

Analysis of splitting schemes for stochastic evolution equations with non-Lipschitz nonlinearities driven by fractional noise *

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Abstract

We propose a novel time-splitting scheme for a class of semilinear stochastic evolution equations driven by cylindrical fractional noise. The nonlinearity is decomposed as the sum of a one-sided, non-globally, Lipschitz continuous function, and of a globally Lipschitz continuous function. The proposed scheme is based on a splitting strategy, where the first nonlinearity is treated using the exact flow of an associated differential equation, and the second one is treated by an explicit Euler approximation. We prove mean-square, strong error estimates for the proposed scheme and show that the order of convergence is $H - 1/4$, where $H \in (1/4, 1)$ is the Hurst index. For the proof, we establish new regularity results for real-valued and infinite dimensional fractional Ornstein-Uhlenbeck process depending on the value of the Hurst parameter H . Numerical experiments illustrate the main result of this manuscript.

Key words: Stochastic partial differential equations; Fractional Brownian motion; splitting schemes; strong error estimates.

1 Introduction

Fractional Brownian motion (fBm) presents both randomness and strong scale-free correlations. Unlike the standard Brownian motion, fractional Brownian motion is a non-Markovian stochastic process characterized by power-law autocorrelation function, which is determined by the Hurst parameter $H \in (0, 1)$. Stochastic differential equations driven by fBm constitute a fundamental class of stochastic systems, and they have been extensively applied in modeling physical phenomena exhibiting long-range dependence and anomalous behavior, such as financial time series exhibiting non-Markovian dynamics [1], complex physical processes demonstrating anomalous diffusion patterns [2], generative inpainting [3], and so on.

The analysis of well-posedness, regularity properties, and numerical approximations for stochastic differential equations with fBm are not only a fundamental topic but also a vitally important research area. In recent years these topics have witnessed a lot of attention, see for instance [4, 5, 6, 7, 8, 9, 10, 11, 12]. It should be noted that, however, significant challenges remain in the study of such equations with non-Lipschitz coefficients driven by fBm [13, 14]. One of the main reasons is that the fBm is neither a Markov process nor a semimartingale, hence the classical stochastic analysis methods can not be applied directly. In this situation, some new stochastic analysis tools need to be further explored.

Stochastic partial differential equations with non-Lipschitz coefficients driven by fractional noise are an important class of stochastic partial differential equations. The typical example is stochastic Allen-Cahn

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equation, which has been applied into modeling a two phase system driven by the Ginzburg–Landau energy and other rare stochastic systems [15, 16, 17]. It is well-known that the main difficulty of dealing with such stochastic partial differential equation is the presence of non globally Lipschitz continuous nonlinearities. In the last decade, progress has been made in the theoretical analysis and numerical approximations, using numerical schemes based on exponential integrator [18], tamed discretization of the nonlinear coefficient [19], Wong-Zakai approximation [20] and full-phase flow splitting method [21, 22, 23], see also [24, 25, 26, 27]. All those references deal with systems driven by standard Wiener processes, i.e. with Hurst parameter $H = 1/2$. However, we note that there remains room for further improvement and refinement. For example, although the full-phase flow splitting method [21, 22, 23] is feasible for the stochastic Allen–Cahn equation, one may not be able to implement such schemes for other models. This suggests the application of semi-phase flow methods, where the drift is decomposed into two parts, one being treated using the exact flow, the other one being treated by an explicit Euler approximation. In addition, the analysis for the fractional case $H \neq 1/2$ needs to be developed.

In this manuscript, we consider stochastic evolution equations driven by fractional noise, which may be written as

$$\begin{cases} dX(t) + AX(t) dt = (F(X(t)) + G(X(t))) dt + dB^H(t), & \forall t \geq 0, \\ X(0) = X_0, \end{cases}$$

where the assumptions on A , F and G will be given in Section 2, $\{B^H(t)\}_{t \in [0, T]}$ is a cylindrical fractional Brownian motion depending on the Hurst parameter H , and X_0 is an initial value. For well-posedness, it is assumed that $H \in (1/4, 1)$. The case $H \neq 1/2$ has already been treated in previous work so it is omitted. The nonlinearity F is non-globally Lipschitz continuous, but it satisfies a one-sided Lipschitz continuity property, whereas the nonlinearity G is assumed to be globally Lipschitz continuous.

We propose to compute approximate the solution of the stochastic evolution equation using the following numerical scheme

$$\begin{cases} X_{n+1} = S(\Delta t)\Phi_{\Delta t}(X_n) + A^{-1}(I - S(\Delta t))G(X_n) + S(\Delta t)\Delta B_n^H, & \forall n \in \{0, \dots, N-1\}, \\ X_0 = x_0, \end{cases}$$

where Δt is the time-step size, ΔB_n^H are increments of the cylindrical fractional Brownian motion, and $\Phi_{\Delta t}$ denotes the phase-flow associated with the nonlinearity F .

The main result of this manuscript is Theorem 4.1: one obtains strong error estimates, with order of convergence $H - 1/4$, with respect to the time-step size Δt . This result is illustrated by numerical experiments in Section 5.

In order to prove Theorem 4.1, we provide auxiliary results, mainly on the regularity properties of the fractional Ornstein–Uhlenbeck and its numerical approximation, obtained when $F = G = 0$. Different arguments are used when $H \in (1/2, 1)$ or when $H \in (1/4, 1/2)$. For completeness, detailed proofs are provided in the appendices.

This article is organized as follows. The setting and assumptions are presented in Section 2. Proposition 4.6 recalls well-posedness and regularity properties of solutions to the stochastic evolution equation. Section 4 is devoted to the presentation of the numerical scheme and to the analysis of its convergence. It contains the statement of Theorem 4.1 and the statement and proofs of several auxiliary results. Section 5 provides numerical experiments which illustrate the main result of this work. Section 6 presents some conclusion and perspectives for future works. Appendices A, B, C and D provide key auxiliary results and their proofs on real-valued fractional Ornstein–Uhlenbeck processes, which play a crucial role in the analysis.

2 Setting

2.1 Notation

Let \mathbb{H} denote the separable Hilbert space $L^2((0, 1); \mathbb{R})$ of square-integrable real-valued functions defined on $(0, 1)$. The associated inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ are defined by

$$\langle x_1, x_2 \rangle = \int_0^1 x_1(\xi) x_2(\xi) d\xi, \quad \|x\| = \left(\int_0^1 x(\xi)^2 d\xi \right)^{1/2}, \quad \forall x_1, x_2, x \in \mathbb{H}.$$

Moreover, let E denote the Banach space $C([0, 1]; \mathbb{R})$ of continuous functions from $[0, 1]$ to \mathbb{R} . The associated norm $\|\cdot\|_E$ is defined by

$$\|x\|_E = \max_{\xi \in [0, 1]} |x(\xi)|.$$

Finally, let $\mathcal{L}(\mathbb{H})$ denote the space of bounded linear operators from \mathbb{H} to \mathbb{H} , and let $\mathcal{L}_2(\mathbb{H})$ denote the space of Hilbert–Schmidt linear operators from \mathbb{H} to \mathbb{H} . The associated norm on $\mathcal{L}(\mathbb{H})$, resp. on $\mathcal{L}_2(\mathbb{H})$, is denoted by $\|\cdot\|_{\mathcal{L}(\mathbb{H})}$, resp. by $\|\cdot\|_{\mathcal{L}_2(\mathbb{H})}$.

2.2 Linear operator

In this work, the linear operator $-A$ is the realization of the Laplace operator with homogeneous Dirichlet boundary conditions on the interval $(0, 1)$.

Assumption 2.1. *The linear operator $A : \mathbb{H} \rightarrow \mathbb{H}$ is the unbounded linear operator on \mathbb{H} with domain*

$$D(A) = H_0^1(0, 1; \mathbb{R}) \cap H^2(0, 1; \mathbb{R}),$$

such that for all $x \in D(A)$, one has

$$Ax = -x''.$$

For all $k \in \mathbb{N}$, set $\lambda_k = k^2\pi^2$ and $e_k = \sqrt{2}\sin(k\pi \cdot)$. Then $(e_k)_{k \in \mathbb{N}}$ is a complete orthonormal system of the Hilbert space \mathbb{H} , and for all $k \in \mathbb{N}$ one has $Ae_k = \lambda_k e_k$.

For all $\alpha \in \mathbb{R}^+$, the linear operator A^α is defined by the domain

$$D(A^\alpha) = \{x \in \mathbb{H}; \sum_{k=1}^{\infty} \lambda_k^{2\alpha} \langle x, e_k \rangle^2 < \infty\},$$

and such that for all $x \in D(A^\alpha)$ one has

$$A^\alpha x = \sum_{k=1}^{\infty} \lambda_k^\alpha \langle x, e_k \rangle e_k.$$

In the sequel the notation

$$\mathbb{H}^\alpha = D(A^{\frac{\alpha}{2}}), \quad \forall \alpha \in \mathbb{R}^+,$$

is employed. The norm $\|\cdot\|_{\mathbb{H}^\alpha}$ is defined by

$$\|x\|_{\mathbb{H}^\alpha}^2 = \|A^{\frac{\alpha}{2}} x\|^2 = \sum_{k \in \mathbb{N}} \lambda_k^\alpha \langle x, e_k \rangle^2, \quad \forall x \in \mathbb{H}^\alpha.$$

Moreover, for all $\alpha \in \mathbb{R}^+$, the linear operator $A^{-\alpha}$ is the bounded linear operator on \mathbb{H} defined by

$$A^{-\alpha} x = \sum_{k=1}^{\infty} \lambda_k^{-\alpha} \langle x, e_k \rangle e_k, \quad \forall x \in \mathbb{H}.$$

The linear operator A generates a semigroup $(S(t))_{t \geq 0}$, where for all $t \geq 0$ the linear operator $S(t)$ is defined by

$$S(t)x = \sum_{k=1}^{\infty} e^{-\lambda_k t} \langle x, e_k \rangle e_k, \quad \forall x \in \mathbb{H}.$$

Proposition 2.2 states standard properties of the semigroup $(S(t))_{t \geq 0}$. Please see for instance [28, 29].

Proposition 2.2. *The semigroup $(S(t))_{t \geq 0}$ satisfies the following properties.*

- For all $t \geq 0$, $S(t)$ is a bounded linear operator on \mathbb{H} , and one has $\|S(t)\|_{\mathcal{L}(\mathbb{H})} \leq 1$.
- For all $t \geq 0$, $S(t)$ is a bounded linear operator on E , and one has $\|S(t)\|_{\mathcal{L}(E)} \leq 1$.
- For all $\alpha \geq 0$, there exists $C_\alpha \in (0, \infty)$ such that one has

$$\|A^\alpha S(t)\|_{\mathcal{L}(\mathbb{H})} \leq C t^{-\alpha}, \quad \forall t > 0.$$

Moreover, for all $\rho \in [0, 1]$, there exists $C_\rho \in (0, \infty)$ such that one has

$$\|A^{-\rho}(I - S(t))\|_{\mathcal{L}(\mathbb{H})} \leq C t^\rho, \quad \forall t \geq 0.$$

- For all $\rho \in [0, 1]$, there exists $C_\rho \in (0, \infty)$ such that for all $t_2 \geq t_1 > 0$ one has

$$\|S(t_2) - S(t_1)\|_{\mathcal{L}(\mathbb{H})} \leq C(t_2 - t_1)^\rho t_1^{-\rho}. \quad (1)$$

It is straightforward to check that the eigenfunctions $\{e_k; k \in \mathbb{N}\}$ satisfy the following properties ([30]): for all $k \in \mathbb{N}$, one has $e_k \in E$, and there exists $C \in (0, \infty)$ such that for all $k \in \mathbb{N}$ one has

$$\max_{\xi \in [0, 1]} |e_k(\xi)| \leq C, \quad \max_{\xi \in [0, 1]} |\nabla e_k(\xi)| \leq C \lambda_k^{1/2}.$$

As a result, for all $\alpha \in [0, 1]$, there exists $C_\alpha \in (0, \infty)$ such that one has

$$|e_k(\xi_2) - e_k(\xi_1)| \leq C_\alpha \lambda_k^{\frac{\alpha}{2}} |\xi_2 - \xi_1|^\alpha, \quad \forall \xi_1, \xi_2 \in [0, 1]. \quad (2)$$

Moreover, one has the following property: for all $\alpha > 1/2$, one has $\mathbb{H}^\alpha \subset E$, and there exists $C_\alpha \in (0, \infty)$ such that

$$\|x\|_E \leq C_\alpha \|x\|_{\mathbb{H}^\alpha}, \quad \forall x \in \mathbb{H}^\alpha. \quad (3)$$

2.3 Nonlinearities

The first nonlinearity $F : E \rightarrow E$ is defined depending on a real-valued continuous mapping $f : \mathbb{R} \rightarrow \mathbb{R}$, such that one has

$$F(x)(\xi) = f(x(\xi)), \quad \forall \xi \in [0, 1], \quad \forall x \in E.$$

The mapping f is not assumed to be globally Lipschitz continuous, instead it is assumed to be one-sided Lipschitz continuous.

Assumption 2.3. *There exists $L_f \in [0, \infty)$ such that one has*

$$(z_2 - z_1)(f(z_2) - f(z_1)) \leq L_f (z_2 - z_1)^2, \quad \forall z_1, z_2 \in \mathbb{R}.$$

Moreover, there exists $q \in \mathbb{N}$ such that one has

$$|f(z_2) - f(z_1)| \leq L_f (1 + |z_1|^{2q} + |z_2|^{2q}) |z_2 - z_1|, \quad \forall z_1, z_2 \in \mathbb{R}.$$

For the construction of the scheme, let us introduce the flow $(\phi_s)_{s \geq 0}$ associated with the nonlinearity f : for any $z \in \mathbb{R}$, the mapping $s \geq 0 \mapsto \phi_s(z) = z_s$ is the solution to the ordinary differential equation

$$\dot{z}_s = f(z_s), \quad s \geq 0; \quad z_0 = z.$$

Under Assumption 2.3, the flow is well-defined. Let us define

$$\psi_s(z) = \frac{\phi_s(z) - z}{s}, \quad \forall s > 0, \forall z \in \mathbb{R},$$

and denote also $\psi_0 = f$.

Lemma 2.4. *There exist $q' \in \mathbb{N}$ and $C \in (0, \infty)$ such that one has*

$$\sup_{s \in [0, 1]} |\phi_s(z_2) - \phi_s(z_1)| \leq e^{Cs} |z_2 - z_1|, \quad \forall z_1, z_2 \in \mathbb{R}, \quad (4)$$

$$\sup_{s \in [0, 1]} (z_2 - z_1)(\psi_s(z_2) - \psi_s(z_1)) \leq C(z_2 - z_1)^2, \quad \forall z_1, z_2 \in \mathbb{R}, \quad (5)$$

$$\sup_{s \in [0, 1]} |\psi_s(z_2) - \psi_s(z_1)| \leq C(1 + |z_1|^{2q'} + |z_2|^{2q'}), \quad \forall z_1, z_2 \in \mathbb{R}, \quad (6)$$

$$|\psi_s(z) - \psi_0(z)| \leq Cs(1 + |z|^{2q'+1}), \quad \forall s \geq 0, \forall z \in \mathbb{R}. \quad (7)$$

For the implementation of the scheme, the flow needs to be known. This is satisfied for instance for polynomial nonlinearities $f(z) = -z^{2q+1}$, for some integer $q \in \mathbb{N}$: in that case one has

$$\Phi_s(z) = \frac{z}{(1 + 2qz^{2q}s)^{\frac{1}{2q}}}, \quad \forall s \geq 0, \forall z \in \mathbb{R}.$$

For any time $s \geq 0$, the nonlinearities $\Phi_s : E \rightarrow E$ and $\Psi_s : E \rightarrow E$ are defined by

$$\Phi_s(x)(\xi) = \phi_s(x(\xi)), \quad \Psi_s(x)(\xi) = \psi_s(x(\xi)), \quad \forall \xi \in [0, 1], \forall x \in E.$$

The second linearity G is defined similarly: there exists a real-valued continuous mapping $g : \mathbb{R} \rightarrow \mathbb{R}$, such that one has

$$G(x)(\xi) = g(x(\xi)), \quad \forall \xi \in [0, 1], \forall x \in E.$$

Contrary to the mapping f , it is assumed that the mapping g is globally Lipschitz continuous.

Assumption 2.5. *There exists $L_g \in [0, \infty)$ such that one has*

$$|g(z_2) - g(z_1)| \leq L_g |z_2 - z_1|, \quad \forall z_1, z_2 \in \mathbb{R}.$$

As a result, the nonlinearity can also be seen as a mapping $G : \mathbb{H} \rightarrow \mathbb{H}$, which is globally Lipschitz continuous.

For instance, one may choose $g(z) = z$ or $g(z) = \sin(z)$.

2.4 Fractional Brownian motion

Let $H \in (0, 1)$ denote the so-called Hurst index. Consider a sequence $(\beta_k^H)_{k \in \mathbb{N}}$ of independent real-valued fractional Brownian motions with Hurst index H defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let us recall that for each $k \in \mathbb{N}$, the process $(\beta_k^H(t))_{t \geq 0}$ is a Gaussian process characterized by its mean $t \geq 0 \mapsto \mathbb{E}[\beta_k^H(t)]$ and its covariance function $t, s \geq 0 \mapsto R^H(t, s) = \mathbb{E}[\beta_k^H(t)\beta_k^H(s)]$, which are given by

$$\mathbb{E}[\beta_k^H(t)] = 0, \quad \forall t \geq 0,$$

$$\mathbb{E}[\beta_k^H(t)\beta_k^H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}), \quad \forall t, s \geq 0.$$

In particular, one has $\beta_k^H(0) = 0$.

Let $(\beta_k)_{k \in \mathbb{N}}$ be a sequence of standard real-valued Brownian motions, which correspond to choosing the Hurst index $H = 1/2$. When $H \neq 1/2$, for each $k \in \mathbb{N}$ the fractional Brownian motion β_k^H may be defined as

$$\beta_k^H(t) = \int_0^t \mathcal{K}_H(t, s) d\beta_k(s), \quad \forall t \geq 0,$$

where \mathcal{K}_H is a kernel, defined differently whether $H \in (1/2, 1)$ or $H \in (0, 1/2)$. Precisely, for all $H \in (0, 1) \setminus \{1/2\}$, set

$$c_H = \begin{cases} \left(\frac{H(2H-1)}{B(2-2H, H-\frac{1}{2})} \right)^{1/2}, & H \in (\frac{1}{2}, 1), \\ - \left(\frac{H(1-2H)}{2B(1-2H, H+\frac{1}{2})} \right)^{1/2}, & H \in (0, \frac{1}{2}), \end{cases}$$

then for all $t > s > 0$ one has

$$K_H(t, s) = \begin{cases} c_H \int_s^t \frac{\tau^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}} (\tau-s)^{H-\frac{3}{2}} d\tau, & H \in (\frac{1}{2}, 1); \\ c_H \left(\frac{t^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}} \frac{(t-s)^{H-\frac{1}{2}}}{H-\frac{1}{2}} - s^{\frac{1}{2}-H} \int_s^t (\tau-s)^{H-\frac{1}{2}} \tau^{H-\frac{3}{2}} d\tau \right), & H \in (0, \frac{1}{2}). \end{cases}$$

Observe that for all values of $H \in (0, 1) \setminus \{1/2\}$ one has

$$\frac{\partial K_H(t, s)}{\partial t} = c_H \frac{t^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}} (t-s)^{H-\frac{3}{2}}, \quad \forall t > s > 0.$$

Let $T \in (0, \infty)$, then the stochastic integral

$$\int_0^T \phi(t) d\beta_k^H(t)$$

may be defined in different ways. In this manuscript, we only need with deterministic integrands $\phi : [0, T] \rightarrow \mathbb{R}$. Let us first assume that ϕ is of class \mathcal{C}^1 . We define

$$\int_0^T \phi(t) d\beta_k^H(t) = \phi(T) \beta_k^H(T) - \int_0^T \phi'(t) \beta_k^H(t) dt.$$

Given the representation of the fractional Brownian motion β_k^H in terms of a standard Brownian motion β_k depending on the kernel K_H , one has the expression

$$\int_0^T \phi(t) d\beta_k^H(t) = \int_0^T \mathcal{K}_{H,T}^* \phi(t) d\beta_k(t),$$

where the linear operator $\mathcal{K}_{H,T}^*$ is given by

$$\mathcal{K}_{H,T}^* \phi(t) = \phi(T) K_H(T, t) - \int_t^T \phi'(s) K_H(s, t) ds, \quad \forall t \in [0, T].$$

It is important to define the stochastic integral for integrands ϕ which are not of class \mathcal{C}^1 . This is made possible using alternative expression for $\mathcal{K}_{H,T}^* \phi$. Using an integration by parts arguments, one obtains for all $t \in [0, T]$

$$\mathcal{K}_{H,T}^* \phi(t) = \begin{cases} \int_t^T \phi(s) \frac{\partial K_H(s, t)}{\partial t} ds, & H \in (\frac{1}{2}, 1); \\ \phi(t) K_H(T, t) + \int_t^T [\phi(s) - \phi(t)] \frac{\partial K_H(s, t)}{\partial t} ds, & H \in (0, \frac{1}{2}). \end{cases}$$

Applying the Itô isometry property for the Itô stochastic integral with respect to standard Brownian motion, one obtains the isometry property

$$\mathbb{E}\left[\left|\int_0^T \phi(t) d\beta_k^H(t)\right|^2\right] = \int_0^T |\mathcal{K}_{H,T}^\star \phi(t)|^2 dt. \quad (8)$$

It is straightforward to generalize the definition of the stochastic integral to integrands ϕ such that the right-hand side of the above isometry property is finite.

