

SYMMETRY FOR THE WAVE EQUATION ON TORUS: SHARP UNIQUE CONTINUATION AND OBSERVABILITY CONDITIONS FOR SPACETIME REGIONS

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ABSTRACT. In this work, we discover a new symmetry structure for the 1D wave equations associated with spacetime observable regions: observable symmetry condition. This structure yields a new conservation law for forced wave equations and provides a necessary condition for unique continuation, observability, and controllability.

Building on this symmetry, we establish a necessary and sufficient condition for unique continuation by introducing a weak GCC. Moreover, this symmetry serves as an essential complement to the classical GCC, allowing us to derive a necessary and sufficient characterization of observability and controllability through spacetime geometric regions.

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1. INTRODUCTION

Let $T > 0$. Let $G \subset [0, T] \times \mathbb{T}$ be a spacetime measurable set with positive measure. Consider the *observability problem* of the wave equation: whether there exists some $C = C(G) > 0$ such that every solution u to

$$(\partial_t^2 - \partial_x^2)u = 0, \quad (u, \partial_t u)|_{t=0} = (u_0, u_1) \in \dot{H}^1(\mathbb{T}) \times L^2(\mathbb{T}), \quad (1.1)$$

satisfies

$$\|u_0\|_{\dot{H}^1(\mathbb{T})}^2 + \|u_1\|_{L^2(\mathbb{T})}^2 \leq C \int_G |\partial_t u(t, x)|^2 dt dx. \quad (1.2)$$

Thanks to the classical Hilbert uniqueness method and an argument to raise regularity (see Appendix A.1 for details on this reduction), the observability of (1.2) is equivalent to the *exact*

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controllability of the controlled wave equation on $[0, T] \times \mathbb{T}$ with control force $f \in L^2(G)$:

$$(\partial_t^2 - \partial_x^2)u = f\mathbf{1}_G, \quad (u, \partial_t u)|_{t=0} = (u_0, u_1) \in \dot{H}^1(\mathbb{T}) \times L^2(\mathbb{T}). \quad (1.3)$$

Our primary objective is to seek the sufficient and necessary geometric conditions on G such that the unique continuation and observability hold.

1.1. History and setting. From the early work of Russell [Rus71], the studies of the controllability and observability for wave equations have been central topics in control theory.

1.1.1. Geometric control condition. In the pioneering work of Rauch and Taylor [RT74], they first related the observability to a geometric condition on the damped region ω and the rays of geometric optics in the boundaryless case. Later, in another pioneering work of Bardos-Lebeau-Rauch [BLR92], the well-known *Geometric control condition* is introduced: for an open observed region ω , every generalized ray should meet ω in a finite time.

Since then, it has become one of the most natural assumptions for the controlled waves. In the existing literature, the observation is most often made on cylindrical domains $G = (0, T) \times \omega$, with ω being an open subset. Under suitable smooth conditions, it is well-known that GCC is sufficient, and depending on the domain, necessary for the observability in the cylindrical domains $(0, T) \times \omega$, see [BLR92, BG97]. When considering stabilization problems, GCC is also a useful condition for the exponential decay of energy. We refer to [RT74, Har89]. Otherwise, one may have logarithmic type of energy decay results [LR97, Bur98]. GCC also plays a role in practical issues, such as sensor designs, tomography techniques used for imaging bodies (see [LRLTT17] for example), etc. For a comprehensive reference of the numerical study, we refer to [Zua05] and its references therein.

Finally, we give a very brief overview of the boundary control case. There are also fruitful results in this direction. For related GCC, we refer to [Leb92]. In particular, for 1D wave equations, one can find many nonlinear results [Li10, LY06] and the references therein.

1.1.2. Unique continuation. A qualitative version of observability is the unique continuation property. The easiest way to ensure this property is to apply the analyticity based on Holmgren's theorem. Besides, Hörmander's pseudo-convexity condition and Carleman estimates are powerful tools dealing with the unique continuation problem. In this direction, there is a large literature such as [RZ98, Tat95, Hör92, Hör97] and more recent work includes [LL19, MS21, FLL25, Sha19]. Here we point out that to construct the appropriate weight to apply Carleman estimates, it is crucial to understand the behavior across the suitable spacetime surfaces, which is more delicate than considering a cylindrical region.

1.1.3. Our setting: spacetime measurable observable region. Recently, researchers started to focus on the spacetime setting [CCM14, LRLTT17, Sha19, Kle25]. Meanwhile, the study of the case where G is a spacetime region is far from complete, even for open regions.

In this paper, we focus on the setting that $G \subset [0, T] \times \mathbb{T}$ is measurable of positive measure. The following condition may be viewed as the natural analogue of the standard geometric control condition in this spacetime measurable setting.

(GCC) Let $T > 0$. A measurable set $G \subset [0, T] \times \mathbb{T}$ is said to satisfy the GCC if there exists a constant $c_0 > 0$ such that for almost every $x \in \mathbb{T}$,

$$\int_0^T \mathbf{1}_G(s, x \pm s) ds \geq c_0. \quad (1.4)$$

From now on, in this paper, (GCC) refers to the preceding definition with respect to (1.4). It is well-known that under the standard setting, namely the observable region is a cylinder $[0, T] \times \omega$, GCC yields several important properties: weak observability and the necessity of observability. These results can be generalized to the spacetime (GCC) for a spacetime measurable observable region. We put the former result in Section 4.1 and the later in Appendix A.3.

1.2. A new symmetry condition. Extend the system 2π -periodically to $x \in \mathbb{R}$, and introduce the null coordinate:

$$\xi = x + t \text{ and } \eta = x - t.$$

Under this new coordinate,

$$2u_\xi = u_x + u_t \text{ and } 2u_\eta = u_x - u_t$$

and the wave equation (1.3) becomes²

$$4\partial_\xi\partial_\eta u = f\mathbf{1}_G.$$

For any $\xi_0 \in \mathbb{R}$, we denote the line $\{(t, x) : x + t = \xi_0\}$ by $L_{\xi=\xi_0}$, and call it the ξ -characteristic. Similarly, we denote the line $\{(t, x) : x - t = \eta_0\}$ by $L_{\eta=\eta_0}$ and call it $L_{\eta=\eta_0}$ the η -characteristic. Thus the (GCC) is equivalent to

$$\text{for a.e. } x \in \mathbb{T}, \text{meas}_{\mathbb{R}}(G \cap L_{\eta=x}) \geq c_0, \quad \text{meas}_{\mathbb{R}}(G \cap L_{\xi=x}) \geq c_0. \quad (1.5)$$

We further define measurable cylinders for any measurable sets $A, B \subset [0, 2\pi]$ as

$$L_{\xi \in A} := \bigcup_{\xi_0 \in A} L_{\xi=\xi_0} \text{ and } L_{\eta \in B} := \bigcup_{\eta_0 \in B} L_{\eta=\eta_0}. \quad (1.6)$$

Introduce the following symmetry condition:

(OSC) Let $A, B \subset \mathbb{T}$ be two measurable subsets. $G \subset [0, T] \times \mathbb{T}$ is said to obey the *observable symmetry condition* for (A, B) if³

$$\text{meas}_{\mathbb{R}^2}([G \cap L_{\xi \in A}] \Delta [G \cap L_{\eta \in B}]) = 0. \quad (1.7)$$

We call a pair (A, B) trivial, if $|A| = |B| = 0$ or $|A| = |B| = 2\pi$. In principle, we are interested in *non-trivial* pairs (A, B) . Since the observable symmetry condition is automatically satisfied for trivial pairs (A, B) .

The observable symmetry condition is very important in our work, see Fig. 1.

The first result is a new conservation law of the forced equation (1.3).

¹This integral is defined in the sense of (1.5).

²We considered the equation on $U(\xi, \eta) := u(t, x)$ and $F(\xi, \eta) = f(t, x)$. For ease of notation, we still denote U by u and F by f .

³As usual, $A \Delta B = (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference between set A and set B . Expression (1.7) means that $G \cap L_{\xi \in A} = G \cap L_{\eta \in B}$ modulo zero measure set in \mathbb{R}^2 .

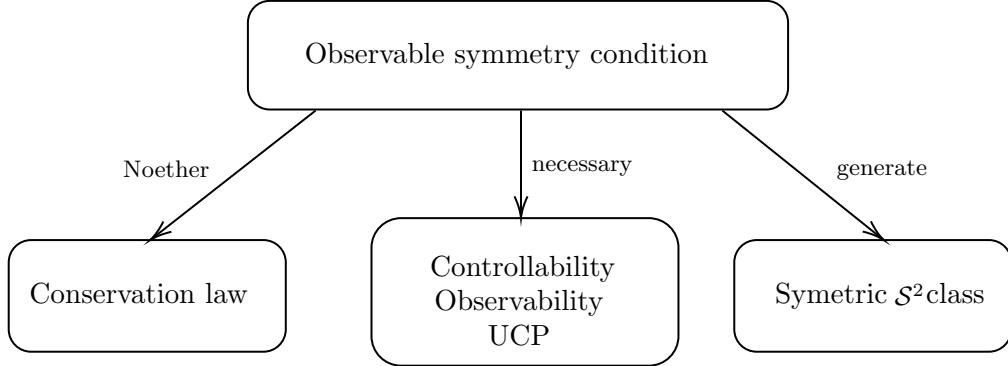


FIGURE 1. The role of (OSC) in this paper

Proposition 1.1 (Conservation law). *Let $A, B \subset \mathbb{T}$. Assume that G satisfies the (A, B) -observable symmetry condition. Let $(u, u_t) \in C([0, T]; \dot{H}^1(\mathbb{T}) \times L^2(\mathbb{T}))$ be a solution to the forced equation (1.3) and define the energy⁴*

$$I(t) = \int_{A-t} (u_x + u_t)(t, x) dx + \int_{B+t} (u_x - u_t)(t, x) dx.$$

Then we have

$$I(t) = I(0), \quad \forall t \in [0, T].$$

In the case $|A| = |B| = 2\pi$, this conservation law is exactly the condition $\int_{\mathbb{T}} u_x(t, x) dx = 0$.

Remark 1.2. *This proposition can be understood as a variant of Noether's theorem⁵ adapted to a controlled wave equation: a geometric symmetry of the forcing region yields a conserved functional of the solution. Indeed, the key hypothesis is that the region G satisfies the observable symmetry condition for (A, B) . The conclusion shows that the energy $I(t)$ is invariant in time, regardless of the force term.*

This new type of symmetry provides a *necessary condition* for the controllability and observability of (1.3), and even a *necessary condition* for the unique continuation of (1.2). See Section 2 for the detailed construction of counterexamples. Moreover, this construction gives rise to two classes of function pairs satisfying the symmetry condition, \mathcal{S}_c^2 and \mathcal{S}^2 , in Definitions 2.7 and 2.9.

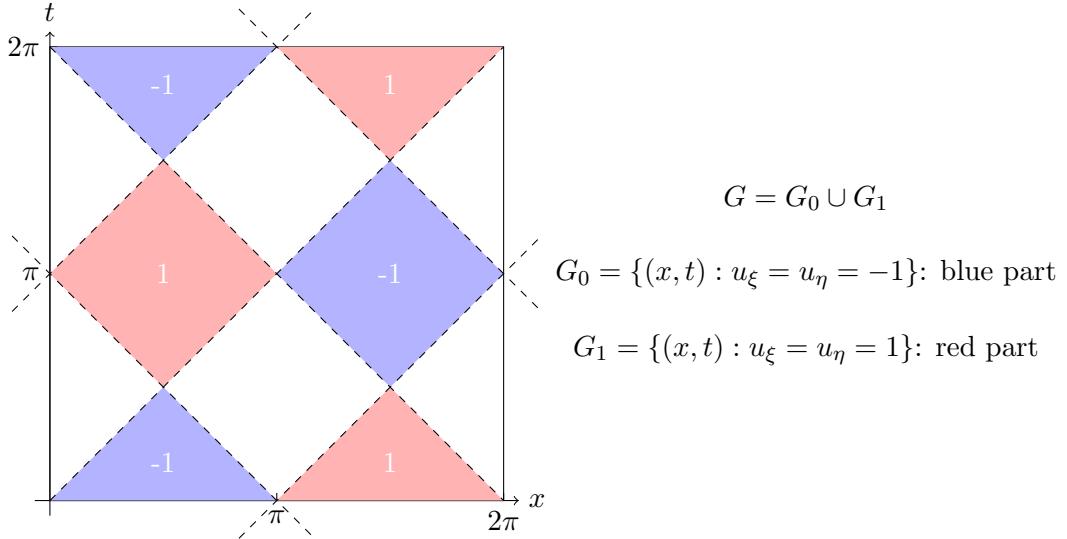
The counterexample. Let $T = 2\pi$. Let $G \subset [0, 2\pi] \times \mathbb{T}$ be given in Fig. 2. Clearly G satisfies (GCC). However, the observable symmetry condition holds for $A = B = (0, \pi)$ or $A = B = (\pi, 2\pi)$. Then one can show that the observability inequality (1.2) fails on G , see Section 2.2.

