

# Calderón splitting and weak solutions for Navier–Stokes equations with initial data in weighted $L^p$ spaces.

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## Abstract

We show the existence of global weak solutions of the 3D Navier–Stokes equations with initial velocity in the weighted spaces  $L^p_{\Phi_\gamma} = L^p(\mathbb{R}^3, \Phi_\gamma(x) dx)$ , where  $2 < p < +\infty$ ,  $0 < \gamma < 2$  and  $\Phi_\gamma(x) = \frac{1}{(1+|x|^2)^{\frac{\gamma}{2}}}$ , using Calderón splitting  $L^p_{\Phi_\gamma} \subset L^2_{\Phi_2} + L^r$  (with some  $r \in (3, +\infty)$ ) and energy controls in  $L^2_{\Phi_2}$ .

**Keywords :** Navier–Stokes equations, weighted spaces, weak solutions, energy controls, Calderón’s splitting

**AMS classification :** 35Q30, 76D05.

## Introduction

We consider the Navier–Stokes equations on  $(0, T) \times \mathbb{R}^3$

$$\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u} - \nabla p \\ \operatorname{div} \mathbf{u} = 0 \\ \lim_{t \rightarrow 0} \mathbf{u}(t, \cdot) = \mathbf{u}_0 \end{cases} \quad (1)$$

when the non-linearity  $\mathbf{u} \cdot \nabla \mathbf{u}$  is rewritten as  $\operatorname{div}(\mathbf{u} \otimes \mathbf{u})$  and the pressure  $p$  is eliminated due to the Leray projection operator  $\mathbb{P}$ , rewriting  $\mathbf{u} \cdot \nabla \mathbf{u} + \nabla p$  as  $\mathbb{P} \operatorname{div}(\mathbf{u} \otimes \mathbf{u})$ .

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We thus shall consider distribution  $\mathbf{u}$  such that  $\mathbf{u} \in L^2((0, T), L^2(\mathbb{R}^3, \frac{dx}{(1+|x|^2)^2}))$ , we first prove that  $\mathbb{P} \operatorname{div}(\mathbf{u} \otimes \mathbf{u})$  is well defined as a distribution on  $(0, T) \times \mathbb{R}^3$ ; assuming moreover that  $\mathbf{u}$  satisfies the Navier–Stokes equations

$$\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - \mathbb{P} \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) \\ \operatorname{div} \mathbf{u} = 0 \end{cases} \quad (2)$$

we shall prove that  $\mathbf{u}$  may be defined as a continuous in time distribution  $t \in [0, T] \mapsto \mathbf{u}(t, \cdot) \in \mathcal{S}'(\mathbb{R}^3)$ , (more precisely, that  $t \mapsto \frac{1}{(1+|x|^2)^2} \mathbf{u}(t, \cdot)$  is continuous from  $[0, T]$  to  $H^{-4}(\mathbb{R}^3)$ ), so that the initial value condition

$$\lim_{t \rightarrow 0} \mathbf{u}(t, \cdot) = \mathbf{u}_0 \quad (3)$$

is meaningful. Those results are described in Proposition 1 and Theorem 2.

There are many examples of solutions such that  $\mathbf{u} \in L^2((0, T), L^2(\mathbb{R}^3, \frac{dx}{(1+|x|^2)^2}))$ . A special subclass of such solutions are the solutions such that

$$\sup_{x_0 \in \mathbb{R}^3} \iint_{(0, T) \times B(x_0, 1)} |\mathbf{u}(s, y)|^2 ds dy < +\infty. \quad (4)$$

In this subclass, one can find

- Kato’s mild solutions in Lebesgue spaces  $\mathbf{u} \in L^\infty((0, T), L^p)$  with  $3 \leq p \leq +\infty$  [5] corresponding to the Cauchy initial value problem with  $\mathbf{u}_0 \in L^p$
- more generally, Koch and Tataru’s mild solutions [6] corresponding to the Cauchy initial value problem with  $\mathbf{u}_0 \in \operatorname{bmo}^{-1}$
- Leray’s weak solutions  $\mathbf{u} \in L^\infty((0, T), L^2)$  [9] corresponding to the Cauchy initial value problem with  $\mathbf{u}_0 \in L^2$
- Calderón’s weak solutions  $\mathbf{u} \in L^\infty((0, T), L^2 + L^\infty)$  [2] corresponding to the Cauchy initial value problem with  $\mathbf{u}_0 \in L^p$  with  $p \geq 2$
- Lemarié-Rieusset’s weak solutions  $\mathbf{u} \in L^\infty((0, T), L^2_{\operatorname{uloc}})$  [7] corresponding to the Cauchy initial value problem with  $\mathbf{u}_0 \in L^2_{\operatorname{uloc}}$

More recently, Fernández-Dalgo & Lemarié-Rieusset [3] and Bradshaw, Kucavica & Tsai [1] considered the Cauchy initial value problem with  $\mathbf{u}_0 \in L^2(\mathbb{R}^3, \frac{dx}{1+|x|^2})$ . Their solutions belong to  $L^\infty((0, T), L^2(\frac{dx}{1+|x|^2}))$  and don’t satisfy condition (4).

Other weak solutions which don’t satisfy condition (4) are the statistically homogeneous solutions of Fursikov and Višik [4], which are proved to belong to  $L^\infty((0, T), L^2(\frac{dx}{(1+|x|^2)^{\gamma/2}}))$  for every  $\gamma > 3$ .

Our main goal in this paper is to prove the following result:

**Theorem 1.** *Let  $2 < p < +\infty$  and  $\gamma \in (0, 2)$ . Let  $\mathbf{u}_0 \in L^p(\mathbb{R}^3, \frac{dx}{1+|x|^2})$  with  $\operatorname{div} \mathbf{u}_0 = 0$ . Then the Navier–Stokes equations*

$$\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - \mathbb{P} \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) \\ \operatorname{div} \mathbf{u} = 0 \\ \lim_{t \rightarrow 0} \mathbf{u}(t, \cdot) = \mathbf{u}_0 \end{cases} \quad (5)$$

*have a solution on  $(0, +\infty) \times \mathbb{R}^3$  with, for every  $0 < T < +\infty$ ,  $\mathbf{u} \in L^\infty((0, T), L^2(\frac{dx}{1+|x|^2})) + L^\infty((0, T), L^r)$  for some  $r \in (3, +\infty)$ .*

The limit case  $L^\infty((0, T), L^2(\frac{dx}{1+|x|^2}))$  corresponds to the results of Fernández-Dalgo & Lemarié-Rieusset [3] and Bradshaw, Kucavica & Tsai [1]. The limit case  $L^\infty((0, T), L^r)$  corresponds to Kato’s mild solutions in  $L^r$  [5] (existence of mild solutions is known only for a finite time  $T \approx \frac{1}{\|\mathbf{u}_0\|_r^{\frac{2r}{r-3}}}$ ). In order to deal with the case  $2 < p < +\infty$ , we shall use Calderón’s method [2] and split the initial value in a sum of two vector fields corresponding to the limit cases which we know how to deal with.

## 1 Weighted Lebesgue and Sobolev spaces

Throughout the paper, we shall deal with weights  $\Phi_\gamma(x) = \Phi(x)^\gamma$ ,  $\gamma \in \mathbb{R}$ , where

$$\Phi(x) = \frac{1}{(1 + |x|^2)^{1/2}} \approx \frac{1}{1 + |x|}.$$

We will work in weighted spaces:

- weighted Lebesgue spaces  $L^p(\Phi_\gamma dx)$  ( $1 \leq p \leq +\infty$ ) with

$$\|u\|_{L^p(\Phi_\gamma dx)} = \|\Phi^{\frac{\gamma}{p}} u\|_p.$$

- weighted Sobolev spaces  $H^s(\Phi_\gamma dx)$  ( $s \in \mathbb{R}$ ) with

$$\|u\|_{H^s(\Phi_\gamma dx)} = \|\Phi^{\frac{\gamma}{2}} u\|_{H^s}.$$

$\mathcal{D}(\mathbb{R}^3)$  is dense in  $L^p(\Phi_\gamma dx)$  for  $1 \leq p < +\infty$  and in  $H^s(\Phi_\gamma dx)$ . We have, for  $u, v \in \mathcal{D}$  (and  $p > 1$ )

$$|\int uv dx| \leq \|u\|_{L^p(\Phi_\gamma dx)} \|v\|_{L^{\frac{p}{p-1}}(\Phi_{-\frac{\gamma}{p-1}} dx)}$$

and

$$|\int uv \, dx| \leq \|u\|_{H^s(\Phi_\gamma dx)} \|v\|_{H^{-s}(\Phi_{-\gamma} dx)}.$$

For  $\gamma \leq \delta$  and  $\sigma \leq s$ , we have the following obvious continuous embeddings:

$$L^p(\Phi_\gamma dx) \subset L^p(\Phi_\delta dx) \text{ and } H^s(\Phi_\gamma dx) \subset H^\sigma(\Phi_\gamma dx).$$

Using the Hölder inequality, we find as well the following embedding

$$L^p(\Phi_\gamma dx) \subset L^q(\Phi_\delta dx) \text{ for } q < p \text{ and } \frac{\delta}{q} > \frac{\gamma}{p} + \frac{3(p-q)}{pq}.$$

Using the Sobolev inequalities, we find that, for  $0 \leq s < \frac{3}{2}$ ,  $\frac{1}{q} = \frac{1}{2} - \frac{s}{3}$  and  $\frac{1}{r} = \frac{1}{2} + \frac{s}{3}$ , we have

$$H^s(\Phi_\gamma dx) \subset L^q(\Phi_{\frac{q}{2}\gamma} dx) \text{ and } L^r(\Phi_{\frac{r}{2}\gamma} dx) \subset H^{-s}(\Phi_\gamma dx).$$

Similarly, for  $s > \frac{3}{2}$ , we have

$$L^1(\Phi_{\frac{\gamma}{2}} dx) \subset H^{-s}(\Phi_\gamma dx).$$

We state two further estimates:

**Lemma 1.** *Let  $u \in H^s(\Phi_\gamma dx)$ . Then*

$$\|\nabla u\|_{H^{s-1}(\Phi_\gamma dx)} \leq C\|u\|_{H^s(\Phi_\gamma dx)}.$$

If  $s \geq 0$ ,

$$\|u\|_{H^s(\Phi_\gamma dx)} \approx \|u\|_{L^2(\Phi_\gamma dx)} + \|\nabla u\|_{H^{s-1}(\Phi_\gamma dx)}.$$

*Proof.* We have

$$\|\nabla u\|_{H^{s-1}(\Phi_\gamma dx)} = \|\Phi^{\frac{\gamma}{2}} \nabla u\|_{H^{s-1}} \leq \|\nabla(\Phi^{\frac{\gamma}{2}} u)\|_{H^{s-1}} + \frac{|\gamma|}{2} \|\Phi^{\frac{\gamma}{2}} u \frac{\nabla \Phi}{\Phi}\|_{H^{s-1}}.$$

Since  $\frac{\nabla \Phi}{\Phi}$  is bounded with all its derivatives, we find that

$$\|\Phi^{\frac{\gamma}{2}} u \frac{\nabla \Phi}{\Phi}\|_{H^{s-1}} \leq C\|\Phi^{\frac{\gamma}{2}} u\|_{H^{s-1}} \leq C\|\Phi^{\frac{\gamma}{2}} u\|_{H^s}$$

and we easily conclude since  $\|\nabla(\Phi^{\frac{\gamma}{2}} u)\|_{H^{s-1}} \leq \|\Phi^{\frac{\gamma}{2}} u\|_{H^s}$ .

If  $s \geq 0$ , we have

$$\|u\|_{H^s(\Phi_\gamma dx)} = \|\Phi^{\frac{\gamma}{2}} u\|_{H^s} \approx \|\Phi^{\frac{\gamma}{2}} u\|_2 + \|\nabla(\Phi^{\frac{\gamma}{2}} u)\|_{H^{s-1}}.$$

We have

$$\|\nabla(\Phi^{\frac{\gamma}{2}}u)\|_{H^{s-1}} \leq \|\nabla u\|_{H^{s-1}(\Phi_\gamma dx)} + C\|\Phi^{\frac{\gamma}{2}}u\|_{H^{s-1}}$$

with

$$\|\Phi^{\frac{\gamma}{2}}u\|_{H^{s-1}} \leq \|\Phi^{\frac{\gamma}{2}}u\|_2$$

if  $0 \leq s \leq 1$  and

$$\|\Phi^{\frac{\gamma}{2}}u\|_{H^{s-1}} \leq \|\Phi^{\frac{\gamma}{2}}u\|_2^{\frac{1}{s}} \|\Phi^{\frac{\gamma}{2}}u\|_{H^s}^{1-\frac{1}{s}} \leq \frac{1}{s\epsilon^s} \|\Phi^{\frac{\gamma}{2}}u\|_2 + \epsilon^{\frac{s}{s-1}} (1 - \frac{1}{s}) \|\Phi^{\frac{\gamma}{2}}u\|_{H^s}$$

if  $s > 1$  and  $\epsilon > 0$ .  $\square$

**Lemma 2.** For  $1 \leq p \leq +\infty$  and  $0 < t \leq 1$  we have

$$\left\| \int \frac{1}{(\sqrt{t} + |x - y|)^4} f(y) dy \right\|_{L^p(\Phi_4 dx)} \leq Ct^{-\frac{1}{2}} \|f\|_{L^p(\Phi_4 dx)}.$$

*Proof.* This is obvious for  $p = +\infty$  since  $\int \frac{1}{(\sqrt{t} + |x - y|)^4} dy = Ct^{-\frac{1}{2}}$ .

For  $p = 1$ , we have to prove that

$$\left\| \frac{1}{(1 + |x|)^4} \int \frac{1}{(\sqrt{t} + |x - y|)^4} (1 + |y|)^4 g(y) dy \right\|_1 \leq C \|g\|_1.$$

As  $(1 + |y|)^4 \leq 16((1 + |x|)^4 + |x - y|^4)$ , we have

$$\int \frac{(1 + |y|^4)}{(1 + |x|)^4 (\sqrt{t} + |x - y|)^4} dx \leq 16 \left( \int \frac{dx}{(1 + |x|)^4} + \int \frac{dx}{(\sqrt{t} + |x - y|)^4} \right) \leq Ct^{-\frac{1}{2}}$$

and we conclude by Fubini.

For  $1 < p < +\infty$ , we conclude by interpolation.  $\square$

Remark: we have as well

$$\left\| \int \frac{1}{(\sqrt{t} + |x - y|)^4} f(y) dy \right\|_{L^p(\Phi_\gamma dx)} \leq Ct^{-\frac{1}{2}} \|f\|_{L^p(\Phi_\gamma dx)}$$

for  $0 \leq \gamma \leq 4$ ,  $1 \leq p \leq +\infty$  and  $0 < t \leq 1$  (by interpolation between  $L^p(dx)$  and  $L^p(\Phi_4 dx)$ ).

Our next result deals with the Leray projection operator acting on the divergence of a tensor in  $L^1(\Phi_4 dx)$ :

**Proposition 1** (Leray projection).

Let  $\mathbb{F} \in L^1(\mathbb{R}^3, \Phi_4 dx)$ . Then there exists a unique pair  $(\mathbf{b}_1, \mathbf{b}_2)$  such that

$$\operatorname{div}(\mathbb{F}) = \mathbf{b}_1 + \mathbf{b}_2$$

with

- $\mathbf{b}_1 \in H^\sigma(\Phi_\gamma dx)$  and  $\operatorname{div} \mathbf{b}_1 = 0$
- $\mathbf{b}_2 \in H^\sigma(\Phi_\gamma dx)$  and  $\nabla \wedge \mathbf{b}_2 = 0$
- $\lim_{\tau \rightarrow +\infty} e^{\tau \Delta} \mathbf{b}_2 = 0$  in  $\mathcal{S}'(\mathbb{R}^3)$ .

for  $\gamma > 7$  and  $\sigma < -\frac{5}{2}$ .