When $H \in (\frac{1}{2}, 1)$, one obtains the following expression for the second-order moment:

$$\int_0^T |\mathcal{K}_{H,T}^\star \phi(t)|^2 dt = H(2H-1) \int_0^T \int_0^T \phi(s_1) \phi(s_2) |s_2 - s_1|^{2H-2} ds_1 ds_2. \quad (9)$$

When $H \in (0, \frac{1}{2})$, there is no similar simple expressions. Instead, the following alternative expression for $\mathcal{K}_{H,T}^\star \phi(t)$ is employed: one has

$$\mathcal{K}_{H,T}^\star \phi(t) = c_H \left(\frac{\phi(t)}{(H - \frac{1}{2})(T-t)^{\frac{1}{2}-H}} - t^{\frac{1}{2}-H} \int_t^T \frac{t^{H-\frac{1}{2}} \phi(t) - s^{H-\frac{1}{2}} \phi(s)}{(s-t)^{\frac{3}{2}-H}} ds \right). \quad (10)$$

To conclude this section, let us introduce the cylindrical fractional Brownian motion $(B^H(t))_{t \geq 0}$ defined by

$$B^H(t) = \sum_{k \in \mathbb{N}} \beta_k^H(t) e_k.$$

The above series does not converge in \mathbb{H} , however it is possible to show that $A^{-\alpha} B^H(t)$ converges if $\alpha > 1/4$: indeed, one has

$$\mathbb{E}[\|A^{-\alpha} B^H(t)\|^2] = t^{2H} \sum_{k \in \mathbb{N}} \lambda_k^{-2\alpha} < \infty, \quad \forall \alpha > 1/4, \forall t \geq 0.$$

More generally, if $\theta \in \mathcal{L}_2(\mathbb{H})$ is an Hilbert–Schmidt linear operator on \mathbb{H} , then the random variable $\theta B^H(t)$ is a well-defined \mathbb{H} -valued Gaussian random variable for all $t \geq 0$, with

$$\mathbb{E}[\|\theta B^H(t)\|^2] = t^{2H} \|\theta\|_{\mathcal{L}_2(\mathbb{H})}^2.$$

For an integrand $\Theta : [0, T] \rightarrow \mathcal{L}_2(\mathbb{H})$, one defines the \mathbb{H} -valued stochastic integral with respect to the cylindrical fractional Brownian motion, as

$$\int_0^T \Theta(t) dB^H(t) = \sum_{k, \ell \in \mathbb{N}} \int_0^T \langle \Theta(t) e_\ell, e_k \rangle d\beta_\ell^H(t) e_k.$$

More details about fBm can be referred to [31, 32, 33].

3 Preliminary properties

3.1 Fractional Ornstein–Uhlenbeck process

Let us introduce the process $(Z^H(t))_{t \geq 0}$ defined by

$$Z^H(t) = \int_0^t S(t-s) dB^H(s), \quad \forall t \geq 0, \quad (11)$$

which is considered as the mild solution to the stochastic evolution equation

$$\begin{cases} dZ^H(t) + AZ^H(t) dt = dB^H(t), \\ Z^H(0) = 0. \end{cases} \quad (12)$$

Below we show that $(Z^H(t))_{t \geq 0}$ is well-defined with values in \mathbb{H} if and only if $H > 1/4$. Moreover, we provide moment bounds in \mathbb{H}^α for appropriate values of α and in E , and temporal regularity results.

Note one can express $Z^H(t)$ as a combination of real-valued Gaussian random variables $\mathcal{Z}_k^H(t)$: for all $t \geq 0$, one has

$$Z^H(t) = \sum_{k \in \mathbb{N}} \mathcal{Z}_k^H(t) e_k, \quad (13)$$

where for all $k \in \mathbb{N}$ the real-valued fractional Ornstein–Uhlenbeck process $(\mathcal{Z}_k^H(t))_{t \geq 0}$ is defined by

$$\mathcal{Z}_k^H(t) = \sum_{\ell \in \mathbb{N}} \int_0^T \langle S(t-s)e_\ell, e_k \rangle d\beta_\ell^H(s) = \int_0^T e^{-\lambda_k(t-s)} d\beta_k^H(s).$$

Therefore the analysis proceeds by proving moment bounds and temporal regularity properties for the real-valued fractional Ornstein–Uhlenbeck processes $(\mathcal{Z}_k^H(t))_{t \geq 0}$. One needs to pay attention to the dependence with respect to λ_k . We refer to Appendices B and C for results on real-valued fractional Ornstein–Uhlenbeck processes when $H \in (1/2, 1)$ and $H \in (0, 1/2)$ respectively.

Verifying that the process $(Z^H(t))_{t \geq 0}$ takes values in the Hilbert space \mathbb{H} requires to assume that $H > 1/4$.

Proposition 3.1. *For all $H \in (1/4, 1/2) \cup (1/2, 1)$, for all $t \geq 0$, almost surely $Z^H(t) \in \mathbb{H}$, and for all $p \in \mathbb{N}$ one has*

$$\sup_{t \geq 0} \mathbb{E}[\|Z^H(t)\|^{2p}] < \infty. \quad (14)$$

Moreover, for all $\alpha \in (0, 2H - 1/2)$, for all $t \geq 0$, almost surely $Z^H(t) \in \mathbb{H}^\alpha$, and for all $p \in \mathbb{N}$ one has

$$\sup_{t \geq 0} \mathbb{E}[\|Z^H(t)\|_{\mathbb{H}^\alpha}^{2p}] < \infty. \quad (15)$$

Proof of Proposition 3.1. Since the considered random variables are Gaussian, it suffices to deal with the case $p = 1$.

Owing to the decomposition (13) of $Z^H(t)$, for all $t \geq 0$ one has

$$\mathbb{E}[\|Z^H(t)\|^2] = \sum_{k \in \mathbb{N}} \mathbb{E}[|\mathcal{Z}_k^H(t)|^2].$$

Applying the inequality (42) from Lemma B.1 when $H \in (1/2, 1)$ or the inequality (46) from Lemma C.1 when $H \in (1/4, 1/2)$, for all $k \in \mathbb{N}$ one has

$$\sup_{t \geq 0} \mathbb{E}[|\mathcal{Z}_k^H(t)|^2] \leq C_H \lambda_k^{-2H}.$$

To obtain the moment bounds (14), it suffices to observe that one has

$$\sum_{k \in \mathbb{N}} \lambda_k^{-2H} < \infty \iff H > 1/4.$$

Next, for all $\alpha \geq 0$ and all $t \geq 0$, one has

$$\mathbb{E}[\|Z^H(t)\|_{\mathbb{H}^\alpha}^{2p}] = \mathbb{E}[\|A^{\alpha/2} Z^H(t)\|^2] = \sum_{k \in \mathbb{N}} \lambda_k^\alpha \mathbb{E}[|\mathcal{Z}_k^H(t)|^2] \leq C_H \sum_{k \in \mathbb{N}} \lambda_k^{\alpha-2H},$$

and one has

$$\sum_{k \in \mathbb{N}} \lambda_k^{\alpha-2H} < \infty \iff 2H - \alpha > 1/2.$$

This concludes the proof of Proposition 3.1. \square

Next, let us provide moment bounds of $Z^H(t)$ in the Banach space E . When $H \in (1/2, 1)$, this is straightforward consequence of Proposition 3.1 due to the embedding property (3). When $H \in (1/4, 1/2)$, one needs to use a version of the Kolmogorov regularity criterion.

Proposition 3.2. *For all $H \in (1/4, 1/2) \cup (1/2, 1)$, for all $t \geq 0$, almost surely $Z^H(t) \in E$. Moreover, for all $p \in \mathbb{N}$ one has*

$$\sup_{t \geq 0} \mathbb{E}[\|Z^H(t)\|_E^{2p}] < \infty. \quad (16)$$

Proof of Proposition 3.2. First, assume that $H \in (1/2, 1)$. In that case, one has $2H - 1/2 > 1/2$, therefore one may choose $\alpha \in (1/2, 2H - 1/2)$, and combining the embedding property (3) and the moment bounds (14) from Proposition 3.1, one obtains

$$\sup_{t \geq 0} \mathbb{E}[\|Z^H(t)\|_E^{2p}] \leq C_\alpha^{2p} \sup_{t \geq 0} \mathbb{E}[\|Z^H(t)\|_{\mathbb{H}^\alpha}^{2p}] < \infty.$$

Second, assume that $H \in (1/4, 1/2)$. For all $t \geq 0$ and all $\xi \in [0, 1]$, one has

$$Z^H(t, \xi) = \sum_{k \in \mathbb{N}} \mathcal{Z}_k^H(t) e_k(\xi).$$

Observe that one has $Z^H(t, 0) = Z^H(t, 1) = 0$ for all $t \geq 0$.

Let $\alpha \in (0, 2H - 1/2)$, then using the inequality (2), for all $\xi_1, \xi_2 \in [0, 1]$, one has for all $t \geq 0$

$$\begin{aligned} \mathbb{E}[\|Z^H(t, \xi_2) - Z^H(t, \xi_1)\|^2] &= \sum_{k \in \mathbb{N}} \mathbb{E}[|\mathcal{Z}_k^H(t)|^2] |e_k(\xi_2) - e_k(\xi_1)|^2 \\ &\leq C_\alpha \sum_{k \in \mathbb{N}} \lambda_k^{\alpha-2H} |\xi_2 - \xi_1|^{2\alpha} \\ &\leq C_{H,\alpha} |\xi_2 - \xi_1|^{2\alpha}, \end{aligned}$$

with $C_{H,\alpha} \in (0, \infty)$.

Since the random variable $Z^H(t, \xi_2) - Z^H(t, \xi_1)$ is Gaussian, for all $p \in \mathbb{N}$ there exists $C_{p,H,\alpha} \in (0, \infty)$ such that

$$\sup_{t \geq 0} \mathbb{E}[\|Z^H(t, \xi_2) - Z^H(t, \xi_1)\|^{2p}] \leq C_{p,H,\alpha} |\xi_2 - \xi_1|^{2p\alpha}.$$

Choosing a sufficiently large $p_\alpha \in \mathbb{N}$, such that $2p_\alpha \alpha > 1$, one may apply the Kolmogorov regularity criterion: for all $p \in \mathbb{N}$ such that $p \geq p_\alpha$, there exists $C_{p,H,\alpha} \in (0, \infty)$ such that

$$\sup_{t \geq 0} \mathbb{E}\left[\sup_{\xi_1, \xi_2 \in [0, 1], \xi_1 \neq \xi_2} \frac{\|Z^H(t, \xi_2) - Z^H(t, \xi_1)\|^{2p}}{|\xi_2 - \xi_1|^{2p\alpha-1}}\right] < \infty.$$

In particular, for all $t \geq 0$ one has $Z^H(t) \in E$. Choosing $\xi_1 = 0$ and $\xi_2 = \xi$ in the above, and recalling that $Z^H(t, 0) = 0$ for all $t \geq 0$, for all $p \geq p_\alpha$ one obtains

$$\sup_{t \geq 0} \mathbb{E}\left[\sup_{\xi \in [0, 1]} \|Z^H(t, \xi)\|^{2p}\right] < \infty.$$

Therefore one obtains the moment bounds (16) and the proof of Proposition 3.2 is completed. \square

Finally, let us provide temporal regularity properties for the process $(Z^H(t))_{t \geq 0}$.

Proposition 3.3. *For all $p \in \mathbb{N}$, $H \in (1/4, 1/2) \cup (1/2, 1)$ and $\alpha \in (0, 2H - 1/2)$, there exists $C_{p,H,\alpha} \in (0, \infty)$ such that one has*

$$\mathbb{E}[\|Z^H(t_2) - Z^H(t_1)\|^{2p}] \leq C_{p,H,\alpha} |t_2 - t_1|^{p\alpha}, \quad \forall t_2 \geq t_1 \geq 0. \quad (17)$$

Proof of Proposition 3.3. Like in the proof of Proposition 3.1, it suffices to consider the case $p = 1$ since the random variables $Z^H(t_2) - Z^H(t_1)$ are Gaussian.

Let $\alpha \in (0, 2H - \frac{1}{2})$. Recall the decomposition 13 of $Z^H(t)$. Applying the inequality (43) from Lemma B.2 when $H \in (1/2, 1)$ or the inequality (51) from Lemma C.2 when $H \in (1/4, 1/2)$, one obtains

$$\begin{aligned}\mathbb{E}[\|Z^H(t_2) - Z^H(t_1)\|^2] &= \sum_{k \in \mathbb{N}} \mathbb{E}[|\mathcal{Z}_k^H(t_2) - \mathcal{Z}_k^H(t_1)|^2] \\ &\leq C_{\alpha, H} \sum_{k \in \mathbb{N}} \lambda_k^{\alpha-2H} |t_2 - t_1|^\alpha.\end{aligned}$$

Note that

$$\sum_{k \in \mathbb{N}} \lambda_k^{\alpha-2H} < \infty.$$

As a result, one obtains the inequality (17) and the proof of Proposition 3.3 is completed. \square

3.2 Well-posedness of the SPDE

Let $T \in (0, \infty)$ and $x_0 \in \mathbb{H}$. The process $(X(t))_{t \in [0, T]}$ is a mild solution to the stochastic evolution equation

$$\begin{cases} dX(t) + AX(t) dt = (F(X(t)) + G(X(t))) dt + dB^H(t), \\ X(0) = x_0, \end{cases} \quad (18)$$

if one has for all $t \in [0, T]$

$$X(t) = S(t)x_0 + \int_0^t S(t-s)(F(X(s)) + G(X(s))) ds + \int_0^t S(t-s) dB(s). \quad (19)$$

Letting

$$Y(t) = X(t) - Z^H(t), \quad \forall t \geq 0,$$

it is equivalent to consider the evolution equation

$$dY(t) + AY(t) dt = (F(Y(t) + Z^H(t)) + G(Y(t) + Z^H(t))) dt.$$

By standard techniques, one obtains the following result.

Proposition 3.4. *Let $H \in (1/4, 1/2) \cup (1/2, 1)$.*

For all $T \in (0, \infty)$ and $x_0 \in E$, the stochastic evolution equation (18) admits a unique mild solution (in the sense of (19)), such that $X(t) \in E$ for all $t \in [0, T]$.

Moreover, for all $p \in \mathbb{N}$, $T \in (0, \infty)$ and $x_0 \in E$, there exists $C_{p,H}(T, \|x_0\|_E) \in (0, \infty)$ such that

$$\sup_{t \in [0, T]} (\mathbb{E}[\|X(t)\|_E^p])^{\frac{1}{p}} \leq C_{p,H}(T, \|x_0\|_E). \quad (20)$$

Finally, for all $p \in \mathbb{N}$, $\alpha \in (0, 2H - 1/2)$, $T \in (0, \infty)$ and $x_0 \in \mathbb{H}^\alpha \cap E$, there exists a positive real number $C_{p,H,\alpha}(T, \|x_0\|_E, \|x_0\|_{\mathbb{H}^\alpha}) \in (0, \infty)$ such that one has

$$(\mathbb{E}[\|X(t_2) - X(t_1)\|^p])^{\frac{1}{p}} \leq C_{p,H,\alpha}(T, \|x_0\|_E, \|x_0\|_{\mathbb{H}^\alpha}) |t_2 - t_1|^{\frac{\alpha}{2}}, \quad \forall t_1, t_2 \in [0, T]. \quad (21)$$

4 Numerical scheme

Let $T \in (0, \infty)$ be an arbitrary time, and define the time-step size $\Delta t = T/N$, for some $M \in \mathbb{N}$.

For all $n \in \{0, \dots, N\}$, set $t_n = n\Delta t$, and for all $n \in \{0, \dots, N\}$ define the increments of the cylindrical fractional Brownian motion

$$\Delta B_n^H = B^H(t_{n+1}) - B^H(t_n).$$

We consider the following numerical scheme: set

$$\begin{cases} X_{n+1} = S(\Delta t)\Phi_{\Delta t}(X_n) + A^{-1}(I - S(\Delta t))G(X_n) + S(\Delta t)\Delta B_n^H, & \forall n \in \{0, \dots, N-1\}, \\ X_0 = x_0. \end{cases} \quad (22)$$

Noting that $\Phi_{\Delta t} = Id + \Delta t\Psi_{\Delta t}$, the scheme can equivalently be written as

$$X_{n+1} = X_n + \Delta t S(\Delta t)\Psi_{\Delta t}(X_n) + A^{-1}(I - S(\Delta t))G(X_n) + S(\Delta t)\Delta B_n^H.$$

The discrete-time process $(X_n)_{0 \leq n \leq N}$ can be described using a discrete-time mild formulation: for all $n \in \{0, \dots, N\}$, one has

$$\begin{aligned} X_n &= S(\Delta t)^n x_0 + \Delta t \sum_{j=0}^{n-1} S(\Delta t)^{n-j} \Psi_{\Delta t}(X_j) + \sum_{j=0}^{n-1} A^{-1}(I - S(\Delta t))S(\Delta t)^{n-j-1} G(X_j) + \sum_{j=0}^{n-1} S(\Delta t)^{n-j} \Delta B_j^H \\ &= S(t_n)x_0 + \Delta t \sum_{j=0}^{n-1} S(t_n - t_j)\Psi_{\Delta t}(X_j) + \sum_{j=0}^{n-1} A^{-1}(I - S(\Delta t))S(t_n - t_{j+1})G(X_j) + \sum_{j=0}^{n-1} S(t_n - t_j)\Delta B_j^H. \end{aligned}$$

The objective of this section is to prove the main result of this manuscript.

Theorem 4.1. *For all $p \in \mathbb{N}$, $\alpha \in (0, 2H - 1/2)$, $T \in (0, \infty)$ and $x_0 \in \mathbb{H}^\alpha \cap E$, there exists a positive real number $C_{p,H,\alpha}(T, \|x_0\|_E, \|x_0\|_{\mathbb{H}^\alpha}) \in (0, \infty)$ such that for all $\Delta t = T/N$ with $N \in \mathbb{N}$, one has*

$$\sup_{n=0, \dots, N} (\mathbb{E}[\|X_n - X(t_n)\|_{\mathbb{H}}^p])^{\frac{1}{p}} \leq C_{p,H,\alpha}(T, \|x_0\|_E, \|x_0\|_{\mathbb{H}^\alpha})\Delta t^{\frac{\alpha}{2}}. \quad (23)$$

The proof of Theorem 4.1 follows from several auxiliary results.