1.3. Necessary and sufficient unique continuation conditon. The second result is about the unique continuation property (UCP). Let $T > 0$ and let $G \in [0, T] \times \mathbb{T}$ be a measurable set.

⁴Indeed, under the (ξ, η) -coordinate this conservation is write as

$$\frac{1}{2} I(t) = \int_{A-t} u_\xi(t, x) dx + \int_{B+t} u_\eta(t, x) dx.$$

⁵Noether's theorem: “Every continuous symmetry of the action (or the equations of motion, in a suitable sense) implies a conserved quantity.”

FIGURE 2. Observability fails on G , though G satisfies GCC.

UCP is the qualitative version of the observability and asks:

(UCP) Let u be a solution of (1.1) and $u_t = 0$ a.e. in $G \implies u \equiv 0$.

Motivated by the geometric features of the system, we formulate the following minimal geometric assumption on the observation region, which is required for unique continuation.

(Weak GCC) Let $T > 0$. A measurable set $G \subset [0, T] \times \mathbb{T}$ is said to satisfy the weak GCC if for almost every $x \in \mathbb{T}$,

$$\int_0^T \mathbf{1}_G(s, x \pm s) ds > 0.$$

This condition is equivalent to

$$\text{for a.e. } x \in \mathbb{T}, \text{meas}_{\mathbb{R}}(G \cap L_{\eta=x}) > 0, \quad \text{meas}_{\mathbb{R}}(G \cap L_{\xi=x}) > 0.$$

Compared to the standard (GCC), it is not required a uniform positive lower bound.

Theorem 1.3. Let $T > 0$ and $G \subset [0, T] \times \mathbb{T}$ be a spacetime measurable set. The following statements are equivalent.

- (1) UCP holds on G .
- (2) G satisfies (weak GCC) and does not obey (OSC) for any non-trivial pair (A, B) ⁶.

Remark 1.4. The essence of this result lies in the role of the observable symmetry condition. Under the minimal (weak GCC) assumption, unique continuation can be established only up to a symmetric function pair belonging to the class \mathcal{S}^2 , which is generated precisely by the observable symmetry condition (see Definitions 2.9). In other words, the (weak GCC) eliminates all obstructions except those arising from the intrinsic symmetry group, and full uniqueness is recovered once solutions are taken modulo this symmetry.

⁶Reminder: for simplicity, when there is no risk of confusion, sometimes we simply write (OSC) indicating that the set G does not obey the observable symmetry condition for any non-trivial pair (A, B) .

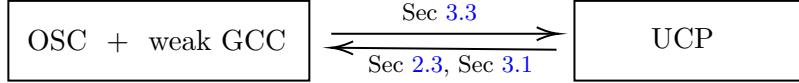


FIGURE 3. Outline of the proof to the equivalence.

1.4. Necessary and sufficient observability/controllability condition. Finally, we obtain the following sharp and complete geometric characterization of observable regions.

Theorem 1.5. *Let $T > 0$ and let $G \subset [0, T] \times \mathbb{T}$ be a measurable set. Then the following two statements are equivalent.*

- (1) *The observability inequality (1.2) holds on G ;*
- (2) *G satisfies (GCC) and does not obey (OSC) for any non-trivial pair (A, B) .*

To the best of our knowledge, this work provides the first necessary and sufficient geometric condition for observability of the wave equation on spacetime region. The observable symmetry condition plays a central role: it completes the classical GCC by capturing the additional geometric structure. This viewpoint also leads to necessary and sufficient conditions in a variety of other important geometric configurations, such as $[0, T] \times \omega$, measurable Cartesian products $E_t \times F_x$, and general spacetime open sets. A detailed comparison is presented in Section 5.

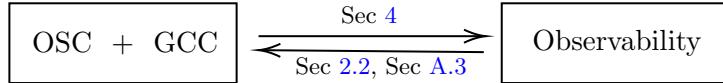


FIGURE 4. Outline of the proof to the equivalence.

Remark 1.6. *The main theorem reveals two structural features of observability that distinguish the wave equation from other important models, for example, the heat equation, [AEWZ14, WWZZ19]. First, the family of observable regions $\mathcal{O}(T)$ depends intrinsically on T ; non-observable sets arise from geometric obstructions that vary with T , leading to a genuinely time-dependent structure for $\{\mathcal{O}(T)\}_{T>0}$. Second, our characterization also suggests that no waiting time is needed for observability, indicating that, for wave equations, the spacetime geometry of the observation region may play a more decisive role than the duration of control. See Section 6.*

Moreover, we believe that the symmetry mechanism uncovered in this work may extend to other instances of wave equations and to a broader range of models, including coupled systems, semilinear equations, and geometric wave equations. These ideas may also provide insight for higher-dimensional problems, where geometric propagation and microlocal structures are considerably more intricate. In addition, the geometric perspectives here may prove useful in questions of sensor placement, optimization of observation regions, and numerical implementations of control and observability.

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2. OBSERVABLE SYMMETRY CONDITION

In this section, we prove a conservation law under the observable symmetry condition, and further show that this symmetry condition provides a necessary condition for controllability and unique continuation.

2.1. A conservation law. Let $T > 0$. Let f defined in $(t, x) \in [0, T] \times \mathbb{T}$ and G be a measurable subset of $[0, T] \times \mathbb{T}$. Assume u is a solution of the wave equation (1.3) with force f . Extend the solution, the set G , and the force f 2π -periodically to $x \in \mathbb{R}$. Thus f is defined in $t \in (0, T), x \in \mathbb{R}$ and u satisfies,

$$(\partial_t^2 - \partial_x^2)u = f\mathbf{1}_G, \quad \forall t \in (0, T), x \in \mathbb{R}. \quad (2.1)$$

Let A, B be subsets of $[0, 2\pi]$. Recall the definition of $L_{\xi \in A}$ and $L_{\eta \in B}$. Define the regions

$$\Omega^+(B; T) := L_{\eta \in B} \cap \{[0, T] \times \mathbb{R}\} = \{(t, x) \mid 0 \leq t \leq T, x \in B + t\},$$

$$\Omega^-(A; T) := L_{\xi \in A} \cap \{[0, T] \times \mathbb{R}\} = \{(t, x) \mid 0 \leq t \leq T, x \in A - t\},$$

where $A - t = \{x - t : x \in A\}, B + t = \{x + t \mid x \in B\}$. First prove the following lemma.

Lemma 2.1. *Let u be a smooth solution of (2.1) and $U = u_x + u_t, V = u_x - u_t$. Let A, B be two measurable subsets of $[0, 2\pi]$. Then*

$$\int_A U(x, 0) dx - \int_{A-T} U(x, T) dx = - \iint_{G \cap L_{\xi \in A}} f dx dt, \quad (2.2)$$

$$\int_{B+T} V(x, T) dx - \int_B V(x, 0) dx = - \iint_{G \cap L_{\eta \in B}} f dx dt. \quad (2.3)$$

Proof. Applying Lemma A.7 to U , we obtain

$$\iint_{\Omega^-(A; T)} (U_x - U_t) dx dt = \int_A U(x, 0) dx - \int_{A-T} U(x, T) dx.$$

By the definition of U , using the equation (2.1), we have

$$U_x - U_t = (\partial_x^2 - \partial_t^2)u = -f\chi_G \quad \text{in } L_{\xi \in A} \cap \{[0, T] \times \mathbb{R}\}.$$

Thus

$$\int_A U(x, 0) dx - \int_{A-T} U(x, T) dx = - \iint_{L_{\xi \in A} \cap \{[0, T] \times \mathbb{R}\}} \chi_G f dx dt = - \iint_{G \cap L_{\xi \in A}} f dx dt.$$

This proves (2.2). Similarly, we obtain (2.3). \square

Proof of Proposition 1.1. By a standard density argument, it suffices to consider smooth solutions. Thus, we assume $u \in C^\infty((0, T) \times \mathbb{T})$ now.

Step 1. We show that $I(T) = I(0)$. If G satisfies the following symmetric property

$$\text{meas}_{\mathbb{R}^2}([G \cap L_{\xi \in A}] \Delta [G \cap L_{\eta \in B}]) = 0,$$

then $G \cap L_{\xi \in A}$ equals to $G \cap L_{\eta \in B}$ up to a zero measure set. Thus for any f we have

$$\iint_{G \cap L_{\xi \in A}} f dx dt = \iint_{G \cap L_{\eta \in B}} f dx dt.$$

This, together with Lemma 2.1, gives

$$\int_A U(x, 0) \, dx - \int_{A-T} U(x, T) \, dx = \int_{B+T} V(x, T) \, dx - \int_B V(x, 0) \, dx,$$

which implies

$$\int_{A-T} U(x, T) \, dx + \int_{B+T} V(x, T) \, dx = \int_A U(x, 0) \, dx + \int_B V(x, 0) \, dx.$$

So $I(T) = I(0)$ holds as desired.

Step 2. We show that $I(T') = I(0)$ for all $T' \in (0, T)$. Fix $T' \in (0, T)$. We define

$$L'_{\xi \in A} := \bigcup_{\xi_0 \in A} L'_{\xi=\xi_0}$$

where

$$L'_{\xi=\xi_0} = \{(t, x) \in [0, T'] \times \mathbb{T} : x + t = \xi_0\}$$

is the restriction of $L_{\xi=\xi_0}$ on the region $[0, T'] \times \mathbb{T}$. Thus

$$G \cap L'_{\xi \in A} \subset G \cap L_{\xi \in A}.$$

With this observation in mind, we find that if

$$\text{meas}_{\mathbb{R}^2}([G \cap L_{\xi \in A}] \Delta [G \cap L_{\eta \in B}]) = 0,$$

then

$$\text{meas}_{\mathbb{R}^2}([G \cap L'_{\xi \in A}] \Delta [G \cap L'_{\eta \in B}]) = 0. \quad (2.4)$$

Indeed, if we set $X = G \cap L_{\xi \in A}$, $Y = G \cap L_{\eta \in B}$, $Z = [0, T'] \times \mathbb{T}$, then

$$[G \cap L'_{\xi \in A}] \Delta [G \cap L'_{\eta \in B}] = (X \cap Z) \Delta (Y \cap Z) = (X \Delta Y) \cap Z.$$

Since $(X \Delta Y)$ has zero measure, we conclude (2.4).

With (2.4) in hand, the result in Step 1 yields that $I(T') = I(0)$ as desired. \square

2.2. Necessary for controllability.

In this part, we prove the following result.