$\mathbf{b}_1$  is called the Leray projection of  $\operatorname{div}(\mathbb{F})$  and we write

$$\mathbf{b}_1 = \mathbb{P} \operatorname{div}(\mathbb{F}).$$

*Proof.* Uniqueness is obvious: if  $\operatorname{div} \mathbb{F} = \mathbf{b}_1 + \mathbf{b}_2 = \mathbf{b}'_1 + \mathbf{b}'_2$  and if  $\mathbf{b} = \mathbf{b}_2 - \mathbf{b}'_2$ , then

$$\Delta \mathbf{b} = \nabla \wedge (\nabla \wedge (\mathbf{b}_2 - \mathbf{b}'_2)) - \nabla(\operatorname{div}(\mathbf{b}_1 - \mathbf{b}'_1)) = 0$$

while

$$\mathbf{b} = \lim_{\tau \rightarrow +\infty} - \int_0^\tau e^{s\Delta} \Delta \mathbf{b} ds = 0.$$

We now construct  $\mathbf{b}_2$ . We want to have

$$\Delta \mathbf{b}_2 = \nabla(\operatorname{div} \mathbf{b}_2) = \nabla(\operatorname{div}(\operatorname{div} \mathbb{F})) = \nabla\left(\sum_{i=1}^3 \sum_{j=1}^3 \partial_i \partial_j F_{i,j}\right)$$

and

$$\mathbf{b}_2 = - \int_0^{+\infty} e^{s\Delta} \nabla(\operatorname{div}(\operatorname{div} \mathbb{F})) ds.$$

Let  $\theta \in \mathcal{D}$  such that  $0 \leq \theta \leq 1$ ,  $\theta$  is supported in the ball  $B(0, 2)$  and  $\sum_{k \in \mathbb{Z}^3} \theta(x - k) = 1$ . For  $\sigma \in \mathbb{R}$  and  $\gamma \in \mathbb{R}$ , we write

$$\|\mathbf{b}_2\|_{H^\sigma(\Phi_\gamma dx)} \leq \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} \left\| \frac{1}{(1 + |x|^2)^{\frac{\gamma}{2}}} \theta(x - k) \int_0^{+\infty} e^{s\Delta} \nabla \operatorname{div}(\operatorname{div}(\theta(x - j)\mathbb{F})) ds \right\|_{H^\sigma}.$$

We take  $\gamma > 7$  and  $\sigma < -\frac{5}{2}$ . When  $|j - k| \geq 8$ , we write

$$\begin{aligned} & \left\| \frac{1}{(1 + |x|^2)^{\frac{\gamma}{2}}} \theta(x - k) \int_0^{+\infty} e^{s\Delta} \nabla \operatorname{div}(\operatorname{div}(\theta(\cdot - j)\mathbb{F})) ds \right\|_{H^\sigma} \\ & \leq C \left\| \frac{1}{(1 + |x|^2)^{\frac{\gamma}{2}}} \theta(x - k) \int_0^{+\infty} e^{s\Delta} \nabla \operatorname{div}(\operatorname{div}(\theta(x - j)\mathbb{F})) ds \right\|_1 \\ & \leq C' \left\| \frac{1}{(1 + |x|^2)^2} \theta(x - k) \int \left( \int_0^{+\infty} |(\nabla \otimes \nabla) \nabla(W_s(x - y))| ds \right) \theta(y - j) |\mathbb{F}(y)| dy \right\|_1 \\ & = C'' \left\| \frac{1}{(1 + |x|^2)^2} \theta(x - k) \int \frac{1}{|x - y|^4} \theta(y - j) |\mathbb{F}(y)| ds \right\|_1 \\ & \leq C''' \left\| \frac{1}{(1 + |x|^2)^2} \theta(x - k) \int \frac{1}{(1 + |x - y|^2)^2} \theta(y - j) |\mathbb{F}(y)| ds \right\|_1 \end{aligned}$$

with

$$\begin{aligned}
& \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} \left\| \frac{1}{(1 + |x|^2)^2} \theta(x - k) \int \frac{1}{(1 + |x - y|^2)^2} \theta(y - j) |\mathbb{F}(y)| \, ds \right\|_1 \\
&= \left\| \frac{1}{(1 + |x|^2)^2} \int \frac{1}{(1 + |x - y|^2)^2} |\mathbb{F}(y)| \, ds \right\|_1 \\
&\leq C' \|\mathbb{F}\|_{L^1(\Phi_4 \, dx)}
\end{aligned}$$

(by Lemma 2).

For  $|j - k| < 8$ , we remark that  $\frac{(1+|k|)^\gamma}{(1+|x|^2)^{\frac{\gamma}{2}}} \theta(x - k)$  is smooth and bounded (with all its derivatives) independently from  $k$ , so that

$$\begin{aligned}
& \left\| \frac{1}{(1 + |x|^2)^{\frac{\gamma}{2}}} \theta(x - k) \int_0^{+\infty} e^{s\Delta} \nabla \operatorname{div}(\operatorname{div}(\theta(\cdot - j)\mathbb{F})) \, ds \right\|_{H^\sigma} \\
&\leq C \frac{1}{(1 + |k|)^\gamma} \|\mathbb{P} \operatorname{div}(\theta(\cdot - j)\mathbb{F})\|_{H^\sigma} \\
&\leq C' \frac{1}{(1 + |k|)^\gamma} \|\theta(\cdot - j)\mathbb{F}\|_{H^{\sigma+1}} \\
&\leq C'' \frac{1}{(1 + |k|)^\gamma} \|\theta(\cdot - j)\mathbb{F}\|_1 \\
&\leq C''' \frac{(1 + |j|)^4}{(1 + |k|)^\gamma} \|\mathbb{F}\|_{L^1(\Phi_4 \, dx)}.
\end{aligned}$$

We may conclude, as

$$\sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3, |j-k| < 8} \frac{(1 + |j|)^4}{(1 + |k|)^\gamma} < +\infty.$$

Finally, we study  $e^{\tau\Delta} \mathbf{b}_2$  when  $\tau \rightarrow +\infty$ . We write

$$\begin{aligned}
\|e^{\tau\Delta} \mathbf{b}_2\|_{L^1(\Phi_4 \, dx)} &\leq C \left\| \frac{1}{(1 + |x|^2)^2} \int \left( \int_0^{+\infty} |(\nabla \otimes \nabla) \nabla(W_{s+\tau}(x - y))| \, ds \right) |\mathbb{F}(y)| \, dy \right\|_1 \\
&\leq C'' \left\| \frac{1}{(1 + |x|^2)^2} \int \frac{1}{(\sqrt{\tau} + |x - y|^2)^2} |\mathbb{F}(y)| \, ds \right\|_1
\end{aligned}$$

with, for  $\tau > 1$ ,

$$\frac{1}{(1 + |x|^2)^2} \frac{1}{(\sqrt{\tau} + |x - y|^2)^2} |\mathbb{F}(y)| \leq \frac{1}{(1 + |x|^2)^2} \frac{1}{(1 + |x - y|^2)^2} |\mathbb{F}(y)| \in L^1(\mathbb{R}^2).$$

By dominated convergence, we find that  $\lim_{\tau \rightarrow +\infty} \|e^{\tau\Delta} \mathbf{b}_2\|_{L^1(\Phi_4 \, dx)} = 0$ .  $\square$

## 2 Weak solutions for the Stokes equations in $L^1((0, T), L^1(\mathbb{R}^3, \frac{dx}{(1+|x|)^4}))$

In this section, we consider the Stokes equations

$$\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - \mathbb{P} \operatorname{div}(\mathbb{F}) \\ \operatorname{div} \mathbf{u} = 0 \\ \lim_{t \rightarrow 0} \mathbf{u}(t, \cdot) = \mathbf{u}_0 \end{cases} \quad (6)$$

where the tensor  $\mathbb{F} = (F_{i,j})_{1 \leq i,j \leq 3}$  belongs to  $L^1((0, T), L^1(\mathbb{R}^3, \frac{dx}{(1+|x|)^4}))$ ,  $\operatorname{div}(\mathbb{F}) = \mathbf{b}$  with  $b_j = \sum_{i=1}^3 \partial_i F_{i,j}$  and where the solution  $\mathbf{u}$  belongs to  $L^1((0, T), L^1(\mathbb{R}^3, \frac{dx}{(1+|x|)^4}))$ . [Remark: we don't study the existence of such a solution, we assume in this section that it exists.]

**Theorem 2** (Stokes equations in weighted Lebesgue space).

Let  $0 < T < +\infty$  and  $\mathbb{F} \in L^1((0, T), L^1(\mathbb{R}^3, \frac{dx}{(1+|x|)^4}))$ . Let  $\mathbf{u}$  be a solution of the Stokes equation

$$\partial_t \mathbf{u} = \Delta \mathbf{u} - \mathbb{P} \operatorname{div}(\mathbb{F}) \quad (7)$$

with  $\mathbf{u} \in L^1((0, T), L^1(\mathbb{R}^3, \frac{dx}{(1+|x|)^4}))$ . Then we have

$$\partial_t \mathbf{u} \in L^1((0, T), H^{-4}(\Phi_8 dx)) \text{ and } \mathbf{u} \in \mathcal{C}([0, T], H^{-4}(\Phi_8 dx)).$$

In particular, if  $\varphi \in \mathcal{D}(\mathbb{R})$  with  $\varphi(0) = 1$  and  $\varphi(T) = 0$ , we have

$$\lim_{t \rightarrow 0} \mathbf{u}(t, \cdot) = - \int_{0 < t < T} \partial_t (\varphi \mathbf{u}) dt. \quad (8)$$

Moreover, we have

$$\operatorname{div}(\mathbf{u}(t, \cdot)) = e^{t\Delta} \operatorname{div}(\mathbf{u}(0, \cdot)).$$

*Proof.* As  $L^1(\Phi_4 dx) \subset H^{-2}(\Phi_8 dx)$ , we have  $\mathbf{u} \in L^1((0, T), H^{-2}(\Phi_8 dx))$  and  $\Delta \mathbf{u} \in L^1((0, T), H^{-4}(\Phi_8 dx))$ . From Proposition 1, we know that, for  $\gamma > 7$  and  $\sigma < -5/2$ ,

$$\|\mathbb{P}(\operatorname{div} \mathbb{F})(t, \cdot)\|_{H^\sigma(\Phi_\gamma dx)} \leq C_{\gamma, \sigma} \|\mathbb{F}(t, \cdot)\|_{L^1(\Phi_4 dx)},$$

hence  $\mathbb{P}(\operatorname{div} \mathbb{F}) \in L^1((0, T), H^{-4}(\Phi_8 dx))$ .

From  $\partial_t \mathbf{u} \in L^1((0, T), H^{-4}(\Phi_8 dx))$  and  $\mathbf{u} \in L^1((0, T), H^{-4}(\Phi_8 dx))$ , we conclude that  $\mathbf{u} \in \mathcal{C}([0, T], H^{-4}(\Phi_8 dx))$ . Finally, we write

$$\partial_t \operatorname{div} \mathbf{u} = \operatorname{div} \partial_t \mathbf{u} = \operatorname{div} \Delta \mathbf{u} = \Delta \operatorname{div} \mathbf{u}$$

so that  $\operatorname{div} \mathbf{u}(t, \cdot) = e^{t\Delta}(\operatorname{div} \mathbf{u}(0, \cdot))$ . □



**Theorem 3** (Solutions bounded in weighted Lebesgue space).

Let  $0 < T < +\infty$ ,  $1 < p < +\infty$  and  $\mathbb{F} \in L^\infty((0, T), L^p(\mathbb{R}^3, \frac{dx}{(1+|x|)^4}))$ . Let  $\mathbf{u}$  be a distribution defined on  $(0, T) \times \mathbb{R}^3$ . Then the following assertions are equivalent:

- assertion A1:  $\mathbf{u}$  is a solution of the Stokes equation

$$\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - \mathbb{P} \operatorname{div}(\mathbb{F}) \\ \operatorname{div} \mathbf{u} = 0 \end{cases} \quad (9)$$

with  $\mathbf{u} \in L^\infty((0, T), L^p(\mathbb{R}^3, \frac{dx}{(1+|x|)^4}))$ .

- assertion A2: there exists a tempered distribution  $\mathbf{u}_0$  on  $\mathbb{R}^3$  such that  $\operatorname{div} \mathbf{u}_0 = 0$ ,  $\mathbf{u}_0 \in L^p(\mathbb{R}^3, \frac{dx}{(1+|x|)^4})$  and

$$\mathbf{u} = e^{t\Delta} \mathbf{u}_0 - \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div}(\mathbb{F}) ds.$$

We then have  $\lim_{t \rightarrow 0} \mathbf{u}(t, \cdot) = \mathbf{u}_0$ .

*Proof.* (A1)  $\implies$  (A2): Let  $\mathbf{u}$  be a solution of the Stokes equation

$$\partial_t \mathbf{u} = \Delta \mathbf{u} - \mathbb{P} \operatorname{div}(\mathbb{F}) \quad (10)$$

with  $\mathbf{u} \in L^\infty((0, T), L^p(\mathbb{R}^3, \frac{dx}{(1+|x|)^4}))$ . As  $\int \frac{dx}{(1+|x|)^4} < +\infty$ , we have  $\mathbf{u} \in L^1((0, T), L^1(\mathbb{R}^3, \frac{dx}{(1+|x|)^4}))$  and  $\mathbb{F} \in L^1((0, T), L^1(\mathbb{R}^3, \frac{dx}{(1+|x|)^4}))$ . By Theorem 2, we know that  $\mathbf{u} \in \mathcal{C}([0, T], H^{-4}(\Phi_8 dx))$ . We may then write  $\lim_{t \rightarrow 0} \mathbf{u}(t, \cdot) = \mathbf{u}_0$  as a strong limit in  $H^{-4}(\Phi_8 dx)$  but as well, since  $\mathbf{u}(t, \cdot)$  is bounded in  $L^p(\Phi_4 dx)$ , as a weak-\* limit in  $L^p(\Phi_4 dx)$ .

(A2)  $\implies$  (A1): From classical estimates on Oseen's tensor (see for instance section 4.5 in [8]), we have

$$|e^{(t-s)\Delta} \mathbb{P} \operatorname{div}(\mathbb{F})| \leq C \int \frac{1}{(\sqrt{t-s} + |x-y|)^4} |\mathbb{F}(s, y)| dy$$

and, by Lemma 2, for  $0 < t < T$ ,

$$\begin{aligned} \left\| \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div}(\mathbb{F}) ds \right\|_{L^p(\Phi_4 dx)} &\leq C \int_0^t \max\left(\frac{1}{\sqrt{t-s}}, 1\right) \|\mathbb{F}(s, \cdot)\|_{L^p(\Phi_4 dx)} ds \\ &\leq C' \sqrt{t}(1 + \sqrt{t}) \|\mathbb{F}\|_{L^\infty((0, T), L^p(\Phi_4 dx))}. \end{aligned}$$

Similarly, we write

$$|e^{t\Delta} \mathbf{u}_0(x)| \leq C \int \frac{\sqrt{t}}{(\sqrt{t} + |x-y|)^4} |\mathbf{u}_0(y)| dy$$

and

$$\|e^{t\Delta} \mathbf{u}_0\|_{L^p(\Phi_4 dx)} \leq C \max(1, \sqrt{t}) \|\mathbf{u}_0\|_{L^p(\Phi_4 dx)} \quad \square$$

### 3 Mollified equations

We want to find a weak solution to the Navier–Stokes equations

$$\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - \mathbb{P} \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) \\ \operatorname{div} \mathbf{u} = 0 \\ \lim_{t \rightarrow 0} \mathbf{u}(t, \cdot) = \mathbf{u}_0 \end{cases} \quad (11)$$

globally in time when  $\mathbf{u}_0$  is divergence free and  $\mathbf{u}_0 \in L^p(\frac{dx}{(1+|x|)^\gamma})$  with  $2 < p < +\infty$  and  $0 < \gamma < 2$ .

Following Leray [9], we replace the Navier–Stokes equations (11) with the mollified equations

$$\begin{cases} \partial_t \mathbf{u}_{\epsilon, \alpha} = \Delta \mathbf{u}_{\epsilon, \alpha} - \mathbb{P} \operatorname{div}((\varphi_\epsilon * (\theta_\alpha \mathbf{u}_{\epsilon, \alpha})) \otimes \mathbf{u}_{\epsilon, \alpha}) \\ \operatorname{div} \mathbf{u}_{\epsilon, \alpha} = 0 \\ \lim_{t \rightarrow 0} \mathbf{u}_{\epsilon, \alpha}(t, \cdot) = \mathbf{u}_0 \end{cases} \quad (12)$$

where  $\varphi \in \mathcal{D}(\mathbb{R}^3)$  with  $\varphi \geq 0$ ,  $\int \varphi(x) dx = 1$ ,  $\varphi(x) = 0$  if  $|x| > 1$  and  $\varphi_\epsilon(x) = \frac{1}{\epsilon^3} \varphi(\frac{x}{\epsilon})$  and where  $\theta \in \mathcal{D}(\mathbb{R}^3)$  with  $0 \leq \theta \leq 1$ ,  $\theta(x) = 1$  if  $|x| \leq 1$ ,  $\theta(x) = 0$  if  $|x| \geq 2$  and  $\theta_\alpha(x) = \theta(\alpha x)$ .