First, define the auxiliary discrete-time process $(Z_n^H)_{n \geq 0}$ defined by

$$Z_n^H = \sum_{j=0}^{n-1} S(\Delta t)^{n-j} \Delta B_j^H = \sum_{j=0}^{n-1} S(t_n - t_j)\Delta B_j^H, \quad \forall n \geq 0, \quad (24)$$

which is the solution to the scheme when the nonlinearities F and G are removed:

$$\begin{cases} Z_{n+1}^H = S(\Delta t)Z_n^H + S(\Delta t)\Delta B_n^H, & \forall n \geq 0, \\ Z_0^H = 0. \end{cases}$$

Note that one can express Z_n^H as a combination of real-valued Gaussian random variables $\mathcal{Z}_{k,n}^H$: one has

$$Z_n^H = \sum_{k \in \mathbb{N}} \mathcal{Z}_{k,n}^H e_k, \quad (25)$$

where for all $k \in \mathbb{N}$ the real-valued discrete-time process $(\mathcal{Z}_{k,n}^H)_{n \geq 0}$ is defined by

$$\mathcal{Z}_{k,n}^H = \sum_{\ell \in \mathbb{N}} \sum_{j=0}^{n-1} \langle S(t_n - t_j)e_\ell, e_k \rangle \Delta B_j^H = \sum_{j=0}^{n-1} e^{-\lambda_k(t_n - t_j)} \Delta \beta_{k,j}^H,$$

with increments of the real-valued fractional Brownian motions denoted by $\Delta \beta_{k,j}^H = \beta_k^H(t_{j+1}) - \beta_k^H(t_j)$.

First, Lemmas D.2 and D.4 show that the discrete-time process $(Z_n^H)_{n \geq 0}$ takes values in \mathbb{H} and in E when it is assumed that $H > 1/4$.

Proposition 4.2. For all $H \in (1/4, 1/2) \cup (1/2, 1)$, for all $\Delta t \in (0, 1)$ and all $n \geq 0$, almost surely $Z_n^H \in \mathbb{H}$ and $Z_n \in E$. Moreover, for all $p \in \mathbb{N}$, one has

$$\sup_{\Delta t \in (0, 1)} \sup_{n \in \mathbb{N}} (\mathbb{E}[\|Z_n^H\|^p])^{\frac{1}{p}} + \sup_{\Delta t \in (0, 1)} \sup_{n \in \mathbb{N}} (\mathbb{E}[\|Z_n^H\|_E^p])^{\frac{1}{p}} < \infty. \quad (26)$$

Proof of Proposition 4.2. Since the considered random variables are Gaussian, it suffices to deal with the case $p = 1$.

Owing to the decomposition (25), for all $n \geq 0$ one has

$$\mathbb{E}[\|Z_n^H\|^2] = \sum_{k \in \mathbb{N}} \mathbb{E}[|\mathcal{Z}_{k,n}^H|^2].$$

Applying the inequality (62) from Lemma D.2 when $H \in (1/2, 1)$ or the inequality (74) from Lemma D.4 when $H \in (1/4, 1/2)$, for all $k \in \mathbb{N}$ one has

$$\sup_{\Delta t \in (0, 1)} \sup_{n \in \mathbb{N}} \mathbb{E}[|\mathcal{Z}_{k,n}^H|^2] \leq C_H \lambda_k^{-2H}.$$

When $H > 1/4$, one has $\sum_{k \in \mathbb{N}} \lambda_k^{-2H} < \infty$, therefore one obtains moment bounds in \mathbb{H} :

$$\sup_{\Delta t \in (0, 1)} \sup_{n \in \mathbb{N}} \mathbb{E}[\|Z_n^H\|^2] < \infty.$$

Similarly, for all $\alpha \in (0, 2H - \frac{1}{2})$, one has

$$\sup_{\Delta t \in (0, 1)} \sup_{n \in \mathbb{N}} \mathbb{E}[\|Z_n^H\|_{\mathbb{H}^\alpha}^2] = \sup_{\Delta t \in (0, 1)} \sup_{n \in \mathbb{N}} \mathbb{E}[\|(-A)^{\frac{\alpha}{2}} Z_n^H\|^2] \leq \sum_{k \in \mathbb{N}} \lambda_k^\alpha \sup_{\Delta t \in (0, 1)} \sup_{n \in \mathbb{N}} \mathbb{E}[|\mathcal{Z}_{k,n}^H|^2] \leq C_J \sum_{k \in \mathbb{N}} \lambda_k^{\alpha-2H} < \infty.$$

To prove moment bounds in E , like in the proof of Proposition 3.2, the two cases $H \in (1/4, 1/2)$ and $H \in (1/2, 1)$ are treated separately.

First, assume that $H \in (1/2, 1)$. In that case, one has $2H - 1/2 > 1/2$, therefore one may choose $\alpha \in (1/2, 2H - 1/2)$, and combining the embedding property (3) and the moment bounds above of Z_n^H in the \mathbb{H}^α norm, one obtains

$$\sup_{\Delta t \in (0, 1)} \sup_{n \in \mathbb{N}} \mathbb{E}[\|Z_n^H\|_E^{2p}] \leq C_\alpha^{2p} \sup_{\Delta t \in (0, 1)} \sup_{n \in \mathbb{N}} \mathbb{E}[\|Z_n^H\|_{\mathbb{H}^\alpha}^{2p}] < \infty.$$

Second, assume that $H \in (1/4, 1/2)$. For all $n \geq 0$ and all $\xi \in [0, 1]$, one has

$$Z_n^H(\xi) = \sum_{k \in \mathbb{N}} \mathcal{Z}_{k,n}^H e_k(\xi).$$

Observe that one has $Z_n^H(0) = Z_n^H(1) = 0$ for all $n \geq 0$.

Let $\alpha \in (0, 2H - 1/2)$, then using the inequality (2), for all $\xi_1, \xi_2 \in [0, 1]$, one has for all $n \geq 0$

$$\begin{aligned} \mathbb{E}[\|Z_n^H(\xi_2) - Z_n^H(\xi_1)\|^2] &= \sum_{k \in \mathbb{N}} \mathbb{E}[|\mathcal{Z}_{k,n}^H|^2] |e_k(\xi_2) - e_k(\xi_1)|^2 \\ &\leq C_\alpha \sum_{k \in \mathbb{N}} \lambda_k^{\alpha-2H} |\xi_2 - \xi_1|^{2\alpha} \\ &\leq C_{H,\alpha} |\xi_2 - \xi_1|^{2\alpha}, \end{aligned}$$

with $C_{H,\alpha} \in (0, \infty)$.

Since the random variable $Z_n^H(\xi_2) - Z_n^H(\xi_1)$ is Gaussian, for all $p \in \mathbb{N}$ there exists $C_{p,H,\alpha} \in (0, \infty)$ such that

$$\sup_{\Delta t \in (0, 1)} \sup_{n \in \mathbb{N}} \mathbb{E}[\|Z_n^H(\xi_2) - Z_n^H(\xi_1)\|^{2p}] \leq C_{p,H,\alpha} |\xi_2 - \xi_1|^{2p\alpha}.$$

Choosing a sufficiently large $p_\alpha \in \mathbb{N}$, such that $2p_\alpha\alpha > 1$, one may apply the Kolmogorov regularity criterion: for all $p \in \mathbb{N}$ such that $p \geq p_\alpha$, there exists $C_{p,H,\alpha} \in (0, \infty)$ such that

$$\sup_{\Delta t \in (0,1)} \sup_{n \in \mathbb{N}} \mathbb{E} \left[\sup_{\xi_1, \xi_2 \in [0,1], \xi_1 \neq \xi_2} \frac{\|Z_n^H(\xi_2) - Z_n^H(\xi_1)\|^{2p}}{|\xi_2 - \xi_1|^{2p\alpha-1}} \right] < \infty.$$

In particular, for all $n \geq 0$ one has $Z_n^H \in E$. Choosing $\xi_1 = 0$ and $\xi_2 = \xi$ in the above, and recalling that $Z_n^H(0) = 0$ for all $n \geq 0$, for all $p \geq p_\alpha$ one obtains

$$\sup_{\Delta t \in (0,1)} \sup_{n \in \mathbb{N}} \mathbb{E}[\|Z_n^H\|_E^{2p}] = \sup_{\Delta t \in (0,1)} \sup_{n \in \mathbb{N}} \mathbb{E} \left[\sup_{\xi \in [0,1]} \|Z_n^H(\xi)\|^{2p} \right] < \infty.$$

The proof of Proposition 4.2 is thus completed. \square

Proposition 4.3. *For all $p \in \mathbb{N}$ and $\alpha \in (0, 2H - 1/2)$, there exists $C_{p,H,\alpha} \in (0, \infty)$ such that for all $\Delta t = T/N$ with $N \in \mathbb{N}$, one has*

$$\sup_{n \in \mathbb{N}} (\mathbb{E}[\|Z^H(t_n) - Z_n^H\|_{\mathbb{H}}^p])^{\frac{1}{p}} \leq C_{p,H,\alpha} \Delta t^{\frac{\alpha}{2}}. \quad (27)$$

Proof of Proposition 4.3. Owing to the decompositions (13) and (25) of $Z^H(t_n)$ and of Z_n^H respectively, for all $n \geq 0$ one has

$$Z^H(t_n) - Z_n^H = \sum_{k \in \mathbb{N}} (\mathcal{Z}_k^H(t_n) - \mathcal{Z}_{k,n}^H) e_k.$$

It suffices to consider the case $p = 1$ since the \mathbb{H} -valued random variable $Z^H(t_n) - Z_n^H$ is Gaussian. One obtains

$$\mathbb{E}[\|Z^H(t_n) - Z_n^H\|^2] = \sum_{k \in \mathbb{N}} \mathbb{E}[|\mathcal{Z}_k^H(t_n) - \mathcal{Z}_{k,n}^H|^2].$$

Let $\alpha \in (0, 2H - 1/2)$. Applying the inequality (61) from Lemma D.1 when $H \in (1/2, 1)$ or the inequality (63) from Lemma D.3 when $H \in (1/4, 1/2)$, one has

$$\mathbb{E}[\|Z^H(t_n) - Z_n^H\|^2] \leq C_{\alpha,H} \sum_{k \in \mathbb{N}} \lambda_k^{\alpha-2H} \Delta t^\alpha.$$

Since one has $\sum_{k \in \mathbb{N}} \lambda_k^{\alpha-2H} < \infty$ under the condition $\alpha \in (0, 2H - 1/2)$, one obtains the inequality (27) and the proof of Proposition 4.3 is thus completed. \square

Proposition 4.4 provides moment bounds in E on the numerical solution X_n .

Proposition 4.4. *For all $p \in \mathbb{N}$, $\alpha \in (0, 2H - 1/2)$, $T \in (0, \infty)$ and $x_0 \in E$, there exists a positive real number $C_{p,H,\alpha}(T, \|x_0\|_E) \in (0, \infty)$ such that, for all $\Delta t = T/N$ with $N \in \mathbb{N}$, one has*

$$\sup_{n=0, \dots, N} (\mathbb{E}[\|X_n\|_E^p])^{\frac{1}{p}} \leq C_{p,H}(T, \|x_0\|_E). \quad (28)$$

Proof of Proposition 4.4. Introduce the auxiliary process $(Y_n)_{0 \leq n \leq N}$ defined by

$$Y_n = X_n - Z_n^H, \quad \forall n \in \{0, \dots, N\}.$$

Recall that $\Phi_{\Delta t} - I = \Delta t \Psi_{\Delta t}$. Then, for all $n \in \{0, \dots, N-1\}$, one has

$$\begin{aligned} Y_{n+1} &= X_{n+1} - Z_{n+1}^H \\ &= S(\Delta t)[\Phi_{\Delta t}(X_n) - Z_n^H] + A^{-1}(I - S(\Delta t))G(X_n) \\ &= S(\Delta t)[\Phi_{\Delta t}(Y_n + Z_n^H) - \Phi_{\Delta t}(Z_n^H)] + A^{-1}(I - S(\Delta t))[G(Y_n + Z_n^H) - G(Z_n^H)] \\ &\quad + \Delta t S(\Delta t) \Psi_{\Delta t}(Z_n^H) + A^{-1}(I - S(\Delta t))G(Z_n^H). \end{aligned}$$

Observe that

$$A^{-1}(I - S(\Delta t)) = \int_0^{\Delta t} S(s) \, ds,$$

therefore

$$\|A^{-1}(I - S(\Delta t))\|_{\mathcal{L}(E)} \leq \int_0^{\Delta t} \|S(s)\|_{\mathcal{L}(E)} \, ds \leq \Delta t. \quad (29)$$

Owing to the global Lipschitz continuity property of $\Phi_{\Delta t}$ and to the local Lipschitz continuity property of $\Psi_{\Delta t}$ stated in Lemma 2.4, and recalling that G is globally Lipschitz continuous, one has the following: there exists $C \in (0, \infty)$ such that one has

$$\|Y_{n+1}\|_E \leq [e^{C\Delta t} + C\Delta t] \|Y_n\|_E + C\Delta t(1 + \|Z_n^H\|_E^{2q'+1}).$$

Note that one has $e^{C\Delta t} + C\Delta t \leq e^{2C\Delta t}$. For all $p \in \mathbb{N}$, owing to the Minkowski inequality, and applying the moment bounds (20) in E for Z_n^H from Proposition 4.2, there exists $C_{p,H} \in (0, \infty)$ such that for all $n \in \{0, \dots, N-1\}$ one has

$$(\mathbb{E}[\|Y_{n+1}\|_E^p])^{\frac{1}{p}} \leq e^{2C\Delta t} (\mathbb{E}[\|Y_n\|_E^p])^{\frac{1}{p}} + C_{p,H} \Delta t.$$

Since $Y_0 = x_0$, it is straightforward to obtain the following inequality: for all $n \in \{0, \dots, N-1\}$ one has

$$(\mathbb{E}[\|Y_n\|_E^p])^{\frac{1}{p}} \leq e^{2Cn\Delta t} \|x_0\|_E + C_{p,H} n \Delta t e^{2CN\Delta t} \leq e^{2CT} \|x_0\|_E + C_{p,H} e^{2CT} T.$$

This yields the inequality (28) and the proof of Proposition 4.4 is completed. \square

To perform the error analysis, let us introduce the auxiliary discrete-time process $(\tilde{X}_n)_{0 \leq n \leq N}$ defined by

$$\tilde{X}_n = S(t_n)x_0 + \Delta t \sum_{j=0}^{n-1} S(t_n - t_j)F(X(t_j)) + \sum_{j=0}^{n-1} A^{-1}(I - S(\Delta t))S(t_n - t_{j+1})G(X(t_j)) + Z_n^H. \quad (30)$$

Note that for all $n \in \{0, \dots, N-1\}$, one has

$$\tilde{X}_{n+1} = S(\Delta t)\tilde{X}_n + \Delta t S(\Delta t)F(X(t_n)) + A^{-1}(I - S(\Delta t))G(X(t_n)) + S(\Delta t)\Delta B_n^H.$$

First, one obtains moment bounds in E for \tilde{X}_n .

Proposition 4.5. *For all $p \in \mathbb{N}$, $\alpha \in (0, 2H - 1/2)$, $T \in (0, \infty)$ and $x_0 \in E$, there exists a positive real number $C_{p,H,\alpha}(T, \|x_0\|_E) \in (0, \infty)$ such that, for all $\Delta t = T/N$ with $N \in \mathbb{N}$, one has*

$$\sup_{n=0, \dots, N} (\mathbb{E}[\|\tilde{X}_n\|_E^p])^{\frac{1}{p}} \leq C_{p,H}(T, \|x_0\|_E). \quad (31)$$

Proof of Proposition 4.5. The mapping f has at most polynomial growth owing to Assumption 2.3, and the mapping g is globally Lipschitz continuous and thus has at most linear growth. For all $p \in \mathbb{N}$, applying the Minkowski inequality and using the upper bound (29), for all $n \in \{0, \dots, N\}$, one has

$$(\mathbb{E}[\|\tilde{X}_n\|_E^p])^{\frac{1}{p}} \leq \|x_0\|_E + C\Delta t \sum_{j=0}^{n-1} \left(1 + (\mathbb{E}[\|X(t_j)\|_E^{2qp}])^{\frac{1}{p}}\right) + C\Delta t \sum_{j=0}^{n-1} \left(1 + (\mathbb{E}[\|X(t_j)\|_E^p])^{\frac{1}{p}}\right) + (\mathbb{E}[\|Z_n^H\|_E^p])^{\frac{1}{p}}.$$

Applying the moment bounds (20) in E for the exact solution $X(t)$ stated in Proposition 3.4 and for the Gaussian numerical solution Z_n^H stated in Proposition 4.2, one obtains the inequality (31) and the proof of Proposition 4.5 is completed. \square

In order to prove Theorem 4.1, it then remains to state and prove Propositions 4.6 and 4.7, which provide error estimates on $\tilde{X}_n - X(t_n)$ and $X_n - \tilde{X}_n$ respectively.

