

PRINCIPAL QUIVER GRASSMANNIANS: CONJECTURES

STANISLAV FEDOTOV AND EVGENY FEIGIN

ABSTRACT. Let P and I be a projective and an injective representations of a Dynkin quiver. We consider quiver Grassmannians of subrepresentations of dimension $\dim P$ inside representations of dimension $\dim P + \dim I$. Based on extensive computer experiments, we formulate several conjectures about the algebro-geometric properties of these quiver Grassmannians.

1. INTRODUCTION

Let Q be a Dynkin quiver with an arbitrary orientation. Let P and I be a projective and an injective representations of Q . Our main objects of study are the quiver Grassmannians $\mathrm{Gr}_{\dim P}(M)$, where M are $\dim P + \dim I$ dimensional representations of Q .

Examples of such varieties naturally show up in various problems of algebraic geometry and representation theory [CI20, CFFFR17, CFFFR20, CFR12]. In particular, classical flag varieties and their PBW degenerations can be realized in this way (see [CL15, Fe11, Fe12-1, FeFi13, FFR17]). Our goal is twofold: first, we develop the software which allows to perform calculations with the quiver representations and quiver Grassmannians. Second, we use the software to check various conjectural properties of our quiver Grassmannians. In what follows, following the suggestion of Markus Reineke, we call quiver Grassmannians $\mathrm{Gr}_{\dim P}(M)$ with $\dim M = \dim P + \dim I$ principal (or PrIncipal), where Pr stands for projective and In for injective (we note that one can not study all possible quiver Grassmannians, since every projective variety can be realized in this way [Re13, Ri18]).

Let Q_0 and Q_1 be the sets of vertices and arrows of Q . Let $\mathbf{d} = \dim P + \dim I$ be the dimension vector of a representations M as above and let $\mathrm{Rep}_{\mathbf{d}}$ be the representation space of Q of dimension \mathbf{d} . A point of the universal quiver Grassmannian $\mathrm{Gr}_{\dim P}(\mathbf{d})$ consists of a point $M \in \mathrm{Rep}_{\mathbf{d}}$ and a collection of $(\dim P)_i$ -dimensional subspaces of M_i , $i \in Q_0$ compatible with the maps of M . One has a natural projection map $\pi : \mathrm{Gr}_{\dim P}(\mathbf{d}) \rightarrow \mathrm{Rep}_{\mathbf{d}}$ whose fibers are the quiver Grassmannians. Our first goal is to describe the reduced scheme structure of these fibers.

By definition, $\mathrm{Gr}_{\dim P}(M)$ are embedded into the product of classical Grassmann varieties $\mathrm{Gr}_{\dim P_i}(M_i)$, $i \in Q_0$ and hence admit the Plücker embedding into the product of projectivized wedge powers. One knows [LW19]

that the quiver Grassmannians can be endowed with the scheme structure defined by quadratic relations labeled by the edges $\alpha \in Q_1$. However, in most cases this scheme structure is not reduced. One can also naturally define quadratic relations for each path in Q . Here is our first conjecture.

Conjecture A. *For every principal quiver Grassmannian the scheme structure defined by the quadratic relations corresponding to all paths in Q is reduced.*

The representation space $\text{Rep}_{\mathbf{d}}$ is naturally acted upon by the group $G_{\mathbf{d}}$, the product of GL_{d_i} for all $i \in Q_0$. Since Q is Dynkin, there is a unique open orbit for this action; let M^0 be an element of this open orbit. Then $\text{Gr}_{\dim P}(M^0)$ is irreducible and of dimension $\langle \dim P, \dim I \rangle$; hence for any \mathbf{d} -dimensional M the dimension of $\text{Gr}_{\dim P}(M^0)$ is at least $\langle \dim P, \dim I \rangle$ (see [CFR12]). We say that M degenerates to N ($M, N \in \text{Rep}_{\mathbf{d}}$), if N is contained in the closure of the $G_{\mathbf{d}}$ orbit of M . We put forward the following conjecture.

Conjecture B. *There exists a representation M^1 of dimension $\dim P$ such that a quiver Grassmannian $\text{Gr}_{\dim P}(N)$, $N \in \text{Rep}_{\mathbf{d}}$ is irreducible of dimension $\langle \dim P, \dim I \rangle$ if and only if N degenerates to M^1 . If $\dim P$ and $\dim Q$ have no zero components, then $M^1 = P \oplus I$.*

From [CFR13-1], we already know that any representation degenerating to $P \oplus I$ is of dimension $\langle \dim P, \dim I \rangle$. Conjecture B is proved in [CFFFR17, CFFFR20] for equioriented type A quiver in certain special cases.

Another natural question to ask is for which representations N the quiver Grassmannians $\text{Gr}_{\dim P}(N)$ are of the expected (minimal possible) dimension $\langle \dim P, \dim I \rangle$. Some special cases of this question were investigated in [CFFFR17], [CFFFR20]. Here is our third conjecture.

Conjecture C. *Let Q be of type A with arbitrary orientation. There exists a representation M^2 of dimension $\dim P$ such that a quiver Grassmannian $\text{Gr}_{\dim P}(N)$, $N \in \text{Rep}_{\mathbf{d}}$ is of dimension $\langle \dim P, \dim I \rangle$ if and only if N degenerates to M^2 .*

In words, Conjecture C states that there exists the deepest quiver Grassmannian of the minimal possible dimension. We give a conjectural description of M^2 in some special cases. We note that if Conjecture C holds true then the subvariety of the universal quiver Grassmannian consisting of fibers over representations degenerating to M^2 is flat ([CFFFR17, CFFFR20]). We also note that Conjecture C does not hold in general: in type D_4 for certain representations P and I there exist three deepest quiver Grassmannians of the minimal possible dimension.

A universal description of minimal dimension locus is conjecturally given, under quite lax restrictions, in terms of dimensions of homomorphism spaces. Here's our next conjecture:

Conjecture D. *Let either of the following two conditions holds: a) Q is of type A and neither $\dim P$ nor $\dim I$ have zero components, b) Q is of type D and $P \oplus I$ contains every indecomposable projective and every indecomposable injective representation as a summand. Then $\dim \text{Gr}_{\dim P}(M) = \langle \dim P, \dim I \rangle$ if and only if*

$$\dim \text{Hom}_Q(M, X) \leq \dim \text{Hom}_Q(P, X) + 1$$

for every non-injective indecomposable representation X .

If Conjecture D holds, M^2 is described as a representation with maximal possible values of $\dim \text{Hom}_Q(N, X)$ not exceeding $\dim \text{Hom}_Q(P, X) + 1$.

Finally, we consider the Plücker algebras (the homogeneous coordinate rings) $\text{Pl}(M)$, $M \in \text{Rep}_{\mathbf{d}}$ of the quiver Grassmannians $\text{Gr}_{\dim P}(M)$. These algebras are defined as the quotients of the polynomial ring in all Plücker variables by the defining ideal $\mathcal{I}(M)$ responsible for the reduced scheme structure of our quiver Grassmannians. Let $\text{Pl}_{\mathbf{m}}(M)$ be the homogeneous components of $\text{Pl}(M)$.

Conjecture E. *Let Q be any Dynkin quiver with any orientation and let P and I be multiplicity free (i.e. each indecomposable summand shows up at most once). Then for $M \in \text{Rep}_{\mathbf{d}}$ one has*

$$\dim \text{Pl}_{\mathbf{m}}(M) = \dim \text{Pl}_{\mathbf{m}}(M^0) \text{ for all } \mathbf{m}$$

if and only if $\text{Gr}_{\dim P}(M)$ is of dimension $\langle \dim P, \dim I \rangle$. For arbitrary M one has $\dim \text{Pl}_{\mathbf{m}}(M) \geq \dim \text{Pl}_{\mathbf{m}}(M^0)$ for all \mathbf{m} .

