On a large deviation principle for 1d cubic NLS with optimal decaying data

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Abstract

In this article, we revisit the work of [7], and prove large deviation principles for more general random initial data for cubic NLS. The Fourier coefficient of our random data admits an optimal polynomial decay.

1 Introduction

1.1 Statement of results

Consider cubic NLS on 1d torus $\mathbb{T}$,

$$\begin{cases} iu_t + \Delta u = \epsilon^2 |u|^2 u, \\ u(0,x) = u_0(x) \end{cases} \tag{1.1}$$

here $\epsilon > 0$ is a parameter.

We will focus on the random initial data of type

$$u_0^\omega(x) = \sum_n c_n g_n^\omega e^{inx} \tag{1.2}$$

where $c_n = \langle n \rangle^{-(\frac{1}{2}+\theta)}, \theta > 0$, and g_n^ω be i.i.d standard complex Gaussian.

Note that $u_0^\omega \in L_x^2$ almost surely, and it is well known that all L^2 initial data generates a global flow, [1]. Our main result is

Theorem 1.1. *Consider* (1.1) *with random initial data* (1.2). *Let c_n in* (1.2) *be $\langle n \rangle^{-(\frac{1}{2}+\theta)}, \theta > 0$.* *Let $T_\epsilon \approx \epsilon^{-1}$. One has the following large deviation principle*

$$\lim_{\epsilon \to 0^+} \epsilon \ln \mathbb{P}(\|u^\omega\|_{L_{t,x}^\infty([0,T_\epsilon]\times\mathbb{T})} > z_0 \epsilon^{-\frac{1}{2}}) = -\frac{z_0^2}{\sum_n |c_n|^2}. \tag{1.3}$$

Remark 1.2. *Upon finishing of this work, we notice a recent preprint of Liang-Wang, [8], which handles Theorem 1.1 in the case $\theta > \frac{1}{2}$. It should be noted that when $\theta > \frac{1}{2}$, the Fourier coefficient*

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is in l^1, which forms an algebra. We handle all $\theta > 0$.

Both [7], [8], and this work actually work for time scale for $c\epsilon^{-1}|\ln \epsilon|$, for some c small but universal. Such improvement comes from standard improvement in Gronwall argument. In [8], they state the time scale as $O(\epsilon^{-1}|\ln \epsilon|)$, however, it seems hard to cover time scale for $C\epsilon^{-1}|\ln \epsilon|$ for C large, see the line below (3.30) in [8].

1.2 Background and motivation

This work is inspired by the pioneering work [7]. Motivated by rogue wave phenomena, Garrido, Grande, Kurianski and Staffilani study (1.1), with random initial data of form (1.2), but for $c_n = ae^{-b|n|}$. They essentially [1] obtained (1.3) for such types of random initial data. It has been remarked in [7, Remark 1.3], *finding optimal family of coefficients (in terms of decay)*, rather than considering exponential decay c_n, remains an interesting open question.

By combining random data type techniques in [2, 3] and some ideas in [4] (which also goes back to Bourgain's 1990s seminal work), we are able to extend the result in [7] to coefficients with polynomial decay $\langle n \rangle^{-\frac{1}{2}-\theta}$, $\theta > 0$. Such constraint is optimal in the sense when $\theta \leqslant 0$, even the initial data will leave L_x^∞. It remains an interesting question to go beyond the time scale T_ϵ. We note that this ϵ^{-1} time scale (neglecting logarithm corrections) is in some sense critical and we refer to [7, Remark 1.7] for more details. We note that both current work and [7], can cover time scale of form $c\frac{|\ln \epsilon|}{\epsilon}$, where c is some small universal number.

The overall proof scheme of current article follows [7], which contains a large deviation principle for the (modified) linear flow and a smoothing estimate for the Duhamel part, (there is some subtle part here since one needs to first slightly perturb the free linear flow, we will discuss it later).

Our main contributions in this work are

- We find a rather straightforward proof for the large deviation principle for the linear flow, which simplifies the arguments in [7], and more importantly, extends to random initial data of form $\sum_n \frac{g_n^\omega}{\langle n \rangle^{\frac{1}{2}+\theta}} e^{inx}$, $\forall \theta > 0$. Our techniques for this part are crucial for our further analysis in the nonlinear smoothing part.

- We adapt the $X^{s,b}$ analysis in [2, 3] in the setting of long time analysis of [7], which gives the desired nonlinear smoothing[2].

We end this session by pointing out ever since the seminal work [2, 3], random initial data theory in nonlinear dispersive PDEs has become a very active research field. Many researchers contribute to this field and it is impossible to survey the field here. We simply mention that in the perspective of regularity in NLS model, the recent breakthrough [5] gives a quite complete answer. We refer to [5], [7] and reference therein for more related reference.

1.3 Notation

Many terms $g_n^\omega, u^\omega, u_0^\omega$ are random, and thus depend on ω. For simplicity, we will hide the implicit ω and simply denote them as g_n, u, u_0.

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[1] They state the result for pointwise in $t \leqslant T_\epsilon \sim \epsilon^{-1}$. We take this chance to cover $\|u\|_{L_{t,x}^\infty}$ rather than estimate pointwise in t.

[2] On one hand, we will use the $X^{s,b}$ analysis in the local random data theory as a black box; on the other hand, the long time analysis we use to get nonlinear smoothing does explicitly use computations of $X^{s,b}$ type, but without going to the explicit form of $X^{s,b}$ analysis.

We introduce $\mathcal{N}_1, \mathcal{N}_2$ as

$$\mathcal{N}_1(f_1, f_2, f_3) = \sum_{n_2 \neq n_1, n_3} \hat{f}_1(n_1) \overline{\hat{f}_2}(n_2) \hat{f}_3(n_3) e^{i(n_1 - n_2 + n_3)x}, \tag{1.4}$$

<div style="text-align:right">`eq: n1`</div>

$$\mathcal{N}_2(f_1, f_2, f_3) = \sum_n \hat{f}_1(n) \overline{\hat{f}_2}(n) \hat{f}_3(n) e^{inx} \tag{1.5}$$

<div style="text-align:right">`eq: n_{2}`</div>

and one has

$$|f|^2 f - 2 \frac{1}{2\pi} (\int |f|^2) f = \mathcal{N}_1(f, f, f) - \mathcal{N}_2(f, f, f) \tag{1.6}$$

<div style="text-align:right">`eq: algebra`</div>

We sometimes short $\mathcal{N}_i(f, f, f)$ as $\mathcal{N}_i(f)$.

Acknowledgement

2 Preliminary

2.1 $X^{s,b}$ space

In this section, we recall that $X^{s,b}$ norm is defined by

$$\|u\|_{X^{s,b}} := \left(\int_{\mathbb{R}} \sum_n \langle \tau + n^2 \rangle^{2b} \langle n \rangle^{2s} |\hat{u}(\tau, n)|^2 d\tau \right)^{\frac{1}{2}},$$

where $\hat{u}(\tau, n)$ is the spacetime Fourier transformation of u. One defines $X^{s,b}[0, T]$ local in time by

$$\|u\|_{X^{s,b}[0,T]} := \inf_{\tilde{u} = u, t \in [0,T]} \|\tilde{u}\|_{X^{s,b}}.$$

These kinds of spaces were first introduced by Bourgain [1] and are of essential use in random data theory, [2], [3]. We will only use them as a black box. We note that

- $X^{s,b}[0, T] \subset C_t H^s([0, T] \times \mathbb{T})$ for any $b > \frac{1}{2}$.

- $X^{s,b}$ space is useful, among others, because of the following $X^{s,b}$ smoothing estimate,

$$\left\| \eta(t) \int_0^t e^{i(t-s)\Delta} F(s) ds \right\|_{X^{s,b+1}[0,1]} \lesssim \|F\|_{X^{s,b}[0,1]}. \tag{2.1}$$

2.2 Hyper-contractivity estimate

We recall the following well-known hypercontractivity of multi-Guassian from [10], see also [3], [6]. Those estimates are widely used in random data theory, and is probably well known in the probability community.

We record it here for the convenience of readers.

Lemma 2.1. *Let $\{g_n^w\}$ be i.i.d complex Gaussian, let $\{c_{n_1,\ldots,n_k}\}$ be (fixed and deterministic) complex numbers. Let*

$$F_k(w) = \sum_{n_1,\ldots,n_k} c_{n_1,\ldots,n_k} g_{n_1}^w \cdots g_{n_k}^w.$$

Then there holds the associated large deviation type estimate

$$\mathbb{P}\{|F_k| > \lambda\|F_k\|_{L^2(\Upsilon)}\} \leqslant e^{-C\lambda^{2/k}}, \quad \forall \lambda > 0.$$

3   An overview of the proof

Let u, u_0 be as in Theorem 1.1. Recall

$$u_0 = \sum_n c_n g_n e^{inx} = \sum_n \frac{g_n}{\langle n \rangle^{\frac{1}{2}+\theta}} e^{inx}. \tag{3.1}$$

We will fix $\theta > 0$. We will only consider $\theta \ll 1$ and the arguments easily extend to the case when θ takes larger value.

For every $\epsilon > 0$, we fix $T_\epsilon \sim \epsilon^{-1}$.

The overall strategy to approach Theorem 1.1, as in [7], is to prove a precise LDP for the (modified) linear flow, and prove that the difference of u and the (modified) linear flow is small with probability close to 1.