Proposition 4.6. *For all $p \in \mathbb{N}$, $\alpha \in (0, 2H - 1/2)$, $T \in (0, \infty)$ and $x_0 \in \mathbb{H}^\alpha \cap E$, there exists a positive real number $C_{p,H,\alpha}(T, \|x_0\|_E, \|x_0\|_{\mathbb{H}^\alpha}) \in (0, \infty)$ such that, for all $\Delta t = T/N$ with $N \in \mathbb{N}$, one has*

$$\sup_{n=0,\dots,N} (\mathbb{E}[\|\tilde{X}_n - X(t_n)\|_{\mathbb{H}}^p])^{\frac{1}{p}} \leq C_{p,H,\alpha}(T, \|x_0\|_E, \|x_0\|_{\mathbb{H}^\alpha}) \Delta t^{\frac{\alpha}{2}}. \quad (32)$$

Proof of Proposition 4.6. Observe that for all $j \in \{0, \dots, N-1\}$, one has

$$A^{-1}(I - S(\Delta t))S(t_n - t_{j+1}) = \int_{t_j}^{t_{j+1}} S(t_{j+1} - s)S(t_n - t_{j+1}) ds = \int_{t_j}^{t_{j+1}} S(t_n - s) ds.$$

As a result, for all $n \in \{0, \dots, N\}$, one has the following expression for \tilde{X}_n :

$$\tilde{X}_n = S(t_n)x_0 + \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} S(t_n - t_j)F(X(t_j)) ds + \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} S(t_n - s)G(X(t_j)) ds + Z_n^H.$$

Moreover, for all $n \in \{0, \dots, N\}$, one has the following expression for $X(t_n)$:

$$X(t_n) = S(t_n)x_0 + \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} S(t_n - s)F(X(s)) ds + \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} S(t_n - s)G(X(s)) ds + Z^H(t_n).$$

Therefore, the error $\tilde{X}_n - X(t_n)$ can be decomposed as follows: one has

$$\tilde{X}_n - X(t_n) = e_n^1 + e_n^2 + e_n^3 + e_n^4, \quad (33)$$

where the error terms e_n^1 , e_n^2 , e_n^3 and e_n^4 are defined as

$$\begin{aligned} e_n^1 &= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} [S(t_n - t_j) - S(t_n - s)]F(X(t_j)) ds \\ e_n^2 &= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} S(t_n - s)[F(X(t_j)) - F(X(s))] ds \\ e_n^3 &= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} S(t_n - s)[G(X(t_j)) - G(X(s))] ds \\ e_n^4 &= Z_n^H - Z^H(t_n). \end{aligned}$$

To deal with the error term e_n^1 , recall that owing to Assumption 2.3 the mapping f has at most polynomial growth. Let $\alpha \in (0, 2H - 1/2)$. Owing to Proposition 2.2, note that for all $j \in \{0, \dots, n-1\}$ and $s \in [t_j, t_{j+1}]$, one has

$$\begin{aligned} \| [S(t_n - t_j) - S(t_n - s)]F(X(t_j)) \|_{\mathbb{H}} &\leq C_\alpha \frac{(s - t_j)^{\frac{\alpha}{2}}}{(t_n - s)^{\frac{\alpha}{2}}} \|F(X(t_j))\|_{\mathbb{H}} \\ &\leq C_\alpha \frac{\Delta t^{\frac{\alpha}{2}}}{(t_n - s)^{\frac{\alpha}{2}}} \left(1 + \|X(t_j)\|_E^{2q}\right). \end{aligned}$$

For all $p \in \mathbb{N}$, applying the Minkowski inequality and the moment bounds (20) in E for the exact solution from Proposition 3.4, one obtains for all $n \in \{0, \dots, N\}$

$$\begin{aligned} (\mathbb{E}[\|e_n^1\|_{\mathbb{H}}^p])^{\frac{1}{p}} &\leq C_\alpha \Delta t^{\frac{\alpha}{2}} \sup_{0 \leq j \leq n} \left(1 + (\mathbb{E}[\|X(t_j)\|_E^{2qp}])^{\frac{1}{p}}\right) \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \frac{1}{(t_n - s)^{\frac{\alpha}{2}}} ds \\ &\leq C_{p,\alpha}(T, \|x_0\|_E) \Delta t^{\frac{\alpha}{2}}. \end{aligned}$$

To deal with the error term e_n^2 , one employs the local Lipschitz continuity property of f from Assumption 2.3: for all $j \in \{0, \dots, N-1\}$ and all $s \in [t_j, t_{j+1}]$ one has

$$\|S(t_n - s)[F(X(t_j)) - F(X(s))]\|_{\mathbb{H}} \leq C(1 + \|X(t_j)\|_E^{2q} + \|X(s)\|_E^{2q})\|X(s) - X(t_j)\|_{\mathbb{H}}.$$

Let $p \in \mathbb{N}$. Applying the Minkowski and the Cauchy–Schwarz inequalities, one has

$$\begin{aligned} (\mathbb{E}[\|e_n^2\|_{\mathbb{H}}^p])^{\frac{1}{p}} &\leq C \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (\mathbb{E}[(1 + \|X(t_j)\|_E^{2qp} + \|X(s)\|_E^{2qp})\|X(s) - X(t_j)\|_{\mathbb{H}}^p])^{\frac{1}{p}} ds \\ &\leq C \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (\mathbb{E}[(1 + \|X(t_j)\|_E^{4qp} + \|X(s)\|_E^{4qp})])^{\frac{1}{2p}} (\mathbb{E}[\|X(s) - X(t_j)\|_{\mathbb{H}}^{2p}])^{\frac{1}{2p}} ds. \end{aligned}$$

Applying the moment bounds (20) in E on the exact solution and the inequality (21) from Proposition 3.4, one obtains the upper bound

$$(\mathbb{E}[\|e_n^2\|_{\mathbb{H}}^p])^{\frac{1}{p}} \leq C_{p,H,\alpha}(T, \|x_0\|_E, \|x_0\|_{\mathbb{H}^\alpha}) \Delta t^{\frac{\alpha}{2}}.$$

To deal with the error term e_n^3 , recall that G is globally Lipschitz continuous owing to Assumption 2.5. Let $p \in \mathbb{N}$. Applying the Minkowski inequality and the inequality (21) from Proposition 3.4, one obtains the upper bounds

$$\begin{aligned} (\mathbb{E}[\|e_n^3\|_{\mathbb{H}}^p])^{\frac{1}{p}} &\leq C \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (\mathbb{E}[\|X(s) - X(t_j)\|_{\mathbb{H}}^p])^{\frac{1}{p}} ds \\ &\leq C_{p,H,\alpha}(T, \|x_0\|_E, \|x_0\|_{\mathbb{H}^\alpha}) \Delta t^{\frac{\alpha}{2}} \end{aligned}$$

Finally, for the error term e_n^4 , it suffices to apply the error estimate (27) from Proposition 4.3: one obtains

$$(\mathbb{E}[\|e_n^4\|_{\mathbb{H}}^p])^{\frac{1}{p}} \leq C_\alpha \Delta t^{\frac{\alpha}{2}}.$$

Gathering the estimates on the error terms e_n^1, e_n^2, e_n^3 and e_n^4 then provides the error estimate (32) and the proof of Proposition 4.6 is completed. \square

Proposition 4.7. *For all $p \in \mathbb{N}$, $\alpha \in (0, 2H - 1/2)$, $T \in (0, \infty)$ and $x_0 \in \mathbb{H}^\alpha \cap E$, there exists a positive real number $C_{p,H,\alpha}(T, \|x_0\|_E, \|x_0\|_{\mathbb{H}^\alpha}) \in (0, \infty)$ such that, for all $\Delta t = T/N$ with $N \in \mathbb{N}$, one has*

$$\sup_{n=0, \dots, N} (\mathbb{E}[\|X_n - \tilde{X}_n\|_{\mathbb{H}}^p])^{\frac{1}{p}} \leq C_{p,H,\alpha}(T, \|x_0\|_E, \|x_0\|_{\mathbb{H}^\alpha}) \Delta t^{\frac{\alpha}{2}}. \quad (34)$$

Proof of Proposition 4.7. Recall that $F = \Psi_0$. For all $n \in \{0, \dots, N-1\}$, one has

$$X_{n+1} - \tilde{X}_{n+1} = S(\Delta t)(X_n - \tilde{X}_n) + \Delta t S(\Delta t)(\Psi_{\Delta t}(X_n) - \Psi_0(X_n)) + A^{-1}(I - S(\Delta t))(G(X_n) - G(X(t_n))).$$

Observe that one has

$$\begin{aligned} \Psi_{\Delta t}(X_n) - \Psi_0(X_n) &= \Psi_{\Delta t}(X_n) - \Psi_{\Delta t}(\tilde{X}_n) + \Psi_{\Delta t}(\tilde{X}_n) - \Psi_{\Delta t}(X(t_n)) + \Psi_{\Delta t}(X(t_n)) - \Psi_0(X(t_n)) \\ G(X_n) - G(X(t_n)) &= G(X_n) - G(\tilde{X}_n) + G(\tilde{X}_n) - G(X(t_n)). \end{aligned}$$

Recalling that $I + \Delta t \Psi_{\Delta t} = \Phi_{\Delta t}$, one thus obtains

$$\begin{aligned} X_{n+1} - \tilde{X}_{n+1} &= S(\Delta t)(\Phi_{\Delta t}(X_n) - \Phi_{\Delta t}(\tilde{X}_n)) \\ &\quad + \Delta t S(\Delta t)(\Psi_{\Delta t}(\tilde{X}_n) - \Psi_{\Delta t}(X(t_n))) \\ &\quad + \Delta t S(\Delta t)(\Psi_{\Delta t}(X(t_n)) - \Psi_0(X(t_n))) \\ &\quad + A^{-1}(I - S(\Delta t))(G(X_n) - G(\tilde{X}_n)) \\ &\quad + A^{-1}(I - S(\Delta t))(G(\tilde{X}_n) - G(X(t_n))). \end{aligned}$$

Note that $\|A^{-1}(I - S(\Delta t))\|_{\mathcal{L}(\mathbb{H})} \leq \Delta t$. Owing to Lemma 2.4, the mapping $\Phi_{\Delta t}$ is globally Lipschitz continuous, uniformly with respect to Δt . Recall also that G is globally Lipschitz continuous. Moreover, applying the properties stated in Lemma 2.4 on the mapping $\Psi_{\Delta t}$, one obtains the following: there exists $C \in (0, \infty)$ such that one has

$$\begin{aligned} \|X_{n+1} - \tilde{X}_{n+1}\|_{\mathbb{H}} &\leq e^{C\Delta t} \|X_n - \tilde{X}_n\|_{\mathbb{H}} \\ &\quad + C\Delta t (1 + \|\tilde{X}_n\|_E^{2q} + \|X(t_n)\|_E^{2q}) \|\tilde{X}_n - X(t_n)\|_{\mathbb{H}} \\ &\quad + C\Delta t^2 (1 + \|X(t_n)\|_E^{2q+1}) \\ &\quad + C\Delta t \|X_n - \tilde{X}_n\|_{\mathbb{H}} \\ &\quad + C\Delta t \|\tilde{X}_n - X(t_n)\|_{\mathbb{H}} \end{aligned}$$

Note that $e^{C\Delta t} + C\Delta t \leq e^{2C\Delta t}$. Let $p \in \mathbb{N}$. Applying the Minkowski and the Cauchy–Schwarz inequalities, for all $n \in \{0, \dots, N-1\}$ one has

$$\begin{aligned} (\mathbb{E}[\|X_{n+1} - \tilde{X}_{n+1}\|_{\mathbb{H}}^p])^{\frac{1}{p}} &\leq e^{2C\Delta t} (\mathbb{E}[\|X_n - \tilde{X}_n\|_{\mathbb{H}}^p])^{\frac{1}{p}} \\ &\quad + C\Delta t \left(1 + (\mathbb{E}[\|\tilde{X}_n\|_E^{4qp}])^{\frac{1}{p}} + (\mathbb{E}[\|X(t_n)\|_E^{4qp}])^{\frac{1}{2p}} \right) (\mathbb{E}[\|\tilde{X}_n - X(t_n)\|_{\mathbb{H}}^{2p}])^{\frac{1}{2p}} \\ &\quad + C\Delta t^2 \left(1 + (\mathbb{E}[\|X(t_n)\|_E^{(2q+1)p}])^{\frac{1}{p}} \right). \end{aligned}$$

Using the moment bounds (31) for \tilde{X}_n from Proposition 4.5 and (20) for $X(t_n)$ from Proposition 3.4, and the inequality (21) from Proposition 3.4, one obtains the following upper bound: for all $n \in \{0, \dots, N-1\}$

$$(\mathbb{E}[\|X_{n+1} - \tilde{X}_{n+1}\|_{\mathbb{H}}^p])^{\frac{1}{p}} \leq e^{2C\Delta t} (\mathbb{E}[\|X_n - \tilde{X}_n\|_{\mathbb{H}}^p])^{\frac{1}{p}} + C_{p,H,\alpha}(T, \|x_0\|_E, \|x_0\|_{\mathbb{H}^\alpha}) \Delta t^{1+\frac{\alpha}{2}}.$$

It is then straightforward to obtain the error estimate (34) and the proof of Proposition 4.7 is completed. \square

Proof of Theorem 4.1. For all $n \in \{0, \dots, N\}$, one has

$$X(t_n) - X_n = X(t_n) - \tilde{X}_n + \tilde{X}_n - X_n.$$

It thus suffices to combine the error estimates (32) and (34) from Propositions 4.6 and 4.7 respectively, in order to establish the error estimate (23). The proof of Theorem 4.1 is then completed. \square

5 Numerical experiments

In this section, we present several numerical experiments which illustrate the convergence result obtained in Theorem 4.1.

We consider the parabolic stochastic partial differential equation driven by fractional noise described as

$$\begin{cases} \frac{\partial X(t, \xi)}{\partial t} = \varepsilon \frac{\partial^2 X(t, \xi)}{\partial \xi^2} + f(X(t, \xi)) + g(X(t, \xi)) + \dot{B}^H(t, \xi), & t \in (0, 1], \quad \xi \in (0, 1), \quad \varepsilon > 0, \\ X(0, \xi) = \sin(\pi\xi), \quad \xi \in (0, 1), \\ X(t, 0) = X(t, 1) = 0, \quad t \in (0, 1]. \end{cases}$$

In all the numerical experiments, the spectral Galerkin method is used for the spatial discretization. The spatial dimension is set equal to $N = 2^k$, where $k \in \{8, 9, 10\}$. The expectations are estimated by the Monte Carlo by averaging over 200 independent realizations. The final time is $T = 1$.

Figures 1, 2 and 3 display in logarithmic scales how the error depends on the time-step size $\Delta t = T/L$, where $L = 2^i$, with $i \in \{3, 4, 5, 6, 7, 8, 9\}$, for different values of the Hurst parameter $H \in \{0.3, 0.5, 0.7, 0.9\}$ and of N . The objective is to illustrate the convergence with rate $H - 1/4$ given by Theorem 4.1, and a

second reference line with slope H is also displayed. The reference solution is computed using the fully discrete scheme with time-step size $\Delta t_{\text{ref}} = T/L_{\text{ref}}$ with $L_{\text{ref}} = 2^{13}$.

First, for Figures 1 and 2, we choose $f(X) + g(X) = -X^3 + X$. One has $\varepsilon = 1$ for Figure 1 and $\varepsilon = 0.01$ for Figure 2. We propose two numerical schemes: a semi-phase flow (SPF) method, with $f(X) = -X^3$ and $g(X) = X$, and a full-phase flow (PPF) method, with $f(X) = -X^3 + X$ and $g(X) = 0$. For all the values of the Hurst parameter H , we observe the convergence with order $H - 1/4$, as predicted by Theorem 4.1, for both methods.

Second, for Figure 3, we choose $f(X) = -X^3$ and $g(X) = X + \sin(X) + 1$. As above, one observes the convergence with order $H - 1/4$ as predicted by Theorem 4.1.

Note that when small values of H , when Δt is small one seems to observe a higher order of convergence H , which is due to using a finite dimensional approximation.

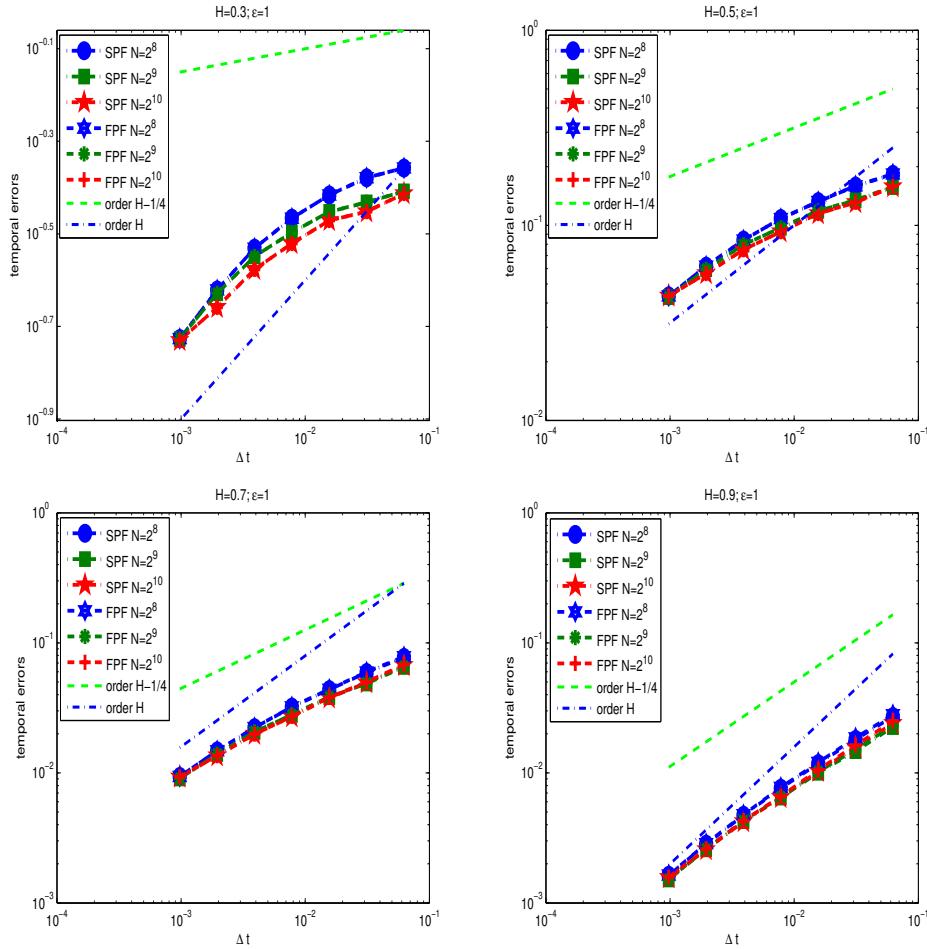


Figure 1: Temporal convergence rate of numerical approximations with $f(t, X) = -X^3 + X$ and $\varepsilon = 1$ for different Hurst parameters $H \in \{0.3, 0.5, 0.7, 0.9\}$.

6 Conclusion

In this work, we have developed and analyzed a new class of splitting schemes based on a partial exact solution of nonlinear term to solve stochastic partial differential equations with non-Lipschitz coefficients

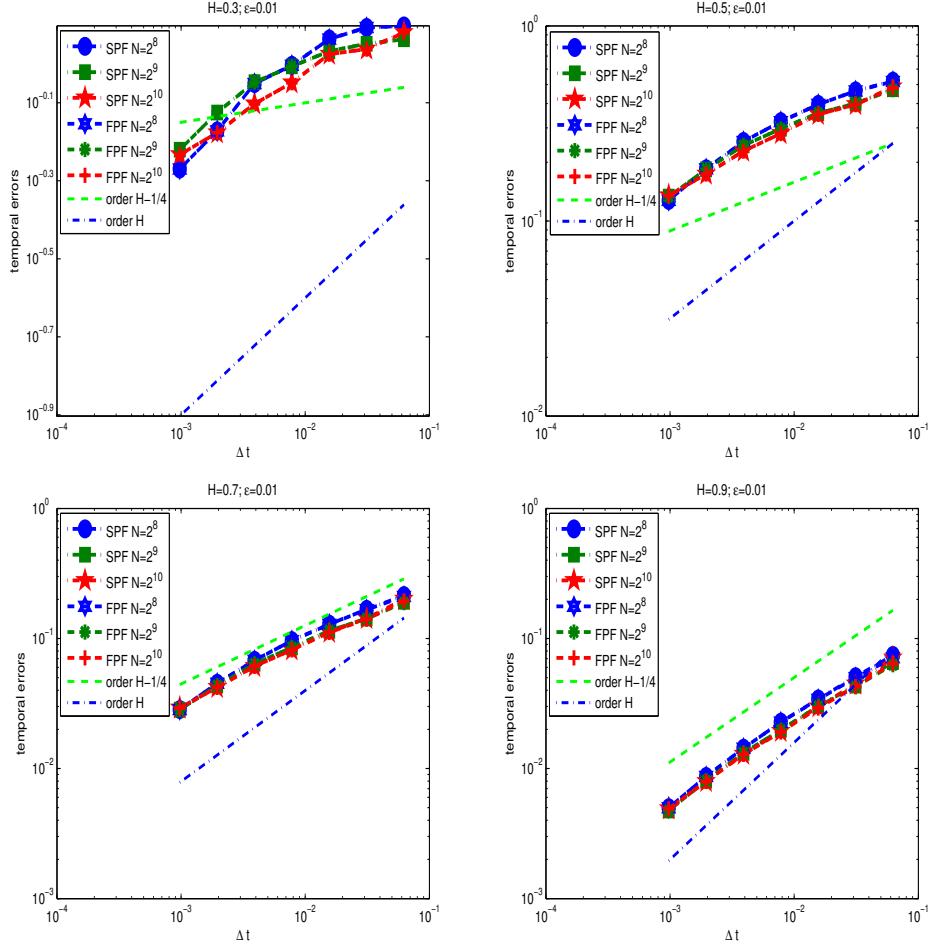


Figure 2: Temporal convergence rate of numerical approximations with $f(t, X) = -X^3 + X$ and $\varepsilon = 0.01$ for different Hurst parameters $H \in \{0.3, 0.5, 0.7, 0.9\}$.

driven by fractional noise when $1/4 < H < 1$. The proposed method is computationally tractable and effective. We have proved that the scheme converges with order $H - 1/4$. This is illustrated by numerical experiments.

However, there are remaining open questions for future work. For example, one may study the weak order of convergence of the proposed numerical approximation. One may also investigate the construction of higher order methods, especially when $1/4 < H < 1/2$. Finally, in this manuscript we have only studied parabolic semilinear stochastic partial differential equations driven by a cylindrical fractional Brownian motion on the one-dimensional domain $(0, 1)$. In future works, one may investigate the case of equations on multidimensional domains $(0, 1)^d$, driven by other classes of noises.