We note that Conjecture E does not hold for arbitrary projective P and injective I (even in type A). At the same time, the conjectural flatness of the equi-dimensional (of minimal dimension) family of quiver Grassmannians $\text{Gr}_{\dim P}(M)$ implies that the Euler characteristic of the natural line bundles induced by the natural maps to $\mathbb{P}(\Lambda^{(\dim P)_i} M_i)$ are constant along the fibers. It is tempting to conjecture that for the multiplicity free P and I the components of the Plücker algebra are isomorphic to the zero cohomology groups of the corresponding line bundles and that the higher cohomology vanish. However, at the moment we do not have enough evidence to conjecture that this indeed holds true.

Finally, let us outline couple of possible further directions. First, the principal quiver Grassmannians for equioriented type A quivers enjoy many nice topological and combinatorial properties [Bi14, CI20, CFFFR20, CFR13-1, Fe12-2]. It would be interesting to extend the study to the case of general quivers. Second, a quiver Grassmannian of subrepresentations of M is naturally acted upon by the group of automorphisms of M . The description of this action and of the induced action on the homogeneous coordinate rings will lead to a better understanding of geometric and algebraic properties of the quiver Grassmannians (see [Ar11, CFR17, Fe23, FFL11, HT99] for some partial results).

Our paper is organized as follows. In section 2 we fix the notation and recall the basic definitions from the theory of quivers. In section 3 we formulate the conjectural description of the reduced scheme structure of the principal quiver Grassmannians. In section 4 we formulate a conjecture on the locus of the irreducible principal quiver Grassmannians of the minimal possible dimension. Sections 5 and 6 treat the case of (possibly reducible) principal quiver Grassmannians of the minimal possible dimension. In Section 7 we discuss the conjectural properties of the homogeneous coordinate rings. Section 8 contains a brief description of the software used to compute examples supporting conjectures given in the paper.

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2. THE SETUP

In this section we recall main definitions from the theory of quiver representations and quiver Grassmannians [ASS06, CB92, CR00, Schi14].

2.1. Generalities. We fix an algebraically closed field \mathbf{k} of characteristic zero. Let Q be a Dynkin quiver with the set of vertices Q_0 and the set of arrows Q_1 . For an arrow α we write $s(\alpha)$ for its source and $t(\alpha)$ for its target. A representation of Q is a collection of vector spaces M_i , $i \in Q_0$ and a collection of linear maps $M_\alpha : M_{s(\alpha)} \rightarrow M_{t(\alpha)}$.

The simple modules of Q are labeled by the vertices of Q . For $i \in Q_0$ we denote by S_i the simple 1-dimensional module supported at the vertex i . There are finitely many iso-classes of indecomposable representations of Q ; these iso-classes are in bijection with positive roots of the simple Lie algebra whose Dynkin diagram is Q . In what follows for a vertex i we denote by P_i the corresponding indecomposable projective module and by I_i the corresponding indecomposable injective module. In particular, one has the canonical embedding $S_i \rightarrow I_i$ and the canonical projection $P_i \rightarrow S_i$. We note that $(P_i)_j$, $i, j \in Q_0$ is non-zero if and only if there exists a path from i to j in Q and $(I_i)_j$, $i, j \in Q_0$ is non-zero if and only if there exists a path from j to i .

For a dimension vector $\mathbf{d} \in \mathbb{Z}_{\geq 0}^{Q_0}$ let $\text{Rep}_{\mathbf{d}}$ be the representation space of Q of dimension \mathbf{d} :

$$\text{Rep}_{\mathbf{d}} = \bigoplus_{\alpha \in Q_1} \text{Hom}(\mathbf{k}^{d_{s(\alpha)}}, \mathbf{k}^{d_{t(\alpha)}}).$$

The space $\text{Rep}_{\mathbf{d}}$ is naturally acted upon (via the base change) by the group $G_{\mathbf{d}}$, which is the product of the general linear groups GL_{d_i} for $i \in Q_0$. The orbits of this action parametrize the isomorphism classes of the \mathbf{d} -dimensional representations of Q . We denote by $\text{Rep}(Q)$ the set of all representations of the quiver Q .

For a point $M \in \text{Rep}_{\mathbf{d}}$ let $\mathcal{O}(M) \subset \text{Rep}_{\mathbf{d}}$ be the orbit $G_{\mathbf{d}}M$. The dimension and codimension of an orbit are computed by the following formulas:

$$\dim \mathcal{O}(M) = \dim G_d - \dim \text{End}_Q(M), \quad \text{codim}_{\text{Rep}_{\mathbf{d}}} \mathcal{O}(M) = \dim \text{Ext}_Q^1(M, M).$$

In particular, an orbit $\mathcal{O}(M)$ is dense in the representation space if and only if M is rigid, i.e. $\text{Ext}_Q^1(M, M) = 0$. For a Dynkin quiver Q there is only finitely many isoclasses of Q modules of a given dimension. Hence, there exists a unique open dense orbit.

We say that M degenerates to N if $\overline{\mathcal{O}(M)} \supset \mathcal{O}(N)$; we write $M \leq N$. One has the following theorem of Bongartz [Bo96]: $M \leq N$ if and only if

$$\dim \text{Hom}_Q(M, X) \leq \dim \text{Hom}_Q(N, X) \quad \forall X \in \text{Rep}(Q)$$

or, equivalently, if

$$\dim \text{Hom}_Q(X, M) \leq \dim \text{Hom}_Q(X, N) \quad \forall X \in \text{Rep}(Q).$$

The degeneration order induces the structure of a poset on the set of isoclasses of Q -modules of a fixed dimension \mathbf{d} : we say that the isoclass of M is less than or equal to the isoclass of N if M degenerates to N . We denote this poset by $\Gamma_{\mathbf{d}}$. In particular, $\Gamma_{\mathbf{d}}$ has a unique minimal element corresponding to the open dense orbit.

Remark 2.1. *Let Q be the equioriented type A quiver. Then the Bongartz criterion can be explicitly formulated as follows: M degenerates to N if and only if, for any path in Q , the rank of the map in M corresponding this path is no smaller than the corresponding rank of the map in N [AdF84]. The general type A case is worked out on [AdF85].*

For two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^{Q_0}$ we denote by $\langle \mathbf{a}, \mathbf{b} \rangle$ the value of the Euler form on these vectors given by

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i \in Q_0} a_i b_i - \sum_{\alpha \in Q_1} a_{s(\alpha)} b_{t(\alpha)}.$$

For representations $M \in \text{Rep}_{\mathbf{a}}$ and $N \in \text{Rep}_{\mathbf{b}}$ one has

$$\langle \mathbf{a}, \mathbf{b} \rangle = \dim \text{Hom}_Q(M, N) - \dim \text{Ext}_Q^1(M, N).$$

2.2. Quiver Grassmannians. Let M be a \mathbf{d} -dimensional representation of Q and let $\mathbf{e} \in \mathbb{Z}_{\geq 0}^{Q_0}$ be a dimension vector such that $e_i \leq d_i$ for all $i \in Q_0$. The universal quiver Grassmannian $\text{Gr}_{\mathbf{e}}(\mathbf{d})$ sits inside the product $\text{Rep}_{\mathbf{d}} \times \prod_{i \in Q_0} \text{Gr}_{e_i}(\mathbf{k}^{d_i})$ and consists of collections $\{f_{\alpha}\}_{\alpha \in Q_1}, (U_i)_{i \in Q_0}$ such that $f_{\alpha}(U_{s(\alpha)}) \subset U_{t(\alpha)}$ for all $\alpha \in Q_1$. One has two natural projections

$$\text{Rep}_{\mathbf{d}} \xleftarrow{\quad} \text{Gr}_{\mathbf{e}}(\mathbf{d}) \xrightarrow{\quad} \prod_{i \in Q_0} \text{Gr}_{e_i}(\mathbf{k}^{d_i});$$

for $M \in \text{Rep}_{\mathbf{d}}$ we denote by $\text{Gr}_{\mathbf{e}}(M)$ the fiber of the projection $\text{Gr}_{\mathbf{e}} \rightarrow \text{Rep}_{\mathbf{d}}$ over the point M .

