

A MIXED LOCAL-NONLOCAL HÉNON PROBLEM IN \mathbb{R}^N

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ABSTRACT. In this article, we study a Hénon-type equation in \mathbb{R}^N driven by a nonlinear operator given by the combination of a local and a nonlocal term. This equation was originally proposed to model spherically symmetric stellar clusters. Here, we prove that, under a suitable relation among the parameters, there exists a threshold separating the existence and non-existence of solutions. Moreover, we establish regularity properties of the solutions.

1. INTRODUCTION

The classical Hénon problem, introduced in [13], consists of the following problem

$$(1.1) \quad -\Delta u = |x|^\alpha u^{q-1} \text{ in } B \subset \mathbb{R}^N, \quad u > 0,$$

with Dirichlet boundary conditions, where B denotes the unit ball in \mathbb{R}^N . The motivation in Hénon's original paper [13] was a model of stationary solutions of the nonlinear Poisson equation describing cluster density under gravitational interaction with inhomogeneous distribution. In this direction, the seminal work of Ni [18] established that the weight $|x|^\alpha$ allows for a wider admissible range of exponents $2 < q < 2_\alpha^* := \frac{2(N+\alpha)}{N-2}$ for which solutions exist. This problem started a huge research on elliptic equations with nonlinear right-hand sides. The literature on the subject is now vast, and providing an exhaustive list of contributions lies beyond the scope of this work. In what follows, we restrict ourselves to highlighting a selection of results that extend the Hénon equation to nonlocal operators or in unbounded domains.

A generalization of (1.1), involving a mixed local–nonlocal operator, was introduced in [21], where existence (and nonexistence) results for positive solutions of

$$\gamma(-\Delta)_p u + (1-\gamma)(-\Delta)_p^s u = |x|^\alpha u^{q-1} \text{ in } B \subset \mathbb{R}^N, \quad u > 0,$$

under exterior Dirichlet boundary conditions were established for an appropriate range of parameters $p < N$ and $p < q < p_\alpha^* := \frac{p(N+\alpha)}{N-sp}$. Here, for $p > 1$, for a function u smooth enough, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} u)$ denotes the usual p -Laplacian of u , and given $s \in (0, 1)$, the fractional p -Laplacian of u is defined, up to a normalization constant, as

$$(-\Delta_p)^s u(x) := \operatorname{p.v.} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} dy.$$

Further generalizations of (1.1) to the fractional Laplacian have been obtained in the unbounded setting, where the ball is replaced by the whole space \mathbb{R}^N . In [2] it was shown that, in contrast

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to the case of a bounded domain, the equation

$$(-\Delta)^s u = |x|^\alpha u^{q-1} \text{ in } \mathbb{R}^N, \quad u > 0,$$

admits no positive solutions in the range $1 < q < 2_\alpha^* := \frac{2(N+\alpha)}{N-2s}$, where $\alpha > -2s$, due to the loss of mass at infinity. Similar nonexistence results for the spectral fractional Laplacian were obtained in [24].

In order to obtain existence of solutions in \mathbb{R}^N , a confining potential can be added to prevent the loss of mass at infinity. In the linear local case, [23] studied the Hénon-type equation

$$(1.2) \quad -\Delta u + |x|^\beta u^{p-1} = |x|^\alpha u^{q-1} \text{ in } \mathbb{R}^N, \quad u > 0,$$

and proved the existence of a positive radial solution for $N \geq 3$ and $p > 1$, under the assumptions $\max\{2, p\} < q < p_\alpha^* := \frac{p(N+\alpha)}{N-2}$ and $\alpha - \beta < \frac{\beta+2(N-1)}{p+2}(q-p)$.

A nonlocal version of (1.2) was studied in [16] for the fractional Laplacian with $\beta = 2$, namely,

$$(1.3) \quad (-\Delta)^s u + |x|^2 u = |x|^\alpha u^{q-1} \text{ in } \mathbb{R}^N, \quad u > 0.$$

In that work, it was shown that for $\frac{1}{2} < s < \frac{N}{2}$ and $q > 1$, under the condition $-N + (\frac{N}{2} - s)q < \alpha < 2 + (q-2)(\frac{N}{2} - s)$, there exists a positive radially symmetric solution to (1.2) that decays to zero at infinity. We note, however, that the technique used in [16] requires $\beta = 2$, and moreover, the resulting range for α does not coincide with the corresponding formula in the local case.

A similar result was proved in [6, Corollary 5.5], where existence of a radial weak solution of (1.3) is guaranteed when $\frac{N}{2} < s$, $q < \frac{2(N+\alpha)}{N-2s}$ and $\alpha + \frac{q(1-2s)-2}{2s} < \frac{s}{2}(q-2)(N-1)$. Moreover, the author in [6] gives a condition for the existence of solution when the right hand side $|x|^2 u$ is replaced by $|x|^\beta u^{p-1}$, namely, $\beta(q-2-2sq) + \alpha(2sp-p+2) < 2s(p-q)(N-1)$ for $\max\{2, p\} < q < \frac{2(N+\alpha)}{N-2s}$.

The goal of this article is to extend existence results for Hénon-type equations in \mathbb{R}^N to operators exhibiting both local and nonlocal behavior, namely,

$$(1.4) \quad \begin{cases} \mathcal{L}_{s,p,\gamma} u + |x|^\beta |u|^{p-2} u = |x|^\alpha |u|^{q-2} u & \text{in } \mathbb{R}^N, \\ u > 0, \end{cases}$$

where u belongs to a suitable functional space of radial functions $\mathcal{W}_\beta^{s,p}(\mathbb{R}^N)$ defined in Section 2 (see (2.1)). For $s \in (0, 1]$ and $\gamma \in [0, 1]$, for a function u smooth enough, the operator $\mathcal{L}_{s,p,\gamma}$ is given by

$$\mathcal{L}_{s,p,\gamma} u := \gamma(-\Delta)_p u + (1-\gamma)(-\Delta)_p^s u.$$

This operator is a mixed local–nonlocal operator, reducing to the classical p –Laplacian when $\gamma = 1$, and to the fractional p –Laplacian when $\gamma = 0$. In the linear case (that is, $p = 2$) it admits a probabilistic interpretation: the local component corresponds to the continuous part of the dynamics, while the nonlocal component models jump behavior.

The existence and qualitative properties of solutions for mixed local–nonlocal operators with nonlinear terms is an active area of research. Related results on bounded domains can be found, for instance, in [21, 4, 5, 14, 17, 12, 25].

Our primary goal is to study the existence of weak solutions to (1.4), defined variationally via the corresponding energy functional.

$$(1.5) \quad \mathcal{J}(u) = \frac{\gamma}{p} \|\nabla u\|_p^p + \frac{1-\gamma}{p} [u]_{s,p}^p + \frac{1}{p} \|u\|_{p,\beta}^p - \frac{1}{q} \|u\|_{q,\alpha}^q$$

defined on a suitable space $\mathcal{W}_\beta^{s,p}(\mathbb{R}^N)$, where $\|u\|_{\beta,p}^p$ denotes the integral $\int_{\mathbb{R}^N} |x|^\beta |u|^p dx$, and $[u]_{s,p}$ is the so-called Gagliardo seminorm (see Section 2 for precise definitions). Indeed, \mathcal{J} is a C^1 functional whose Frechét derivative is given by

$$\begin{aligned} \langle \mathcal{J}'(u), v \rangle &= \gamma \int_{\mathbb{R}^N} \Phi_p(\nabla u) \cdot \nabla v dx + (1-\gamma) \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\Phi_p(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+sp}} dxdy \\ &\quad + \int_{\mathbb{R}^N} |x|^\beta \Phi_p(u)v dx - \int_{\mathbb{R}^N} |x|^\alpha \Phi_q(u)v dx \quad \forall u, v \in \mathcal{W}_\beta^{s,p}(\mathbb{R}^N), \end{aligned}$$

where $\Phi_p(t) := |t|^{p-2}t$. Therefore, any critical point of \mathcal{J} is a weak solution of (1.4).

