a and 3.5.3.b.

3.5.4. With the previous simplification of the axioms for an orientable MW-premodule M , is it possible to extend the definition of Section 3.3 over a base scheme S , without requiring the existence of a pinning. Indeed, under this hypothesis we can define general residue maps as follows:

- (1) Let \mathcal{O} be a 1-dimensional local domain which is essentially of finite type over S , with fraction field F and residue field κ . Then \mathcal{O} is an excellent ring. Let A be the integral closure of \mathcal{O} , which is regular, semi-local and finite over \mathcal{O} . Let $\{z_1, \dots, z_n\}$ be the closed points of A , which correspond to valuations on F with residue fields $\kappa(z_1), \dots, \kappa(z_n)$ (indeed, the normalization morphism is finite because S is excellent). Denote by $\phi_i : \kappa \rightarrow \kappa(z_i)$ the induced finite morphism of S -fields. We then define a map

$$(3.5.4.a) \quad \partial_{\mathcal{O}} : M_n(F) \xrightarrow[3.5.3.b]{(\partial_{z_i})_{i=1}^n} \bigoplus_{i=1}^n M_{n-1}(\kappa(z_i)) \xrightarrow[3.5.3.a]{\sum_{i=1}^n \phi_{i*}} M_{n-1}(\kappa).$$

- (2) Let X be an S -scheme essentially of finite type, and let x and y be two points on X . We define a map

$$(3.5.4.b) \quad \partial_y^x : M_*(x) \rightarrow M_{*-1}(y)$$

as follows: let Z be the reduced Zariski closure of x in X , and if $y \in Z^{(1)}$, then the map (3.5.4.b) is defined as the map $\partial_{\mathcal{O}_{Z,y}}$ in (3.5.4.a) associated to the 1-dimensional local domain $\mathcal{O}_{Z,y}$; otherwise we put $\partial_y^x = 0$.

Definition 3.5.5. Consider the above notation. An *orientable Milnor-Witt cycle module* over S is an orientable Milnor-Witt cycle premodule (Section 3.2.1) which in addition satisfies the conditions (fd) and (c)

- (fd): **FINITE SUPPORT OF DIVISORS.** Let X be an irreducible normal S -scheme with generic point ξ and let ρ be an element of $M(\xi)$. Then $\partial_x^\xi(\rho) = 0$ for all but finitely many $x \in X^{(1)}$.

(c): CLOSEDNESS. Let X be a local, integral S -scheme of dimension 2, with generic point ξ and closed point x_0 . Then

$$(3.5.5.a) \quad 0 = \sum_{x \in X^{(1)}} \partial_{x_0}^x \circ \partial_x^\xi : M_*(\xi) \rightarrow M_{*-2}(x_0).$$

We also call such an object a *Milnor cycle module* over S , or simply *M-module*. We denote by $\mathbf{M}\text{-Mod}_S$ the category of Milnor cycle modules over S .

Example 3.5.6. According to Section 3.4.4(1), Milnor K-theory defines a Milnor cycle module over any base scheme S equipped with a pinning. This follows from the fact that K_*^{MW} is a MW-module over such bases, which follows the proof of [Kat86]. In fact, based on *loc. cit.* one deduces that K_*^M is a Milnor cycle module over any base scheme S , without requiring the existence of a pinning.

In [Ros96], Rost introduced the notion of cycle modules, focusing on schemes essentially of finite type over a given base field $S = \text{Spec}(k)$. However, it is straightforward to extend all the definitions of *loc. cit.* to the setting of a base scheme S equipped with a dimension function δ , which replaces the Krull dimension used by Rost. In fact, our notion of Milnor cycle modules, as defined in Section 3.5.5, agrees with the original and the extended definitions of Rost cycle modules:

Proposition 3.5.7. *Let S be a base scheme. There is a canonical equivalence between the category $\mathbf{M}\text{-Mod}_S$ and the category of Rost cycle modules over S .*

Proof. The data in Definition 3.2.1, as well as the four operations (D1), (D2), (D3) and (D4), correspond precisely to Rost's data for cycle modules ([Ros96, Definition 1.1]). Therefore it suffices to check that the relations in Definition 3.5.5 corresponds to Rost's relations. While most of the relations can be checked verbatim, we need to show that our axioms imply the following a priori stronger versions of Rost:

(R1cRost) Let $\phi : E \rightarrow F$ and $\psi : E \rightarrow L$ be morphisms of fields with ϕ finite. Let A be the artinian ring $F \otimes_E L$. For each prime $p \in \text{Spec } A$, let $\phi_p : L \rightarrow A/p$ and $\psi_p : F \rightarrow A/p$ be the morphisms induced by ϕ and ψ . Let l_p be the length of the local ring A/p . One has:

$$\psi_* \circ \phi^! = \sum_{p \in \text{Spec } A} l_p \cdot (\phi_p)^! \circ (\psi_p)_*.$$

(R3aRost) Let $\phi : E \rightarrow F$ be a morphism of S -fields. Let w be an S -valuation on F which restricts to a non-trivial valuation v on E of ramification index e , and denote by $\bar{\phi} : \kappa(v) \rightarrow \kappa(w)$ the induced morphism of S -fields. Then

$$(3.5.7.a) \quad \partial_w \circ \phi_* = e \cdot \bar{\phi}_* \circ \partial_v.$$

The axioms (R1cRost) and (R3aRost) can be proved using the methods in [Fel21b, §4], which finishes the proof. \square

3.5.8. In the homological case, we define a *homological Milnor cycle module* as an η -trivial homological cycle module M (Definitions 1.2.2 and 3.4.2), also called *M-homomodule*. Equivalently, M is M-hopremodule that satisfy the conditions (FD') and (C') of Section 1.2.2. We denote by M-Homod_S the category of homological Milnor cycle modules over S , considered as a full subcategory of the category of homological MW-cycle modules MW-Homod_S .

The following result states a duality between cohomological and homological versions of Milnor cycle modules:

Theorem 3.5.9 (Duality, oriented version). *Let S be a base scheme. Given a Milnor cycle module M over S , for every S -field $\text{Spec } E$, we let*

$$(3.5.9.a) \quad \mathcal{D}(M)(E, n) = M_n(E).$$

Reciprocally, given a homological Milnor cycle module \mathcal{M} over S , we let

$$(3.5.9.b) \quad \mathcal{D}(\mathcal{M})_n(E) = \mathcal{M}(E, n).$$

Then these two constructions establish two functors that are equivalences of categories inverse to each other

$$(3.5.9.c) \quad \mathcal{D} : \text{M-Mod}_S \simeq \text{M-Homod}_S : \mathcal{D}.$$

Proof. As all the twists have disappeared, the proof is straightforward with the functorialities 3.5.3.a and (3.5.4.a). We leave the details to the reader (see also the proof of Theorem 3.3.4 below). \square

Example 3.5.10. Combining Examples 1.1.7, 1.3.6, one deduces that Quillen algebraic K '-theory defines an M-homomodule K_* over S . The associated dual M-module K_*^M , obtained from the above theorem, readily corresponds to the Quillen algebraic K -theory restricted to S -fields.

Note that the canonical *symbol map* induces a morphism of M-modules:

$$(3.5.10.a) \quad K_*^M \rightarrow K_*$$

which is an isomorphism in degree at most 2, according to Matsumoto's theorem.

3.5.11. Consider the above notation and fix a M-module M . As in the case if MW-modules Section 3.5.11, one defines the homological (resp. cohomological) complex of cycles of X with coefficients in M as follows:

$$(3.5.11.a) \quad C_p(X, M_q) = C_p(X, \mathcal{D}(M)_q) = \bigoplus_{x \in X^{(p)}} M_{q+p}(x)$$

$$(3.5.11.b) \quad C^p(X, M_q) = C^p(X, \mathcal{D}(M)_q) = \bigoplus_{x \in X^{(p)}} M_{q-p}(x).$$

see respectively (1.3.3.a) and (1.3.7.a). Note that there also exists a definition twisted by a line bundle \mathcal{L} , but according to (3.5.1.a), this definition does not depend on the chosen twists and agrees with the above one. Note also that it follows from our definition that, under the equivalence of Section 3.5.7, the above definition agrees with the one given in [Ros96, §5].

Taking homology (resp. cohomology), one deduces a twisted Borel-Moore homology theory $A_*(X, M)$ and respectively a cohomology theory $A^*(X, M)$ satisfying the expected axioms over S , as proved in the previous sections. Beware that the twists now only comes from the grading of M , as the twist by line bundles are no longer relevant. Therefore, the axioms satisfied by the above pair of homology and cohomology theories are precisely that of [BO74] — except the base S can be arbitrary according to the conventions of this paper, rather than just a field. Finally, using Section 2.4, one also define a cohomology with support, and Thom isomorphisms in the untwisted form, which shows that the associated cohomology theory is canonically (GL-)oriented. This justifies the terminology orientable MW-module.

Finally, we note that Section 3.4.3 gives the following comparison result between our general notion of MW-homodule and the extension of Rost's definition of cycle modules over a base (with a dimension function).

Corollary 3.5.12. *Let S be a base scheme (admitting a dimension function δ). Then the category of Rost cycle modules over S is a full abelian sub-category of that of MW-modules MW-Homod_S , and the inclusion functor admits a right adjoint. Using the notation from above:*

$$\text{M-Mod}_S = \text{MW-Mod}_S^{\eta=0} \xrightleftharpoons[\text{D}]{\text{D}} \text{MW-Homod}_S^{\eta=0} \xrightleftharpoons[\text{Q}_\eta]{j_\eta} \text{MW-Homod}_S$$

4. APPLICATIONS

4.1. Homological Chow-Witt groups.

4.1.1. We will now illustrate the theory obtained by specializing to Milnor-Witt K-theory K_*^{MW} , seen as a MW-module according to Section 3.3.6. As in *loc. cit.*, we fix a base scheme S , endowed with a dimension function δ and a pinning λ (Definition 3.1.3).¹⁵

To set up the notation, let us consider the MW-modules over S introduced in Section 3.4.4, or Section 3.4.5:

$$(4.1.1.a) \quad M \in \{K_*^{MW}, K_*^M = K_*^{MW}/\eta, 2K_*^M, K_*^M/2, W = K_*^{MW}[\eta^{-1}], I^* = K_*^{MW}/h\}.$$

