

**JACOB'S LADDERS, OUR OLD FORMULA (1985) AND NEW  
 $\zeta$ -EQUIVALENT OF THE FERMAT-WILES THEOREM ON  
TWO-PARAMETRIC SET OF LEMNISCATES OF BERNOULLI**

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**ABSTRACT.** In our paper from 1985 we have constructed two integrals of the Riemann's function  $Z^2(t)$  over two disconnected sets with asymptotically equal measures such that these two integrals differ by considerably big excess. In the present paper we use the formula for that excess to construct a new  $\zeta$ -equivalent of the Fermat-Wiles theorem on a two-parametric set of lemniscates of Bernoulli.

1. INTRODUCTION

**1.1.** Let us remind the Hardy-Littlewood integral<sup>1</sup>

$$(1.1) \quad J = \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt, \quad T > 0.$$

We have obtained in our paper [2] some new facts about the structure of the increments

$$(1.2) \quad \int_T^{T+U} \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt, \quad U = T^{5/12} \ln^3 T$$

of the Hardy-Littlewood integral as it follows.

**1.2.** At first, let us remind the following notions connected with the Riemann's function

$$(1.3) \quad Z(t) = e^{i\vartheta(t)} \zeta\left(\frac{1}{2} + it\right); \quad Z(t) \in \mathbb{R} \text{ if } t \in \mathbb{R},$$

where<sup>2</sup>

$$(1.4) \quad \vartheta(t) = -\frac{t}{2} \ln \pi + \operatorname{Im} \ln \Gamma\left(\frac{1}{4} + i\frac{t}{2}\right) = \frac{t}{2} \ln \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + \mathcal{O}\left(\frac{1}{t}\right).$$

Of course, it is true that

$$(1.5) \quad \left| \zeta\left(\frac{1}{2} + it\right) \right| = |Z(t)|.$$

Let us put (see (1.4))

$$\vartheta_1(t) = \frac{t}{2} \ln \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8}.$$

In our paper [3] we have defined the following set of sequences

$$(1.6) \quad \{g_\nu(\tau)\}_{\nu=1}^\infty, \quad \tau \in [-\pi, \pi],$$

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*Key words and phrases.* Riemann zeta-function.

<sup>1</sup>See [1]

<sup>2</sup>See [11], (35), (44), (62), comp. [12], p. 98.

where, see [3], (6),

$$(1.7) \quad \vartheta_1[g_\nu(\tau)] = \frac{\pi}{2}\nu + \frac{\tau}{2}; \quad g_\nu(0) = g_\nu.$$

Next, we have defined<sup>3</sup> two systems of disconnected sets

$$(1.8) \quad \begin{aligned} G_3(v) &= G_3(v; T, U) = \\ &\bigcup_{\substack{T \leq g_{2\nu} \leq T+U}} \{t : g_{2\nu}(-v) < t < g_{2\nu}(v)\}, \quad 0 \leq v \leq \frac{\pi}{2}, \\ G_4(w) &= G_4(w; T, U) = \\ &\bigcup_{\substack{T \leq g_{2\nu+1} \leq T+U}} \{t : g_{2\nu+1}(-w) < t < g_{2\nu+1}(w)\}, \quad 0 \leq w \leq \frac{\pi}{2}. \end{aligned}$$

**1.3.** Now, let us put, see (1.5), (1.8),

$$(1.9) \quad \begin{aligned} \sum_{T \leq g_{2\nu} \leq T+U} \int_{g_{2\nu}(-v)}^{g_{2\nu}(v)} Z^2(t) dt &= \int_{G_3(v)} Z^2(t) dt, \\ \sum_{T \leq g_{2\nu+1} \leq T+U} \int_{g_{2\nu+1}(-w)}^{g_{2\nu+1}(w)} Z^2(t) dt &= \int_{G_4(w)} Z^2(t) dt. \end{aligned}$$

In this paper we use as a basic formula a variant of our formula [3], (16). Namely, we use the following result.

**Formula 1.**

$$(1.10) \quad \begin{aligned} \int_{G_3(v)} Z^2(t) dt - \int_{G_4(w)} Z^2(t) dt &= \\ \frac{4}{\pi} U \sin v + \mathcal{O}(v T^{5/12} \ln^3 T) & \end{aligned}$$

for two continuum systems  $\{G_3(v)\}$ ,  $\{G_4(w)\}$  of disconnected sets with almost equal measures<sup>4</sup>:

$$(1.11) \quad m[G_3(v)] = m[G_4(w)] + \mathcal{O}\left(\frac{v}{\ln T}\right), \quad T \rightarrow \infty,$$

where

$$(1.12) \quad U = T^{5/12} \ln^3 T, \quad 0 < v \leq \frac{\pi}{2}.$$

*Remark 1.* That is, we have proved in [3] that there is considerable difference between two basic integrals (1.9), namely there is a big excess of the first one over the other.

**1.4.** Now, let us remind the lemniscate of Bernoulli. It is the planar locus traced out by a point  $M$  that moves in such a way that the product of its distances from two fixed points

$$(1.13) \quad F_1 = (a, 0), \quad F_2 = (-a, 0)$$

is constant

$$(1.14) \quad |F_1 M| |F_2 M| = a^2.$$

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<sup>3</sup>See [3], p. 25.