Our first result establishes a range of parameters for which a weak solution of (1.4) exists. For further reference, we let for $\gamma \in [0, 1]$,

$$p_{\gamma,s,\alpha}^* := \begin{cases} \frac{p(N+\alpha)}{N-sp}, & \text{if } \gamma = 0, \\ \frac{p(N+\alpha)}{N-p}, & \text{if } \gamma \in (0, 1]. \end{cases}$$

Precisely, we have:

Theorem 1.1. *Let $s \in (0, 1]$ such that $1/p < s < N/p$, $\alpha > -sp$ and $p < q < p_{\gamma,s,\alpha}^*$. Moreover, assume that*

$$(1.6) \quad \alpha - \beta + (q-p) \left(\frac{1-N}{p} \right) < 0.$$

Then, problem (1.4) admits a radial weak solution.

The proof of Theorem 1.1 relies mainly on establishing a suitable compact embedding of the fractional Sobolev space of weighted radial functions into an appropriate weighted Lebesgue space (see Theorem 3.7). This compactness result is the key ingredient that enables the application of a Mountain Pass–type existence theorem (see Lemma 6.1).

We note that our result extends the existence obtained in [6, 23, 16] to the setting of local–nonlocal nonlinear operators on unbounded domains. Furthermore, the range of parameters given in (1.6) improves upon that obtained in [16] in the particular case $p = 2$ and $\beta = 2$.

Regarding regularity, in the following result, by employing a De Giorgi’s iteration scheme, we find that radial weak solutions of (1.4) are indeed bounded.

Theorem 1.2. *Let $s \in (0, 1]$ be such that $1/p < s < N/p$, $\alpha > -sp$ and $p < q < p_{\gamma,s,\alpha}^*$. Moreover, assume condition (1.6). Then, for any radial weak solution u to (1.4), there is a constant $C > 0$ such that*

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leq C.$$

Finally, by applying a generalization of Pohozaev’s existence result established in [20], we show that the value $\frac{p(N+\alpha)}{N-sp}$ serves as a threshold for the existence of bounded solutions with a weighted control of the gradient.

Theorem 1.3. *Let $s \in (0, 1]$ be such that $1/p < s < N/p$, $\alpha > -sp$ and $\beta > p(1-s)$. Then, for all $q > p_{\gamma,s,\alpha}^*$, problem (1.4) has no solutions $u \in \tilde{\mathcal{X}}(\mathbb{R}^N)$, being*

$$\tilde{\mathcal{X}}(\mathbb{R}^N) = \{u \in \mathcal{W}_\beta^{s,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \text{ such that } |\nabla u(x)| |x| \in L^r(\mathbb{R}^N) \text{ for some } r > 1\}.$$

The paper is organized as follows. In Section 2 we provide the functional framework and preliminary definitions. In the next Section 3, we quote the main properties of radial functions related to our problem, specially, Strauss' type lemmas, generalized Rother's Lemma and compactness of embeddings between fractional Sobolev spaces of radial functions and appropriate Lebesgue weighted spaces. In Section 4, we state the existence of solutions to problem (1.4) through variational methods, and in Section 5 we prove the boundedness of radial solutions. Finally, Section 6 is devoted to prove our non-existence result. For convenience of the reader, we add an Appendix with the Brezis-Nirenberg version of the Mountain Pass Theorem and some monotonicity properties of operators that we employ in the paper.

2. PRELIMINARIES

2.1. Functional framework. Given $p > 1$, $\beta > 0$ and $s \in (0, 1]$ we consider the weighted fractional Sobolev space

$$W_\beta^{s,p}(\mathbb{R}^N) = \{u \in L_\beta^p(\mathbb{R}^N) : [u]_{s,p} < \infty\}$$

endowed with the norm

$$\|u\|_{s,p,\beta} := \|u\|_{p,\beta} + [u]_{s,p},$$

where $\|u\|_p := (\int_{\mathbb{R}^N} |u|^p dx)^{1/p}$ and the seminorm is given by

$$[u]_{s,p} = \begin{cases} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{sp+N}} dx dy \right)^{1/p} & \text{if } s \in (0, 1) \\ \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^{1/p} & \text{if } s = 1. \end{cases}$$

Here, the following weighted Lebesgue spaces are used

$$L_\beta^p(\mathbb{R}^N) := \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} : \int_{\mathbb{R}^N} |x|^\beta |u|^p dx < \infty \right\}$$

with the corresponding norm

$$\|u\|_{p,\beta} = \left(\int_{\mathbb{R}^N} |x|^\beta |u|^p dx \right)^{1/p}.$$

When $\beta = 0$ we just write $\|u\|_{s,p}$ and $\|u\|_p$ instead of $\|u\|_{s,p,0}$ and $\|u\|_{p,0}$, respectively. Moreover, we write $W^{s,p}(\mathbb{R}^N)$ in place of $W_0^{s,p}(\mathbb{R}^N)$.

The corresponding subset of radial functions is defined as

$$W_{rad,\beta}^{s,p}(\mathbb{R}^N) = \{u \in W_\beta^{s,p}(\mathbb{R}^N) : u \text{ is radial}\}.$$

With the aforementioned definitions, the natural framework for introducing weak solutions of (1.4) is the space

$$(2.1) \quad \mathcal{W}_\beta^{s,p}(\mathbb{R}^N) = \begin{cases} W_{rad,\beta}^{s,p}(\mathbb{R}^N) & \text{if } \gamma = 0 \\ W_{rad,\beta}^{1,p}(\mathbb{R}^N) & \text{if } \gamma \in (0, 1]. \end{cases}$$

Once the functional spaces have been introduced, the relation between the parameter $\gamma \in [0, 1]$ in the Hénon equation (1.4) and the fractional parameter $s \in (0, 1]$ becomes clear: we have $s = 1$ whenever $\gamma \in [0, 1)$, whereas $s \in (0, 1)$ when $\gamma = 0$.

2.2. The operator $\mathcal{L}_{s,p,\gamma}$. Given $\gamma \in [0, 1]$, $s \in (0, 1)$ and $1 < p < q$, we define the mixed local-nonlocal operator

$$\mathcal{L}_{s,p,\gamma} u := \gamma(-\Delta)_p u + (1 - \gamma)(-\Delta)_p^s u$$

for all $u \in \mathcal{W}_\beta^{s,p}(\mathbb{R}^N)$. Here, we denote the fractional p -Laplacian of order $s \in (0, 1]$ as

$$(-\Delta_p)^s u(x) := \begin{cases} \text{p.v.} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} dy & \text{if } s \in (0, 1) \\ -\operatorname{div}(|\nabla u(x)|^{p-2}\nabla u(x)) & \text{if } s = 1. \end{cases}$$

Thus, $(-\Delta_p)^1 u$ stands for $-\Delta_p u$.

