

# THE DE JONG FUNDAMENTAL GROUP OF A NON-TRIVIAL ABELIAN VARIETY IS NON-ABELIAN

SEAN HOWE

**ABSTRACT.** We show the de Jong fundamental group of any non-trivial abelian variety over a complete algebraically closed extension of  $\mathbb{Q}_p$  is non-abelian.

## 1. INTRODUCTION

Let  $C$  be an algebraically closed field of characteristic zero. If  $A/C$  is an abelian variety and  $0 \in A(C)$  is the identity element, then the étale fundamental group  $\pi_{1,\text{ét}}(A, 0)$  is identified with  $T_{\hat{\mathbb{Z}}} A = \varprojlim_n A[n](C)$  by its translation action on the universal profinite étale cover  $\tilde{A} = \varprojlim_n A$  (in both limits the indices are positive integers ordered by divisibility, and the transition map for  $m|n$  is multiplication by  $n/m$ ). If  $C$  is a complete algebraically closed extension of  $\mathbb{Q}_p$ , then we can also pass to the associated rigid analytic space  $A^{\text{rig}}/C$ . Its category of finite étale covers is equivalent to the category of finite étale covers of  $A$ , so that

$$\pi_{1,\text{ét}}(A^{\text{rig}}, 0) = \pi_{1,\text{ét}}(A, 0) = T_{\hat{\mathbb{Z}}} A.$$

In particular,  $\pi_{1,\text{ét}}(A^{\text{rig}}, 0)$  is an abelian group.

However, in the rigid analytic setting, there are more general notions of covering spaces that mix properties of topological and finite étale coverings. One such notion is that of de Jong coverings as introduced in [2], and there is an associated de Jong fundamental group  $\pi_{1,\text{dJ}}(A^{\text{rig}}, 0)$ , a pro-discrete topological group whose continuous actions on discrete sets correspond to de Jong coverings of  $A^{\text{rig}}$ . It is well known that de Jong fundamental groups can be much larger than classical étale fundamental groups: for example, although  $\pi_{1,\text{ét}}(\mathbb{P}_C^{n,\text{rig}}) = \pi_{1,\text{ét}}(\mathbb{P}_C^n) = \{1\}$ , for  $n \geq 1$  the de Jong fundamental group  $\pi_{1,\text{dJ}}(\mathbb{P}_C^{n,\text{rig}})$  is large: a connected component over  $C$  of the height  $n+1$  Lubin-Tate tower furnishes, via the Gross-Hopkins period map, a tower of connected de Jong coverings of  $\mathbb{P}_C^{n,\text{rig}}$ . For  $y \in \mathbb{P}^{n,\text{rig}}(C)$ , this tower is classified by

$$(1) \quad \rho_n : \pi_{1,\text{dJ}}(\mathbb{P}_C^{n,\text{rig}}, y) \rightarrow \text{SL}_{n+1}(\mathbb{Q}_p),$$

and this map is a surjection by [2, Proposition 7.4].

For abelian varieties over  $C$  of bad reduction, there are well-known connected de Jong coverings with infinite discrete fibers coming from Raynaud uniformization. These are all abelian and, moreover, the associated covering groups are free  $\mathbb{Z}$ -modules of finite rank so that, up to profinite completion, these covers are detected already by the classical étale fundamental group. The main point of this note is that there are abundant de Jong covers beyond this construction. In particular:

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**Theorem 1.** *Let  $C/\mathbb{Q}_p$  be a complete algebraically closed extension, let  $A/C$  be an abelian variety, and let  $0 \in A(C)$  be the identity element. If the dimension of  $A$  is at least 1, then  $\pi_{1,\text{dJ}}(A^{\text{rig}}, 0)$  is not an abelian group.*

In particular, in the setting of Theorem 1, the natural map  $\pi_{1,\text{dJ}}(A^{\text{rig}}, 0) \rightarrow \pi_{1,\text{ét}}(A^{\text{rig}}, 0)$  is not injective and thus finite étale coverings do not detect all de Jong coverings. This is in contrast to the complex analytic setting where the topological fundamental group of a complex abelian variety is a finite free  $\mathbb{Z}$ -module and thus injects into its profinite completion, the étale fundamental group.