<sup>4</sup>See [3], (13).

*Remark 2.* Of course, the lemniscate of Bernoulli is a special case of the ovals of Cassini:  $a^2 \rightarrow b^2$  in (1.14).

Further, let us remind the rationals of Fermat:

$$(1.15) \quad \frac{x^n + y^n}{z^n}, \quad x, y, z, n \in \mathbb{N}, \quad n \geq 3.$$

We shall use the two parametric set of the lemniscates of Bernoulli<sup>5</sup>:

$$(1.16) \quad a = \left( \frac{4}{\pi} \sin v \right)^{6/5} \left( \frac{1}{8l_3} \right)^{1/2}; \quad l_3 > 0, \quad 0 < v \leq \frac{\pi}{2}.$$

As an example of a new result obtained in this paper we can formulate the following statement: The  $\zeta$ -condition

$$(1.17) \quad \begin{aligned} & \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \left\{ \sum_{\substack{g_{2\nu} \leq \frac{x^n+y^n}{z^n} \tau + U(\frac{x^n+y^n}{z^n} \tau) \\ g_{2\nu} \geq \frac{x^n+y^n}{z^n} \tau}} \int_{g_{2\nu}(-v)}^{g_{2\nu}(v)} Z^2(t) dt - \right. \\ & \left. \sum_{\substack{g_{2\nu+1} \leq \frac{x^n+y^n}{z^n} \tau + U(\frac{x^n+y^n}{z^n} \tau) \\ g_{2\nu+1} \geq \frac{x^n+y^n}{z^n} \tau}} \int_{g_{2\nu+1}(-v)}^{g_{2\nu+1}(v)} Z^2(t) dt \right\}^{12/5} \times \\ & \left\{ \int_{[\frac{x^n+y^n}{z^n} \tau]^4}^{[\frac{x^n+y^n}{z^n} \tau + 2|F_1 M|]^4} \prod_{r=0}^3 Z^2(\varphi_1^r(t)) dt \times \right. \\ & \left. \int_{[\frac{x^n+y^n}{z^n} \tau]^3}^{[\frac{x^n+y^n}{z^n} \tau + 2|F_2 M|]^3} \prod_{r=0}^2 Z^2(\varphi_1^r(t)) dt \right\}^{-1} \times \\ & \left\{ \int_{[\frac{x^n+y^n}{z^n} \tau]^1}^{[\frac{x^n+y^n}{z^n} \tau + 2l_3]^1} Z^2(t) dt \right\}^{-1/5} \neq 1; \\ & [G]^r = \varphi_1^{-r}(G), \end{aligned}$$

on the set of all Fermat's rationals and for every fixed

$$(1.18) \quad l_3 > 0, \quad 0 < v \leq \frac{\pi}{2},$$

expresses the  $\zeta$ -equivalent of the Fermat-Wiles theorem on the two parametric set of corresponding lemniscates of Bernoulli.

*Remark 3.* Let us remind that C. F. Gauss introduced the lemniscate functions  $\text{sl } \tau$  and  $\text{cl } \tau$ :

$$\int_0^{\text{sl } \tau} \frac{dx}{\sqrt{1-x^4}} = \tau, \quad \int_{\text{cl } \tau}^1 \frac{dx}{\sqrt{1-x^4}} = \tau$$

as the first elliptic functions on the Gauss plane.

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<sup>5</sup>See (1.13) and (1.14).

## 2. JACOB'S LADDERS: NOTIONS AND BASIC GEOMETRICAL PROPERTIES

**2.1.** In this paper we use the following notions of our works [5] – [9]:

- (a) Jacob's ladder  $\varphi_1(T)$ ,
- (b) direct iterations of Jacob's ladders

$$\begin{aligned}\varphi_1^0(t) &= t, \quad \varphi_1^1(t) = \varphi_1(t), \quad \varphi_1^2(t) = \varphi_1(\varphi_1(t)), \dots, \\ \varphi_1^k(t) &= \varphi_1(\varphi_1^{k-1}(t))\end{aligned}$$

for every fixed natural number  $k$ ,

- (c) reverse iterations of Jacob's ladders

$$\begin{aligned}(2.1) \quad \varphi_1^{-1}(T) &= \frac{1}{T}, \quad \varphi_1^{-2}(T) = \varphi_1^{-1}(\frac{1}{T}) = \frac{2}{T}, \dots, \\ \varphi_1^{-r}(T) &= \varphi_1^{-1}(\frac{r-1}{T}) = \frac{r}{T}, \quad r = 1, \dots, k,\end{aligned}$$

where, for example,

$$(2.2) \quad \varphi_1^r(T) = \frac{r-1}{T}$$

for every fixed  $k \in \mathbb{N}$  and every sufficiently big  $T > 0$ . We also use the properties of the reverse iterations listed below.