Given an S -scheme X (essentially of finite type) and a line bundle \mathcal{L} over X , we can then use the definition of Section 3.3.7 and introduce the homology of X with coefficients in M and twist \mathcal{L} , over S :

$$(4.1.1.b) \quad C_p(X/S, M_q, \mathcal{L}) := C_p(X, M_q, \mathcal{L}) \quad := \bigoplus_{x \in X_{(p)}} M_{q+p}(\kappa(x), \lambda_x \otimes \mathcal{L}_x),$$

$$(4.1.1.c) \quad A_p(X/S, M_q, \mathcal{L}) := H_p(C_*(X/S, M_q, \mathcal{L})) \quad := H_p(C_*(X, M_q, \mathcal{L})).$$

While it is possible to follow the conventions of Section 3.4.7, we have chosen to indicate the base scheme S in the notations here, seeing this theory as the Borel-Moore theory of X/S with

¹⁵For convenient explicit choices, see Section 4.1.5.

coefficients in the MW-homodule $\mathcal{M} = \mathcal{D}_\lambda(M)$ defined over S .¹⁶ Recall that when M is oriented (e.g. $M \in \{K_*^M, 2K_*^M\}$), the group $A_p(X/S, M_q, \mathcal{L})$ can be defined without assuming the existence of a pinning: see Section 3.5.11; in the presence of a pinning, it does not depend on \mathcal{L} or on the choice of pinning. Finally, in the case $M = W$, the above group does not depend on q , and we will simply write:

$$(4.1.1.d) \quad A^p(X/S, W, \mathcal{L}) = A^p(X/S, K_q^{MW}[\eta^{-1}], \mathcal{L}), \text{ any } q \in \mathbf{Z}.$$

In any case, the above theory satisfies the following functoriality properties, for $\pi : X' \rightarrow X$ proper and $f : Y \rightarrow X$ lci smoothable:

$$\pi_* : A_p(X'/S, M_q, \pi^* \mathcal{L}) \rightarrow A_p(X/S, M_q, \mathcal{L}), \quad \text{Section 1.4.1}$$

$$f^! : A_p(X/S, M_q, \mathcal{L}) \rightarrow A_{p+d}(Y/S, M_q, \omega_f \otimes f^* \mathcal{L}), \quad \text{Section 2.2.5.}$$

Moreover, one has the *localization exact sequence* (1.5.8.a) associated with any closed immersion $i : Z \rightarrow X$ of S -schemes.

An interesting part of this complex arises when $q = -p$, giving the following 3-term complex: (4.1.1.e)

$$\bigoplus_{y \in X_{(p+1)}} K_1^{MW}(\kappa(y), \lambda_y \otimes \mathcal{L}_y) \xrightarrow{d_{p+1}} \bigoplus_{x \in X_{(p)}} \text{GW}(\kappa(x), \lambda_x \otimes \mathcal{L}_x) \xrightarrow{d_p} \bigoplus_{s \in X_{(p-1)}} W(\kappa(s), \lambda_s \otimes \mathcal{L}_s).$$

Definition 4.1.2. Consider the above notation. One defines the *homological Chow-Witt and Chow groups* $\widetilde{\text{CH}}_p(X/S, \mathcal{L}) = A_p(X/S, K_{-p}^{MW}, \mathcal{L})$ and $\text{CH}_p(X/S) = A_p(X/S, K_{-p}^M)$, associated with (X, \mathcal{L}) and X in degree p , relative to (S, δ, λ) . In the case of $\widetilde{\text{CH}}_p$ this is the homology in the middle of the complex (4.1.1.e), while for CH_p , it is the usual definition.¹⁷

One also considers the group of (*homological*) *quadratic cycles* in degree p as $\widetilde{Z}_p(X/S, \mathcal{L}) = \text{Ker}(d_p)$. Two such quadratic cycles are *rationally equivalent* if their difference belongs to the image of d_{p-1} .

The rank map, or equivalently the map F of Section 3.4.5, induces a canonical projection: $\widetilde{Z}_p(X, \mathcal{L}) \rightarrow Z_p(X/S)$, where the right hand-side are made of algebraic cycles of X , graded by δ . This map is compatible with rational equivalence, therefore inducing a canonical map, compatible with all functorialities:

$$(4.1.2.a) \quad F : \widetilde{\text{CH}}_p(X/S, \mathcal{L}) \rightarrow \text{CH}_p(X/S).$$

¹⁶Beware however that it is not clear that the bigraded theory $A_*(X/S, M, \mathcal{L})$ satisfies the axioms of a twisted bivariant theory as in [DJK21]: one misses the arbitrary base change map in S , though it can be defined for slci (or flat) morphisms (see [FJ25]). In a forthcoming work, we will establish such a formalism in full generality by showing the representability of this theory in the motivic stable homotopy category, and clarify its relations with the (perverse) homotopy t-structure.

¹⁷See [Ros96, Remark 5.1], except we use the dimension function δ for the grading.

Similarly, as the MW-module W is the η -localisation of K^{MW} , one gets a canonical map, compatible with all functorialities:

$$(4.1.2.b) \quad \widetilde{CH}_p(X/S, \mathcal{L}) \rightarrow A_p(X/S, W, \mathcal{L}),$$

where the right hand-side is the Witt homology of X/S , (4.1.1.d). It is also obtained by localizing the left-hand side with respect to the endomorphism η built in Section 1.4.5. The next proposition extends a corollary Section 3.4.5 and extends a basic computation known over a field.

Proposition 4.1.3. *Consider the above notation. Then the canonical map:*

$$\widetilde{CH}_p(X/S, \mathcal{L}) \rightarrow CH_p(X/S) \times A_p(X/S, W, \mathcal{L})$$

made of (4.1.2.a) and (4.1.2.b) becomes an isomorphism after inverting 2.

Obviously, the same result holds with cohomological conventions (4.2.1.a).

Remark 4.1.4. (1) While one can always twist a pinning by a line bundle, beware that the underlying pinning λ is defined over S while the twist \mathcal{L} considered above has only to be defined over an S -scheme X . This justifies our choice of notation.

(2) One can introduce several other numbering conventions, following the motivic literature:

- standard motivic notation: $\widetilde{A}_{p,q}(X/S, M, \mathcal{L}) := A_{p-q}(X/S, M_{-q}, \mathcal{L})$.
- Bloch numbering: $\widetilde{A}_p(X/S, M, \mathcal{L}, q) := A_{p+q}(X, M_{-q}, \mathcal{L})$.
- numbering with respect to \mathbf{G}_m -twists: $\widetilde{A}_p(X, M, \mathcal{L}, \{q\}) := A_p(X, M_{-q}, \mathcal{L})$.

4.1.5. Canonical choices. Recall that, over the base S , both choices of the pinning λ and the dimension function δ can be determined by the single choice of a dualizing complex K_S (Section 3.1.5).¹⁸

When S is regular, a natural choice for the homological conventions is $K_S = \mathcal{O}_S[\dim(S)]$. Then the corresponding dimension function δ is the *Krull dimension*, and the pinning λ over an S -point $x : \text{Spec}(E) \rightarrow S$ is defined by:

$$(4.1.5.a) \quad \lambda_x = \omega_{x/s} \otimes \nu_s^\vee = \omega_x$$

where $s \in S$ is the image of x , $\omega_{x/s} = \det(\mathbb{L}_{x/s})$ is the determinant of the cotangent complex of the (finitely generated) field extension $E/\kappa(s)$ and $\nu_s = \det(\mathcal{N}_{s/S(s)})$ is the determinant of the conormal sheaf of s in the local scheme $S_{(s)}$. With these particular choices, one gets:

- the Chow-Witt group $\widetilde{CH}_p(X/S, \mathcal{L})$ is a sub-quotient of $\bigoplus_{x \in X_{(p)}} \text{GW}(\kappa(x), \omega_{x/S} \otimes \mathcal{L}_x)$. It is made of rational equivalence classes of quadratic cycles (Section 4.1.2) whose support is of pure Krull dimension p .
- The Chow-group $CH_0(X/S)$ corresponds to classes of algebraic cycles made of closed points of X .

¹⁸In case the reader wants to be precise about those choices, we propose to replace \mathcal{L} in the previous notation by $K_S\{\mathcal{L}\}$.

- When S is the spectrum of a perfect field k , a MW-module M is a Milnor-Witt module in the sense of [Fel20] and the Chow-Witt group $A_p(X/k, M_q, \mathcal{L})$ defined here agrees with that of *loc. cit.* Definition 5.4 taken into account the convention of §4.1.

Example 4.1.6. Quadratic degree. Note that pushforwards exist at the level of cycles, while only smooth pullbacks exist at the level of cycles. In [FJ25], *flat pullbacks* are defined and do exist at the cycle level.

In particular, we get a **quadratic degree map** associated with any proper S -scheme X :

$$\widetilde{\deg}_{X/S} : \bigoplus_{x \in X_{(0)}} \mathrm{GW}(\kappa(x), \omega_{x/s} \otimes \nu_s^\vee) = \widetilde{Z}_0(X/S) \rightarrow \widetilde{Z}_0(S) = \bigoplus_{s \in S_{(0)}} \mathrm{GW}(\kappa(s), \nu_s^\vee).$$

In fact, this quadratic degree map is obtained by taking the sum over point $x \in X$ of the Scharlau transfer map associated with the differential trace map $tr_{x/s}^\omega : \omega_{x/s} \rightarrow \kappa(s)$ defined out a Grothendieck duality (see [Dég25, Def. 6.2.4]):

$$tr_{x/s} : \mathrm{GW}(\kappa(x), \omega_{x/s} \otimes \nu_s^\vee) \rightarrow \mathrm{GW}(\kappa(s), \nu_s^\vee), \omega \mapsto tr_{x/s*}^\omega(\omega)$$

Explicitly, $tr_{x/s*}^\omega(\omega)$ is (the class of) the composite map: $E \times E \xrightarrow{\omega} \omega_{x/s} \otimes \nu_s^\vee \xrightarrow{tr_{x/s}^\omega \otimes \mathrm{Id}} \nu_s^\vee$.

Note that similarly to the classical intersection theory ([Ful98, Example 1.4.2]), the quadratic degree trace map is defined even without assuming that X is proper over S , but in the non proper case it does not preserve rational equivalence.

4.2. Cohomological Chow-Witt groups.

4.2.1. We consider the notation of Section 4.1.1. In particular, S is a base scheme equipped with both a dimension function δ and a pinning λ , and M is one of the MW-module of (4.1.1.a).

Still following Section 3.3.7, we now introduce our cohomological conventions. For an S -scheme X which is now assumed to be essentially slci, with canonical sheaf ω_X , one puts:

$$(4.2.1.a) \quad C^p(X/S, M_q, \mathcal{L}) := C^p(X, M_q, \mathcal{L}) \quad := \bigoplus_{x \in X^{(p)}} M_{q-p}(x, \omega_{X,x} \otimes \lambda_x \otimes \mathcal{L}_x^\vee),$$

$$(4.2.1.b) \quad A^p(X/S, M_q, \mathcal{L}) := H^p(C^*(X/S, M_q, \mathcal{L})) \quad := H^p(C^*(X, M_q, \mathcal{L})).$$

so that we have the (trivial) *duality isomorphism* (3.3.7.c). When $X = S$, we simply write: $A^p(X, M_q, \mathcal{L})$ for the above cohomology group.