A Appendix: notation

Let $(\beta^H(t))_{t \geq 0}$ be a real-valued fractional Brownian motion with Hurst index $H \in (0, 1)$, given by

$$\beta^H(t) = \int_0^t \mathcal{K}_H(t, s) d\beta(s), \quad \forall t \geq 0,$$

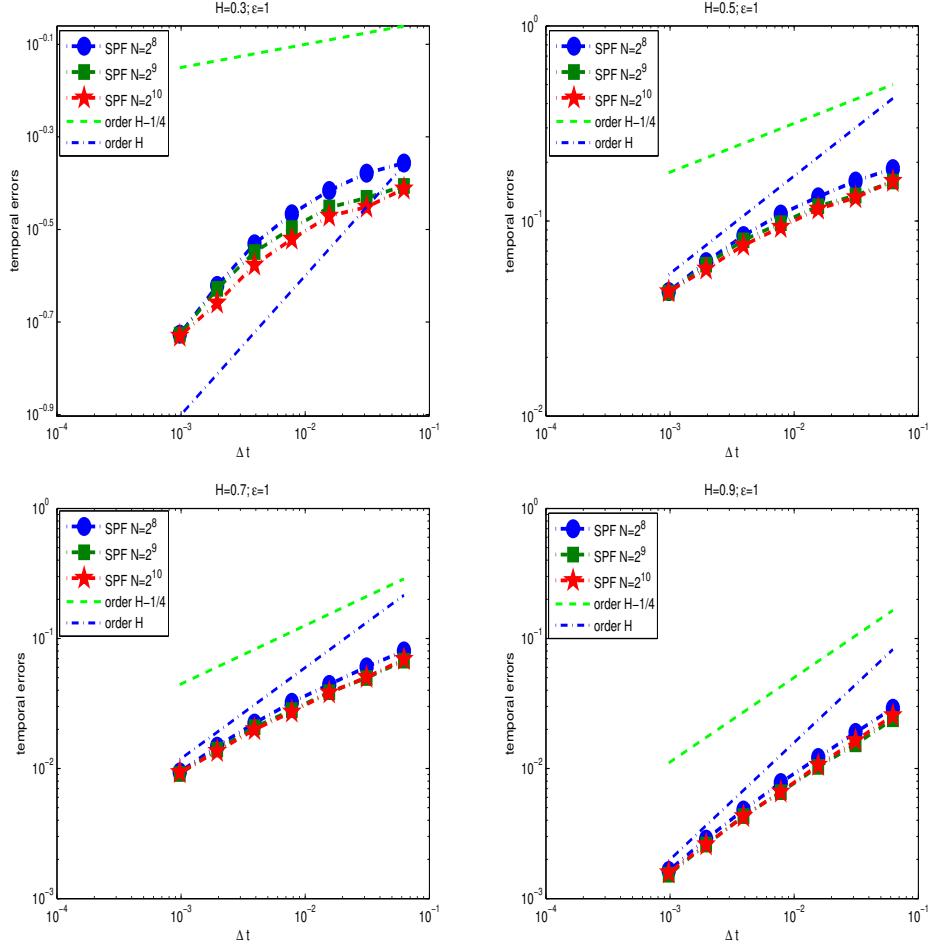


Figure 3: Temporal convergence rate of numerical approximations with $f(t, X) = -X^3 + X + \sin(X) + 1$ and $\varepsilon = 1$ for different Hurst parameters $H \in \{0.3, 0.5, 0.7, 0.9\}$.

where $(\beta(t))_{t \geq 0}$ is a standard real-valued Brownian motion.

For all $\lambda \in (0, \infty)$, let us consider the real-valued fractional Ornstein–Uhlenbeck process $(\mathcal{Z}^{H,\lambda}(t))_{t \geq 0}$ defined by

$$\mathcal{Z}^{H,\lambda}(t) = \int_0^t e^{-\lambda(t-s)} d\beta^H(s), \quad \forall t \geq 0. \quad (35)$$

For all $\lambda \in (0, \infty)$ and all $t \geq 0$, introduce the auxiliary function $\phi_{\lambda,t}$ defined by

$$\phi_{\lambda,t}(s) = e^{-\lambda(t-s)} \mathbb{1}_{s \in [0,t]}.$$

Then for all $t \geq 0$ one has

$$\mathcal{Z}^{H,\lambda}(t) = \int_0^t \phi_{\lambda,t}(s) d\beta^H(s). \quad (36)$$

Moreover, for all $\lambda \in (0, \infty)$ and all $t_2 \geq t_1 \geq 0$, introduce the auxiliary function ψ_{λ,t_1,t_2} defined by

$$\psi_{\lambda,t_1,t_2}(s) = e^{-\lambda(t_2-s)} \mathbb{1}_{s \in [t_1,t_2]}.$$

Then for all $t \geq 0$, one has

$$\mathcal{Z}^{H,\lambda}(t_2) - \mathcal{Z}^{H,\lambda}(t_1) = (e^{-\lambda(t_2-t_1)} - 1)\mathcal{Z}^{H,\lambda}(t_1) + \int_{t_1}^{t_2} e^{-\lambda(t_2-s)} d\beta^H(s) \quad (37)$$

$$= (e^{-\lambda(t_2-t_1)} - 1)\mathcal{Z}^{H,\lambda}(t_1) + \int_0^{t_2} \psi_{\lambda,t_1,t_2}(s) d\beta^H(s). \quad (38)$$

Since the case $H = 1/2$ is well-known, we assume that $H \neq 1/2$. The cases $H \in (1/2, 1)$ and $H \in (0, 1/2)$ are treated separately, since different techniques are employed in the proofs.

Given the time-step size $\Delta t \in (0, 1)$, recall that $t_n = n\Delta t$ for all $n \in \mathbb{N}_0$. Let the increments of the real-valued fractional Brownian motion $(\beta^H(t))_{t \geq 0}$ on the interval $[t_n, t_{n+1}]$ be denoted by $\Delta\beta_n^H = \beta^H(t_{n+1}) - \beta^H(t_n)$, for all $n \geq 0$. Moreover, introduce the auxiliary mapping $\ell : [0, \infty) \rightarrow \mathbb{N}_0$, defined by $\ell(t) = \lfloor t/\Delta t \rfloor$, where $\lfloor \cdot \rfloor$ denotes the integer part, i.e. one has $\ell_{\ell(s)} = t_n$ for all $s \in [t_n, t_{n+1}]$.

For all $\lambda \in (0, \infty)$, let us consider the discrete-time approximation $(\mathcal{Z}_n^{H,\lambda})_{n \geq 0}$ of the real-valued fractional Ornstein–Uhlenbeck process defined by (35): set $\mathcal{Z}_0^{H,\lambda} = 0$, and for all $n \geq 0$

$$\mathcal{Z}_{n+1}^{H,\lambda} = e^{-\lambda\Delta t}(\mathcal{Z}_n^{H,\lambda} + \Delta\beta_n^H). \quad (39)$$

One obtains the following expression: for all $n \geq 0$ one has

$$\mathcal{Z}_n^{H,\lambda} = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} e^{-\lambda(t_n-t_j)} d\beta^H(s) = \int_0^{t_n} e^{-\lambda(t_n-t_{\ell(s)})} d\beta^H(s). \quad (40)$$

For all $\lambda \in (0, \infty)$, $\Delta t > 0$ and $n \geq 0$, introduce the auxiliary function $\varepsilon_{\lambda,t_n}^{\Delta t}$ defined by

$$\varepsilon_{\lambda,t_n}^{\Delta t}(s) = (e^{-\lambda(t_n-s)} - e^{-\lambda(t_n-t_{\ell(s)})}) \mathbf{1}_{[0,t_n]}(s).$$

Combining (35) and (40), the error $\mathcal{Z}^{H,\lambda}(t_n) - \mathcal{Z}_n^{H,\lambda}$ is expressed as

$$\mathcal{Z}^{H,\lambda}(t_n) - \mathcal{Z}_n^{H,\lambda} = \int_0^{t_n} \varepsilon_{\lambda,t_n}^{\Delta t}(s) d\beta^H(s). \quad (41)$$

B Appendix: properties of the one-dimensional fractional Ornstein–Uhlenbeck process when $H \in (1/2, 1)$

B.1 Moment bounds

Lemma B.1. *For all $H \in (1/2, 1)$, there exists $C_H \in (0, \infty)$ such that for all $\lambda \in (0, \infty)$*

$$\sup_{t \geq 0} \mathbb{E}[|\mathcal{Z}^{H,\lambda}(t)|^2] \leq C_H \lambda^{-2H}. \quad (42)$$

Proof of Lemma B.1. Using the identity (36) and applying the Itô isometry formula (8) and its formulation (9) when $H \in (1/2, 1)$, one has

$$\begin{aligned} \mathbb{E}[|\mathcal{Z}^{H,\lambda}(t)|^2] &= \int_0^t |\mathcal{K}_{H,t}^* \phi_{\lambda,t}(s)|^2 ds \\ &= H(2H-1) \int_0^t \int_0^t \phi_{\lambda,t}(s_1) \phi_{\lambda,t}(s_2) |s_2 - s_1|^{2H-2} ds_1 ds_2 \\ &= H(2H-1) \int_0^t \int_0^t e^{-\lambda(t-s_1)} e^{-\lambda(t-s_2)} |s_2 - s_1|^{2H-2} ds_1 ds_2. \end{aligned}$$

The integral appearing in the right-hand side above can be written as

$$\begin{aligned} \int_0^t \int_0^t e^{-\lambda(t-s_1)} e^{-\lambda(t-s_2)} |s_2 - s_1|^{2H-2} ds_1 ds_2 &= \int_0^t \int_0^t e^{-\lambda s_1} e^{-\lambda s_2} |s_2 - s_1|^{2H-2} ds_1 ds_2 \\ &= \lambda^{-2H} \int_0^{\lambda t} \int_0^{\lambda t} e^{-r_1} e^{-r_2} |r_2 - r_1|^{2H-2} dr_1 dr_2, \end{aligned}$$

using changes of variables $s_1 = \lambda r_1$ and $s_2 = \lambda r_2$.

Applying the Fubini theorem, one obtains

$$\begin{aligned} \int_0^{\lambda t} \int_0^{\lambda t} e^{-r_1} e^{-r_2} |r_2 - r_1|^{2H-2} dr_1 dr_2 &= 2 \int_0^{\lambda t} \int_{r_1}^{\lambda t} e^{-2r_1} e^{-(r_2-r_1)} (r_2 - r_1)^{2H-2} dr_2 dr_1 \\ &= 2 \int_0^{\lambda t} e^{-2r_1} \int_0^{\lambda(t-r_1)} e^{-r} r^{2H-2} dr dr_1 \\ &\leq \int_0^{+\infty} e^{-r} r^{2H-2} dr < \infty. \end{aligned}$$

Due to the condition $H > 1/2$ the integral in the right-hand side above is finite. Therefore there exists $C_H \in (0, \infty)$ such that the inequality (42) holds, and the proof of Lemma B.1 is completed. \square

B.2 Temporal regularity

Lemma B.2. *For all $H \in (1/2, 1)$ and all $\alpha \in [0, 2H]$, there exists $C_{\alpha, H} \in (0, \infty)$ such that for all $\lambda \in (0, \infty)$, and all $t_1, t_2 \geq 0$, one has*

$$\mathbb{E}[|\mathcal{Z}^{H, \lambda}(t_2) - \mathcal{Z}^{H, \lambda}(t_1)|^2] \leq C_H \lambda^{\alpha-2H} |t_2 - t_1|^\alpha. \quad (43)$$

Proof of Lemma B.2. Recall the decomposition (38) of $\mathcal{Z}^{H, \lambda}(t_2) - \mathcal{Z}^{H, \lambda}(t_1)$.

For the first term in (38), using the inequality

$$|1 - e^{-z}| \leq z^\gamma, \quad \forall z \geq 0, \quad \forall \gamma \in [0, 1],$$

and applying the moment bounds (42) from Lemma B.1, one has

$$\mathbb{E}[|(e^{-\lambda(t_2-t_1)} - 1)\mathcal{Z}^{H, \lambda}(t_1)|^2] \leq C_H \lambda^{\alpha-2H} (t_2 - t_1)^\alpha. \quad (44)$$

For the second term in (38), applying the Itô isometry formula (8) and its formulation (9) when $H \in (1/2, 1)$, one has

$$\begin{aligned} \mathbb{E}\left[\left|\int_{t_1}^{t_2} e^{-\lambda(t_2-s)} d\beta^H(s)\right|^2\right] &= \mathbb{E}\left[\left|\int_0^{t_2} \psi_{\lambda, t_1, t_2}(s) d\beta^H(s)\right|^2\right] \\ &= \int_0^{t_2} |\mathcal{K}_{H, t_2}^* \psi_{\lambda, t_1, t_2}(s)|^2 ds \\ &= H(2H-1) \int_0^{t_2} \int_0^{t_2} \psi_{\lambda, t_1, t_2}(s_1) \psi_{\lambda, t_1, t_2}(s_2) |s_2 - s_1|^{2H-2} ds_1 ds_2 \\ &= H(2H-1) \int_{t_1}^{t_2} \int_{t_1}^{t_2} e^{-\lambda(t_2-s_1)} e^{-\lambda(t_2-s_2)} |s_2 - s_1|^{2H-2} ds_1 ds_2 \\ &= H(2H-1) \int_0^{t_2-t_1} \int_0^{t_2-t_1} e^{-\lambda s_1} e^{-\lambda s_2} |s_2 - s_1|^{2H-2} ds_1 ds_2. \end{aligned}$$

Using changes of variables $r_1 = \lambda s_1$ and $r_2 = \lambda s_2$, one obtains

$$\begin{aligned}
& \int_0^{t_2-t_1} \int_0^{t_2-t_1} e^{-\lambda s_1} e^{-\lambda s_2} |s_2 - s_1|^{2H-2} ds_1 ds_2 \\
&= \lambda^{-2H} \int_0^{\lambda(t_2-t_1)} \int_0^{\lambda(t_2-t_1)} e^{-r_1} e^{-r_2} |r_2 - r_1|^{2H-2} dr_1 dr_2 \\
&= 2\lambda^{-2H} \int_0^{\lambda(t_2-t_1)} \int_{r_1}^{\lambda(t_2-t_1)} e^{-2r_1} e^{-(r_2-r_1)} |r_2 - r_1|^{2H-2} dr_2 dr_1 \\
&= 2\lambda^{-2H} \int_0^{\lambda(t_2-t_1)} e^{-2r_1} dr_1 \int_0^{\lambda(t_2-t_1)} e^{-r} r^{2H-2} dr.
\end{aligned}$$

To proceed, one needs to treat separately the cases $\lambda(t_2 - t_1) \leq 1$ and $\lambda(t_2 - t_1) > 1$.

On the one hand, when $\lambda(t_2 - t_1) \leq 1$, one has

$$\begin{aligned}
\int_0^{\lambda(t_2-t_1)} e^{-2r_1} dr_1 \int_0^{\lambda(t_2-t_1)} e^{-r} r^{2H-2} dr &\leq \int_0^{\lambda(t_2-t_1)} dr_1 \int_0^{\lambda(t_2-t_1)} r^{2H-2} dr \\
&\leq (\lambda(t_2 - t_1))^{\frac{2H-1}{2H-1}} = \frac{(\lambda(t_2 - t_1))^{2H}}{2H-1}.
\end{aligned}$$

On the other hand, when $\lambda(t_2 - t_1) > 1$, one has

$$\int_0^{\lambda(t_2-t_1)} e^{-2r_1} dr_1 \int_0^{\lambda(t_2-t_1)} e^{-r} r^{2H-2} dr \leq \int_0^\infty e^{-2r_1} dr_1 \int_0^\infty e^{-r} r^{2H-2} dr < \infty.$$

Combining the two cases, there exist $C_{0,H}, C_{1,H} \in (0, \infty)$ such that one has the two inequalities

$$\begin{aligned}
\int_0^{\lambda(t_2-t_1)} e^{-2r_1} dr_1 \int_0^{\lambda(t_2-t_1)} e^{-r} r^{2H-2} dr &\leq C_{0,H} \\
&\leq C_{1,H} (\lambda(t_2 - t_1))^{2H}.
\end{aligned}$$

By an interpolation interpolation, for all $\alpha \in [0, 2H]$, there exists $C_{\alpha,H} \in (0, \infty)$ such that one has

$$\int_0^{\lambda(t_2-t_1)} e^{-2r_1} dr_1 \int_0^{\lambda(t_2-t_1)} e^{-r} r^{2H-2} dr \leq C_{\alpha,H} (\lambda(t_2 - t_1))^\alpha. \quad (45)$$

Gathering the estimates (44) and (45), and recalling the decomposition (38), one obtains (43) and the proof of Lemma B.2 is completed. \square

C Appendix: properties of the one-dimensional fractional Ornstein–Uhlenbeck process when $H \in (0, 1/2)$

C.1 Moment bounds

Lemma C.1. *For all $H \in (0, \frac{1}{2})$, there exists $C_H \in (0, \infty)$ such that for all $\lambda \in (0, \infty)$*

$$\sup_{t \geq 0} \mathbb{E}[|\mathcal{Z}^{H,\lambda}(t)|^2] \leq C_H \lambda^{-2H}. \quad (46)$$

Proof of Lemma C.1. Using the identity (36), applying the Itô isometry formula (8), and using the expression (10) for $\mathcal{K}_{H,T}^* \phi(t)$ when $H \in (0, \frac{1}{2})$, one has

$$\begin{aligned}\mathbb{E}[|\mathcal{Z}^{H,\lambda}(t)|^2] &= \int_0^t |\mathcal{K}_{H,t}^* \phi_{\lambda,t}(s)|^2 ds \\ &\leq C \int_0^t \frac{|\phi_{\lambda,t}(s)|^2}{(t-s)^{1-2H}} ds + C \int_0^t \left| s^{\frac{1}{2}-H} \int_s^t \frac{s^{H-\frac{1}{2}} \phi_{\lambda,t}(s) - \tau^{H-\frac{1}{2}} \phi_{\lambda,t}(\tau)}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds.\end{aligned}$$

For the first term, for all $t \geq 0$ one has

$$\begin{aligned}\int_0^t \frac{|\phi_{\lambda,t}(s)|^2}{(t-s)^{1-2H}} ds &= \int_0^t \frac{e^{-2\lambda(t-s)}}{(t-s)^{1-2H}} ds = \int_0^t \frac{e^{-2\lambda s}}{s^{1-2H}} ds \\ &= \lambda^{-2H} \int_0^{\lambda t} \frac{e^{-2r}}{r^{1-2H}} dr \leq \lambda^{-2H} \int_0^\infty \frac{e^{-2r}}{r^{1-2H}} dr,\end{aligned}$$

with a change of variable $r = \lambda s$. Owing to the condition $H > 0$ one obtains the following result: there exists $C_H \in (0, \infty)$ such that for all $\lambda \in (0, \infty)$ and $t \geq 0$ one has

$$\int_0^t \frac{|\phi_{\lambda,t}(s)|^2}{(t-s)^{1-2H}} ds \leq C_H \lambda^{-2H}. \quad (47)$$