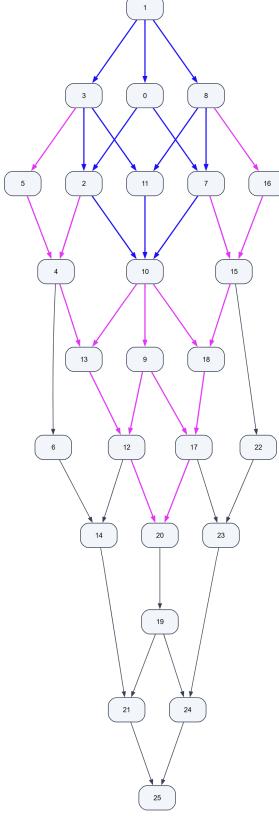


FIGURE 1. Degeneration picture for $Q = \bullet \rightarrow \bullet \leftarrow \bullet$, $\mathbf{d} = (3, 4, 3)$.

Let us fix a projective representation P and an injective representation I ; thus P is a direct sum of the modules P_i with certain multiplicities and I is a direct sum of the modules I_i with certain multiplicities. Let

$$\mathbf{d} = \dim P + \dim I.$$

We are interested in quiver Grassmannians $\text{Gr}_{\dim P}(M)$ for $M \in \text{Rep}_{\mathbf{d}}$ being \mathbf{d} -dimensional Q -module. In other words, we want to study the fibers of the universal quiver Grassmannian $\text{Gr}_{\dim P}(P + I)$ (over the representation space $\text{Rep}_{\mathbf{d}}$). Following suggestion of Markus Reineke we call these quiver Grassmannians *PrIncipal* (*Pr* for projectives and *In* for injectives).

Since Q is Dynkin, the group $G_{\mathbf{d}}$ has a finite number of orbits in $\text{Rep}_{\mathbf{d}}$. Hence there exists a unique open orbit $\mathcal{O}(M^0)$ (for a rigid representation M^0), which degenerates to any other orbit and a special fiber $\text{Gr}_{\dim P}(M^0)$ of $\text{Gr}_{\dim P}(\mathbf{d})$. Yet another special fiber is given by $\text{Gr}_{\dim P}(P \oplus I)$ (since $\mathbf{d} = \dim P + \dim I$).

Remark 2.2. *For any $M \in \text{Rep}_{\mathbf{d}}$ the quiver Grassmannian $\text{Gr}_{\dim P}(M)$ is non-empty. In fact, one knows [Scho92] that this statement is implied by the inequalities $\langle \mathbf{a}, \dim I \rangle \geq 0$ for any dimension vector \mathbf{a} , which holds*

true because I is injective. Furthermore, if Q is of type A , then it follows from [CEFR21, CEFF22] that for any $M \in \text{Rep}_{\mathbf{d}}$ the quiver Grassmannian $\text{Gr}_{\dim P}(M)$ is connected. More precisely, one shows that all quiver Grassmannians for the type A quivers are connected. It is tempting to conjecture that the same holds true for other Dynkin quivers as well: all the type D quiver Grassmannians $\text{Gr}_{\dim P}(M)$ we've seen in our computations are indeed connected.

The following statements were proved in [CFR12].

- $\text{Gr}_{\dim P}(M^0)$ and $\text{Gr}_{\dim P}(P \oplus Q)$ are irreducible,
- $\dim \text{Gr}_{\dim P}(M^0) = \dim \text{Gr}_{\dim P}(M^0) = \langle \dim P, \dim I \rangle$,
- $\text{Gr}_{\dim P}(P \oplus Q)$ is a flat degeneration of $\text{Gr}_{\dim P}(M^0)$,
- if M degenerates to $P \oplus Q$, then $\text{Gr}_{\dim P}(M)$ is irreducible of dimension $\langle \dim P, \dim I \rangle$.

2.3. Equioriented type A case. Let Q be the equioriented quiver of type A_n . We denote the vertices of Q by $1, 2, \dots, n$ with arrows given by $i \rightarrow i + 1$. The indecomposable representations are labeled by pairs i, j with $1 \leq i \leq j \leq n$; we denote the corresponding representation by $U(i, j)$. In particular, all non-trivial components of $U(i, j)$ are one-dimensional and the support of $U(i, j)$ consists of vertices between i and j . One has $S_i = U(i, i)$, $P_i = U(i, n)$, $I_i = U(1, i)$. The representation M^0 is equal to $U(1, n)^{\oplus n+1}$.

Let us consider the special case $P = \bigoplus_{i=1}^n P_i$, $I = \bigoplus_{i=1}^n I_i$. Then P is isomorphic (as a Q -module) to the path algebra and I to its dual. One has (recall $\mathbf{d} = \dim P + \dim I$)

$$\dim A = (1, \dots, n), \quad \dim A^* = (n, \dots, 1), \quad \mathbf{d} = (n+1, \dots, n+1).$$

The scalar product $\langle \dim A, \dim A^* \rangle$ is equal to $n(n+1)/2$. One observes that $\text{Gr}_{\dim A}(M^0)$ is isomorphic to the classical full flag variety for the group SL_{n+1} (see e.g. [Fu97]) and $\text{Gr}_{\dim A}(P \oplus Q)$ is isomorphic to the PBW degenerate flag variety for the group SL_{n+1} (see e.g. [Fe11, Fe12-1, Fe23]).

We also introduce one more representation of dimension \mathbf{d} :

$$M^2 = \bigoplus_{i=1}^n U(i, n) \oplus \bigoplus_{i=1}^n U(i, i) \oplus \bigoplus_{i=2}^n U(1, i-1).$$

The following is proved in [CFFFR17] (see also [CFFFR20]).

- $\text{Gr}_{\dim A}(M)$ is irreducible of dimension $n(n+1)/2$ if and only if M degenerates to M^1 ,
- $\text{Gr}_{\dim A}(M)$ is of dimension $n(n+1)/2$ if and only if M degenerates to M^2 .

Remark 2.3. The results of [CFFFR17] have to do with flat and flat irreducible loci of the universal quiver Grassmannian $\text{Gr}_{\dim A}(\mathbf{d})$.

3. REDUCED SCHEME STRUCTURE

For a number $n \in \mathbb{Z}_{>0}$ we write $[n]$ for the set $\{1, \dots, n\}$.