The functoriality properties can be translated from the homological case, using (3.3.7.c). Let us be more explicit for the comfort of the reader. Given an slci morphism $f : Y \rightarrow X$ (resp. proper morphism $\pi : X' \rightarrow X$) of essentially slci S -schemes, one gets:

$$f^* : A^p(X/S, M_q, \mathcal{L}) \rightarrow A^p(Y/S, M_q, f^* \mathcal{L}), \quad (2.4.0.a).$$

$$\pi_! : A^p(X'/S, M_q, \omega_\pi \otimes \pi^* \mathcal{L}) \rightarrow A^p(X/S, M_q, \mathcal{L}), \quad \text{Section 1.4.1.}$$

Note that ω_π is well-defined as π is quasi-smooth under our assumptions. In fact, $A^p(-/S, M_q, \mathcal{L})$ is a cohomology theory on any of the sites defined in Section 2.2.7. Moreover, it is SL-oriented according to Section 2.4.2, and admits an action of Euler classes according to Section 2.3.1.

4.2.2. Canonical choices (regular case). It is relevant to consider a simpler notation when $X = S$ is a regular scheme. As in Section 4.1.5, we consider the canonical pinning λ such that $\lambda_x = \omega_x$, for any point $x : \text{Spec}(E) \rightarrow X$ with canonical sheaf ω_x ; and the canonical dimension function $\delta = -\text{codim}_X$. Note that these choices correspond to the choice of dualizing complex $K_X = \mathcal{O}_X$ on X (see Section 3.1.5).

Let M be a the MW-module as in (4.1.1.a), and \mathcal{L} a line bundle over X . With the preceding choices, the definition of the previous paragraph in the case $X = S$ allow to define the cohomological Chow-Witt groups of X with twists \mathcal{L} and with coefficients in M :

$$C^p(X, M_q, \mathcal{L}) := \bigoplus_{x \in X^{(p)}} M_{q-p}(x, \nu_x \otimes \mathcal{L}_x^\vee), \quad A^p(X, M_q, \mathcal{L}) = H_p(C^p(X, M_q, \mathcal{L})).$$

where $x \in X^{(p)}$ now really is a codimension p point of X , and $\nu_x = \det(\mathcal{N}_{x/S(x)})$. One also defines the codimension p cycles on X with coefficients in M as $Z^p(X, M_q, \mathcal{L}) \subset C^p(X, M_q, \mathcal{L})$ defined by the kernel of the differential of the complex. Let us introduce an explicit definition.

Definition 4.2.3. Consider the above notation: X a regular scheme, \mathcal{L} a line bundle over X , and the choices of pinning and dimension function are normalized by the choice of dualizing complex $K_X = \mathcal{O}_X$. Specializing the above definition by taking $M = K_*^{MW}$, $q = p$, one defines the *Chow-Witt group*

$$\widetilde{\text{CH}}^p(X, \mathcal{L}) = A^p(X, K_p^{MW}, \mathcal{L}).$$

It is a quotient of the group of *quadratic cycles* $\tilde{Z}^p(X, \mathcal{L}) \subset \bigoplus_{x \in X^{(p)}} \text{GW}(\kappa(x), \nu_x \otimes \mathcal{L}_x^\vee)$, by the image of the differential operator of the complex of cycles with coefficients in K_*^{MW} ; we call the corresponding equivalence relation on quadratic cycles the *rational equivalence*.

4.2.4. Fasel and Srinivas, defined a twisted Chow-Witt group associated with a regular $\mathbf{Z}[1/2]$ -scheme in [FS09, Definition 32] when X is a $\mathbf{Z}[1/2]$. It follows from our definition and from the cartesian square (3.4.5.f) that their definition coincides with ours. To be precise, there is a slight difference in conventions and one has to change \mathcal{L} by \mathcal{L}^\vee to get exactly the same definition. However, by definition of twists, there is a canonical isomorphism between the \mathcal{L} -twisted group and the $(\mathcal{L})^\vee$ -twisted one. Our choice is motivated by the duality isomorphism below.

Example 4.2.5. Comparison with classical Chow groups. Taking $M = K_*^M$, $q = p$ gives the Chow-group $\text{CH}^p(X)$ is made of the usual codimension p algebraic cycle classes of X . Specializing (3.4.6.b) to our notation gives an exact sequence:

$$A^d(X, I^{d+1}, \mathcal{L}) \rightarrow \widetilde{\text{CH}}^d(X, \mathcal{L}) \xrightarrow{F} \text{CH}^d(X) \rightarrow 0$$

where d is the dimension of X . More generally, we get as in Section 4.1.3, a canonical map

$$\widetilde{\text{CH}}^p(X, \mathcal{L}) \rightarrow \widetilde{\text{CH}}^p(X) \times A^p(X, W, \mathcal{L})$$

which is an isomorphism after inverting 2 (use Section 3.4.5).

One can rewrite the duality isomorphism (3.3.7.c) with the conventions of Section 4.1.5 and Section 4.2.3 as follows.

Proposition 4.2.6. *Let $f : X \rightarrow S$ be a morphism essentially of finite type between regular schemes. Then, under the choice of conventions for homological and cohomological Chow-Witt groups from Section 4.1.5 and Section 4.2.3, there is a duality isomorphism:*

$$\widetilde{\mathrm{CH}}_p(X/S, \mathcal{L}) \simeq \widetilde{\mathrm{CH}}^{d-p}(X, \omega_{X/S} \otimes \mathcal{L}^\vee).$$

Assume moreover that X is a $\mathbf{Z}[1/2]$ -scheme. Then the Chow-Witt group $\widetilde{\mathrm{CH}}^p(X, \mathcal{L}^\vee)$ defined above precisely coincides with the Chow-Witt group defined by Fasel and Srinivas in [FS09, Definition 32].

In particular, we have extended Fasel-Srinivas theory over regular schemes to a Borel-Moore theory defined for S -schemes X essentially of finite type, according to Section 4.1.2).

Example 4.2.7. As a corollary, one deduces from [FS09, Theorem 34] that the group $\widetilde{\mathrm{CH}}^p(X, \mathcal{L}^\vee)$ can be identified with the E_2 -term, in degree $(p, 0)$, of the coniveau spectral sequence of a regular $\mathbf{Z}[1/2]$ -scheme X with coefficients in hermitian K-theory. Based on the recent advances on [BYN22], this can be extended to arbitrary regular schemes. One can also obtain this comparison by using the homological Chow-Witt groups and the hermitian G-theory, based on coherent \mathcal{O}_X -modules rather than perfect complexes.

Let us consider again the choices of Section 4.1.5 for homology, $K_S = \mathcal{O}_S[\dim(S)]$. Then for any S -scheme X , the \mathbf{Z} -graded group $A_*(X/S, W, \mathcal{L})$ of (4.1.1.d) coincides with the E_2 -term of the Gersten-Witt spectral sequence associated with the dualizing complex K_S as in [JX25, §3.1]. See *loc. cit.* Th. 4.2.5 for a proof, which actually holds for an arbitrary choice of dualizing complex K_S . In particular, when $X = S$ is regular separated, and $K_X = \mathcal{O}_X[\dim(X)]$, 2 invertible on X , this is also the E_2 -term of the Gersten-Witt spectral sequence of Balmer and Walter, [BW02].

4.3. Unramified sheaves and Bloch formulas.

4.3.1. We introduce a weaker, relative, version of homotopy modules ([Fel21a, Def. 4.1.2]) over a base scheme S . A *homotopy premodule* over S will be a \mathbf{Z} -graded Nisnevich sheaf $F = F_*$ of abelian groups on Sm_S equipped with isomorphisms $\epsilon_n : F_n \rightarrow (F_{n+1})_{-1}$ where the right-hand side is obtained by Voevodsky's construction (*loc. cit.* 4.1.1). Recall that by adjunction, the map ϵ_n corresponds to the so-called *suspension maps*

$$(4.3.1.a) \quad \sigma_n : \mathbf{G}_m \otimes F_n \rightarrow F_{n+1}.$$

It is not difficult to see that homotopy premodules over S , with obvious morphisms, form a Grothendieck abelian category. We say that F is a *homotopy module* if it is \mathbf{A}^1 -local, that is, if its cohomology is \mathbf{A}^1 -invariant.

Let S be a scheme such that $\delta = -\mathrm{codim}_S$ is a dimension function on S . For example, this is the case if S is normal and quasi-excellent ([ILO14, XIV, 2.3.3]), or if S is universally catenary, equidimensional, and its irreducible components are equicodimensional ([GD67, 0_{IV}, 14.3.3], [ILO14, footnote before Lemme 2.4.2]).

Given an arbitrary MW-homodule \mathcal{M} over S , the Rost-Schmid complex of an S -scheme X with line bundle \mathcal{L} induces a left-augmented bounded complex of \mathbf{Z} -graded abelian groups:

$$(4.3.1.b) \quad 0 \rightarrow A^0(X, \mathcal{M}, \mathcal{L}) \rightarrow C^0(X, \mathcal{M}, \mathcal{L}) \xrightarrow{d^0} C^1(X, \mathcal{M}, \mathcal{L}) \rightarrow \dots$$

This complex is natural with respect to essentially smooth pullbacks (Section 1.4.2).

Lemma 4.3.2. *Consider the above assumptions and notation. Then the functor*

$$(4.3.2.a) \quad \underline{\mathcal{M}}(X) = A^0(X, \mathcal{M}, \mathcal{L}).$$

for $X \in Sm_S$ has a canonical structure of a homotopy premodule over S , which is \mathbf{A}^1 -invariant. Moreover, the functor $\mathcal{M} \mapsto \underline{\mathcal{M}}$ is left exact.

Proof. The fact $\underline{\mathcal{M}}$ really is a presheaf on Sm_S follows from Section 2.2.6. The remaining of the proof is the same as that of [Fel21a, 4.1.7], as in our more general case, we also have the localization exact sequence (1.5.8.a), and the \mathbf{A}^1 -invariance result Section 2.1.4. The left exactness follows from the long exact sequence associated with an exact sequence of MW-hommodules. \square

Example 4.3.3. For these examples, we assume in addition that S admits pinning λ . Then the lemma applies to any MW-module M by considering the associated MW-homodule $\mathcal{D}_\lambda(M)$ as in Section 3.3.4. We let $\underline{M} := \underline{\mathcal{D}_\lambda(M)}$. Our main examples come from Section 3.4.4. We therefore obtain the homotopy premodules over S : \underline{K}_*^{MW} , \underline{K}_*^M , \underline{W} and \underline{I}^* obtained by the applying the previous lemma to the MW-cycle module $M = K_*^{MW}$ over S , and respectively to its η -quotient, η -localization and h -quotient.