For the second term, note that one has the decomposition

$$s^{H-\frac{1}{2}} \phi_{\lambda,t}(s) - \tau^{H-\frac{1}{2}} \phi_{\lambda,t}(\tau) = (s^{H-\frac{1}{2}} - \tau^{H-\frac{1}{2}}) \phi_{\lambda,t}(s) + \tau^{H-\frac{1}{2}} (\phi_{\lambda,t}(s) - \phi_{\lambda,t}(\tau)),$$

therefore one obtains the upper bound

$$\int_0^t \left| s^{\frac{1}{2}-H} \int_s^t \frac{s^{H-\frac{1}{2}} \phi_{\lambda,t}(s) - \tau^{H-\frac{1}{2}} \phi_{\lambda,t}(\tau)}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds \leq 2(\mathcal{I}_{H,\lambda,t}^1 + \mathcal{I}_{H,\lambda,t}^2),$$

where $\mathcal{I}_{H,\lambda,t}^1$ and $\mathcal{I}_{H,\lambda,t}^2$ are defined as

$$\begin{aligned}\mathcal{I}_{H,\lambda,t}^1 &= \int_0^t \left| s^{\frac{1}{2}-H} \phi_{\lambda,t}(s) \int_s^t \frac{s^{H-\frac{1}{2}} - \tau^{H-\frac{1}{2}}}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds \\ &= \int_0^t \left| s^{\frac{1}{2}-H} e^{-\lambda(t-s)} \int_s^t \frac{s^{H-\frac{1}{2}} - \tau^{H-\frac{1}{2}}}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds \\ \mathcal{I}_{H,\lambda,t}^2 &= \int_0^t \left| s^{\frac{1}{2}-H} \int_s^t \frac{\tau^{H-\frac{1}{2}} (\phi_{\lambda,t}(s) - \phi_{\lambda,t}(\tau))}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds \\ &= \int_0^t \left| s^{\frac{1}{2}-H} \int_s^t \frac{\tau^{H-\frac{1}{2}} (e^{-\lambda(t-s)} - e^{-\lambda(t-\tau)})}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds.\end{aligned}$$

To deal with the term $\mathcal{I}_{H,\lambda,t}^1$, we prove the following auxiliary inequality: there exists $C_H \in (0, \infty)$ such that one has

$$\int_s^t \frac{s^{H-\frac{1}{2}} - \tau^{H-\frac{1}{2}}}{(\tau-s)^{\frac{3}{2}-H}} d\tau \leq C_H s^{2H-1}, \quad \forall t \geq s \geq 0. \quad (48)$$

The inequality (48) is proved as follows: using a change of variable $\tau = s(1+r)$, one has

$$\int_s^t \frac{s^{H-\frac{1}{2}} - \tau^{H-\frac{1}{2}}}{(\tau-s)^{\frac{3}{2}-H}} d\tau = \int_0^{\frac{t-s}{s}} s^{H-\frac{1}{2}} \frac{1 - (1+r)^{H-\frac{1}{2}}}{s^{\frac{3}{2}-H} r^{\frac{3}{2}-H}} s dr \leq s^{2H-1} \int_0^\infty \frac{1 - (1+r)^{H-\frac{1}{2}}}{r^{\frac{3}{2}-H}} dr.$$

Owing to the condition $H \in (0, \frac{1}{2})$ one has

$$\begin{aligned} \int_0^\infty \frac{1 - (1+r)^{H-\frac{1}{2}}}{r^{\frac{3}{2}-H}} dr &\leq \int_0^1 \frac{1 - (1+r)^{H-\frac{1}{2}}}{r^{\frac{3}{2}-H}} dr + \int_1^\infty \frac{1}{r^{\frac{3}{2}-H}} dr \\ &\leq (\frac{1}{2} - H) \int_0^1 \frac{1}{r^{\frac{1}{2}-H}} dr + \int_1^\infty \frac{1}{r^{\frac{3}{2}-H}} dr < \infty, \end{aligned}$$

and the proof of the inequality (48) is completed.

Applying the inequality (48), for the term $\mathcal{I}_{H,\lambda,t}^1$ one then obtains the upper bound

$$\int_0^t \left| s^{\frac{1}{2}-H} e^{-\lambda(t-s)} \int_s^t \frac{s^{H-\frac{1}{2}} - \tau^{H-\frac{1}{2}}}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds \leq C_H \int_0^t s^{2H-1} e^{-2\lambda(t-s)} ds.$$

Moreover, by a change of variable $\lambda s = r$, one has

$$\int_0^t s^{2H-1} e^{-2\lambda(t-s)} ds = \lambda^{-2H} \int_0^{\lambda t} r^{2H-1} e^{-2(\lambda t-r)} dr.$$

Observe that the integral in the right-hand side above satisfies for all $t \geq 0$

$$\begin{aligned} \int_0^{\lambda t} r^{2H-1} e^{-2(\lambda t-r)} dr &\leq \int_0^1 r^{2H-1} dr + \mathbb{1}_{\lambda t \geq 1} \int_1^{\lambda t} e^{-2(\lambda t-r)} dr \\ &\leq \int_0^1 r^{2H-1} dr + \int_0^\infty e^{-2r'} dr' < \infty. \end{aligned}$$

As a result, there exists $C_H \in (0, \infty)$ such that for all $\lambda \in (0, \infty)$ and $t \geq 0$ one has

$$\mathcal{I}_{H,\lambda,t}^1 = \int_0^t \left| s^{\frac{1}{2}-H} e^{-\lambda(t-s)} \int_s^t \frac{s^{H-\frac{1}{2}} - \tau^{H-\frac{1}{2}}}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds \leq C_H \lambda^{-2H}. \quad (49)$$

It remains to deal with the term $\mathcal{I}_{H,\lambda,t}^2$. First, one has the upper bound

$$\mathcal{I}_{H,\lambda,t}^2 = \int_0^t \left| s^{\frac{1}{2}-H} \int_s^t \frac{\tau^{H-\frac{1}{2}} (e^{-\lambda(t-s)} - e^{-\lambda(t-\tau)})}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds \leq \int_0^t \left| \int_s^t \frac{(e^{-\lambda(t-\tau)} - e^{-\lambda(t-s)})}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds.$$

By changes of variables $u = t - s$ and $v = t - \tau$, one has

$$\begin{aligned} \int_0^t \left| \int_s^t \frac{(e^{-\lambda(t-\tau)} - e^{-\lambda(t-s)})}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds &= \int_0^t \left| \int_0^u \frac{e^{-\lambda v} - e^{-\lambda u}}{(u-v)^{\frac{3}{2}-H}} dv \right|^2 du \\ &= \int_0^t \left| \int_0^u e^{-\lambda(u-w)} \frac{1 - e^{-\lambda w}}{w^{\frac{3}{2}-H}} dw \right|^2 du, \end{aligned}$$

where the last identity is obtained by the change of variable $w = v - u$ and writing

$$e^{-\lambda v} - e^{-\lambda u} = e^{-\lambda(u-w)} - e^{-\lambda u} = e^{-\lambda(u-w)}(1 - e^{-\lambda w}).$$

Applying the Cauchy-Schwarz inequality and the Fubini theorem, one has

$$\begin{aligned} \int_0^t \left| \int_0^u e^{-\lambda(u-w)} \frac{1 - e^{-\lambda w}}{w^{\frac{3}{2}-H}} dw \right|^2 du &\leq \int_0^t \left(\int_0^u e^{-\lambda(u-w)} dw \right) \left(\int_0^u e^{-\lambda(u-w)} \frac{(1 - e^{-\lambda w})^2}{w^{3-2H}} dw \right) du \\ &\leq \lambda^{-1} \int_0^t \int_w^t e^{-\lambda(u-w)} du \frac{(1 - e^{-\lambda w})^2}{w^{3-2H}} dw \\ &\leq \lambda^{-2} \int_0^t \frac{(1 - e^{-\lambda w})^2}{w^{3-2H}} dw. \end{aligned}$$

Finally, using a change of variables $r = \lambda w$, one has

$$\int_0^t \frac{(1 - e^{-\lambda w})^2}{w^{3-2H}} dw = \lambda^{2-2H} \int_0^{\lambda t} \frac{(1 - e^{-r})^2}{r^{3-2H}} dr \leq \lambda^{2-2H} \int_0^\infty \frac{(1 - e^{-r})^2}{r^{3-2H}} dr,$$

where using the condition $H \in (0, 1/2)$ one has

$$\int_0^\infty \frac{(1 - e^{-r})^2}{r^{3-2H}} dr \leq \int_0^1 r^{2H-1} dr + \int_0^\infty \frac{1}{r^{3-2H}} dr < \infty.$$

As a result, there exists $C_H \in (0, \infty)$ such that for all $\lambda \in (0, \infty)$ and $t \geq 0$ one has

$$\mathcal{I}_{H,\lambda,t}^2 = \int_0^t \left| s^{\frac{1}{2}-H} \int_s^t \frac{\tau^{H-\frac{1}{2}}(e^{-\lambda(t-s)} - e^{-\lambda(t-\tau)})}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds \leq C_H \lambda^{-2H}. \quad (50)$$

Combining the auxiliary bounds (47), (49) and (50), one obtains (46) and the proof of Lemma C.1 is completed. \square

C.2 Temporal regularity

Lemma C.2. *For all $H \in (0, \frac{1}{2})$ and for all $\alpha \in (0, 2H)$, there exists $C_{\alpha,H} \in (0, \infty)$ such that for all $\lambda \in (0, \infty)$, and all $t_1, t_2 \geq 0$, one has*

$$\mathbb{E}[|\mathcal{Z}^{H,\lambda}(t_2) - \mathcal{Z}^{H,\lambda}(t_1)|^2] \leq C_{\alpha,H} \lambda^{\alpha-2H} |t_2 - t_1|^\alpha. \quad (51)$$

Remark C.3. Note that the inequality (51) from Lemma C.2 is proved only for $\alpha < 2H$ (when $H \in (0, 1/2)$), whereas the inequality (43) from Lemma B.2 is proved for $\alpha \leq 2H$ (when $H \in (1/2, 1)$).

Proof of Lemma C.2. Recall the decomposition (38) of $\mathcal{Z}^{H,\lambda}(t_2) - \mathcal{Z}^{H,\lambda}(t_1)$.

For the first term in (38), using the inequality

$$|1 - e^{-z}| \leq z^\gamma, \quad \forall z \geq 0, \quad \forall \gamma \in [0, 1],$$

and applying the moment bounds (46) from Lemma C.1, one has

$$\mathbb{E}[|(e^{-\lambda(t_2-t_1)} - 1)\mathcal{Z}^{H,\lambda}(t_1)|^2] \leq C_H \lambda^{\alpha-2H} (t_2 - t_1)^\alpha. \quad (52)$$

For the second term in (38), applying the Itô isometry formula (8) and using the expression (10) for $\mathcal{K}_{H,T}^* \phi(t)$ when $H \in (0, \frac{1}{2})$, one has

$$\begin{aligned} \mathbb{E}\left[\left|\int_{t_1}^{t_2} e^{-\lambda(t_2-s)} d\beta^H(s)\right|^2\right] &= \int_0^{t_2} |\mathcal{K}_{H,t_2}^* \psi_{\lambda,t_1,t_2}(s)|^2 ds \\ &\leq C_H \int_0^{t_2} \frac{|\psi_{\lambda,t_1,t_2}(s)|^2}{(t_2-s)^{1-2H}} ds \\ &\quad + C_H \int_0^{t_2} \left| s^{\frac{1}{2}-H} \int_s^{t_2} \frac{s^{H-\frac{1}{2}} \psi_{\lambda,t_1,t_2}(s) - \tau^{H-\frac{1}{2}} \psi_{\lambda,t_1,t_2}(\tau)}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds. \end{aligned}$$

For the first term, for all $t_2 \geq t_1 \geq 0$, one has

$$\begin{aligned} \int_0^{t_2} \frac{|\psi_{\lambda,t_1,t_2}(s)|^2}{(t-s)^{1-2H}} ds &= \int_{t_1}^{t_2} \frac{e^{-2\lambda(t_2-s)}}{(t_2-s)^{1-2H}} ds = \int_0^{t_2-t_1} \frac{e^{-2\lambda s}}{s^{1-2H}} ds \\ &= \lambda^{-2H} \int_0^{\lambda(t_2-t_1)} \frac{e^{-2r}}{r^{1-2H}} dr, \end{aligned}$$

using a change of $r = \lambda s$. There exists $C_H \in (0, \infty)$ such that the following holds: for all $\alpha \in [0, 1]$ and all $\theta \geq 0$, one has

$$\int_0^\theta \frac{e^{-2r}}{r^{1-2H}} dr \leq C_H \theta^{2H} \mathbb{1}_{\theta \leq 1} + \mathbb{1}_{\theta \geq 1} \int_0^\infty \frac{e^{-2r}}{r^{1-2H}} dr \leq C_H \min(\theta, 1) \leq C_H \theta^\alpha.$$

Applying the inequality above with $\theta = \lambda(t_2 - t_1)$, one obtains the following inequality: there exists $C_H \in (0, \infty)$ such that for all $\alpha \in [0, 2H]$, $\lambda \in (0, \infty)$ and $t_2 \geq t_1 \geq 0$, one has

$$\int_0^{t_2} \frac{|\psi_{\lambda,t_1,t_2}(s)|^2}{(t-s)^{1-2H}} ds \leq C_H \lambda^{\alpha-2H} (t_2 - t_1)^\alpha. \quad (53)$$

For the second term, note that one has the decomposition

$$s^{H-\frac{1}{2}} \psi_{\lambda,t_1,t_2}(s) - \tau^{H-\frac{1}{2}} \psi_{\lambda,t_1,t_2}(\tau) = (s^{H-\frac{1}{2}} - \tau^{H-\frac{1}{2}}) \psi_{\lambda,t_1,t_2}(s) + \tau^{H-\frac{1}{2}} (\psi_{\lambda,t_1,t_2}(s) - \psi_{\lambda,t_1,t_2}(\tau)),$$

therefore one obtains the upper bound

$$\int_0^{t_2} \left| s^{\frac{1}{2}-H} \int_s^{t_2} \frac{s^{H-\frac{1}{2}} \psi_{\lambda,t_1,t_2}(s) - \tau^{H-\frac{1}{2}} \psi_{\lambda,t_1,t_2}(\tau)}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds \leq 2(\mathcal{J}_{H,\lambda,t_1,t_2}^1 + \mathcal{J}_{H,\lambda,t_1,t_2}^2),$$

where $\mathcal{J}_{H,\lambda,t_1,t_2}^1$ and $\mathcal{J}_{H,\lambda,t_1,t_2}^2$ are defined by

$$\begin{aligned} \mathcal{J}_{H,\lambda,t_1,t_2}^1 &= \int_0^{t_2} \left| s^{\frac{1}{2}-H} \psi_{\lambda,t_1,t_2}(s) \int_s^{t_2} \frac{s^{H-\frac{1}{2}} - \tau^{H-\frac{1}{2}}}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds \\ \mathcal{J}_{H,\lambda,t_1,t_2}^2 &= \int_0^{t_2} \left| s^{\frac{1}{2}-H} \int_s^{t_2} \frac{\tau^{H-\frac{1}{2}} (\psi_{\lambda,t_1,t_2}(s) - \psi_{\lambda,t_1,t_2}(\tau))}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds. \end{aligned}$$

To deal with the term $\mathcal{J}_{H,\lambda,t_1,t_2}^1$, applying the inequality (48) (see the proof of Lemma C.1), one obtains

$$\begin{aligned} \int_0^{t_2} \left| s^{\frac{1}{2}-H} \psi_{\lambda,t_1,t_2}(s) \int_s^{t_2} \frac{s^{H-\frac{1}{2}} - \tau^{H-\frac{1}{2}}}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds &\leq C_H \int_0^{t_2} s^{2H-1} \psi_{\lambda,t_1,t_2}(s)^2 ds \\ &\leq C_H \int_{t_1}^{t_2} s^{2H-1} e^{-2\lambda(t_2-s)} ds \\ &\leq C_H \lambda^{-2H} \int_{\lambda t_1}^{\lambda t_2} r^{2H-1} e^{-2(\lambda t_2-r)} dr, \end{aligned}$$

using a change of variables $r = \lambda s$.

We prove another auxiliary inequality: for all $\alpha \in [0, 2H]$, there exists $C_{\alpha,H} \in (0, \infty)$ such that for all one has

$$\int_{\lambda t_1}^{\lambda t_2} r^{2H-1} e^{-2(\lambda t_2-r)} dr \leq C_{\alpha,H} \lambda^\alpha (t_2 - t_1)^\alpha, \quad \forall t_2 \geq t_1 \geq 0. \quad (54)$$

The inequality (54) is proved as follows. First, for all $t_2 \geq t_1 \geq 0$, one has

$$\begin{aligned} \int_{\lambda t_1}^{\lambda t_2} r^{2H-1} e^{-2(\lambda t_2-r)} dr &\leq \int_0^{\lambda t_2} r^{2H-1} e^{-2(\lambda t_2-r)} dr \\ &\leq \int_0^1 r^{2H-1} e^{-2(\lambda t_2-r)} dr + \mathbb{1}_{\lambda t_2 \geq 1} \int_1^{\lambda t_2} r^{2H-1} e^{-2(\lambda t_2-r)} dr \\ &\leq \int_0^1 r^{2H-1} dr + \int_0^{\lambda t_2} e^{-2(\lambda t_2-r)} dr \\ &\leq \frac{1}{2H} + \frac{1}{2}. \end{aligned}$$

Second, assuming that $\lambda(t_2 - t_1) \leq 1$, one has

$$\int_{\lambda t_1}^{\lambda t_2} r^{2H-1} e^{-2(\lambda t_2 - r)} dr \leq \int_{\lambda t_1}^{\lambda t_2} r^{2H-1} dr = \frac{(\lambda t_2)^{2H} - (\lambda t_1)^{2H}}{2H} \leq C_H (\lambda(t_2 - t_1))^{2H}.$$

Combining the two upper bounds above, one obtains the following inequality: there exists $C_H \in (0, \infty)$ such that, for all $\lambda \in (0, \infty)$, $\alpha \in [0, 2H]$ and $t_2 \geq t_1 \geq 0$, one has

$$\int_{\lambda t_1}^{\lambda t_2} r^{2H-1} e^{-2(\lambda t_2 - r)} dr \leq C_H \min(1, \lambda(t_2 - t_1))^{2H} \leq C_H \min(1, \lambda(t_2 - t_1))^\alpha \leq C_H \lambda^\alpha (t_2 - t_1)^\alpha,$$

and the proof of the inequality (54) is completed.