**Remark 1.** From a classical perspective, an abelian variety having a non-abelian fundamental group is surprising. On the other hand, it is perhaps not as surprising as the non-triviality (and worse... [4])) of the fundamental group of  $\mathbb{P}_C^{n,\text{rig}}$  furnished already by the covers of Eq. (1), and we use these covers in our proof of Theorem 1.

For  $n = \dim A$ , we establish Theorem 1 by pulling back the cover of  $\mathbb{P}_C^n$  in Eq. (1) along a generically étale map  $A \rightarrow \mathbb{P}_C^n$  and then using geometric Sen theory to show the monodromy of this pullback remains non-abelian. The proof also implies that the monodromy is infinite, an observation we first learned from Sasha Petrov.

**Question 1.** This method of proof does not work for a general abeloid variety  $A/C$  — indeed, such an  $A$  may not admit *any* rational functions, in which case the covers of Eq. (1) cannot be used to produce any candidate non-abelian covers of  $A$ . It is thus natural to ask: is the de Jong fundamental group of a non-trivial abeloid non-abelian? We note that, for  $A/C$  abeloid, the usual profinite étale fundamental group is “as expected,” i.e. equal to  $T_{\mathbb{Z}} A$  (see [1, Appendix A]).

The key ingredients in the proof of Theorem 1 are the injectivity of the geometric Sen morphism for the Lubin-Tate tower (Lemma 2, which follows from a computation of the geometric Sen morphism given in [3]), and the observation that, if the monodromy is abelian, then the geometric Sen morphism is constant (Lemma 3). We treat these preliminaries in §2 and then use them to prove Theorem 1 in §3.

**Remark 2.** Our proof does not use the surjectivity of Eq. (1); all that is needed is the pointwise injectivity of the associated geometric Sen morphism (a “softer” fact than the actual monodromy computation for Eq. (1)).

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## 2. PRELIMINARIES

In this section we establish the technical results needed in the proof of Theorem 1.

**2.1. The geometric Sen morphism and its functoriality.** Recall from [6, Theorem 1.0.4] that, for  $X/C$  a smooth rigid analytic variety and  $K$  a  $p$ -adic Lie

group with Lie algebra  $\tilde{\mathfrak{k}}$ , any pro-étale  $K$ -torsor  $\tilde{X}/X$  gives rise to a geometric Sen morphism

$$\theta_{\tilde{X}} \in \left( \Omega_{X/C}^1 \otimes_{\mathcal{O}} \left( \tilde{\mathfrak{k}} \otimes_{\underline{\mathbb{Q}_p}} \hat{\mathcal{O}}(-1) \right) \right) (X),$$

where  $\tilde{\mathfrak{k}} = \tilde{X} \times^K \underline{\mathfrak{k}}$  is the twisted form of the constant local system  $\underline{\mathfrak{k}}$  associated to  $\tilde{X}$  and the adjoint action of  $K$  on  $\mathfrak{k}$ . Dually, we view this as a morphism on  $X_{\text{proét}}$ ,

$$\kappa_{\tilde{X}} : T_{X/C} \otimes_{\mathcal{O}} \hat{\mathcal{O}} \rightarrow \tilde{\mathfrak{k}} \otimes_{\underline{\mathbb{Q}_p}} \hat{\mathcal{O}}(-1).$$

We recall also from the statement of [6, Theorem 1.0.4] that  $\kappa_{\tilde{X}}$  is natural for pullbacks and push-outs of torsors:

**Lemma 1** (Functoriality of  $\kappa$ ). *Let  $K$  be a  $p$ -adic Lie group.*

- (1) *If  $f : X \rightarrow Y$  is a map of smooth rigid analytic varieties over  $C$  and  $\tilde{Y}/Y$  is a pro-étale  $K$ -torsor, then  $\kappa_{f^*\tilde{Y}} = f^* \kappa_{\tilde{X}} \circ df$ , where  $df : T_{X/C} \rightarrow \pi^* T_{Y/C}$  is the derivative of  $f$ .*
- (2) *If  $H$  is a  $p$ -adic Lie group and  $\rho : H \rightarrow K$  is a continuous homomorphism, then for any smooth rigid analytic variety  $X/C$  and pro-étale  $H$ -torsor  $\tilde{X}/X$ ,  $\kappa_{\tilde{X} \times^{\rho} H} = d\rho \circ \kappa_{\tilde{X}}$ , where  $\mathfrak{h} := \text{Lie}H$  and  $d\rho : \mathfrak{h} \rightarrow \mathfrak{k}$  is the derivative.*