Definition 4.3.4. Under the assumptions of the previous lemma, we call $\underline{\mathcal{M}}$ the unramified sheaf associated with the MW-homodule \mathcal{M} . The same definition applies to an M-module, or more generally to an MW-module when we assuming the existence of a pinning on S .

The following theorem is a reinforcement of the fundamental result of [BHP23], the Gersten resolution for Milnor-Witt K-theory of a local essentially smooth scheme over a Dedekind ring. What our theory brings is the construction of the Nisnevich sheaf \underline{M} , and in particular its contravariant functoriality.

Theorem 4.3.5. *Assume that S is a regular base scheme of dimension ≤ 1 , with codimension function $\delta = -\text{codim}_S$ and consider the pinning given by \mathcal{O}_S on S . Let M be an MW-module obtained by e -localization or e -trivialization of K^{MW} .*

Then \underline{M} is a homotopy module over S . Moreover, for any essentially smooth S -scheme X and line bundle \mathcal{L} over X , and any topology $t = \text{Zar}, \text{Nis}$, one has an isomorphism of \mathbf{Z} -graded abelian groups, natural with respect to essentially smooth pullbacks:

$$A^p(X, M, \mathcal{L}^\vee) \simeq H_t^p(X, \underline{M}_X\{\mathcal{L}\})$$

where we have put $\underline{M}_X\{\mathcal{L}\} = \underline{M}|_{X_t} \otimes_{\mathbf{G}_m} \mathcal{L}^\times$, as sheaves over the small site X_t , using the action given by the suspension map (4.3.1.a).

In particular, we obtain Bloch's formula and its extension to Chow-Witt groups, as defined in Section 4.2.3: for any essentially smooth S -scheme X , and any integer p , $t = \text{Zar}, \text{Nis}$,

$$(4.3.5.a) \quad \text{CH}^p(X) \simeq H_t^p(X, \underline{K}_p^M),$$

$$(4.3.5.b) \quad \widetilde{\text{CH}}^p(X) \simeq H_t^p(X, \underline{K}_p^{MW}).$$

Proof. The sequence (4.3.1.b) defines an augmented complex of \mathbf{Z} -graded abelian sheaves on X_t . Moreover $C^*(-, \mathcal{D}_\lambda(M), \mathcal{L})$ is a complex which satisfies the Brown-Gersten property: *i.e.* it is Nisnevich local and therefore flasque (Zariski-local). Therefore, one only needs to prove that (4.3.1.b) is exact whenever X is essentially smooth and local over S . Then the canonical map $X \rightarrow S$ factors through the localization $j : S_{(s)} \rightarrow S$ where s is the image of the closed point of X . As the map j is pro-open, one reduces to the case $S = S_{(s)}$. Then S is an excellent discrete valuation ring and one can apply [BHP23, Th. 1.1] to the MW-module M . Indeed, it is known that K^{MW} satisfy the property (R5) (Def. 2.13) of that paper (Th. 1.4), and one deduces that M does satisfy (R5) as for any field E , $M(E, \mathcal{L})$ is obtained by tensoring with $K^{MW}(E)[e^{-1}]$ or $K^{MW}(E)/(e)$ respectively. \square

4.3.6. Bloch's formula is no longer true if X is singular. For example, let X be the spectrum of a 1-dimensional noetherian domain A . Then by definition, the group $\text{CH}^1(X) = \text{Cl}(A)$ is trivial if and only if A is principal.

In particular, if A is in addition local, $\text{CH}^1(A) = 0$ if and only if A is regular (*i.e.* a discrete valuation ring). So under these assumptions on A , Bloch's formula for CH^1 and the Zariski topology is true if and only if A is regular. In particular, any strict order A in a number ring localized at a prime \mathfrak{p} dividing its conductor is a counter-example to Bloch's formula for the Zariski topology. To get a counter-example for the Nisnevich topology, one can take the henselization $A_{\mathfrak{p}}^h$. As a conclusion, one may highlight the following result which shows that the assumptions of the above theorem are somewhat optimal.

Scholium 4.3.7. Let S be an integral 1-dimensional base scheme, and $t = \text{Zar}, \text{Nis}$. Then the following conditions are equivalent:

- (1) S is regular.
- (2) For every essentially smooth S -scheme X , we have $\text{CH}^1(X) \simeq H_{\text{Zar}}^1(X, \underline{K}_1^M)$.
- (3) For every localization X of S , we have $\text{CH}^1(X) \simeq H_{\text{Zar}}^1(X, \underline{K}_1^M)$.
- (4) For every essentially smooth S -scheme X , we have $\text{CH}^1(X) \simeq H_{\text{Nis}}^1(X, \underline{K}_1^M)$.
- (5) For every Nis-localization X of S , we have $\text{CH}^1(X) \simeq H_{\text{Nis}}^1(X, \underline{K}_1^M)$.

We will see that the same results hold for the Milnor-Witt extension of Bloch's formula (4.3.5.b). Let us now draw some corollaries of Theorem 4.3.5.

Corollary 4.3.8. Consider the assumptions and notation of Theorem 4.3.5. Then the homotopy module \underline{M} defines an object, denoted by $\mathbf{H}\underline{M}_S$, of the (Nisnevich-local) \mathbf{A}^1 -derived category $\text{D}_{\mathbf{A}^1}(S)$. The object $\mathbf{H}\underline{M}_S$ admits a canonical SL -orientation.¹⁹

¹⁹*i.e.* an MSL -module structure.

Let $p : X \rightarrow S$ be a morphism of schemes essentially of finite type, and v be a virtual vector bundle over X of rank r . If X is essentially smooth over S , one has the following canonical isomorphism:

$$H^{n,v}(X, \mathbf{H}\underline{M}_S) := [\Sigma^\infty \mathbf{Z}_S(X), \mathbf{H}\underline{M}_S \otimes \mathrm{Th}_S(v)[n]]_{\mathrm{D}_{\mathbf{A}^1}(S)} \simeq A^{n+r}(X, M_r, \det(v)^\vee),$$

which is natural with respect to contravariant functoriality and Gysin morphisms. Moreover, in any case, one has a canonical isomorphism:

$$H_{n,v}^{BM}(X/S, \mathbf{H}\underline{M}_S) := [\Sigma^\infty \mathrm{Th}_X(v)[n], p^!(\mathbf{H}\underline{M}_S)]_{\mathrm{D}_{\mathbf{A}^1}(S)} \simeq A_{n+r}(X, M_{-r}, \det(v)),$$

which is natural with respect to proper covariance and Gysin maps.

Proof. The first isomorphism, at least with $v = 0$ follows from Theorem 4.3.5. In fact, replacing $\Sigma^\infty X$ in the formula with $\Sigma^\infty \mathbf{Z}(X/X - Z)$, for any closed subscheme $Z \subset X$, one even obtain an isomorphism with the Chow-Witt group of X with support in Z and coefficients in $\mathcal{D}_\lambda(M)$, as defined in Section 2.4.1.. Then the first isomorphism follow from Section 2.4.2. This also shows that $\mathbf{H}\underline{M}_S$ is SL-orientable. To get the second isomorphism, one uses that both $A_*(-, M, *)$ and $H_{**}^{BM}(-/S, \mathbf{H}\underline{M}_S)$ satisfies duality (1.3.7.c) and admit localization exact sequences (1.5.8.a). \square

Example 4.3.9. As a particular case, we obtain the representability of Chow-Witt groups, Chow groups, unramified Witt groups and unramified I^* -cohomology over an S as in the previous theorem, in the stable \mathbf{A}^1 -derived category, and therefore in the motivic stable homotopy category. This was previously known over a field ([Mor12]), and for Witt groups over a Dedekind $\mathbf{Z}[1/2]$ -scheme by [Bac22].

Corollary 4.3.10. *Under the assumptions and notation of Theorem 4.3.5, one has exact sequences of homotopy modules over S :*

$$(4.3.10.a) \quad 0 \rightarrow \underline{I}^*\{1\} \rightarrow \underline{K}_*^{MW} \xrightarrow{F} \underline{K}_*^M \rightarrow 0$$

$$(4.3.10.b) \quad 0 \rightarrow 2\underline{K}_*^M \xrightarrow{H} \underline{K}_*^{MW} \xrightarrow{P} \underline{I}^* \rightarrow 0$$

where F (resp. P) is canonical projection to the cokernel of the endomorphism of \underline{K}_*^{MW} obtained by multiplication with η (resp. h).

Proof. Section 3.4.5 implies that both sequences are left exact and we are left to prove surjectivity of the maps denoted by F and P . This follows from the long exact sequences of Section 3.4.6 and Theorem 4.3.5. \square

By being more precise, one can (partially) extend the previous result.

Proposition 4.3.11. *Let S be a normal base scheme, with $\delta = -\mathrm{codim}_S$.*

(1) *The following canonical map is an isomorphism of Nisnevich sheaves over S :*²⁰

$$(4.3.11.a) \quad \mathbf{G}_m \rightarrow \underline{K}_1^M$$

²⁰Remark that according to Section 3.5.6, one does not need that S admits a pinning for this assertion.

(2) Assume in addition that S admits a pinning. Then the following short sequence of \mathbf{A}^1 -invariant Nisnevich sheaves over Sm_S is exact:

$$(4.3.11.b) \quad 0 \rightarrow \underline{\mathbb{I}}^2 \rightarrow \underline{K}_1^{MW} \xrightarrow{F} \mathbf{G}_m \rightarrow 0$$

where the map F is induced by the projection to the cokernel of $\gamma_\eta : \underline{K}_2^{MW} \rightarrow \underline{K}_1^{MW}$.

Proof. The first point follows as a normal noetherian domain A with fraction field K is the intersection of the discrete valuation rings V such that $A \subset V \subset K$. For the second point, taking into account the first exact sequence of Section 3.4.6, one is left to prove that the map denoted by F in the above proposition is surjective. This can be checked over the spectrum of a local essentially smooth S -ring A , say with fraction field K . By construction, $\underline{K}_1^{MW}(A)$ is the sub-group of $K_1^{MW}(K, \lambda_K)$ made of unramified elements, where λ_K is given by the chosen pinning over S . Let us choose a non-zero element $l \in \lambda_K$. Given a unit $a \in A^\times$, the element $x_a = [a] \otimes l$ in $K_1^{MW}(K, \lambda_K)$ is unramified at every place of A . Hence it defines an element $x_a \in K_1^{MW}(A)$, which obviously satisfies $F(x_a) = a$. \square

Remark 4.3.12. If S is singular, the first assertion is not true. For example if X is the normal crossing divisor in \mathbf{P}^2 given by the union of the two projective lines $x = 0, y = 0$, $\mathbf{G}_m(X) = k^\times$ but $\underline{K}_1^M(X) = k^\times \times k^\times$.