Let modified linear flow u_{app} be defined as

$$u_{app}(t,x) := e^{-2it\epsilon^2 \frac{1}{2\pi}\|u_0\|_{L_x^2}^2} \sum_n c_n g_n e^{inx} e^{it\epsilon^2 c_n^2 |g_n|^2 - itn^2}. \tag{3.2}$$

(Strictly speaking, u_{app} depends on ϵ, for notation simplicity, we hide this ϵ.)
Note that u_{app} solves

$$\begin{cases} i\partial_t u_{app} + \Delta u_{app} = -\epsilon^2 \mathcal{N}_2(u_{app}) + 2\epsilon^2 \frac{1}{2\pi}\|u_0\|_{L_x^2} u_{app}, \\ u_{app}(0,x) = u_0(x) \end{cases} \tag{3.3}$$

We refer to (1.4), (1.5), (1.6) for the notation $\mathcal{N}_2$.
Theorem 1.1 follows from the following two lemmas.

Lemma 3.1. *Let $z_0 > 0$, $T_\epsilon = O(\epsilon^{-1})$ and u_{app} be defined as (3.2), one has the large deviation principle,*

$$\lim_{\epsilon \to 0^+} \epsilon \ln \mathbb{P}(\|u_{app}\|_{L_{t,x}^\infty([0,T_\epsilon] \times \mathbb{T})} > z_0 \epsilon^{-\frac{1}{2}}) = -\frac{z_0^2}{\sum_n c_n^2} \tag{3.4}$$

Remark 3.2. *One can see from our proof that the result actually holds for $T_\epsilon = e^{o(\epsilon^{-1})}$.*

and

Lemma 3.3. *Let u_0 be as in (3.1), and u_{app} be as in (3.2). Let u be the associated solution to (1.1) with initial data u_0. Let $\theta > 0$ be fixed. Let $T_\epsilon = O(\epsilon^{-1})$. Let $C_0 \gg 1$. Then there exists $s > \frac{1}{2}$, and $\delta_1 > 0$, so that*

$$-\ln \mathbb{P}(\|u - u_{app}\|_{L_t^\infty H^s([0,T_\epsilon] \times \mathbb{T})} \geqslant \epsilon^{-\frac{1}{2}+\delta_1}) \geqslant C_0 \epsilon^{-1} \tag{3.5}$$

Once Lemma 3.1 and Lemma 3.3 are established, Theorem 1.1 follows by the similar argument as in [7].

Proof of Theorem 1.1 assuming Lemma 3.1 and Lemma 3.3. We first explain the overall idea.
Since we are computing large deviation for $\|u\|_{L^\infty}$ at scale $\epsilon^{-1/2}$, any error of form of $\epsilon^{-1/2+\delta}, \delta > 0$ is irrelevant. It does take a probability $e^{-C_0\epsilon^{-1}}$ to reduce u to u_{app}. But here C_0 can be chosen as arbitrary large.
Note that for any c fixed, $-\epsilon \ln(e^{-c\epsilon^{-1}} + e^{-C_0\epsilon^{-1}}) \to c$ for all C_0 large than c.

We present more details for the convenience of readers.

We start with an upper bound for $\mathbb{P}\left(\sup\limits_{t\in[0,T_\epsilon]}\sup\limits_{x\in\mathbb{T}}|u(t,x)| > z_0\epsilon^{-1/2}\right)$. It is bounded by

$$\mathbb{P}\left(\|u_{\mathrm{app}}\|_{L^\infty_{x,t}} > z_0\epsilon^{-1/2} - \epsilon^{-1/2+\delta_1}\right) + \mathbb{P}\left(\|u - u_{\mathrm{app}}\|_{C_t[0,T_\epsilon]H^s} > \epsilon^{-1/2+\delta_1}\right) \qquad (3.6)$$

<div style="text-align:right">`eq:  twoupper`</div>

Via Lemma 3.1, the first term in (3.6) is bounded by

$$e^{-\frac{\epsilon^{-1}z_0^2}{\sum\limits_{k\in\mathbb{Z}}c_k^2}+o(\epsilon^{-1})}. \qquad (3.7)$$

Via Lemma 3.3, the second term is bounded by $e^{-C_0\epsilon^{-1}}$ for C_0 large.
Thus, we have

$$\mathbb{P}\left(\sup\limits_{t\in[0,T_\epsilon]}\sup\limits_{x\in\mathbb{T}}|u(t,x)| > z_0\epsilon^{-1/2}\right) \leqslant 2e^{-\frac{\epsilon^{-1}z_0^2}{\sum\limits_{k\in\mathbb{Z}}c_k^2}+o(\epsilon^{-1})} \qquad (3.8)$$

<div style="text-align:right">`eq:  upperbound`</div>

Lower bound is similar, (the point is that we use LDP for the modified linear flow and only need an upper bound of probability to control the error by Lemma 3.3) and we get

$$\mathbb{P}\left(\sup\limits_{t\in[0,T_\epsilon]}\sup\limits_{x\in\mathbb{T}}|u(t,x)| > z_0\epsilon^{-1/2}\right) \geqslant \frac{1}{2}e^{-\frac{\epsilon^{-1}z_0^2}{\sum\limits_{k\in\mathbb{Z}}c_k^2}+o(\epsilon^{-1})} \qquad (3.9)$$

<div style="text-align:right">`eq:  lowerbound`</div>

Combine (3.9) and (3.8), the desired LDP estimate for u follows.

<div style="text-align:right">□</div>

By [7, Lemma 4.2] or [9, Lemma 4.2], η is a standard complex Gaussian if and only $\eta e^{it|\eta|^2}$ is a standard complex Gaussian. Our proof for Lemma 3.1 actually yields to the same result[3] for the linear free flow.

`pforlinearflow` **Lemma 3.4.** *Let $z_0 > 0$, $T_\epsilon = O(\epsilon^{-1})$ and u_0 be as in (3.1), one has the large deviation principle*

$$\lim_{\epsilon\to0^+}\epsilon\ln\mathbb{P}(\|e^{it\Delta}u_0\|_{L^\infty_{t,x}([0,T_\epsilon]\times\mathbb{T})} > z_0\epsilon^{-\frac{1}{2}}) = -\frac{z_0^2}{\sum_n c_n^2} \qquad (3.10)$$

<div style="text-align:right">`eq:  linearldp`</div>

We will prove Lemma 3.1 and Lemma 3.4 in Section 4.
Lemma 3.3 mainly follows from the following two lemmas.
The first one is random data local theory

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[3]We will not use this result in the proof, and strictly speaking those two results, though same, are not equivalent, but we feel it is good to record this estimate for linear flow here.

Lemma 3.5. *Let u, u_{app}, T_ϵ be as in Lemma 3.3. For all (fixed) $0 < \delta_1 \ll 1$ and $\theta > 0$. There exists some $s > \frac{1}{2}$, so that the following holds.*

Assume that for some $t_0 \leqslant T_\epsilon$, one has

$$\|u(t_0) - u_{app}(t_0)\|_{H_x^s} \leqslant \epsilon^{-\frac{1}{2}}, \tag{3.11}$$

then one has up to (extra) probability $e^{-\epsilon^{-1-\delta_2}}$, for some $\delta_2 > 0$,

$$\|e^{2it\epsilon^2 \frac{1}{2\pi} \|u_0\|_2^2} u(t) - e^{2it_0\epsilon^2 \frac{1}{2\pi} \|u_0\|_2^2} e^{i(t-t_0)\Delta} u(t_0)\|_{X^{s,b}[t_0, t_0+1]} \leqslant \epsilon^{\frac{1}{2}-\delta_1} \tag{3.12}$$

Proof. Lemma 3.5 lies essentially in [2, 3], see in particular [4], Proposition 4.6, and Lemma 4.9. All aforementioned works focus on the regularity issue, i.e. the part $s > \frac{1}{2}$. However, its easy to see the size also matches. Since our initial data is at size $\epsilon^{-1/2}$, the nonlinearity is of cubic form $\epsilon|u|^2 u$, and **crucially**, we only need to control error at size $\epsilon^{1/2-\delta_1}$ rather than critical scale $\epsilon^{1/2}$. This actually freedom, combines the hypercontractivity estimate, Lemma 2.1, allows a loss of probability $e^{-\epsilon^{-1-\delta_2}}$ rather than $e^{-\epsilon^{-1}}$, the latter would be too large for our application. $\square$

The second one is a normal form type transformation, which is also the key in [7], except that here we needs to combine it with the analysis based on Lemma 3.5, via computations of same nature as the $X^{s,b}$ type computations in [3].

Lemma 3.6. *Let $u, u_{app}, u_0, T_\epsilon$ be as in Lemma 3.3. Let $C_1 \gg C_0 \gg 1$ and $C_2 \gg 1$, $s > \frac{1}{2}$. Let $0 < \delta_1 \ll 1$. Let $F_\epsilon = \{\omega| \|u_0\|_2 \leqslant C_1 \epsilon^{-1/2}\}$. Then there exists a set E_ϵ with probability measure $e^{-\epsilon^{-1-\delta_3}}$, for some $\delta_3 > 0$, such that the following holds for all $\omega \in F_\epsilon - E_\epsilon$:*

If there holds bootstrap hypothesis, for all $t \leqslant T \leqslant T_\epsilon$

$$\|u - u_{app}\|_{C_t H^s([0,T]\times\mathbb{T})} \leqslant \epsilon^{-\frac{1}{2}-\delta_1}, \tag{3.13}$$

then there holds bootstrap estimate

$$\|u - u_{app}\|_{C_t H^s([0,T]\times\mathbb{T})} \leqslant \epsilon^{\frac{1}{2}-C_2\delta_1} \tag{3.14}$$

We will present the proof of Lemma 3.3 assuming Lemma 3.5 and Lemma 3.6 in Section 5.

4 LDP for the linear flow and modified linear flow

We prove Lemma 3.4 and Lemma 3.1 in this section. The proof are of same nature. We start with Lemma 3.4, which is slightly easier.

4.1 Proof of Lemma 3.4

We first start with the following classical pointwise Large Deviations Principle for $e^{it\Delta} u_0(x)$.