We shall prove that equations (12) have a solution

$$\mathbf{u}_{\epsilon, \alpha} \in \cap_{0 < T < +\infty} L^\infty((0, T), L^p(\Phi_\gamma dx)).$$

However, the control of  $\mathbf{u}_{\epsilon, \alpha}$  with respect to  $\epsilon$  and  $\alpha$  is not good enough when  $(\epsilon, \alpha)$  goes to  $(0, 0)$ : we find that

$$\sup_{0 < t < T} \|\mathbf{u}_{\epsilon, \alpha}(t, \cdot)\|_{L^p(\Phi_\gamma dx)} \leq C_{\epsilon, \alpha, T, \mathbf{u}_0} \text{ with } \lim_{\epsilon, \alpha \rightarrow (0, 0)} C_{\epsilon, \alpha, T, \mathbf{u}_0} = +\infty.$$

Instead of  $L^p(\Phi_\gamma dx)$ , we follow Calderón [2] and we shall work in  $L^2(\Phi_2 dx) + L^r$  (with  $3 < r < \infty$ ) and prove that, for every  $T > 0$ , we have

$$\sup_{0 < t < T} \|\mathbf{u}_{\epsilon, \alpha}(t, \cdot)\|_{L^2(\Phi_2 dx) + L^r} \leq C_{T, \mathbf{u}_0} < +\infty \quad (13)$$

and

$$\|\mathbf{u}_{\epsilon, \alpha}(t, \cdot)\|_{L^2((0, T), H^{\frac{1}{2}}(\Phi_4 dx))} \leq C_{T, \mathbf{u}_0} < +\infty. \quad (14)$$

We shall see that estimates (13) and (14) are sufficient to grant some sequence  $\mathbf{u}_{\epsilon_n, \alpha_n}$  is weakly convergent in  $L^2((0, T), \Phi_4 dx)$  to a solution  $\mathbf{u}$  of the Navier–Stokes equations.

### 3.1 First estimates in the norm of $L^p(\Phi_\gamma dx)$ .

In this section,  $2 \leq p < +\infty$  and  $2 \leq \gamma \leq 4$ .

**Lemma 3.** *If  $f \in L^p(\Phi_\gamma dx)$  then*

$$\|\varphi_\epsilon * (\theta_\alpha f)\|_\infty \leq \|\varphi\|_{\frac{p}{p-1}} \frac{1}{\epsilon^{\frac{3}{p}}} (1 + \frac{2}{\alpha})^{\frac{\gamma}{p}} \|f\|_{L^p(\Phi_\gamma dx)}.$$

*Proof.* Just write

$$\|\varphi_\epsilon * (\theta_\alpha f)\|_\infty \leq \|\varphi_\epsilon\|_{\frac{p}{p-1}} \|\theta_\alpha \Phi^{-\frac{\gamma}{p}}\|_\infty \|\Phi^{\frac{\gamma}{p}} f\|_p. \quad \square$$

**Lemma 4.** *If  $\mathbf{u}_0 \in L^p(\Phi_\gamma dx)$ , then  $e^{t\Delta} \mathbf{u}_0 \in \mathcal{C}([0, +\infty[, L^p(\Phi_\gamma dx))$  and*

$$\|e^{t\Delta} \mathbf{u}_0\|_{L^p(\Phi_\gamma dx)} \leq C \max(1, \sqrt{t}) \|\mathbf{u}_0\|_{L^p(\Phi_\gamma dx)}.$$

*Proof.* We write again

$$|e^{t\Delta} \mathbf{u}_0(x)| \leq C \int \frac{\sqrt{t}}{(\sqrt{t} + |x - y|)^4} |\mathbf{u}_0(y)| dy$$

and

$$\|e^{t\Delta} \mathbf{u}_0\|_{L^p(\Phi_\gamma dx)} \leq C \max(1, \sqrt{t}) \|\mathbf{u}_0\|_{L^p(\Phi_\gamma dx)}.$$

Thus, for  $0 < T < +\infty$ , convolution with the heat kernel is a bounded map from  $L^p(\Phi_\gamma dx)$  to  $L^\infty((0, T), L^p(\Phi_\gamma dx))$ .

We then remark that  $L^p(dx)$  is dense in  $L^p(\Phi_\gamma dx)$ , and that  $W^{2,p}$  is dense in  $L^p$ . If  $\mathbf{u}_0 \in W^{2,p}$ , then, for  $0 \leq t \leq \tau$ , we have

$$\|e^{t\Delta} \mathbf{u}_0 - e^{\tau\Delta} \mathbf{u}_0\|_{L^p(\Phi_\gamma dx)} \leq \|e^{t\Delta} \mathbf{u}_0 - e^{\tau\Delta} \mathbf{u}_0\|_p \leq \int_t^\tau \|e^{s\Delta} \Delta \mathbf{u}_0\|_p ds \leq (\tau - t) \|\Delta \mathbf{u}_0\|_p.$$

Thus, convolution with the heat kernel is a bounded map from  $W^{2,p}$  to  $\mathcal{C}([0, T], L^p(\Phi_\gamma dx))$  for the  $W^{2,p}$  norm and from  $W^{2,p}$  to  $L^\infty([0, T], L^p(\Phi_\gamma dx))$  for the  $L^p(\Phi_\gamma dx)$  norm. Thus, it is a bounded map from  $L^p(\Phi_\gamma dx)$  to  $\mathcal{C}([0, T], L^p(\Phi_\gamma dx))$ .  $\square$

**Lemma 5.** *If  $\mathbb{F} \in L^\infty((0, T), L^p(\Phi_\gamma dx))$  (where  $0 < T < +\infty$ ), then*

$$\int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} \mathbb{F} ds \in \mathcal{C}([0, T], L^p(\Phi_\gamma dx))$$

and, for  $0 \leq t \leq T$ ,

$$\left\| \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} \mathbb{F} ds \right\|_{L^p(\Phi_\gamma dx)} \leq C \max(t, \sqrt{t}) \|\mathbb{F}\|_{L^\infty((0, T), L^p(\Phi_\gamma dx))}.$$

*Proof.* We write

$$|\int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} \mathbb{F} ds| \leq C \int_0^t \int \frac{1}{\sqrt{t-s} + |x-y|} |\mathbb{F}(s, y)| dy$$

and

$$\begin{aligned} \|\int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} \mathbb{F} ds\|_{L^p(\Phi_\gamma dx)} &\leq C \int_0^t \max(1, \frac{1}{\sqrt{t-s}}) \|\mathbb{F}(s, \cdot)\|_{L^p(\Phi_\gamma dx)} ds \\ &\leq C'(t + \sqrt{t}) \|\mathbb{F}\|_{L^\infty((0, T), L^p(\Phi_\gamma dx))}. \end{aligned}$$

Thus, for  $0 < T < +\infty$ , the operator  $\mathcal{L}$  defined by

$$\mathbb{F} \mapsto \mathcal{L}(\mathbb{F}) = \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} \mathbb{F} ds$$

is a bounded map from  $L^\infty((0, T), L^p(\Phi_\gamma dx))$  to  $L^\infty((0, T), L^p(\Phi_\gamma dx))$ .

We then remark that  $L^\infty((0, T), L^p(\Phi_\gamma dx))$  is embedded in  $L^4((0, T), L^p(\Phi_\gamma dx))$  and that  $L^\infty((0, T), W^{2,p})$  is dense in  $L^4((0, T), L^p(\Phi_\gamma dx))$ . The operator  $\mathcal{L}$  is a bounded map from  $L^4((0, T), L^p(\Phi_\gamma dx))$  to  $L^\infty((0, T), L^p(\Phi_\gamma dx))$ :

$$\begin{aligned} \|\int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} \mathbb{F} ds\|_{L^p(\Phi_\gamma dx)} &\leq C \int_0^t \max(1, \frac{1}{\sqrt{t-s}}) \|\mathbb{F}(s, \cdot)\|_{L^p(\Phi_\gamma dx)} ds \\ &\leq C \left( \int_0^t \max(1, \frac{1}{\sqrt{t-s}})^{4/3} ds \right)^{3/4} \|\mathbb{F}\|_{L^4((0, T), L^p(\Phi_\gamma dx))} \\ &\leq C' \max(t^{3/4}, t^{1/4}) \|\mathbb{F}\|_{L^4((0, T), L^p(\Phi_\gamma dx))}. \end{aligned}$$

If  $\mathbb{F} \in L^\infty((0, T), W^{2,p})$ , then, for  $0 \leq t \leq T$ , we have

$$\begin{aligned} \|\partial_t \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} \mathbb{F} ds\|_{L^p(\Phi_\gamma dx)} &\leq \|\partial_t \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} \mathbb{F} ds\|_p \\ &= \|\mathbb{P} \operatorname{div} \mathbb{F}(t, \cdot) + \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} \Delta \mathbb{F} ds\|_p \\ &\leq C(\|\mathbb{F}(t, \cdot)\|_{W^{2,p}} + \int_0^t \frac{1}{\sqrt{t-s}} \|\Delta \mathbb{F}(s, \cdot)\|_p ds) \\ &\leq C'(1 + \sqrt{t}) \|\mathbb{F}\|_{L^\infty((0, T), W^{2,p})}. \end{aligned}$$

Thus,  $\mathcal{L}$  is a bounded map from  $L^\infty((0, T), W^{2,p})$  to  $\mathcal{C}([0, T], L^p(\Phi_\gamma dx))$  for the  $L^\infty((0, T), W^{2,p})$  norm and from  $L^\infty((0, T), W^{2,p})$  to  $L^\infty([0, T], L^p(\Phi_\gamma dx))$  for the  $L^4((0, T), L^p(\Phi_\gamma dx))$  norm; we find that  $\mathcal{L}$  is a bounded map from  $L^\infty((0, T), L^p(\Phi_\gamma dx))$  to  $\mathcal{C}([0, T], L^p(\Phi_\gamma dx))$ .  $\square$

**Proposition 2.** *Let  $2 \leq p < +\infty$ ,  $2 \leq \gamma \leq 4$  and  $\mathbf{u}_0 \in L^p(\Phi_\gamma dx)$ . The mollified equations*

$$\begin{cases} \partial_t \mathbf{u}_{\epsilon, \alpha} = \Delta \mathbf{u}_{\epsilon, \alpha} - \mathbb{P} \operatorname{div}((\varphi_\epsilon * (\theta_\alpha \mathbf{u}_{\epsilon, \alpha})) \otimes \mathbf{u}_{\epsilon, \alpha}) \\ \lim_{t \rightarrow 0} \mathbf{u}_{\epsilon, \alpha}(t, \cdot) = \mathbf{u}_0 \end{cases} \quad (15)$$

*have a unique (maximal) solution in  $\mathcal{C}([0, T_{\epsilon, \alpha}[, L^p(\Phi_\gamma dx))$ .*

*If the maximal time of existence  $T_{\epsilon, \alpha}$  is finite, then  $\lim_{t \rightarrow T_{\epsilon, \alpha}} \|\mathbf{u}_\epsilon(t, \cdot)\|_{L^2(\Phi_4 dx)} = +\infty$ .*

*There exists a constant  $C_0 > 0$  such that*

$$T_{\epsilon, \alpha} > \min(1, \frac{\epsilon^3 \alpha^4}{C_0 \|\mathbf{u}_0\|_{L^p(\Phi_\gamma dx)}^2 (2 + \alpha)^4}).$$

*Proof.* . We consider the fixed point problem in  $\mathcal{C}([0, T], L^p(\Phi_\gamma dx))$

$$\mathbf{u} = e^{t\Delta} \mathbf{u}_0 - B_{\epsilon, \alpha}(\mathbf{u}, \mathbf{u}),$$

where

$$B_{\epsilon, \alpha}(\mathbf{v}, \mathbf{w}) = \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div}((\varphi_\epsilon * (\theta_\alpha \mathbf{v})) \otimes \mathbf{w}) ds.$$

. We define

$$R = \sup_{t \in [0, T]} \|e^{t\Delta} \mathbf{u}_0\|_{L^p(\Phi_\gamma dx)}$$

and we want to prove that the map  $\mathbf{u} \mapsto e^{t\Delta} \mathbf{u}_0 - B_{\epsilon, \alpha}(\mathbf{u}, \mathbf{u})$  is a contraction in  $B_R = \{\mathbf{u} \in \mathcal{C}([0, T], L^p(\Phi_\gamma dx)) / \sup_{t \in [0, T]} \|\mathbf{u}(t, \cdot)\|_{L^p(\Phi_\gamma dx)} \leq 2R\}$ .

Fist, we use Lemma 4 and get that  $R \leq C_1 \max(1, \sqrt{T}) \|\mathbf{u}_0\|_{L^p(\Phi_\gamma dx)}$ .

As  $L^p(\Phi_\gamma dx) \subset L^4(\Phi_2 dx)$ , we can use Lemmas 3 and 5 and find, for  $\mathbf{v}, \mathbf{w}$  in  $\mathcal{C}([0, T], L^p(\Phi_\gamma dx))$ ,

$$\|\varphi_\epsilon * (\theta_\alpha \mathbf{v}(t, \cdot))\|_\infty \leq \|\varphi\|_2 \frac{1}{\epsilon^{\frac{3}{2}}} (\frac{2 + \alpha}{\alpha})^2 \|\mathbf{v}(t, \cdot)\|_{L^p(\Phi_\gamma dx)}$$

and, for  $0 < t < T$ ,

$$\begin{aligned} & \|B_\epsilon(\mathbf{v}, \mathbf{w})(t, \cdot)\|_{L^p(\Phi_\gamma dx)} \\ & \leq C_2 (T + \sqrt{T}) \frac{1}{\epsilon^{\frac{7}{2}}} (2 + \epsilon)^2 \sup_{t \in [0, T]} \|\mathbf{v}(t, \cdot)\|_{L^p(\Phi_\gamma dx)} \sup_{t \in [0, T]} \|\mathbf{w}(t, \cdot)\|_{L^p(\Phi_\gamma dx)}. \end{aligned}$$

Thus, we will have a contraction in  $B_R$  if

$$4C_2 (T + \sqrt{T}) \frac{1}{\epsilon^{\frac{3}{2}}} (\frac{2 + \alpha}{\alpha})^2 R < 1,$$

in particular if

$$4C_1C_2\sqrt{T}(1+\sqrt{T})^2\frac{1}{\epsilon^{\frac{3}{2}}}(\frac{2+\alpha}{\alpha})^2\|\mathbf{u}_0\|_{L^p(\Phi_\gamma dx)} < 1.$$

This proves that that

$$T_{\epsilon,\alpha} > \min(1, \frac{\epsilon^3\alpha^4}{C_0\|\mathbf{u}_0\|_{L^p(\Phi_\gamma dx)}^2(2+\alpha)^4}).$$

If  $T_{\epsilon,\alpha}$  is finite and  $T < T_{\epsilon,\alpha}$ , considering the initial value problem at initial time  $T$ , we find

$$T_{\epsilon,\alpha} - T > \min(1, \frac{\epsilon^3\alpha^4}{C_0\|\mathbf{u}_{\epsilon,\alpha}(T, \cdot)\|_{L^p(\Phi_\gamma dx)}^2(2+\alpha)^4}).$$

Thus,  $\lim_{T \rightarrow T_{\epsilon,\alpha}} \|\mathbf{u}_{\epsilon,\alpha}(T, \cdot)\|_{L^p(\Phi_\gamma dx)} = +\infty$ . We prove as well that

$$\lim_{T \rightarrow T_{\epsilon,\alpha}} \|\mathbf{u}_{\epsilon,\alpha}(T, \cdot)\|_{L^4(\Phi_2 dx)} = +\infty.$$

If this was not the case, then the maximal existence time  $\tilde{T}_{\epsilon,\alpha}$  for  $\mathbf{u}_{\epsilon,\alpha}$  in  $\mathcal{C}([0, \tilde{T}_{\epsilon,\alpha}), L^2(\Phi_4 dx))$  would satisfy  $\tilde{T}_{\epsilon,\alpha} > T_{\epsilon,\alpha}$  and thus  $\mathbf{u}_{\epsilon,\alpha}$  would be bounded in  $L^2(\Phi_4 dx)$  on  $[0, T_{\epsilon,\alpha})$ . For  $0 < T < t < T_{\epsilon,\alpha}$ , we would have

$$\begin{aligned} & \|\mathbf{u}_{\epsilon,\alpha}(t, \cdot)\|_{L^p(\Phi_\gamma dx)} \\ & \leq \|e^{(t-T)\Delta}\mathbf{u}_{\epsilon,\alpha}(T, \cdot)\|_{L^p(\Phi_\gamma dx)} + \left\| \int_T^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div}((\varphi_\epsilon * (\theta_\alpha \mathbf{u}_{\epsilon,\alpha})) \otimes \mathbf{u}_{\epsilon,\alpha}) ds \right\|_{L^p(\Phi_\gamma dx)} \\ & \leq C \max(1, \sqrt{T_{\epsilon,\alpha} - T}) \|\mathbf{u}_{\epsilon,\alpha}(T, \cdot)\|_{L^p(\Phi_\gamma dx)} \\ & + C_2(T_{\epsilon,\alpha} - T + \sqrt{T_{\epsilon,\alpha} - T}) \frac{1}{\epsilon^{\frac{3}{2}}} (\frac{2+\alpha}{\alpha})^2 \sup_{s \in [0, T_{\epsilon,\alpha}]} \|\mathbf{u}_{\epsilon,\alpha}(s, \cdot)\|_{L^2(\Phi_4 dx)} \sup_{s \in [T, t]} \|\mathbf{u}_{\epsilon,\alpha}(s, \cdot)\|_{L^p(\Phi_\gamma dx)}. \end{aligned}$$