4.3.13. Given an arbitrary locally noetherian normal scheme S , one deduces from [GD67, 21.6.10] that the particular case of Bloch's formula (4.3.5.a):

$$Pic(S) = H_{\text{Nis}}^1(S, \mathbf{G}_m) \simeq CH^1(S)$$

is valid if and only if S is locally factorial. The latter condition is strictly weaker than regular, at least in dimension bigger than 2. The Milnor-Witt case is much more difficult to establish. One can still obtain the following extension of Section 4.3.5, using the Gersten conjecture for the Witt group, due to Balmer and Walter [BW02].

Proposition 4.3.14. *Let S be a regular base $\mathbf{Z}[1/2]$ -scheme with $\delta = -\text{codim}_S$ of dimension less than 4. Then the canonical map*

$$H_{\text{Nis}}^1(S, \underline{K}_1^{MW}) \rightarrow \widetilde{CH}^1(S)$$

is an isomorphism.

One simply uses the previously mentioned results, together with the exact sequence (3.4.5.b).

4.4. The graded Chow-Witt groups of a 1-dimensional scheme.

4.4.1. We first generalize the definition of the abelian group $SK_1(A)$ of a regular ring A to the case of possibly singular schemes X . We assume that X is a scheme such that $\delta = -\text{codim}_X$ is a dimension function. According to Section 4.3.4, this allows us to define the unramified K -theory (resp. Milnor K -theory) sheaf \underline{K}_* (resp. \underline{K}_*^M), associated with the M -homomodule \mathcal{K}_* (resp. the M -module K_*^M); see Section 1.3.6. According to Matsumoto's theorem, these two \mathbf{Z} -graded sheaves coincide in degrees < 3 .

Next, we consider the coniveau spectral sequence of X associated with K' -theory,

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}} K_{-p-q}(\kappa(x)) \implies K'_{-p-q}(X).$$

Since $K_i(\kappa(x)) = 0$ for $i < 0$, this spectral sequence is concentrated in the region $p \geq 0$, $p + q \leq 0$. Its E_2 -page can be identified with

$$E_2^{p,q} \simeq H^p(X, \underline{K}_{-q}).$$

In particular,

$$E_2^{0,-1} = H^0(X, \underline{K}_1) = \underline{K}_1(X) = \underline{K}_1^M(X).$$

The coniveau spectral sequence yields a decreasing filtration $F^\bullet K'_1(X)$ whose associated graded pieces are the groups $E_\infty^{p,q}$ with $p + q = -1$. The corresponding edge morphism of the spectral sequence is a natural homomorphism

$$(4.4.1.a) \quad \phi : K'_1(X) \longrightarrow E_2^{0,-1} = \underline{K}_1(X).$$

If moreover X has dimension ≤ 1 , then $E_r^{p,q} = 0$ for $p \geq 2$, so for $r \geq 2$ there is no non-trivial differential with source or target $E_r^{0,-1}$. Hence $E_2^{0,-1} = E_\infty^{0,-1}$ in this case. The group $E_\infty^{0,-1}$ is the first graded piece of the coniveau filtration on $K'_1(X)$, and it follows that the edge morphism (4.4.1.a) is an epimorphism.

Definition 4.4.2. Consider the preceding assumptions, one defines the group $SK'_1(X)$ as the kernel of the map (4.4.1.a).

Remark 4.4.3. (1) If X is normal and excellent, then $\underline{K}_1(X) \simeq \mathbf{G}_m(X)$, and by Proposition 4.3.11 the map (4.4.1.a) is the usual determinant map $K'_1(X) \rightarrow \mathbf{G}_m(X)$. In particular, when X is regular (so that $K'_1(X) \simeq K_1(X)$), the group $SK'_1(X)$ identifies with the classical group $SK_1(X)$. (2) Let X be a (possibly singular) scheme of dimension 1. One deduces from the localization exact sequence in K' -theory the isomorphism:

$$(4.4.3.a) \quad A_0(X, \underline{K}_1^M) \simeq SK'_1(X).$$

4.4.4. Let us fix a 1-dimensional base scheme X with a pinning λ , and $\delta = \dim$.

We will use the homological conventions of Section 4.1.1, with $X = S$. In particular, according to Section 4.1.2 (resp. Section 4.3.3) for any line bundle \mathcal{L} on X , one defines the Chow-Witt group $\widetilde{CH}_0(X, \mathcal{L})$ of (Milnor-Witt) 0-cycles (resp. unramified first Milnor-Witt first K-group $\underline{K}_1^{MW}(X, \mathcal{L})$) as the cokernel (resp. kernel) of the (quadratic) divisor class map:

$$(4.4.4.a) \quad \bigoplus_{\eta \in X_{(1)}} \underline{K}_1^{MW}(\kappa(\eta), \mathcal{L}_\eta) \xrightarrow{\text{div}_X} \bigoplus_{x \in X_{(0)}} \text{GW}(\kappa(x), \lambda_x \otimes \mathcal{L}_x)$$

Replacing Milnor-Witt K-theory by Milnor K-theory, or in fact Quillen K-theory, one similarly gets the Chow group $CH_0(X)$ of 0-cycles on X (resp. unramified (Milnor) K-theory $\underline{K}_1^M(X)$). Then (4.1.2.a) provides a canonical “forgetful map”

$$(4.4.4.b) \quad F : \widetilde{CH}_0(X, \mathcal{L}) \rightarrow CH_0(X).$$

Example 4.4.5. In general, the map (4.4.4.b) is not always an isomorphism. Here are a few examples:

- (1) If X is the spectrum of a field F , then the map (4.4.4.b) is canonically identified with the rank map $\mathrm{GW}(F, \mathcal{L}) \rightarrow \mathbf{Z}$, which is not always an isomorphism.
- (2) If F is a field and \mathcal{L} is an odd degree line bundle over the projective line \mathbf{P}_F^1 , Fasel ([Fas13]) shows that $\widetilde{\mathrm{CH}}^1(\mathbf{P}_F^1, \mathcal{L}) \simeq \mathrm{GW}(F)$ (see also [Dég25, Th. 3.4.4] for an elementary proof), and the rank map $\mathrm{GW}(F) \rightarrow \mathbf{Z}$ need not be an isomorphism as above.
- (3) Here is an example of a smooth affine curve. We know that the map (4.4.4.b) together with the map P in (4.3.10.b) induce a canonical map

$$(4.4.5.a) \quad \widetilde{\mathrm{CH}}_0(X, \mathcal{L}) \rightarrow \mathrm{CH}_0(X) \oplus H_0^{BM}(X_r, \mathcal{L})$$

where $H_0^{BM}(X_r, \mathcal{L})$ is the Borel-Moore real étale homology ([JX25]), and the map (4.4.5.a) is an isomorphism after inverting 2. If $X = \mathrm{Spec} R$ where $R = \mathbf{R}[X, Y]/(X^2 + Y^2 - 1)$ is a Dedekind domain, then the topological space $X(\mathbf{R})$ is simply the sphere S^1 , and we have

$$H_0^{BM}(X_r, \mathcal{O}_X) \simeq H_0^{BM}(X(\mathbf{R})) \simeq H_0^{BM}(S^1) \simeq \mathbf{Z},$$

so the map (4.4.4.b) is not an isomorphism.

The goal of this section is to give a criterion for the map (4.4.4.b) to be an isomorphism. The following result is a consequence of all the theory we have developed, together with the Milnor conjecture as restated in Section 3.4.5.

Theorem 4.4.6. *Consider the above notation and assume that the following conditions hold:*

- (1) *The residue field $\kappa(x)$ of every closed point $x \in X_{(0)}$ satisfies $I^2(\kappa(x)) = 0$.*
- (2) *The group $\mathrm{SK}'_1(X)$, namely the kernel of the map (4.4.1.a), is 2-divisible.*

Then the map (4.4.4.b) is an isomorphism.

Proof. Under our assumptions and notation, the exact sequence (3.4.6.b) gives rise to the following short exact sequence:

$$A_0(X, I^1, \mathcal{L}) \rightarrow \widetilde{\mathrm{CH}}_0(X, \mathcal{L}) \xrightarrow{F} \mathrm{CH}_0(X) \rightarrow 0.$$

Therefore, it suffices to prove that the abelian group $A_0(X, I^1, \mathcal{L})$ vanishes. Next, the exact sequence (3.4.6.a) gives the following short exact sequence:

$$A_0(X, I^2, \mathcal{L}) \rightarrow A_0(X, I^1, \mathcal{L}) \rightarrow A_0(X, K_1^M/2, \mathcal{L}) \rightarrow 0.$$

where $A_0(X, I^2, \mathcal{L})$ is a sub-quotient of $\bigoplus_{x \in X_{(0)}} I^2(\kappa(x), \lambda_x \otimes \mathcal{L}_x)$. Therefore, assumption (1) implies that $A_0(X, I^2, \mathcal{L}) = 0$, and it follows that

$$(4.4.6.a) \quad A_0(X, I, \mathcal{L}) \simeq A_0(X, K_1^M/2, \mathcal{L}) \simeq A_0(X, K_1^M, \mathcal{L})/2 \stackrel{(4.4.3.a)}{\simeq} \mathrm{SK}'_1(X)/2.$$

Thus assumption (2) allows us to conclude. \square

Example 4.4.7. We now give examples where the assumptions of Theorem 4.4.6 are fulfilled.

(1) The first assumption of Theorem 4.4.6 is concerned with fields k such that

$$I^2(k) = 0.$$

Given the separation of the I -adic filtration of the Witt group of k , this amounts to require that $I^2(k) = I^3(k)$.

If k has characteristic different from 2, the two Milnor conjectures (see [Voe03,], [OVV07, Th. 4.1]) gives an isomorphism $I^2(k)/I^3(k) \simeq H^2(k, \mu_2)$ with étale cohomology. In particular, if k is a field of étale cohomological 2-dimension at most 2, then $I^2(k) = 0$. This is the case if k is finite, or even quasi-finite.