Proposition 2.2 (Controllability implies OSC). *If the system (1.3) is exactly controllable on some observation set $G \subset [0, T] \times \mathbb{T}$, then for any non-trivial pair (A, B) , G does not obey the (OSC).*

Proof. We argue by contradiction. Suppose that there are two non-trivial sets A, B such that

$$\text{meas}_{\mathbb{R}^2}([G \cap L_{\xi \in A}] \Delta [G \cap L_{\eta \in B}]) = 0.$$

According to Proposition 1.1, we have $I(T) = I(0)$, where

$$I(t) = \int_{A-t} (u_x + u_t)(t, x) \, dx + \int_{B+t} (u_x - u_t)(t, x) \, dx.$$

Since A, B are both non-trivial, we can always choose initial data (u_0, u_1) such that

$$I(0) = \int_A (u_{0x} + u_1)(t, x) \, dx + \int_B (u_{0x} - u_1)(t, x) \, dx \neq 0.$$

In fact, we can choose $(u_0, u_1) \in \dot{H}^1 \times L^2$ such that

$$u_{0x} + u_1 = \chi_A + a\chi_{\mathbb{T} \setminus A}, \quad u_{0x} - u_1 = \chi_B + a\chi_{\mathbb{T} \setminus B} \quad (2.5)$$

where a is chosen as

$$a = -\frac{\text{meas}_{\mathbb{R}}(A) + \text{meas}_{\mathbb{R}}(B)}{4\pi - (\text{meas}_{\mathbb{R}}(A) + \text{meas}_{\mathbb{R}}(B))} \quad (2.6)$$

to ensure that

$$\int_{\mathbb{T}} u_{0x} dx = \text{meas}_{\mathbb{R}}(A) + \text{meas}_{\mathbb{R}}(B) + a(2\pi - \text{meas}_{\mathbb{R}}(A)) + a(2\pi - \text{meas}_{\mathbb{R}}(B)) = 0.$$

This selection is legitimate since $4\pi - (\text{meas}_{\mathbb{R}}(A) + \text{meas}_{\mathbb{R}}(B)) > 0$. Moreover, noting both A and B has positive measures, we find

$$I(0) = \text{meas}_{\mathbb{R}}(A) + \text{meas}_{\mathbb{R}}(B) > 0.$$

According to the conservation law, we must have $I(T) \neq 0$ no matter what control f is. So the solution can not be steered to 0 at time $t = T$. Thus, the system is not exactly controllable. \square

Proof of the Counterexample in Fig. 2. Set

$$\begin{aligned} \partial_{\eta} u|_{t=0, x \in (0, \pi)} &= -1, \quad \partial_{\eta} u|_{t=0, x \in (\pi, 2\pi)} = 1, \\ \partial_{\xi} u|_{t=0, x \in (0, \pi)} &= -1, \quad \partial_{\xi} u|_{t=0, x \in (\pi, 2\pi)} = 1. \end{aligned}$$

This choice ensures that

$$\int_{\mathbb{T}} \partial_x u(t=0, x) dx = \int_{\mathbb{T}} (\partial_{\xi} u + \partial_{\eta} u)(t=0, x) dx = 0.$$

Thus, we can solve the wave equation $\partial_{\xi} \partial_{\eta} u = 0$ on the domain $(t, x) \in (0, 2\pi) \times \mathbb{T}$. Based on this special solution, we further define

$$G := \{(\xi, \eta); \partial_{\eta} u = \partial_{\xi} u = -1, \text{ or } \partial_{\xi} u = \partial_{\eta} u = 1\}.$$

This region is divided into two parts, $G = G_0 \cup G_1$ as shown in Fig. 2:

$$G_0 := \{(\xi, \eta); \partial_{\eta} u = \partial_{\xi} u = -1\}, \quad G_1 := \{(\xi, \eta); \partial_{\eta} u = \partial_{\xi} u = 1\}.$$

Clearly, the set G satisfies (GCC) with $c_0 = \sqrt{2}\pi$. But

$$\int_G |\partial_t u|^2 dx dt = \int_G |\partial_{\eta} u - \partial_{\xi} u|^2 d\xi d\eta = 0$$

and the corresponding initial data satisfies

$$\|\partial_x u_0\|_{L^2(\mathbb{T})}^2 + \|u_1\|_{L^2(\mathbb{T})}^2 = 2 \left(\|\partial_{\eta} u|_{t=0, x \in \mathbb{T}}\|_{L^2(\mathbb{T})}^2 + \|\partial_{\xi} u|_{t=0, x \in \mathbb{T}}\|_{L^2(\mathbb{T})}^2 \right) = 8\pi > 0.$$

Thus, the observability (1.2) does not hold. \square

Example 2.3. Let G_0, G_1 be the blue and red part of the square $[0, 2\pi]^2$, and let $G = G_0 \cup G_1$, see Fig. 5. The set G is constructed similarly as that in Fig. 2. Then G satisfies the (GCC), but the observability fails on G .

2.3. Necessary for unique continuation. Proposition 2.2 is equivalent to the necessity of observability by duality.

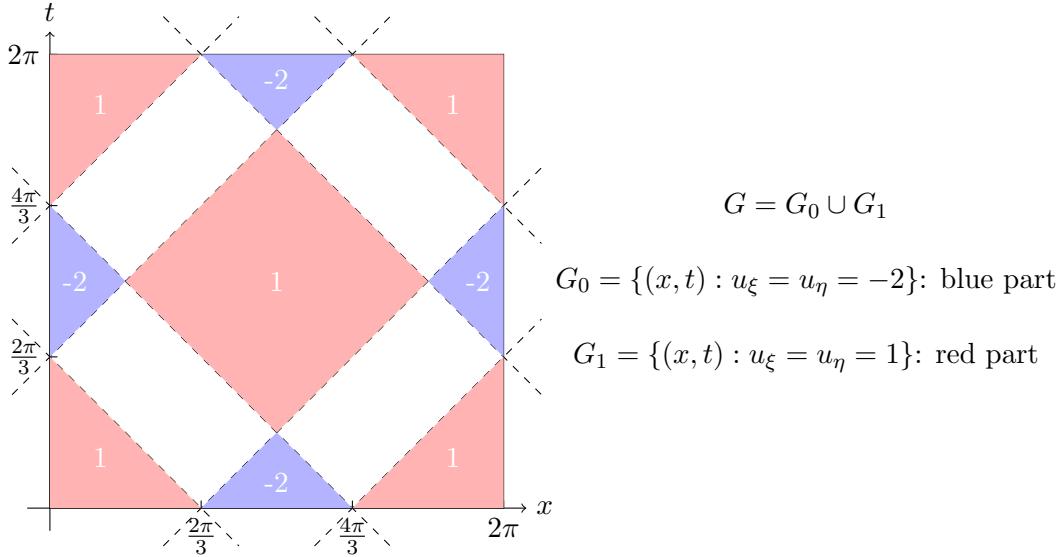


FIGURE 5. The set G satisfies GCC, but on which the observability fails.

Proposition 2.4 (Observability implies OSC). *If the observability (1.2) holds on some observation set $G \subset [0, T] \times \mathbb{T}$, then for all non-trivial pair (A, B) , G does not obey the (OSC) for (A, B) .*

Indeed, we can even show that OSC is necessary for the unique continuation property.

Proposition 2.5 (UCP implies OSC). *Assume that the following unique continuation property holds*

Let u be a solution of (1.1) and $u_t = 0$ a.e. in $G \implies u \equiv 0$.

Then for any non-trivial pair (A, B) , G does not obey the (OSC) for (A, B) .

Proof. We argue by contradiction. Assume that there are two non-trivial sets A, B such that

$$\text{meas}_{\mathbb{R}^2}([G \cap L_{\xi \in A}] \Delta [G \cap L_{\eta \in B}]) = 0.$$

This implies that (see Fig. 6)

$$G \subset (L_{\xi \in A} \cap L_{\eta \in B}) \cup \left(([0, T] \times \mathbb{T}) \setminus (L_{\xi \in A} \cup L_{\eta \in B}) \right). \quad (2.7)$$

We are going to construct nonzero solutions such that $u_t = 0$ on G . The construction is the same as in (2.5). Let

$$u_\eta|_{t=0} = \chi_A + a\chi_{\mathbb{T} \setminus A}, \quad u_\xi|_{t=0} = \chi_B + a\chi_{\mathbb{T} \setminus B} \quad (2.8)$$

where a is given by (2.6). Then

$$\int_{\mathbb{T}} u_{0x} dx = \int_{\mathbb{T}} (u_\eta + u_\xi)|_{t=0} dx = 0.$$

It is easy to see that

$$u_\eta = 1 \text{ on } L_{\xi \in A}, \quad u_\eta = a \text{ on } L_{\xi \in \mathbb{T} \setminus A}$$

and

$$u_\xi = 1 \text{ on } L_{\eta \in B}, \quad u_\xi = a \text{ on } L_{\eta \in \mathbb{T} \setminus B}.$$

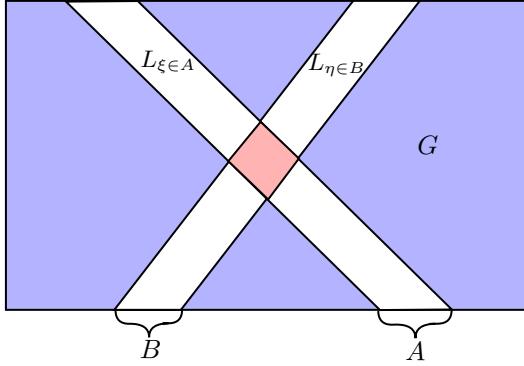


FIGURE 6. A typical region G (blue part plus red part) satisfies the observable symmetry condition for the pair (A, B) .

It follows that

$$u_t = \frac{1}{2}(u_\xi - u_\eta) = 0 \text{ on } (L_{\xi \in A} \cap L_{\eta \in B}) \cup (L_{\xi \in \mathbb{T} \setminus A} \cap L_{\eta \in \mathbb{T} \setminus B}) \quad (2.9)$$

and

$$u_t = -1 \text{ on } L_{\xi \in A} \setminus L_{\eta \in B}, \quad u_t = 1 \text{ on } L_{\eta \in A} \setminus L_{\xi \in B}. \quad (2.10)$$

Thanks to (2.7) and (2.9), we find

$$u_t = 0 \text{ on } G$$

but (2.10) shows that u is not identically zero on $[0, T] \times \mathbb{T}$, which leads a contradiction with the UCP. \square

2.4. Generated symmetric function pairs. Motivated by the symmetry condition (OSC) as well as the above examples, we define the following.

Definition 2.6 (Decomposition pair). We say $\{(A_k, B_k), k = 1, \dots, K\}$ is a decomposition pair, if $A_k, B_k \subset \mathbb{T}$ are positive measure sets such that

$$\{A_k : k = 1, \dots, K\} \subset \mathbb{T} \text{ are disjoint measurable sets,}$$

$$\{B_k : k = 1, \dots, K\} \subset \mathbb{T} \text{ are disjoint measurable sets,}$$

$$\cup_{1 \leq k \leq K} A_k = \cup_{1 \leq k \leq K} B_k = \mathbb{T}.$$

Definition 2.7 (Symmetric function pair). We say (f, g) is a symmetric function pair if there exists a decomposition pair $\{(A_k, B_k), k = 1, \dots, K\}$, a sequence of different numbers $\{s_k\}_{k=1}^K$ such that

$$f = \sum_{1 \leq k \leq K} s_k \chi_{A_k}, \quad g = \sum_{1 \leq k \leq K} s_k \chi_{B_k},$$

and

$$\int_{\mathbb{T}} (f + g)(x) dx = 0.$$

The set of all symmetric function pairs is denoted by \mathcal{S}_c^2 .

The essential feature is that f and g take the same values s_k on the respective positive-measure sets A_k and B_k . Such symmetric pairs are closely related to (OSC) and, in particular, are used to study the relation between the (GCC) and the UCP.

In what follows, we introduce slightly weaker notions, the weak decomposition pair and the weak symmetric function pair, which will be used to investigate the corresponding relationship between the (weak GCC) and UCP.

Definition 2.8 (Weak decomposition pair). Let I be a countable index set (it can be finite). We say $\{(A_k, B_k), k \in I\}$ is a weak decomposition pair, if $A_k, B_k \subset \mathbb{T}$ are positive measure sets such that

$$\{A_k : k \in I\} \subset \mathbb{T} \text{ are disjoint measurable sets,}$$

$$\{B_k : k \in I\} \subset \mathbb{T} \text{ are disjoint measurable sets,}$$

$$\cup_{k \in I} A_k = \cup_{k \in I} B_k = \mathbb{T}.$$

Definition 2.9 (Weak symmetric function pair). We say (f, g) is a weak symmetric function pair if there exists a weak decomposition pair $\{(A_k, B_k), k \in I\}$, a sequence of different numbers $\{s_k\}_{k \in I}$ such that

$$f = \sum_{k \in I} s_k \chi_{A_k}, \quad g = \sum_{k \in I} s_k \chi_{B_k} \quad \text{in } L^2(\mathbb{T})$$

and

$$\int_{\mathbb{T}} (f + g)(x) dx = 0.$$

The set of all weak symmetric function pairs is denoted by \mathcal{S}^2 .

3. SHARP UNIQUE CONTINUATION CONDITION

In this section, we aim to establish a sufficient and necessary condition to ensure the unique continuation property. We begin with the necessity in Section 3.1, where we show that the (weak GCC) is required for unique continuation. The sufficiency is then proved in two steps. On the one hand, we obtain unique continuation under (OSC) and (GCC) in Section 3.2. On the other hand, in Section 3.3 we generalize this result and prove unique continuation under (OSC) and (weak GCC).

Although the latter already suffices to build the sufficient part, the former offers a cleaner setting in which to present the main ideas in a simpler way, and it will also be used later to complete the proof of wave observability in the next section. For these reasons, we separate the argument into two subsections.

3.1. Unique continuation implies weak GCC. We demonstrate that the (weak GCC) introduced here is the minimal geometric requirement for establishing unique continuation.