Let us denote the components of the dimension vector $\dim P$ by $(a_i)_{i \in Q_0}$. A quiver Grassmannian $\mathrm{Gr}_{\dim P}(M)$ sits inside the product of classical Grassmannians $\mathrm{Gr}_{a_i}(M_i)$, $i \in Q_0$. Using the standard Plücker embeddings of the Grassmann varieties one arrives at the embedding

$$\mathrm{Gr}_{\dim P}(M) \subset \prod_{i \in Q_0} \mathrm{Gr}_{a_i}(M_i) \subset \prod_{i \in Q_0} \mathbb{P}(\Lambda^{a_i}(M_i)).$$

We call the composition of these embedding the Plücker embedding for quiver Grassmannians. Let $\Delta_J^{(i)}$, $J \in \binom{[d_i]}{a_i}$, be the Plücker coordinates in the wedge space $\Lambda^{a_i}(M_i)$. We denote by the same symbols the homogeneous coordinates on the projective space $\mathbb{P}(\Lambda^{a_i}(M_i))$. Let $\mathcal{I}(M)$ be the multi-homogeneous ideal in the polynomial ring in all the variables $\Delta_J^{(i)}$ (for all $i \in Q_0$) consisting of all multi-homogeneous polynomials vanishing on $\mathrm{Gr}_{\dim P}(M)$. In order to give a (conjectural) explicit description of $\mathcal{I}(M)$ we introduce some notation.

Let us fix a basis $v_p^{(i)}$, $1 \leq p \leq d_i$ in each component M_i of the representation M . For an arrow $\alpha : i \rightarrow j$, $\alpha \in Q_1$ let $m_{\alpha,p,q}$ be the matrix of the map M_α written in the fixed bases. For a number $p \in [d_i]$ and a subset $I \subset [d_i]$ we write $\epsilon(p, I) = \#\{p' \in I : p' \leq p\}$. The following relations are labeled by an arrow $\alpha : i \rightarrow j$, a set $I \subset [d_i]$ of cardinality $a_i - 1$ and a set $J \subset [d_j]$ of cardinality $[d_j + 1]$:

$$(3.1) \quad R(\alpha, I, J) = \sum_{\substack{p \in [d_i] \setminus I \\ q \in J}} (-1)^{\epsilon(p, I) + \epsilon(q, J)} m_{\alpha,p,q} \Delta_{I \cup p}^{(i)} \Delta_{J \setminus \{q\}}^{(j)}.$$

It is shown in [LW19] that the relations above cut out $\mathrm{Gr}_{\dim P}(M)$ pointwise inside the product of projectivized wedge powers. Similarly to relations (3.1) one gets a relation $R(\pi, I, J)$ for any path π in Q with $s(\pi) = i$, $t(\pi) = j$ and $I \in \binom{[d_i]}{a_i - 1}$, $J \in \binom{[d_j]}{a_j + 1}$:

$$(3.2) \quad R(\pi, I, J) = \sum_{\substack{p \in [d_i] \setminus I \\ q \in J}} (-1)^{\epsilon(p, I) + \epsilon(q, J)} m_{\pi,p,q} \Delta_{I \cup p}^{(i)} \Delta_{J \setminus \{q\}}^{(j)},$$

with $m_{\pi,p,q}$ being matrix coefficients of the map M_π (the composition of maps corresponding to the arrows in π).

Recall the ideal $\mathcal{I}(M)$ defining the reduced scheme structure of $\mathrm{Gr}_{\dim P}(M)$. We put forward the following conjecture:

Conjecture 3.1. *For any $M \in \mathrm{Rep}_\mathbf{d}$ the ideal $\mathcal{I}(M)$ is generated by the quadratic relations $R(\pi, I, J)$ for all path π in Q and all $I \subset [d_{s(\pi)}]$, $\#I = a_{s(\pi)-1}$ and $J \subset [d_{t(\pi)}]$, $\#J = a_{t(\pi)} + 1$.*

In particular, the ideal generated by all the relations $R(\pi, I, J)$ is saturated (with respect to each group of Plücker variables $\Delta_\bullet^{(i)}$) and is prime.

We note that Conjecture 3.1 is far from being true for general quiver Grassmannians.

4. MINIMAL DIMENSION LOCUS: IRREDUCIBLE GRASSMANNIANS

Recall that $\dim \text{Gr}_{\dim P}(M^0) = \dim \text{Gr}_{\dim P}(P \oplus Q) = \langle \dim P, \dim I \rangle$ and both varieties are irreducible. Moreover, if a representation $M \in \text{Rep}_d$ degenerates to $P \oplus Q$, then the quiver Grassmannian $\text{Gr}_{\dim P}(M)$ is also irreducible and of the same dimension.

Let's denote by $\Gamma_d(1)$ the subset of the degeneration poset Γ_d that consists of representations $M \in \text{Rep}_d$ whose Grassmannian $\text{Gr}_{\dim P}(M)$ is irreducible of dimension $\langle \dim P, \dim I \rangle$.

Clearly, the lower ideal of Γ_d consisting of representations M that degenerate to $P \oplus Q$ is contained in $\Gamma_d(1)$. It turns out that they often coincide.

Example 4.1. Let Q be the A_3 quiver of the form $\bullet \rightarrow \bullet \leftarrow \bullet$. Let P and I be the direct sums of all projective (resp., injective) indecomposable representations. Then the poset $\Gamma_d(1)$ is visualized at Figure 2 below. Its single sink is exactly $P \oplus Q$.

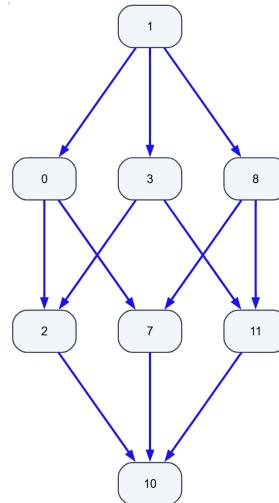


FIGURE 2. $\Gamma_d(1)$ for $Q = \bullet \rightarrow \bullet \leftarrow \bullet$, $P = P_1 \oplus P_2 \oplus P_3$, $I = I_1 \oplus I_2 \oplus I_3$.

Conjecture 4.2. There exists a representation M^1 such that for a representation $M \in \text{Rep}_d$ the quiver Grassmannian $\text{Gr}_{\dim P}(M)$ is irreducible of dimension $\langle \dim P, \dim I \rangle$ if and only if M degenerates to M^1 (i.e. $M \in \Gamma_d(1)$ if and only if $M^0 \leq M \leq M^1$). If, furthermore, no component of either $\dim P$ and $\dim I$ is zero, $M^1 = P \oplus I$.

The statement is proved in [CFFFR17, CFFFR20] for certain P and I for the equioriented type A quivers.

The condition that $\dim P$ and $\dim I$ do not have zero components seems to be crucial. In all the cases we checked, where some of the components vanish (both for A_n and D_n quivers), M^1 was different from $P \oplus I$ and thus $\{M : M^0 \leq M \leq P \oplus I\}$ was a strict subgraph of $\Gamma_{\mathbf{d}}(1)$.

Example 4.3. Let Q be the A_3 quiver of the form $\bullet \rightarrow \bullet \leftarrow \bullet$. Let $P = P_1 \oplus P_2 \oplus P_3$ and $I = I_1 \oplus I_3$. Then $\mathbf{d} = (2, 3, 2)$, $\dim I = (1, 0, 1)$, and $\dim P = (1, 3, 1)$.

In this case, whatever the representation M is, any one-dimensional subspace of M_1 and any one-dimensional subspace of M_3 define a subrepresentation of dimension $\dim P$. Thus, $\mathrm{Gr}_{\dim P}(M) \cong \mathbb{P}^1 \times \mathbb{P}^1$ for every M . So, $\Gamma_{\mathbf{d}}(1) = \Gamma_{\mathbf{d}}$ and $M^1 = 2S_1 \oplus 3S_2 \oplus 2S_3$ which is not $P \oplus I$.