Assume that k is a field of characteristic 2. Then Kato proved in [Kat82, Introduction, Theorem (2)] that $I^2(k)/I^3(k) = \nu(2)_k$ where $\nu(2)_k$ is a sub-group of $\Omega^2(k/k^2)$, the absolute Kähler 2-differential forms of k . One deduces that $I^2(k) = 0$ if k is either perfect, or the function field of a curve over a perfect base field, or a local field with perfect residue field.

(2) The second assumption of Theorem 4.4.6 is satisfied when X is the spectrum of a regular domain R such that $\mathrm{SK}_1(R) = 0$. In particular, this is the case when R is either euclidean, or if $R = \mathcal{O}_{F,S}$ for a number field F ([Mil71, Corollary 16.3, page 159]), or if R is a Dedekind domain of finite type over a finite field ([Bas68, VI Corollary 8.5, page 337]).

In summary, Theorem 4.4.6 applies to the following cases:

Corollary 4.4.8. *Let $X = \mathrm{Spec} R$, where R is either a Dedekind domain of finite type over \mathbf{Z} , or a Euclidean domain such that all the residue fields of its closed points satisfy $I^2(k) = 0$. Then the map $\widetilde{\mathrm{CH}}_0(X, \mathcal{L}) \rightarrow \mathrm{CH}_0(X)$ in (4.4.4.b) is an isomorphism.*

The following lemma is useful to compute SK'_1 in the singular case:

Lemma 4.4.9. *Let X be a 1-dimensional, reduced, Noetherian Nagata scheme. Let us consider the cartesian square:*

$$\begin{array}{ccc} T & \longrightarrow & \widetilde{X} \\ q \downarrow & \Delta & \downarrow p \\ Z & \dashrightarrow & X \end{array}$$

such that Z is the singular locus of X , and \widetilde{X} the normalization of X . Then Z is a finite set of closed points of X , and T/Z is finite.

Moreover, if $\mathrm{SK}_1(\widetilde{X}) = 0$, then there exists an exact sequence:

$$\bigoplus_{y \in T} \kappa(y)^\times \xrightarrow{\sum_{y/x} N_{y/x}} \bigoplus_{x \in Z} \kappa(x)^\times \xrightarrow{i_*} \mathrm{SK}'_1(X) \rightarrow 0$$

where the sum runs over the points $y/x \in T/Z$, and $N_{y/x}$ denotes the norm map of the finite field extension $\kappa(y)/\kappa(x)$.

Proof. As X is Nagata, the normalization map p is finite. Since X is 1-dimensional Noetherian reduced, its singular locus Z has positive codimension. The first claim follows. Assume that $\mathrm{SK}_1(\widetilde{X}) = 0$. According to (4.4.3.a), one gets: $\mathrm{SK}'_1(X) \simeq A_0(X, K_2^M)$, $\mathrm{SK}_1(\widetilde{X}) \simeq A_0(\widetilde{X}, K_2^M)$.

The square Δ is a cdh-square, so it induces a long exact sequence for Chow groups with coefficients in the cycle module K_*^M over X according to Section 1.5.9. The end of this long exact sequence is precisely the desired exact sequence. \square

Example 4.4.10. If A is an order in a number ring \mathcal{O}_F , then one has $\mathrm{SK}'_1(A) = 0$. Indeed, the ideal \mathfrak{f} of Z (resp. T) in $X = \mathrm{Spec}(A)$ (resp. $\tilde{X} = \mathrm{Spec}(\mathcal{O}_F)$) corresponds to the conductor of A in \mathcal{O}_F . One concludes by applying Lemma 4.4.9 to this situation, using the case of \mathcal{O}_F in Section 4.4.7, the surjectivity of q and the fact that the norm map $N_{y/x}$ is surjective for extensions of finite fields.

In contrast, the group $\mathrm{SK}_1(A)$ vanishes if and only if \mathfrak{f} is square free.

Corollary 4.4.11. *Let $A \subset \mathcal{O}_F$ be an order in a number ring, $X = \mathrm{Spec}(A)$. Then the forgetful map (4.4.4.b) is an isomorphism. In particular, for a finite set S of finite places of F , and any invertible $\mathcal{O}_{F,S}$ -module L , we have a canonical isomorphism²¹*

$$\widetilde{\mathrm{CH}}^1(\mathcal{O}_{F,S}, L) \simeq \mathrm{Cl}(\mathcal{O}_{F,S}).$$

Proof. The first (resp. second) condition of Section 4.4.6 is true according to [MH73, Chap. V] (resp. Section 4.4.10). \square

In fact, more original examples follow from Section 4.4.6:

Corollary 4.4.12. *Let $S = \mathrm{Spec}(\mathcal{O}_F)$ be the spectrum of a number ring, $\mathfrak{p} \subset \mathcal{O}_F$ a prime, and $U = S - \{\mathfrak{p}\}$. Consider one of the following situations:*

- (1) *Consider the non-separated (i.e. non-affine in this case) Dedekind scheme $X = S \sqcup_U S$. This is the spectrum of a number ring with the point \mathfrak{p} doubled.*
- (2) *Consider the pinching $X = S \sqcup_{\{\mathfrak{p}\}} S = \mathrm{Spec}(\mathcal{O}_F \times_{\kappa(\mathfrak{p})} \mathcal{O}_F)$ as defined in [Fer03]. This is a normal crossing scheme made of two copies of the Dedekind scheme S glued at the point \mathfrak{p} .*

In each case, the forgetful map (4.4.4.b) is an isomorphism.

Proof. We treat the two cases separately.

First note that in both situations X is a one-dimensional, Noetherian scheme finite over $S = \mathrm{Spec}(\mathcal{O}_F)$. In particular, every closed point $x \in X_{(0)}$ lies over a non-zero prime of \mathcal{O}_F , hence $\kappa(x)$ is a finite field. For a finite field κ the fundamental ideal $I(\kappa) \subset W(\kappa)$ satisfies $I(\kappa)^2 = 0$. This verifies hypothesis (1) of Theorem 4.4.6 for all the schemes under consideration.

Case (1): the non-separated Dedekind scheme $X = S \sqcup_U S$. Let $U = S \setminus \{\mathfrak{p}\}$ and let X be obtained by gluing two copies S_1, S_2 of S along U :

$$X = S_1 \sqcup_U S_2.$$

Then X is a (possibly non-separated) Dedekind scheme: it is regular, one-dimensional, and its local rings are DVRs away from the generic point. All residue fields at closed points are again finite, so hypothesis (1) of Theorem 4.4.6 holds for S and for any open subscheme of S .

²¹Some computations of the Chow group of 0-cycles of orders in a number ring can be found in [KK24].

By Example 4.4.7 and its extension to orders in \mathcal{O}_F , the hypotheses of Theorem 4.4.6 are satisfied on S and on any affine open subscheme $V \subset S$ (viewed as Spec of an order in F), and hence the forgetful map (4.4.4.b) is an isomorphism on S and on every such V . Since the Chow and Chow-Witt groups in question are computed as the homology of Rost-Schmid type complexes, they satisfy Mayer-Vietoris for open coverings. It follows that the map (4.4.4.b) is an isomorphism on any open subscheme of S , in particular on U .

Now cover X by the two open subsets X_1, X_2 corresponding to the two copies of S , with intersection $X_1 \cap X_2 \simeq U$. For this cover there are Mayer-Vietoris exact sequences for the Chow-Witt groups and for the Chow groups, and the forgetful map (4.4.4.b) is compatible with these sequences. We have just seen that (4.4.4.b) is an isomorphism on $X_1 \simeq S$, on $X_2 \simeq S$, and on their intersection $X_1 \cap X_2 \simeq U$. A diagram chase (or the five lemma applied to the Mayer-Vietoris long exact sequences) shows that the induced map (4.4.4.b) on X is also an isomorphism.

Case (2): the pinching $X = S \sqcup_{\{\mathfrak{p}\}} S$. Let X be the Ferrand pinching

$$X = S \sqcup_{\{\mathfrak{p}\}} S = \text{Spec}(\mathcal{O}_F \times_{\kappa(\mathfrak{p})} \mathcal{O}_F).$$

Then X is a reduced, Noetherian, Nagata scheme of dimension 1, with normalization

$$p : \tilde{X} \longrightarrow X, \quad \tilde{X} \simeq S \sqcup S,$$

and singular locus $Z = \{x\}$ equal to the image of the closed point \mathfrak{p} . The fibre $T = Z \times_X \tilde{X}$ consists of two closed points y_1, y_2 with residue fields

$$\kappa(y_1) \simeq \kappa(y_2) \simeq \kappa(x) \simeq \kappa(\mathfrak{p}).$$

By the Bass-Milnor-Serre theorem we have $\text{SK}_1(\mathcal{O}_F) = 0$, hence $\text{SK}'_1(S) = 0$ and therefore

$$\text{SK}'_1(\tilde{X}) = \text{SK}'_1(S) \oplus \text{SK}'_1(S) = 0.$$

Thus the hypothesis $\text{SK}_1(\tilde{X}) = 0$ of Lemma 4.4.9 is satisfied, and that lemma yields an exact sequence

$$\kappa(\mathfrak{p})^\times \oplus \kappa(\mathfrak{p})^\times \xrightarrow{N_{y_1/x} + N_{y_2/x}} \kappa(\mathfrak{p})^\times \longrightarrow \text{SK}'_1(X) \longrightarrow 0,$$

and each norm map $N_{y_i/x}$ is the identity on $\kappa(\mathfrak{p})^\times$. Hence the first arrow is the multiplication map

$$\kappa(\mathfrak{p})^\times \times \kappa(\mathfrak{p})^\times \longrightarrow \kappa(\mathfrak{p})^\times, \quad (a, b) \mapsto ab,$$

which is surjective. It follows that

$$\text{SK}'_1(X) = 0,$$

so $\text{SK}'_1(X)$ is in particular 2-divisible. Together with the verification of (1) above, both hypotheses of Theorem 4.4.6 are satisfied by this X , and the forgetful map (4.4.4.b) is an isomorphism.

This completes the proof in both cases. \square

4.4.13. In fact, in the case of an excellent scheme X of dimension 1, we can extend the previous techniques to get a complete computation of all the \mathbf{Z} -graded Chow-Witt groups with coefficients in the main MW-modules of Section 3.4.4. In practice, the localization long exact sequence allows us to reduce to the case where $X = \operatorname{Spec}(R)$ is the spectrum of a Dedekind domain of finite type over \mathbf{Z} , which we will adopt for the following next results. We start with the case $M = K_*^M$.