If  $T$  is close enough to  $T_{\epsilon,\alpha}$ , so that

$$C_2(T_{\epsilon,\alpha} - T + \sqrt{T_{\epsilon,\alpha} - T}) \frac{1}{\epsilon^{\frac{3}{2}}} (\frac{2+\alpha}{\alpha})^2 \sup_{s \in [0, T_{\epsilon,\alpha}]} \|\mathbf{u}_{\epsilon,\alpha}(s, \cdot)\|_{L^2(\Phi_4 dx)} < \frac{1}{2},$$

we would get

$$\sup_{T \leq t < T_{\epsilon,\alpha}} \|\mathbf{u}_{\epsilon,\alpha}(t, \cdot)\|_{L^p(\Phi_\gamma dx)} \leq 2C \max(1, \sqrt{T_{\epsilon,\alpha} - T}) \|\mathbf{u}_{\epsilon,\alpha}(T, \cdot)\|_{L^p(\Phi_\gamma dx)}$$

in contradiction with  $\lim_{t \rightarrow T_{\epsilon,\alpha}} \|\mathbf{u}_{\epsilon,\alpha}(t, \cdot)\|_{L^p(\Phi_\gamma dx)} = +\infty$ .  $\square$

### 3.2 Calderón's splitting.

Now, for  $2 < p < +\infty$  and  $0 < \gamma < 2$ , we want to study the mollified equations when the initial data  $\mathbf{u}_0$  belongs to  $L^p(\Phi_\gamma dx) \subset L^2(\Phi_4 dx)$  with  $\operatorname{div} \mathbf{u}_0 = 0$ . Following Calderón [2], we will split the solution  $\mathbf{u}_{\epsilon,\alpha}$  as a sum  $\mathbf{u}_{\epsilon,\alpha} = \mathbf{v}_{\eta,\epsilon,\alpha} + \mathbf{b}_{\eta,\epsilon,\alpha} \in L^\infty((0, T_{(\eta)}), L^2(\Phi_2 dx)) + L^\infty((0, T_{(\eta)}), L^r) \subset L^\infty((0, T_{(\eta)}), L^2(\Phi_4 dx))$  for some  $r \in (3, +\infty)$ . The aim is to get a minoration of the existence time  $T_{(\eta)}$  independent of  $\epsilon$  and  $\alpha$ .

**Lemma 6.** *Let  $2 < p < +\infty$  and  $0 < \gamma < 2$ . Let  $r_0 = 2\frac{p-\gamma}{2-\gamma}$  and  $\max(r_0, 3) < r < +\infty$ . Let  $\mathbf{u}_0 \in L^p(\mathbb{R}^3, \frac{dx}{(1+|x|^2)^2})$  with  $\operatorname{div} \mathbf{u}_0 = 0$ . Then, for every  $\eta > 0$  there exists  $\mathbf{v}_{0,\eta}$  and  $\mathbf{b}_{0,\eta}$  such that*

$$\mathbf{u}_0 = \mathbf{v}_{0,\eta} + \mathbf{b}_{0,\eta}$$

with

$$\mathbf{v}_{0,\eta} \in L^2(\mathbb{R}^3, \frac{dx}{(1+|x|^2)}), \operatorname{div} \mathbf{v}_{0,\eta} = 0$$

and

$$\mathbf{b}_{0,\eta} \in L^r(\mathbb{R}^3), \operatorname{div} \mathbf{b}_{0,\eta} = 0, \|\mathbf{b}_{0,\eta}\|_r < \eta.$$

*Proof.* We have the interpolation result  $L^p(\Phi_\gamma dx) = [L^2(\Phi_2 dx), L^{r_0}]_{[1-\frac{\gamma}{p}]}$ , so that  $L^p(\Phi_\gamma dx) \subset L^2(\Phi_2 dx) + L^{r_0}$ . Moreover,

$$L^{r_0} = [L^2, L^r]_{[\frac{\frac{1}{2}-\frac{1}{r_0}}{\frac{1}{2}-\frac{1}{r}}]} \subset [L^2, L^r]_{[\frac{\frac{1}{2}-\frac{1}{r_0}}{\frac{1}{2}-\frac{1}{r}}, \infty]}.$$

Let  $\delta = \frac{\frac{1}{2}-\frac{1}{r_0}}{\frac{1}{2}-\frac{1}{r}}$ . We may split  $\mathbf{u}_0$  in  $\mathbf{u}_0 = \mathbf{u}_1 + \mathbf{u}_2$  with  $\mathbf{u}_1 \in L^2(\Phi_2 dx)$  and  $\mathbf{u}_2 \in L^{r_0}$  and, for every  $A > 0$ , we may split  $\mathbf{u}_2$  into  $\mathbf{v}_A + \mathbf{b}_A$  with

$$\|\mathbf{v}_A\|_{L^2(\Phi_2 dx)} \leq CA^\delta \|\mathbf{u}_2\|_{r_0} \text{ and } \|\mathbf{b}_A\|_r \leq CA^{\delta-1} \|\mathbf{u}_2\|_{r_0}.$$

Moreover, as  $\Phi_\gamma$  is a Muckenhoupt weight in the class  $\mathcal{A}_p$  and  $\Phi_2$  is a Muckenhoupt weight in the class  $\mathcal{A}_2$ , we may apply the Leray projection operator and write

$$\mathbf{u}_0 = \mathbb{P}(\mathbf{u}_0) = \mathbb{P}(\mathbf{u}_1 + \mathbf{v}_A) + \mathbb{P}(\mathbf{b}_A)$$

with

$$\mathbb{P}(\mathbf{u}_1 + \mathbf{v}_A) \in L^2(\mathbb{R}^3, \frac{dx}{(1+|x|^2)}), \operatorname{div} \mathbb{P}(\mathbf{u}_1 + \mathbf{v}_A) = 0$$

and

$$\mathbb{P}(\mathbf{b}_A) \in L^r(\mathbb{R}^3), \operatorname{div} \mathbb{P}(\mathbf{b}_A) = 0, \|\mathbb{P}(\mathbf{b}_A)\|_r < CA^{\delta-1} \|\mathbf{u}_2\|_{r_0}.$$

We conclude by taking  $A$  large enough. □

For some  $T$ , we then split the solution  $\mathbf{u}_{\epsilon,\alpha}$  (defined in  $\mathcal{C}([0, T_{\epsilon,\alpha}], L^2(\Phi_4 dx))$ ) in  $\mathbf{u}_{\epsilon,\alpha} = \mathbf{v}_{\eta,\epsilon,\alpha} + \mathbf{b}_{\eta,\epsilon,\alpha}$  where  $\mathbf{b}_{\eta,\epsilon,\alpha}$  is a solution in  $L^\infty((0, T), L^r)$  of

$$\mathbf{b}_{\eta,\epsilon,\alpha} = e^{t\Delta} \mathbf{b}_{0,\eta} - B_{\epsilon,\alpha}(\mathbf{b}_{\eta,\epsilon,\alpha}, \mathbf{b}_{\eta,\epsilon,\alpha})$$

and  $\mathbf{v}_{\eta,\epsilon,\alpha}$  is a solution in  $\bigcap_{S < T_{\epsilon,\alpha}} L^\infty((0, \min(S, T), L^2(\Phi_4 dx))$  of

$$\mathbf{v}_{\eta,\epsilon,\alpha} = e^{t\Delta} \mathbf{v}_{0,\eta} - B_{\epsilon,\alpha}(\mathbf{b}_{\eta,\epsilon,\alpha}, \mathbf{v}_{\eta,\epsilon,\alpha}) - B_{\epsilon,\alpha}(\mathbf{v}_{\eta,\epsilon,\alpha}, \mathbf{b}_{\eta,\epsilon,\alpha}) - B_{\epsilon,\alpha}(\mathbf{v}_{\eta,\epsilon,\alpha}, \mathbf{v}_{\eta,\epsilon,\alpha}).$$

### 3.3 Local estimates in the norm of $L^r$ .

We write  $W_t$  for the heat kernel  $W_t(x) = \frac{1}{(4\pi t)^{\frac{3}{2}}} e^{-\frac{|x|^2}{4t}}$ .

**Proposition 3.** *Let  $r > 3$ . There exists a constant  $C_1 > 0$  such that, for every  $\epsilon > 0$ ,  $\alpha > 0$ ,  $\eta > 0$  and every  $\mathbf{b}_{0,\eta} \in L^r$  with  $\|\mathbf{b}_{0,\eta}\|_r \leq \eta$ , the mollified equations*

$$\begin{cases} \partial_t \mathbf{b}_{\eta,\epsilon,\alpha} = \Delta \mathbf{b}_{\eta,\epsilon,\alpha} - \mathbb{P} \operatorname{div}((\varphi_\epsilon * (\theta_\alpha \mathbf{b}_{\eta,\epsilon,\alpha})) \otimes \mathbf{b}_{\eta,\epsilon,\alpha}) \\ \lim_{t \rightarrow 0} \mathbf{b}_{\eta,\epsilon,\alpha}(t, \cdot) = \mathbf{b}_{0,\eta} \end{cases} \quad (16)$$

have a unique solution on  $(0, T_{[\eta]}) \times \mathbb{R}^3$  (with  $T_{[\eta]}^{\frac{1}{2} - \frac{3}{2r}} = \frac{1}{C_1 \eta}$ ) such that

- $\mathbf{b}_{\eta,\epsilon,\alpha} \in \mathcal{C}([0, T_{[\eta]}], L^2(\Phi_4 dx))$ ,
- $\sup_{0 \leq t \leq T_{[\eta]}} \|\mathbf{b}_{\eta,\epsilon,\alpha}(t, \cdot)\|_r \leq 2\eta$ ,
- $\sup_{0 \leq t \leq T_{[\eta]}} t^{\frac{1}{2}} \|\nabla \otimes \mathbf{b}_{\eta,\epsilon,\alpha}(t, \cdot)\|_r \leq 2\|\nabla W_1\|_1 \eta$ ,
- $\sup_{0 < t \leq T_{[\eta]}} t^{\frac{3}{2r}} \|\mathbf{b}_{\eta,\epsilon,\alpha}(t, \cdot)\|_\infty \leq 2\|W_1\|_{\frac{r}{r-1}} \eta$ .

*Proof.* First, we remark that  $L^r(\mathbb{R}^3) \subset L^2(\Phi_4 dx)$ . We have the obvious results for the heat kernel operating on  $L^\infty$ :

$$\|e^{t\Delta} f\|_{L^r(dx)} = \|W_t * f\|_r \leq \|W_t\|_1 \|f\|_r = \|f\|_r,$$

$$\|\nabla e^{t\Delta} f\|_{L^r(dx)} = \|\nabla W_t * f\|_r \leq \|\nabla W_t\|_1 \|f\|_r = \frac{1}{\sqrt{t}} \|\nabla W_1\|_1 \|f\|_r$$

and

$$\|e^{t\Delta} f\|_{L^\infty(dx)} = \|W_t * f\|_\infty \leq \|W_t\|_{\frac{r}{r-1}} \|f\|_r = \frac{1}{t^{\frac{3}{2r}}} \|W_1\|_{\frac{r}{r-1}} \|f\|_r.$$



We consider the fixed point problem

$$\mathbf{b} = e^{t\Delta} \mathbf{b}_{0,\eta} - B_{\epsilon,\alpha}(\mathbf{b}, \mathbf{b}).$$

We want to prove that the map  $\mathbf{b} \mapsto e^{t\Delta} \mathbf{b}_{0,\eta} - B_{\epsilon,\alpha}(\mathbf{b}, \mathbf{b})$  is a contraction in

$$\begin{aligned} B_\eta = \{ \mathbf{b} \in \mathcal{C}([0, T], L^2(\Phi_4 dx)) \mid & \sup_{t \in [0, T]} \|\mathbf{b}(t, \cdot)\|_r \leq 2\eta, \\ & \sup_{0 < t \leq T} \sqrt{t} \|\nabla \otimes \mathbf{b}(t, \cdot)\|_r \leq 2\|W_1\|_1 \eta, \\ & \sup_{0 < t \leq T} t^{\frac{3}{2r}} \|\mathbf{b}(t, \cdot)\|_\infty \leq 2\|W_1\|_{\frac{r}{r-1}} \eta \}. \end{aligned}$$

Let  $\mathbf{v}, \mathbf{w} \in \mathcal{C}([0, T], L^2(\Phi_4 dx))$  with

- $\sup_{0 < t < T} \|\mathbf{v}(t, \cdot)\|_r < +\infty$ ,
- $\sup_{0 < t < T} \|\mathbf{w}(t, \cdot)\|_r < +\infty$ ,
- $\sup_{0 < t < T} t^{\frac{3}{2r}} \|\mathbf{v}(t, \cdot)\|_\infty < +\infty$ ,
- $\sup_{0 < t < T} t^{\frac{3}{2r}} \|\mathbf{w}(t, \cdot)\|_\infty < +\infty$ ,
- $\sup_{0 < t < T} \sqrt{t} \|\nabla \otimes \mathbf{v}(t, \cdot)\|_r < +\infty$ ,
- $\sup_{0 < t < T} \sqrt{t} \|\nabla \otimes \mathbf{w}(t, \cdot)\|_r < +\infty$ .

We have the inequalities

$$\begin{aligned} \|B_{\epsilon,\alpha}(\mathbf{v}, \mathbf{w})\|_r &\leq C \int_0^t \frac{1}{\sqrt{t-s}} \|\varphi_\epsilon * (\theta_\alpha \mathbf{v}(s, \cdot))\|_r \|\mathbf{w}(s, \cdot)\|_\infty ds \\ &\leq C \int_0^t \frac{1}{\sqrt{t-s}} \frac{1}{s^{\frac{3}{2r}}} ds \sup_{0 < t < T} \|\mathbf{v}(s, \cdot)\|_r \sup_{0 < s < T} s^{\frac{3}{2r}} \|\mathbf{w}(s, \cdot)\|_\infty \\ &\leq C' t^{\frac{1}{2}(1-\frac{3}{r})} \sup_{0 < t < T} \|\mathbf{v}(s, \cdot)\|_r \sup_{0 < s < T} s^{\frac{3}{2r}} \|\mathbf{w}(s, \cdot)\|_\infty, \end{aligned}$$

$$\begin{aligned} \|B_{\epsilon,\alpha}(\mathbf{v}, \mathbf{w})\|_\infty &\leq C \int_0^t \frac{1}{\sqrt{t-s}} \|\varphi_\epsilon * (\theta_\alpha \mathbf{v}(s, \cdot))\|_\infty \|\mathbf{w}(s, \cdot)\|_\infty ds \\ &\leq C \int_0^t \frac{1}{\sqrt{t-s}} \frac{1}{s^{\frac{3}{r}}} ds \sup_{0 < t < T} s^{\frac{3}{2r}} \|\mathbf{v}(s, \cdot)\|_\infty \sup_{0 < s < T} s^{\frac{3}{2r}} \|\mathbf{w}(s, \cdot)\|_\infty \\ &\leq C' t^{-\frac{3}{2r}} t^{\frac{1}{2}(1-\frac{3}{r})} \sup_{0 < t < T} s^{\frac{3}{2r}} \|\mathbf{v}(s, \cdot)\|_\infty \sup_{0 < s < T} s^{\frac{3}{2r}} \|\mathbf{w}(s, \cdot)\|_\infty, \end{aligned}$$

and, for  $1 \leq j \leq 3$ , remarking that  $\|\partial_j \theta_\alpha\|_3 = \|\partial_j \theta\|_3$ ,

$$\begin{aligned}
\|\partial_j B_{\epsilon, \alpha}(\mathbf{v}, \mathbf{w})\|_r &\leq C \int_0^t \frac{1}{\sqrt{t-s}} \|\varphi_\epsilon * (\theta_\alpha \mathbf{v}(s, \cdot))\|_\infty \|\partial_j \mathbf{w}(s, \cdot)\|_r ds \\
&\quad + C \int_0^t \frac{1}{\sqrt{t-s}} \|\varphi_\epsilon * (\theta_\alpha \partial_j \mathbf{v}(s, \cdot))\|_r \|\mathbf{w}(s, \cdot)\|_\infty ds \\
&\quad + C \int_0^t \frac{1}{(t-s)^{1-\frac{3}{2r}}} \|\varphi_\epsilon * (\partial_j \theta_\alpha \mathbf{v}(s, \cdot))\|_3 \|\mathbf{w}(s, \cdot)\|_\infty ds \\
&\leq C' t^{-\frac{3}{2r}} \left( \sup_{0 < t < T} s^{\frac{3}{2r}} \|\mathbf{v}(s, \cdot)\|_\infty + \sup_{0 < s < T} \sqrt{s} \|\nabla \otimes \mathbf{v}(s, \cdot)\|_r \right) \\
&\quad \left( \sup_{0 < s < T} s^{\frac{3}{2r}} \|\mathbf{w}(s, \cdot)\|_\infty + \sup_{0 < s < T} \sqrt{s} \|\nabla \otimes \mathbf{w}(s, \cdot)\|_r \right).
\end{aligned}$$