$$(2.3) \quad \frac{r}{T} - \frac{r-1}{T} \sim (1-c)\pi(T); \quad \pi(T) \sim \frac{T}{\ln T}, \quad r = 1, \dots, k, \quad T \rightarrow \infty,$$

$$(2.4) \quad \frac{0}{T} = T < \frac{1}{T}(T) < \frac{2}{T}(T) < \dots < \frac{k}{T}(T),$$

and

$$(2.5) \quad T \sim \frac{1}{T} \sim \frac{2}{T} \sim \dots \sim \frac{k}{T}, \quad T \rightarrow \infty.$$

*Remark 4.* The asymptotic behaviour of the points

$$\{\frac{1}{T}, \frac{2}{T}, \dots, \frac{k}{T}\}$$

is as follows: at  $T \rightarrow \infty$  these points recede unboundedly each from other and all together are receding to infinity. Hence, the set of these points behaves at  $T \rightarrow \infty$  as one-dimensional Friedmann-Hubble expanding Universe.

**2.2.** Let us remind that we have proved<sup>6</sup> the existence of almost linear increments

$$\begin{aligned}(2.6) \quad \int_{\frac{r-1}{T}}^{\frac{r}{T}} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt &\sim (1-c) \frac{r-1}{T}, \\ r &= 1, \dots, k, \quad T \rightarrow \infty, \quad \frac{r}{T} = \frac{r}{T}(T) = \varphi_1^{-r}(T)\end{aligned}$$

for the Hardy-Littlewood integral (1918)

$$(2.7) \quad J(T) = \int_0^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt.$$

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<sup>6</sup>See [9], (3.4).

For completeness, we give here some basic geometrical properties related to Jacob's ladders. These are generated by the sequence

$$(2.8) \quad T \rightarrow \left\{ \begin{matrix} r \\ T(T) \end{matrix} \right\}_{r=1}^k$$

of reverse iterations of the Jacob's ladders for every sufficiently big  $T > 0$  and every fixed  $k \in \mathbb{N}$ .

**Property 1.** The sequence (2.8) defines a partition of the segment  $[T, T]$  as follows

$$(2.9) \quad |[T, T]| = \sum_{r=1}^k |[\frac{T}{r}, \frac{T}{r}]|$$

on the asymptotically equidistant parts

$$(2.10) \quad \frac{T}{r} - \frac{T}{r-1} \sim \frac{T}{r+1} - \frac{T}{r}, \\ r = 1, \dots, k-1, T \rightarrow \infty.$$

**Property 2.** Simultaneously with the Property 1, the sequence (2.8) defines the partition of the integral

$$(2.11) \quad \int_T^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt$$

into the parts

$$(2.12) \quad \int_T^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt = \sum_{r=1}^k \int_{\frac{T}{r-1}}^{\frac{T}{r}} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt,$$

that are asymptotically equal

$$(2.13) \quad \int_{\frac{T}{r-1}}^{\frac{T}{r}} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt \sim \int_{\frac{T}{r}}^{\frac{T}{r+1}} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt, T \rightarrow \infty.$$

It is clear, that (2.10) follows from (2.3) and (2.5) since

$$(2.14) \quad \frac{T}{r} - \frac{T}{r-1} \sim (1-c) \frac{T}{\ln T} \sim (1-c) \frac{T}{\ln T}, \quad r = 1, \dots, k,$$

while our eq. (2.13) follows from (2.6) and (2.5).

### 3. SOME NEXT REMARKS ABOUT THE RESULT (1.10) OF THE PAPER [3]

Since the paper [3] has been published 40 years ago in almost unknown journal, we give here some complements to formulae (1.8) – (1.12).

**3.1.** We start with the following remarks.

*Remark 5.* Since, see (1.17),

$$(3.1) \quad g_{2\nu} \left( \frac{\pi}{2} \right) = g_{2\nu+1} \left( -\frac{\pi}{2} \right),$$

then we have the systems of two disconnected sets

$$(3.2) \quad \{G_3(v)\}, \{G_4(w)\}, \quad 0 < v, w \leq \frac{\pi}{2}$$

with the property<sup>7</sup>

$$(3.3) \quad G_3(v) \cap G_4(w) = \emptyset.$$

*Remark 6.* We have used the symbols  $x, y$  in our paper [3] instead of  $v, w$  used in the present paper, where  $x$  and  $y$  are reserved for the contact with the Fermat-Wiles theorem.

**3.2.** Since<sup>8</sup>

$$(3.4) \quad \begin{aligned} g_{2\nu}(v) - g_{2\nu}(-v) &= \frac{2v}{\ln \frac{T}{2\pi}} + \mathcal{O}\left(\frac{vU}{T \ln^2 T}\right), \\ g_{2\nu+1}(w) - g_{2\nu+1}(-w) &= \frac{2w}{\ln \frac{T}{2\pi}} + \mathcal{O}\left(\frac{wU}{T \ln^2 T}\right), \end{aligned}$$

where

$$(3.5) \quad g_{2\nu}(-v), g_{2\nu+1}(-w) \in [T, T+U], \quad 0 < v, w \leq \frac{\pi}{2},$$

and<sup>9</sup>

$$\sum_{T \leq g_{2\nu} \leq T+U} 1 = \frac{1}{\pi} U \ln \frac{T}{2\pi} + \mathcal{O}(1); \quad \sum_{g_{2\nu}} = \sum_{g_{2\nu+1}} + O(1),$$

then for measures of the sets  $G_3(v)$  and  $G_4(w)$  we have the following formulae

$$(3.6) \quad \begin{aligned} m[G_3(v)] &= \frac{v}{\pi} U + \mathcal{O}\left(\frac{v}{\ln T}\right), \\ m[G_4(w)] &= \frac{w}{\pi} U + \mathcal{O}\left(\frac{w}{\ln T}\right), \end{aligned}$$

where, of course<sup>10</sup>,

$$(3.7) \quad G_3(v), G_4(w) \subset [T, T+U], \quad 0 < v, w \leq \frac{\pi}{2}.$$