**2.2. Representations of the de Jong fundamental group.** Suppose  $X/C$  is a smooth connected rigid analytic variety and  $x \in X(C)$ . If  $K$  is a  $p$ -adic Lie group and  $\rho : \pi_{1,\text{dJ}}(X, x) \rightarrow K$  is a continuous homomorphism, then the discrete  $\pi_{1,\text{dJ}}(X, x)$ -sets  $K/U$ , as  $U$  varies over open subgroups of  $K$ , give rise to a tower  $\tilde{X}_{\rho} = (X_{K/U, \rho})_U$  of étale covers of  $X$ . Viewed as an object of  $X_{\text{proét}}$ ,  $\tilde{X}_{\rho}$  is a  $K$ -torsor. In this case, we write  $\kappa_{\rho} = \kappa_{\tilde{X}_{\rho}}$  for the associated geometric Sen morphism.

In particular, we can apply this to the  $\rho_n$  of Eq. (1), whose associated torsor is a geometric connected component of the infinite level Lubin-Tate tower.

**Lemma 2.** *For  $\rho_n$  as in Eq. (1), the geometric Sen morphism  $\kappa_{\rho_n}$  is an injection at every geometric point.*

*Proof.* By Lemma 1-(2), it suffices to treat  $\rho_n$  as a map to  $\text{GL}_{n+1}(\mathbb{Q}_p)$ . We note that over  $\mathbb{P}^n$  we have a universal filtration on the trivial  $\text{GL}_{n+1}$ -torsor and thus, via the adjoint representation, a filtration on  $\mathfrak{gl}_{n+1} \otimes_{\underline{\mathbb{Q}_p}} \hat{\mathcal{O}}$ . We also have the Hodge-Tate filtration on  $\tilde{\mathfrak{gl}}_{n+1} \otimes_{\underline{\mathbb{Q}_p}} \hat{\mathcal{O}}$ . As a specific case of the computation given for local Shimura varieties in [3, §4.3],  $\kappa_{\rho_n}$  is then canonically identified with the map

$$T_{\mathbb{P}_C^n} \otimes_{\mathcal{O}} \hat{\mathcal{O}} = \text{gr}_{\text{univ}}^{-1} \left( \mathfrak{gl}_{n+1} \otimes_{\underline{\mathbb{Q}_p}} \hat{\mathcal{O}} \right) = \text{gr}_{\text{HT}}^1 \left( \tilde{\mathfrak{gl}}_{n+1} \otimes_{\underline{\mathbb{Q}_p}} \hat{\mathcal{O}} \right) (-1)$$

where the second equality is the Hodge-Tate comparison and we note that, because the filtrations are minuscule, the last term is equal to

$$\text{Fil}_{\text{HT}}^1 \left( \tilde{\mathfrak{gl}}_{n+1} \otimes_{\underline{\mathbb{Q}_p}} \hat{\mathcal{O}} \right) (-1) \subseteq \tilde{\mathfrak{gl}}_{n+1} \otimes_{\underline{\mathbb{Q}_p}} \hat{\mathcal{O}}(-1).$$

Alternatively, this computation follows from a more general computation relating the geometric Sen and Kodaira-Spencer morphisms for de Rham torsors [5]. Indeed, this  $\text{GL}_{n+1}(\mathbb{Q}_p)$ -torsor is the base change to  $C$  of a de Rham  $\text{GL}_{n+1}(\mathbb{Q}_p)$  torsor on  $\mathbb{P}_{\mathbb{Q}_p}^{n, \text{rig}}$  whose associated filtered  $\text{GL}_{n+1}$ -torsor with integrable connection is, by construction, the trivial torsor with the trivial connection and universal filtration.  $\square$

**2.3. Abelian torsors.** If  $H$  is an abelian  $p$ -adic Lie group with Lie algebra  $\mathfrak{h}$ , then the adjoint action on  $\mathfrak{h}$  is trivial. In particular, if  $X/C$  is a smooth rigid analytic variety and  $\tilde{X}/X$  is an  $H$ -torsor,  $\tilde{\mathfrak{h}}$  is simply the constant local system  $\underline{\mathfrak{h}}$  and

$$(2) \quad \kappa_{\tilde{X}} \in \text{Hom}(T_{X/C} \otimes_{\mathcal{O}} \hat{\mathcal{O}}, \mathfrak{h} \otimes_{\mathbb{Q}_p} \hat{\mathcal{O}}(-1)) = H^0(X, \Omega_{X/C}) \otimes_{\mathbb{Q}_p} \mathfrak{h}(-1).$$

