

A simple geometric proof for the characterisation of e-merging functions

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Abstract

E-values offer a powerful framework for aggregating evidence across different (possibly dependent) statistical experiments. A fundamental question is to identify e-merging functions, namely mappings that merge several e-values into a single valid e-value. A simple and elegant characterisation of this function class was recently obtained by Wang [2025], though via technically involved arguments. This note gives a short and intuitive geometric proof of the same characterisation, based on a supporting hyperplane argument applied to concave envelopes. We also show that the result holds even without imposing monotonicity in the definition of e-merging functions, which was needed for the existing proof. This shows that any non-monotone merging rule is automatically dominated by a monotone one, and hence extending the definition beyond the monotone case brings no additional generality.

Introduction

Testing with *e-values* is emerging as a flexible alternative to classical *p-value*-based hypothesis testing [Ramdas et al., 2023]. One of the main advantages of e-values is their ability to aggregate evidence across experiments more effectively than standard p-values, a feature that is particularly valuable in multiple and sequential testing (see, e.g., Vovk and Wang 2021, Wang and Ramdas 2022, Hartog and Lei 2025). Formally, an e-value is the observed realisation of an e-variable, defined as a non-negative random variable whose expectation under the null hypothesis is upper bounded by one. Functions that take several e-values as inputs and combine them into a single “summary” e-value are called *e-merging* functions. A key strength of this framework is that e-merging can be easily performed without imposing any assumptions on the dependence structure of the individual e-values.

In a recent work, Wang [2025] provided a complete characterisation of e-merging functions, by showing that every such function must be dominated by an affine map, and hence yielding a remarkably simple final description. Despite the simplicity of the result, the proof proposed by Wang [2025] is technically involved, relying on a minimax theorem and tools from optimal transport duality, and proceeding through several auxiliary results. In this short note, we show that the same characterisation can be obtained by a much more concise, and arguably more intuitive, geometric argument. Moreover, our approach yields a slightly stronger result: the same conclusion holds without explicitly assuming monotonicity in the definition of an e-merging function. In particular, any potentially non-monotone e-merging function is automatically dominated by a non-decreasing one. This refinement does not follow directly from the arguments in Wang [2025], where the monotonicity assumption is needed in the proof.

Characterisation of the e-merging functions

Given a measurable space \mathcal{X} , and a collection \mathcal{H} of probability measures on \mathcal{X} (the so-called *null hypothesis*), an *e-variable* for \mathcal{H} is a non-negative measurable function $E : \mathcal{X} \rightarrow [0, \infty)$, whose expectation is upper bounded by 1 under every $P \in \mathcal{H}$. For $K \geq 1$ integer, following Wang [2025] we define an *e-merging function* as a component-wise non-decreasing mapping $F : [0, \infty)^K \rightarrow [0, \infty)$ such that, for any K -tuple $\mathbf{E} = (E_1, \dots, E_K)$ of e-variables for \mathcal{H} , the composition

$$F \circ \mathbf{E} : x \mapsto F(E_1(x), \dots, E_K(x))$$

is again an e-variable for \mathcal{H} . As a side remark, e-variables are typically allowed to take the value infinity, and one would therefore wish to consider mappings $[0, \infty]^K \rightarrow [0, \infty]$ in the definition of e-merging

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functions. However, as noted by Vovk and Wang [2021], despite the conceptual importance of infinite e-values, it is sufficient to restrict attention to finite e-values when discussing e-merging functions (see Appendix C therein).

Wang [2025] proved the following simple and direct characterisation of e-merging functions. A non-decreasing mapping $F : [0, \infty)^K \rightarrow [0, \infty)$ is an e-merging function if, and only if, there is a K -tuple $\mathbf{w} = (w_1, \dots, w_K)$ of non-negative coefficients that sum up to at most 1, such that, for any $\mathbf{u} \in [0, \infty)^K$,

$$F(\mathbf{u}) \leq 1 + \sum_{k=1}^K w_k(u_k - 1).$$

The proof of Wang [2025] is based on a clever and non-trivial application of optimal transport duality. The key idea is to express the worst-case expectation of a merging function over all dependence structures as the solution of a dual optimization problem involving separable dominating functions. This reduction effectively decouples the coordinates. The problem is then reduced to a one-dimensional bound, which is handled using a sharp characterization of admissible one-dimensional e-merging functions. While the final result is simple, the proof requires several auxiliary intermediate results and a delicate minimax argument.

Here we present a much simpler proof for the characterisation of e-merging functions. The key idea is to consider the concave envelope of F and use a classical hyperplane separation argument to show that it is dominated by a suitable affine function. Unlike the approach of Wang [2025], this proof does not rely on the monotonicity assumption in the definition of an e-merging function.¹ For this reason, we formulate the result in a slightly more general setting. Specifically, we call a *generalised* e-merging function any Borel measurable mapping $F : [0, \infty)^K \rightarrow [0, \infty)$ such that $F \circ \mathbf{E}$ is an e -variable for every K -tuple \mathbf{E} of e -variables.

Theorem 1. *Let $K \geq 1$ be an integer and $F : [0, \infty)^K \rightarrow [0, \infty)$ a generalised e-merging function. Then, there is a K -tuple $\mathbf{w} \in [0, 1]^K$ such that $\sum_{k=1}^K w_k \leq 1$ and, for every $\mathbf{u} \in [0, \infty)^K$,*

$$F(\mathbf{u}) \leq 1 + \sum_{k=1}^K w_k(u_k - 1).$$

We remark that the monotonicity requirement in the definition of e-merging functions is nevertheless natural from a statistical perspective: larger e-values correspond to stronger evidence against the null, and one would therefore expect this ordering to be preserved under merging. However Theorem 1 shows that this restriction is not essential at a mathematical level: any generalised e-merging function is always dominated by a (monotone) e-merging function. In this sense, allowing for non-monotonicity does not enlarge the effective class of admissible merging procedures, and there is therefore effectively no gain in extending the definition beyond the monotone case.