Recall $u_0 = \sum_n \frac{g_n}{\langle n \rangle^{\frac{1}{2}+\theta}} e^{inx}$.

Lemma 4.1. *Fix t, x, then*

$$\mathbb{P}(|e^{it\Delta} u_0(x)| > z_0 \epsilon^{-\frac{1}{2}}) = e^{-\frac{z_0^2 \epsilon^{-1}}{\sum_{n\in\mathbb{Z}} |c_n|^2}}. \tag{4.1}$$

Proof. Fix t, x, it is well-known that $e^{it\Delta}u_0(x)$ is still a Guassian with mean 0 and variance $\sum_{n\in\mathbb{Z}} |c_n|^2$.

Therefore, $e^{it\Delta}u_0(x)$ follows a Rayleigh distibution and (4.1) holds, see also [7]. $\square$

Now we apply Lemma 4.1 to obtain Lemma 3.4. We fix $z_0, \epsilon > 0$ throughout.

Applying (4.1), we have the desired lower bound

$$\mathbb{P}(\sup_{t\in[0,T]} \sup_{x\in\mathbb{T}} |e^{it\Delta}u_0(x)| > z_0\epsilon^{-\frac{1}{2}}) \geqslant e^{-\frac{z_0^2\epsilon^{-1}}{\sum\limits_{n\in\mathbb{Z}} |c_n|^2}}.$$

Next we turn to the upper bound. Those types of computations originates from [3], and play an important role in [4], in particular proof of Prop 4.1, see also [6].

Let us fix dyadic $N \sim \epsilon^{-\frac{1000}{\theta}}$ large.

We note that up to probability $e^{-N^{\theta/100}}$, we have

$$|g_n| \leqslant \langle n\rangle^{\frac{\theta}{100}}, |n| \geqslant N. \tag{4.2}$$

<div style="text-align: right">eq: tamegaussi</div>

and

$$|g_n| \leqslant N, |n| \leqslant N \tag{4.3}$$

<div style="text-align: right">eq: tameguassi</div>

We claim

<div style="text-align: left">: uniformhigh</div>

Lemma 4.2. *Take dyadic $N \sim \epsilon^{-\frac{1000}{\theta}}$ large, up to probability $e^{-N^{\theta/200}} \ll e^{-\epsilon^{-2}}$, one has for all dyadic $M \geqslant N$*

$$\sup_{t\in[0,T_\epsilon]} \|P_M e^{it\Delta}u_0\|_{L^\infty} \lesssim M^{-\theta/100} \tag{4.4}$$

<div style="text-align: right">eq: uniformhig</div>

and in particular

$$\sup_{t\in[0,T_\epsilon]} \|P_{\geqslant N} e^{it\Delta}u_0\|_{L^\infty} \lesssim N^{-\theta/100}. \tag{4.5}$$

<div style="text-align: right">eq: uniformhig</div>

Furthermore, we claim

<div style="text-align: left">lowfrequency</div>

Lemma 4.3. *Under the same assumptions of Lemma 4.2, up to probability $e^{-N} \ll e^{-\epsilon^{-2}}$, we have*

$$\|P_{\leqslant N}u_0\|_{H^{10}} \lesssim N^{100}. \tag{4.6}$$

<div style="text-align: right">eq: lowhs</div>

and in particular

$$\|\nabla_{t,x}P_{\leqslant N}e^{it\Delta}u_0\|_{L_t^\infty L_x^\infty} \lesssim N^{100} \tag{4.7}$$

<div style="text-align: right">eq: derivative</div>

We first prove Lemma 3.4 assuming Lemma 4.2, Lemma 4.3.

Proof. We short $e^{it\Delta}u_0$ via u_{lin} First observe, since $N \sim \epsilon^{-1000/\theta}$. For any fixed points $(x_1, t_1), ..., (x_L, t_L)$, with $L \lesssim N^{1000}$, one has (since polynomial always dominated by exponential), thanks to Lemma 4.1,

$$-\ln\mathbb{P}\left(\sup_{i=1,...,L} |u_{lin}(t_i, x_i)| > z_0\epsilon^{-1/2}\right) = \frac{z_0^2\epsilon^{-1}}{\sum_n |c_n|^2} + o(\epsilon^{-1}) \tag{4.8}$$

<div style="text-align: right">eq: multildppo</div>

Secondly, by Lemma 4.2, (since $N^{-\theta/100} \sim \epsilon^{10} \ll \epsilon^{-1/2}$), we have

$$-\ln\mathbb{P}\left(\sup_{i=1,...,L} |P_{\leqslant N}u_{lin}(t_i, x_i)| > z_0\epsilon^{-1/2}\right) = \frac{z_0^2\epsilon^{-1}}{\sum_n |c_n|^2} + o(\epsilon^{-1}) \tag{4.9}$$

Now, we pose $(x_1, t_1), ..., (x_L, t_L)$ even on $[0, T_\epsilon] \times \mathbb{T}$, thus for any $t \in [0, T_\epsilon]$, one can find (essentially unique) $|(t_i, x_i) - (t, x)| \lesssim N^{-500}$, thus by (4.7), we have the desired

$$-\ln \mathbb{P}\left(\sup_t \|u_{lin}(t, x)\|_{L_x^\infty} > z_0 \epsilon^{-1/2}\right) = \frac{z_0^2 \epsilon^{-1}}{\sum_n |c_n|^2} + o(\epsilon^{-1}) \tag{4.10}$$

`eq:  finalldp1:`

□

We are left with the proof of Lemma 4.2 and Lemma 4.3.

We first prove Lemma 4.3. It is almost direct since (4.3) implies (4.6), and by Sobolev embedding and the equation of linear Schrodinger, we obtain (4.7).

We now prove Lemma 4.2.

It is enough to prove for all $M \geqslant N$, one has up to probability $e^{-M^{\theta/150}}$, there holds (4.4).

This part is similar to our previous proof of Lemma 3.4 assuming Lemma 4.2, Lemma 4.3.

Note that for any $(x_1, t_1), ..., (x_L, t_L)$, with $L \lesssim M^{1000}$, we have

$$\mathbb{P}(\sup_l |P_M u_{lin}(x_l, t_l)| > M^{-\theta/100}) \lesssim e^{-M^{\theta/100}} \tag{4.11}$$

Furthermore, we have, up to probability e^{-M}

$$\|P_M u_0\|_{H^{10}} \leqslant M^{100} \tag{4.12}$$

and thus

$$\|\nabla_{t,x} e^{it\Delta} P_M u_0\|_{L_{t,x}^\infty} \lesssim M^{100}. \tag{4.13}$$

Pose $(x_1, t_1), ..., (x_L, t_L)$ evenly on $[0, T_\epsilon] \times \mathbb{T}$, and the desired estimates follows as we do for (4.10).

4.2   Proof of Lemma 3.1

Lemma 3.1 follows almost same as the argument in previous section.

It suffices to show (3.4) for $b(t, x) := e^{2it\epsilon^2 \frac{1}{2\pi} \|u_0\|_2^2} u_{app} = \sum_k c_k g_k e^{e^{ikx + it\epsilon^2 c_k^2 |g_k|^2 - itk^2}}$. Fix t, noting that $\{g_k e^{it\epsilon^2 c_k^2 |g_k|^2}\}_{k \in \mathbb{Z}}$ are still i.i.d complex Guassian normal random variables (see [7, Lemma 4.2]), thus Lemma 4.1 remains valid.

The Sobolev norm control in space is same as the free linear flow. The time derivative control is slightly different. But observe, in frequency k, the time derivative for the free flow gives an extra k^2, $(\partial_t(e^{-itk^2})e^{ikx} = -ik^2 \partial_t(e^{-itk^2})e^{ikx})$. But for the modified linear flow, the time derivative at frequency k gives an extra $\epsilon^2 c_k^2 |g_k|^2$, noting that we are doing ldp for g_k at level of $\epsilon^{-1/2}$, so this part is neglectable compared to k^2, and previous arguments works line by line same.

5   Error control

We first prove Lemma 3.3 assuming Lemma 3.5 and Lemma 3.6.

To start, we first observe Lemma 3.5 implies the following (one may need to slightly change δ_2),

Lemma 5.1. *Let u, u_{app}, T_ϵ be as in Lemma 3.3. For all (fixed) $\theta > 0$. There exists some $s > \frac{1}{2}$, so that the following holds.*

Assume that for some $t_0 \leqslant T_\epsilon$, one has

$$\|u(t_0) - u_{app}(t_0)\|_{H_x^s} \leqslant \epsilon^{-\frac{1}{2}}, \tag{5.1}$$

`eq:  close2`

then one has up to (extra) probability $e^{-\epsilon^{-1-\delta_2}}$, for some $\delta_2 > 0$,

$$\|u(t) - u_{app}(t)\|_{L_t^\infty H^s[t_0,t_0+1]} \lesssim \epsilon^{-\frac{1}{2}} \tag{5.2}$$

`eq: hsmain`

Proof of Lemma 5.1. Recall $X^{s,b}$ embeds into $L_t^\infty H^s$, estimate (3.12) implies

$$\|e^{2it\epsilon^2 \frac{1}{2\pi}\|u_0\|_2^2}u(t) - e^{2it_0\epsilon^2 \frac{1}{2\pi}\|u_0\|_2^2}e^{i(t-t_0)\Delta}u(t_0)\|_{L_t^\infty H^s[t_0,t_0+1]} \leqslant \epsilon^{\frac{1}{2}-\delta_1} \tag{5.3}$$

`eq: hswork`

Now we note that

$$\begin{aligned}
e^{2it\epsilon^2 \frac{1}{2\pi}\|u_0\|_{L^2}^2}(u-upp)(t) =& e^{2it\epsilon^2 \frac{1}{2\pi}\|u_0\|_{L^2}^2}u(t) - e^{2it_0\epsilon^2 \frac{1}{2\pi}\|u_0\|_{L^2}^2}u(t_0) \\
&+ e^{2it_0\epsilon^2 \frac{1}{2\pi}\|u_0\|_{L^2}^2}e^{i(t-t_0)\Delta}u(t_0) - e^{i(t-t_0)\Delta}u_{app}(t_0) \\
&+ e^{i(t-t_0)\Delta}u_{app}(t_0) - e^{2it\epsilon^2\|u_0\|_2^2}u_{app(t)}
\end{aligned} \tag{5.4}$$

`eq: workdeco`

Note that phase $e^{2it\epsilon^2 \frac{1}{2\pi}\|u_0\|_{L^2}^2}$ does not change H^s. The first term in RHS of (5.4) is controlled via (5.3). The second term is controlled by (5.1).