In particular, when  $X$  is an abelian variety, we immediately obtain:

**Lemma 3.** *Suppose  $A/C$  is an abelian variety,  $H$  is an abelian  $p$ -adic Lie group, and  $\tilde{A}/A$  is a pro-étale  $H$ -torsor. Then, the geometric Sen morphism*

$$\kappa_{\tilde{A}} : \text{Lie}A \otimes_C \hat{\mathcal{O}} = T_{A/C} \otimes_{\mathcal{O}} \hat{\mathcal{O}} \rightarrow \mathfrak{h} \otimes_{\mathbb{Q}_p} \hat{\mathcal{O}}(-1)$$

*is constant, i.e., it is induced by the base change from  $C$  to  $\hat{\mathcal{O}}$  of a  $C$ -linear map*

$$\text{Lie}A \rightarrow \mathfrak{h} \otimes_{\mathbb{Q}_p} C(-1).$$

**Remark 3.** The simplification in the case of abelian covers as in Eq. (2) was observed already in [1] where, in particular, we computed the geometric Sen morphism for all profinite étale torsors over abelian varieties (which are necessarily abelian).

### 3. PROOF OF THEOREM 1

We now prove Theorem 1.

*Proof of Theorem 1.* Let  $n = \dim A$ . We may choose an embedding of  $A$  into  $\mathbb{P}_C^m$ ,  $m > n$ , and then project onto a generic  $n$ -plane  $\mathbb{P}_C^n \subseteq \mathbb{P}_C^m$  to obtain a generically étale map  $f : A \rightarrow \mathbb{P}_C^n$ . We claim that the pullback of the  $\text{SL}_{n+1}(\mathbb{Q}_p)$ -torsor associated to  $f$  has non-abelian monodromy, i.e.,  $\rho_n \circ f_*$  has non-abelian image, where  $f_* : \pi_{1,\text{dJ}}(A^{\text{rig}}, 0) \rightarrow \pi_{1,\text{dJ}}(\mathbb{P}_C^{n,\text{rig}}, \pi(0))$  is the map induced by  $f$ .

To see this, first observe that the map  $f$  is proper as  $A/C$  is proper and  $\mathbb{P}_C^n/C$  is separated. Thus  $f$  cannot be étale since  $\mathbb{P}_C^n$  has no non-trivial connected finite étale covers. Thus  $df$  is not an isomorphism, so there exists a nonzero vector  $v \in \text{Lie}A = H^0(A, T_{A/C})$  and a point  $x \in A(C)$  such that  $df_x(v) = 0$ , where  $df_x$  is the derivative at that point, a map of finite dimensional  $C$ -vector spaces  $T_{A/C,x} \rightarrow T_{\mathbb{P}_C^n/C,f(x)}$ . Since  $f$  is generically étale, there also exists a point  $y \in A(C)$  such that  $df_y(v) \neq 0$ .

Now, suppose  $\rho_n \circ \pi_*$  has abelian image, and write  $H \leq \text{SL}_n(\mathbb{Q}_p)$  for the (abelian) closure of its image, a  $p$ -adic Lie group. Write  $\mathfrak{h} := \text{Lie}H$ , and write  $\rho$  for the induced map  $\pi_{1,\text{dJ}}(A^{\text{rig}}, 0) \rightarrow H$ . Then, by Lemma 3,  $\kappa_\rho$  is a constant map. Thus it cannot annihilate the vector  $v \in \text{Lie}A$  at one point but not another. However, for  $\iota : H \rightarrow \text{SL}_{n+1}(\mathbb{Q}_p)$  the inclusion, the functorialities of Lemma 1 give

$$d\iota \circ \kappa_\rho = \kappa_{\rho_n \circ \pi_*} = \pi^* \kappa_{\rho_n} \circ d\pi$$

In particular, since  $d\iota$  and  $\pi^* \kappa_{\rho_n}$  are injective at every geometric point (the latter by Lemma 2), the kernels of  $\kappa_\rho$  and  $d\pi$  agree at every geometric point. Thus we obtain a contradiction since  $v$  is in the kernel of  $d\pi_x$  but not in the kernel of  $d\pi_y$ .  $\square$

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