Example 4.4. Let Q be the A_4 quiver of the form $\bullet \rightarrow \bullet \leftarrow \bullet \leftarrow \bullet$. Let $P = \bigoplus_{i=1}^4 P_i$ and $I = I_1 \oplus I_3 \oplus I_4$. Then $\mathbf{d} = (2, 4, 3, 3)$, $\dim I = (1, 0, 1, 2)$, and $\dim P = (1, 4, 2, 1)$.

In this case $\Gamma_{\mathbf{d}}(1) \neq \Gamma_{\mathbf{d}}$, but

$$M^1 = 2U(1, 1) + 4U(2, 2) + U(3, 3) + 2U(3, 4) + U(4, 4),$$

which differs from

$$P \oplus I = U(1, 1) + U(1, 2) + U(2, 2) + U(2, 3) + U(2, 4) + U(3, 4) + U(4, 4).$$

In the degenerate examples that we studied, the representation M^1 can be constructed from $P \oplus I$ in the following way. Let's call a vertex k *deficient* if $(\dim P)_k = 0$ or $(\dim I)_k = 0$. For every direct summand of $P \oplus I$, “split” it at every deficient vertex by vanishing the maps incidental to this vertex. The resulting sum will be M^1 .

In the previous A_4 example, the deficient vertex is 2, and we have four summands M in $P \oplus I$ such that $M_2 \neq 0$. Let's split those supported not only at 2:

$$\begin{aligned} U(1, 2) &\mapsto U(1, 1) + U(2, 2), \\ U(2, 3) &\mapsto U(2, 2) + U(3, 3), \\ U(2, 4) &\mapsto U(2, 2) + U(3, 4). \end{aligned}$$

The resulting sum

$$\begin{aligned} U(1, 1) + (U(1, 1) + U(2, 2)) + U(2, 2) + (U(2, 2) + U(3, 3)) + \\ (U(2, 2) + U(3, 4)) + U(3, 4) + U(4, 4) \end{aligned}$$

is exactly M^1 .

5. MINIMAL DIMENSION LOCUS

In this section we study the locus of representations M such that the quiver Grassmannian $\mathrm{Gr}_{\dim P}(M)$ is of the expected (minimal possible) dimension $\langle \dim P, \dim I \rangle$, but may have more than one irreducible component. We denote by $\Gamma_{\mathbf{d}}(2)$ the poset of isoclasses of such representations M .

Clearly, if $N \in \Gamma_{\mathbf{d}}(2)$ and $M \leq N$, then $M \in \Gamma_{\mathbf{d}}(2)$ and hence $\Gamma_{\mathbf{d}}(2) \subset \Gamma_{\mathbf{d}}$ is a lower ideal, which contains a subideal $\Gamma_{\mathbf{d}}(1)$.

Example 5.1. Let Q be the A_3 quiver of the form $\bullet \rightarrow \bullet \leftarrow \bullet$. Let P and I be the direct sums of all projective (resp., injective) indecomposable representations. Then the poset $\Gamma_{\mathbf{d}}(2)$ is visualized at Figure 3 below.

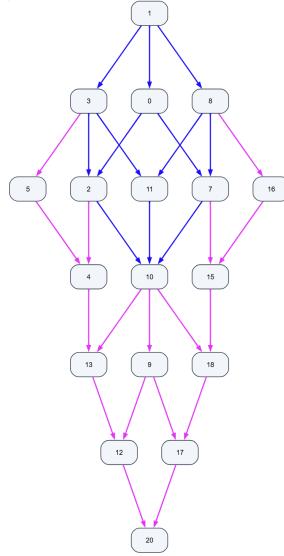


FIGURE 3. $\Gamma_{\mathbf{d}}(2)$ for $Q = \bullet \rightarrow \bullet \leftarrow \bullet$, $P = P_1 \oplus P_2 \oplus P_3$, $I = I_1 \oplus I_2 \oplus I_3$.

5.1. Type A. Let Q be a type A quiver with arbitrary orientation. We put forward the following conjecture.

Conjecture 5.2. There exists a unique representation $M^2 \in \text{Rep}_{\mathbf{d}}$ such that $\text{Gr}_{\dim P}(M)$ is of dimension $\langle \dim P, \dim I \rangle$ if and only if M degenerates to M^2 . If M degenerates to M^2 , then $\text{Gr}_{\dim P}(M)$ is equidimensional.

The conjecture above can be rephrased in the following way: the poset $\Gamma_{\mathbf{d}}(2)$ consists of elements M such that $M^0 \leq M \leq M^2$ for certain element $M^2 \in \Gamma_{\mathbf{d}}$.

In general we expect that the following conjecture holds true.

Conjecture 5.3. If $\dim \text{Gr}_{\dim P}(M) = \langle \dim P, \dim I \rangle$, then $\dim \text{Gr}_{\dim P}(M)$ is equidimensional.

Recall (see [CFFFR17]) that for the equioriented type A_n quiver and $P = \bigoplus_{i \in Q_0} P_i$, $I = \bigoplus_{i \in Q_0} I_i$ the representation M^2 as above does exist and is isomorphic to

$$(5.1) \quad M^2 = P \oplus S \oplus I/S, \quad S = \bigoplus_{i \in Q_0} S_i.$$

The corresponding quiver Grassmannian $\mathrm{Gr}_{\dim P}(M^2)$ is equidimensional, of expected dimension $n(n+1)/2$ and has the Catalan number irreducible components.

Let us adjust the description (5.1) above to the conjectural answer in type A for general orientation. We decompose Q into the union of several equioriented type A quivers $Q = Q(1) \cup \dots \cup Q(r)$ such that for each ℓ the quivers $Q(\ell)$ and $Q(\ell+1)$ have a unique common vertex $i(\ell) \in Q_0$. Let $M^2(\ell)$ be the deepest representation (5.1) corresponding to the equioriented quiver $Q(\ell)$. In particular, each $M^2(\ell)$ contains summands of the form $2S(i)$ for i being the leftmost and the rightmost vertices of $Q(\ell)$.

We put forward the following conjectures:

Conjecture 5.4. *Let Q be of type A with the decomposition $Q = \bigcup_{\ell=1}^r Q(\ell)$, $Q(\ell) \cap Q(\ell+1) = i_\ell$. Let also $P = \bigoplus_{j \in Q_0} P_j$ and $I = \bigoplus_{j \in Q_0} I_j$. Then*

$$M^2 = \bigoplus_{\ell=1}^r M^2(\ell) - 2 \bigoplus_{a=1}^{r-1} S_{i_a},$$

where the subtraction means that we remove two summands of the form S_{i_a} , $1 \leq a < r$ from the decomposition of $\bigoplus_{\ell=1}^r M^2(\ell)$ into the direct sum of indecomposable representations.

Conjecture 5.5. *Let $P = \bigoplus_j u_j^P P_j$, $I = \bigoplus_j u_j^I I_j$ with all $u_j^P, u_j^I \neq 0$. Let also $M^2(P, I)$ be the corresponding deepest representation with minimal dimension Grassmannian. Then*

$$M^2(P \oplus P', I \oplus I') = M^2(P, I) \oplus P' \oplus I'.$$

In the case if some u_j^P or u_j^I are zero, the exact form of M^2 is not entirely clear for us yet. In some cases Conjecture 5.5 still holds, in some it doesn't.

We provide some concrete examples below.

5.2. Zig-zag case. We continue using the notation $U(i, j)$ for the indecomposable A_n module supported on vertices between i and j (for arbitrary orientation). In the following Conjecture we consider quiver A_n with alternating orientation.