Corollary 1. *Let $K \geq 1$ be an integer and $F : [0, \infty)^K \rightarrow [0, \infty)$ Borel-measurable. Then, F is a generalised e-merging function if, and only if, there is a K -tuple $\mathbf{w} \in [0, 1]^K$ such that $\sum_{k=1}^K w_k \leq 1$ and, for every $\mathbf{u} \in [0, \infty)^K$,*

$$F(\mathbf{u}) \leq 1 + \sum_{k=1}^K w_k(u_k - 1).$$

In particular, every generalised e-merging function is dominated by a (monotone) e-merging function.

Proof. Fix a \mathbf{w} as in the statement, and let $F_{\mathbf{w}} : u \mapsto 1 + \sum_{k=1}^K w_k(u_k - 1)$. It is straightforward to check that this is a (non-decreasing) e-merging function. Then, any Borel $F : [0, \infty)^K \rightarrow [0, \infty)$ dominated by $F_{\mathbf{w}}$ is clearly a generalised e-merging function. The other direction is precisely Theorem 1. \square

Geometric proof of the characterisation

The idea to prove Theorem 1 is rather simple. We consider the concave envelope \hat{F} of the generalised e-merging function F and we show that $\hat{F}(\mathbf{1}) \leq 1$, where $\mathbf{1}$ denotes the vector $(1, \dots, 1) \in \mathbb{R}^K$. Then, the supporting hyperplane theorem will immediately yield the desired conclusion.

First, let us establish a simple preliminary lemma, which will be the key ingredient of our proof.

¹A key technical step in the proof of Wang [2025] is that the analysis can be restricted to upper semi-continuous e-merging functions. This reduction relies on a lemma from Vovk and Wang [2021], which shows that every (monotone) e-merging function is dominated by an upper semi-continuous e-merging function. To prove this result, Vovk and Wang [2021] crucially exploited the monotonicity assumption.

Lemma 1. Fix an integer $n \geq 1$, consider n vectors $\mathbf{u}^{(1)}, \dots, \mathbf{u}^{(n)} \in [0, \infty)^K$, and a weight vector $(q_1, \dots, q_n) \in [0, 1]^n$, such that $\sum_{j=1}^n q_j = 1$. We have the following implication:

$$\sum_{j=1}^n q_j \mathbf{u}^{(j)} = \mathbf{1} \quad \implies \quad \sum_{j=1}^n q_j F(\mathbf{u}^{(j)}) \leq 1.$$

Proof. Let $\mathcal{X} = \{x_1, \dots, x_n\}$ be a set made of n distinct elements, and consider the hypothesis $\mathcal{H} = \{Q\}$ on \mathcal{X} , with $Q = \sum_{j=1}^n q_j \delta_{x_j}$. Define K random variables E_1, \dots, E_K on \mathcal{X} via $E_k(x_j) = u_k^{(j)}$. Then,

$$\mathbb{E}_Q[E_k] = \sum_{j=1}^n q_j E_k(x_j) = \sum_{j=1}^n q_j u_k^{(j)} = 1,$$

and so each E_k is an e-variable for \mathcal{H} . We have

$$\sum_{j=1}^n q_j F(\mathbf{u}^{(j)}) = \sum_{j=1}^n q_j F(E_1(x_j), \dots, E_K(x_j)) = \mathbb{E}_Q[F(E_1, \dots, E_K)] \leq 1,$$

since F being a generalised e-merging function implies that $F(E_1, \dots, E_K)$ is an e-variable for \mathcal{H} . \square

We now recall that the concave envelope \hat{F} of F is defined as the infimum among all concave functions that dominate F . A classical result (see, *e.g.*, Section B.2.5 in [Hiriart-Urruty and Lemaréchal 2001](#)) gives

$$\hat{F}(\mathbf{u}) = \sup_{n \geq 1} \sup_{q \in \Delta_n} \sup \left\{ \sum_{j=1}^n q_j F(\mathbf{u}^{(j)}) : \mathbf{u}^{(j)} \in [0, \infty)^K \forall j, \sum_{j=1}^n q_j \mathbf{u}^{(j)} = \mathbf{u} \right\},$$

where $\Delta_n = \{q \in [0, 1]^n : \sum_{j=1}^n q_j = 1\}$. Combined with Lemma 1, this directly yields that

$$\hat{F}(\mathbf{1}) \leq 1.$$

Now, applying the supporting hyperplane theorem (see, *e.g.*, Section A.4.2 in [Hiriart-Urruty and Lemaréchal 2001](#)) to the hypograph of \hat{F} (which is a closed convex set) we easily deduce that \hat{F} is dominated by an affine function G such that $G(\mathbf{1}) = 1$, namely a G of the form

$$G(\mathbf{u}) = 1 + \sum_{k=1}^K w_k (u_k - 1),$$

with $\mathbf{w} \in \mathbb{R}^K$. Note that F (and so \hat{F}) is non-negative on the whole domain $[0, \infty)^K$, and so G as well must be non-negative. This implies that the components of \mathbf{w} are non-negative and have sum at most 1, as required. Since \hat{F} dominates F , we conclude.

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