The free wave equation (1.1), under the null coordinate, becomes $\partial_\xi \partial_\eta u = 0$. Recall the definitions of $L_{\xi=\xi_0}$, $L_{\xi \in A}$, $L_{\eta=\eta_0}$, and $L_{\eta \in B}$. Throughout this section, since the value of $T > 0$ is fixed, we simplify notation by writing

$$L_{\xi=\xi_0} \cap \{[0, T] \times \mathbb{R}\}, \quad L_{\xi \in A} \cap \{[0, T] \times \mathbb{R}\}, \quad L_{\eta=\eta_0} \cap \{[0, T] \times \mathbb{R}\}, \quad L_{\eta \in B} \cap \{[0, T] \times \mathbb{R}\},$$

simply as $L_{\xi=\xi_0}$, $L_{\xi \in A}$, $L_{\eta=\eta_0}$, and $L_{\eta \in B}$.

If u is a smooth solution to (1.1), then the classical theory of transport equation shows that $\partial_\xi u$ is invariant along the ξ -characteristics and $\partial_\eta u$ is invariant along the η -characteristics. In our setting with $\partial_\xi u, \partial_\eta u \in L^2(\mathbb{T})$, one has $\partial_\xi u, \partial_\eta u$ are constant almost everywhere on characteristic lines; its proof is left in Section A.2.

Lemma 3.1. *Let u be a solution to (1.1). Then for almost every ξ (resp. η), $\partial_\xi u$ (resp. $\partial_\eta u$) is a constant almost everywhere on the ξ -characteristics (resp. η -characteristics).*

Proposition 3.2 (Unique continuation implies weak GCC). *Let $T > 0$ and $G \in [0, T] \times \mathbb{T}$. Assume that the following unique continuation property holds*

$$\text{Let } u \text{ be a solution of (1.1) and } u_t = 0 \text{ a.e. in } G \implies u \equiv 0.$$

Then G satisfies the (weak GCC).

Proof. We argue by contradiction. Suppose that G does not satisfy the (weak GCC), there exists a subset $A \subset \mathbb{T}$ such that $\text{meas}_{\mathbb{R}}(A) > 0$ and

$$\text{meas}_{\mathbb{R}}(G \cap L_{\eta \in A}) = 0 \quad \text{or} \quad \text{meas}_{\mathbb{R}}(G \cap L_{\xi \in A}) = 0.$$

Without loss of generality, we only consider the case

$$\text{meas}_{\mathbb{R}}(G \cap L_{\eta \in A}) = 0. \quad (3.1)$$

Since A has positive measure, one can find a non-zero function ϕ such that

$$\int_A \phi(x) dx = 0, \quad \int_A |\phi| dx = \text{meas}_{\mathbb{R}}(A) > 0. \quad (3.2)$$

Set

$$u_{0x} = -u_1 = \chi_A \phi.$$

This is possible since $\int_{\mathbb{T}} u_{0,x} dx = \int_A \phi dx = 0$. Then we have

$$\partial_\xi u|_{t=0} = u_{0x} + u_1 = 0, \quad \partial_\eta u|_{t=0} = u_{0x} - u_1 = 2\chi_A \phi.$$

It follows that

$$\partial_\xi u = 0 \quad \text{on } [0, T] \times \mathbb{T}$$

and

$$\partial_\eta u = 0 \quad \text{on } L_{\eta \in \mathbb{T} \setminus A}.$$

Thus we have

$$u_t = \frac{1}{2}(\partial_\xi u - \partial_\eta u) = 0 \quad \text{on } L_{\eta \in \mathbb{T} \setminus A}.$$

Thanks to (3.1), we see $G \subset L_{\eta \in \mathbb{T} \setminus A}$, and thus $u_t = 0$ on G . By the UCP, we must have $\partial_\eta u|_{t=0} = 2\chi_A \phi \equiv 0$, which leads a contradiction with (3.2). \square

3.2. OSC and GCC imply unique continuation. We already know that (weak GCC) is necessary for unique continuation. Now we turn to analyze the difference between these two conditions: (GCC) and (OSC). We start by proving that (GCC) implies unique continuation up to a symmetry function pair from the class \mathcal{S}_c^2 .

Proposition 3.3 (GCC implies UCP up to \mathcal{S}_c^2). *Let $T > 0$. Let $G \in [0, T] \times \mathbb{T}$ satisfy (GCC). If u solves the wave equation (1.1) and $u_t = 0$ on G , then its initial state $(\partial_\xi u, \partial_\eta u)|_{t=0}$ belongs to the symmetry function pair class \mathcal{S}_c^2 .*

Proof. Since $\partial_t u = \partial_\xi u - \partial_\eta u$, the assumption $u_t = 0$ almost everywhere on G implies that

$$\partial_\xi u = \partial_\eta u \text{ a.e. on } G. \quad (3.3)$$

Let (ξ_0, η_0) be a point such that (3.3) holds, namely $\partial_\xi u(\xi_0, \eta_0) = \partial_\eta u(\xi_0, \eta_0)$. Thanks to the equation $\partial_\xi \partial_\eta u = 0$, using Lemma 3.1 one has

$$\partial_\eta u|_{L_{\eta=\eta_0}} = \partial_\eta u(\xi_0, \eta_0), \quad \partial_\xi u|_{L_{\xi=\xi_0}} = \partial_\xi u(\xi_0, \eta_0)$$

hold almost everywhere on the lines $L_{\eta=\eta_0}$ and $L_{\xi=\xi_0}$, respectively. It follows that

$$\partial_\eta u|_{L_{\eta=\eta_0}} = \partial_\xi u|_{L_{\xi=\xi_0}} \tag{3.4}$$

up to sets with zero measure.

On the other hand, if we define $\mathcal{U}_{\eta=\eta_0} := \{\xi; (\xi, \eta_0) \in G\}$, and on the η -characteristic line $L_{\eta=\eta_0}$ we define

$$G_{\eta=\eta_0} := \{(\xi, \eta) \in G; \eta = \eta_0\} = G \cap L_{\eta=\eta_0}.$$

Then, from the (GCC) with lower bound c_0 we know that $\sqrt{2}|\mathcal{U}_{\eta=\eta_0}| = |G_{\eta=\eta_0}| \geq c_0$.

Hence, by the observation (3.4) one has

$$\partial_\eta u|_{L_{\eta=\eta_0}} = \partial_\xi u|_{L_{\xi=\xi_0}} \text{ for almost every } \xi_0 \in \mathcal{U}_{\eta=\eta_0}$$

up to sets with zero measure. In particular, we have

$$\partial_\eta u|_{\{t=0\} \cap L_{\eta=\eta_0}} = \partial_\eta u(\xi_0, \eta_0) = \partial_\xi u|_{\{t=0\} \cap L_{\xi \in \mathcal{U}_{\eta=\eta_0}}} \tag{3.5}$$

up to sets with zero measure. Namely, the value of $\partial_\eta u(\xi_0, \eta_0)$ determines almost every value of $\partial_\xi u|_{\{t=0\} \cap L_{\xi \in \mathcal{U}_{\eta=\eta_0}}}$ for $|\mathcal{U}_{\eta=\eta_0}| \geq c_0/\sqrt{2}$. This means that $\partial_\xi u|_{\{t=0\} \cap L_{\xi \in \mathcal{U}_{\eta=\eta_0}}}$ is a constant almost everywhere. Repeating this process if necessary, there exists an integer $N \in \mathbb{N}^*$, smaller than $1 + \sqrt{2}(2\pi + T)/c_0$, and finite distinct numbers $\{a_j\}_{1 \leq j \leq N}$ and finite disjoint measurable sets $\{\mathcal{U}_j\}_{1 \leq j \leq N} \subset \mathbb{T}$ such that

$$\partial_\xi u|_{t=0} = \sum_{j=1}^N \mathbf{1}_{\mathcal{U}_j}(x) a_j, \quad a.e.x \in \mathbb{T} \tag{3.6}$$

with $\bigcup_j \mathcal{U}_j = \mathbb{T}$, $|\mathcal{U}_j| \geq c_0/\sqrt{2}$. Similarly, there exists an integer M , smaller than $1 + \sqrt{2}(2\pi + T)/c_0$ and distinct constants $\{b_k\}_{1 \leq k \leq M}$ and disjoint measurable sets $\{\mathcal{V}_k\}_{1 \leq k \leq M}$ such that

$$\partial_\eta u|_{t=0} = \sum_{k=1}^M \mathbf{1}_{\mathcal{V}_k}(x) b_k, \quad a.e.x \in \mathbb{T} \tag{3.7}$$

with $\bigcup_k \mathcal{V}_k = \mathbb{T}$, $|\mathcal{V}_k| \geq c_0/\sqrt{2}$.

Note that, for almost every point $(\xi_0, \eta_0) \in G$, we know that $\xi_0 \in \bigcup_{j=1}^N \mathcal{U}_j$ and $\eta_0 \in \bigcup_{k=1}^M \mathcal{V}_k$. Let us assume that $(\xi_0, \eta_0) \in \mathcal{U}_j \times \mathcal{V}_k = L_{\xi \in \mathcal{U}_j} \cap L_{\eta \in \mathcal{V}_k}$. Similar to the previous discussion, we obtain

$$\begin{aligned} a_j &= \partial_\xi u|_{\{t=0, \xi_0 \in \mathcal{U}_j\}} = \partial_\xi u|_{L_{\xi=\xi_0}} = \partial_\xi u(\xi_0, \eta_0) \\ &= \partial_\eta u|_{L_{\eta=\eta_0}} = \partial_\xi u|_{\{t=0, \eta_0 \in \mathcal{V}_k\}} = b_k. \end{aligned}$$

This shows that, for every set \mathcal{U}_j , there exists a unique set \mathcal{V}_k corresponding to it. Thus, after relabeled \mathcal{V}_k , we can rewrite (3.6) and (3.7) as

$$\partial_\xi u|_{t=0} = \sum_{j=1}^K \mathbf{1}_{\mathcal{U}_j}(x) a_j, \quad a.e.x \in \mathbb{T} \tag{3.8}$$

and

$$\partial_\eta u|_{t=0} = \sum_{j=1}^K \mathbf{1}_{\mathcal{V}_j}(x) a_j, \quad a.e.x \in \mathbb{T}. \quad (3.9)$$

This proves that $(\partial_\xi u|_{t=0}, \partial_\eta u|_{t=0}) \in \mathcal{S}_c^2$. Indeed, $\{\mathcal{V}_k, \mathcal{U}_k\}_{1 \leq k \leq M}$ is a decomposition pair in Definition 2.6. \square

Remark 3.4. *Proposition 3.3 says that if the solution to (1.4) satisfies $u_t = 0$ on some G with (GCC), then the initial data $(u_{0x} + u_1, u_{0x} - u_1)$ is a symmetric simple function pair. This result can be understood as a unique continuation property up to the set \mathcal{S}_c^2 .*

Corollary 3.5 (Unique continuation). *Let $T > 0$. Assume that $G \in [0, T] \times \mathbb{T}$ satisfies (GCC) and does not obey the (OSC) for any non-trivial pair (A, B) . If u solves the wave equation (1.1) and $u_t = 0$ on G , then $u \equiv 0$.*

Proof. Thanks to Proposition 3.3, we know that $(\partial_\xi u|_{t=0}, \partial_\eta u|_{t=0}) \in \mathcal{S}_c^2$. So we can assume that (3.8) and (3.9) hold. We split the discussion into two cases.

Case (1). We assume $K = 1$. In this case, by (3.8) and (3.9) we have $\partial_\xi u|_{t=0} = \partial_\eta u|_{t=0} = a_1$ for almost every $x \in \mathbb{T}$. It follows that

$$\partial_x u(t, x)|_{t=0} = \frac{1}{2}(\partial_\xi u|_{t=0} + \partial_\eta u|_{t=0}) = a_1.$$

Note that $u(0, x) = u_0 \in \dot{H}^1(\mathbb{T})$, we have

$$a_1 = \partial_x u(0, x) = \sum_{k \neq 0} \widehat{\partial_x u_0}(k) e^{ikx}$$

where $\widehat{\partial_x u_0}(k)$ is the k -th Fourier coefficient of $\partial_x u_0$. It follows that $a_1 = 0$ and thus $u_0 \equiv 0$. This also implies that $u_1 \equiv 0$. Hence, the solution $u \equiv 0$.

Case (2). We assume $K \geq 2$. In the sequel, we shall show that this case will never happen, due to our assumption that G does not obey the (OSC) for any non-trivial pair (A, B) . In fact, thanks to (3.3), we know

$$G \subset \{(t, x) \in [0, T] \times \mathbb{T} : \partial_\xi u = \partial_\eta u\}.$$

This, together with (3.8)-(3.9), implies that

$$G \subset \bigcup_{l=1}^K (L_{\xi \in \mathcal{U}_l} \cap L_{\eta \in \mathcal{V}_l}) \quad (3.10)$$

up to a set with zero measure.