Conjecture 5.6. *Let Q be of type A_n with alternating orientation (a “zig-zag quiver”). Then M^2 is written as $\bigoplus_{i=1}^{n-1} U(i, i+1) \oplus 2 \bigoplus_{i \in Q_0} S_i$. The quiver Grassmannian $\mathrm{Gr}_{\dim P}(M^2)$ has 2^{n-1} irreducible components each of dimension $2n-1$.*

5.3. A_5 example. Let us consider an $n = 5$ example. Let Q be as follows: $1 \rightarrow 2 \leftarrow 3 \leftarrow 4 \leftarrow 5$. Then the representation M^2 is given by the following formula:

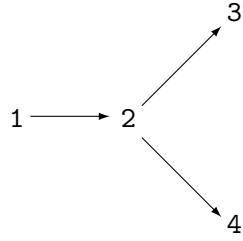
$$\begin{aligned} M^2 = & 2U(1, 1) + U(1, 2) + 2U(2, 2) + U(2, 3) + U(3, 3) + U(2, 4) + \\ & U(4, 4) + U(2, 5) + U(3, 5) + U(4, 5) + 2U(5, 5). \end{aligned}$$

We note that the right hand side can be rewritten as

$$\left(2U(1,1) + U(1,2) + 2U(2,2)\right) + \left(2U(2,2) + U(2,3) + U(3,3) + U(2,4) + U(4,4) + U(2,5) + U(3,5) + U(4,5) + 2U(5,5)\right) - 2U(2,2),$$

where the sums in brackets correspond to the representations M^2 for the quivers $1 \rightarrow 2$ and $2 \leftarrow 3 \leftarrow 4 \leftarrow 5$ (with $-2U(2,2)$ coming as gluing at vertex 2).

5.4. Type D. Let Q be the following D_4 quiver



Computer experiments support the following proposition.

Proposition 5.7. *Let Q be a quiver with vertices $1, 2, 3, 4$ and arrows $1 \rightarrow 2$, $2 \rightarrow 3$ and $2 \rightarrow 4$. Let $P = \bigoplus_{i \in Q_0} P_i$, $I = \bigoplus_{i \in Q_0} I_i$. Then there exist three representations $M^2(1)$, $M^2(2)$, $M^2(3)$ in $\text{Rep}_{\mathbf{d}}$ such that $\dim \text{Gr}_{\dim P}(M) = \langle \dim P, \dim I \rangle$ if and only if M degenerates to one of $M^2(i)$, $i = 1, 2, 3$. None of the representations $M^2(1)$, $M^2(2)$, $M^2(3)$ degenerate one to another.*

Let us provide more details. One has the following explicit formulas:

$$\mathbf{d} = (5, 5, 4, 4), \dim P = (1, 2, 3, 3), \langle \dim P, \dim I \rangle = 9.$$

In order to write down the representations $M^2(i)$ explicitly, we denote by $V(x_1, x_2, x_3, x_4)$ the indecomposable Q module of dimension (x_1, x_2, x_3, x_4) . Then one has

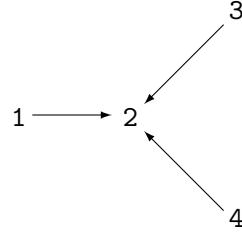
$$\begin{aligned} M^2(1) = & 2V(1, 0, 0, 0) \oplus V(1, 1, 0, 0) \oplus V(0, 1, 0, 0) \oplus V(1, 1, 1, 0) \oplus \\ & 2V(0, 0, 1, 0) \oplus V(0, 1, 0, 1) \oplus 2V(0, 0, 0, 1) \oplus V(1, 1, 1, 1), \end{aligned}$$

$$\begin{aligned} M^2(2) = & 2V(1, 0, 0, 0) \oplus V(1, 1, 0, 0) \oplus V(0, 1, 0, 0) \oplus V(0, 1, 1, 0) \oplus \\ & 2V(0, 0, 1, 0) \oplus V(1, 1, 0, 1) \oplus 2V(0, 0, 0, 1) \oplus V(1, 1, 1, 1), \end{aligned}$$

$$\begin{aligned} M^2(3) = & 2V(1, 0, 0, 0) \oplus V(1, 1, 0, 0) \oplus V(1, 1, 1, 0) \oplus V(0, 1, 1, 0) \oplus \\ & 2V(0, 0, 1, 0) \oplus V(1, 1, 0, 1) \oplus V(0, 1, 0, 1) \oplus 2V(0, 0, 0, 1). \end{aligned}$$

All three quiver Grassmannians $\text{Gr}_{\dim P}(M^2(i))$ have 13 irreducible components of dimension 9.

Now let us consider the D_4 quiver with all vertices pointing to the central vertex



Proposition 5.8. *Let Q be a quiver with vertices $1, 2, 3, 4$ and arrows $1 \rightarrow 2$, $3 \rightarrow 2$ and $4 \rightarrow 2$. Let $P = \bigoplus_{i \in Q_0} P_i$, $I = \bigoplus_{i \in Q_0} I_i$. Then there exists a unique representation M^2 in $\text{Rep}_{\mathbf{d}}$ such that $\dim \text{Gr}_{\dim P}(M) = \langle \dim P, \dim I \rangle$ if and only if M degenerates to M^2 .*

Let us provide the details in this case. One has

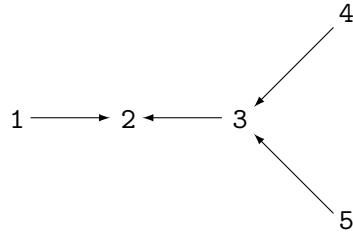
$$\mathbf{d} = (3, 5, 3, 3), \dim P = (1, 4, 1, 1), \langle \dim P, \dim I \rangle = 7$$

and

$$\begin{aligned} M^2 = 2V(1, 0, 0, 0) \oplus V(1, 1, 0, 0) \oplus 2V(0, 1, 0, 0) \oplus V(0, 1, 1, 0) \oplus \\ 2V(0, 0, 1, 0) \oplus V(0, 1, 0, 1) \oplus 2V(0, 0, 0, 1). \end{aligned}$$

The quiver Grassmannian $\text{Gr}_{\dim P}(M^2)$ has 8 irreducible components of dimension 7.

Finally, let us mention that for Q of type D_5 of the form



one also gets three deepest representations M^2 such that the corresponding quiver Grassmannian is of the expected (minimal possible) dimension.

6. THE HOMOMORPHISM DIMENSION CRITERION

The goal of this section is to formulate a criterion for a representation N to belong to $\Gamma_{\mathbf{d}}(2)$ in terms of dimensions of certain homomorphism spaces.

We start with the following observation, which has to do with the description of $\Gamma_{\mathbf{d}}(1)$:

- for every indecomposable injective $X = I_j$, $\dim \text{Hom}_Q(M^1, X) = \mathbf{d}_j$,

- for every indecomposable non-injective X , $\dim \text{Hom}_Q(P \oplus I, X) = \dim \text{Hom}_Q(P, X)$

(the first equality follows from the general fact that $\dim \text{Hom}_Q(M, I_j) = \mathbf{d}_j$ for every $M \in \text{Rep}_{\mathbf{d}}(Q)$, and the second holds true since there are no homomorphisms from I to X). Assume that the dimension vectors $\dim P$ and $\dim I$ never vanish. In this case we expect (see section 4) that $M^1 = P \oplus Q$ and hence we can characterize $\Gamma_{\mathbf{d}}(1)$ as the set of all representations M such that, for every non-injective indecomposable representation X ,

$$\dim \text{Hom}_Q(M, X) \leq \dim \text{Hom}_Q(P, X).$$

It turns out that M^2 allows for a similar description.