The operator $\mathcal{L}_{s,p,\gamma}$ is well defined between $W^{s,p}(\mathbb{R}^N)$ and its dual space, and the following representation formula holds: given $u \in W^{s,p}(\mathbb{R}^N)$

$$\langle \mathcal{L}_{s,p,\gamma} u, v \rangle = \gamma \int_{\mathbb{R}^N} \Phi_p(\nabla u) \cdot \nabla v \, dx + (1 - \gamma) \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\Phi_p(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp}} \, dxdy$$

for all $v \in W^{s,p}(\mathbb{R}^N)$, being $\Phi_p(t) := |t|^{p-2}t$ for $t \in \mathbb{R}$. Therefore, we say that $u \in \mathcal{W}_\beta^{s,p}(\mathbb{R}^N)$ is a *weak solution* of (1.4) if

$$\langle \mathcal{L}_{s,p,\gamma} u, v \rangle + \int_{\mathbb{R}^N} |x|^\beta \Phi_p(u)v \, dx = \int_{\mathbb{R}^N} |x|^\alpha \Phi_q(u)v \, dx \quad \forall v \in \mathcal{W}_\beta^{s,p}(\mathbb{R}^N).$$

Observe that weak solutions are critical points of the functional

$$\mathcal{J}(u) := \frac{\gamma}{p} \|\nabla u\|_p^p + \frac{1 - \gamma}{p} [u]_{s,p} + \frac{1}{p} \|u\|_{p,\beta} - \frac{1}{q} \|u\|_{q,\alpha}$$

defined on $\mathcal{W}_\beta^{s,p}(\mathbb{R}^N)$, whose Frechét derivative is given by

$$\langle \mathcal{J}'(u), v \rangle = \langle \mathcal{L}u, v \rangle + \int_{\mathbb{R}^N} |x|^\beta \Phi_p(u)v \, dx - \int_{\mathbb{R}^N} |x|^\alpha \Phi_q(u)v \, dx \quad \forall v \in \mathcal{W}_\beta^{s,p}(\mathbb{R}^N).$$

3. BEHAVIOR OF FUNCTIONS IN $W_{rad,\beta}^{s,p}(\mathbb{R}^N)$

The Strauss Radial Lemma is a powerful tool that quantifies the decay of radial functions at infinity. The following version is derived from Theorems 10 and 13 in [22].

Lemma 3.1 (Strauss' Lemma). *Assume $1/p < s \leq 1$ and $sp < N$. Then, there is a constant $C > 0$ such that for any $u \in W_{rad}^{s,p}(\mathbb{R}^N)$,*

$$(3.1) \quad |u(x)| \leq C|x|^{(1-N)/p}\|u\|_{s,p},$$

for all $x \in \mathbb{R}^N$.

Remark 3.2. We observe that in the previous lemma, we may replace the norm $\|u\|_{s,p}$ with the norm $\|u\|_{s,p,\beta}$. Indeed, for $u \in W_\beta^{s,p}(\mathbb{R}^N)$ with compact support,

$$\begin{aligned} \|u\|_p^p &= \int_{|x|<1} |u|^p dx + \int_{|x|\geq 1} |u|^p dx \\ (3.2) \quad &\leq \left(\int_{|x|<1} |u|^{p^*} \right)^{p/p^*} (|B|)^{N/s} + \int_{|x|\geq 1} |x|^\beta |u|^p dx \\ &\leq C(N, s) \|u\|_{L^{p^*}(\mathbb{R}^N)}^p + \||x|^\beta u\|_p^p \\ &\leq C(N, s, p) [u]_{s,p}^p + \||x|^\beta u\|_p^p \leq C \|u\|_{s,p,\beta}^p, \end{aligned}$$

where we have used the inequality ([8, Theorem 6.5])

$$\|u\|_{L^{p^*}(\mathbb{R}^N)}^p \leq C(N, s, p) [u]_{s,p}^p.$$

By density, this proves that $\|u\|_{s,p} \leq C \|u\|_{s,p,\beta}$ in $W_\beta^{s,p}(\mathbb{R}^N)$ and the continuity of the embedding $W_\beta^{s,p}(\mathbb{R}^N) \subset W^{s,p}(\mathbb{R}^N)$.

Remark 3.3. Recall that (see for instance [9, Lemma 4.2]), for $u \in W^{1,p}(\mathbb{R}^N)$, $1 \leq p < \infty$ and $s \in (0, 1)$, it holds that

$$(3.3) \quad [u]_{s,p}^p \leq \frac{N\omega_N}{p} \left(\frac{2^p}{s} \|u\|_p^p + \frac{1}{1-s} \|\nabla u\|_p^p \right).$$

Then, by (3.3) and Remark 3.2,

$$(3.4) \quad [u]_{s,p}^p \leq C(\|u\|_p^p + \|\nabla u\|_p^p) \leq C(\|u\|_{1,p,\beta}^p + \|\nabla u\|_p^p) \leq C \|u\|_{1,p,\beta}^p.$$

We quote the following continuity embedding from [7, Theorem 6.2]:

Lemma 3.4. *Let $0 < s < N/p$, $c > -N$, $(1-sp)c \leq (N-1)ps$ and let $q = \frac{p(N+c)}{N-sp}$. Then, there is $C > 0$ such that*

$$(3.5) \quad \left(\int_{\mathbb{R}^N} |x|^c |u|^q dx \right)^{1/q} \leq C \|u\|_{H^{s,p}},$$

for all radial $u \in H^{s,p}(\mathbb{R}^N)$.

Here $\|u\|_{H^{s,p}}$ denotes the norm of u in the Bessel potential space $H^{s,p}(\mathbb{R}^N)$ (see, for instance, [1] for a precise definition). We next show that we may replace $\|u\|_{H^{s,p}}$ in the previous lemma by the norm $\|u\|_{s,p}$.

Lemma 3.5. *Let $s \in (0, 1]$ such that $0 < s < N/p$. Let $\alpha \in \mathbb{R}$ be such that $-sp < \alpha$, $(1-sp)\alpha \leq (N-1)sp$, and $p < q < \frac{p(N+\alpha)}{N-sp}$. Then, there are $C > 0$ and $c = (q, s)$ with $-sp < c < \alpha$, such that*

$$(3.6) \quad \left(\int_{\mathbb{R}^N} |x|^c |u|^q dx \right)^{1/q} \leq C \|u\|_{s,p},$$

for all $u \in W_{rad}^{s,p}(\mathbb{R}^N)$.

Proof. Let $s \in (0, 1]$ such that $0 < s < N/p$, and consider α, q such that $-sp < \alpha \leq \frac{(N-1)sp}{1-sp}$ and $p < q < \frac{p(N+\alpha)}{N-sp}$. Let $u \in W_{rad}^{s,p}(\mathbb{R}^N)$. We choose $0 < s' < s$ closed enough to s so that it satisfies the following conditions. First, s' satisfies the relation

$$q < \frac{p(N+\alpha)}{N - s'p},$$

which is true since the function

$$g(r, t) := \frac{p(N+r)}{N - tp}$$

is increasing in r and t , and s' is close to s . Moreover, letting

$$\delta := sp - s'p > 0,$$

we require s' close enough to s so that

$$(3.7) \quad \left(\frac{1 - s'p}{1 - s'p - \delta} \right) (Ns'p - s'p + \delta N) < (N - 1)s'p.$$

Now, by taking s' even closer to s , there is $c = c(q, s) < \alpha$ such that

$$(3.8) \quad q = \frac{p(N+c)}{N - s'p}.$$

In order to apply Lemma 3.4, we need to check that

$$(3.9) \quad (1 - s'p)c \leq (N - 1)s'p.$$

Observe that from (3.8), we have

$$c = \frac{q(N - s'p)}{p} - N,$$

and so recalling the assumptions on q and α , we have

$$\begin{aligned}
(1 - s'p)c &= (1 - s'p) \left(\frac{q(N - s'p)}{p} - N \right) \\
&< (1 - s'p) \left(\frac{(N - s'p)}{p} \cdot \frac{p(N + \alpha)}{N - sp} - N \right) \\
(3.10) \quad &\leq (1 - s'p) \left(\frac{N - s'p}{N - sp} \left(N + \frac{(N - 1)sp}{1 - sp} \right) - N \right) \\
&= \frac{1 - s'p}{1 - sp}(Nsp - s'p) \\
&= \frac{1 - s'p}{1 - s'p - \delta}(Ns'p - s'p + \delta N) < (N - 1)s'p,
\end{aligned}$$

where we have used the assumption on δ given in (3.7). This proves that c satisfies (3.9).