Now we define $G_l := G \cap L_{\xi \in \mathcal{U}_l} \cap L_{\eta \in \mathcal{V}_l}$, $l \in \{1, \dots, K\}$. Then, using (3.10), we have

$$G = G \bigcap \bigcup_{l=1}^K (L_{\xi \in \mathcal{U}_l} \cap L_{\eta \in \mathcal{V}_l}) = \bigcup_{l=1}^K G_l \quad (3.11)$$

up to a set with zero measure. With $K \geq 2$ in mind, since $|G| > 0$, there exist at least two sets $G_{l_0}, G_{l'_0}$ such that $\text{meas}_{\mathbb{R}^2}(G_{l_0}) > 0, \text{meas}_{\mathbb{R}^2}(G_{l'_0}) > 0$. Thus, the sets \mathcal{U}_{l_0} and \mathcal{V}_{l_0} associated with G_{l_0} satisfy that

$$0 < \text{meas}_{\mathbb{R}}(\mathcal{U}_{l_0}), \text{meas}_{\mathbb{R}}(\mathcal{V}_{l_0}) < 2\pi.$$

In other words, they are non-trivial sets. However, for these \mathcal{U}_{l_0} and \mathcal{V}_{l_0} , using (3.11), we have

$$G \cap L_{\xi \in \mathcal{U}_{l_0}} = \bigcup_{l=1}^K (G_l \cap L_{\xi \in \mathcal{U}_{l_0}}) = G_{l_0} \cap L_{\xi \in \mathcal{U}_{l_0}} \cap L_{\eta \in \mathcal{V}_{l_0}} = \bigcup_{l=1}^K (G_l \cap L_{\eta \in \mathcal{V}_{l_0}}) = G \cap L_{\eta \in \mathcal{V}_{l_0}}$$

up to a set with zero measure. This implies that

$$\text{meas}_{\mathbb{R}^2}([G \cap L_{\xi \in \mathcal{U}_{l_0}}] \Delta [G \cap L_{\eta \in \mathcal{V}_{l_0}}]) = 0,$$

and thus G obeys (OSC) with the non-trivial pair $(L_{\xi \in \mathcal{U}_{l_0}}, L_{\eta \in \mathcal{V}_{l_0}})$, which contradicts to our assumptions.

The above analysis shows that case (2) is impossible, and in case (1) we must have $\partial_\eta u|_{t=0} = \partial_\xi u|_{t=0} \equiv 0$. Hence, we obtain

$$\partial_t u(0, x) = 0, \quad \partial_x u(0, x) = 0.$$

This implies that the solution u to (1.1) is identically 0. \square

3.3. OSC and weak GCC imply unique continuation. In this sequel, we generalize the result in Proposition 3.3, which forms the following proposition.

Proposition 3.6 (Weak GCC implies UCP up to \mathcal{S}^2). *Let $T > 0$. Let $G \in [0, T] \times \mathbb{T}$ satisfy the (weak GCC). If u solves the wave equation (1.1) and $u_t = 0$ on G , then its initial state $(\partial_\xi u, \partial_\eta u)|_{t=0}$ belongs to the symmetry function pair class \mathcal{S}^2 .*

Proof. We use the same notation in the proof of Proposition 3.3. Recall that we have the relation (3.5), namely

$$\partial_\eta u|_{\{t=0\} \cap L_{\eta=\eta_0}} = \partial_\eta u(\xi_0, \eta_0) = \partial_\xi u|_{\{t=0\} \cap L_{\xi \in \mathcal{U}_{\eta=\eta_0}}}$$

up to sets with zero measure. Namely, the value of $\partial_\eta u(\xi_0, \eta_0)$ determines almost every value of $\partial_\xi u|_{\{t=0\} \cap L_{\xi \in \mathcal{U}_{\eta=\eta_0}}}$. Since G satisfies the (weak GCC), we know

$$\text{meas}_{\mathbb{R}}(\{t = 0\} \cap L_{\xi \in \mathcal{U}_{\eta=\eta_0}}) > 0.$$

In other words, $\partial_\xi u|_{\{t=0\}}$ equals to $\partial_\eta u(\xi_0, \eta_0)$ at least on a positive measure subset set of \mathbb{T} . Thus the set

$$\mathcal{U}_{(\xi_0, \eta_0)} := \{x_0 \in \mathbb{T} : \partial_\xi u|_{\{t=0, x=x_0\}} = \partial_\eta u(\xi_0, \eta_0)\}$$

has positive measure. Repeating this process for other point $(\xi'_0, \eta'_0) \in [0, T] \times \mathbb{T}$, we shall find that, there exists a family sets $\mathcal{U}_\ell (\ell \in I)$, I is a index set, such that

$$\mathcal{U}_\ell \cap \mathcal{U}_{\ell'} = \emptyset \text{ for } \ell \neq \ell', \quad \text{meas}_{\mathbb{R}}(\mathcal{U}_\ell) > 0 \text{ for } \ell \in I, \quad \mathbb{T} = \cup_{\ell \in I} \mathcal{U}_\ell \quad (3.12)$$

and $\partial_\xi u|_{\{t=0\}}$ is a constant on each \mathcal{U}_ℓ . One can show that, under the restrictions (3.12), the index set I is at most countable, thus, after relabeling if necessary, $\{\mathcal{U}_\ell, \ell \geq 1\}$ is a weak decomposition of \mathbb{T} . Moreover, there exists a sequence of different numbers $\{s_\ell\}$ such that

$$\partial_\xi u|_{\{t=0\}} = s_\ell \text{ on } \mathcal{U}_\ell.$$

Note that $\partial_\xi u|_{\{t=0\}} \in L^2(\mathbb{T})$, and \mathcal{U}_ℓ are disjoint for different ℓ , we find

$$\|\partial_\xi u|_{\{t=0\}}\|_{L^2(\mathbb{T})}^2 = \sum_{\ell} |s_\ell|^2 \text{meas}_{\mathbb{R}}(\mathcal{U}_\ell) < \infty.$$

Thus we obtain

$$\partial_\xi u|_{\{t=0\}} = \sum_{\ell \geq 1} s_\ell \chi_{\mathcal{U}_\ell} \quad \text{in } L^2(\mathbb{T}).$$

Similarly, we can find a weak decomposition $\mathcal{V}_\ell (\ell \geq 1)$ such that

$$\partial_\eta u|_{\{t=0\}} = \sum_{\ell \geq 1} s_\ell \chi_{\mathcal{V}_\ell} \quad \text{in } L^2(\mathbb{T}).$$

In summary, we find that $(\partial_\xi u|_{\{t=0\}}, \partial_\eta u|_{\{t=0\}}) \in \mathcal{S}^2$. \square

With Proposition 3.6 in hand, the following result follows the same way as the proof of Corollary 3.5.

Corollary 3.7 (OSC and weak GCC imply UCP). *Assume that G satisfies the (weak GCC) and does not obey the (OSC) for any non-trivial pair (A, B) . If u solves the wave equation (1.1) and $u_t = 0$ on G , then $u \equiv 0$.*

4. SHARP OBSERVABILITY AND CONTROLLABILITY CONDITION

This section is devoted to proving our Theorem 1.5. Inspired by the standard compactness-uniqueness approach, we prove a weak observability up to a compact term in Section 4.1. Then, based on the unique continuation property that we established in the preceding section, we complete the proof of Theorem 1.5. For the necessity of (GCC), it is a direct consequence of the transport equations; we include this part in the Appendix A.3.

4.1. Observability of wave equation up to a compact term. In this section, we show that (GCC) is sufficient for the observability for the wave equation at high frequencies. The proof uses the observability inequality for transport equations.

Proposition 4.1. *Let $G \subset [0, T] \times \mathbb{T}$ be a measurable set with positive measure and G satisfy (GCC). There exists $N > 0$ and $C > 0$ such that for every $(u_0, u_1) \in \dot{H}^1(\mathbb{T}) \times L^2(\mathbb{T})$ with $\text{supp}(\widehat{u_0}) \subset \{|k| > N\}$ and $\text{supp}(\widehat{u_1}) \subset \{|k| > N\}$, we have*

$$\|u_0\|_{\dot{H}^1(\mathbb{T})}^2 + \|u_1\|_{L^2(\mathbb{T})}^2 \leq C \int_G |\partial_t u(t, x)|^2 dt dx, \quad (4.1)$$

where u is the solution to (1.3) with initial data (u_0, u_1) .

Proof. For the simplicity of presentation, we set $T = 2\pi$. Note that the value of T does not affect the calculation in high frequency. Suppose $\text{supp}(\widehat{u_0}) \subset \{|k| > N\}$ and $\text{supp}(\widehat{u_1}) \subset \{|k| > N\}$. We write

$$u_0(x) = \sum_{|k|>N} \widehat{u_0}(k) e^{ikx}, \quad u_1(x) = \sum_{|k|>N} \widehat{u_1}(k) e^{ikx}.$$

Therefore, the solution to (1.3) has the decomposition

$$u(t, x) = \sum_{|k|>N} \left(\frac{i k \widehat{u_0}(k) + \widehat{u_1}(k)}{2ik} e^{ik(t+x)} + \frac{i k \widehat{u_0}(k) - \widehat{u_1}(k)}{2ik} e^{-ik(t-x)} \right).$$

Then we can rewrite the right-hand side of (4.1) as

$$\begin{aligned} \|\partial_t u\|_{L^2(G)}^2 &= \int_G \left| \sum_{|k|>N} \left(ik \frac{\widehat{ku_0}(k) + \widehat{u_1}(k)}{2ik} e^{ik(t+x)} - ik \frac{\widehat{ku_0}(k) - \widehat{u_1}(k)}{2ik} e^{-ik(t-x)} \right) \right|^2 dx dt \\ &= \frac{1}{4} \int_G \left| \sum_{|k|>N} \left((ik\widehat{u_0}(k) + \widehat{u_1}(k))e^{ik(t+x)} - (ik\widehat{u_0}(k) - \widehat{u_1}(k))e^{-ik(t-x)} \right) \right|^2 dx dt. \end{aligned}$$

We expand the square term as

$$\int_G \left| \sum_{|k|>N} \left((ik\widehat{u_0}(k) + \widehat{u_1}(k))e^{ik(t+x)} - (ik\widehat{u_0}(k) - \widehat{u_1}(k))e^{-ik(t-x)} \right) \right|^2 dx dt = I_1 + I_2 + I_3 + I_4,$$

where $I_j, j = 1, 2, 3, 4$, are given by

$$\begin{aligned} I_1 &= \int_G \left| \sum_{|k|>N} (ik\widehat{u_0}(k) + \widehat{u_1}(k))e^{ik(t+x)} \right|^2 dx dt, \\ I_2 &= - \int_{\mathbb{T} \times \mathbb{T}} \mathbf{1}_G(t, x) \sum_{|k|, |l|>N} (ik\widehat{u_0}(k) - \widehat{u_1}(k)) \overline{(il\widehat{u_0}(l) + \widehat{u_1}(l))} e^{-il(t+x)} e^{-ik(t-x)} dx dt, \\ I_3 &= - \int_{\mathbb{T} \times \mathbb{T}} \mathbf{1}_G(t, x) \sum_{|k|, |l|>N} \overline{(ik\widehat{u_0}(k) - \widehat{u_1}(k))} (il\widehat{u_0}(l) + \widehat{u_1}(l)) e^{il(t+x)} e^{ik(t-x)} dx dt, \\ I_4 &= \int_G \left| \sum_{|k|>N} (ik\widehat{u_0}(k) - \widehat{u_1}(k))e^{-ik(t-x)} \right|^2 dx dt. \end{aligned}$$

We first note that the dominating terms are I_1 and I_4 . Using the estimate (A.9) and (A.10), we obtain

$$I_1 > 2\pi c \sum_{|k|>N} |ik\widehat{u_0}(k) + \widehat{u_1}(k)|^2, \quad I_4 > 2\pi c \sum_{|k|>N} |ik\widehat{u_0}(k) - \widehat{u_1}(k)|^2.$$

It follows that $I_1 + I_4 > 4\pi c \sum_{|k|>N_0} (|k|^2 |\widehat{u_0}(k)|^2 + |\widehat{u_1}(k)|^2)$. Next we turn to the mixed terms I_2 and I_3 . For I_2 , we rewrite

$$I_2 = - \sum_{|k|, |l|>N} (ik\widehat{u_0}(k) - \widehat{u_1}(k)) \overline{(il\widehat{u_0}(l) + \widehat{u_1}(l))} \widehat{\mathbf{1}_G}(l+k, l-k).$$

By Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} |I_2| &\leq \left(\sum_{|k|, |l|>N} |(ik\widehat{u_0}(k) - \widehat{u_1}(k)) \overline{(il\widehat{u_0}(l) + \widehat{u_1}(l))}|^2 \right)^{\frac{1}{2}} \left(\sum_{|k|, |l|>N} |\widehat{\mathbf{1}_G}(l+k, l-k)|^2 \right)^{\frac{1}{2}} \\ &\leq 2 \sum_{|k|>N} (|k|^2 |\widehat{u_0}(k)|^2 + |\widehat{u_1}(k)|^2) \left(\sum_{|k|, |l|>N} |\widehat{\mathbf{1}_G}(l+k, l-k)|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Let $k + l = \alpha_1, l - k = \alpha_2$, then

$$\{(\alpha_1, \alpha_2) \in \mathbb{Z}^2 : |\alpha_1 + \alpha_2| > 2N, |\alpha_1 - \alpha_2| > 2N\} \subset \{(\alpha_1, \alpha_2) \in \mathbb{Z}^2 : \alpha_1^2 + \alpha_2^2 > 2N^2\}.$$

Using the Plancherel theorem, we know that $\left(\sum_{\alpha \in \mathbb{Z}^2} |\widehat{\mathbf{1}_G}(\alpha)|^2\right)^{\frac{1}{2}} = \|\mathbf{1}_G\|_{L^2(\mathbb{T})} = |G|^{\frac{1}{2}} < \infty$. Thus, there exists a constant $N > 0$ depending only on G such that

$$\left(\sum_{|\alpha|^2 > 2N, \alpha \in \mathbb{Z}^2} |\widehat{\mathbf{1}_G}(\alpha)|^2\right)^{\frac{1}{2}} \leq \frac{c}{10}.$$

With this choice of N , we can bound I_2 as $|I_2| \leq \frac{c}{5} \sum_{|k| > N} (|k|^2 |\widehat{u_0}(k)|^2 + |\widehat{u_1}(k)|^2)$. Similarly, we have the same estimate for I_3 , $|I_3| \leq \frac{c}{5} \sum_{|k| > N} (|k|^2 |\widehat{u_0}(k)|^2 + |\widehat{u_1}(k)|^2)$. Combining the above bounds together, we infer the high-frequency estimate

$$\begin{aligned} & \int_G \left| \sum_{|k| > N} \left((\mathrm{i}k \widehat{u_0}(k) + \widehat{u_1}(k)) e^{\mathrm{i}k(t+x)} - (\mathrm{i}k \widehat{u_0}(k) - \widehat{u_1}(k)) e^{-\mathrm{i}k(t-x)} \right) \right|^2 dx dt \\ & \geq I_1 + I_4 - |I_2| - |I_3| \geq (4\pi c - \frac{2c}{4}) (|k|^2 |\widehat{u_0}(k)|^2 + |\widehat{u_1}(k)|^2). \end{aligned} \quad (4.2)$$

By the Plancherel theorem, we compute the left-hand side of (4.1) as

$$\|u_0\|_{\dot{H}^1(\mathbb{T})}^2 + \|u_1\|_{L^2(\mathbb{T})}^2 = \sum_{|k| > N} (|k|^2 |\widehat{u_0}(k)|^2 + |\widehat{u_1}(k)|^2).$$

This, together with (4.2), gives the desired conclusion. \square

Based on the high-frequency estimates in Proposition 4.1, we can obtain an observability inequality in G up to a compact term.

Corollary 4.2. *There exists a constant $C > 0$ such that,*

$$\|u_0\|_{\dot{H}^1(\mathbb{T})}^2 + \|u_1\|_{L^2(\mathbb{T})}^2 \leq C \left(\int_G |\partial_t u(t, x)|^2 dx dt + \|u_0\|_{L^2(\mathbb{T})}^2 + \|u_1\|_{\dot{H}^{-1}(\mathbb{T})}^2 \right), \quad (4.3)$$

holds for every solution u to the wave equation (1.1) with $(u_0, u_1) \in \dot{H}^1(\mathbb{T}) \times L^2(\mathbb{T})$.

Proof. Let N be the same as in Proposition 4.1. For $(u_0, u_1) \in \dot{H}^1(\mathbb{T}) \times L^2(\mathbb{T})$, we decompose them into

$$u_0 = u_0^N + u_{0,N}; \quad u_1 = u_1^N + u_{1,N},$$

where $u_j^N (j = 0, 1)$ denotes the truncated high-frequency part. Let u^N and u_N be the solution to (1.1) with initial states (u_0^N, u_1^N) and $(u_{0,N}, u_{1,N})$, respectively. Therefore, we obtain

$$\|u_0\|_{\dot{H}^1(\mathbb{T})}^2 + \|u_1\|_{L^2(\mathbb{T})}^2 = \|u_0^N\|_{\dot{H}^1(\mathbb{T})}^2 + \|u_1^N\|_{L^2(\mathbb{T})}^2 + \|u_{0,N}\|_{\dot{H}^1(\mathbb{T})}^2 + \|u_{1,N}\|_{L^2(\mathbb{T})}^2.$$

Using the high-frequency estimate (4.1), we have

$$\|u_0\|_{\dot{H}^1(\mathbb{T})}^2 + \|u_1\|_{L^2(\mathbb{T})}^2 \leq C \int_G |\partial_t u^N(t, x)|^2 dx dt + \|u_{0,N}\|_{\dot{H}^1(\mathbb{T})}^2 + \|u_{1,N}\|_{L^2(\mathbb{T})}^2.$$

Due to the linear superposition of $u = u^N + u_N$, we derive that

$$\|u_0\|_{\dot{H}^1(\mathbb{T})}^2 + \|u_1\|_{L^2(\mathbb{T})}^2 \leq 2C \int_G |\partial_t u(t, x)|^2 dx dt + 2C \int_G |\partial_t u_N(t, x)|^2 dx dt + \|u_{0,N}\|_{\dot{H}^1(\mathbb{T})}^2 + \|u_{1,N}\|_{L^2(\mathbb{T})}^2.$$

Using the low-frequency truncation and energy estimates of wave equations, we deduce that

$$\begin{aligned} 2C \int_G |\partial_t u_N(t, x)|^2 dx dt + \|u_{0,N}\|_{\dot{H}^1(\mathbb{T})}^2 + \|u_{1,N}\|_{L^2(\mathbb{T})}^2 & \leq (4\pi C + 1) \left(\|u_{0,N}\|_{\dot{H}^1(\mathbb{T})}^2 + \|u_{1,N}\|_{L^2(\mathbb{T})}^2 \right) \\ & \leq (4\pi C + 1) N^2 \left(\|u_{0,N}\|_{L^2(\mathbb{T})}^2 + \|u_{1,N}\|_{\dot{H}^{-1}(\mathbb{T})}^2 \right). \end{aligned}$$

As a consequence, we obtain the weak observability up to a compact term:

$$\|u_0\|_{\dot{H}^1(\mathbb{T})}^2 + \|u_1\|_{L^2(\mathbb{T})}^2 \leq (4\pi C + 1)N^2 \left(\int_G |\partial_t u(t, x)|^2 dx dt + \|u_{0,N}\|_{L^2(\mathbb{T})}^2 + \|u_{1,N}\|_{\dot{H}^{-1}(\mathbb{T})}^2 \right).$$

□

4.2. Sufficient part of Theorem 1.5. Now we prove that (OSC) and (GCC) are sufficient to establish the observability inequality (1.2). We argue by contradiction.

Assume that (1.2) fails. Then there exists a sequence of initial data $(u_{0,n}, u_{1,n}) \in \dot{H}^1(\mathbb{T}) \times L^2(\mathbb{T})$ for $n = 1, 2, \dots$, such that

$$\|\partial_x u_{0,n}\|_{L^2}^2 + \|u_{1,n}\|_{L^2}^2 = 1,$$

and

$$\int_G |\partial_t u_n(t, x)|^2 dx dt \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (4.4)$$

where u_n is the solution of (1.1) with initial data $(u_{0,n}, u_{1,n})$. Therefore, we may extract a subsequence, still denoted by $(u_{0,n}, u_{1,n})$, that converges weakly in $\dot{H}^1(\mathbb{T}) \times L^2(\mathbb{T})$ to some (u_0, u_1) . Correspondingly, $(u_n, \partial_t u_n)$ converges weakly in $\dot{H}^1(\mathbb{T}) \times L^2(\mathbb{T})$ to a solution $(u, \partial_t u)$ of (1.1).

On one hand, from (4.4) we obtain for the limit

$$\int_G |\partial_t u|^2 dx dt = 0,$$

which implies $\partial_t u = 0$ almost everywhere in G . Then, the unique continuation property, Corollary 3.5 yields $u \equiv 0$.

On the other hand, since u_n solves (1.1), Corollary 4.2 gives the weak observability estimate

$$1 = \|\partial_x u_{0,n}\|_{L^2}^2 + \|u_{1,n}\|_{L^2}^2 \leq C \left(\int_G |\partial_t u_n(t, x)|^2 dx dt + \|u_{0,n}\|_{L^2(\mathbb{T})}^2 + \|u_{1,n}\|_{\dot{H}^{-1}(\mathbb{T})}^2 \right). \quad (4.5)$$

Because the embedding $\dot{H}^1 \times L^2 \hookrightarrow L^2 \times \dot{H}^{-1}$ is compact, the convergence $(u_{0,n}, u_{1,n}) \rightarrow (u_0, u_1)$ holds strongly in $L^2(\mathbb{T}) \times \dot{H}^{-1}(\mathbb{T})$. Passing to the limit $n \rightarrow \infty$ in (4.5) yields

$$1 \leq C(\|u_0\|_{L^2(\mathbb{T})}^2 + \|u_1\|_{\dot{H}^{-1}(\mathbb{T})}^2),$$

which contradicts the fact that $u \equiv 0$ (and hence $u_0 = 0$, $u_1 = 0$ in the relevant spaces). Consequently, the observability inequality (1.2) must hold.

5. OTHER IMPORTANT GEOMETRIC CONFIGURATIONS

In this work, we have provided a necessary and sufficient condition on a spacetime measurable observable region for the wave equation. In the literature, many other important geometric configurations are considered, such as measurable Cartesian products $E_t \times F_x$, general spacetime open sets, and especially $[0, T] \times \omega$ with open set ω . In this section, we comment on our condition in these cases.

5.1. Cylinder observable region $[0, T] \times \omega$. This is the most classical setting. Let us now restrict the geometric region G to the following class: for a given $T > 0$, consider

$$G \in \mathcal{D}_1(T) := \{[0, T] \times \omega : \omega \in \mathbb{T} \text{ open}\}.$$

Our criterion clearly provides a sufficient condition for the observability on such regions. However, the observable symmetry condition automatically fails for “many” non-trivial pairs $(A, B) \in [0, 2\pi]$. This is evident from the geometry illustrated in Fig. 7. Therefore, there is no need to consider (OSC) in this setting.

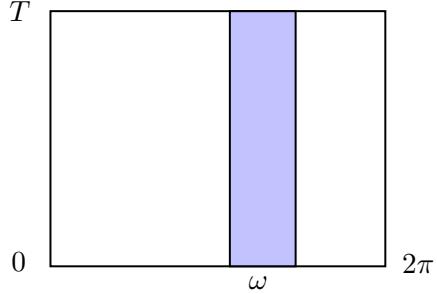


FIGURE 7. Observation on $G = [0, T] \times \omega$

5.2. Measurable Cartesian products $E_t \times F_x$. Next, we restrict G to the following class: for a given $T > 0$, consider (see Fig. 8)

$$G \in \mathcal{D}_2(T) := \{E_t \times F_x : E_t \in \mathbb{T}, F_x \in \mathbb{T} \text{ are measurable sets}\}.$$

Although such regions appear much more flexible, still the observable symmetry condition cannot be recovered on many non-trivial measurable product sets, since the time and space components decouple in a way that destroys the underlying symmetry mechanism revealed by our main result.