Conjecture 6.1. *Let $P \oplus I$ contains every indecomposable projective and every indecomposable injective representation as a summand. Then a representation M lies in $\Gamma_{\mathbf{d}}(2)$ if and only if, for every non-injective indecomposable representation X ,*

$$\dim \text{Hom}_Q(M, X) \leq \dim \text{Hom}_Q(P, X) + 1.$$

Remark 6.2. *Dually, this can be formulated as: a representation M lies in $\Gamma_{\mathbf{d}}(2)$ if and only if, for every non-projective indecomposable representation X ,*

$$\dim \text{Hom}_Q(X, M) \leq \dim \text{Hom}_Q(X, I) + 1 /$$

This provides a general way of describing M^2 for both A and D types. For A type quivers, there exists a representation M^2 satisfying, for all indecomposable non-injective representations X ,

$$\dim \text{Hom}_Q(M^2, X) = \dim \text{Hom}_Q(P, X) + 1.$$

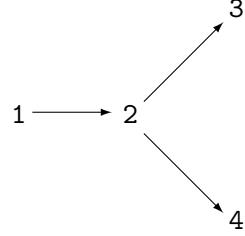
For D type quivers such a representation M^2 does not always exist. That's why in some cases we have three sinks $M^2(i)$ in $\Gamma_{\mathbf{d}}(2)$. However,

$$\max_i \dim \text{Hom}_Q(M^2(i), X) = \dim \text{Hom}_Q(P, X) + 1$$

for all indecomposable non-injective X (assuming the conditions of Conjecture 6.1 are satisfied). Note that this perfectly aligns with Conjecture 5.5. Indeed, if the above equation holds, then

$$\max_i \dim \text{Hom}_Q(M^2(i) \oplus P' \oplus I', X) = \dim \text{Hom}_Q(P \oplus P', X) + 1.$$

If some of P_j or I_j are missing in $P \oplus I$, then Conjecture 6.1 sometimes fails. For example, let $Q = D_4$ with the arrow orientation illustrated below, $P = P_1 \oplus P_3 \oplus P_4$, $I = I_1 \oplus I_2 \oplus I_3 \oplus I_4$.



Then $\Gamma_{\mathbf{d}}(2)$ has a single sink

$$\begin{aligned} M^2 = & 2V(1, 0, 0, 0) \oplus V(1, 1, 0, 0) \oplus V(0, 1, 0, 0) \oplus V(1, 1, 1, 0) \oplus \\ & 2V(0, 0, 1, 0) \oplus V(1, 1, 0, 1) \oplus 2V(0, 0, 0, 1), \end{aligned}$$

such that

$$\dim \text{Hom}_Q(M^2, V(0, 1, 1, 1)) = 4 = \dim \text{Hom}_Q(P, V(0, 1, 1, 1)) + 2.$$

This case has one more peculiarity: the set of representations N such that $\dim \text{Hom}_Q(N, X) \leq \dim \text{Hom}_Q(P, X) + 1$, for all non-injective indecomposable X , has four sinks instead of one or three. At the same time, M^1 in this case coincides with $P \oplus I$.

On the other hand, if Q and I are the same and $P = P_1 \oplus P_2 \oplus P_4$, the conjecture holds.

7. PLÜCKER ALGEBRAS AND LINE BUNDLES

Recall the notation $a_i = \dim P_i$ and the standard closed embedding $\text{Gr}_{\dim P}(M) \hookrightarrow \prod_{i \in Q_0} \text{Gr}_{a_i}(M_i)$. Each Grassmann variety admits the Plücker embedding into the projective space $\mathbb{P}(\Lambda^{a_i})(M_i)$. Let $\mathcal{O}_i(m)$ be the line bundle on $\text{Gr}_{a_i}(M_i)$ obtained as a pullback of $\mathcal{O}(m)$ on $\mathbb{P}(\Lambda^{a_i}(M_i))$. For $\mathbf{m} \in \mathbb{Z}_{\geq 0}^{Q_0}$ we denote by $\mathcal{O}(\mathbf{m})$ the exterior tensor product $\prod_{i \in Q_0} \mathcal{O}_i(m_i)$, which is a line bundle on the product of Grassmannians. We use the same symbol $\mathcal{O}(\mathbf{m})$ to denote the restriction to the quiver Grassmannian $\text{Gr}_{\dim P}(M)$.

Recall the multi-homogeneous ideals $\mathcal{I}(M)$ inside the polynomial ring in Plücker variables $\Delta_J^{(i)}$ which defines the reduced scheme structure of the quiver Grassmannians $\text{Gr}_{\dim P}(M)$. The multi-graded Plücker algebra admits the decomposition

$$\text{Pl}(M) = \mathbb{C}[\Delta_J^{(i)}]/\mathcal{I}(M) = \bigoplus_{\mathbf{m} \in \mathbb{Z}_{\geq 0}^{Q_0}} \text{Pl}_{\mathbf{m}}(M),$$

where the homogeneous component $\text{Pl}_{\mathbf{m}}(M)$ is spanned by all the monomials of the form

$$\prod_{i \in Q_0} \Delta_{J(1)}^{(i)} \dots \Delta_{J(m_i)}^{(i)}, \quad J(r) \subset [d_i], \quad |J(r)| = a_i \quad (r \in [m_i]).$$

For example, for Q being the equioriented type A quiver the homogeneous components $\text{Pl}_m(M^0)$ are identified with the dual irreducible highest weight \mathfrak{sl}_n modules and $\text{Pl}_m(M^1)$ are identified with dual PBW degenerate representations [Fe23]. We put forward the following conjectures.

Conjecture 7.1. *Let $P = \bigoplus_{i \in Q_0} P_i$, $I = \bigoplus_{j \in Q^0} I_j$. For any $M \in \text{Rep}_{\mathbf{d}}$ and any $\mathbf{m} \in \mathbb{Z}_{\geq 0}^{Q_0}$ one has the inequality $\dim \text{Pl}_m(M) \geq \dim \text{Pl}_m(M^0)$. The equality $\dim \text{Pl}_m(M) = \dim \text{Pl}_m(M^0)$ holds for all \mathbf{m} if and only if $\dim \text{Gr}_{\dim P}(M) = \langle \dim P, \dim I \rangle$.*

Remark 7.2. *Conjecture 7.1 does not hold for general P and I even for Q of type A . More precisely, if one allows arbitrary multiplicities of indecomposable projective and injective modules in the decomposition of P and I , then the dimensions $\dim \text{Pl}_m(M)$ may be different from $\dim \text{Pl}_m(M^0)$ even for M degenerating to M^1 .*

Conjecture 7.3. *Let $P = \bigoplus_{i \in Q_0} P_i$, $I = \bigoplus_{j \in Q^0} I_j$. Then for any M such that $\text{Gr}_{\dim P}(M)$ is of minimal dimension, one has for all $\mathbf{m} \in \mathbb{Z}_{\geq 0}^{Q_0}$*

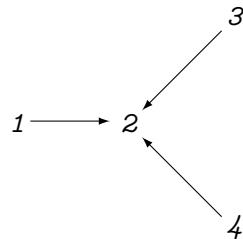
$$\dim H^0(\text{Gr}_{\dim P}, \mathcal{O}(\mathbf{m})) = \dim \text{Pl}_m(M), \quad H^{>0}(\text{Gr}_{\dim P}, \mathcal{O}(\mathbf{m})) = 0.$$