*Remark 7.* Let us denote

$$(3.8) \quad G_3\left(\frac{\pi}{2}\right) = \bar{G}_3, \quad G_4\left(\frac{\pi}{2}\right) = \bar{G}_4.$$

Then, see (3.6),

$$(3.9) \quad m[\bar{G}_3] + m[\bar{G}_4] = U + \mathcal{O}\left(\frac{1}{\ln T}\right), \quad T \rightarrow \infty.$$

**3.3.** Finally, we present, for completeness, next two results of the paper [3]. Namely, the formulae

$$(3.10) \quad \int_{G_3(v)} Z^2(t) dt \sim \int_{G_4(v)} Z^2(t) dt \sim \frac{v}{\pi} U \ln T, \quad T \rightarrow \infty,$$

and also the formula for the difference of the two integrals

$$(3.11) \quad \int_{G_3(v)} Z^2(t) dt - \int_{G_4(v)} Z^2(t) dt \sim \frac{4}{\pi} U \sin v, \quad T \rightarrow \infty.$$

<sup>7</sup>See the symbols of inequalities in curly brackets in (1.8).

<sup>8</sup>See [3], (11).

<sup>9</sup>See [2], (21).

<sup>10</sup>See (3.5).

**4. THE FUNCTIONAL GENERATED BY THE FORMULA (1.10) AND NEXT  
 $\zeta$ -EQUIVALENT OF THE FERMAT-WILES THEOREM**

**4.1.** We rewrite the formula (1.10) into the following form, see (1.5), (1.12),

$$(4.1) \quad \begin{aligned} & \sum_{g_{2\nu} \geq T}^{g_{2\nu} \leq T+U(T)} \int_{g_{2\nu}(-v)}^{g_{2\nu}(v)} Z^2(t) dt - \\ & \sum_{g_{2\nu+1} \geq T}^{g_{2\nu+1} \leq T+U(T)} \int_{g_{2\nu+1}(-v)}^{g_{2\nu+1}(v)} Z^2(t) dt = \\ & \left\{ 1 + \mathcal{O}\left(\frac{1}{\ln T}\right) \right\} \frac{4}{\pi} \sin v T^{5/12} \ln^3 T, \quad T \rightarrow \infty, \end{aligned}$$

and further

$$(4.2) \quad \begin{aligned} & \left\{ \sum_{g_{2\nu} \geq T}^{g_{2\nu} \leq T+U(T)} \int_{g_{2\nu}(-v)}^{g_{2\nu}(v)} Z^2(t) dt - \sum_{g_{2\nu+1} \geq T}^{g_{2\nu+1} \leq T+U(T)} \int_{g_{2\nu+1}(-v)}^{g_{2\nu+1}(v)} Z^2(t) dt \right\}^{12/5} = \\ & \left\{ 1 + \mathcal{O}\left(\frac{1}{\ln T}\right) \right\} \left( \frac{4}{\pi} \sin v \right)^{12/5} T \ln^4 T \ln^{1/5} T, \\ & U(T) = T^{5/12} \ln^3 T, \quad 0 < v \leq \frac{\pi}{2}. \end{aligned}$$

Next, we use in (4.2) our formula<sup>11</sup>

$$(4.3) \quad 2l \ln^k T = \left\{ \int_T^{\widehat{T+2l}} \prod_{r=0}^{k-1} Z^2(\varphi_1^r(t)) dt \right\} \times \left\{ 1 + \mathcal{O}\left(\frac{\ln \ln T}{\ln T}\right) \right\},$$

$$k \in \mathbb{N}, \quad l > 0; \quad \widehat{T} = [T]^k = \varphi_1^{-k}(T), \dots$$

for  $k = 4, 3, 1$ , where we use the partition  $7 = 4 + 3$  (for example), and obtain the following

$$(4.4) \quad \begin{aligned} & \left\{ \sum_{g_{2\nu} \geq T}^{g_{2\nu} \leq T+U(T)} \int_{g_{2\nu}(-v)}^{g_{2\nu}(v)} Z^2(t) dt - \sum_{g_{2\nu+1} \geq T}^{g_{2\nu+1} \leq T+U(T)} \int_{g_{2\nu+1}(-v)}^{g_{2\nu+1}(v)} Z^2(t) dt \right\}^{12/5} = \\ & \left\{ 1 + \mathcal{O}\left(\frac{\ln \ln T}{\ln T}\right) \right\} \left( \frac{4}{\pi} \sin v \right)^{12/5} \frac{1}{8l_1 l_2 l_3} T \times \\ & \int_T^{\widehat{T+2l_1}} \prod_{r=0}^3 Z^2(\varphi_1^r(t)) dt \times \int_T^{\widehat{T+2l_2}} \prod_{r=0}^2 Z^2(\varphi_1^r(t)) dt \times \\ & \left\{ \int_T^{\widehat{T+2l_1}} Z^2(t) dt \right\}^{1/5} \end{aligned}$$