The third term can be controlled via the explicit formula of u_{app},

$$\begin{aligned}
\|e^{i(t-t_0)\Delta}u_{app}(t_0) - e^{2it\epsilon^2\|u_0\|_2^2}u_{app(t)}\|_{H^s}^2 &\lesssim \sum_n |c_n g_n|^2 |t-t_0|^2 (\epsilon^2 c_n^2 |g_n|^2)^2 n^{2s} \\
&\lesssim \sum_n \langle n \rangle^{-3-6\theta}\langle n \rangle^{2s}|g_n|^6 \epsilon^4
\end{aligned} \tag{5.5}$$

The desired estimates follows now from Lemma 2.1, (or by computing the probability $|g_n| \ll < n >^{0+} \epsilon^{-(\frac{1}{2}+)}$) □

We now apply Lemma 5.1 and (assuming) Lemma 3.6 to close the proof of Lemma 3.3.

Proof. It follows standard continuity argument. Note that we are doing an upper bound estimate for $\mathbb{P}(\|u - u_{app}\|_{L_t^\infty H^s([0,T_\epsilon]\times\mathbb{T})} \geqslant \epsilon^{-\frac{1}{2}+\delta_1})\mathbb{P}(\|u - u_{app}\|_{L_t^\infty H^s([0,T_\epsilon]\times\mathbb{T})} \geqslant \epsilon^{-\frac{1}{2}+\delta_1})$, and F_ϵ^c and E_ϵ in Lemma 3.6 admits small enough probability for the estimate in Lemma 3.3. Everytime, we apply Lemma 5.1 to extend the solution for extra time interval of Length 1, which will force us to drop $e^{-\epsilon^{-1-\delta_2}}$. We need to drop at most $\sim 1/\epsilon$ times, and such a loss of probability is allowed. We leave the left to interested readers.

□

We now turn to the proof of Lemma 3.6.
Lemma 3.6 can be reduced to the following Lemma via a Gronwall argument.

`m: normalform`

Lemma 5.2. *Under the assumption of Lemma 3.6. Assuming bootstrap hypothesis (3.13) for $[0,T]$. There exists a set E_ϵ with probability measure $e^{-\epsilon^{-1-\delta_3}}$ for some $\delta_3 > 0$, such that for all $\omega \in F_\epsilon - E_\epsilon$: one has for some $C, C_3 > 0$*

$$\|(u-u_{app})(t)\|_{H^s} \leqslant C\epsilon^2\|u_0\|_{L_x^2}^2 \int_0^t \|u(\tau) - u_{app}(\tau)\|_{H^s}d\tau + C\epsilon^{1/2-C_3\delta_3} \tag{5.6}$$

`eq: normalform`

for all $t \leqslant T$.

We first prove Lemma 3.6 assuming Lemma 5.2.

Proof. It is standard gronwall argument.

Let $A(t) := \|(u - u_{app})(t)\|_{H^s}$ and $B(t) := \int_0^t A(\tau)d\tau$. By Lemma 5.2, for all $t \leqslant T$ and $\omega \in F_\epsilon - E_\epsilon$, we have

$$A(t) \leqslant CC_1^2\epsilon \int_0^t A(\tau)d\tau + C\epsilon^{1/2 - C_3\delta_3}. \tag{5.7}$$ `eq: gronwall`

Thus, we have

$$B(t)' \leqslant CC_1^2\epsilon B(t) + C\epsilon^{1/2 - C_3\delta_3},$$

which implies

$$\frac{d}{dt}(e^{-CC_1^2\epsilon t}B(t)) \leqslant C\epsilon^{1/2 - C_3\delta_3}e^{-CC_1^2\epsilon t}.$$

Note that $B(0) = 0$. For all $t \leqslant T$ and $\omega \in F_\epsilon - E_\epsilon$, we have

$$B(t) \leqslant C_1^{-2}\epsilon^{-1/2 - C_3\delta_3}(e^{CC_1^2\epsilon t} - 1). \tag{5.8}$$ `eq: gcontrol`

Inserting (5.8) into (5.7), yields (3.14) if we take $C_2 = 2C_3$ and $\delta_1 = \delta_3$. ☐

Remark 5.3. *We can obtain $T_\epsilon \leqslant c\epsilon^{-1}|\ln \epsilon|$ for some universal constant with $0 < c \ll 1$ as in [7].*

We now turn to the proof of Lemma 5.2, which is the most technical part of the article.

Proof. Let $w(t, x) = e^{2it\epsilon^2 \frac{1}{2\pi}\|u_0\|_{L_x^2}^2 - it\Delta}u(t, x) = \sum_k w_k(t)e^{ikx}$. From (1.1), we have

$$\begin{cases} i\partial_t w_k = -\epsilon^2|w_k|^2 w_k + \epsilon^2 \sum_{R_k(k_1, k_2, k_3)} w_{k_1}\overline{w_{k_2}}w_{k_3}e^{-it\Omega} \\ w_k(0) = \frac{g_k}{\langle k \rangle^{\frac{1}{2} + \theta}} \end{cases}, \tag{5.9}$$ `eq: eqforw`

where $\Omega = k_1^2 - k_2^2 + k_3^2 - k^2$, $R_k(k_1, k_2, k_3) = \{(k_1, k_2, k_3) \in \mathbb{Z}^3 : k = k_1 - k_2 + k_3, \Omega \neq 0\}$, see also [7, (4.5)]. Let $a(t, x) = e^{2it\epsilon^2 \frac{1}{2\pi}\|u_0\|_{L_x^2}^2 - it\Delta}u_{app}(t, x) = \sum_k a_k(t)e^{ikx}$. From (3.3), we have

$$\begin{cases} i\partial_t a_k = -\epsilon^2|a_k|^2 a_k \\ a_k(0) = \frac{g_k}{\langle k \rangle^{\frac{1}{2} + \theta}} \end{cases}, \tag{5.10}$$ `eq: eqfora`

see also [7, (4.6)]. We subtract (5.10) from (5.9), then integrate it from 0 to t and use the fact that $\|(u - u_{app})(t)\|_{H^s} = \|(w - a)(t)\|_{H^s}$, the square of the LHS of (5.6) is bounded by

$$\epsilon^4 \sum_k \left|\int_0^t \left(|a_k|^2 a_k - |w_k|^2 w_k\right)(\tau)d\tau\right|^2 \langle k \rangle^{2s} + \epsilon^4 \sum_k \left|\sum_{R_k(k_1, k_2, k_3)} \int_0^t (w_{k_1}\overline{w_{k_2}}w_{k_3})(\tau)e^{-i\tau\Omega}d\tau\right|^2 \langle k \rangle^{2s}. \tag{5.11}$$ `eq: fisrtredu`

Next, Minkowski's inequality implies that the first term of (5.11) is controlled by

$$\lesssim \epsilon^4 \sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} \left(\int_0^t \left(|a_k|^2 + |w_k|^2\right)|a_k - w_k|d\tau\right)^2 \lesssim \epsilon^4\|u_0\|_{L_x^2}^4 \left(\int_0^t \|u(\tau) - u_{app}(\tau)\|_{H^s}d\tau\right)^2,$$

which accounts for the first term in (5.6). In order to get the other term, following [7], we first integrate by parts

$$\int_0^t w_{k_1}\overline{w_{k_2}}w_{k_3}e^{-i\tau\Omega}d\tau = -\frac{1}{i\Omega}w_{k_1}\overline{w_{k_2}}w_{k_3}e^{-i\tau\Omega}\Big|_{\tau=0}^{\tau=t} + \frac{1}{i\Omega}\int_0^t \partial_\tau \left(w_{k_1}\overline{w_{k_2}}w_{k_3}\right)e^{-i\tau\Omega}d\tau. \tag{5.12}$$ `eq: integrateb`

Therefore, we reduce (5.6) to the following two estimates: for all $\omega \in F_\epsilon - E_\epsilon$, one has

$$\sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} \left| \sum_{R_k(k_1,k_2,k_3)} \frac{1}{\Omega} \int_0^t \partial_\tau (w_{k_1} \overline{w_{k_2}} w_{k_3})(\tau) e^{-i\tau\Omega} d\tau \right|^2 \lesssim \epsilon^{-3-2C_3\delta_1} \qquad (5.13) \quad \boxed{\text{eq: \quad derivative}}$$

and

$$\sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} \left| \sum_{R_k(k_1,k_2,k_3)} \frac{1}{\Omega} \left[(w_{k_1} \overline{w_{k_2}} w_{k_3})(t) e^{-it\Omega} - (w_{k_1} \overline{w_{k_2}} w_{k_3})(0) \right] \right|^2 \lesssim \epsilon^{-3-2C_3\delta_1}. \qquad (5.14) \quad \boxed{\text{eq: \quad boundary}}$$

