THE DE JONG FUNDAMENTAL GROUP OF $\mathbb{P}^1_C$ DEPENDS ON C AND IS NOT ALWAYS TOPOLOGICALLY COUNTABLY GENERATED.

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Abstract. For $C/\mathbb{Q}_p$ a complete algebraically closed field, we construct a collection of non-isomorphic rank two $\mathbb{Q}_p$-local systems on $\mathbb{P}^1_C$ indexed by C. This implies that the de Jong fundamental group $\pi_{1,\mathrm{dJ}}(\mathbb{P}^1_C)$ depends on C and, if C has cardinality $> 2^{\mathbb{N}}$, that $\pi_{1,\mathrm{dJ}}(\mathbb{P}^1_C)$ is not topologically countably generated. The argument in fact applies to any connected rigid analytic variety over C with a non-constant function to $\mathbb{P}^1_C$.

Let $C/\mathbb{Q}_p$ be a complete algebraically closed extension and consider $\mathbb{P}^1_C$ as a rigid analytic variety over C. For $t \in C$, we have a map $\varphi_t : \mathbb{P}^1_C \to \mathbb{P}^1_C,\ (t, [x:y]) \mapsto [x^2 : (x-ty)^2]$. On the open subvariety $\mathbb{A}^1_C = \mathbb{P}^1_C \backslash \{[1:0]\}$ with coordinate $z = x/y$, φ_t is the map $z \mapsto (z-t)^2$. Let $\mathbb{L}$ be the Lubin-Tate rank two $\mathbb{Q}_p$-local system on the rigid analytic variety $\mathbb{P}^1_C$ described in [1, Proposition 7.2] and let $\mathbb{L}_t := \varphi_t^* \mathbb{L}$.

Lemma 1. *If $t_1 \neq t_2$, then $\mathbb{L}_{t_1}$ is not isomorphic to $\mathbb{L}_{t_2}$.*

Proof. Let $\kappa : T_{\mathbb{P}^1_C} \to \mathcal{E}nd(\mathbb{L} \otimes \hat{\mathcal{O}})(-1)$ be the geometric Sen morphism associated to $\mathbb{L} \otimes \hat{\mathcal{O}}$ by [4, Theorem 1.0.3]. From its computation in [2, §4.3], it follows that κ is injective at every geometric point of $\mathbb{P}^1_C$ (cf. [3, Lemma 2]). Now, for $t \in C$, functoriality of the geometric Sen morphism [4, Theorem 1.0.3-(5)] implies that the geometric Sen morphism of $\mathbb{L}_t$ is $\varphi_t^* \kappa \circ d\varphi_t$. In particular, writting κ_i for the geometric Sen morphism of $\mathbb{L}_{t_i}$, we find κ_1 is zero on the fiber $T_{\mathbb{P}^1_C,[t_1:1]}$ whereas κ_2 is not (the derivative of $z \mapsto (z-t)^2$ on $\mathbb{A}^1_C$ vanishes exactly at t). Thus $\mathbb{L}_1 \not\cong \mathbb{L}_2$. ☐

Theorem 1. *Let $\pi_{1,\mathrm{dJ}}(\mathbb{P}^1_C, [1:0])$ be the de Jong fundamental group of [1]. If $S \subseteq \pi_{1,\mathrm{dJ}}(\mathbb{P}^1_C, [1:0])$ is infinite and 2^S has cardinality less than C, then S is not a set of topological generators for $\pi_{1,\mathrm{dJ}}(\mathbb{P}^1_C, [1:0])$. In particular, if C has cardinality $> 2^{\mathbb{N}}$ then $\pi_{1,\mathrm{dJ}}(\mathbb{P}^1_C, [1:0])$ is not topologically countably generated.*

Proof. By Lemma 1, $\{\mathbb{L}_t\}_{t\in C}$ is a collection of non-isomorphic rank two $\mathbb{Q}_p$-local systems indexed by C. On the other hand, by [1, Theorem 4.2], the isomorphism classes of rank two local systems on $\mathbb{P}^1_C$ are the same as isomorphism classes of continuous representations of $\pi_{1,\mathrm{dJ}}(\mathbb{P}^1_C, [1:0])$ on 2-dimensional $\mathbb{Q}_p$-vector spaces. If S is a set of topological generators, then the cardinality of this latter set is bounded by the cardinality of the set $\mathrm{GL}_2(\mathbb{Q}_p)^S$. Since $\mathrm{GL}_2(\mathbb{Q}_p)$ has cardinality $2^{\mathbb{N}}$, this is the same cardinality as $(2^{\mathbb{N}})^S = 2^{\mathbb{N}\times S}$. Since we assume S is infinite, this is the same cardinality as 2^S, and thus we conclude. ☐

Corollary 1. *$\pi_{1,\mathrm{dJ}}(\mathbb{P}^1_C, [1:0])$ depends on the choice of C.*

Proof. For a fixed C, take C' of cardinality larger than the power set of $\pi_{1,\mathrm{dJ}}(\mathbb{P}^1_C, [1:0])$. Then, Theorem 1 implies $\pi_{1,\mathrm{dJ}}(\mathbb{P}^1_{C'}, [1:0]) \not\cong \pi_{1,\mathrm{dJ}}(\mathbb{P}^1_C, [1:0])$ (since $S = \pi_{1,\mathrm{dJ}}(\mathbb{P}^1_C, [1:0])$ generates $\pi_{1,\mathrm{dJ}}(\mathbb{P}^1_C, [1:0])$). ☐

Remark 1. Suppose Z/C is a connected reduced rigid analytic variety admitting a non-constant rational function, i.e. a non-constant map $f : Z \to \mathbb{P}^1_C$ (e.g., Z could be quasi-projective). Let Z^{sm} be the open dense smooth locus. By composing with an automorphism of $\mathbb{P}^1_C$, we can assume there exists a point $z \in Z^{\mathrm{sm}}(C)$ in the fiber of f over $[0:1]$ such that df is surjective at z; by the implicit function theorem, the same then holds over any point in a sufficiently small ball B around $[0:1]$ in $\mathbb{P}^1(C)$. The local systems $f^*\mathbb{L}_t$, $t \in B$, are then pairwise non-isomorphic rank two $\mathbb{Q}_p$-local systems on Z indexed by B — indeed, arguing as in the proof of Lemma 1 using the geometric Sen morphism of $f^*\mathbb{L}_t|_{Z^{\mathrm{sm}}}$, one finds that there is a point in the fiber over $s \in B$ where this geometric Sen morphism is non-zero exactly when $s \neq t$. Using this result in place of Lemma 1, we find the statements of Theorem 1 and Corollary 1 hold also for Z in place of $\mathbb{P}^1_C$.

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