for every fixed

$$(4.5) \quad l_1, l_2, l_3 > 0, \quad 0 < v \leq \frac{\pi}{2}.$$

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<sup>11</sup>See [10], (3.18);  $f_m \equiv 1$ .

**4.2.** Next, we introduce the following condition, see (4.4),

$$(4.6) \quad \left( \frac{4}{\pi} \sin v \right)^{12/5} \frac{1}{8l_1 l_2 l_3} = 1,$$

that is, for example,

$$(4.7) \quad l_1 l_2 = \left( \frac{4}{\pi} \sin v \right)^{12/5} \frac{1}{8l_3}, \quad l_1, l_2, l_3 > 0, \quad 0 < v \leq \frac{\pi}{2},$$

or

$$(4.8) \quad l_1 l_2 = a^2, \quad a = \left( \frac{4}{\pi} \sin v \right)^{6/5} \left( \frac{1}{8l_3} \right)^{1/2}.$$

Now, we put, for example<sup>12</sup>

$$(4.9) \quad l_1 = |F_1 M|, \quad l_2 = |F_2 M|$$

into (4.8) and we obtain<sup>13</sup> the following result.

**Lemma 1.** It is true that

$$(4.10) \quad \begin{aligned} & \left\{ \sum_{g_{2\nu} \geq T}^{g_{2\nu} \leq T+U(T)} \int_{g_{2\nu}(-v)}^{g_{2\nu}(v)} Z^2(t) dt - \sum_{g_{2\nu+1} \geq T}^{g_{2\nu+1} \leq T+U(T)} \int_{g_{2\nu+1}(-v)}^{g_{2\nu+1}(v)} Z^2(t) dt \right\}^{12/5} = \\ & \left\{ 1 + \mathcal{O} \left( \frac{\ln \ln T}{\ln T} \right) \right\} T \int_T^{\widehat{T+2|F_1 M|}} \prod_{r=0}^3 Z^2(\varphi_1^r(t)) dt \times \\ & \int_T^{\widehat{T+2|F_2 M|}} \prod_{r=0}^2 Z^2(\varphi_1^r(t)) dt \times \left\{ \int_T^{\widehat{T+2l_3}} Z^2(t) dt \right\}^{1/5} \end{aligned}$$

for every fixed

$$(4.11) \quad l_3 > 0, \quad 0 < v \leq \frac{\pi}{2},$$

and every point  $M \in \mathcal{L}[l_3, v]$  i.e. arbitrary point of the lemniscate of Bernoulli (4.8)  $= \mathcal{L}[l_3, v]$ .

**4.3.** Next, we make the substitution

$$(4.12) \quad T = x\tau, \quad x > 0; \quad \{T \rightarrow +\infty\} \Leftrightarrow \{\tau \rightarrow +\infty\}$$

in (4.10) and we obtain, as usually in our theory, the following functional.

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<sup>12</sup>See (1.13), (1.14), (1.16).

<sup>13</sup>See (4.4), (4.6) – (4.8).

**Theorem 1.** It is true that

(4.13)

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \times$$

$$\left\{ \sum_{g_{2\nu} \geq x\tau}^{g_{2\nu} \leq x\tau + U(x\tau)} \int_{g_{2\nu}(-v)}^{g_{2\nu}(v)} Z^2(t) dt - \sum_{g_{2\nu+1} \geq x\tau}^{g_{2\nu+1} \leq x\tau + U(x\tau)} \int_{g_{2\nu+1}(-v)}^{g_{2\nu+1}(v)} Z^2(t) dt \right\}^{12/5} \times$$

$$\left\{ \int_{[x\tau]^4}^{[x\tau+2|F_1M|]^4} \prod_{r=0}^3 Z^2(\varphi_1^r(t)) dt \times \int_{[x\tau]^3}^{[x\tau+2|F_2M|]^3} \prod_{r=0}^2 Z^2(\varphi_1^r(t)) dt \right\}^{-1} \times$$

$$\left\{ \int_{[x\tau]^1}^{[x\tau+2l_3]^1} Z^2(t) dt \right\}^{-1/5} = x$$

for every fixed

$$(4.14) \quad x > 0, \quad l_3 > 0, \quad 0 < v \leq \frac{\pi}{2},$$

and for every point  $M \in \mathcal{L}[l_3, v]$ .