For the norm

$$\|\mathbf{v}\|_* = \max \left( \sup_{0 < s < T} \|\mathbf{v}(s, \cdot)\|_r, \sup_{0 < s < T} \sqrt{s} \|\nabla \otimes \mathbf{v}(s, \cdot)\|_r, \sup_{0 < s < T} s^{\frac{3}{2r}} \|\mathbf{v}(s, \cdot)\|_\infty \right),$$

we find

$$\|B_{\epsilon, \alpha}(\mathbf{v}, \mathbf{w})\|_* \leq C T^{\frac{1}{2} - \frac{3}{2r}} \|\mathbf{v}\|_* \|\mathbf{w}\|_*$$

(where  $C$  doesn't depend on  $\epsilon$  nor on  $\alpha$ ), which proves that  $B_{\epsilon, \alpha}$  is a contraction on  $B_\eta$  for  $T^{\frac{1}{2} - \frac{3}{2r}} \eta$  small enough.  $\square$

### 3.4 Local estimates in the norm of $L^2(\Phi_2 dx)$ .

We now want to estimate  $\mathbf{v}_{\eta, \epsilon, \alpha}(t, \cdot)$  in  $L^2(\Phi_2 dx)$  for  $0 \leq t \leq T < \min(T_{\epsilon, \alpha}, T_{[\eta]})$ . We have

$$\mathbf{v}_{\eta, \epsilon, \alpha}(t, \cdot) = e^{t\Delta} \mathbf{v}_{0, \eta} - \mathcal{L}(\mathbb{F}_{\eta, \epsilon, \alpha})$$

where we defined the operator  $\mathcal{L}$  by

$$\mathbb{F} \mapsto \mathcal{L}(\mathbb{F}) = \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} \mathbb{F} ds$$

and where

$$\mathbb{F}_{\eta, \epsilon, \alpha} = (\varphi_\epsilon * (\theta_\alpha \mathbf{u}_{\epsilon, \alpha})) \otimes \mathbf{u}_{\epsilon, \alpha} - (\varphi_\epsilon * (\theta_\alpha \mathbf{b}_{\eta, \epsilon, \alpha})) \otimes \mathbf{b}_{\eta, \epsilon, \alpha}.$$

**Lemma 7.**  $\mathbf{v}_{\eta, \epsilon, \alpha} \in \mathcal{C}([0, T], L^2(\Phi_2 dx)) \cap L^2((0, T), H^1(\Phi_2 dx))$  and  $\partial_t \mathbf{v}_{\eta, \epsilon, \alpha} \in L^2((0, T), H^{-1}(\Phi_2 dx))$ . In particular, we have

$$\begin{aligned}
&\int |\mathbf{v}_{\eta, \epsilon, \alpha}(t, x)|^2 \Phi_2(x) dx - \int |\mathbf{v}_{0, \eta}(x)|^2 \Phi_2(x) dx \\
&\quad = 2 \int_0^t \int \mathbf{v}_{\eta, \epsilon, \alpha}(s, x) \cdot \partial_t \mathbf{v}_{\eta, \epsilon, \alpha}(s, x) \Phi_2(x) ds dx \\
&\quad = 2 \int_0^t \int \mathbf{v}_{\eta, \epsilon, \alpha}(s, x) \cdot (\Delta \mathbf{v}_{\eta, \epsilon, \alpha}(s, x) - \mathbb{P} \operatorname{div} \mathbb{F}_{\eta, \epsilon, \alpha}(s, x)) \Phi_2(x) ds dx.
\end{aligned}$$

*Proof.* First, we recall that  $\|f\|_{H^s(\Phi_2 dx)} = \|\Phi f\|_{H^s}$ . As  $\Phi$  is bounded with all its derivatives, we have  $\|\Phi f\|_{H^s} \leq C_s \|f\|_{H^s}$  for every  $s \in \mathbb{R}$ . As  $\Phi_2$  is a weight in the Muckenhoupt class  $\mathcal{A}_2$  we have

$$\|e^{t\Delta} f\|_{L^2(\Phi_2 dx)} \leq \|\mathcal{M}_f\|_{L^2(\Phi_2 dx)} \leq C \|f\|_{L^2(\Phi_2 dx)}$$

where  $\mathcal{M}_f$  is the Hardy–Littlewood maximal function of  $f$ .

If  $\mathbf{w}_0 \in L^2$ , we have  $e^{t\Delta} \mathbf{w}_0 \in \mathcal{C}([0, T], L^2) \cap L^2((0, T), H^1)$  and  $\partial_t(e^{t\Delta} \mathbf{w}_0) \in L^2((0, T), H^{-1})$ ; in particular,  $\Phi e^{t\Delta} \mathbf{w}_0 \in L^2((0, T), H^1)$  and  $\partial_t(\Phi e^{t\Delta} \mathbf{w}_0) \in L^2((0, T), H^{-1})$ . We thus have, for  $0 < t < T$ ,

$$\begin{aligned} \|\Phi e^{t\Delta} \mathbf{w}_0\|_2^2 - \|\Phi \mathbf{w}_0\|_2^2 &= 2 \int_0^t \int (\Phi e^{s\Delta} \mathbf{w}_0)(s, x) \cdot \partial_t(\Phi e^{s\Delta} \mathbf{w}_0)(s, x) ds dx \\ &= 2 \int_0^t \int \Phi(x)^2 e^{s\Delta} \mathbf{w}_0(s, x) \cdot \Delta e^{s\Delta} \mathbf{w}_0(s, x) ds dx \\ &= \int_0^t \int \Phi(x)^2 (\Delta |e^{s\Delta} \mathbf{w}_0(s, x)|^2 - 2|\nabla e^{s\Delta} \mathbf{w}_0(s, x)|^2) ds dx. \end{aligned}$$

so that, as  $|\Delta \Phi_2| \leq C \Phi_2$ ,

$$\begin{aligned} 2 \int_0^t \|\nabla e^{s\Delta} \mathbf{w}_0(s, \cdot)\|_{L^2(\Phi_2 dx)}^2 ds \\ = -\|\Phi e^{t\Delta} \mathbf{w}_0\|_2^2 + \|\Phi \mathbf{w}_0\|_2^2 + \int_0^t \int |e^{s\Delta} \mathbf{w}_0(s, x)|^2 \Delta \Phi_2(x) ds dx \\ \leq C \|\mathbf{w}_0\|_{L^2(\Phi_2 dx)}^2 (1 + t). \end{aligned}$$

By density of  $L^2$  in  $L^2(\Phi_2 dx)$ , we find that  $\mathbf{w}_0 \mapsto e^{t\Delta} \mathbf{w}_0$  is a bounded map from  $L^2(\Phi_2 dx)$  to  $\mathcal{C}([0, T], L^2(\Phi_2 dx)) \cap L^2((0, T), H^1(\Phi_2 dx))$ . Moreover, since  $\partial_t e^{t\Delta} \mathbf{w}_0 = \Delta e^{t\Delta} \mathbf{w}_0$ , we have that  $\partial_t e^{t\Delta} \mathbf{w}_0 \in L^2((0, T), H^{-1}(\Phi_2 dx))$ .

A classical result on the heat kernel states that, if  $\mathbb{F} \in L^2((0, T), L^2)$ , then  $\int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} \mathbb{F} \in \mathcal{C}([0, T], L^2) \cap L^2((0, T), H^1)$  and thus  $\int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} \mathbb{F} \in \mathcal{C}([0, T], L^2(\Phi_2 dx)) \cap L^2((0, T), H^1(\Phi_2 dx))$ .

As we have

$$\begin{aligned} \|\mathbb{F}_{\eta, \epsilon, \alpha}(t, \cdot)\|_2 &\leq \|\varphi_\epsilon * (\theta_\alpha \mathbf{u}_{\epsilon, \alpha})\|_\infty \left( \int_{|x| \leq \frac{2}{\alpha} + \epsilon} |\mathbf{u}_{\epsilon, \alpha}(t, x)|^2 dx \right)^{\frac{1}{2}} \\ &\quad + \|\varphi_\epsilon * (\theta_\alpha \mathbf{b}_{\epsilon, \alpha})\|_\infty \left( \int_{|x| \leq \frac{2}{\alpha} + \epsilon} |\mathbf{b}_{\epsilon, \alpha}(t, x)|^2 dx \right)^{\frac{1}{2}}, \end{aligned}$$

we find that

$$\mathcal{L}(\mathbb{F}_{\eta, \epsilon, \alpha}) \in \mathcal{C}([0, T], L^2) \cap L^2((0, T), H^1) \subset \mathcal{C}([0, T], L^2(\Phi_2 dx)) \cap L^2((0, T), H^1(\Phi_2 dx)).$$

Moreover,  $\partial_t \mathcal{L}(\mathbb{F}_{\eta, \epsilon, \alpha}) = \Delta \mathcal{L}(\mathbb{F}_{\eta, \epsilon, \alpha}) - \mathbb{P} \operatorname{div} \mathbb{F}_{\eta, \epsilon, \alpha}$  with

$$\Delta \mathcal{L}(\mathbb{F}_{\eta, \epsilon, \alpha}) \in L^2((0, T), H^{-1}(\Phi_2 dx))$$

and

$$\mathbb{P} \operatorname{div}(\mathbb{F}_{\eta, \epsilon, \alpha}) \in L^2((0, T), H^{-1}) \subset L^2((0, T), H^{-1}(\Phi_2 dx)).$$

Finally, as  $\Phi \mathbf{v}_{\eta, \epsilon, \alpha} \in L^2((0, T), H^1)$  and  $\partial_t(\Phi \mathbf{v}_{\eta, \epsilon, \alpha}) \in L^2((0, T), H^{-1})$ , one has

$$\|\Phi \mathbf{v}_{\eta, \epsilon, \alpha}(t, \cdot)\|_2^2 - \|\Phi \mathbf{v}_{\eta, \epsilon, \alpha}(0, \cdot)\|_2^2 = 2 \int_0^t \int (\Phi \mathbf{v}_{\eta, \epsilon, \alpha})(s, x) \cdot \partial_t(\Phi \mathbf{v}_{\eta, \epsilon, \alpha})(s, x) ds dx.$$

□

Lemma 7 will be a key ingredient for controlling the norm of  $\mathbf{v}_{\eta, \epsilon, \alpha}$  in  $\mathcal{C}([0, T], L^2(\Phi_2 dx)) \cap L^2((0, T), H^1(\Phi_2 dx))$  and the existence time of the solution  $\mathbf{v}_{\eta, \epsilon, \alpha}$  independently from  $\epsilon$  and  $\alpha$ .

**Proposition 4.** *Let  $2 < p < +\infty$  and  $0 < \gamma < 2$ . There exists a constant  $C_2 > 1$  such that, for every  $\epsilon > 0$ , every  $\alpha > 0$ , every  $\eta > 0$  and every  $\mathbf{u}_0 \in L^p(\Phi_\gamma dx)$  with  $\operatorname{div} \mathbf{u}_0 = 0$ , writing  $\mathbf{u}_0 = \mathbf{b}_{0, \eta} + \mathbf{v}_{0, \eta}$  with  $\mathbf{v}_{0, \eta} \in L^2(\Phi_2 dx)$ ,  $\operatorname{div} \mathbf{v}_{0, \eta} = 0$  and  $\|\mathbf{b}_{0, \eta}\|_r < \eta$ ,  $\operatorname{div} \mathbf{b}_{0, \eta} = 0$ , and writing  $\mathbf{u}_{\epsilon, \alpha} = \mathbf{v}_{\eta, \epsilon, \alpha} + \mathbf{b}_{\eta, \epsilon, \alpha}$  with  $\mathbf{u}_{\epsilon, \alpha} \in \mathcal{C}([0, T_{\epsilon, \alpha}], L^p(\Phi_\gamma dx))$  (as described in Proposition 2) and  $\mathbf{b}_{\eta, \epsilon, \alpha} \in L^\infty((0, T_{[\eta]}), L^r)$  (as described in Proposition 3), the mollified equation*

$$\mathbf{v}_{\eta, \epsilon, \alpha} = e^{t\Delta} \mathbf{v}_{0, \eta} - B_{\epsilon, \alpha}(\mathbf{b}_{\eta, \epsilon, \alpha}, \mathbf{v}_{\eta, \epsilon, \alpha}) - B_{\epsilon, \alpha}(\mathbf{v}_{\eta, \epsilon, \alpha}, \mathbf{b}_{\eta, \epsilon, \alpha}) - B_{\epsilon, \alpha}(\mathbf{v}_{\eta, \epsilon, \alpha}, \mathbf{v}_{\eta, \epsilon, \alpha}) \quad (17)$$

has a unique solution on  $(0, T_{\eta, \epsilon, \alpha}) \times \mathbb{R}^3$  such that

- $T_{\eta, \epsilon, \alpha} = \min(T_{\epsilon, \alpha}, \frac{1}{C_2} T_{[\eta]}, \frac{1}{C_2} \frac{1}{1 + \max(1, \alpha)^6 \|\mathbf{v}_{0, \eta}\|_{L^2(\Phi_2 dx)}^4})$
- $\mathbf{v}_{\eta, \epsilon, \alpha} \in \mathcal{C}([0, T_{\eta, \epsilon, \alpha}], L^2(\Phi_2 dx))$  with

$$\sup_{0 \leq t \leq T_{\eta, \epsilon, \alpha}} \|\mathbf{v}_{\eta, \epsilon, \alpha}(t, \cdot)\|_{L^2(\Phi_2 dx)} \leq 2 \|\mathbf{v}_{0, \eta}\|_{L^2(\Phi_2 dx)},$$

- $\mathbf{v}_{\eta, \epsilon, \alpha} \in L^2((0, T_{\eta, \epsilon, \alpha}), H^1(\Phi_2 dx))$  with

$$\|\mathbf{v}_{\eta, \epsilon, \alpha}(t, \cdot)\|_{L^2((0, T_{\eta, \epsilon, \alpha})(H^1(L^2(\Phi_2 dx)))} \leq C_2 \|\mathbf{v}_{0, \eta}\|_{L^2(\Phi_2 dx)}.$$

In particular, we have

$$T_{\epsilon, \alpha} \geq \min\left(\frac{1}{C_2} \left(\frac{1}{C_1 \|\mathbf{b}_{0, \eta}\|_r}\right)^{\frac{2r}{r-3}}, \frac{1}{C_2} \frac{1}{1 + \max(1, \alpha)^6 \|\mathbf{v}_{0, \eta}\|_{L^2(\Phi_2 dx)}^4}\right).$$

*Proof.* We write

$$\begin{aligned}
\partial_t \mathbf{v}_{\eta, \epsilon, \alpha} &= \Delta \mathbf{v}_{\eta, \epsilon, \alpha} - \mathbb{P} \operatorname{div}((\varphi_\epsilon * (\theta_\alpha \mathbf{v}_{\eta, \epsilon, \alpha})) \otimes \mathbf{b}_{\eta, \epsilon, \alpha}) - \mathbb{P} \operatorname{div}((\varphi_\epsilon * (\theta_\alpha \mathbf{b}_{\eta, \epsilon, \alpha})) \otimes \mathbf{v}_{\eta, \epsilon, \alpha}) \\
&\quad - \mathbb{P} \operatorname{div}((\varphi_\epsilon * (\theta_\alpha \mathbf{v}_{\eta, \epsilon, \alpha})) \otimes \mathbf{v}_{\eta, \epsilon, \alpha}) \\
&= \Delta \mathbf{v}_{\eta, \epsilon, \alpha} - \mathbb{P} \operatorname{div}((\varphi_\epsilon * (\theta_\alpha \mathbf{v}_{\eta, \epsilon, \alpha})) \otimes \mathbf{b}_{\eta, \epsilon, \alpha}) - \mathbb{P} \operatorname{div}((\varphi_\epsilon * (\theta_\alpha \mathbf{b}_{\eta, \epsilon, \alpha})) \otimes \mathbf{v}_{\eta, \epsilon, \alpha}) \\
&\quad - \operatorname{div}((\varphi_\epsilon * (\theta_\alpha \mathbf{v}_{\eta, \epsilon, \alpha})) \otimes \mathbf{v}_{\eta, \epsilon, \alpha}) - \nabla q_{\eta, \epsilon, \alpha}
\end{aligned}$$

with

$$q_{\eta, \epsilon, \alpha} = \sum_{1 \leq i \leq 3} \sum_{1 \leq j \leq 3} R_i R_j ((\varphi_\epsilon * (\theta_\alpha \mathbf{v}_{\eta, \epsilon, \alpha, i})) \mathbf{v}_{\eta, \epsilon, \alpha, j}).$$