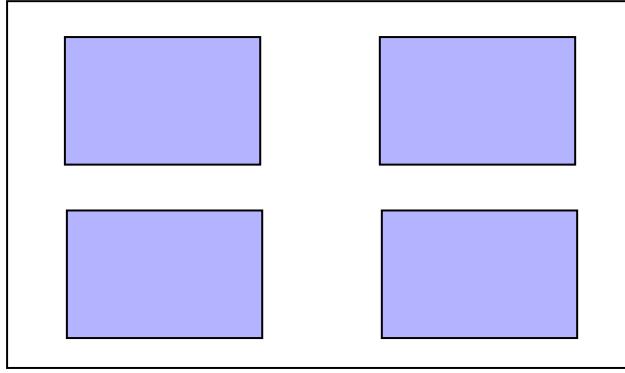


FIGURE 8. The observation region $G = E \times F$ with two intervals E, F

Lemma 5.1. *Let $T > 0$. Let $G \in \mathcal{D}_2(T)$. If $G \in \mathcal{D}_2(T)$ satisfies (GCC), then it does not obey (OSC) for any non-trivial pair (A, B) in $[0, 2\pi]$.*

Proof. From the (GCC) assumption, one easily deduces that $|E| + |F| \geq 2\pi$. Next, one shows that if for a non-trivial pair (A, B) in $[0, 2\pi]$, the set G satisfies the (OSC) for (A, B) , then $G \subset L_{\xi \in A} \cap L_{\eta \in B}$. By the non-triviality of (A, B) , at least one of the two sets has measure smaller than 2π . This contradicts the (GCC) assumption. \square

Consequently, as an application of Theorem 1.5, one has

Corollary 5.2. *Let $T > 0$. Let $G \in \mathcal{D}_2(T)$. Then the wave equation is observable on G if and only if (GCC) is satisfied.*

Remark 5.3. *This example suggests a viewpoint for wave equations: in some sense, the optimal shape of the observation region should be considered in spacetime rather than as a Cartesian product. On the one hand, the natural GCC manifests its intrinsic geometric nature in the spacetime setting. On the other hand, the symmetry structure underlying our characterization becomes absent in Cartesian product regions.*

5.3. General spacetime open regions. One may also consider general spacetime open observation regions. For a given $T > 0$, consider

$$\mathcal{D}_3(T) := \{G \in [0, T] \times \mathbb{T} : \text{open}\}.$$

In this setting, the tools of microlocal analysis are particularly powerful.

On the one hand, our criterion again provides a sufficient condition for observability on any such region. On the other hand, it is also a necessary condition. Indeed, the same counterexample presented in Figure 2 demonstrates that the observable symmetry condition cannot be omitted: even for open spacetime sets, a failure of symmetry leads to a loss of observability.

Motivated by this observation, we obtain an additional sufficient condition for observability in the class $\mathcal{D}_3(T)$.

Corollary 5.4. *Let $T > 0$. Let $G \in \mathcal{D}_3(T)$. Then the wave equation is observable on G if $\exists c_0 > 0$ such that for “every” $x \in \mathbb{T}$*

$$\int_0^T \mathbf{1}_G(s, x \pm s) ds \geq c_0. \quad (5.1)$$

Proof. It suffices to demonstrate the symmetry condition. Assume that there is a non-trivial pair (A, B) in $[0, 2\pi]$ satisfying the (OSC). Since the region G is open, by continuity, we can further assume that A and B are open intervals. Assume that $A = \mathbb{T}$ or $(0, 2\pi)$, while $|B| < 2\pi$, then by the (GCC) assumption and by considering a characteristic starting from $\mathbb{T} \setminus B$, this is not possible. Assume that $0 < |A|, |B| < 2\pi$, and assume that $A = (a_1, a_2)$. By the geometry and the fact that G satisfies the (OSC) for (A, B) , we deduce that the line $L_{\xi=a_1}$ intersects with G has zero measure. This is in contradiction to the assumption. \square

Remark 5.5. *This example shows that the uniform lower bound of “almost every” geodesic intersects G is essential in the assumption of (GCC), even when G is an arbitrary spacetime open region.*

6. FURTHER COMMENTS AND PERSPECTIVES

In this final section, we discuss several additional comments arising from our results and outline a number of directions for future research.

(1) Structure of the observable sets $\mathcal{O}(T)$. Our results provide a complete characterization of observable regions $\mathcal{O}(T) \subset [0, T] \times \mathbb{T}$ for every $T > 0$. In sharp contrast with the case of the heat equation, see for instance [AEWZ14, WWZZ19] and the references therein, the family $\mathcal{O}(T)$ for the wave equation exhibits a genuine dependence on T , as is clearly reflected in the construction of non-observable regions.

It would be very interesting to investigate possible structural properties of $\{\mathcal{O}(T) : T \in (0, +\infty)\}$, including inclusion relations, invariance properties, and potential dynamical features.

(2) Absence of waiting time. This characterization shows that, for the wave equation under spacetime setting, no waiting time is required to achieve observability or controllability. In classical studies on cylindrical regions $[0, T] \times \omega$, it is typical to impose a minimal control time linked to geometric propagation considerations [BLR92, BZ04, BDE20, DZZ08]. Our results suggest a complementary perspective: the geometric structure of the observation region itself may play a more decisive role than the control duration. In particular, once the spacetime region satisfies the symmetry condition identified here, observability holds for every $T > 0$. Conversely, when this geometric requirement is not met, increasing the control time does not by itself restore observability.

In this sense, our work offers a possible viewpoint on the interplay between geometry and control time, suggesting that the geometric configuration of the observation set may, in some cases, play a more fundamental role than the duration of the control.

(3) Coupled transport equations and degenerate control interpretation. The observable symmetry condition may be reformulated in terms of two coupled transport equations supplemented by a degenerate control. This perspective suggests natural extensions to broader classes of PDEs, including coupled wave systems (see [LN26]), where similar hidden symmetries may play a decisive role.

(4) Stability of spacetime damped waves. Due to the standard energy estimate. The criterion developed in this paper leads directly to stability results for the damped wave equation with spacetime damping regions, providing a unified geometric viewpoint for both damping and observation mechanisms. For related works, one can find [DLZ03, Sun23, LR97, Bur98].

(5) Some practical considerations. The geometric viewpoint developed in this work may also have implications for some practical problems, such as sensor placement [PTZ13], optimization of observation regions [Tré18], and numerical strategies for control and observability [BDE25, EZ13]. In such contexts, identifying the minimal geometric requirements is often essential, and the symmetry-based perspective introduced here may serve as a useful guiding principle.

(6) Other cases. We expect that the same symmetric mechanism extends to wave equations on intervals, half-lines, or the entire real line (see, for instance, [LY06, Li10]). In particular, unique continuation and observability on such domains may also be governed by analogous symmetry structures together with (weak) GCC.

Recently, there are more related works investigating the observability from the spacetime measurable sets in different dispersive models. We refer to [BZ25a, BZ25b, NWX25a, NWX25b]. However, since the principal symbol exhibits different features, the wave equation is distinguished from other typical models.

(7) Higher-dimensional waves. The symmetry structure uncovered in the 1D torus case may offer insight into how geometric features influence wave propagation at the spacetime level. At the same time, we wish to emphasize that extending these ideas to higher dimensions remains substantially more challenging. The geometric and microlocal structures involved become significantly richer, and it is not yet clear how the symmetry mechanisms identified here

might generalize. Nevertheless, we hope that the viewpoint developed in this work may serve as a useful starting point for further explorations of observability and unique continuation in higher-dimensional settings. We also notice that there is recent progress in Schrödinger setting in [BZ25a].

(8) Semilinear equations and geometric wave models. An interesting direction is to investigate whether the mechanisms identified here can be extended to semilinear or geometric wave equations. Recently, the controlled flow of geometric maps has been initiated by the last author and his co-authors in [KX24, CX25, CKX25a, CKX25b], highlighting the interplay among PDE, geometry, and control. In geometric wave settings, nonlinear effects may interact with wave propagation in subtle ways, and it remains unclear how the symmetry structure introduced here should be adapted. A better understanding of how spacetime symmetries interact with nonlinear or geometric features may open further possibilities.

APPENDIX A.

A.1. HUM: Observability and controllability. In this part, we will present two observability inequalities in different functional spaces and their associated controllability result based on the classic Hilbert uniqueness method (HUM). This is a preparation to introduce our sharp geometric conditions.

Let us consider the following adjoint equation:

$$(\partial_t^2 - \partial_x^2)v = 0, \quad (v, \partial_t v)|_{t=0} = (v_0, v_1) \in L^2(\mathbb{T}) \times \dot{H}^{-1}(\mathbb{T}). \quad (\text{A.1})$$

By the standard approach of HUM, the next lemma states the equivalence of controllability and its associated observability in the appropriate Sobolev spaces. We omit its proof and refer to [Cor07, Sec. 2.4].

Lemma A.1. *Let $f \in L^2(G)$ and u be the solution*

$$(\partial_t^2 - \partial_x^2)u = f\mathbf{1}_G. \quad (\text{A.2})$$

Then, u is exactly controllable in $\dot{H}^1(\mathbb{T}) \times L^2(\mathbb{T})$ if and only if

$$\|v_0\|_{L^2(\mathbb{T})}^2 + \|v_1\|_{\dot{H}^{-1}(\mathbb{T})}^2 \leq C \int_G |v(t, x)|^2 dt dx \quad (\text{A.3})$$

holds for all solutions v for (A.1).

In fact, we prove the equivalence of the two observability (1.2) and (A.3) in Lemma A.2, which is inspired by [RLLT17, Sec. 2B]. While they proved the equivalence for an open observation domain, for the case of measurable observed sets, the proof follows almost line by line.

Lemma A.2. *The observability (1.2) is equivalent to having the same constant $C > 0$ such that (A.3) holds for every solution v for (A.1). Equivalently, the solution u to (A.2) is exactly controllable in $\dot{H}^1(\mathbb{T}) \times L^2(\mathbb{T})$ if and only if (1.2) holds for all solutions u for (1.1).*

A.2. Some results on transport equations. We first show that, if u solves $\partial_\xi \partial_\eta u = 0$, then $\partial_\xi u, \partial_\eta u$ are constant almost everywhere on characteristic lines. We start with a lemma.

Lemma A.3. *Let $f \in L^2(\mathbb{T})$. Then the unique solution $u \in C(\mathbb{R}, L^2(\mathbb{T}))$ of*

$$\partial_t u \pm \partial_x u = 0, \quad u|_{t=0} = f(x)$$

is given by

$$u(t, x) = f(x \mp t). \quad \text{for a.e. } (t, x) \in \mathbb{R} \times \mathbb{T}.$$

Proof. We only consider the equation $\partial_t u + \partial_x u = 0$. The other case is similar. Assume that f has the Fourier expansion in $L^2(\mathbb{T})$ sense, namely

$$f(x) = \sum_k a_k e^{ikx} \quad \text{in } L^2(\mathbb{T}). \quad (\text{A.4})$$

Then the unique solution is given by

$$u(t, x) = \sum_k a_k e^{ik(x-t)} \quad \text{in } L^2(\mathbb{T}) \quad (\text{A.5})$$

for every $t \in \mathbb{R}$. It is easy to see that it is the unique solution and $u \in C(\mathbb{R}, L^2(\mathbb{T}))$. Since $f \in L^2(\mathbb{T})$, by a famous theorem of Carleson, the $L^2(\mathbb{T})$ -norm convergency in (A.4) can be strengthened to pointwise convergence, namely

$$f(x) = \sum_k a_k e^{ikx} \quad \text{a.e. } x \in \mathbb{T}. \quad (\text{A.6})$$

Combining (A.5) and (A.6), we find

$$u(t, x) = f(x - t) \quad \text{for a.e. } (t, x) \in \mathbb{R} \times \mathbb{T}$$

as required. \square

Now we prove Lemma 3.1.

Proof of Lemma 3.1. We only prove the result for $W := \partial_\eta U$. According to the wave equation, we know $\partial_\xi W = 0$, or equivalently

$$\partial_t W + \partial_x W = 0, \quad W|_{t=0} = f(x)$$

where $f(x) = \partial_\eta W|_{t=0} = (\partial_t - \partial_x)u|_{t=0} = u_1 - u_0 \in L^2(\mathbb{T})$. Thanks to Lemma A.3, we have

$$W(t, x) = f(x - t) \quad \text{for a.e. } (t, x) \in \mathbb{R} \times \mathbb{T}.$$

In other words, the set

$$\Sigma = \{(t, x) \in \mathbb{R} \times \mathbb{T} : u(t, x) \neq f(x - t)\}$$

satisfies $\text{meas}_{\mathbb{R}^2}(\Sigma) = 0$. For fixed $\eta \in \mathbb{T}$, the η -characteristic line L_η can be parameterized by $t \in \mathbb{R}$ as

$$(t, x) = (t, t + \eta).$$

Define the slice of Σ along L_η as

$$\Sigma_\eta = \{t \in \mathbb{R} : (t, t + \eta) \in \Sigma\}.$$

Then, for $(t, x) \in L_\eta$, we have $u(t, x) = f(\eta)$ if and only if $t \notin \Sigma_\eta$.