**4.4.** We obtain from (4.13) the following consequence for Fermat's rationals<sup>14</sup>

**Corollary.**

$$(4.15) \quad \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \times$$

$$\left\{ \sum_{g_{2\nu} \geq \frac{x^n+y^n}{z^n}\tau}^{g_{2\nu} \leq \frac{x^n+y^n}{z^n}\tau + U(\frac{x^n+y^n}{z^n}\tau)} \int_{g_{2\nu}(-v)}^{g_{2\nu}(v)} Z^2(t) dt - \right.$$

$$\left. \sum_{g_{2\nu+1} \geq \frac{x^n+y^n}{z^n}\tau}^{g_{2\nu+1} \leq \frac{x^n+y^n}{z^n}\tau + U(\frac{x^n+y^n}{z^n}\tau)} \int_{g_{2\nu+1}(-v)}^{g_{2\nu+1}(v)} Z^2(t) dt \right\}^{12/5} \times$$

$$\left\{ \int_{[\frac{x^n+y^n}{z^n}\tau]^4}^{[\frac{x^n+y^n}{z^n}\tau + 2|F_1M|]^4} \prod_{r=0}^3 Z^2(\varphi_1^r(t)) dt \times \right.$$

$$\left. \int_{[\frac{x^n+y^n}{z^n}\tau]^3}^{[\frac{x^n+y^n}{z^n}\tau + 2|F_2M|]^3} \prod_{r=0}^2 Z^2(\varphi_1^r(t)) dt \right\}^{-1} \times$$

$$\left\{ \int_{[\frac{x^n+y^n}{z^n}\tau]^1}^{[\frac{x^n+y^n}{z^n}\tau + 2l_3]^1} Z^2(t) dt \right\}^{-1/5} = \frac{x^n + y^n}{z^n}$$

for every fixed

$$(4.16) \quad l_3 > 0, \quad 0 < v \leq \frac{\pi}{2},$$

and for every point  $M \in \mathcal{L}[l_3, v]$ .

Consequently, we obtain from (4.15) the following result.

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<sup>14</sup>See (1.15).

**Theorem 2.** The  $\zeta$ -condition

$$(4.17) \quad \begin{aligned} & \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \times \\ & \left\{ \begin{array}{l} g_{2\nu} \leq \frac{x^n + y^n}{z^n} \tau + U(\frac{x^n + y^n}{z^n} \tau) \\ g_{2\nu+1} \geq \frac{x^n + y^n}{z^n} \tau \end{array} \right. \int_{g_{2\nu}(-v)}^{g_{2\nu}(v)} Z^2(t) dt - \\ & \left. \begin{array}{l} g_{2\nu+1} \leq \frac{x^n + y^n}{z^n} \tau + U(\frac{x^n + y^n}{z^n} \tau) \\ g_{2\nu+1} \geq \frac{x^n + y^n}{z^n} \tau \end{array} \right. \int_{g_{2\nu+1}(-v)}^{g_{2\nu+1}(v)} Z^2(t) dt \left. \right\}^{12/5} \times \\ & \left\{ \int_{[\frac{x^n + y^n}{z^n} \tau]^4}^{[\frac{x^n + y^n}{z^n} \tau + 2|F_1 M|]^4} \prod_{r=0}^3 Z^2(\varphi_1^r(t)) dt \times \right. \\ & \left. \int_{[\frac{x^n + y^n}{z^n} \tau]^3}^{[\frac{x^n + y^n}{z^n} \tau + 2|F_2 M|]^3} \prod_{r=0}^2 Z^2(\varphi_1^r(t)) dt \right\}^{-1} \times \\ & \left\{ \int_{[\frac{x^n + y^n}{z^n} \tau]^1}^{[\frac{x^n + y^n}{z^n} \tau + 2l_3]^1} Z^2(t) dt \right\}^{-1/5} \neq 1 \end{aligned}$$

on the set of all Fermat's rationals and for every fixed

$$(4.18) \quad l_3 > 0, \quad 0 < v \leq \frac{\pi}{2},$$

and also for every point  $M \in \mathcal{L}[l_3, v]$ , expresses the new  $\zeta$ -equivalent of the Fermat-Wiles theorem (on two parametric set of lemniscates of Bernloulli, see (1.13), (1.14), (1.16).)

## 5. A FACTORIZATION FORMULA FOR THE EXCESS (1.10)

**5.1.** Next, we shall use in (4.4) our almost linear formula, see [8], (3.4), (3.6),  $r = 1$ , in the form

$$(5.1) \quad (1 - c)T = \{1 + \mathcal{O}(T^{-1/3+\delta})\} \int_T^{\frac{1}{T}} Z^2(t) dt,$$

that gives us the following formula.