Then, by Lemma 3.4, it follows that

$$(3.11) \quad \left(\int_{\mathbb{R}^N} |x|^c |v|^q dx \right)^{1/q} \leq C \|v\|_{H^{s',p}},$$

for all radial $v \in H^{s',p}(\mathbb{R}^N)$. Recall that $H^{s',p}(\mathbb{R}^N)$ denotes the Bessel potential space. Since $s' < s$, by Theorem 11.1 in [1], the following inclusion holds:

$$W^{s,p}(\mathbb{R}^N) \subset H^{s',p}(\mathbb{R}^N) \quad \text{with continuity.}$$

Thus, from (3.11), we conclude (3.6). This ends the proof. \blacksquare

Next, we prove a continuity embedding between the spaces $W_{rad,\beta}^{s,p}(\mathbb{R}^N)$ and $L_\alpha^q(\mathbb{R}^N)$, for any $p < q < \frac{p(N+\alpha)}{N-sp}$.

Lemma 3.6. *Let $s \in (0, 1]$ such that $1/p < s < N/p$, $-sp < \alpha$ and $p < q < \frac{p(N+\alpha)}{N-sp}$. Moreover, assume that*

$$(3.12) \quad \alpha - \beta + (q - p) \left(\frac{1 - N}{p} \right) < 0.$$

Then, for any $u \in W_{rad,\beta}^{s,p}(\mathbb{R}^N)$, there holds

$$(3.13) \quad \int_{\mathbb{R}^N} |x|^\alpha |u|^q dx \leq C \|u\|_{s,p}^\eta \|u\|_{p,\beta}^\omega,$$

for some $\eta, \omega > 0$ depending on N, p, q and s .

Proof. Let $\varepsilon > 0$. Since $q < \frac{p(N+\alpha)}{N-sp}$, by Lemma 3.5, there is $c < \alpha$ such that

$$(3.14) \quad \left(\int_{\mathbb{R}^N} |x|^c |u|^q dx \right)^{1/q} \leq C \|u\|_{s,p}.$$

Now,

$$(3.15) \quad \int_{|x| \leq \varepsilon} |x|^\alpha |u|^q dx = \int_{|x| \leq \varepsilon} |x|^{\alpha-c} |x|^c |u|^q dx \leq C \varepsilon^{\alpha-c} \|u\|_{s,p}^q.$$

Next, by Strauss' Lemma 3.1 and Remark 3.2, we get that

$$\begin{aligned}
 \int_{|x|>\varepsilon} |x|^\alpha |u|^q dx &= \int_{|x|>\varepsilon} |x|^{\alpha-\beta} |x|^\beta |u|^{q-p} |u|^p dx \\
 (3.16) \quad &\leq C \int_{|x|>\varepsilon} |x|^\beta |u|^p |x|^{\alpha-\beta} \left(|x|^{\frac{1-N}{p}} \|u\|_{s,p} \right)^{q-p} dx \\
 &= C \|u\|_{s,p}^{q-p} \int_{|x|>\varepsilon} |x|^\beta |u|^p |x|^{\alpha-\beta+(q-p)\frac{1-N}{p}} dx.
 \end{aligned}$$

Condition (3.12) yields that

$$(3.17) \quad \int_{|x|>\varepsilon} |x|^\alpha |u|^q dx \leq C \varepsilon^{\alpha-\beta+(q-p)\frac{1-N}{p}} \|u\|_{L_\beta^p}^p \|u\|_{s,p}^{q-p}.$$

Therefore, combining (3.15) and (3.17),

$$(3.18) \quad \int_{\mathbb{R}^N} |x|^\alpha |u|^q dx \leq C \left(\varepsilon^{\alpha-c} \|u\|_{s,p}^p + C \varepsilon^{\alpha-\beta+(q-p)\frac{1-N}{p}} \|u\|_{L_\beta^p}^p \|u\|_{s,p}^{q-p} \right).$$

Maximizing the auxiliary function

$$g(\varepsilon) := C_1 \varepsilon^{e_1} + C_2 \varepsilon^{-e_2},$$

with

$$C_1 = C \|u\|_{s,p}^p, \quad C_2 = C \|u\|_{L_\beta^p}^p \|u\|_{s,p}^{q-p}$$

and

$$(3.19) \quad e_1 = \alpha - c, \quad -e_2 = \alpha - \beta + (q - p) \frac{1 - N}{p},$$

we finally get (3.13) with

$$\eta = \frac{p e_2 + (q - p) e_1}{e_1 + e_2} \quad \text{and} \quad \omega = \frac{p e_1}{e_1 + e_2}.$$

Observe that η is positive since by (3.12),

$$\eta = \frac{p(\beta - \alpha) + (q - p)(N - 1) + (q - p)(\alpha - c)}{e_1 + e_2} > \frac{(q - p)(\alpha - c)}{e_1 + e_2} > 0.$$

This concludes the proof. ■

We are now ready to prove the compactness of the embedding $W_{rad,\beta}^{s,p}(\mathbb{R}^N)$ into $L_\alpha^q(\mathbb{R}^N)$.

Theorem 3.7. *Let $s \in (0, 1]$ such that $1/p < s < N/p$, $\alpha > -sp$ and $p < q < \frac{p(N+\alpha)}{N-sp}$. Moreover, assume that*

$$(3.20) \quad \alpha - \beta + (q - p) \left(\frac{1 - N}{p} \right) < 0.$$

Then, the following embedding is compact

$$W_{rad,\beta}^{s,p}(\mathbb{R}^N) \subset L_\alpha^q(\mathbb{R}^N).$$

Proof. Suppose that $\{u_n\}_{n \in \mathbb{N}} \subset W_{rad,\beta}^{s,p}(\mathbb{R}^N)$ and $u_n \rightharpoonup 0$. Let $M > 0$ such that $\|u\|_{s,p,\beta} \leq M$. Given $\varepsilon \in (0, 1]$, we write

$$(3.21) \quad \|u_n\|_{q,\alpha}^q = \int_{|x|<\varepsilon} |u_n|^q |x|^\alpha dx + \int_{\varepsilon \leq |x| < 1/\varepsilon} |u_n|^q |x|^\alpha dx + \int_{|x|\geq 1/\varepsilon} |u_n|^q |x|^\alpha dx := (i) + (ii) + (iii).$$

Regarding (i), we first observe that since $q < \frac{p(N+\alpha)}{N-sp}$, then by Lemma 3.5 that there is $c \in (-sp, \alpha)$ such that

$$q = \frac{p(N+c)}{N-sp}$$

and

$$\int_{\mathbb{R}^N} |x|^c |u_n|^q dx \leq C \|u_n\|_{s,p}^q \leq C \|u_n\|_{s,p,\beta}^q \leq CM^q,$$

where we have appeal to Remark 3.2. Hence,

$$(3.22) \quad (i) = \int_{|x|<\varepsilon} |x|^{\alpha-c} |x|^c |u_n|^q dx \leq C \varepsilon^{\alpha-c} M^q.$$

To deal with (ii), we let $\Omega_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon \leq |x| < \varepsilon^{-1}\}$. Observe that

$$\int_{\Omega_\varepsilon} |u_n|^p dx = \int_{\Omega_\varepsilon} |x|^{-\beta} |x|^\beta |u_n|^p dx \leq \varepsilon^{-\beta} \int_{\Omega_\varepsilon} |x|^\beta |u_n|^p dx \leq M^p \varepsilon^{-\beta}.$$

By Remark 3.2 and the compact embedding $W^{s,p} \subset L^p$ in bounded domains (see, for instance [8, Theorem 7.1]), it follows, up to a subsequence that we do not relabel, that

$$u_n \rightarrow 0 \quad \text{in } L^p(\Omega_\varepsilon).$$

Thus, by Lemma 3.1

$$(3.23) \quad (ii) \leq C(\varepsilon) \|u_n\|_{L^\infty(\Omega_\varepsilon)}^{q-p} \int_{\Omega_\varepsilon} |u_n|^p dx \leq C(\varepsilon, M) \int_{\Omega_\varepsilon} |u_n|^p dx \rightarrow 0,$$

as $n \rightarrow \infty$.