Applying the inequality (54), for the term $\mathcal{J}_{H,\lambda,t_1,t_2}^1$, one then obtains the following upper bound: there exists $C_H \in (0, \infty)$ such that for all $\alpha \in [0, 2H]$, $\lambda \in (0, \infty)$ and $t_2 \geq t_1 \geq 0$ one has

$$\mathcal{J}_{H,\lambda,t_1,t_2}^1 = \int_0^{t_2} \left| s^{\frac{1}{2}-H} \psi_{\lambda,t_1,t_2}(s) \int_s^{t_2} \frac{s^{H-\frac{1}{2}} - \tau^{H-\frac{1}{2}}}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds \leq C_{\alpha,H} \lambda^{\alpha-2H} (t_2 - t_1)^\alpha. \quad (55)$$

It remains to deal with the error term $\mathcal{J}_{H,\lambda,t_1,t_2}^2$. First, one has the upper bound

$$\int_0^{t_2} \left| s^{\frac{1}{2}-H} \int_s^{t_2} \frac{\tau^{H-\frac{1}{2}} (\psi_{\lambda,t_1,t_2}(s) - \psi_{\lambda,t_1,t_2}(\tau))}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds \leq \int_0^{t_2} \left| \int_s^{t_2} \frac{|\psi_{\lambda,t_1,t_2}(s) - \psi_{\lambda,t_1,t_2}(\tau)|}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds.$$

For all $\tau \in [s, t_2]$, one has

$$\begin{aligned} \psi_{\lambda,t_1,t_2}(s) - \psi_{\lambda,t_1,t_2}(\tau) &= e^{-\lambda(t_2-s)} \mathbb{1}_{s \geq t_1} - e^{-\lambda(t_2-\tau)} \mathbb{1}_{\tau \geq t_1} \\ &= e^{-\lambda(t_2-s)} (\mathbb{1}_{s \geq t_1} - \mathbb{1}_{\tau \geq t_1}) + (e^{-\lambda(t_2-s)} - e^{-\lambda(t_2-\tau)}) \mathbb{1}_{\tau \geq t_1}, \end{aligned}$$

and note that the following identity is satisfied:

$$\mathbb{1}_{\tau \geq s} (\mathbb{1}_{s \geq t_1} - \mathbb{1}_{\tau \geq t_1}) = -\mathbb{1}_{\tau \geq t_1 > s}.$$

As a result, the error term $\mathcal{J}_{H,\lambda,t_1,t_2}^2$ can be treated as follows: one has

$$\begin{aligned} \mathcal{J}_{H,\lambda,t_1,t_2}^2 &= \int_0^{t_2} \left| \int_s^{t_2} \frac{|\psi_{\lambda,t_1,t_2}(s) - \psi_{\lambda,t_1,t_2}(\tau)|}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds \\ &\leq 2 \int_0^{t_2} \left| \int_s^{t_2} \frac{e^{-\lambda(t_2-s)} \mathbb{1}_{\tau \geq t_1 > s}}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds + 2 \int_0^{t_2} \left| \int_s^{t_2} \frac{|e^{-\lambda(t_2-s)} - e^{-\lambda(t_2-\tau)}| \mathbb{1}_{\tau \geq t_1}}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds \\ &\leq 2(\mathcal{J}_{H,\lambda,t_1,t_2}^{2,1} + \mathcal{J}_{H,\lambda,t_1,t_2}^{2,2} + \mathcal{J}_{H,\lambda,t_1,t_2}^{2,3}), \end{aligned}$$

where $\mathcal{J}_{H,\lambda,t_1,t_2}^{2,1}$, $\mathcal{J}_{H,\lambda,t_1,t_2}^{2,2}$ and $\mathcal{J}_{H,\lambda,t_1,t_2}^{2,3}$ are defined by

$$\begin{aligned} \mathcal{J}_{H,\lambda,t_1,t_2}^{2,1} &= 2 \int_0^{t_1} \left| \int_{t_1}^{t_2} \frac{e^{-\lambda(t_2-s)}}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds \\ \mathcal{J}_{H,\lambda,t_1,t_2}^{2,2} &= 2 \int_0^{t_1} \left| \int_{t_1}^{t_2} \frac{|e^{-\lambda(t_2-s)} - e^{-\lambda(t_2-\tau)}|}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds \\ \mathcal{J}_{H,\lambda,t_1,t_2}^{2,3} &= 2 \int_{t_1}^{t_2} \left| \int_s^{t_2} \frac{|e^{-\lambda(t_2-s)} - e^{-\lambda(t_2-\tau)}|}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds. \end{aligned}$$

For the term $\mathcal{J}_{H,\lambda,t_1,t_2}^{2,1}$, one has

$$\begin{aligned} \int_0^{t_1} \left| \int_{t_1}^{t_2} \frac{e^{-\lambda(t_2-s)}}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds &= \int_0^{t_1} e^{-2\lambda(t_2-s)} \left| \int_{t_1}^{t_2} \frac{1}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds \\ &= \int_0^{t_1} e^{-2\lambda(t_2-s)} \left| \frac{(t_2-s)^{H-\frac{1}{2}} - (t_1-s)^{H-\frac{1}{2}}}{H-\frac{1}{2}} \right|^2 ds \\ &\leq C_H e^{-2\lambda(t_2-t_1)} \int_0^{t_1} e^{-2\lambda(t_1-s)} [(t_2-t_1+t_1-s)^{H-\frac{1}{2}} - (t_1-s)^{H-\frac{1}{2}}]^2 ds \\ &\leq C_H \int_0^{t_1} e^{-2\lambda s} [(t_2-t_1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}]^2 ds. \end{aligned}$$

Using a change of variable $r = \lambda s$, one has

$$\int_0^{t_1} e^{-2\lambda s} [(t_2-t_1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}]^2 ds = \lambda^{-2H} \int_0^{\lambda t_1} e^{-2r} [(\lambda(t_2-t_1)+r)^{H-\frac{1}{2}} - r^{H-\frac{1}{2}}]^2 dr.$$

Combining the inequalities

$$|(\theta+r)^{H-\frac{1}{2}} - r^{H-\frac{1}{2}}| \leq 2r^{H-\frac{1}{2}}, \quad \forall \theta, r > 0,$$

and

$$|(\theta+r)^{H-\frac{1}{2}} - r^{H-\frac{1}{2}}| \leq C_H \theta r^{H-\frac{3}{2}}, \quad \forall \theta, r > 0,$$

for all $\alpha \in [0, 1]$ one obtains

$$|(\theta+r)^{H-\frac{1}{2}} - r^{H-\frac{1}{2}}| \leq C_{\alpha,H} \frac{\theta^\alpha}{r^{\alpha+\frac{1}{2}-H}}.$$

As a result, if the condition $\alpha < 2H$ is satisfied, one obtains

$$\begin{aligned} \int_0^{\lambda t_1} e^{-2r} [(\lambda(t_2-t_1)+r)^{H-\frac{1}{2}} - r^{H-\frac{1}{2}}]^2 dr &\leq C_{\alpha,H} (\lambda(t_2-t_1))^\alpha \int_0^{\lambda t_1} \frac{1}{r^{\alpha+1-2H}} dr \\ &\leq C_{\alpha,H} \int_0^\infty \frac{1}{r^{\alpha+1-2H}} dr \lambda^\alpha (t_2-t_1)^\alpha. \end{aligned}$$

Therefore, for all $\alpha \in [0, 2H]$, there exists $C_{\alpha,H} \in (0, \infty)$ such that for all $\lambda \in (0, \infty)$ and $t_2 \geq t_1 \geq 0$, one has

$$\mathcal{J}_{H,\lambda,t_1,t_2}^{2,1} \leq C_{\alpha,H} \lambda^{\alpha-2H} (t_2-t_1)^\alpha. \quad (56)$$

For the term $\mathcal{J}_{H,\lambda,t_1,t_2}^{2,2}$, applying the Cauchy-Schwarz inequality, one has

$$\begin{aligned} \mathcal{J}_{H,\lambda,t_1,t_2}^{2,2} &= \int_0^{t_1} \left| \int_{t_1}^{t_2} \frac{|e^{-\lambda(t_2-s)} - e^{-\lambda(t_2-\tau)}|}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds \\ &= \int_0^{t_1} \left| \int_{t_1}^{t_2} \frac{e^{-\lambda(t_2-\tau)}(1-e^{-\lambda(\tau-s)})}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds \\ &\leq \int_0^{t_1} \left(\int_{t_1}^{t_2} e^{-\lambda(t_2-\tau)} d\tau \right) \left(\int_{t_1}^{t_2} e^{-\lambda(t_2-\tau)} \frac{(1-e^{-\lambda(\tau-s)})^2}{(\tau-s)^{3-2H}} d\tau \right) ds \\ &\leq \frac{1-e^{-\lambda(t_2-t_1)}}{\lambda} \int_0^{t_1} \int_{t_1}^{t_2} e^{-\lambda(t_2-\tau)} \frac{(1-e^{-\lambda(\tau-s)})^2}{(\tau-s)^{3-2H}} d\tau ds. \end{aligned}$$

Note that for all $\alpha \in [0, 1]$, one has

$$1 - e^{-\lambda(t_2-t_1)} \leq (\lambda(t_2-t_1))^\alpha.$$

Using changes of variables $u = \lambda\tau$ and $v = \lambda s$, then $u' = \lambda t_2 - u$ and $v' = \lambda t_2 - v$, one obtains

$$\begin{aligned}
\int_0^{t_1} \int_{t_1}^{t_2} e^{-\lambda(t_2-\tau)} \frac{(1-e^{-\lambda(\tau-s)})^2}{(\tau-s)^{3-2H}} d\tau ds &= \lambda^{1-2H} \int_0^{\lambda t_1} \int_{\lambda t_1}^{\lambda t_2} e^{-(\lambda t_2-u)} \frac{(1-e^{-(u-v)})^2}{(u-v)^{3-2H}} du dv \\
&= \lambda^{1-2H} \int_{\lambda(t_2-t_1)}^{\lambda t_2} \int_0^{\lambda(t_2-t_1)} e^{-u'} \frac{(1-e^{-(v'-u')})^2}{(v'-u')^{3-2H}} du' dv' \\
&\leq \lambda^{1-2H} \int_0^{\lambda(t_2-t_1)} e^{-u'} \int_{\lambda(t_2-t_1)}^{\lambda t_2} \frac{(1-e^{-(v'-u')})^2}{(v'-u')^{3-2H}} dv' du' \\
&\leq \lambda^{1-2H} \int_0^\infty e^{-u} du \int_0^\infty \frac{(1-e^{-w})^2}{w^{3-2H}} dw \\
&\leq C_H \lambda^{1-2H}.
\end{aligned}$$

Therefore, for all $\alpha \in [0, 2H]$, there exists $C_{\alpha,H} \in (0, \infty)$ such that for all $\lambda \in (0, \infty)$ and $t_2 \geq t_1 \geq 0$, one has

$$\mathcal{J}_{H,\lambda,t_1,t_2}^{2,2} \leq C_{\alpha,H} \lambda^{\alpha-2H} (t_2 - t_1)^\alpha. \quad (57)$$

For the term $\mathcal{J}_{H,\lambda,t_1,t_2}^{2,3}$, one has

$$\begin{aligned}
\int_{t_1}^{t_2} \left| \int_s^{t_2} \frac{|e^{-\lambda(t_2-s)} - e^{-\lambda(t_2-\tau)}|}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds &= \int_{t_1}^{t_2} \left| \int_s^{t_2} \frac{e^{-\lambda(t_2-\tau)} (1-e^{-\lambda(\tau-s)})}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds \\
&\leq \int_{t_1}^{t_2} \left(\int_s^{t_2} e^{-\lambda(t_2-\tau)} d\tau \right) \left(\int_s^{t_2} e^{-\lambda(t_2-\tau)} \frac{(1-e^{-\lambda(\tau-s)})^2}{(\tau-s)^{3-2H}} d\tau \right) ds.
\end{aligned}$$

Note that for all $t_2 \geq s \geq 0$ and $\lambda \in (0, \infty)$, one has

$$\int_s^{t_2} e^{-\lambda(t_2-\tau)} d\tau = \lambda^{-1} (1 - e^{-\lambda(t_2-s)}) \leq \lambda^{-1}.$$

Thus, using a change of variables $u = \lambda\tau$ and $v = \lambda s$, one obtains

$$\begin{aligned}
\int_{t_1}^{t_2} \left| \int_s^{t_2} \frac{|e^{-\lambda(t_2-s)} - e^{-\lambda(t_2-\tau)}|}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds &\leq \lambda^{-1} \int_{t_1}^{t_2} \int_s^{t_2} e^{-\lambda(t_2-\tau)} \frac{(1-e^{-\lambda(\tau-s)})^2}{(\tau-s)^{3-2H}} d\tau ds \\
&\leq \lambda^{-2H} \int_{\lambda t_1}^{\lambda t_2} \int_v^{\lambda t_2} e^{-(\lambda t_2-u)} \frac{(1-e^{-(u-v)})^2}{(u-v)^{3-2H}} du dv \\
&\leq \lambda^{-2H} \int_{\lambda t_1}^{\lambda t_2} \int_{\lambda t_1}^u e^{-(\lambda t_2-u)} \frac{(1-e^{-\lambda(u-v)})^2}{(u-v)^{3-2H}} dv du \\
&\leq \lambda^{-2H} \int_{\lambda t_1}^{\lambda t_2} e^{-(\lambda t_2-u)} du \int_0^\infty \frac{(1-e^{-w})^2}{w^{3-2H}} dw.
\end{aligned}$$

Note that for all $\alpha \in [0, 2H]$ one has

$$\int_{\lambda t_1}^{\lambda t_2} e^{-(\lambda t_2-u)} du \leq (\lambda(t_2 - t_1))^\alpha.$$

Therefore, for all $\alpha \in [0, 2H]$, there exists $C_{\alpha,H} \in (0, \infty)$ such that for all $\lambda \in (0, \infty)$ and $t_2 \geq t_1 \geq 0$, one has

$$\mathcal{J}_{H,\lambda,t_1,t_2}^{2,3} \leq C_{\alpha,H} \lambda^{\alpha-2H} (t_2 - t_1)^\alpha. \quad (58)$$

Recalling that

$$\mathcal{J}_{H,\lambda,t_1,t_2}^2 \leq 2(\mathcal{J}_{H,\lambda,t_1,t_2}^{2,1} + \mathcal{J}_{H,\lambda,t_1,t_2}^{2,2} + \mathcal{J}_{H,\lambda,t_1,t_2}^{2,3})$$

and combining the inequalities (56), (57) and (58) yields the following inequality: for all $\alpha \in [0, 2H]$, there exists $C_{\alpha,H} \in (0, \infty)$ such that for all $\lambda \in (0, \infty)$ and $t_2 \geq t_1 \geq 0$, one has

$$\mathcal{J}_{H,\lambda,t_1,t_2}^2 = \int_0^{t_2} \left| \int_s^{t_2} \frac{|\psi_{\lambda,t_1,t_2}(s) - \psi_{\lambda,t_1,t_2}(\tau)|}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds \leq C_{\alpha,H} \lambda^{\alpha-2H} (t_2 - t_1)^\alpha. \quad (59)$$

Finally, combining the inequalities (53), (55) and (59), one obtains the following inequality: for all $\alpha \in [0, 2H]$, there exists $C_{\alpha,H} \in (0, \infty)$ such that for all $\lambda \in (0, \infty)$ and $t_2 \geq t_1 \geq 0$, one has

$$\mathbb{E}[|\mathcal{Z}^{H,\lambda}(t_2) - \mathcal{Z}^{H,\lambda}(t_1)|^2] \leq C_{\alpha,H} \lambda^{\alpha-2H} (t_2 - t_1)^\alpha \quad (60)$$

Gathering the estimates (52) and (60) and recalling the decomposition (38), one obtains (51) and the proof of Lemma C.2 is completed. \square

D Appendix: error estimates for the numerical approximation of the fractional Ornstein–Uhlenbeck process

D.1 Case $H \in (1/2, 1)$

Lemma D.1. *For all $H \in (1/2, 1)$, there exists $C_H \in (0, \infty)$ such that for all $\lambda \in (0, \infty)$ and all $\Delta t \in (0, 1)$ one has*

$$\sup_{n \in \mathbb{N}} \mathbb{E}[|\mathcal{Z}^{H,\lambda}(t_n) - \mathcal{Z}_n^{H,\lambda}|^2] \leq C_H \lambda^{-2H} \min(1, \lambda \Delta t)^2. \quad (61)$$

Proof of Lemma D.1. Recall the expression (41) of the error $\mathcal{Z}^{H,\lambda}(t_n) - \mathcal{Z}_n^{H,\lambda}$. Applying the Itô isometry formula (8) and its formulation (9) when $H \in (1/2, 1)$, for all $n \geq 0$ one has

$$\begin{aligned} \mathbb{E}[|\mathcal{Z}^{H,\lambda}(t_n) - \mathcal{Z}_n^{H,\lambda}|^2] &= \int_0^{t_n} |\mathcal{K}_{H,t_n}^* \varepsilon_{\lambda,t_n}^{\Delta t}(t)|^2 dt \\ &= H(2H-1) \int_0^{t_n} \int_0^{t_n} \varepsilon_{\lambda,t_n}^{\Delta t}(s_1) \varepsilon_{\lambda,t_n}^{\Delta t}(s_2) |s_2 - s_1|^{2H-2} ds_1 ds_2. \end{aligned}$$

Note that for all $s \in [0, t_n]$, one has

$$\varepsilon_{\lambda,t_n}^{\Delta t}(s) = e^{-\lambda(t_n-s)} (1 - e^{-\lambda(s-t_\ell(s))}),$$

and that one has

$$(1 - e^{-\lambda(s-t_\ell(s))}) \leq \min(1, \lambda \Delta t).$$

As a result, one obtains the inequality

$$\mathbb{E}[|X^{H,\lambda}(t_n) - x_n^{H,\lambda}|^2] \leq \min(1, \lambda \Delta t)^2 H(2H-1) \int_0^t \int_0^t e^{-\lambda(t-s_1)} e^{-\lambda(t-s_2)} |s_2 - s_1|^{2H-2} ds_1 ds_2.$$

Applying the upper bounds obtained in the proof of Lemma B.1, the inequality (61) holds, and the proof of Lemma D.1 is completed. \square

Note that combining Lemma B.1 and D.1, one obtains the following moment bounds for the discrete-time approximation $(\mathcal{Z}_n^{H,\lambda})_{n \geq 0}$ of $(\mathcal{Z}^{H,\lambda}(t_n))_{n \geq 0}$.

Lemma D.2. *For all $H \in (1/2, 1)$, there exists $C_H \in (0, \infty)$ such that for all $\lambda \in (0, \infty)$ one has*

$$\sup_{\Delta t \in (0,1)} \sup_{n \geq 0} \mathbb{E}[|\mathcal{Z}_n^{H,\lambda}|^2] \leq C_H \lambda^{-2H}. \quad (62)$$

D.2 Case $H \in (0, 1/2)$

Lemma D.3. For all $H \in (0, \frac{1}{2})$ and for all $\alpha \in [0, 2H]$, there exists $C_{\alpha, H} \in (0, \infty)$ such that for all $\lambda \in (0, \infty)$ and all $\Delta t \in (0, 1)$ one has

$$\sup_{n \in \mathbb{N}} \mathbb{E}[|\mathcal{Z}^{H, \lambda}(t_n) - \mathcal{Z}_n^{H, \lambda}|^2] \leq C_H \lambda^{-2H} (\lambda \Delta t)^{2\alpha}. \quad (63)$$

Proof of Lemma D.3. Recall the expression (41) of the error $\mathcal{Z}^{H, \lambda}(t_n) - \mathcal{Z}_n^{H, \lambda}$. Applying the Itô isometry formula (8) and using the expression (10) for $\mathcal{K}_{H, T}^* \phi(t)$ when $H \in (0, \frac{1}{2})$, one has

$$\begin{aligned} \mathbb{E}[|\mathcal{Z}^{H, \lambda}(t_n) - \mathcal{Z}_n^{H, \lambda}|^2] &= \int_0^{t_n} |\mathcal{K}_{H, t_n}^* \varepsilon_{\lambda, t_n}^{\Delta t}(s)|^2 dt \\ &\leq 2c_H^2 \int_0^{t_n} \frac{|\varepsilon_{\lambda, t_n}^{\Delta t}(s)|^2}{(t_n - s)^{1-2H}} ds + 2c_H^2 \int_0^{t_n} \left| s^{\frac{1}{2}-H} \int_s^{t_n} \frac{s^{H-\frac{1}{2}} \varepsilon_{\lambda, t_n}^{\Delta t}(\tau) - \tau^{H-\frac{1}{2}} \varepsilon_{\lambda, t_n}^{\Delta t}(\tau)}{(\tau - s)^{\frac{3}{2}-H}} d\tau \right|^2 ds. \end{aligned}$$

For the first term, note that one has

$$|\varepsilon_{\lambda, t_n}^{\Delta t}(s)| \leq \min(1, \lambda \Delta t) e^{-\lambda(t_n - s)}, \quad \forall s \in [0, t_n], \quad (64)$$

and as a result one obtains

$$\int_0^{t_n} \frac{|\varepsilon_{\lambda, t_n}^{\Delta t}(s)|^2}{(t_n - s)^{1-2H}} ds \leq \min(1, \lambda \Delta t)^2 \int_0^{t_n} \frac{e^{-2\lambda(t_n - s)}}{(t_n - s)^{1-2H}} ds.$$

Proceeding as in the proof of Lemma C.1, one obtains the following inequality: there exists $C_H \in (0, \infty)$ such that for all $\lambda \in (0, \infty)$, all $\Delta t \in (0, 1)$ and all $n \geq 0$ one has

$$\int_0^{t_n} \frac{|\varepsilon_{\lambda, t_n}^{\Delta t}(s)|^2}{(t_n - s)^{1-2H}} ds \leq C_H \lambda^{-2H} \min(1, \lambda \Delta t)^2. \quad (65)$$