**Lemma 2.**

$$\begin{aligned}
 & \left\{ \sum_{g_{2\nu} \geq T}^{g_{2\nu} \leq T+U(T)} \int_{g_{2\nu}(-v)}^{g_{2\nu}(v)} Z^2(t) dt - \sum_{g_{2\nu+1} \geq T}^{g_{2\nu+1} \leq T+U(T)} \int_{g_{2\nu+1}(-v)}^{g_{2\nu+1}(v)} Z^2(t) dt \right\}^{12/5} = \\
 & \left\{ 1 + \mathcal{O} \left( \frac{\ln \ln T}{\ln T} \right) \right\} \left( \frac{4}{\pi} \sin v \right)^{12/5} \frac{1}{8(1-c)l_1 l_2 l_3} \times \int_T^T Z^2(t) dt \times \\
 (5.2) \quad & \left\{ \int_{\widehat{T+2l_3}}^{\widehat{T+2l_3}} Z^2(t) dt \right\}^{1/5} \times \int_T^{\widehat{T+2l_2}} \prod_{r=0}^2 Z^2(\varphi_1^r(t)) dt \times \\
 & \int_T^{\widehat{T+2l_3}} \prod_{r=0}^4 Z^2(\varphi_1^r(t)) dt
 \end{aligned}$$

for every fixed

$$(5.3) \quad l_1, l_2, l_3 > 0, \quad 0 < v \leq \frac{\pi}{2}.$$

**5.2.** Now, we use the following condition for the asymptotic equality<sup>15</sup>

$$(5.4) \quad \left( \frac{4}{\pi} \sin v \right)^{12/5} \frac{1}{8(1-c)l_1 l_2 l_3} = 1$$

i.e., for example<sup>16</sup>,

$$(5.5) \quad l_1 l_2 \bar{a}^2, \quad \bar{a} = \left( \frac{4}{\pi} \sin v \right)^{6/5} \left( \frac{1}{8(1-c)l_3} \right)^{1/2}.$$

Consequently, we put, for example<sup>17</sup>

$$(5.6) \quad F_3 = F_3(\bar{a}, 0), \quad F_4 = F_4(-\bar{a}, 0)$$

and after this we substitute the values

$$(5.7) \quad l_1 = |F_3 M|, \quad l_2 = |F_4 M|$$

into (5.2) and we obtain<sup>18</sup> the following result.

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<sup>15</sup>Comp. (4.6).

<sup>16</sup>Comp. (4.7), (4.8).

<sup>17</sup>Comp. (1.13) – (1.16).

<sup>18</sup>Comp. (4.6) – (4.9).

**Theorem 3.**

$$\begin{aligned}
 & \sum_{g_{2\nu} \geq T}^{g_{2\nu} \leq T+U(T)} \int_{g_{2\nu}(-v)}^{g_{2\nu}(v)} Z^2(t) dt - \sum_{g_{2\nu+1} \geq T}^{g_{2\nu+1} \leq T+U(T)} \int_{g_{2\nu+1}(-v)}^{g_{2\nu+1}(v)} Z^2(t) dt \sim \\
 & \left\{ \int_T^{\frac{1}{T}} Z^2(t) dt \right\}^{5/12} \times \left\{ \int_T^{\frac{1}{T+2l_3}} Z^2(t) dt \right\}^{1/12} \times \\
 (5.8) \quad & \left\{ \int_T^{\frac{3}{T+2|F_4M|}} \prod_{r=0}^2 Z^2(\varphi_1^r(t)) dt \right\}^{5/12} \times \\
 & \left\{ \int_T^{\frac{4}{T+2|F_3M|}} \prod_{r=0}^3 Z^2(\varphi_1^r(t)) dt \right\}^{5/12}
 \end{aligned}$$

for every fixed

$$(5.9) \quad l_3 > 0, \quad 0 < v \leq \frac{\pi}{2},$$

and every point  $M \in \bar{\mathcal{L}}[l_3, v]$ , i.e. the corresponding lemniscate of Bernoulli (5.5)  $= \bar{\mathcal{L}}[l_3, v]$ .

*Remark 8.* The formula (5.8) expresses the asymptotic factorization of the excess (1.10) by means of the basic set of integrals

$$\begin{aligned}
 (5.10) \quad & \left\{ \int_T^{[T]^1} Z^2(t) dt, \int_{[T]^1}^{[T+2l_3]^1} Z^2(t) dt \right. \\
 & \left. \int_{[T]^3}^{[T+2l_2]^3} \prod_{r=0}^2 Z^2(\varphi_1^r(t)) dt, \int_{[T]^4}^{[T+2l_1]^4} \prod_{r=0}^3 Z^2(\varphi_1^r(t)) dt \right\},
 \end{aligned}$$

where, of course,

$$[G]^r = \varphi_1^r(G),$$

on the other two-parametric system of lemniscates of Bernoulli  $\bar{\mathcal{L}}[l_3, v]$ .

## 6. ON THE STRUCTURE OF THE SET OF INTERVALS UNDER QUESTION

**6.1.** Our results listed above contain integrations over the intervals

$$\begin{aligned}
 (6.1) \quad & \{(T, [T]^1), ([T]^1, [T + 2l_3]^1), ([T]^3, [T + 2l_2]^3), \\
 & ([T]^4, [T + 2l_1]^4)\}.
 \end{aligned}$$

We use the formula<sup>19</sup>

$$(6.2) \quad [T]^r - [T]^{r-1} \sim (1 - c) \frac{T}{\ln T}, \quad r = 1, \dots, k$$

where

$$(6.3) \quad [T]^r = \frac{r}{T} = \varphi_1^{-r}(T),$$

in the following cases:

---

<sup>19</sup>Comp. (2.14).