Finally, we apply the Strauss' Lemma 3.1 to get

$$(3.24) \quad (iii) \leq C \int_{|x|\geq 1/\varepsilon} |x|^\beta |x|^{\alpha-\beta} |u_n|^p \left(|x|^{(1-N)/p} \|u_n\|_{s,p,\beta} \right)^{q-p} dx \leq CM^{p-q} \int_{|x|\geq 1/\varepsilon} |x|^\delta |x|^\beta |u_n|^p dx,$$

where $\delta := \alpha - \beta - (p - q) \frac{N-1}{p} < 0$ by assumption. Hence

$$(3.25) \quad (iii) \leq CM^{2p-q} \varepsilon^{-\delta}.$$

Finally, the conclusion follows combining (3.22), (3.23) and (3.25). ■

4. EXISTENCE OF SOLUTIONS

Proof of Theorem 1.1. Let us check the conditions of the Brezis-Nirenberg lemma given in Lemma 6.1.

(i) Case $\gamma = 0$. By (2.1), the underlying space is $W_{rad,\beta}^{s,p}(\mathbb{R}^N)$. Let u such that $\|u\|_{s,p,\beta} \ll 1$. Then, by Theorem 3.7,

$$\begin{aligned}\mathcal{J}(u) &= \frac{1}{p}[u]_{s,p}^p + \frac{1}{p}\|u\|_{p,\beta}^p - \frac{1}{q}\|u\|_{q,\alpha}^q \\ &\geq \frac{1}{p}([u]_{s,p}^p + \|u\|_{p,\beta}^p) - C\|u\|_{s,p,\beta}^q \\ &\geq \frac{1}{p2^p}([u]_{s,p} + \|u\|_{p,\beta})^p - C\|u\|_{s,p,\beta}^q = \frac{1}{p2^p}\|u\|_{s,p,\beta}^p - C\|u\|_{s,p,\beta}^q\end{aligned}$$

where we have used that $a^p + b^p \geq \max\{a^p, b^p\} \geq (\frac{a+b}{2})^p$. Since $p < q$, we have that $\mathcal{J}(u) > 0$ for $\|u\|_{s,p,\beta} = r$ small enough.

Similarly, when $\gamma \in (0, 1]$, by (2.1), the underlying space is now $W_{rad,\beta}^{1,p}(\mathbb{R}^N)$. Hence, using Theorem 3.7 yields

$$\begin{aligned}\mathcal{J}(u) &= \frac{\gamma}{p}\|\nabla u\|_p^p + \frac{1-\gamma}{p}[u]_{s,p}^p + \frac{1}{p}\|u\|_{p,\beta}^p - \frac{1}{q}\|u\|_{q,\alpha}^q \\ &\geq \frac{\gamma}{p}(\|\nabla u\|_p^p + \|u\|_{p,\beta}^p) - C\|u\|_{1,p,\beta}^q \\ &\geq \frac{\gamma}{p2^p}(\|\nabla u\|_p + \|u\|_{p,\beta})^p - C\|u\|_{1,p,\beta}^q = \frac{\gamma}{p2^p}\|u\|_{1,p,\beta}^p - C\|u\|_{1,p,\beta}^q,\end{aligned}$$

obtaining the same conclusion than before.

(ii) Let $\gamma \in [0, 1]$. Let $w > 0$ fixed, and define $v := tw$, $t \in \mathbb{R}$. Then

$$\mathcal{J}(v) = \frac{\gamma t^p}{p}\|\nabla w\|_p^p + \frac{(1-\gamma)t^p}{p}[w]_{s,p}^p + \frac{t^p}{p}\|w\|_{p,\beta}^p - \frac{t^q}{q}\|w\|_{q,\alpha}^q \leq C_1 t^p - C_2 t^q.$$

Since $p < q$, there exists t such that $\mathcal{J}(v) < 0$ and $\|v\| > r$.

(iii) Let $\gamma \in [0, 1]$. Let $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{W}_\beta^{s,p}(\mathbb{R}^N)$ be such that $\mathcal{J}'(u_n) \rightarrow 0$ and $\mathcal{J}(u_n) \rightarrow c$.

Since $\mathcal{J}'(u_n) \rightarrow 0$, we get that

$$|\langle \mathcal{J}'(u_n), u_n \rangle| = |\gamma\|\nabla u_n\|_p^p + (1-\gamma)[u_n]_{s,p}^p + \|u_n\|_{p,\beta}^p - \|u_n\|_{q,\alpha}^q| \leq \|u_n\|_{s,p,\beta}$$

for n large enough. Since $|\mathcal{J}(u_n)| \leq C$, we have that

$$\left| \frac{\gamma}{p}\|\nabla u_n\|_p^p + \frac{1-\gamma}{p}[u_n]_{s,p}^p + \frac{1}{p}\|u_n\|_{p,\beta}^p - \frac{1}{q}\|u_n\|_{q,\alpha}^q \right| \leq C.$$

The last two expression give that

$$\begin{aligned}\gamma\|\nabla u_n\|_p^p + (1-\gamma)[u_n]_{s,p}^p + \|u_n\|_{p,\beta}^p &\leq pC + \frac{p}{q}\|u_n\|_{q,\alpha}^q \\ &\leq pC + \frac{p}{q}\gamma\|\nabla u_n\|_p^p + \frac{p}{q}(1-\gamma)[u_n]_{s,p}^p + \frac{p}{q}\|u_n\|_{p,\beta}^p + \frac{p}{q}\|u_n\|_{s,p,\beta}\end{aligned}$$

from where, since $p < q$,

$$\left(1 - \frac{p}{q}\right) (\gamma \|\nabla u_n\|_p^p + (1 - \gamma)[u_n]_{s,p}^p + \|u_n\|_{p,\beta}^p) \leq pC + \frac{p}{q} \|u_n\|_{s,p,\beta}.$$

When $\gamma = 0$, this gives that

$$\left(1 - \frac{p}{q}\right) ([u_n]_{s,p}^p + \|u_n\|_{p,\beta}^p) \leq pC + \frac{p}{q} \|u_n\|_{s,p,\beta}.$$

When $\gamma \in (0, 1]$, we get

$$\gamma \left(1 - \frac{p}{q}\right) (\|\nabla u\|_p^p + \|u_n\|_{p,\beta}^p) \leq pC + \frac{p}{q} \|u_n\|_{1,p,\beta},$$

that is, for any $\gamma \in [0, 1]$ we have the following relation:

$$\left(1 - \frac{p}{q}\right) \|u_n\|_{s,p,\beta}^p \leq C'(p + \|u_n\|_{s,p,\beta}),$$

since $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ holds for $a, b > 0$. Hence, for $\gamma \in [0, 1]$, the sequence $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $\mathcal{W}_\beta^{s,p}(\mathbb{R}^N)$. Then, up to a subsequence, $u_n \rightharpoonup u$ weakly in $\mathcal{W}_\beta^{s,p}(\mathbb{R}^N)$. From the compact embedding given in Theorem 3.7, we get that $u_n \rightarrow u$ strongly in $L_\alpha^q(\mathbb{R}^N)$, which implies

$$\int_{\mathbb{R}^N} |x|^\alpha |u_n - u|^q dx \rightarrow 0.$$

Therefore

$$\begin{aligned} \left| \langle \mathcal{L}_{s,p,\gamma} u_n, u_n - u \rangle + \int_{\mathbb{R}^N} |x|^\beta |u_n|^{p-2} u_n (u_n - u) dx \right| &= \left| \langle \mathcal{J}'(u_n), u_n - u \rangle + \int_{\mathbb{R}^N} |x|^\alpha |u_n|^{q-2} u_n (u_n - u) dx \right| \\ &\leq \|\mathcal{J}'(u_n)\|_{\mathcal{W}_\beta^{s,p}(\mathbb{R}^N)'} \|u_n - u\|_{s,p,\beta} + \int_{\mathbb{R}^N} |x|^\alpha |u_n|^{q-1} |u_n - u| dx, \end{aligned}$$

where $\mathcal{W}_\beta^{s,p}(\mathbb{R}^N)'$ denotes the dual space of $\mathcal{W}_\beta^{s,p}(\mathbb{R}^N)$. The first term goes to 0 since $\mathcal{J}'(u_n) \rightarrow 0$ and the sequence $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $\mathcal{W}_\beta^{s,p}(\mathbb{R}^N)$. For the second term, observe that by Hölder's inequality and (3.13), it vanishes as $n \rightarrow \infty$. Indeed,