5.1   Proof of (5.13)

We deal with (5.13) first. Using Hölder's inequality, we can bound LHS of (5.13) by

$$t \int_0^t \sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} \left(\sum_{R_k(k_1,k_2,k_3)} \frac{1}{|\Omega|} |\partial_\tau (w_{k_1} \overline{w_{k_2}} w_{k_3})(\tau)| \right)^2 d\tau.$$

Since $t \lesssim T_\epsilon \lesssim \epsilon^{-1}$, we can reduce (5.13) to: for all $\omega \in F_\epsilon - E_\epsilon$ (for some E_ϵ with admissible small probability), one has

$$\sup_{0 \leqslant \tau \leqslant t} \sum_{R_k(k_1,k_2,k_3)} \frac{1}{|\Omega|} |\partial_\tau (w_{k_1} \overline{w_{k_2}} w_{k_3})(\tau)| \lesssim \langle k \rangle^{-s-1/2-\delta_3} \epsilon^{-1/2-C_3\delta_1} \qquad (5.15) \quad \boxed{\text{eq: \quad dersiredpp}}$$

for some $\delta_3 > 0$. By (5.9) and the symmetry of $k_1 \leftrightarrow k_3$, we can bound LHS of (5.15) pointwise in τ by

$$\epsilon^2 \sum_{R_k(k_1,k_2,k_3)} \frac{1}{|\Omega|} |w_{k_1}(\tau) w_{k_2}(\tau)| \cdot |w_{k_3}(\tau)|^3 + \epsilon^2 \sum_{R_k(k_1,k_2,k_3)} \frac{1}{|\Omega|} |w_{k_1}(\tau) w_{k_3}(\tau)| \cdot |w_{k_2}(\tau)|^3$$

$$+ \epsilon^2 \sum_{R_k(k_1,k_2,k_3)} \frac{|w_{k_1}(\tau) w_{k_2}(\tau)|}{|\Omega|} \left| \sum_{R_{k_3}(k_1',k_2',k_3')} w_{k_1'} \overline{w_{k_2'}} w_{k_3'} e^{-i\tau\Omega'} \right|$$

$$+ \epsilon^2 \sum_{R_k(k_1,k_2,k_3)} \frac{|w_{k_1}(\tau) w_{k_3}(\tau)|}{|\Omega|} \left| \sum_{R_{k_2}(k_1',k_2',k_3')} w_{k_1'} \overline{w_{k_2'}} w_{k_3'} e^{-i\tau\Omega''} \right|$$

$$=: \spadesuit_1 + \spadesuit_2 + \spadesuit_3 + \spadesuit_4,$$

where $\Omega' = (k_1')^2 - (k_2')^2 + (k_3')^2 - k_3^2$ and $\Omega'' = (k_1')^2 - (k_2')^2 + (k_3')^2 - k_2^2$. Therefore, we reduce (5.15) to: for all $\omega \in F_\epsilon - E_\epsilon$, one has

$$\sup_{0 \leqslant \tau \leqslant t} (\spadesuit_1 + \spadesuit_2 + \spadesuit_3 + \spadesuit_4) \lesssim \langle k \rangle^{-s-1/2-\delta_3} \epsilon^{-1/2-C_3\delta_1}. \qquad (5.16) \quad \boxed{\text{eq: \quad intergoal}}$$

From the bootstrap hypothesis (3.13), we can write

$$w(t,x) = \sum_n c_n g_n e^{inx} e^{it\epsilon^2 c_n^2 |g_n|^2} + h(t,x) = \sum_n w_n(t) e^{inx} \qquad (5.17) \quad \boxed{\text{eq: \quad bhdeco}}$$

with $\|h\|_{C_t[0,T]H^s} < \epsilon^{-1/2-\delta_1}$. Therefore, if we denote

$$G_\epsilon^c := \{\omega : |c_k g_k| \lesssim \epsilon^{-1/2-\delta_1} \langle k \rangle^{-\frac{1}{2}-\theta+\delta_4}, k \in \mathbb{Z}\} \qquad (5.18) \quad \boxed{\text{eq: \quad gepsic}}$$

for some $0 < \delta_4 \ll 1$ to be chosen later, then we have $\mathbb{P}(G_\epsilon) \leqslant e^{-\epsilon^{-1-\delta_1}}$. Further, for any $\omega \in F_\epsilon - G_\epsilon$, we have

$$\sup_{0 \leqslant \tau \leqslant t} |w_k(\tau)| \lesssim \epsilon^{-1/2 - \delta_1} \langle k \rangle^{-s}, k \in \mathbb{Z} \tag{5.19}$$

<div align="right"><code>eq: wktaupoint</code></div>

thanks to the bootstrap hypothesis (3.13). Next, we claim that we can reduce (5.16) to:

For every n, there exists a set $E_{\epsilon,n}$ with probability measure $e^{-\langle n \rangle^{\delta_5} \epsilon^{-1-2\delta_1}}$ and $\boxed{0 < \delta_5 < \theta}$ to be chosen later such that for all $\omega \in F_\epsilon - G_\epsilon - E_{\epsilon,n}$, one has

$$\sup_{0 \leqslant \tau \leqslant t} \left| \sum_{R_n(n_1, n_2, n_3)} w_{n_1} \overline{w_{n_2}} w_{n_3} e^{-i\tau(n_1^2 - n_2^2 + n_3^2 - n^2)} \right| \lesssim \langle n \rangle^{-\theta + \delta_5} \epsilon^{-3/2 - 3\delta_1}. \tag{5.20}$$

<div align="right"><code>eq: crucialred</code></div>

Indeed, if (5.20) holds, then for all $\omega \in F_\epsilon - G_\epsilon - \bigcup_n E_{\epsilon,n}$, using (5.19), the LHS of (5.16) is bounded by

$$\lesssim \sum_{R_k(k_1, k_2, k_3)} \frac{\epsilon^{-1/2 - 5\delta_1}}{|\Omega| \langle k_1 \rangle^s} \left(\frac{1}{\langle k_2 \rangle^s \langle k_3 \rangle^{3s}} + \frac{1}{\langle k_2 \rangle^s \langle k_3 \rangle^{\theta - \delta_5}} + \frac{1}{\langle k_3 \rangle^s \langle k_2 \rangle^{3s}} + \frac{1}{\langle k_3 \rangle^s \langle k_2 \rangle^{\theta - \delta_5}} \right)$$
$$\lesssim \sum_{R_k(k_1, k_2, k_3)} \frac{\epsilon^{-1/2 - 5\delta_1}}{|\Omega| \langle k_1 \rangle^s \langle k_2 \rangle^s \langle k_3 \rangle^{\theta - \delta_5}} + \frac{\epsilon^{-1/2 - 5\delta_1}}{|\Omega| \langle k_1 \rangle^s \langle k_3 \rangle^s \langle k_2 \rangle^{\theta - \delta_5}}. \tag{5.21}$$

Now let $E_\epsilon = \bigcup_n E_{\epsilon,n} \bigcap G_\epsilon$. We have

$$\mathbb{P}(E_\epsilon) \leqslant \mathbb{P}(G_\epsilon) + \sum_n \mathbb{P}(E_{\epsilon,n}) \leqslant e^{-\epsilon^{-1-\delta_1}} + \sum_n e^{-\langle n \rangle^{\delta_5} \epsilon^{-1-2\delta_1}} \lesssim e^{-\epsilon^{-1-\delta_1}}.$$

So if (5.20) holds and we take $\boxed{C_3 \geqslant 5}$, then we reduce (5.16) to showing following two inequalities:

$$\sum_{R_k(k_1, k_2, k_3)} \frac{1}{|\Omega| \langle k_1 \rangle^s \langle k_2 \rangle^s \langle k_3 \rangle^{\theta - \delta_5}} \lesssim \langle k \rangle^{-s - 1/2 - \delta_3} \tag{5.22}$$

<div align="right"><code>eq: key1</code></div>

and

$$\sum_{R_k(k_1, k_2, k_3)} \frac{1}{|\Omega| \langle k_1 \rangle^s \langle k_3 \rangle^s \langle k_2 \rangle^{\theta - \delta_5}} \lesssim \langle k \rangle^{-s - 1/2 - \delta_3} \tag{5.23}$$

<div align="right"><code>eq: key2</code></div>

We rewrite LHS of (5.22) as

$$\sum_{N_1, N_2, N_3} \sum_{S_k(k_1, k_2, k_3)} \frac{1}{|\Omega| \langle k_1 \rangle^s \langle k_2 \rangle^s \langle k_3 \rangle^{\theta - \delta_5}}, \tag{5.24}$$

<div align="right"><code>eq: dyadickey1</code></div>

where N_1, N_2, N_3 are dyadic numbers and for fixed k, we denote

$$S_k(k_1, k_2, k_3) := \{(k_1, k_2, k_3) \in \mathbb{Z}^3 : k = k_1 - k_2 + k_3, k_2 \neq k_1, k_3, |k_i| \sim N_i, i = 1, 2, 3\}.$$

(Restrictly speaking, $S_k(k_1, k_2, k_3)$ defined above depends on N_1, N_2, N_3, we hide this dependence for simplicity.) We always assume $N_1 \leqslant N_3$ due to the symmetry of $k_1 \leftrightarrow k_3$.