For the second term, note that one has the decomposition

$$s^{H-\frac{1}{2}} \varepsilon_{\lambda, t_n}^{\Delta t}(s) - \tau^{H-\frac{1}{2}} \varepsilon_{\lambda, t_n}^{\Delta t}(\tau) = (s^{H-\frac{1}{2}} - \tau^{H-\frac{1}{2}}) \varepsilon_{\lambda, t_n}^{\Delta t}(s) + \tau^{H-\frac{1}{2}} (\varepsilon_{\lambda, t_n}^{\Delta t}(s) - \varepsilon_{\lambda, t_n}^{\Delta t}(\tau)),$$

therefore one obtains the upper bound

$$\int_0^{t_n} \left| s^{\frac{1}{2}-H} \int_s^{t_n} \frac{s^{H-\frac{1}{2}} \varepsilon_{\lambda, t_n}^{\Delta t}(\tau) - \tau^{H-\frac{1}{2}} \varepsilon_{\lambda, t_n}^{\Delta t}(\tau)}{(\tau - s)^{\frac{3}{2}-H}} d\tau \right|^2 ds \leq 2(\mathcal{K}_{H, \lambda, \Delta t, n}^1 + \mathcal{K}_{H, \lambda, \Delta t, n}^2),$$

where $\mathcal{K}_{H, \lambda, \Delta t, n}^1$ and $\mathcal{K}_{H, \lambda, \Delta t, n}^2$ are defined as

$$\begin{aligned} \mathcal{K}_{H, \lambda, \Delta t, n}^1 &= \int_0^{t_n} \left| s^{\frac{1}{2}-H} \varepsilon_{\lambda, t_n}^{\Delta t}(s) \int_s^{t_n} \frac{s^{H-\frac{1}{2}} - \tau^{H-\frac{1}{2}}}{(\tau - s)^{\frac{3}{2}-H}} d\tau \right|^2 ds \\ \mathcal{K}_{H, \lambda, \Delta t, n}^2 &= \int_0^{t_n} \left| s^{\frac{1}{2}-H} \int_s^{t_n} \frac{\tau^{H-\frac{1}{2}} (\varepsilon_{\lambda, t_n}^{\Delta t}(s) - \varepsilon_{\lambda, t_n}^{\Delta t}(\tau))}{(\tau - s)^{\frac{3}{2}-H}} d\tau \right|^2 ds. \end{aligned}$$

To deal with the term $\mathcal{K}_{H, \lambda, \Delta t, n}^1$, owing to the inequality (64), one has

$$\begin{aligned} \int_0^{t_n} \left| s^{\frac{1}{2}-H} \varepsilon_{\lambda, t_n}^{\Delta t}(s) \int_s^{t_n} \frac{s^{H-\frac{1}{2}} - \tau^{H-\frac{1}{2}}}{(\tau - s)^{\frac{3}{2}-H}} d\tau \right|^2 ds &\leq \min(1, \lambda \Delta t)^2 \int_0^{t_n} \left| s^{\frac{1}{2}-H} e^{-2\lambda(t_n - s)} \int_s^{t_n} \frac{s^{H-\frac{1}{2}} - \tau^{H-\frac{1}{2}}}{(\tau - s)^{\frac{3}{2}-H}} d\tau \right|^2 ds \\ &\leq \min(1, \lambda \Delta t)^2 \mathcal{I}_{H, \lambda, t_n}^1, \end{aligned}$$

where $\mathcal{I}_{H,\lambda,t_n}^1$ is defined in the proof of Lemma C.1. Applying the inequality (49) then yields the following inequality: there exists $C_H \in (0, \infty)$ such that for all $\lambda \in (0, \infty)$, all $\Delta t \in (0, 1)$ and all $n \geq 0$ one has

$$\mathcal{K}_{H,\lambda,\Delta t,n}^1 \leq C_H \lambda^{-2H} \min(1, \lambda \Delta t)^2. \quad (66)$$

It remains to deal with the term $\mathcal{K}_{H,\lambda,\Delta t,n}^2$. Owing to the decomposition

$$\begin{aligned} \varepsilon_{\lambda,t_n}^{\Delta t}(s) - \varepsilon_{\lambda,t_n}^{\Delta t}(\tau) &= e^{-\lambda(t_n-s)}(1 - e^{-\lambda(s-t_{\ell(s)})}) - e^{-\lambda(t_n-\tau)}(1 - e^{-\lambda(\tau-t_{\ell(\tau)})}) \\ &= (e^{-\lambda(t_n-s)} - e^{-\lambda(t_n-\tau)})(1 - e^{-\lambda(\tau-t_{\ell(\tau)})}) \\ &\quad + e^{-\lambda(t_n-s)}(e^{-\lambda(\tau-t_{\ell(\tau)})} - e^{-\lambda(s-t_{\ell(s)})}), \end{aligned}$$

one has

$$\mathcal{K}_{H,\lambda,\Delta t,n}^2 \leq 2(\mathcal{K}_{H,\lambda,\Delta t,n}^{2,1} + \mathcal{K}_{H,\lambda,\Delta t,n}^{2,2}),$$

where $\mathcal{K}_{H,\lambda,\Delta t,n}^{2,1}$ and $\mathcal{K}_{H,\lambda,\Delta t,n}^{2,2}$ are defined as

$$\begin{aligned} \mathcal{K}_{H,\lambda,\Delta t,n}^{2,1} &= \int_0^{t_n} \left| \int_s^{t_n} \frac{(e^{-\lambda(t_n-s)} - e^{-\lambda(t_n-\tau)})(1 - e^{-\lambda(\tau-t_{\ell(\tau)})})}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds \\ \mathcal{K}_{H,\lambda,\Delta t,n}^{2,2} &= \int_0^{t_n} \left| \int_s^{t_n} \frac{e^{-\lambda(t_n-s)}(e^{-\lambda(\tau-t_{\ell(\tau)})} - e^{-\lambda(s-t_{\ell(s)})})}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds. \end{aligned}$$

For the term $\mathcal{K}_{H,\lambda,\Delta t,n}^{2,1}$, using the upper bound

$$|1 - e^{-\lambda(\tau-t_{\ell(\tau)})}| \leq \min(1, \lambda \Delta t),$$

one obtains the upper bound

$$\mathcal{K}_{H,\lambda,\Delta t,n}^{2,1} \leq \min(1, \lambda \Delta t)^2 \int_0^{t_n} \left| \int_s^{t_n} \frac{(e^{-\lambda(t_n-s)} - e^{-\lambda(t_n-\tau)})(1 - e^{-\lambda(\tau-t_{\ell(\tau)})})}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds.$$

Applying the inequality (50) on the term $\mathcal{I}_{H,\lambda,t_n}^2$ from the proof of Lemma C.1 then yields the following inequality: there exists $C_H \in (0, \infty)$ such that for all $\lambda \in (0, \infty)$, all $\Delta t \in (0, 1)$ and all $n \geq 0$ one has

$$\mathcal{K}_{H,\lambda,\Delta t,n}^{2,1} \leq C_H \lambda^{-2H} \min(1, \lambda \Delta t)^2. \quad (67)$$

The treatment of the error term $\mathcal{K}_{H,\lambda,\Delta t,n}^{2,2}$ is more delicate.

One has the inequalities

$$|e^{-\lambda(\tau-t_{\ell(\tau)})} - e^{-\lambda(s-t_{\ell(s)})}| \leq \min(1, \lambda \Delta t) \quad (68)$$

and

$$|e^{-\lambda(\tau-t_{\ell(\tau)})} - e^{-\lambda(s-t_{\ell(s)})}| \leq \min(1, \lambda(\tau-s)) + \mathbb{1}_{\tau \in [t_{\ell(s)+1}, t_{\ell(s)+2}]} \min(1, \lambda \Delta t) \quad (69)$$

are satisfied. To prove the inequality (68), it suffices to observe that $\tau - t_{\ell(\tau)} \leq \Delta t$ and $s - t_{\ell(s)} \leq \Delta t$. To prove the inequality (69), three cases are considered:

- if $\tau > t_{\ell(s)+2}$, one has $\tau - s \geq \Delta t$, therefore

$$|e^{-\lambda(\tau-t_{\ell(\tau)})} - e^{-\lambda(s-t_{\ell(s)})}| \leq \min(1, \lambda \Delta t) \leq \min(1, \lambda(\tau-s));$$

- if $\tau < t_{\ell(s)+1}$, one has $t_{\ell(\tau)} = t_{\ell(s)}$, thus

$$|e^{-\lambda(\tau-t_{\ell(\tau)})} - e^{-\lambda(s-t_{\ell(s)})}| = (1 - e^{-\lambda(\tau-s)}) e^{-\lambda(s-t_{\ell(s)})} \leq \min(1, \lambda(\tau-s));$$

- if $\tau \in [t_{\ell(s)+1}, t_{\ell(s)+2}[$, one has $t_{\ell(\tau)} = t_{\ell(s)+1} = t_{\ell(s)} + \Delta t$

$$\begin{aligned}
|e^{-\lambda(\tau-t_{\ell(\tau)})} - e^{-\lambda(s-t_{\ell(s)})}| &\leq |e^{-\lambda(\tau-t_{\ell(s)}-\Delta t)} - e^{-\lambda(s-t_{\ell(s)})}| \\
&\leq |e^{-\lambda(\tau-t_{\ell(s)}-\Delta t)} - e^{-\lambda(\tau-t_{\ell(s)})}| + |e^{-\lambda(\tau-t_{\ell(s)})} - e^{-\lambda(s-t_{\ell(s)})}| \\
&\leq |1 - e^{-\lambda\Delta t}| + |1 - e^{-\lambda(\tau-s)}| \\
&\leq \min(1, \lambda\Delta t) + \min(1, \lambda(\tau-s)).
\end{aligned}$$

Gathering the inequalities obtained in the three cases yields the inequality (69).

Let $\alpha \in [0, 2H]$. Combining the inequalities (68) and (69), one obtains

$$\begin{aligned}
|e^{-\lambda(\tau-t_{\ell(\tau)})} - e^{-\lambda(s-t_{\ell(s)})}| &\leq |e^{-\lambda(\tau-t_{\ell(\tau)})} - e^{-\lambda(s-t_{\ell(s)})}|^{1-\alpha} |e^{-\lambda(\tau-t_{\ell(\tau)})} - e^{-\lambda(s-t_{\ell(s)})}|^\alpha \\
&\leq \left(\min(1, \lambda(\tau-s)) + \mathbb{1}_{\tau \in [t_{\ell(s)+1}, t_{\ell(s)+2}[} \min(1, \lambda\Delta t) \right)^{1-\alpha} \left(\min(1, \lambda\Delta t) \right)^\alpha \\
&\leq \min(1, \lambda(\tau-s))^{1-\alpha} \min(1, \lambda\Delta t)^\alpha + \mathbb{1}_{\tau \in [t_{\ell(s)+1}, t_{\ell(s)+2}[} \min(1, \lambda\Delta t).
\end{aligned}$$

As a result, one obtains the upper bound

$$\begin{aligned}
\left| \int_s^{t_n} \frac{(e^{-\lambda(\tau-t_{\ell(\tau)})} - e^{-\lambda(s-t_{\ell(s)})})}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right| &\leq (\lambda\Delta t)^\alpha \int_s^{t_n} \frac{\min(1, \lambda(\tau-s))^{1-\alpha}}{(\tau-s)^{\frac{3}{2}-H}} d\tau \\
&\quad + \min(1, \lambda\Delta t) \int_{t_{\ell(s)+1}}^{t_{\ell(s)+2}} \frac{1}{(\tau-s)^{\frac{3}{2}-H}} d\tau.
\end{aligned}$$

As a result, one obtains

$$\mathcal{K}_{H,\lambda,\Delta t,n}^{2,2} \leq 2 \left(\mathcal{K}_{H,\lambda,\Delta t,n}^{2,2,1} + \mathcal{K}_{H,\lambda,\Delta t,n}^{2,2,2} \right),$$

where $\mathcal{K}_{H,\lambda,\Delta t,n}^{2,2}$ and $\mathcal{K}_{H,\lambda,\Delta t,n}^{2,2,2}$ are defined by

$$\begin{aligned}
\mathcal{K}_{H,\lambda,\Delta t,n}^{2,2,1} &= (\lambda\Delta t)^{2\alpha} \int_0^{t_n} e^{-2\lambda(t_n-s)} \left| \int_s^{t_n} \frac{\min(1, \lambda(\tau-s))^{1-\alpha}}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds \\
\mathcal{K}_{H,\lambda,\Delta t,n}^{2,2,2} &= \min(1, \lambda\Delta t)^2 \int_0^{t_n} e^{-2\lambda(t_n-s)} \left| \int_{t_{\ell(s)+1}}^{t_{\ell(s)+2}} \frac{1}{(\tau-s)^{\frac{3}{2}-H}} d\tau \right|^2 ds
\end{aligned}$$

For the term $\mathcal{K}_{H,\lambda,\Delta t,n}^{2,2,1}$, one has

$$\begin{aligned}
\mathcal{K}_{H,\lambda,\Delta t,n}^{2,2,1} &\leq (\lambda\Delta t)^{2\alpha} \int_0^{t_n} e^{-2\lambda(t_n-s)} \left(\int_0^\infty \frac{\min(1, \lambda\tau')^{1-\alpha}}{(\tau')^{\frac{3}{2}-H}} d\tau' \right)^2 ds \\
&\leq \lambda^{-2H} (\lambda\Delta t)^{2\alpha} \int_0^\infty e^{-2r} dr \left(\int_0^\infty \frac{\min(1, r)^{1-\alpha}}{r^{\frac{3}{2}-H}} dr \right)^2,
\end{aligned}$$

after changes of variables. Assuming that $\alpha < 2H$ with $H \in (0, 1/2)$ ensures that $\alpha - H < 1/2$, and thus that one has $\int_0^\infty \frac{\min(1, r)^{1-\alpha}}{r^{\frac{3}{2}-H}} dr < \infty$. Therefore one obtains the following inequality: for all $\alpha \in [0, 2H]$, there exists $C_{\alpha,H} \in (0, \infty)$ such that for all $\lambda \in (0, \infty)$, all $\Delta t \in (0, 1)$ and all $n \geq 0$ one has

$$\mathcal{K}_{H,\lambda,\Delta t,n}^{2,2,1} \leq C_{\alpha,H} \lambda^{-2H} (\lambda\Delta t)^{2\alpha}. \quad (70)$$

To deal with the term $\mathcal{K}_{H,\lambda,\Delta t,n}^{2,2,2}$, note that one has the following identity

$$\int_{t_{\ell(s)+1}}^{t_{\ell(s)+2}} \frac{1}{(\tau-s)^{\frac{3}{2}-H}} d\tau = \frac{1}{\frac{1}{2}-H} \left(\frac{1}{(t_{\ell(s)+1}-s)^{\frac{1}{2}-H}} - \frac{1}{(t_{\ell(s)+2}-s)^{\frac{1}{2}-H}} \right),$$

which provides two inequalities:

$$\int_{t_{\ell(s)+1}}^{t_{\ell(s)+2}} \frac{1}{(\tau-s)^{\frac{3}{2}-H}} d\tau \leq \frac{1}{\frac{1}{2}-H} \frac{1}{(t_{\ell(s)+1}-s)^{\frac{1}{2}-H}}, \quad \int_{t_{\ell(s)+1}}^{t_{\ell(s)+2}} \frac{1}{(\tau-s)^{\frac{3}{2}-H}} d\tau \leq \frac{1}{\frac{1}{2}-H} \frac{\Delta t^{\frac{\alpha}{2}}}{(t_{\ell(s)+1}-s)^{\frac{3}{2}-H}}.$$

For all $\alpha \in [0, 2H)$, combining the two upper bounds above by an interpolation argument, one has

$$\int_{t_{\ell(s)+1}}^{t_{\ell(s)+2}} \frac{1}{(\tau-s)^{\frac{3}{2}-H}} d\tau \leq \frac{1}{\frac{1}{2}-H} \frac{\Delta t^{\frac{\alpha}{2}}}{(t_{\ell(s)+1}-s)^{\frac{1}{2}-H+\frac{\alpha}{2}}}.$$

Therefore, one obtains

$$\begin{aligned} \mathcal{K}_{H,\lambda,\Delta t,n}^{2,2,2} &\leq C_H \Delta t^\alpha \min(1, \lambda \Delta t)^2 \int_0^{t_n} e^{-2\lambda(t_n-s)} \frac{1}{(t_{\ell(s)+1}-s)^{1-2H+\alpha}} ds \\ &\leq C_H \Delta t^\alpha \min(1, \lambda \Delta t)^2 \sum_{k=0}^{n-1} e^{-2\lambda(t_n-t_{k+1})} \int_{t_k}^{t_{k+1}} \frac{1}{(t_{\ell(s)+1}-s)^{1-2H+\alpha}} ds \\ &\leq C_H \Delta t^\alpha \min(1, \lambda \Delta t)^2 \sum_{k=0}^{n-1} e^{-2\lambda(t_n-t_{k+1})} \int_0^{\lambda \Delta t} \frac{\lambda^{\alpha-2H}}{r^{1-2H+\alpha}} dr \\ &\leq C_H \lambda^{-2H} (\lambda \Delta t)^\alpha \frac{\min(1, \lambda \Delta t)^2}{1 - e^{-2\lambda \Delta t}} \int_0^\infty \frac{1}{r^{1-2H+\alpha}} dr. \end{aligned}$$

Assuming that $\alpha < 2H$ ensures that $\int_0^\infty \frac{1}{r^{1-2H+\alpha}} dr < \infty$. In addition, one has

$$\sup_{z \in (0, \infty)} \frac{\min(1, z)}{1 - e^{-2z}} < \infty.$$

Therefore one obtains the following inequality: for all $\alpha \in [0, 2H)$, there exists $C_{\alpha,H} \in (0, \infty)$ such that for all $\lambda \in (0, \infty)$, all $\Delta t \in (0, 1)$ and all $n \geq 0$ one has

$$\mathcal{K}_{H,\lambda,\Delta t,n}^{2,2,2} \leq C_{\alpha,H} \lambda^{-2H} (\lambda \Delta t)^{2\alpha}. \quad (71)$$

Combining the inequalities (70) and (71) then yields the following inequality: for all $\alpha \in [0, 2H)$, there exists $C_{\alpha,H} \in (0, \infty)$ such that for all $\lambda \in (0, \infty)$, all $\Delta t \in (0, 1)$ and all $n \geq 0$ one has

$$\mathcal{K}_{H,\lambda,\Delta t,n}^{2,2} \leq C_{\alpha,H} \lambda^{-2H} (\lambda \Delta t)^{2\alpha}. \quad (72)$$

Gathering the estimates (67) and (72) then yields the following inequality: for all $\alpha \in [0, 2H)$, there exists $C_{\alpha,H} \in (0, \infty)$ such that for all $\lambda \in (0, \infty)$, all $\Delta t \in (0, 1)$ and all $n \geq 0$ one has

$$\mathcal{K}_{H,\lambda,\Delta t,n}^2 \leq C_{\alpha,H} \lambda^{-2H} (\lambda \Delta t)^{2\alpha}. \quad (73)$$

Finally, gathering the inequalities (65), (66) and (73), one obtains the inequality (63) and the proof of Lemma D.3 is completed. \square

Note that combining Lemma C.1 and D.3, one obtains the following moment bounds for the discrete-time approximation $(\mathcal{Z}_n^{H,\lambda})_{n \geq 0}$ of $(\mathcal{Z}^{H,\lambda}(t_n))_{n \geq 0}$.

Lemma D.4. *For all $H \in (0, 1/2)$, there exists $C_H \in (0, \infty)$ such that for all $\lambda \in (0, \infty)$ one has*

$$\sup_{\Delta t \in (0, 1)} \sup_{n \geq 0} \mathbb{E}[|\mathcal{Z}_n^{H,\lambda}|^2] \leq C_H \lambda^{-2H}. \quad (74)$$

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