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^\alpha |u_n|^{q-1} |u_n - u| dx &= \int_{\mathbb{R}^N} |x|^{\frac{\alpha(q-1)}{q}} |u_n|^{q-1} |x|^{\frac{\alpha}{q}} |u_n - u| dx \\ &\leq \left(\int_{\mathbb{R}^N} |x|^\alpha |u_n|^q dx \right)^{\frac{q-1}{q}} \left(\int_{\mathbb{R}^N} |x|^\alpha |u_n - u|^q dx \right)^{\frac{1}{q}} \\ &\leq C \|u_n\|_{s,p}^{\eta_1} \left(\int_{\mathbb{R}^N} |x|^\beta |u_n|^p dx \right)^{\omega_1} \left(\int_{\mathbb{R}^N} |x|^\alpha |u_n - u|^q dx \right)^{\frac{1}{q}} \\ &\leq C_1 \|u_n\|_{s,p,\beta}^{\eta_2} \left(\int_{\mathbb{R}^N} |x|^\alpha |u_n - u|^q dx \right)^{\frac{1}{q}} \\ &\leq C_2 \left(\int_{\mathbb{R}^N} |x|^\alpha |u_n - u|^q dx \right)^{\frac{1}{q}} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

where $C, C_1, C_2, \eta_1, \eta_2$ and ω_1 are positive constants depending only of N, s, p, q .

Therefore, since by Remark 4.1 below, the operator $\tilde{\mathcal{L}}$ defined on $\mathcal{W}_\beta^{s,p}(\mathbb{R}^N) \times \mathcal{W}_\beta^{s,p}(\mathbb{R}^N)$ as

$$\langle \tilde{\mathcal{L}}u, v \rangle := \langle \mathcal{L}_{s,p,\gamma} u, v \rangle + \int_{\mathbb{R}^N} |x|^\beta |u|^{p-2} uv \, dx$$

fulfills the (S) -property of compactness, due to Proposition 6.3 and the previous computations, we get that $u_n \rightarrow u$ strongly in $\mathcal{W}_\beta^{s,p}(\mathbb{R}^N)$.

(iv) To obtain a positive solution, we choose the mapping P defined as $P(u) = |u|$. This map is continuous, $\mathcal{J}(P(u)) = \mathcal{J}(u)$ for all $u \in \mathcal{W}_\beta^{s,p}(\mathbb{R}^N)$, $P(0) = 0$ and, taking $P(v) = v$ when $v \in \mathcal{W}_\beta^{s,p}(\mathbb{R}^N)$ is positive.

Therefore, the conclusion follow from Lemma 6.1. This concludes the proof. \blacksquare

Remark 4.1. Given $u, v \in \mathcal{W}_\beta^{s,p}(\mathbb{R}^N)$, $s \in (0, 1]$, by Hölder's inequality

$$\begin{aligned} \langle \mathcal{L}_{s,p,\gamma} u, v \rangle &\leq \gamma \left(\int_{\mathbb{R}^N} (|\nabla u|^{p-1})^{p'} \right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}^N} |\nabla v|^p \right)^{\frac{1}{p}} \\ &\quad + (1 - \gamma) \left(\iint_{\mathbb{R}^{2N}} \left(\frac{|u(x) - u(y)|}{|x - y|^{\frac{N+sp}{p'}}} \right)^{p'} dx dy \right)^{\frac{1}{p'}} \left(\iint_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{\frac{1}{p}} \\ &= \gamma \|\nabla u\|_p^{p-1} \|\nabla v\|_p + (1 - \gamma) [u]_{s,p}^{p-1} [v]_{s,p}. \end{aligned}$$

If $\gamma = 0$, and hence $s \in (0, 1)$, then

$$(4.1) \quad \langle \mathcal{L}_{s,p,\gamma} u, v \rangle \leq [u]_{s,p}^{p-1} [v]_{s,p} \leq \|u\|_{s,p,\beta}^{p-1} \|v\|_{s,p,\beta}.$$

If $\gamma \in (0, 1]$, and hence $s = 1$, using (3.3), we get that

$$(4.2) \quad \langle \mathcal{L}_{s,p,\gamma} u, v \rangle \leq \gamma \|\nabla u\|_p^{p-1} \|\nabla v\|_p + (1 - \gamma) [u]_{s,p}^{p-1} [v]_{s,p} \leq C \|u\|_{1,p,\beta}^{p-1} \|v\|_{1,p,\beta}.$$

Moreover,

$$(4.3) \quad \int_{\mathbb{R}^N} |x|^\beta |u|^{p-2} uv \, dx \leq \int_{\mathbb{R}^N} |x|^{\frac{\beta(p-1)}{p}} |u|^{p-1} |x|^{\frac{\beta}{p}} v \, dx \leq \left(\int_{\mathbb{R}^N} |x|^\beta |u|^p \, dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^N} |x|^\beta |v|^p \, dx \right)^{\frac{1}{p}}.$$

Hence, from (4.1), (4.2) and (4.3), we get that the operator $\tilde{\mathcal{L}}$ defined as

$$\langle \tilde{\mathcal{L}}_{s,p,\gamma} u, v \rangle := \langle \mathcal{L}_{s,p,\gamma} u, v \rangle + \int_{\mathbb{R}^N} |x|^\beta |u|^{p-2} uv \, dx$$

fulfills the following relation

$$\langle \tilde{\mathcal{L}}_{s,p,\gamma} u, v \rangle \leq C \|u\|_{s,p,\beta}^{p-1} \|v\|_{s,p,\beta}.$$

Moreover, by definition

$$\langle \tilde{\mathcal{L}}_{s,p,\gamma} u, v \rangle = \langle \mathcal{L}_{s,p,\gamma} u, u \rangle + \int_{\mathbb{R}^N} |x|^\beta |u|^p \, dx \leq C \|u\|_{s,p,\beta}^p.$$

Therefore, in light of Proposition 6.3, the operator $\tilde{\mathcal{L}}_{s,p,\gamma}$ satisfies the (S) -property of compactness according to Definition 6.2.

5. BOUNDEDNESS OF SOLUTIONS

In this section, we prove that any radial solution u of problem (1.4) is bounded.

Proof of Theorem 1.2. In what follows, we will apply a De Giorgi's iteration scheme to control the level sets of a solution u to problem (1.4).