This gives (as  $\operatorname{div} \mathbf{v}_{\eta, \epsilon, \alpha} = 0$ )

$$\begin{aligned}
2\partial_t \mathbf{v}_{\eta, \epsilon, \alpha} \cdot \mathbf{v}_{\eta, \epsilon, \alpha} &= \Delta(|\mathbf{v}_{\eta, \epsilon, \alpha}|^2) - 2|\nabla \otimes \mathbf{v}_{\eta, \epsilon, \alpha}|^2 \\
&\quad - 2\mathbf{v}_{\eta, \epsilon, \alpha} \cdot (\mathbb{P} \operatorname{div}((\varphi_\epsilon * (\theta_\alpha \mathbf{v}_{\eta, \epsilon, \alpha})) \otimes \mathbf{b}_{\eta, \epsilon, \alpha}) + \mathbb{P} \operatorname{div}((\varphi_\epsilon * (\theta_\alpha \mathbf{b}_{\eta, \epsilon, \alpha})) \otimes \mathbf{v}_{\eta, \epsilon, \alpha})) \\
&\quad - \operatorname{div}(|\mathbf{v}_{\eta, \epsilon, \alpha}|^2 (\varphi_\epsilon * (\theta_\alpha \mathbf{v}_{\eta, \epsilon, \alpha}))) + 2|\mathbf{v}_{\eta, \epsilon, \alpha}|^2 \operatorname{div}(\varphi_\epsilon * (\theta_\alpha \mathbf{v}_{\eta, \epsilon, \alpha})) \\
&\quad - 2 \operatorname{div}(q_{\eta, \epsilon, \alpha} \mathbf{v}_{\eta, \epsilon, \alpha})
\end{aligned}$$

with

$$\operatorname{div}(\varphi_\epsilon * (\theta_\alpha \mathbf{v}_{\eta, \epsilon, \alpha})) = \varphi_\epsilon * (\mathbf{v}_{\eta, \epsilon, \alpha} \cdot \nabla \theta_\alpha).$$

Integrating against  $\Phi(x)^2 dx$ , we obtain

$$\frac{d}{dt} \|\mathbf{v}_{\eta, \epsilon, \alpha}\|_{L^2(\Phi_2 dx)}^2 + 2\|\nabla \otimes \mathbf{v}_{\eta, \epsilon, \alpha}\|_{L^2(\Phi_2 dx)}^2 = \sum_{k=1}^6 A_k$$

with:

- $A_1 = \int \Phi^2 \Delta(|\mathbf{v}_{\eta, \epsilon, \alpha}|^2) dx = \int |\mathbf{v}_{\eta, \epsilon, \alpha}|^2 \Delta(\Phi^2) dx \leq C \|\mathbf{v}_{\eta, \epsilon, \alpha}\|_{L^2(\Phi_2 dx)}^2$  (since  $|\Delta(\Phi^2)| \leq C\Phi^2$ );
- $A_2 = -2 \int \Phi^2 \mathbf{v}_{\eta, \epsilon, \alpha} \cdot \mathbb{P} \operatorname{div}((\varphi_\epsilon * (\theta_\alpha \mathbf{v}_{\eta, \epsilon, \alpha})) \otimes \mathbf{b}_{\eta, \epsilon, \alpha}) dx$ ; since the Riesz transforms are bounded on  $L^2(\Phi_2 dx)$  and since

$$|\int \Phi^2 f \partial_j g dx| \leq \|\Phi f\|_{H^1} \|\Phi \partial_j g\|_{H^{-1}} \leq C(\|\Phi f\|_2 + \|\Phi \nabla f\|_2) \|\Phi g\|_2,$$

we have (since  $\|\varphi_\epsilon * f\|_{L^2(\Phi_2 dx)} \leq C\|\mathcal{M}_f\|_{L^2(\Phi_2 dx)} \leq C'\|f\|_{L^2(\Phi_2 dx)}$ )

$$\begin{aligned}
A_2 &\leq C(\|\mathbf{v}_{\eta, \epsilon, \alpha}\|_{L^2(\Phi_2 dx)} + \|\nabla \otimes \mathbf{v}_{\eta, \epsilon, \alpha}\|_{L^2(\Phi_2 dx)}) \|(\varphi_\epsilon * (\theta_\alpha \mathbf{v}_{\eta, \epsilon, \alpha})) \otimes \mathbf{b}_{\eta, \epsilon, \alpha}\|_{L^2(\Phi_2 dx)} \\
&\leq C'(\|\mathbf{v}_{\eta, \epsilon, \alpha}\|_{L^2(\Phi_2 dx)} + \|\nabla \otimes \mathbf{v}_{\eta, \epsilon, \alpha}\|_{L^2(\Phi_2 dx)}) \|\mathbf{v}_{\eta, \epsilon, \alpha}\|_{L^2(\Phi_2 dx)} \|\mathbf{b}_{\eta, \epsilon, \alpha}\|_\infty \\
&\leq C''(\|\mathbf{v}_{\eta, \epsilon, \alpha}\|_{L^2(\Phi_2 dx)} + \|\nabla \otimes \mathbf{v}_{\eta, \epsilon, \alpha}\|_{L^2(\Phi_2 dx)}) \|\mathbf{v}_{\eta, \epsilon, \alpha}\|_{L^2(\Phi_2 dx)} \eta t^{-\frac{3}{2r}} \\
&\leq \frac{1}{10} \|\nabla \otimes \mathbf{v}_{\eta, \epsilon, \alpha}\|_{L^2(\Phi_2 dx)}^2 + C''' \|\mathbf{v}_{\eta, \epsilon, \alpha}\|_{L^2(\Phi_2 dx)}^2 (1 + \eta^2 t^{-\frac{3}{r}});
\end{aligned}$$

- $A_3 = -2 \int \Phi^2 \mathbf{v}_{\eta,\epsilon,\alpha} \cdot \mathbb{P} \operatorname{div}((\varphi_\epsilon * (\theta_\alpha \mathbf{b}_{\eta,\epsilon,\alpha})) \otimes \mathbf{v}_{\eta,\epsilon,\alpha}) dx$ ; with similar computations as for  $A_2$  we find

$$A_3 \leq \frac{1}{10} \|\nabla \otimes \mathbf{v}_{\eta,\epsilon,\alpha}\|_{L^2(\Phi_2 dx)}^2 + C''' \|\mathbf{v}_{\eta,\epsilon,\alpha}\|_{L^2(\Phi_2 dx)}^2 (1 + \eta^2 t^{-\frac{3}{r}});$$

- $A_4 = - \int \Phi^2 \operatorname{div}(|\mathbf{v}_{\eta,\epsilon,\alpha}|^2 (\varphi_\epsilon * (\theta_\alpha \mathbf{v}_{\eta,\epsilon,\alpha}))) dx$ ; since  $|\nabla \Phi^2| \leq C \Phi^3$ , we have

$$A_4 \leq C \int \Phi^3 |\mathbf{v}_{\eta,\epsilon,\alpha}|^2 |\varphi_\epsilon * (\theta_\alpha \mathbf{v}_{\eta,\epsilon,\alpha})| dx;$$

as  $\Phi \mathbf{v}_{\eta,\epsilon,\alpha} \in L^2$ , we have  $\Phi(\varphi_\epsilon * (\theta_\alpha \mathbf{v}_{\eta,\epsilon,\alpha})) \in L^2$ ; on the other hand, we have  $\Phi \mathbf{v}_{\eta,\epsilon,\alpha} \in L^2$ , we have  $\Phi \mathbf{v}_{\eta,\epsilon,\alpha} \in H^1$ , so that  $\Phi \mathbf{v}_{\eta,\epsilon,\alpha} \in L^3 \cap L^6$ ; thus, we have

$$\begin{aligned} A_4 &\leq C \|\Phi \mathbf{v}_{\eta,\epsilon,\alpha}\|_6 \|\Phi \mathbf{v}_{\eta,\epsilon,\alpha}\|_3 \|\Phi(\varphi_\epsilon * (\theta_\alpha \mathbf{v}_{\eta,\epsilon,\alpha}))\|_2 \\ &\leq C' \|\Phi \mathbf{v}_{\eta,\epsilon,\alpha}\|_6 \|\Phi \mathbf{v}_{\eta,\epsilon,\alpha}\|_3 \|\Phi \mathbf{v}_{\eta,\epsilon,\alpha}\|_2 \\ &\leq C'' (\|\mathbf{v}_{\eta,\epsilon,\alpha}\|_{L^2(\Phi_2 dx)} + \|\nabla \otimes \mathbf{v}_{\eta,\epsilon,\alpha}\|_{L^2(\Phi_2 dx)})^{\frac{3}{2}} \|\mathbf{v}_{\eta,\epsilon,\alpha}\|_{L^2(\Phi_2 dx)}^{\frac{3}{2}} \\ &\leq \frac{1}{10} \|\nabla \otimes \mathbf{v}_{\eta,\epsilon,\alpha}\|_{L^2(\Phi_2 dx)}^2 + C''' \|\mathbf{v}_{\eta,\epsilon,\alpha}\|_{L^2(\Phi_2 dx)}^2 (1 + \|\mathbf{v}_{\eta,\epsilon,\alpha}\|_{L^2(\Phi_2 dx)}^4); \end{aligned}$$

- $A_5 = -2 \int \Phi^2 \operatorname{div}(q_{\eta,\epsilon,\alpha} \mathbf{v}_{\eta,\epsilon,\alpha}) dx$ ; similarly, we have

$$A_5 \leq C \int \Phi^3 |q_{\eta,\epsilon,\alpha}| |\mathbf{v}_{\eta,\epsilon,\alpha}| dx \leq \|\Phi \mathbf{v}_{\eta,\epsilon,\alpha}\|_6 \|\Phi^2 q_{\eta,\epsilon,\alpha}\|_{6/5};$$

we recall that

$$q_{\eta,\epsilon,\alpha} = \sum_{1 \leq i \leq 3} \sum_{1 \leq j \leq 3} R_i R_j ((\varphi_\epsilon * (\theta_\alpha \mathbf{v}_{\eta,\epsilon,\alpha,i})) \mathbf{v}_{\eta,\epsilon,\alpha,j})$$

with  $\Phi \mathbf{v}_{\eta,\epsilon,\alpha} \in L^3$  and  $\Phi(\varphi_\epsilon * (\theta_\alpha \mathbf{v}_{\eta,\epsilon,\alpha})) \in L^2$ ; thus,

$$\Phi^2((\varphi_\epsilon * (\theta_\alpha \mathbf{v}_{\eta,\epsilon,\alpha})) \otimes \mathbf{v}_{\eta,\epsilon,\alpha}) \in L^{\frac{6}{5}}$$

or, equivalently,  $(\varphi_\epsilon * (\theta_\alpha \mathbf{v}_{\eta,\epsilon,\alpha})) \otimes \mathbf{v}_{\eta,\epsilon,\alpha} \in L^{\frac{6}{5}}(\Phi^{\frac{12}{5}} dx)$ ; as  $\frac{12}{5} < 3$ ,  $\Phi^{\frac{12}{5}}$  belongs to the Muckenhoupt class  $\mathcal{A}_{\frac{6}{5}}$ , and the Riesz transforms are bounded on  $L^{\frac{6}{5}}(\Phi^{\frac{12}{5}} dx)$ ; we thus get

$$\begin{aligned} A_5 &\leq C \|\Phi \mathbf{v}_{\eta,\epsilon,\alpha}\|_6 \|\Phi \mathbf{v}_{\eta,\epsilon,\alpha}\|_3 \|\Phi(\varphi_\epsilon * (\theta_\alpha \mathbf{v}_{\eta,\epsilon,\alpha}))\|_2 \\ &\leq \frac{1}{10} \|\nabla \otimes \mathbf{v}_{\eta,\epsilon,\alpha}\|_{L^2(\Phi_2 dx)}^2 + C' \|\mathbf{v}_{\eta,\epsilon,\alpha}\|_{L^2(\Phi_2 dx)}^2 (1 + \|\mathbf{v}_{\eta,\epsilon,\alpha}\|_{L^2(\Phi_2 dx)}^4); \end{aligned}$$

- $A_6 = 2 \int \Phi^2 |\mathbf{v}_{\eta,\epsilon,\alpha}|^2 (\varphi_\epsilon * (\mathbf{v}_{\eta,\epsilon,\alpha} \cdot \nabla \theta_\alpha)) dx$ ; we have

$$\begin{aligned} A_6 &\leq C \|\Phi \mathbf{v}_{\eta,\epsilon,\alpha}\|_6 \|\Phi \mathbf{v}_{\eta,\epsilon,\alpha}\|_3 \|\varphi_\epsilon * (\mathbf{v}_{\eta,\epsilon,\alpha} \cdot \nabla \theta_\alpha)\|_2 \\ &\leq C \|\Phi \mathbf{v}_{\eta,\epsilon,\alpha}\|_6 \|\Phi \mathbf{v}_{\eta,\epsilon,\alpha}\|_3 \|\mathbf{v}_{\eta,\epsilon,\alpha} \cdot \nabla \theta_\alpha\|_2; \end{aligned}$$

as  $|\nabla \theta_\alpha(x)| \leq C \max(\alpha, 1) \Phi(x)$ , we find

$$A_6 \leq \frac{1}{10} \|\nabla \otimes \mathbf{v}_{\eta,\epsilon,\alpha}\|_{L^2(\Phi_2 dx)}^2 + C' \|\mathbf{v}_{\eta,\epsilon,\alpha}\|_{L^2(\Phi_2 dx)}^2 (1 + \max(\alpha, 1)^6 \|\mathbf{v}_{\eta,\epsilon,\alpha}\|_{L^2(\Phi_2 dx)}^4).$$

Thus, integrating on  $(0, t)$  for  $0 < t < T$  with  $T < \min(T_{\epsilon,\alpha}, T_{[\eta]})$ , we obtain, for  $C_T = \sup_{0 < t < T} \|\mathbf{v}_{\eta,\epsilon,\alpha}(t, \cdot)\|_{L^2(\Phi_2 dx)}^2$ ,

$$C_T \leq \|\mathbf{v}_{0,\eta}\|_{L^2(\Phi_2 dx)}^2 + CTC_T + CC_T T^{1-\frac{3}{r}} \eta^2 + CT \max(1, \alpha)^6 C_T^3.$$

. We obtain  $C_T \leq 4 \|\mathbf{v}_{0,\eta}\|_{L^2(\Phi_2 dx)}^2$  if

$$CT(1 + 16 \max(1, \alpha)^6 \|\mathbf{v}_{0,\eta}\|_{L^2(\Phi_2 dx)}^4) < \frac{1}{4}$$

and

$$CT^{1-\frac{3}{r}} \eta^2 < \frac{1}{2}.$$

□

### 3.5 Rescaling and global estimates

The minoration on  $T_{\epsilon,\alpha}$  given in Proposition 4 depends on  $\eta$ , i.e. on  $\|\mathbf{b}_{0,\eta}\|_r$  and on  $\|\mathbf{v}_{0,\eta}\|_{L^2(\Phi_2 dx)}$ . As a matter of fact, following Fernández-Dalgo & Lemarié-Rieusset [3] and Bradshaw, Kucavica & Tsai [1], we can partly get rid of this restriction:

**Theorem 4.** *Let  $2 < p < +\infty$  and  $0 < \gamma < 2$ . For every  $\mathbf{u}_0 \in L^p(\Phi_\gamma dx)$  with  $\operatorname{div} \mathbf{u}_0 = 0$ , let  $\mathbf{u}_{\epsilon,\alpha}$  be the (maximal) solution of the mollified equations*

$$\begin{cases} \partial_t \mathbf{u}_{\epsilon,\alpha} = \Delta \mathbf{u}_{\epsilon,\alpha} - \mathbb{P} \operatorname{div}((\varphi_\epsilon * (\theta_\alpha \mathbf{u}_\epsilon)) \otimes \mathbf{u}_{\epsilon,\alpha}) \\ \lim_{t \rightarrow 0} \mathbf{u}_{\epsilon,\alpha}(t, \cdot) = \mathbf{u}_0 \end{cases} \quad (18)$$

*in  $\mathcal{C}([0, T_\epsilon], L^p(\Phi_\gamma dx))$ . For every  $T > 0$ , there exists  $\alpha_{T,\mathbf{u}_0} > 0$  such that, for every  $0 < \epsilon$  and every  $0 < \alpha < \alpha_{T,\mathbf{u}_0}$ , one has  $T_{\epsilon,\alpha} > T$ . Moreover, one may split  $\mathbf{u}_{\epsilon,\alpha}$  as  $\mathbf{u}_{\epsilon,\alpha} = \mathbf{b}_{\epsilon,\alpha,T} + \mathbf{v}_{\epsilon,\alpha,T}$  on  $(0, T) \times \mathbb{R}^3$  with*

- $\mathbf{b}_{\epsilon,\alpha,T} \in \mathcal{C}([0, T], L^2(\Phi_4 dx))$ ,
- $\sup_{0 \leq t \leq T} \|\mathbf{b}_{\epsilon,\alpha,T}(t, \cdot)\|_r \leq C_{3,T,\mathbf{u}_0}$ ,

- $\sup_{0 < t \leq T} t^{\frac{1}{2}} \|\nabla \otimes \mathbf{b}_{\epsilon, \alpha, T}(t, \cdot)\|_r \leq C_{3, T, \mathbf{u}_0},$
- $\sup_{0 < t \leq T} t^{\frac{3}{2r}} \|\mathbf{b}_{\epsilon, \alpha, T}(t, \cdot)\|_\infty \leq C_{3, T, \mathbf{u}_0},$
- $\mathbf{v}_{\epsilon, \alpha, T} \in \mathcal{C}([0, T], L^2(\Phi_2 dx))$  with

$$\sup_{0 \leq t \leq T} \|\mathbf{v}_{\epsilon, \alpha, T}(t, \cdot)\|_{L^2(\Phi_2 dx)} \leq C_{3, T, \mathbf{u}_0},$$

- $\mathbf{v}_{\eta, \epsilon, \alpha} \in L^2((0, T), H^1(\Phi_2 dx))$  with

$$\|\mathbf{v}_{\epsilon, \alpha, T}(t, \cdot)\|_{L^2((0, T)(H^1(L^2(\Phi_2 dx)))} \leq C_{3, T, \mathbf{u}_0}$$

where  $C_{3, T, \mathbf{u}_0}$  doesn't depend on  $\epsilon$  nor on  $\alpha$ .