- If $N_3 \gg N_2$ or $N_3 \sim N_2 \gg N_1$, we have $N_3 \gtrsim \langle k \rangle$ and $|\Omega| \sim N_3 |k_2 - k_1|$, thus,

$$
\begin{aligned}
\sum_{N_1, N_2, N_3} \sum_{S_k(k_1, k_2, k_3)} \frac{1}{|\Omega| \langle k_1 \rangle^s \langle k_2 \rangle^s \langle k_3 \rangle^{\theta - \delta_5}} &\lesssim \sum_{N_1, N_2, N_3} N_3^{-\theta - 1 + \delta_5} (N_2 N_1)^{-s} \sum_{S_k(k_1, k_2, k_3)} |k_2 - k_1|^{-1} \\
&\lesssim \sum_{N_1, N_2, N_3} N_3^{-\theta - 1 + \delta_5} (N_2 N_1)^{-s} (N_2 \wedge N_1) \ln(N_1 \vee N_2) \\
&\lesssim \langle k \rangle^{-\theta - 1 + \delta_5} \\
&\lesssim \langle k \rangle^{-s - 1/2 - \delta_3},
\end{aligned}
$$

where the last inequality we take δ_3, δ_5 such that $\boxed{\delta_3 + \delta_5 < \frac{1}{2} + \theta - s}$.

- If $N_2 \gg N_3$, we have $N_2 \gtrsim \langle k \rangle$ and $|\Omega| \sim N_2^2$ and

$$
\sum_{N_1, N_2, N_3} \sum_{S_k(k_1, k_2, k_3)} \frac{1}{|\Omega| \langle k_1 \rangle^s \langle k_2 \rangle^s \langle k_3 \rangle^{\theta - \delta_5}} \lesssim \sum_{N_1, N_2, N_3} N_2^{-s - 2} N_1^{1 - s} N_3^{1 + \delta_5 - \theta} \lesssim \langle k \rangle^{-2s} \lesssim \langle k \rangle^{-s - 1/2 - \delta_3},
$$

where the last inequality we choose δ_3 such that $\boxed{\delta_3 < s - \frac{1}{2}}$.

- If $N_3 \sim N_2 \sim N_1$, we have $N_3 \gtrsim \langle k \rangle$ and

$$
\begin{aligned}
\sum_{N_1, N_2, N_3} \sum_{S_k(k_1, k_2, k_3)} \frac{1}{|\Omega| \langle k_1 \rangle^s \langle k_2 \rangle^s \langle k_3 \rangle^{\theta - \delta_5}} &\lesssim \sum_{N_1, N_2, N_3} N_3^{-\theta - 2s + \delta_5} \ln^2 N_2 \\
&\lesssim \langle k \rangle^{-\theta - 2s + \delta_5} \ln^4 \langle k \rangle \\
&\lesssim \langle k \rangle^{-s - 1/2 - \delta_3},
\end{aligned}
$$

where the last inequality we take δ_3, δ_5 such that $\boxed{\delta_3 + \delta_5 < s - \frac{1}{2} + \theta}$.

Therefore, (5.22) holds.

For (5.23), LHS of (5.23) is bounded by

$$
\sum_{N_1, N_2, N_3} \sum_{S_k(k_1, k_2, k_3)} \frac{1}{|\Omega| \langle k_1 \rangle^s \langle k_3 \rangle^s}. \tag{5.25}
$$

We also assume $N_1 \leqslant N_3$ due to the symmetry of $k_1 \leftrightarrow k_3$.

- If $N_2 \gg N_3$, we have $N_2 \gtrsim \langle k \rangle$ and $|\Omega| \sim N_2^2$ and

$$
\sum_{N_1, N_2, N_3} \sum_{S_k(k_1, k_2, k_3)} \frac{1}{|\Omega| \langle k_1 \rangle^s \langle k_3 \rangle^s} \lesssim \sum_{N_1, N_2, N_3} N_2^{-2} (N_3 N_1)^{1 - s} \lesssim \langle k \rangle^{-2s} \lesssim \langle k \rangle^{-s - 1/2 - \delta_3},
$$

where the last inequality we choose $\boxed{\delta_3 < s - \frac{1}{2}}$.

- If $N_3 \gg N_2$, we have $N_3 \gtrsim \langle k \rangle$ and $|\Omega| \sim N_3 |k_3 - k|$ and

$$
\begin{aligned}
\sum_{N_1, N_2, N_3} \sum_{S_k(k_1, k_2, k_3)} \frac{1}{|\Omega| \langle k_1 \rangle^s \langle k_3 \rangle^s} &\lesssim \sum_{N_1, N_2, N_3} N_3^{-s - 1} N_1^{-s} \sum_{S_k(k_1, k_2, k_3)} |k_3 - k|^{-1} \\
&\lesssim \sum_{N_1, N_2, N_3} N_3^{-s - 1} N_1^{-s} (N_1 \wedge N_2) \ln N_3 \\
&\lesssim \langle k \rangle^{-2s+} \\
&\lesssim \langle k \rangle^{-s - 1/2 - \delta_3},
\end{aligned}
$$

where the last inequality we choose $\boxed{\delta_3 < s - \frac{1}{2}}$.

- If $N_3 \sim N_2 \gg N_1$, we have $N_3 \gtrsim \langle k \rangle$ and $|\Omega| \sim N_2 |k_2 - k_3|$ and

$$
\sum_{N_1,N_2,N_3} \sum_{S_k(k_1,k_2,k_3)} \frac{1}{|\Omega|\langle k_1 \rangle^s \langle k_3 \rangle^s} \lesssim \sum_{N_1,N_2,N_3} N_3^{-s} N_2^{-1} N_1^{-s} \sum_{S_k(k_1,k_2,k_3)} |k_2 - k_3|^{-1}
$$
$$
\lesssim \sum_{N_1,N_2,N_3} N_3^{-s-1} N_1^{1-s} \ln N_3
$$
$$
\lesssim \langle k \rangle^{-2s+}
$$
$$
\lesssim \langle k \rangle^{-s-1/2-\delta_3},
$$

where the last inequality we choose that $\boxed{\delta_3 < s - \frac{1}{2}}$.

- If $N_3 \sim N_2 \sim N_1$, we have $N_3 \gtrsim \langle k \rangle$ and

$$
\sum_{N_1,N_2,N_3} \sum_{S_k(k_1,k_2,k_3)} \frac{1}{|\Omega|\langle k_1 \rangle^s \langle k_3 \rangle^s} \lesssim \sum_{N_1,N_2,N_3} N_3^{-s} N_1^{-s} \ln^2 N_3
$$
$$
\lesssim \langle k \rangle^{-2s+}
$$
$$
\lesssim \langle k \rangle^{-s-1/2-\delta_3},
$$

where the last inequality we choose that $\boxed{\delta_3 < s - \frac{1}{2}}$.

Therefore, (5.23) holds.

Hence, we reduce (5.16) to (5.20).

Note that we have $w_k(\tau) = c_k g_k e^{i\tau\epsilon^2 c_k^2 |g_k|^2} + h_k =: I_k + II_k$ for all $k \in \mathbb{Z}$. We can divide LHS of (5.20) into 8 terms and denote by $J_i = I$ if we choose $c_{k_i'} g_{k_i'} e^{i\tau\epsilon^2 c_{k_i'}^2 |g_{k_i'}|^2}$ for the decomposition term of $w_{k_i'}$, and denote by $J_i = II$ otherwise, for $i = 1, 2, 3$.

5.1.1   Case 1: $J_1 = J_2 = J_3 = I$

We note that the subtle point here is that we need to get estimate in the form of $\sup_\tau$, but hypercontractivity estimates only works well for (finite) fixed τ, so the trick we prove for linear LDP matters here.

In this case, we reduce (5.20) to proving that there exists a set $E_{\epsilon,n}$ with probability measure $e^{-\langle n \rangle^{\delta_5} \epsilon^{-1-2\delta_1}}$, for all $\omega \in F_\epsilon - G_\epsilon - E_{\epsilon,n}$, one has

$$
\sup_{0 \leqslant \tau \leqslant t} \sum_{N_1,N_2,N_3} \left| \sum_{S_n(n_1,n_2,n_3)} \frac{g_{n_1} e^{i\tau\epsilon^2 c_{n_1}^2 |g_{n_1}|^2} \overline{g_{n_2}} e^{-i\tau\epsilon^2 c_{n_2}^2 |g_{n_2}|^2} g_{n_3} e^{i\tau\epsilon^2 c_{n_3}^2 |g_{n_3}|^2}}{\langle n_1 \rangle^{\frac{1}{2}+\theta} \langle n_2 \rangle^{\frac{1}{2}+\theta} \langle n_3 \rangle^{\frac{1}{2}+\theta}} e^{-i\tau(n_1^2 - n_2^2 + n_3^2 - n^2)} \right|
$$
$$
\lesssim \langle n \rangle^{-\theta+\delta_5} \epsilon^{-3/2-3\delta_1}.
$$
$$(5.26) \quad \boxed{\text{eq: estimateI}}$$

Denote

$$
F_{n,N_1,N_2,N_3,\epsilon}(\tau) := \sum_{S_n(n_1,n_2,n_3)} \frac{g_{n_1} e^{i\tau\epsilon^2 c_{n_1}^2 |g_{n_1}|^2} \overline{g_{n_2}} e^{-i\tau\epsilon^2 c_{n_2}^2 |g_{n_2}|^2} g_{n_3} e^{i\tau\epsilon^2 c_{n_3}^2 |g_{n_3}|^2}}{\langle n_1 \rangle^{\frac{1}{2}+\theta} \langle n_2 \rangle^{\frac{1}{2}+\theta} \langle n_3 \rangle^{\frac{1}{2}+\theta}} e^{-i\tau(n_1^2 - n_2^2 + n_3^2 - n^2)}.
$$

Let $\tilde{N}_1, \tilde{N}_2, \tilde{N}_3$ be the decreasing order of N_1, N_2, N_3. Therefore, we have $\tilde{N}_1 \gtrsim \langle n \rangle$. So for all $\omega \in F_\epsilon - G_\epsilon - E_{\epsilon,n}$, we reduce (5.26) to the following estimate

$$\sup_{0 \leqslant \tau \leqslant t} \sum_{\tilde{N}_1, \tilde{N}_2, \tilde{N}_3} \left| F_{n, \tilde{N}_1, \tilde{N}_2, \tilde{N}_3, \epsilon}(\tau) \right| \lesssim \langle n \rangle^{-\theta + \delta_5} \epsilon^{-3/2 - 3\delta_1}. \tag{5.27}$$