For a positive integer k , define

$$w_k := (u - (1 - 2^{-k}))_+.$$

Then, as in [11], the following holds

$$(5.1) \quad w_{k+1} \leq w_k \text{ in } \mathbb{R}^N, \quad u(x) < (2^{k+1} - 1)w_k \text{ in } \{w_{k+1} > 0\}, \quad \text{and } \{w_{k+1} > 0\} \subset \{w_k > 2^{-(k+1)}\}.$$

Also, $0 \leq w_k \leq |u| + 1 \in L^1(\mathbb{R}^N)$ and $w_k(x) \rightarrow (u(x) - 1)_+$ a.e. in \mathbb{R}^N , so by dominated convergence theorem,

$$(5.2) \quad \lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} |x|^\alpha |w_k|^q dx = \int_{\mathbb{R}^N} |x|^\alpha |(u - 1)_+|^q dx.$$

Now,

$$\begin{aligned} (5.3) \quad & \frac{1}{p} \|w_{k+1}\|_{s,p,\beta} \leq C(p, \gamma) \left(\gamma \int_{\mathbb{R}^N} |\nabla w_{k+1}|^{p-2} \nabla w_{k+1} \cdot \nabla w_{k+1} dx \right. \\ & + (1 - \gamma) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w_{k+1}(x) - w_{k+1}(y)|^{p-2} (w_{k+1}(x) - w_{k+1}(y)) (w_{k+1}(x) - w_{k+1}(y))}{|x - y|^{N+sp}} dx dy \Big) \\ & + \int_{\mathbb{R}^N} |x|^\beta |w_{k+1}|^{p-1} w_{k+1} dx \\ & \leq C(p, \gamma) \left(\gamma \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla w_{k+1} dx \right. \\ & + (1 - \gamma) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (w_{k+1}(x) - w_{k+1}(y))}{|x - y|^{N+sp}} dx dy \Big) \\ & + \int_{\mathbb{R}^N} |x|^\beta |u|^{p-1} w_{k+1} dx \\ & = C(p, \gamma) \int_{\mathbb{R}^N} |x|^\alpha |u|^{q-1} w_{k+1} dx, \end{aligned}$$

where we have used the fact that

$$|v_+(x) - v_+(y)|^p \leq (v(x) - v(y))(v_+(x) - v_+(y))|v_+(x) - v_+(y)|^{p-2}$$

and also that u is a solution of (1.4). Now, by (5.1), we get

$$(5.4) \quad \int_{\mathbb{R}^N} |x|^\alpha |u|^{q-1} w_{k+1} dx \leq \int_{\mathbb{R}^N} |x|^\alpha |(2^{k+1} - 1)w_k|^{q-1} (2^{k+1} - 1)w_k dx = C_{k+1} \int_{\mathbb{R}^N} |x|^\alpha |w_k|^q dx,$$

with $C_{k+1} = (2^{k+1} - 1)^q$. By Lemma 3.6 applied to $w_{k+1} \in \mathcal{W}_\beta^{s,p}(\mathbb{R}^N)$ (recall also the notation (3.19)) and by (5.3) and (5.4),

$$(5.5) \quad \int_{\mathbb{R}^N} |x|^\alpha |w_{k+1}|^q dx \leq C \|w_{k+1}\|_{s,p,\beta}^{\frac{pe_2+qe_1}{e_1+e_2}} \leq C_{k+1} \left(\int_{\mathbb{R}^N} |x|^\alpha |w_k|^q dx \right)^{\frac{pe_2+qe_1}{pe_1+pe_2}} = C_{k+1} \left(\int_{\mathbb{R}^N} |x|^\alpha |w_k|^q dx \right)^{1+\delta}$$

for some $\delta > 0$ since $q > p$. Therefore, by the numerical lemma [10, Lemma 13], we obtain that there is $\varepsilon \in (0, 1)$ such that if

$$(5.6) \quad \int_{\mathbb{R}^N} |x|^\alpha |w_0|^q dx < \varepsilon$$

then

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} |x|^\alpha |w_k|^q dx = 0.$$

Hence, (5.2) implies $\|u\|_{L^\infty(\mathbb{R}^N)} \leq 1$. Observe that

$$\int_{\mathbb{R}^N} |x|^\alpha |w_0|^q dx = \int_{\mathbb{R}^N} |x|^\alpha u_+^q dx$$

so the result is valid under the assumption

$$\int_{\mathbb{R}^N} u_+^q |x|^\alpha dx < \varepsilon.$$

If now $\xi = \int_{\mathbb{R}^N} u_+^q |x|^\alpha dx > 0$ is arbitrary, then we choose $C > 1$ large enough so that

$$(5.7) \quad \int_{\mathbb{R}^N} \left(\frac{u_+}{C} \right)^q |x|^\alpha dx < \varepsilon.$$

Moreover, u/C also satisfies (5.3) with an extra constant at the end. Hence, from the above argument

$$\|u/C\|_{L^\infty(\mathbb{R}^N)} \leq 1 \text{ or } \|u\|_{L^\infty(\mathbb{R}^N)} \leq C.$$

This ends the proof of the theorem. ■

6. NON-EXISTENCE OF SOLUTIONS

We begin this section with the following computation, which justifies our use of the non-existence result [20, Proposition 1.4], originally stated for bounded domains. Given $\lambda > 1$, we can write

$$u(\lambda x) - u(x) = \int_1^\lambda \frac{d}{dt} u(tx) dt = \int_1^\lambda \nabla u(tx) \cdot x dt.$$

Therefore,

$$\left| \frac{u(\lambda x) - u(x)}{\lambda - 1} \right|^r \leq \frac{1}{(\lambda - 1)^r} \left(\int_1^\lambda |\nabla u(tx)| |x| dt \right)^r.$$

Using Jensen's inequality yields

$$\left| \frac{u(\lambda x) - u(x)}{\lambda - 1} \right|^r \leq \frac{1}{\lambda - 1} \int_1^\lambda |\nabla u(tx)|^r |x|^r dt.$$

Then, integrating on \mathbb{R}^N , using Fubini's Theorem and changing variables we get

$$\begin{aligned}
 \int_{\mathbb{R}^N} \left| \frac{u(\lambda x) - u(x)}{\lambda - 1} \right|^r dx &\leq \frac{1}{\lambda - 1} \int_1^\lambda \frac{1}{t^{N+r}} \int_{\mathbb{R}^N} |\nabla u(y)|^r |y|^r dy dt \\
 (6.1) \quad &= \frac{1}{N+r-1} \frac{1 - \lambda^{-(N+r-1)}}{\lambda - 1} \int_{\mathbb{R}^N} |\nabla u(y)|^r |y|^r dy \\
 &\leq C_{N,r} \int_{\mathbb{R}^N} |\nabla u(y)|^r |y|^r dy,
 \end{aligned}$$

since $\lambda > 1$ and $N + r - 1 > 0$. Therefore, if $|\nabla u(x)| |x| \in L^r(\mathbb{R}^N)$, for some $r > 1$, (6.1) enable us to apply [20, Lemma 4.2], and hence [20, Proposition 1.4] holds with $\Omega = \mathbb{R}^N$.

Proof of Theorem 1.3. Let $s \in (0, 1]$ and $\beta > 0$. Given $u \in \tilde{\mathcal{X}}(\mathbb{R}^N)$ denote $u_\lambda(x) = u(\lambda x)$ for $\lambda > 1$. An easy computation gives that, when $\beta > p(1-s)$,

$$\|u_\lambda\|_{s,p,\beta} = \|u_\lambda\|_{p,\beta} + [u_\lambda]_{s,p} \leq \lambda^{-\frac{N-sp}{p}} [u]_{s,p} + \lambda^{-\frac{N+\beta-p}{p}} \|u\|_{p,\beta} \leq \lambda^{-\tau} \|u\|_{s,p,\beta},$$

where $\tau := \frac{N-sp}{p}$.