In particular, we have

- $\sup_{0 < \epsilon, 0 < \alpha < \alpha_T} \sup_{0 < t < T} \|\mathbf{u}_{\epsilon, \alpha}(t, \cdot)\|_{L^2(\Phi_{\frac{7}{2}} dx)} \leq C_{4, T, \mathbf{u}_0}$
- $\sup_{0 < \epsilon, 0 < \alpha < \alpha_T} \|\mathbf{u}_{\epsilon, \alpha}(t, \cdot)\|_{L^2((0, T), H^{1/2}(\Phi_2 dx))} \leq C_{4, T, \mathbf{u}_0}$
- $\sup_{0 < \epsilon, 0 < \alpha < \alpha_T} \|\partial_t \mathbf{u}_{\epsilon, \alpha}(t, \cdot)\|_{L^2((0, T), H^{-4}(\Phi_8 dx))} \leq C_{4, T, \mathbf{u}_0}$

*Proof.* We first recall the results on  $\mathbf{u}_{\epsilon, \alpha}$  obtained in Propositions 2, 3, 4. Let  $T > 0$ . Following Proposition 4, we choose  $\eta_T$  with  $\eta_T \leq \frac{1}{C_1(C_2 T)^{\frac{r-3}{2r}}}$  and we split  $\mathbf{u}_0$  into  $\mathbf{u}_0 = \mathbf{b}_{0, \eta_T} + \mathbf{v}_{0, \eta_T}$  with  $\|\mathbf{b}_{0, \eta_T}\|_\infty < \eta_T$  and  $\mathbf{v}_{0, \eta_T} \in L^2(\Phi_2 dx)$ . We then split  $\mathbf{u}_{\epsilon, \alpha}$  into  $\mathbf{u}_{\epsilon, \alpha} = \mathbf{b}_{\eta_T, \epsilon, \alpha} + \mathbf{v}_{\eta_T, \epsilon, \alpha}$ . We rescale  $\mathbf{u}_{\epsilon, \alpha}$  into

$$\begin{aligned} \mathbf{u}_{\lambda, \epsilon, \alpha}(t, x) &= \frac{1}{\lambda} \mathbf{u}_{\epsilon, \alpha}\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) = \frac{1}{\lambda} \mathbf{b}_{\eta_T, \epsilon, \alpha}\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) + \frac{1}{\lambda} \mathbf{v}_{\eta_T, \epsilon, \alpha}\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) \\ &= \mathbf{b}_{\eta_T, \lambda, \epsilon, \alpha}(t, x) + \mathbf{v}_{\eta_T, \lambda, \epsilon, \alpha}(t, x). \end{aligned}$$

We write similarly

$$\mathbf{u}_{\lambda, 0}(x) = \frac{1}{\lambda} \mathbf{u}_0\left(\frac{x}{\lambda}\right), \quad \mathbf{v}_{\eta_T, \lambda, 0}(x) = \frac{1}{\lambda} \mathbf{v}_{0, \eta_T}\left(\frac{x}{\lambda}\right), \quad \mathbf{b}_{\eta_T, \lambda, 0}(x) = \frac{1}{\lambda} \mathbf{b}_{0, \eta_T}\left(\frac{x}{\lambda}\right).$$

We have

$$\begin{aligned} \partial_t \mathbf{u}_{\lambda, \epsilon, \alpha}(t, x) &= \frac{1}{\lambda^3} (\partial_t \mathbf{u}_{\epsilon, \alpha})\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) \\ &= \frac{1}{\lambda^3} (\Delta \mathbf{u}_{\epsilon, \alpha} - \mathbb{P} \operatorname{div}((\varphi_\epsilon * (\theta_\alpha \mathbf{u}_{\epsilon, \alpha})) \otimes \mathbf{u}_{\epsilon, \alpha}))\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) \\ &= \Delta \mathbf{u}_{\lambda, \epsilon, \alpha}(t, x) - \mathbb{P} \operatorname{div}\left(\frac{1}{\lambda} (\varphi_\epsilon * (\theta_\alpha \mathbf{u}_{\epsilon, \alpha}))\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) \otimes \mathbf{u}_{\lambda, \epsilon, \alpha}(t, x)\right) \\ &= \Delta \mathbf{u}_{\lambda, \epsilon, \alpha}(t, x) - \mathbb{P} \operatorname{div}((\varphi_{\lambda \epsilon} * (\theta_{\frac{\alpha}{\lambda}} \mathbf{u}_{\lambda, \epsilon, \alpha}))(t, x) \otimes \mathbf{u}_{\lambda, \epsilon, \alpha}(t, x)) \end{aligned}$$



Thus,  $\mathbf{u}_{\lambda,\epsilon,\alpha}$  is the solution of the mollified equation

$$\mathbf{u}_{\lambda,\epsilon,\alpha} = e^{t\Delta} \mathbf{u}_{\lambda,0} - B_{\lambda\epsilon, \frac{\alpha}{\lambda}}(\mathbf{u}_{\lambda,\epsilon,\alpha}, \mathbf{u}_{\lambda,\epsilon,\alpha}).$$

By Proposition 4, we know that the existence time  $T_{\lambda,\epsilon,\alpha}$  of  $\mathbf{u}_{\lambda,\epsilon,\alpha}$  can be controlled by below as

$$T_{\lambda,\epsilon,\alpha} \geq \min\left(\frac{1}{C_2} \frac{1}{(C_1 \|\mathbf{b}_{\eta_T, \lambda, 0}\|_r)^{\frac{2r}{r-3}}}, \frac{1}{C_2} \frac{1}{1 + \max(1, \frac{\alpha}{\lambda})^6 \|\mathbf{v}_{\eta_T, \lambda, 0}\|_{L^2(\Phi_2 dx)}^4}\right).$$

Thus, we have (since  $\lambda^2 \|\mathbf{b}_{\eta_T, \lambda, 0}\|_r^{\frac{2r}{r-3}} = \|\mathbf{b}_{0, \eta_T}\|_r^{\frac{2r}{r-3}}$ )

$$\begin{aligned} T_{\epsilon,\alpha} &= \frac{1}{\lambda^2} T_{\lambda,\epsilon,\alpha} \\ &\geq \min\left(\frac{1}{C_2} \frac{1}{(C_1 \|\mathbf{b}_{0, \eta_T}\|_r)^{\frac{2r}{r-3}}}, \frac{1}{C_2 \lambda^2} \frac{1}{1 + \max(1, \frac{\alpha}{\lambda})^6 \|\mathbf{v}_{\eta_T, \lambda, 0}\|_{L^2(\Phi_2 dx)}^4}\right). \end{aligned}$$

We have

$$\lambda^2(1 + \|\mathbf{v}_{\eta_T, \lambda, 0}\|_{L^2(\Phi_2 dx)}^4) = \lambda^2 + \left(\int |\mathbf{v}_{0, \eta_T}(x)|^2 \frac{1}{1 + |x|^2} \frac{\lambda^2 + \lambda^2 |x|^2}{1 + \lambda^2 |x|^2} dx\right)^2.$$

For every  $x \in \mathbb{R}^3$ , we have

$$\sup_{0 < \lambda < 1} \frac{\lambda^2 + \lambda^2 |x|^2}{1 + \lambda^2 |x|^2} = 1 \text{ and } \lim_{\lambda \rightarrow 0} \frac{\lambda^2 + \lambda^2 |x|^2}{1 + \lambda^2 |x|^2} = 0.$$

Thus, by dominated convergence, we get that

$$\lim_{\lambda \rightarrow 0} \lambda^2(1 + \|\mathbf{v}_{\eta_T, \lambda, 0}\|_{L^2(\Phi_2 dx)}^4) = 0.$$

We take a  $\lambda_T \in (0, 1)$  such that  $\lambda_T^2(1 + \|\mathbf{v}_{\eta_T, \lambda_T, 0}\|_{L^2(\Phi_2 dx)}^4) < \frac{1}{C_2 T}$  and we get that  $T_{\epsilon,\alpha} = \frac{1}{\lambda_T^2} T_{\lambda_T, \epsilon, \alpha} \geq \max(1, \frac{\alpha}{\lambda_T})^6 T$ .

In particular, we have  $T_{\epsilon,\alpha} > T$  provided that  $\alpha < \alpha_{T, \mathbf{u}_0} = \frac{1}{\lambda_T}$ .

With those values of  $\eta_T$  and  $\lambda_T$ , writing  $\mathbf{b}_{\epsilon, \alpha, T} = \mathbf{b}_{\eta_T, \epsilon, \alpha}$  and  $\mathbf{v}_{\epsilon, \alpha, T}(t, x) = \mathbf{v}_{\eta_T, \epsilon, \alpha}(t, x) = \lambda_T \mathbf{v}_{\eta_T, \lambda_T, \epsilon, \alpha}(\lambda_T^2 t, \lambda_T x)$ , we use Proposition 3 and Proposition 4 to get the following estimates on the solution  $\mathbf{u}_{\epsilon, \alpha}$  on  $(0, T)$ :

- $\sup_{0 \leq t \leq T} \|\mathbf{b}_{\epsilon, \alpha, T}(t, \cdot)\|_r \leq 2 \|\mathbf{b}_{0, \eta_T}\|_r,$
- $\sup_{0 < t \leq T} t^{\frac{1}{2}} \|\nabla \otimes \mathbf{b}_{\epsilon, \alpha, T}(t, \cdot)\|_r \leq 2 \|\nabla W_1\|_1 \|\mathbf{b}_{0, \eta_T}\|_r,$
- $\sup_{0 < t \leq T} t^{\frac{3}{2r}} \|\mathbf{b}_{\epsilon, \alpha, T}(t, \cdot)\|_\infty \leq C_{3, T, \mathbf{u}_0} \|\mathbf{b}_{0, \eta_T}\|_r,$

- $\sup_{0 \leq t \leq T} \|\mathbf{v}_{\epsilon, \alpha, T}(t, \cdot)\|_{L^2(\Phi_2 dx)} \leq \frac{2}{\lambda_T} \|\mathbf{v}_{0, \eta_T}\|_{L^2(\Phi_2 dx)},$
- $\|\mathbf{v}_{\epsilon, \alpha, T}(t, \cdot)\|_{L^2((0, T)(H^1(L^2(\Phi_2 dx)))} \leq \frac{C_2}{\lambda_T} \|\mathbf{v}_{0, \eta_T}\|_{L^2(\Phi_2 dx)}.$

(where we use the inequality

$$\lambda \|f\|_{L^2(\phi_2 dx)}^2 \leq \int \frac{1}{\lambda^2} |f(\frac{x}{\lambda})|^2 \Phi_2(x) dx = \int |f(x)|^2 \frac{\lambda}{1 + |\lambda x|^2} dx \leq \frac{1}{\lambda} \|f\|_{L^2(\phi_2 dx)}^2$$

for  $0 < \lambda \leq 1$ ).

In particular, we have

- $\|\mathbf{u}_{\epsilon, \alpha}(t, \cdot)\|_{L^2(\Phi_{\frac{T}{2}} dx)} \leq \|\mathbf{v}_{\epsilon, \alpha, T}(t, \cdot)\|_{L^2(\Phi_2 dx)} + \|\mathbf{b}_{\epsilon, \alpha, T}(t, \cdot)\|_r$ , so that

$$\sup_{0 < \epsilon, 0 < \alpha < \alpha_T} \sup_{0 < t < T} \|\mathbf{u}_{\epsilon, \alpha}(t, \cdot)\|_{L^2(\Phi_{\frac{T}{2}} dx)} \leq C_{4, T, \mathbf{u}_0}$$

- We have

$$\begin{aligned} \|\Phi^2 \mathbf{b}_{\epsilon, \alpha, T}\|_{H^{1/2}} &\leq \sqrt{\|\Phi^2 \mathbf{b}_{\epsilon, \alpha, T}\|_2 \|\Phi^2 \mathbf{b}_{\epsilon, \alpha, T}\|_{H^1}} \\ &\leq \sqrt{\|\mathbf{b}_{\epsilon, \alpha, T}\|_{L^2(\Phi_4 dx)}} \sqrt{\|\mathbf{b}_{\epsilon, \alpha, T}\|_{L^2(\Phi_4 dx)} + \|\nabla \otimes \mathbf{b}_{\epsilon, \alpha, T}\|_{L^2(\Phi_4 dx)}} \\ &\leq C \sqrt{\|\mathbf{b}_{\epsilon, \alpha, T}\|_r} \sqrt{(\|\mathbf{b}_{\epsilon, \alpha, T}\|_r + \|\nabla \otimes \mathbf{b}_{\epsilon, \alpha, T}\|_r)} \\ &\leq C' (1 + \sqrt{C_{3, T, \mathbf{u}_0}} t^{-\frac{3}{4r}}) \|\mathbf{b}_{0, \eta_T}\|_r. \end{aligned}$$

As

$$\begin{aligned} \|\mathbf{u}_{\epsilon, \alpha}(t, \cdot)\|_{L^2((0, T), H^{1/2}(\Phi_4 dx))} \\ \leq \|\mathbf{v}_{\epsilon, \alpha}(t, \cdot)\|_{L^2((0, T), H^1(\Phi_2 dx))} + \|\mathbf{b}_{\epsilon, \alpha}(t, \cdot)\|_{L^2((0, T), H^{1/2}(\Phi_4 dx))}, \end{aligned}$$

we get that

$$\sup_{0 < \epsilon, 0 < \alpha < \alpha_T} \|\mathbf{u}_{\epsilon, \alpha}(t, \cdot)\|_{L^2((0, T), H^{1/2}(\Phi_2 dx))} \leq C_{4, T, \mathbf{u}_0}$$

- From the proof of Theorem 2, we see that  $\mathbb{P} \operatorname{div}((\varphi_\epsilon * (\theta_\alpha \mathbf{u}_{\epsilon, \alpha})) \otimes \mathbf{u}_{\epsilon, \alpha})$  and  $\Delta \mathbf{u}_{\epsilon, \alpha}$  are bounded in  $L^2((0, T), H^{-4}(\Phi_8 dx))$ . Thus, we have

$$\sup_{0 < \epsilon, 0 < \alpha < \alpha_T} \|\partial_t \mathbf{u}_{\epsilon, \alpha}(t, \cdot)\|_{L^2((0, T), H^{-4}(\Phi_8 dx))} \leq C_{4, T, \mathbf{u}_0}.$$

Theorem 4 is proved.  $\square$

## 4 Solutions to the Navier–Stokes equations.

In this final section, we prove Theorem 1. The key tool for going from mollified equations to Navier–Stokes equations will be the following lemma (a simpler variant of the Aubin–Lions theorem):