Proposition 4.4.14. *Let $X = \operatorname{Spec}(R)$ be the spectrum of a Dedekind domain of finite type²² over \mathbf{Z} , with function field K , and let r be the number of real places of K (so $r = 0$ if $\operatorname{char}(K) > 0$). Then one has:*

$$A_0(X, K_q^M) = \begin{cases} \operatorname{Cl}(R) & q = 0, \\ 0 & \text{otherwise,} \end{cases} \quad A_1(X, K_{q-1}^M) = \begin{cases} \mathbf{Z} & q = 0, \\ R^\times & q = 1, \\ \underline{K}_2^M(R) & q = 2, \\ (\mathbf{Z}/2)^r & q > 2. \end{cases}$$

Proof. The computation stated in the first column follows by the very definition for $q = 0$, Section 4.4.3 for $q = 1$, and the vanishing of $K_q^M(k)$ for k a finite field and $q > 1$. For the second column, the cases $q = 0, 1$ are obvious, $q = 2$ is by definition while the case $q > 2$ follows from [BT73, II, §2, Th. (2.1)]. \square

Remark 4.4.15. Note that the abelian group $\underline{K}_2^M(R)$ appearing in the previous theorem was explicitly computed by Tate in most of the cases. We refer the reader to [BT73, Appendix].

4.4.16. Let us continue with the case of Witt K-theory. We first introduce a new abelian group used in the next proposition. Given an arbitrary 1-dimensional scheme X of finite type over \mathbf{Z} , we consider the map

$$\partial : \bigoplus_{\mathfrak{p}, 2 \nmid \mathfrak{p}} \mathbf{Z}/2 \rightarrow \underline{K}_2^M(X)/2, (\epsilon_{\mathfrak{p}})_{\mathfrak{p}} \mapsto \sum_{\mathfrak{p}, \epsilon_{\mathfrak{p}} \neq 0} \{-\pi_{\mathfrak{p}}, \pi_{\mathfrak{p}}\} \bmod 2$$

and denote by $\underline{k}_2^M(X)$ its cokernel.

Proposition 4.4.17. *Let $X = \operatorname{Spec}(R)$ be the spectrum of a Dedekind domain of finite type over \mathbf{Z} , with function field K , and let r be the number of real places of K . Let \mathcal{L} be a line bundle over $\operatorname{Spec}(R)$, corresponding to a projective R -module L of rank 1. Then one gets:*

$$A_0(X, K_q^W, \mathcal{L}) = \begin{cases} 0 & q > 0, \\ \operatorname{Cl}(R)/2 & q \leq 0, \end{cases} \quad A_1(X, K_{q-1}^W, \mathcal{L}) = \begin{cases} \underline{W}(R, L) & q \leq 0, \\ \underline{I}^q(R, L) & q > 0. \end{cases}$$

Moreover, $\underline{W}(R, L)$ is a sub-ring of $W(K, L_K)$ and one has a chain of ideals

$$(4.4.17.a) \quad \underline{W}(R, L) \supset \underline{I}(R, L) \supset \underline{I}^2(R, L) \supset \dots$$

²²More generally, this should work if X is the spectrum of a Dedekind domain with finite residue fields when $q \leq 2$.

In general, $\underline{I}^q(R, L) \subset I^q(K, L_K)$ as ideals of $W(K, L_K)$. One has exact sequences:

$$(4.4.17.b) \quad 0 \rightarrow \underline{I}(R, L) \rightarrow \underline{W}(R, L) \rightarrow \mathbf{Z}/2 \rightarrow 0$$

$$(4.4.17.c) \quad 0 \rightarrow \underline{I}^2(R, L) \rightarrow \underline{I}(R, L) \rightarrow R^\times/2 \rightarrow 0$$

$$(4.4.17.d) \quad 0 \rightarrow \underline{I}^3(R, L) \rightarrow \underline{I}^2(R, L) \rightarrow \underline{k}_2^M(R) \rightarrow 0.$$

If $q \geq 3$,

$$(4.4.17.e) \quad \underline{I}^q(R, L) \stackrel{(1)}{=} I^q(K, L_K) \stackrel{(2)}{=} [(2^q)W(K, L_K)]^r \simeq \mathbf{Z}^r, r \text{ number of real places of } K.$$

If the field K has characteristic 2, $\underline{I}^q(R, L) = 0$ when $q \geq 2$.

Proof. We consider the first two stated computations. The second column follows from our notation, and the fact $K_q^W(k) = W(k)$ for any field k and any $q \leq 0$. Consider the first column. The case $q > 1$ follows as $I^q(k) = 0$ for a finite field k and an integer $q > 1$. The case $q = 1$ follows from (4.4.6.a). The case $q = 0$ follows from the case $q = 1$ together with the exact sequence (3.4.6.a) for $p = q = 0$. For $q = -1$, the same exact sequence gives:

$$\underline{W}(X, \mathcal{L}) \rightarrow \mathbf{Z}/2 \rightarrow A_0(X, K_0^W, \mathcal{L}) \xrightarrow{\nu} A_0(X, K_{-1}^W, \mathcal{L}) \rightarrow 0$$

which concludes, as $\underline{W}(X, \mathcal{L})$ is not the zero ring. The case $q < -1$ then follows as multiplication by η induces an isomorphism on K_*^W in negative degrees.

Next, the sequence of inclusion (4.4.17.a), and the accompanying assertions, together with the exact sequence (4.4.17.b) all follows from the part of the exact sequence (3.4.6.a) concerning $p = 1$. Similarly, (4.4.17.c) follows from the same exact sequence, the vanishing of $A_0(X, K_2^W, \mathcal{L})$, and the isomorphism $K_1^M/2(R) \simeq \underline{K}_1^M(R)/2$. The latter follows from definitions and an easy application of the snake lemma. The exact sequence (4.4.17.d) finally follows from the definitions of the unramified sheaves \underline{I}^2 , \underline{K}^M and another application of the snake lemma.

Let us consider the computation (4.4.17.e). The identification (1) follows from the vanishing $I^2(k)$ for a finite field k . The identification (2) follows from the separation of I -adic filtration of the Witt ring $W(K, L)$, and the Milnor conjecture proved by Voevodsky:

$$I^q(K, \mathcal{L})/I^{q+1}(K, \mathcal{L}) = H^q(K, \mu_2) = (\mathbf{Z}/2)^r.$$

See [CP84], Theorem I.2.5 and the discussion that follows. Finally, the vanishing when K is of the characteristic 2 case follows from the Milnor conjecture, this time due to Kato, and the vanishing of $H^q(K, \mu_2)$ for $q > 1$. \square

4.4.18. The previous result yields an explicit computation of the unramified Witt group $\underline{W}(R, L)$ in terms of the units of R modulo square, the number of real places of K , and the group $\underline{k}_2^M(R)$. Thanks to the purity theorem of Balmer and Walter, [BW02, Th. 10.1], this immediately yields the same computation for the symmetric Witt group $W(R, L)$ when R is a $\mathbf{Z}[1/2]$ -algebra. In fact, one can obtain this computation without the restriction to characteristic not two.

Theorem 4.4.19 (Purity for symmetric Witt groups). *Let X be a regular scheme of dimension less or equal to one, and \mathcal{L} be a line bundle over X .*

Then the canonical map:

$$W(X, \mathcal{L}) \rightarrow \underline{W}(X, \mathcal{L})$$

where the left hand-side is the \mathcal{L} -twisted (symmetric) Witt group of X and the right hand-side is the unramified Witt group (see Section 4.3.4 applied to the MW-homodule W), is an isomorphism.

In particular, the computations of the preceding proposition can be extended to the (symmetric) Witt group of an arbitrary regular scheme of dimension 1 and of finite type over \mathbf{Z} . One can also avoid the regular assumption by using the coherent symmetric Witt group.

Proof. The following proof is valid in arbitrary characteristic, contrary to [BW02]. According to [CDH⁺20, Ex. 1.3.7], one gets higher symmetric Witt groups $W_n^{[i]}$ — also known as L-groups according to *op. cit.* — for arbitrary schemes together satisfying the localization long exact sequence for regular schemes. Our bigraduation follows the convention of Schlichting's higher (Grothendieck-)Witt groups. Note that with this notation, one gets a periodicity result: $W_n^{[i]} = W_{n+1}^{[i+1]}$.

For a regular scheme X , one deduces from the devissage result of [CDH⁺20, Th. 2.2.4] that the coniveau spectral sequence has the following E_1 -term: $E_1^{p,q} = \bigoplus_{x \in X^{(p)}} W_{-p-q}^{[-p]}(\kappa(x), \omega_x \otimes \mathcal{L}_x)$, and converges to $W_{-p-q}(X, \mathcal{L})$. Remark that, according to the previous periodicity result, one gets: $W_{-p-q}^{[-p]} = W_{-q} = W^{[q]}$. In particular, the above spectral sequence can be identified, in characteristic not 2, with that of [BW02, Th. 7.2]

When X has dimension 1, the spectral sequence degenerate at E_2 . Moreover, according to [CDH⁺20, Cor. 1.3.8], one deduces that $W_{-p-q}^{[-p]}(\kappa(x), \omega_x \otimes \mathcal{L}_x)$ vanishes for q odd. In particular, when X has dimension 1, the only contribution to $W_0^{[0]}(X, \mathcal{L})$ is given by the E_2 -term $E_2^{0,0}$, which is precisely $\underline{W}_0^{[0]}(X, \mathcal{L})$. \square

We end-up with a complete computation of all Chow-Witt groups with coefficients in K_*^{MW} in the 1-dimensional case.

Theorem 4.4.20. *Let $X = \text{Spec}(R)$ be the spectrum of a Dedekind domain of finite type over \mathbf{Z} , with function field K , and let r be the number of real places of K . Let \mathcal{L} be a line bundle over $\text{Spec}(R)$, corresponding to a projective R -module L of rank 1. Then one gets:*

$$A_0(X, K_q^{MW}, \mathcal{L}) = \begin{cases} 0 & q > 0, \\ \text{Cl}(R) & q = 0, \\ \text{Cl}(R)/2 & q < 0, \end{cases} \quad A_1(X, K_q^{MW}, \mathcal{L}) = \begin{cases} \underline{W}(R, L) & q < 0, \\ \underline{GW}(R, L) & q = 0, \\ \underline{K}_1^{MW}(R, L) & q = 1, \\ \mathbf{Z}^r & q > 1. \end{cases}$$

Moreover, one has exact sequences:

$$(4.4.20.a) \quad 0 \rightarrow \underline{I}(R, L) \rightarrow \underline{GW}(R, L) \rightarrow \mathbf{Z} \rightarrow 0$$

$$(4.4.20.b) \quad 0 \rightarrow \underline{I}^2(R, L) \rightarrow \underline{K}_1^{MW}(R, L) \rightarrow R^\times \rightarrow 0$$

Proof. All assertions follow from the computations of the two previous propositions and the exact sequence (3.4.6.b). Note in particular that, for $q > 1$, the existence of the exact sequence:

$$0 \rightarrow \mathbf{Z}^r \rightarrow A_1(X, K_q^{MW}, \mathcal{L}) \rightarrow (\mathbf{Z}/2)^r \rightarrow 0$$

forces the middle group to be \mathbf{Z}^r , as for each real place of K , the image of the corresponding factor on the left column is necessarily 2-divisible. \square

4.5. Finiteness results.

4.5.1. Recall the following conjecture (see [Wei13, IV, 6.8] and [Kah05, Conj. 36]):

Conjecture 4.5.2 (Bass). *For any scheme X of finite type over $\text{Spec } \mathbf{Z}$, and any integer $n \geq 0$, the K' -theory (also called G -theory) group $K'_n(X)$ is finitely generated.*

This conjecture is known if X is of dimension at most 1 (see [Kah05, Prop. 38]).