Let $f(x, t) = |x|^\alpha |t|^{q-2} t$ and $F(x, u) = \int_0^u f(x, t) dt = \frac{1}{q} |x|^\alpha |t|^q$, being $u \in \mathcal{W}_\beta^{s,p}(\mathbb{R}^N)$ a weak solution of (1.4). It is straightforward to see that

$$\tau t f(x, t) > nF(x, t) + x \cdot F_x(x, t) \quad \text{for all } t \in \mathbb{R}^N, t \neq 0$$

whenever $q > \frac{N+\alpha}{\tau}$, that is, when

$$q > \begin{cases} \frac{p(N+\alpha)}{N-sp} & \text{if } \gamma = 0 \\ \frac{p(N+\alpha)}{N-p} & \text{if } \gamma \in (0, 1]. \end{cases}$$

Moreover, recall that $u \in L^\infty(\mathbb{R}^N)$ by definition of the space $\tilde{\mathcal{X}}(\mathbb{R}^N)$. So, by [20, Proposition 1.4] we have that $u \equiv 0$. ■

APPENDIX

The following variant of the Mountain-Pass Theorem will be of use for our arguments. See [3, Theorem 10].

Lemma 6.1 (Brezis–Nirenberg). *Let E be a Banach space and $\Phi \in C^1(E, \mathbb{R})$ satisfy $\Phi(0) = 0$ and the following conditions:*

- (i) *there exists $r > 0$ such that $\inf_{\|u\|=r} \Phi(u) > 0$;*
- (ii) *there exists v with $\|v\| > r$ such that $\Phi(v) < 0$;*
- (iii) *the Palais–Smale condition: if $\Phi'(u_n) \rightarrow 0$ and $\Phi(u_n) \rightarrow c$, then (u_n) has a subsequence converging to $u \in E$;*
- (iv) *there exists a continuous mapping $P : E \rightarrow E$ such that $P(0) = 0$, $P(v) = v$, and $\Phi(P(u)) \leq \Phi(u)$ for every $u \in E$.*

Then there exists a critical point $u^ \in \overline{P(E)}$ such that*

$$\Phi(u^*) \geq \inf_{\|u\|=r} \Phi(u).$$

Definition 6.2. *The functional \mathcal{J} defined on E satisfies the (S) -property if $\{u_n\}_{n \in \mathbb{N}}$ is a sequence in E such that $u_n \rightharpoonup u$ weakly in E and $\langle \mathcal{J}'(u_n), u_n - u \rangle \rightarrow 0$, then $u_n \rightarrow u$ strongly in E .*

The following result characterizes the (S) -property. See [19, Proposition 1.3].

Proposition 6.3. *Let E be a uniformly convex Banach space and let $A_p \in C^1(E, \mathbb{R})$ be such that*

- (i) $\langle A_p(u), v \rangle \leq r \|u\|_E^{p-1} \|v\|_E$
- (ii) $\langle A_p(u), u \rangle = r \|u\|_E^p$

for some $r > 0$, for all $u, v \in E$. Then A_p satisfies the (S) -property.

REFERENCES

- [1] Aronszajn, N., Mulla, F., Szeptycki, P., *On spaces of potential connected with L^p classes*. Ann. Inst. Fourier 13 (1963), 211–306. [6](#) [8](#)
- [2] Barrios, B. , Quaas, A. , *The sharp exponent in the study of the nonlocal Hénon equation in \mathbb{R}^N : a Liouville theorem and an existence result*. Calc. Var. PDE 59 (2020), 114. [1](#)
- [3] Berestycki, H., Capuzzo-Dolcetta, I. and Nirenberg L., *Variational methods for indefinite superlinear homogeneous elliptic problems*. NoDEA 2 (1995), 553–572. [16](#)
- [4] Biagi, S., Dipierro, S., Valdinoci, E., Vecchi, E., *A Brezis–Nirenberg type result for mixed local and nonlocal operators*. NoDEA 32 (2025), 62. [2](#)
- [5] da Silva, J. V., Fiscella, A., Viloria, V., *Mixed local–nonlocal quasilinear problems with critical nonlinearities*. J. Differ. Equ. 408 (2024), 494–536. [2](#)
- [6] De Nápoli, P., *Symmetry breaking for an elliptic equation involving the fractional Laplacian*. Differ. Integral Equ. 31 (2018), 75–94. [2](#) [3](#)
- [7] De Nápoli, P. and Drelichman, I., *Elementary proofs of embedding theorems for potential spaces of radial functions*. Methods of Fourier analysis and approximation theory, Appl. Numer. Harmon. Anal. (2016), 115–138. [6](#)
- [8] Di Nezza, E., Palatucci, G. Valdinoci, E., *Hitchhiker’s guide to the fractional Sobolev spaces*. Bull. Sci. Math. 136 (2012), 521–573. [6](#) [10](#)
- [9] Fernández Bonder, J., Salort, A., *Fractional order Orlicz–Sobolev spaces*. J. Funct. Anal. 277 (2019), 333–367. [6](#)
- [10] Fernández Bonder, J., Salort, A., Vivas, H., *Global Hölder regularity for eigenfunctions of the fractional g -Laplacian*. J. Math. Anal. Appl. 526 (2023), 127332. [15](#)
- [11] Franzina, G., Palatucci, G. *Fractional p -eigenvalues*. Riv. Mat. Univ. Parma 5 (2014), 373–386. [14](#)
- [12] Garain, P., Lindgren, E., *Higher Hölder regularity for mixed local and nonlocal degenerate elliptic equations*. Calc. Var. PDE 62 (2023), 67. [2](#)
- [13] Hénon, M., *Numerical experiments on the stability of spherical stellar systems*. Astron. Astrophys. 24 (1973), 229–238. [1](#)
- [14] Huang, S., Tian, Q., Zha, X. Uniform boundedness results of solutions to mixed local and nonlocal elliptic operator. AIMS Math. 8 (2023), 20665–20678. [2](#)
- [15] Le, P. , *Liouville theorem for Hénon–Hardy systems in the unit ball*. Rev. Mat. Complut. 36 (2023), 827–840.
- [16] Ma, L., *On nonlocal Hénon type problems with the fractional Laplacian*. Nonlinear Anal. 203 (2021), 112190. [2](#) [3](#)
- [17] Maione, A. , Mugnai, D. , Vecchi, E., *Variational methods for nonpositive mixed local–nonlocal operators*. Fract. Calc. Appl. Anal. 26 (2023), 943–961. [2](#)
- [18] W.-M. Ni, *A nonlinear Dirichlet problem on the unit ball and its applications*. Indiana Univ. Math. J. 31 (1982), 801–807. [1](#)
- [19] Perera, K., Agarwal, R., O’Regan, D., *Morse theoretic aspects of p -Laplacian type operators*. Amer. Math. Soc., Providence, RI, 2010. [17](#)
- [20] Ros-Oton, X. , Serra, J., *Nonexistence results for nonlocal equations with critical and supercritical nonlinearities*. Commun. Partial Differ. Equ. 40 (2015), 115–133. [3](#) [15](#) [16](#)

- [21] Salort, A., Vecchi, E. *On the mixed local–nonlocal Hénon equation.* Differ. Integral Equ. 35 (2022), 795–818. [1](#), [2](#)
- [22] Sickel, W., Skrzypczak, L., Vybiral, J., *On the interplay of regularity and decay in case of radial functions I. Non-homogeneous spaces.* Commun. Contemp. Math. 14 (2012). [5](#)
- [23] Sintzoff, P., *Symmetry of solutions of a semilinear elliptic equation with unbounded coefficients.* Differ. Integral Equ. 16 (2003), 769–786. [2](#), [3](#)
- [24] Sire, Y., Wei, J., *On a fractional Hénon equation and applications.* Math. Res. Lett. 22 (2015), 1791–1804. [2](#)
- [25] Su, X., Valdinoci, E., Wei, Y., Zhang, J., *Regularity results for solutions of mixed local and nonlocal elliptic equations.* Math. Z. 302 (2022), 1855–1878. [2](#)

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