**Lemma 8** (Rellich–Lions lemma). *Let  $\sigma < 0 < s$ ,  $0 < T < +\infty$  and  $R > 0$ . Let  $(u_n)_{n \in \mathbb{N}}$  be a sequence of functions on  $(0, T) \times \mathbb{R}^3$  such that  $u_n$  is bounded in  $L^2((0, T), H^s)$  and  $\partial_t u_n$  is bounded in  $L^2((0, T), H^\sigma)$  and  $u_n(t, x) = 0$  for  $|x| > R$ . Then there exists a subsequence  $(u_{n_k})_{k \in \mathbb{N}}$  and  $u_\infty \in L^2((0, T) \times \mathbb{R}^3)$  such that  $u_{n_k}$  is strongly convergent to  $u_\infty$  in  $L^2((0, T), L^2)$ .*

*Proof.* Extend  $u_n$  to  $\mathbb{R} \times \mathbb{R}^3$  in the following way: let  $\omega \in \mathcal{D}(\mathbb{R})$  with  $\omega(t) = 1$  for  $|t - \frac{T}{2}| \leq \frac{3T}{4}$  and  $\omega(t) = 0$  for  $|t - \frac{T}{2}| \geq \frac{7T}{8}$ . Define  $v_n$  on  $\mathbb{R} \times \mathbb{R}^3$  by  $v_n(t, x) = u_n(t, x)$  if  $0 \leq t \leq T$ ,  $= \omega(t)u_n(-t, x)$  if  $-T \leq t \leq 0$ ,  $= u_n(2T - t)$  if  $T \leq t \leq 2T$  and  $= 0$  if  $t \notin [-T, 2T]$ . Then  $v_n$  is bounded in  $L^2(\mathbb{R}, H^s)$  and  $\partial_t v_n$  is bounded in  $L^2(\mathbb{R}, H^\sigma)$ . Taking the Fourier transform on  $\mathbb{R} \times \mathbb{R}^3$  defined by

$$\tilde{F}(\tau, \xi) = \iint F(t, x) e^{-i(t\tau + x \cdot \xi)} dt dx,$$

we find that  $(1 + |\xi|^2)^{s/2} \tilde{v}_n$  is bounded in  $L^2(\mathbb{R} \times \mathbb{R}^3)$  and  $\tau(1 + |\xi|^2)^{\sigma/2} \tilde{v}_n$  is bounded in  $L^2(\mathbb{R} \times \mathbb{R}^3)$ . As  $\sigma < s$ ,  $(1 + |\tau|^2)^{1/2} (1 + |\xi|^2)^{\sigma/2} \tilde{v}_n$  is bounded in  $L^2(\mathbb{R} \times \mathbb{R}^3)$ . Let  $s_0 = \frac{s}{1 - \sigma + s}$ ; we have  $0 < s_0 < 1$  and  $s_0 = (1 - s_0)s + s_0\sigma$ , so that

$$\begin{aligned} (1 + \tau^2 + |\xi|^2)^{s_0/2} &\leq (1 + \tau^2)^{s_0/2} (1 + |\xi|^2)^{s_0/2} \\ &= ((1 + \tau^2)^{1/2} (1 + |\xi|^2)^{\sigma/2})^{s_0} (1 + |\xi|^2)^{s/2})^{1-s_0} \end{aligned}$$

and thus  $(1 + \tau^2 + |\xi|^2)^{s_0/2} \tilde{v}_n$  is bounded in  $L^2(\mathbb{R} \times \mathbb{R}^3)$ , or equivalently  $v_n$  is bounded in  $H^{s_0}(\mathbb{R} \times \mathbb{R}^3)$ . As  $s_0 > 0$  and as all the  $v_n$  are supported in the compact set  $[-T/2, 2T] \times \overline{B(0, R)}$ , we may apply Rellich's theorem and get that there exists a subsequence  $(v_{n_k})_{k \in \mathbb{N}}$  that is strongly convergent in  $L^2(\mathbb{R} \times \mathbb{R}^3)$ . The subsequence  $(u_{n_k})_{k \in \mathbb{N}}$  is then strongly convergent in  $L^2((0, T), L^2(\mathbb{R}^3))$ .  $\square$

Theorem 1 will then be proved in the following way:

**Proposition 5.** *There exists a sequence  $(\epsilon_n, \alpha_n)_{n \in \mathbb{N}}$  and a vector field  $\mathbf{u}$  such that:*

- $\lim_{n \rightarrow +\infty} \epsilon_n = \lim_{n \rightarrow +\infty} \alpha_n = 0$ ,
- $\mathbf{u}_{\epsilon_n, \alpha_n}$  is strongly convergent to  $\mathbf{u}$  in  $L^2((0, T), \Phi_4 dx)$  for every  $T > 0$ .

Moreover,  $\mathbf{u}$  is a solution on  $(0, +\infty) \times \mathbb{R}^3$  of the Navier–Stokes equations

$$\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - \mathbb{P} \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) \\ \operatorname{div} \mathbf{u} = 0 \\ \lim_{t \rightarrow 0} \mathbf{u}(t, \cdot) = \mathbf{u}_0 \end{cases} \quad (19)$$

and, for every  $0 < T < +\infty$ , we have  $\mathbf{u} \in L^\infty((0, T), L^2(\frac{dx}{1+|x|^2})) + L^\infty((0, T), L^r)$ .

*Proof.* Let  $\mathbb{N}^2$  be enumerated as  $\{(j_n, k_n) / n \geq 1\}$ . We start with a sequence  $(\epsilon_{0,n}, \alpha_{0,n})_{n \in \mathbb{N}}$  which converges to  $(0, 0)$ . We choose sequences  $(\epsilon_{q,n}, \alpha_{q,n})_{n \in \mathbb{N}}$  by induction on  $q$ , such that the sequence  $(\epsilon_{q+1,n}, \alpha_{q+1,n})_{n \in \mathbb{N}}$  will be a subsequence of the sequence  $(\epsilon_{q,n}, \alpha_{q,n})_{n \in \mathbb{N}}$ .

Assume that we have chosen the sequence  $(\epsilon_{q,n}, \alpha_{q,n})_{n \in \mathbb{N}}$  for some  $q \geq 0$ . We take  $\omega_{q+1} \in \mathcal{D}(\mathbb{R}^3)$  such that  $\omega_{q+1}(x) = 1$  for  $|x| \leq R_{q+1} = 2^{k_{q+1}}$  and  $= 0$  for  $|x| > 2R_{q+1}$ . Let  $T_{q+1} = 2^{j_{q+1}}$ . For  $\alpha_{q,n} < \alpha_{T_{q+1}, \mathbf{u}_0}$ ,  $\omega_{q+1} \mathbf{u}_{\epsilon_{q,n}, \alpha_{q,n}}$  is bounded in  $L^2((0, T_{q+1}), H^{1/2})$  and supported in  $\{x \in \mathbb{R}^3 / |x| \leq 2R_{q+1}\}$ , while  $\partial_t(\omega_{q+1} \mathbf{u}_{\epsilon_{q,n}, \alpha_{q,n}})$  is bounded in  $L^2((0, T_{q+1}), H^{-4})$ . We may apply Lemma 8 and choose a subsequence  $(\epsilon_{q+1,n}, \alpha_{q+1,n})_{n \in \mathbb{N}}$  such that  $\omega_{q+1} \mathbf{u}_{\epsilon_{q+1,n}, \alpha_{q+1,n}}$  is strongly convergent in  $L^2((0, T_{q+1}), L^2)$  and thus  $\mathbf{u}_{\epsilon_{q+1,n}, \alpha_{q+1,n}}$  is strongly convergent in  $L^2((0, T_{q+1}) \times B_{R_{q+1}})$ .

We then use Cantor's diagonal argument and define  $(\epsilon_n, \alpha_n) = (\epsilon_{n,n}, \alpha_{n,n})$ . We have

- $\lim_{n \rightarrow +\infty} \epsilon_n = \lim_{n \rightarrow +\infty} \alpha_n = 0$ ,
- $\mathbf{u}_{\epsilon_n, \alpha_n}$  is strongly convergent in  $L^2((0, T), L^2(B_R))$  for every  $T > 0$  and every  $R > 0$ .

With similar arguments, we can grant that, for any  $T_k$ , we have that  $\mathbf{u}_{\epsilon_n, \alpha_n} = \mathbf{v}_{\epsilon_n, \alpha_n, T_k} + \mathbf{b}_{\epsilon_n, \alpha_n, T_k}$  on  $(0, T_k) \times \mathbb{R}^3$  with the strong convergence of  $\mathbf{v}_{\epsilon_n, \alpha_n, T_k}$  to  $\mathbf{v}_{T_k}$  and of  $\mathbf{b}_{\epsilon_n, \alpha_n, T_k}$  to  $\mathbf{b}_{T_k}$  in  $L^2((0, T_k), L^2(B_R))$  for every  $R > 0$ .

Let  $\mathbf{u}$  be the limit of  $\mathbf{u}_{\epsilon_n, \alpha_n}$ . We have, for  $R > 0$  and  $\chi_R$  the characteristic function of the ball  $B_R$  (and for  $\alpha_n < \alpha_{T_k, \mathbf{u}_0}$ ),

$$\begin{aligned} [\bullet] & \| \mathbf{b}_{T_k} - \mathbf{b}_{\epsilon_n, \alpha_n, T_k} \|_{L^2((0, T_k), L^2(\Phi_4 dx))} \\ & \leq \frac{1}{R^{\frac{1}{4}}} \| \mathbf{b}_{T_k} - \mathbf{b}_{\epsilon_n, \alpha_n, T_k} \|_{L^2((0, T_k), L^2(\Phi_{7/2} dx))} + \| \chi_R (\mathbf{b}_{T_k} - \mathbf{b}_{\epsilon_n, \alpha_n, T_k}) \|_{L^2((0, T_k), L^2)} \\ & \leq \frac{C}{R^{\frac{1}{4}}} \| \mathbf{b}_{T_k} - \mathbf{b}_{\epsilon_n, \alpha_n, T_k} \|_{L^2((0, T_k), L^r)} + \| \mathbf{b}_{T_k} - \mathbf{b}_{\epsilon_n, \alpha_n, T_k} \|_{L^2((0, T_k) \times B_R)}. \end{aligned}$$

As  $\| \mathbf{b}_{T_k} - \mathbf{b}_{\epsilon_n, \alpha_n, T_k} \|_{L^\infty((0, T_k), L^r)} \leq C_{T_k, \mathbf{u}_0}$ , we find that

$$\limsup_{n \rightarrow +\infty} \| \mathbf{b}_{T_k} - \mathbf{b}_{\epsilon_n, \alpha_n, T_k} \|_{L^2((0, T_k), L^2(\Phi_4 dx))} \leq C C_{T_k, \mathbf{u}_0} \sqrt{T_k} \frac{1}{R^{\frac{1}{4}}}.$$

Letting  $R$  go to  $+\infty$ , we find  $\lim_{n \rightarrow +\infty} \|\mathbf{b}_{T_k} - \mathbf{b}_{\epsilon_n, \alpha_n, T_k}\|_{L^2((0, T_k), L^2(\Phi_4 dx))} = 0$ .  
[•] Similarly, we have

$$\begin{aligned} & \|\mathbf{v}_{T_k} - \mathbf{v}_{\epsilon_n, \alpha_n, T_k}\|_{L^2((0, T_k), L^2(\Phi_4 dx))} \\ & \leq \frac{1}{R} \|\mathbf{v}_{T_k} - \mathbf{v}_{\epsilon_n, \alpha_n, T_k}\|_{L^2((0, T_k), L^2(\Phi_2 dx))} + \|\chi_R(\mathbf{v}_{T_k} - \mathbf{v}_{\epsilon_n, \alpha_n, T_k})\|_{L^2((0, T_k), L^2)}. \end{aligned}$$

As  $\|\mathbf{v}_{T_k} - \mathbf{v}_{\epsilon_n, \alpha_n, T_k}\|_{L^\infty((0, T_k), L^2(\Phi_2 dx))} \leq C_{T_k, \mathbf{u}_0}$ , we find

$$\lim_{n \rightarrow +\infty} \|\mathbf{v}_{T_k} - \mathbf{v}_{\epsilon_n, \alpha_n, T_k}\|_{L^2((0, T_k), L^2(\Phi_4 dx))} = 0.$$

[•] We remark that we have the inequalities

$$\begin{aligned} & \|\varphi_{\epsilon_n} * (\theta_{\alpha_n} f)\|_r \leq \|f\|_r, \\ & \|\varphi_{\epsilon_n} * (\theta_{\alpha_n} f)\|_{L^2(\Phi_2 dx)} \leq \|\mathcal{M}_f\|_{L^2(\Phi_2 dx)} \leq C\|f\|_{L^2(\Phi_2 dx)} \text{ (as } \Phi_2 \text{ is a Muckenhoupt weight),} \\ & \|\varphi_{\epsilon_n} * (\theta_{\alpha_n} f)\|_{L^2(B_R)} = \|\varphi_{\epsilon_n} * (\theta_{\alpha_n} \chi_{R+1} f)\|_{L^2(B_R)} \leq \|f\|_{L^2(B_{R+1})} \text{ (for } \epsilon_n < 1), \end{aligned}$$

so that we have

$$\lim_{n \rightarrow +\infty} \|\varphi_{\epsilon_n} * (\theta_{\alpha_n} (\mathbf{b}_{T_k} - \mathbf{b}_{\epsilon_n, \alpha_n, T_k}))\|_{L^2((0, T_k), L^2(\Phi_4 dx))} = 0$$

and

$$\lim_{n \rightarrow +\infty} \|\varphi_{\epsilon_n} * (\theta_{\alpha_n} (\mathbf{v}_{T_k} - \mathbf{v}_{\epsilon_n, \alpha_n, T_k}))\|_{L^2((0, T_k), L^2(\Phi_4 dx))} = 0.$$

[•] Similarly, we have the inequalities

$$\begin{aligned} & \|f - \varphi_{\epsilon_n} * (\theta_{\alpha_n} f)\|_r = \|\varphi_{\epsilon_n} * ((1 - \theta_{\alpha_n})f)\|_r \leq \|f\|_r, \\ & \|f - \varphi_{\epsilon_n} * (\theta_{\alpha_n} f)\|_{L^2(\Phi_2 dx)} \leq \|\mathcal{M}_{(1 - \theta_{\alpha_n})f}\|_{L^2(\Phi_2 dx)} \leq C\|f\|_{L^2(\Phi_2 dx)}, \\ & \|f - \varphi_{\epsilon_n} * (\theta_{\alpha_n} f)\|_{L^2(B_R)} = \|\varphi_{\epsilon_n} * ((1 - \theta_{\alpha_n})\chi_{R+1} f)\|_{L^2(B_R)} \leq \|(1 - \theta_{\alpha_n})\chi_{R+1} f\|_2 \end{aligned}$$

(for  $\epsilon_n < 1$ ).

As we have  $(1 - \theta_{\alpha_n})\chi_{R+1} = 0$  for  $\alpha_n < \frac{1}{R+1}$ , we find that

$$\lim_{n \rightarrow +\infty} \|\mathbf{b}_{T_k} - \varphi_{\epsilon_n} * (\theta_{\alpha_n} \mathbf{b}_{T_k})\|_{L^2((0, T_k), L^2(\Phi_4 dx))} = 0$$

and

$$\lim_{n \rightarrow +\infty} \|\mathbf{v}_{T_k} - \varphi_{\epsilon_n} * (\theta_{\alpha_n} \mathbf{v}_{T_k})\|_{L^2((0, T_k), L^2(\Phi_4 dx))} = 0.$$

Combining all those estimates, we find that for every  $T > 0$

$$\lim_{n \rightarrow +\infty} \|\mathbf{u} \otimes \mathbf{u} - (\varphi_{\epsilon_n} * (\theta_{\alpha_n} \mathbf{u}_{\epsilon_n, \alpha_n})) \otimes \mathbf{u}_{\epsilon_n, \alpha_n}\|_{L^1((0, T), L^2(\Phi_4 dx))} = 0.$$

From the proof of Theorem 2, we see that

$$\lim_{n \rightarrow +\infty} \Delta \mathbf{u}_{\epsilon_n, \alpha_n} - \mathbb{P} \operatorname{div}((\varphi_{\epsilon_n} * (\theta_{\alpha_n} \mathbf{u}_{\epsilon_n, \alpha_n})) \otimes \mathbf{u}_{\epsilon_n, \alpha_n}) = \Delta \mathbf{u} - \mathbb{P} \operatorname{div}(\mathbf{u} \otimes \mathbf{u})$$

in  $L^1((0, T), H^{-4}(\Phi_8 dx))$ .  $\mathbf{u}$  is a solution of the Navier–Stokes equations (19).  $\square$

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