(a) Since the summation of (6.2) for  $r = 1, 2, 3$  gives

$$(6.4) \quad [T]^3 - T \sim 3(1 - c) \frac{T}{\ln T},$$

and

$$(6.5) \quad [T]^3 - [T + 2l_3]^1 - T - 2l_3 \sim (1 - c) \frac{T}{\ln T},$$

then we obtain the following difference

$$(6.6) \quad [T]^3 - [T + 2l_3]^1 - T - 2l_3 \sim 2(1 - c) \frac{T}{\ln T}, \quad T \rightarrow \infty,$$

for every fixed  $l_3 > 0$ . Consequently, it is true that

$$(6.7) \quad [T]^3 > [T + 2l_3]^1, \quad T \rightarrow \infty,$$

and the inequality holds true with a big reserve.

(b) Since the summation of (6.2) for  $r = 1, 2, 3, 4$  gives

$$(6.8) \quad [T]^4 - T \sim 4(1 - c) \frac{T}{\ln T},$$

and the summation for  $r = 1, 2, 3$  with the substitution

$$(6.9) \quad T \rightarrow T + 2l_2; \quad \frac{T + 2l_2}{\ln(T + 2l_2)} \sim \frac{T}{\ln T}, \quad T \rightarrow \infty,$$

for every fixed  $l_2 > 0$  gives

$$(6.10) \quad [T + 2l_2]^3 - T \sim 2l_2 + 3(1 - c) \frac{T}{\ln T} \sim 3(1 - c) \frac{T}{\ln T},$$

then we obtain<sup>20</sup>

$$(6.11) \quad [T]^4 - [T + 2l_2]^3 \sim (1 - c) \frac{T}{\ln T}.$$

Consequently, it is true that

$$(6.12) \quad [T]^4 > [T + 2l_2]^3, \quad T \rightarrow \infty,$$

and the inequality holds true with a big reserve.

(c) Only 0-adjacent interval corresponds to the first and the second of intervals in (6.1).

*Remark 9.* It is true that the set of intervals (6.1) is the disconnected set with the big measures of the second and the third adjacent intervals, comp. (6.6) and (6.11).

## 7. CONCLUDING REMARKS

**7.1.** Let us remind that the oval of Cassini is the locus traced out by a point  $M$  moving so that the product of its distances from two given points  $F_1$  and  $F_2$  is constant

$$(7.1) \quad |F_1M||F_2M| = b^2,$$

where

$$(7.2) \quad F_1 = F_1(a, 0), \quad F_2 = F_2(-a, 0).$$

(α) In the critical case  $a = b$  the curve is known as lemniscate of Bernoulli.

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<sup>20</sup>See (6.8), (6.10).

( $\beta$ ) In the case  $b < a$  the curve consists of two separated ovals. Each of points  $F_1, F_2$  is surrounded by one of these ovals.

( $\gamma$ ) In the case  $b > a$  the curve is an oval with both points  $F_1, F_2$  in its interior.

**7.2.** Since we wish to keep  $a^2$  with

$$(7.3) \quad a = \left( \frac{4}{\pi} \sin v \right)^{6/5} \left( \frac{1}{8l_3} \right)^{1/2}, \quad l_3 > 0, \quad 0 < v \leq \frac{\pi}{2}$$

on the right-hand side of (7.1), we put

$$(7.4) \quad |F_5M||F_6M| = a^2,$$

where

$$(7.5) \quad F_5 = F_5(c, 0), \quad F_6 = F_6(-c, 0).$$

We, however, do not wish to introduce a new parameter  $c$ , so we shall consider only following two possibilities

$$(7.6) \quad c = \frac{1}{2}a, \quad c = 2a$$

in (7.4), (7.5).

*Remark 10.* Since<sup>21</sup>

$$(7.7) \quad a > 0,$$

then we have

(a) for

$$(7.8) \quad c = \frac{1}{2}a \rightarrow \mathfrak{O}_1[l_3, v]$$

the continuum set of Cassini ovals  $\mathfrak{O}_1[l_3, v]$  of the type ( $\gamma$ ),

(b) for

$$(7.9) \quad c = 2a \rightarrow \mathfrak{O}_2[l_3, v]$$

the continuum set of Cassini ovals  $\mathfrak{O}_2[l_3, v]$  of the type ( $\beta$ ).

*Remark 11.* Finally, if we substitute the values

$$(7.10) \quad |F_7M|, \quad |F_8M|$$

with

$$(7.11) \quad F_7 = F_7(\frac{1}{2}a, 0), \quad F_8 = F_8(-\frac{1}{2}a, 0),$$

and also substitute the values

$$(7.12) \quad |F_9M|, \quad |F_{10}M|$$

with

$$(7.13) \quad F_9 = F_9(2a, 0), \quad F_{10} = F_{10}(-2a, 0),$$

in (4.13) and (4.17) instead of  $|F_1M|$  and  $|F_2M|$ , respectively, and next in (5.8) instead of  $|F_3M|$  and  $|F_4M|$ , then we obtain the corresponding new Theorem on the continuum sets of Cassini ovals  $\mathfrak{O}_1[l_3, v]$  and  $\mathfrak{O}_2[l_3, v]$ .

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<sup>21</sup>See (7.3).

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