Inspired by the Bass Conjecture 4.5.2, we introduce the following finiteness conditions on MW-homomorphisms.

Definition 4.5.3. Let S be a base scheme essentially of finite type over \mathbf{Z} , and let R be a ring.

- (1) Let \mathcal{M} be an R -linear MW-homomorphism over S . Given an S -scheme X essentially of finite type, we say that \mathcal{M} is *R -homologically finite relative to X* or satisfies the property (FT_X^R) , if for all pair of integers p, q , and every line bundle \mathcal{L} over X , $A_p(X, \mathcal{M}_q, \mathcal{L})$ is a finitely generated R -module.

We say that \mathcal{M} is *R -homologically finite* or satisfies the property (FT^R) if \mathcal{M} satisfies (FT_X^R) for any S -scheme X of finite type.

- (2) Endow S with the pinning λ associated to $\mathcal{O}_{\text{Spec } \mathbf{Z}}$ (Example 3.1.5), and let M be an R -linear MW-module over S . We say that M satisfies the property (FT_X^R) for an S -scheme X of finite type (resp. satisfies the property (FT^R)) if $\mathcal{D}_\lambda(M)$ does so.

Remark 4.5.4. (1) If one uses induction on the dimension and the localization sequence 1.5.8, property (FT^R) can be reduced to (FT_X^R) for finite type regular affine schemes X over \mathbf{Z} .

- (2) The property $(\text{FT}^{\mathbf{Z}})$ implies (FT^R) for any noetherian ring R , by the Universal Coefficient Theorem.
- (3) For the classical MW-(ho)modules (namely the ones enlisted in Conjecture 4.5.5 below), one should not expect the homological finiteness to hold over general schemes. For example, for an infinite field k one has $A_0(k, K_1^M) = k^*$ which is not a finitely generated abelian group. Instead, this property incarnates an arithmetic finiteness property, which is expected to hold for arithmetic schemes such as number rings or finite fields, and in

practice one should restrict to schemes of finite type over $\text{Spec } \mathbf{Z}$. However, it is useful to have some flexibility, as in Definition 4.5.3.

- (4) The properties (FT_X^R) and (FT^R) do not depend on the underlying choice of the dimension function δ on S . In the case of MW-modules, these properties are independent of the choice of pinning, if one only considers pinnings associated to dualizing complexes (Example 3.1.5).

We conjecture the following arithmetic finiteness property for some usual MW-modules:

Conjecture 4.5.5. *The MW-modules K_*^{MW} , K_*^M , I^* , \mathcal{GW}_* , \mathcal{W}_* and \mathcal{K}_* over $\text{Spec } \mathbf{Z}$ (see Examples 1.3.6, 3.3.6, 3.5.6 and 4.3.3) satisfy the property $(\text{FT}^{\mathbf{Z}})$.*

Remark 4.5.6. (1) Note that the MW-module \mathcal{K}_* is only partially related to K_*^M : the associated Milnor cycle modules over \mathbf{Z} coincide only up to degree 3. Nonetheless, this coincidence is sufficient to ensure that both cycle modules define the *same* Chow groups. In particular, the property $(\text{FT}^{\mathbf{Z}})$ for any of the two would imply that Chow groups of finite type \mathbf{Z} -schemes are finitely generated abelian groups.

- (2) The MW-homodule \mathcal{GW}_* satisfies (FT_X^R) if both MW-hommodules \mathcal{K}_* and \mathcal{W}_* satisfy (FT_X^R) . By the short exact sequence (3.4.5.b), the MW-module K_*^{MW} satisfies (FT_X^R) if both MW-modules K_*^M and I^* satisfy (FT_X^R) .
- (3) Since $\mathcal{W}_*[1/2] \simeq I^*[1/2]$, the property $(\text{FT}^{\mathbf{Z}[1/2]})$ for these two MW-modules are equivalent after inverting 2. We will actually prove this property later in Proposition 4.5.14.

4.5.7. We now provide some evidence for Conjecture 4.5.5. We start by explaining how the case of the MW-homodule \mathcal{K}_* associated to Quillen K-theory is related to the Bass Conjecture 4.5.2:

Lemma 4.5.8. *Let X be a scheme of finite type over $\text{Spec } \mathbf{Z}$. Consider the following conditions*

- (i) *For any integer n , $K'_n(X)$ is finitely generated.*
- (ii) *The MW-homodule \mathcal{K}_* satisfies the property $(\text{FT}_X^{\mathbf{Z}})$.*

Then (ii) implies (i). When X has dimension 1, then (i) implies (ii).

Proof. This follows from the δ -niveau spectral sequence for K-theory, as recalled in Section 1.3.6: the identification of the E_1 -term with the complex computing Chow(-Witt) groups of X with coefficients in \mathcal{K}_* implies the following computation of the E_2 -term:

$$E_{p,q}^{2,\delta} = A_p(X, \mathcal{K}_q).$$

Both assertions follow from the fact the δ -niveau spectral sequence is concentrated in the region $p \in [\delta_-(X), \delta_+(X)]$. \square

In particular, by Lemma 4.5.8, the property $(\text{FT}^{\mathbf{Z}})$ for Quillen K-theory \mathcal{K}_* implies the Bass conjecture in general. Conversely, in the 1-dimensional case, one deduces from known cases of the Bass conjecture the following corollary.

Corollary 4.5.9. *Let X be a scheme of finite type over $\text{Spec } \mathbf{Z}$ of dimension at most 1. For example, X can be the spectrum of an order in a number ring, or a curve over a finite field. Then the MW-module \mathcal{K}_* satisfies the property $(\text{FT}_X^{\mathbf{Z}})$.*

Remark 4.5.10. Lemma 4.5.8 can be extended to any of the MW-hommodules of Section 1.3.6, provided one considers the whole family of associated MW-hommodules. The case of Quillen K-theory is simpler: indeed, Bott periodicity implies that the twisted Borel-Moore theory associated to K-theory is single-graded instead of being bigraded, and therefore there is a single associated MW-hommodule up to shifts of degree.

Remark 4.5.11. Compared to the Bass conjecture 4.5.2, there are reasons to believe in the a priori stronger property $(FT^{\mathbb{Z}})$ for Quillen K-theory \mathcal{K}_* . For example, let X be a scheme of finite type over $\text{Spec } \mathbb{Z}$ of dimension 2. While not directly implied by the Bass conjecture, all the Chow groups $\text{CH}_i(X) = A_i(X, \mathcal{K}_i)$ are known to be finitely generated for $i = 0, 1, 2$. Indeed, by the localization sequence 1.5.8 (or [Ful98, Proposition 1.8]) and noetherian induction, we may assume X is in addition regular. Then we can use the following known results:

- $\text{CH}_2(X) = \mathbb{Z}^n$, where n is the number of irreducible components of X of dimension 2;
- $\text{CH}_1(X) = \text{Pic}(X)$ is finitely generated, by a theorem of Samuel (see [Kah06]);
- $\text{CH}_0(X)$ is finitely generated, by a theorem of Kato and Saito (see [KS83]).

As a further evidence for Conjecture 4.5.5, we use the computations of the previous section to show that it is true in dimension at most 1:

Theorem 4.5.12. *The MW-modules \mathcal{K}_*^{MW} , \mathcal{K}_*^M , \mathcal{I}^* , \mathcal{GW}_* , \mathcal{W}_* and \mathcal{K}_* over $\text{Spec } \mathbb{Z}$ are all homologically finite in dimension less or equal to 1. In other words, they satisfy the property $(FT_X^{\mathbb{Z}})$ for any scheme X of finite type over \mathbb{Z} of dimension at most 1.*

Proof. By the localization sequence 1.5.8 and noetherian induction, we are reduced to the case where X is regular and affine. The case of \mathcal{K}_*^M (resp. \mathcal{K}_*^W , \mathcal{W} , \mathcal{K}_*^{MW}) follows from the computation of Proposition 4.4.14 (resp. Proposition 4.4.17; Theorem 4.4.20) and the finiteness of the class group of R , Section 4.4.7(2) for $\text{SK}_1(R)$, and [BT73, II, §1, Cor. (1.2)]. The case of \mathcal{K}_* is Corollary 4.5.9, which implies the case of \mathcal{GW}_* . \square

4.5.13. We end this section by proving the homological finiteness for $\mathcal{W}_{*, \mathbb{Z}[1/2]} \simeq \mathcal{I}_{\mathbb{Z}[1/2]}^*$:

Proposition 4.5.14. *Let X be a scheme of finite type over $\text{Spec } \mathbb{Z}$ and let \mathcal{L} be a line bundle over X .*

Then for any integers n and p , the group $A_p(X, \mathcal{I}^\{n\}, \mathcal{L})_{\mathbb{Z}[1/2]}$ is a finitely generated $\mathbb{Z}[1/2]$ -module. In other words, the MW-module \mathcal{I}^* satisfies $(FT^{\mathbb{Z}[1/2]})$.*

Proof. By the localization sequence (1.5.8.a) and noetherian induction, we may assume X quasi-projective over $\text{Spec } \mathbb{Z}$, and that \mathcal{L} is trivial. Denote by $f : X \rightarrow \text{Spec } \mathbb{Z}$ the structural morphism. After inverting 2, the group $A_p(X, \mathcal{I}^*\{n\})_{\mathbb{Z}[1/2]}$ agrees with the Borel-Moore real étale homology $H_n(X_r, f^! \mathcal{O}_{\mathbb{Z}})$ ([Jac17], [JX25]). Since X is quasi-projective over $\text{Spec } \mathbb{Z}$, we are reduced to show that for any scheme Y smooth over $\text{Spec } \mathbb{Z}$ and any line bundle M over Y , the real étale cohomology group $H^n(Y_r, \mathbb{Z}(M))$ is finitely generated, which is proven in [Jin22, Corollary 2.11]. \square

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