Remark 7.4. *Conjecture 7.3 is not expected to hold for general P and I (although we do not have a program computing the dimensions of the cohomology groups). The reason comes from Remark 7.2: over a flat family the Euler characteristic of $\mathcal{O}(\mathbf{m})$ is preserved, however the dimensions of the homogeneous components of the Plücker algebra vary. This might be an indicator that the higher cohomology groups do not vanish in general.*

Example 7.5. *Let Q be of the form $1 \rightarrow 2 \leftarrow 3$. Then for $u_i = m_i + 1$:*

$$\dim \text{Pl}_m(M^0) = \frac{1}{12} u_1 u_2 u_3 (3u_1 u_2 + 3u_1 u_3 + 3u_2 u_3 + 2u_2^2 + 1).$$

Example 7.6. *Let Q be of the form*



Then

$$\begin{aligned} \dim \text{Pl}_m(M^0) = & \frac{1}{24} u_1 u_2 u_3 u_4 (3u_1 u_2 u_3 + 3u_1 u_2 u_4 + 3u_1 u_3 u_4 + 3u_2 u_3 u_4 + \\ & u_2^3 + 2u_2^2(u_1 + u_3 + u_4) + 2u_2 + u_1 + u_3 + u_4) \end{aligned}$$

(as above, $u_i = m_i + 1$).

8. SOFTWARE

Suggesting the above conjectures was made possible after extensive computer experiments with the **quiver-representation** library we've developed. It's open source and available on github [Fed25].

The library's main language is Python 3. It also relies on:

- Macaulay 2 <https://macaulay2.com/> to derive algebro-geometric properties of quiver Grassmannians from their equations.
- GNU Parallel for batch computations.
- Graphviz <https://graphviz.org/> for visualization of Γ_d .

The library allows to perform a wide range of module-theoretic computations for quiver representations over the field of complex numbers or over any finite field, including:

- Inferring indecomposable simple, projective and injective representations.
- Computing direct sums, kernels and cokernels of morphisms, radicals and socles, projective covers and injective hulls.
- Finding a basis of $\text{Hom}_Q(M, N)$ for given M and N .
- Enumerating indecomposables of A_n and D_n ; creating a representation for a bag of intervals.

The library also has the full pipeline of checking the hypotheses mentioned in this paper: from enumerating all Plücker and incidence relations for a given quiver Grassmannian to constructing the graph Γ_d and to computing dimensions of irreducible components and Hilbert functions with Macaulay 2.

Despite the library supports module-theoretic computations over \mathbb{C} , we advise you to use it with caution. Floating-point computations are unavoidably imprecise; for example, a tiny computational error might make any matrix full-rank, which would lead to incorrect kernels, cokernels, etc.

Computations over finite fields lack this problem. So, in our experiments, the calculation of $\dim \text{Hom}_Q(\cdot, \cdot)$ was done over \mathbb{F}_{107} . The data needed to assess our hypotheses using the pipeline can be found in the `examples` folder: <https://github.com/st-fedotov/quiver/tree/main/examples>.

The pipeline tends to suffer from memory exhaustion due to combinatorial explosion when a quiver contains paths of length ≥ 5 or if the multiplicities of indecomposable projectives or injectives in $P \oplus I$ are greater than 1. This can be somewhat relieved by increasing the `gc_heap_size` parameter in the experimental config provided you have enough RAM. Still, the computations might take some time.

The bulk of our experiments were done on a Nebius' virtual machine with 128 CPU and 512Gb RAM. Depending on the memory requirements, we used between 30 and 120 parallel computational streams. Please check the library's `readme` for installation guidance and further details.

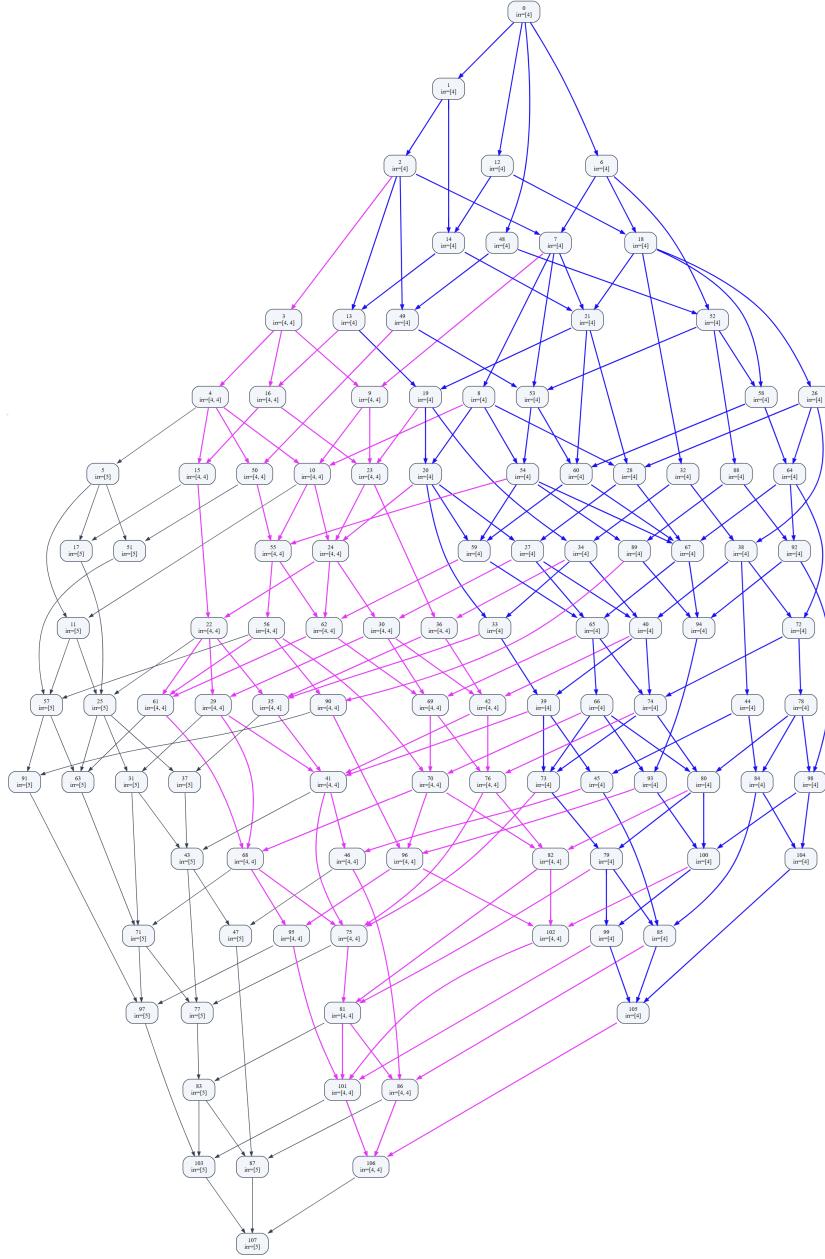


FIGURE 4. $\Gamma_d(2)$ for $Q = \bullet \rightarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet$, $P = P_1 \oplus P_2 \oplus P_3$, $I = I_1 \oplus I_2 \oplus I_3$.

The library was created with the help of coding agents: GPT-5 in chat mode <https://chatgpt.com/>, Codex <https://chatgpt.com/codex/>, and Claude Code <https://www.claude.com/product/clause-code>.

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NEBIUS, LONDON, UK

Email address: st.n.fedotov@gmail.com

SCHOOL OF MATHEMATICAL SCIENCES, TEL AVIV UNIVERSITY, TEL AVIV, 69978,
ISRAEL

Email address: evgfeig@gmail.com