Firstly, fix any τ, noting that $g_n e^{i\tau \epsilon^2 c_n^2 |g_n|^2 - i\tau n^2}$ are still $i.i.d$ standard complex Guassian [7], we have

$$\mathbb{E}\left(\left| F_{n, \tilde{N}_1, \tilde{N}_2, \tilde{N}_3, \epsilon}(\tau) \right|^2\right) = 2 \sum_{S_n(n_1, n_2, n_3)} \frac{1}{\langle n_1 \rangle^{1+2\theta} \langle n_2 \rangle^{1+2\theta} \langle n_3 \rangle^{1+2\theta}} \lesssim \tilde{N}_1^{-1-2\theta} \tilde{N}_2^{-2\theta} \tilde{N}_3^{-2\theta},$$

Then applying the hypercontractivity of (multi) Guassians, up to $e^{-\epsilon^{-1-2\delta_1} \tilde{N}_1^{2\delta_5}}$, we have

$$\left| F_{n, \tilde{N}_1, \tilde{N}_2, \tilde{N}_3, \epsilon}(\tau) \right| \lesssim \tilde{N}_1^{-1/2 - \theta + 3\delta_5} \tilde{N}_2^{-\theta} \tilde{N}_3^{-\theta} \epsilon^{-3/2 - 3\delta_1}.$$

Now we decompose $[0, t] = \cup_{i=1}^{t\tilde{N}_1^{10}} I_i$ such that $I_j \cap I_k = \varnothing$ if $|j - k| > 1$ and $I_j \cap I_{j+1} = t_j$ and the length of each I_i equal to $\tilde{N}_1^{-10}$. Therefore, there exists a set $E_{\epsilon, n, \tilde{N}_1, \tilde{N}_2, \tilde{N}_3}$ with probability measure $t\tilde{N}_1^{10} e^{-\epsilon^{-1-2\delta_1} \tilde{N}_1^{2\delta_5}}$ such that for all $\omega \in (E_{\epsilon, n, \tilde{N}_1, \tilde{N}_2, \tilde{N}_3})^c$, we have

$$\left| F_{n, \tilde{N}_1, \tilde{N}_2, \tilde{N}_3, \epsilon}(t_j) \right| \lesssim \tilde{N}_1^{-1/2 - \theta + 3\delta_5} \tilde{N}_2^{-\theta} \tilde{N}_3^{-\theta} \epsilon^{-3/2 - 3\delta_1}, \quad j = 1, \cdots, t\tilde{N}_1^{10}. \tag{5.28}$$

Since for all $\omega \in (G_\epsilon)^c$, we have

$$\epsilon^2 c_k^2 |g_k|^2 \lesssim \epsilon^{1-2\delta_1} \ll 1, k \in \mathbb{Z}.$$

Therefore, for all $\omega \in F_\epsilon - G_\epsilon$, we have

$$\begin{aligned}
&\sup_{\tau > 0} \left| \partial_\tau F_{n, \tilde{N}_1, \tilde{N}_2, \tilde{N}_3, \epsilon}(\tau) \right| \\
&\leqslant \sum_{S_n(n_1, n_2, n_3)} \frac{|g_{n_1} g_{n_2} g_{n_3}| \, |\epsilon^2 c_{n_1}^2 |g_{n_1}|^2 - \epsilon^2 c_{n_2}^2 |g_{n_2}|^2 + \epsilon^2 c_{n_3}^2 |g_{n_3}|^2 - (n_1^2 - n_2^2 + n_3^2 - n^2)|}{\langle n_1 \rangle^{\frac{1}{2} + \theta} \langle n_2 \rangle^{\frac{1}{2} + \theta} \langle n_3 \rangle^{\frac{1}{2} + \theta}} \\
&\lesssim \sum_{S_n(n_1, n_2, n_3)} |n_1^2 - n_2^2 + n_3^2 - n^2| \|u_0\|_2^3 \\
&\lesssim \tilde{N}_1^4 \epsilon^{-3/2}.
\end{aligned} \tag{5.29}$$

So from (5.28) and (5.29), for all $\omega \in F_\epsilon - G_\epsilon - E_{\epsilon, n, \tilde{N}_1, \tilde{N}_2, \tilde{N}_3}$, we have

$$\sup_j \sup_{\tau \in I_j} \left| F_{n, \tilde{N}_1, \tilde{N}_2, \tilde{N}_3, \epsilon}(\tau) \right| \lesssim \sup_j \left| F_{n, \tilde{N}_1, \tilde{N}_2, \tilde{N}_3, \epsilon}(t_j) \right| + \tilde{N}_1^4 \epsilon^{-3/2} |I_j| \lesssim \tilde{N}_1^{-1/2 - \theta + 3\delta_5} \tilde{N}_2^{-\theta} \tilde{N}_3^{-\theta} \epsilon^{-3/2 - 3\delta_1}. \tag{5.30}$$

If we set $E_{\epsilon, n} := \bigcup_{\tilde{N}_1, \tilde{N}_2, \tilde{N}_3} E_{\epsilon, n, \tilde{N}_1, \tilde{N}_2, \tilde{N}_3}$, then we have

$$\mathbb{P}(E_{\epsilon, n}) \leqslant \sum_{\tilde{N}_1, \tilde{N}_2, \tilde{N}_3} \mathbb{P}(E_{\epsilon, n, \tilde{N}_1, \tilde{N}_2, \tilde{N}_3}) \lesssim \sum_{\tilde{N}_1, \tilde{N}_2, \tilde{N}_3} t\tilde{N}_1^{10} e^{-\epsilon^{-1-2\delta_1} \tilde{N}_1^{2\delta_5}} \lesssim \epsilon^{-1} e^{-\epsilon^{-1-2\delta_1} \langle n \rangle^{3\delta_5/2}} \lesssim e^{-\epsilon^{-1-2\delta_1} \langle n \rangle^{\delta_5}}.$$

Further, (5.30) implies for all $\omega \in F_\epsilon - G_\epsilon - E_{\epsilon, n}$, one has

$$\sup_{0 \leqslant \tau \leqslant t} \sum_{\tilde{N}_1, \tilde{N}_2, \tilde{N}_3} \left| F_{n, \tilde{N}_1, \tilde{N}_2, \tilde{N}_3, \epsilon}(\tau) \right| \lesssim \sum_{\tilde{N}_1, \tilde{N}_2, \tilde{N}_3} \tilde{N}_1^{-1/2 - \theta + 3\delta_5} \tilde{N}_2^{-\theta} \tilde{N}_3^{-\theta} \epsilon^{-3/2 - 3\delta_1} \lesssim \langle n \rangle^{-\theta + \delta_5} \epsilon^{-3/2 - 3\delta_1} \tag{5.31}$$

if $\boxed{\delta_5 < \frac{1}{4}}$, where the last inequality we use the fact that $\tilde{N}_1 \gtrsim \langle n \rangle$. Therefore, (5.31) implies (5.27).

5.1.2   Case 2: $J_1 = J_2 = I, J_3 = II$

In this case, we we reduce (5.20) to proving that for all $\omega \in F_\epsilon - G_\epsilon$, one has

$$\sup_{0 \leqslant \tau \leqslant t} \sum_{R_n(n_1,n_2,n_3)} \frac{|g_{n_1} g_{n_2} h_{n_3}|}{\langle n_1 \rangle^{\frac{1}{2}+\theta} \langle n_2 \rangle^{\frac{1}{2}+\theta}} \lesssim \langle n \rangle^{-\theta+\delta_5} \epsilon^{-3/2-3\delta_1}. \tag{5.32}$$

eq: I-I-II

Using Cauchy-Schwarz inequality and (5.18), for all $\omega \in F_\epsilon - G_\epsilon$, we have

$$\sup_{0 \leqslant \tau \leqslant t} \sum_{R_n(n_1,n_2,n_3)} \frac{|g_{n_1} g_{n_2} h_{n_3}|}{\langle n_1 \rangle^{\frac{1}{2}+\theta} \langle n_2 \rangle^{\frac{1}{2}+\theta}}$$

$$\lesssim \|h\|_{C_t[0,T]H^s} \left(\sum_{n_3} \langle n_3 \rangle^{-2s} \left| \sum_{R_{n_3,n}(n_1,n_2)} \frac{\epsilon^{-1-2\delta_1}}{\langle n_1 \rangle^{1/2+\theta-\delta_4} \langle n_2 \rangle^{1/2+\theta-\delta_4}} \right|^2 \right)^{\frac{1}{2}} \tag{5.33}$$

eq: uniformI-1

$$\lesssim \epsilon^{-3/2-3\delta_1} \left(\sum_{n_3} \langle n_3 \rangle^{-2s} \left| \sum_{R_{n_3,n}(n_1,n_2)} \frac{1}{\langle n_1 \rangle^{1/2+\theta-\delta_4} \langle n_2 \rangle^{1/2+\theta-\delta_4}} \right|^2 \right)^{\frac{1}{2}},$$

where for fixed n, n_3, we denote $R_{n_3,n}(n_1,n_2) := \{(n_1,n_2) \in \mathbb{Z}^2 : n = n_1 - n_2 + n_3, n_2 \neq n_1, n_3\}$. Next, we aim to reduce (5.32) to following