Let χ_Σ be the characteristic function of Σ . Then we have

$$0 = \iint_{\mathbb{R}^2} \chi_\Sigma(t, x) dt dx = \iint_{\mathbb{R}^2} \chi_\Sigma(t, t + \eta) dt d\eta.$$

By Fubini's Theorem,

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \chi_\Sigma(t, t + \eta) dt \right) d\eta = 0.$$

The inner integral is exactly the one-dimensional Lebesgue measure of Σ_η , $\text{meas}_{\mathbb{R}}(\Sigma_\eta)$. Hence,

$$\int_{\mathbb{R}} \text{meas}_{\mathbb{R}}(\Sigma_\eta) d\eta = 0.$$

It follows that

$$\text{meas}_{\mathbb{R}}(\Sigma_\eta) = 0 \quad \text{for almost every } \eta \in \mathbb{R}.$$

This gives the desired conclusion. \square

Next, we investigate the observability for the transport equation:

$$(\partial_t - \partial_x)u = 0, \quad u|_{t=0} = u_0 \in L^2(\mathbb{T}), \quad (\text{A.7})$$

$$(\partial_t + \partial_x)u = 0, \quad u|_{t=0} = u_0 \in L^2(\mathbb{T}). \quad (\text{A.8})$$

Since the proofs of the two observability inequalities for (A.7) and (A.8) are almost the same, we only present the proof for (A.7). Let $\widehat{u}(k)$ be the k -th Fourier coefficient of u defined as $\widehat{u}(k) = \frac{1}{2\pi} \int_0^{2\pi} u(x) e^{-ikx} dx$. We write the initial state u_0 and its associated solution u to (A.7) into Fourier series

$$u_0(x) = \sum_{k \in \mathbb{Z}} \widehat{u}_0(k) e^{ikx}, \quad u(t, x) = \sum_{k \in \mathbb{Z}} \widehat{u}_0(k) e^{ik(t+x)}.$$

Then the observability $\|u_0\|_{L^2(\mathbb{T})}^2 \leq C \int_G |u(t, x)|^2 dx dt$ for the transport equation (A.7) is equivalent to

$$\sum_{k \in \mathbb{Z}} |\widehat{u}_0(k)|^2 \leq C \int_G \left| \sum_{k \in \mathbb{Z}} \widehat{u}_0(k) e^{ik(t+x)} \right|^2 dx dt.$$

Proposition A.4. *Let $G \subset \mathbb{T}^2$ be a measurable set with positive measure and G satisfy (GCC). Then for any subset $\Lambda \subset \mathbb{Z}$ and any $\{a_k\}_{k \in \Lambda} \in l^2$, we have*

$$2\pi c_0 \sum_{k \in \Lambda} |a_k|^2 \leq \int_G \left| \sum_{k \in \Lambda} a_k e^{ik(t+x)} \right|^2 dx dt, \quad (\text{A.9})$$

$$2\pi c_0 \sum_{k \in \Lambda} |a_k|^2 \leq \int_G \left| \sum_{k \in \Lambda} a_k e^{-ik(t-x)} \right|^2 dx dt. \quad (\text{A.10})$$

Proof. We prove (A.9) by direct computation (similarly for (A.10)). Let $A_k := \begin{cases} a_k, & k \in \Lambda, \\ 0, & \text{otherwise.} \end{cases}$

$$\int_G \left| \sum_{k \in \mathbb{Z}} A_k e^{ik(t+x)} \right|^2 dx dt = \int_{\mathbb{T}^2} \left| \sum_{k \in \mathbb{Z}} A_k e^{ik(t+x)} \right|^2 \mathbf{1}_G(t, x) dx dt = \int_{\mathbb{T}} \left| \sum_{k \in \mathbb{Z}} A_k e^{iky} \right|^2 \int_{\mathbb{T}} \mathbf{1}_G(t, y-t) dt dy.$$

Using (GCC), we know that for a.e. $y \in \mathbb{T}$, $\int_{\mathbb{T}} \mathbf{1}_G(t, y-t) dt \geq c_0 > 0$. Therefore, we derive that

$$\left| \sum_{k \in \mathbb{Z}} A_k e^{iky} \right|^2 \int_{\mathbb{T}} \mathbf{1}_G(t, y-t) dt \geq c_0 \left| \sum_{k \in \mathbb{Z}} A_k e^{iky} \right|^2, \quad \text{for a.e. } y \in \mathbb{T}.$$

By Plancherel's theorem, $\int_{\mathbb{T}} \left| \sum_{k \in \mathbb{Z}} A_k e^{iky} \right|^2 dy = 2\pi \sum_{k \in \mathbb{Z}} |A_k|^2 = 2\pi \sum_{k \in \Lambda} |a_k|^2$. As a consequence, we obtain the observability (A.9):

$$2\pi c_0 \sum_{k \in \Lambda} |a_k|^2 \leq \int_G \left| \sum_{k \in \mathbb{Z}} A_k e^{ik(t+x)} \right|^2 dx dt = \int_G \left| \sum_{k \in \Lambda} a_k e^{ik(t+x)} \right|^2 dx dt.$$

The observability (A.10) follows similarly. \square

The following theorem gives a sufficient and necessary condition of the observable set for the transport equation. We only state the result for (A.7), the reader easily figures out the the necessary modifications for (A.8).

Theorem A.5. *Let $T > 0$ and G be a measurable subset of $[0, T] \times \mathbb{T}$ with positive measure. Then the observability inequality*

$$\|u_0\|_{L^2(\mathbb{T})}^2 \leq C \int_G |u(t, x)|^2 dx dt \quad (\text{A.11})$$

holds with a positive constant $C > 0$ for all solutions to the transport equation (A.7) if and only if there exists a constant $c_0 > 0$ such that

$$\int_0^T \mathbf{1}_G(s, x-s) ds \geq c_0 \quad \text{for a.e. } x \in \mathbb{T} \quad (\text{A.12})$$

Proof. The direction (A.12) \Rightarrow (A.11) follows clearly from Proposition A.4. To show the inverse direction, namely (A.11) \Rightarrow (A.12), we use the contradiction argument. Suppose that (A.12) is not true, then for any $\varepsilon > 0$, there exists a set $E_\varepsilon \subset \mathbb{T}$ with positive measure $|E_\varepsilon| > 0$ such that

$$\int_0^T \mathbf{1}_G(s, x-s) ds < \varepsilon \quad \text{for a.e. } x \in E_\varepsilon. \quad (\text{A.13})$$

Let $u_{0\varepsilon}(x) = \mathbf{1}_{E_\varepsilon}(x)$. Then $u_{0\varepsilon} \in L^2(\mathbb{T})$ and its support is contained in E_ε . Then, similar to the proof in Proposition A.4, we have

$$\int_G |u(t, x)|^2 dx dt = \int_{\mathbb{T}} |u_{0\varepsilon}(x)|^2 \int_{\mathbb{T}} \mathbf{1}_G(t, x-t) dt dx \leq \varepsilon \|u_{0\varepsilon}\|_{L^2(\mathbb{T})}^2.$$

Taking $\varepsilon > 0$ small enough, say $\varepsilon = C/2$, we obtain a contradiction with (A.11). \square

A.3. GCC is necessary for the wave observability.

Corollary A.6. *If the observability (1.2) holds on some observation set $G \subset [0, T] \times \mathbb{T}$, then G needs to satisfies (GCC).*

Proof. The idea is to reduce the proof to that of transport equations. Assume that

$$u_0(x) = f(x) = \sum_{k \in \mathbb{Z}} a_k e^{ikx}, \quad u_1(x) = g(x) = \sum_{k \in \mathbb{Z}} b_k e^{ikx}. \quad (\text{A.14})$$

Then the solution to the wave equation (1.1) can be expressed as

$$u(t, x) = a_0 + b_0 t + \sum_{k \in \mathbb{Z} \setminus \{0\}} (c_k e^{ik(x+t)} + d_k e^{ik(x-t)}) \quad (\text{A.15})$$

with c_k, d_k given by

$$c_k = \frac{1}{2}(a_k + \frac{b_k}{k}), \quad d_k = \frac{1}{2}(a_k - \frac{b_k}{k}), \quad k \in \mathbb{Z} \setminus \{0\}. \quad (\text{A.16})$$

This implies d'Alembert's formula on torus

$$u(t, x) = \frac{1}{2}(f(x+t) + f(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} g(s) ds.$$

Assume that the observability inequality holds. Then

$$\|f_x\|_{L^2}^2 + \|g\|_{L^2}^2 \lesssim \int_G |u_t|^2 dx dt \quad \text{for all } (f, g) \in \dot{H}^1(\mathbb{T}) \times L^2(\mathbb{T}).$$

Using d'Alembert's formula, this is equivalent to

$$\|f_x\|_{L^2}^2 + \|g\|_{L^2}^2 \lesssim \int_G |f'(x+t) - f'(x-t) + g(x+t) + g(x-t)|^2 dx dt \quad \text{for all } (f, g) \in \dot{H}^1(\mathbb{T}) \times L^2(\mathbb{T}).$$

Replacing f' by f , we have

$$\|f\|_{L^2}^2 + \|g\|_{L^2}^2 \lesssim \int_G |f(x+t) - f(x-t) + g(x+t) + g(x-t)|^2 dx dt \quad \text{for all } (f, g) \in L^2(\mathbb{T}) \times L^2(\mathbb{T}).$$

Letting $f = g$ and $f = -g$, we obtain

$$\|g\|_{L^2}^2 \lesssim \int_G |g(x+t)|^2 dx dt \quad \text{for all } g \in L^2(\mathbb{T})$$

and

$$\|g\|_{L^2}^2 \lesssim \int_G |g(x-t)|^2 dx dt \quad \text{for all } g \in L^2(\mathbb{T})$$

respectively. According to the necessary condition of observability of transport equation, see Theorem A.5, we find that G is necessary to satisfy (GCC). \square

A.4. A basic integration lemma.

Lemma A.7. *Let $T > 0$. Let $A, B \subset \mathbb{R}$ be measurable sets. Assume that $u \in C^1(\mathbb{R}^2)$ (or at least u is continuously differentiable on an open set containing Ω). Then*

$$\begin{aligned} \iint_{\Omega^+(B;T)} (u_x + u_t) dx dt &= \int_{T+B} u(x, T) dx - \int_B u(x, 0) dx, \\ \iint_{\Omega^-(A;T)} (u_x - u_t) dx dt &= \int_A u(x, 0) dx - \int_{A-T} u(x, T) dx. \end{aligned}$$

Proof. We only prove the first equation, since the second one can be proved similarly. Perform the change of variables

$$\eta = x - t, \quad \tau = t,$$

with inverse transformation $x = \eta + \tau$, $t = \tau$. The Jacobian determinant equals 1. This map transforms Ω into

$$\Phi(\Omega^+) = \{(\eta, \tau) \mid 0 \leq \tau \leq T, \eta \in B\} = B \times [0, T].$$

Define $v(\eta, \tau) = u(\eta + \tau, \tau)$. Then

$$\frac{\partial v}{\partial \tau} = u_x(\eta + \tau, \tau) \cdot 1 + u_t(\eta + \tau, \tau) = u_x + u_t.$$

By the change-of-variables formula,

$$\iint_{\Omega^+} (u_x + u_t) dx dt = \iint_{B \times [0, T]} \frac{\partial v}{\partial \tau} d\eta d\tau.$$

Using Fubini's theorem on the product region,

$$\iint_{A \times [0, T]} \frac{\partial v}{\partial \tau} d\eta d\tau = \int_A \left(\int_0^T \frac{\partial v}{\partial \tau} d\tau \right) d\eta.$$

For fixed η , the fundamental theorem of calculus gives

$$\int_0^T \frac{\partial v}{\partial \tau} d\tau = v(\eta, T) - v(\eta, 0) = u(\eta + T, T) - u(\eta, 0).$$

Hence

$$\int_B (u(\eta + T, T) - u(\eta, 0)) d\eta = \int_B u(\eta + T, T) d\eta - \int_B u(\eta, 0) d\eta.$$

In the first integral set $x = \eta + T$. When $\eta \in B$, we have $x \in T + B$ and $d\eta = dx$; therefore

$$\int_B u(\eta + T, T) d\eta = \int_{T+B} u(x, T) dx.$$

Consequently,

$$\iint_{\Omega^+} (u_x + u_t) dx dt = \int_{T+B} u(x, T) dx - \int_B u(x, 0) dx,$$

which completes the proof. \square

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