$$\sum_{R_{n_3,n}(n_1,n_2)} \frac{1}{\langle n_1 \rangle^{1/2+\theta-\delta_4} \langle n_2 \rangle^{1/2+\theta-\delta_4}} \lesssim \langle n - n_3 \rangle^{-2\theta+2\delta_5}. \tag{5.34}$$

eq: mainreduct

If (5.34) holds, noting that

$$\sum_{|n_3| \ll |n|} \langle n_3 \rangle^{-2s} \langle n - n_3 \rangle^{-4\theta+4\delta_5} \lesssim \langle n \rangle^{-4\theta+4\delta_5}$$

and

$$\sum_{|n_3| \gg |n|} \langle n_3 \rangle^{-2s} \langle n - n_3 \rangle^{-4\theta+4\delta_5} \sim \sum_{|n_3| \gg |n|} \langle n_3 \rangle^{-2s-4\theta+4\delta_5} \lesssim \langle n \rangle^{-4\theta+4\delta_5}$$

and

$$\sum_{|n_3| \sim |n|} \langle n_3 \rangle^{-2s} \langle n - n_3 \rangle^{-4\theta+4\delta_5} \sim \langle n \rangle^{-2s} \sum_{|n_3| \sim |n|} \langle n_3 - n \rangle^{-4\theta+4\delta_5} \lesssim \langle n \rangle^{1-2s-4\theta+4\delta_5},$$

then we have

$$\sum_{n_3} \langle n_3 \rangle^{-2s} \langle n - n_3 \rangle^{-4\theta+4\delta_5} \lesssim \langle n \rangle^{-4\theta+4\delta_5},$$

which, along with (5.34) and (5.33), yields (5.32). So we reduce (5.32) to (5.34) if we take $\boxed{\delta_5 < \theta}$. Similarly, noting that

$$\sum_{|n_1| \ll |n-n_3|} \langle n_1 \rangle^{\delta_4-\frac{1}{2}-\theta} \langle n - n_3 - n_1 \rangle^{\delta_4-\frac{1}{2}-\theta} \lesssim \langle n - n_3 \rangle^{\delta_4-\frac{1}{2}-\theta} \sum_{|n_1| \ll |n-n_3|} \langle n_1 \rangle^{\delta_4-\frac{1}{2}-\theta} \lesssim \langle n - n_3 \rangle^{2\delta_4-2\theta}$$

and

$$\sum_{|n_1| \gg |n-n_3|} \langle n_1 \rangle^{\delta_4-\frac{1}{2}-\theta} \langle n - n_3 - n_1 \rangle^{\delta_4-\frac{1}{2}-\theta} \sim \sum_{|n_1| \gg |n-n_3|} \langle n_1 \rangle^{2\delta_4-1-2\theta} \lesssim \langle n - n_3 \rangle^{-2\theta+2\delta_4}$$

and

$$\sum_{|n_1|\sim|n-n_3|} \langle n_1 \rangle^{\delta_4-\frac{1}{2}-\theta} \langle n-n_3-n_1 \rangle^{\delta_4-\frac{1}{2}-\theta} \sim \langle n-n_3 \rangle^{\delta_4-\frac{1}{2}-\theta} \sum_{|n_1|\sim|n-n_3|} \langle n-n_3-n_1 \rangle^{\delta_4-\frac{1}{2}-\theta} \lesssim \langle n-n_3 \rangle^{-2\theta+2\delta_4},$$

then we have (5.34) if we take $\boxed{\delta_5 \leqslant \delta_4}$.

5.1.3 Case 3: $J_3 = J_2 = I, J_1 = II$

Similar to Case 2.

5.1.4 Case 4: $J_1 = J_3 = I, J_2 = II$

Similar to Case 2.

5.1.5 Case 5: $J_1 = I, J_2 = J_3 = II$

In this case, we we reduce (5.20) to proving that for all $\omega \in F_\epsilon - G_\epsilon$, one has

$$\sup_{0\leqslant\tau\leqslant t} \sum_{R_n(n_1,n_2,n_3)} \frac{|g_{n_1} h_{n_2} h_{n_3}|}{\langle n_1 \rangle^{\frac{1}{2}+\theta}} \lesssim \langle n \rangle^{-\theta+\delta_5} \epsilon^{-3/2-3\delta_1}. \qquad (5.35) \quad \boxed{\texttt{I-II-II}}$$

Using Cauchy-Schwarz inequality and (5.18), for all $\omega \in F_\epsilon - G_\epsilon$, we have

$$\begin{aligned}
&\sup_{0\leqslant\tau\leqslant t} \sum_{R_n(n_1,n_2,n_3)} \frac{|g_{n_1} h_{n_2} h_{n_3}|}{\langle n_1 \rangle^{\frac{1}{2}+\theta}} \\
&\lesssim \sup_{0\leqslant\tau\leqslant t} \|h\|_{C_t[0,T]H^s} \left(\sum_{n_3} \langle n_3 \rangle^{-2s} \left| \sum_{R_{n_3,n}(n_1,n_2)} \frac{\epsilon^{-1/2-\delta_1}|h_{n_2}|}{\langle n_1 \rangle^{1/2+\theta-\delta_4}} \right|^2 \right)^{\frac{1}{2}} \\
&\lesssim \epsilon^{-1/2-\delta_1} \|h\|^2_{C_t[0,T]H^s} \left(\sum_{R_n(n_1,n_2,n_3)} \frac{1}{\langle n_1 \rangle^{1+2\theta-2\delta_4} \langle n_2 \rangle^{2s} \langle n_3 \rangle^{2s}} \right)^{\frac{1}{2}} \\
&\lesssim \epsilon^{-3/2-3\delta_1} \left(\sum_{N_1,N_2,N_3} \sum_{S_n(n_1,n_2,n_3)} \frac{1}{\langle n_1 \rangle^{2s} \langle n_2 \rangle^{2s} \langle n_3 \rangle^{2s}} \right)^{\frac{1}{2}}
\end{aligned} \qquad (5.36) \quad \boxed{\texttt{uniformI-II-II}}$$

if $\boxed{\delta_4 < \frac{1}{2} + \theta - s}$. Since

$$\sum_{N_1,N_2,N_3} \sum_{S_n(n_1,n_2,n_3)} \frac{1}{\langle n_1 \rangle^{2s} \langle n_2 \rangle^{2s} \langle n_3 \rangle^{2s}} \lesssim \sum_{\tilde{N}_1,\tilde{N}_2,\tilde{N}_3} \tilde{N}_1^{-2s} (\tilde{N}_2 \tilde{N}_3)^{1-2s} \lesssim \langle n \rangle^{-2s},$$

where $\tilde{N}_1, \tilde{N}_2, \tilde{N}_3$ are the decreasing order of N_1, N_2, N_3, the LHS of (5.36) is bounded by $\epsilon^{-3/2-3\delta_1} \langle n \rangle^{-s}$, which implies (5.35).

5.1.6 Case 6: $J_2 = I, J_1 = J_3 = II$

Similar to Case 5.

5.1.7 Case 7: $J_3 = I, J_1 = J_2 = II$

Similar to Case 5.

5.1.8 Case 8: $J_1 = J_2 = J_3 = II$

In this case, we we reduce (5.20) to proving that for all $\omega \in F_\epsilon - G_\epsilon$, one has

$$\sup_{0 \leqslant \tau \leqslant t} \sum_{R_n(n_1,n_2,n_3)} |h_{n_1} h_{n_2} h_{n_3}| \lesssim \langle n \rangle^{-\theta + \delta_5} \epsilon^{-3/2 - 3\delta_1}. \tag{5.37}$$

`eq: II-II-II`

Using Cauchy-Schwarz inequality and (5.18) as in (5.36), we have

$$\sup_{0 \leqslant \tau \leqslant t} \sum_{R_n(n_1,n_2,n_3)} |h_{n_1} h_{n_2} h_{n_3}| \lesssim \epsilon^{-3/2 - 3\delta_1} \left(\sum_{N_1, N_2, N_3} \sum_{S_n(n_1,n_2,n_3)} \frac{1}{\langle n_1 \rangle^{2s} \langle n_2 \rangle^{2s} \langle n_3 \rangle^{2s}} \right)^{\frac{1}{2}}. \tag{5.38}$$

`eq: pointwise`

Therefore, similar to Case 5, (5.38) is enough to imply (5.37).

Therefore, combining the above 8 cases, we obtain (5.20) and thus we complete the proof of (5.13).

Finally, we reduce Lemma 5.2 to (5.14).

5.2 Proof of (5.14)

It suffices to prove (5.14) for all $\omega \in F_\epsilon - G_\epsilon$, there exists some $\delta_6 > 0$ such that

$$\sup_{0 \leqslant \tau \leqslant t} \sum_{R_k(k_1,k_2,k_3)} \frac{|w_{k_1} \overline{w_{k_2}} w_{k_3}(\tau)|}{|\Omega|} \lesssim \epsilon^{-3/2 - C_3 \delta_1} \langle k \rangle^{-s - 1/2 - \delta_6}. \tag{5.39}$$

`eq: uniformbou`

Using (5.19), for all $\omega \in F_\epsilon - G_\epsilon$, LHS of (5.39) is bounded by

$$\epsilon^{-3/2 - 3\delta_1} \sum_{R_k(k_1,k_2,k_3)} \frac{1}{|\Omega| \langle k_1 \rangle^s \langle k_2 \rangle^s \langle k_3 \rangle^s}. \tag{5.40}$$

`eq: uniformbou`

Therefore, since we have taken $\boxed{C_3 \geqslant 5}$, we reduce (5.39) to bound (5.40) by $\epsilon^{-3/2 - 3\delta_1} \langle k \rangle^{-s - 1/2 - \delta_6}$. This can be implied by (5.22) and (5.23) if we take $\delta_6 = \delta_4$.

Hence, we complete the proof of Lemma 5.2.

$\square$

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