ADS216D Assignment 3 Total marks:55

Due date: 24 May 2021 (12PM) **Submission platform: D2L** **Submission format: PDF**

Ensure that you follow submission guidelines as specified in the study guide regarding naming of the document, font size etc.

Question 1 **(15)**

Epp Ex 4.1 p161 (189), Ex 4.2 p 168 (196)

1.1 Epp Ex 4.1 32 (3)

32. If a is any odd integer and b is any even integer, then, $2a + 3b$ is even.

Let a b any arbitrary odd integer, then there exists k such that a = 2k+1 (definition odd).

Let b be any arbitrary even integer, then there exists m such that b = 2m (definition even)

Then 2a + 3b = 2(2k+1) + 3 (2m) substitute values for a and b

= 4k + 2 + 6m distr law

= 2 (2k+1 + 3m) distr law

= 2d; with d = 2k+1 + 3m; an integer since integers closed under multiplication and addition

` So 2a + 3b is even (Definition even)

1.2 Epp Ex 4.1 37 (prove that statement is false) (3)

37. There exists an integer $k \geq 4$ such that $2k^2 - 5k + 2$ is prime.

$2k^2 – 5k + 2 = (2k - 1)(k - 2)$ factorization

Now k>=4, so k-2 >=2 and 2k – 1 >= 7, meaning that

$2k^2 – 5k + 2$ can be written as the product of two terms, both of them being >1,

Which implies that $2k^2 – 5k + 2$ is composite (definition composite number)

And therefore not prime.

1.3 Epp Ex 4.2 20 (3)

H 18. If r and s are any two rational numbers, then $\frac{r+s}{2}$ is rational.

H 19. For all real numbers a and b, if $a < b$ then $a < \frac{a+b}{2} < b$. (You may use the properties of inequalities in T17–T27 of Appendix A.)

20. Given any two rational numbers r and s with $r < s$, there is another rational number between r and s. (*Hint:* Use the results of exercises 18 and 19.)

In this problem, we do not start with definitions of rational numbers, since we have two previous proofs (18 and 19) that simplify the problem.

If r and s are rational, then $\frac{r+s}{2}$ is rational (Problem 18 result)

According to Problem 19, (for real numbers), if r < s, then r < $\frac{r+s}{2}$ < s

The result of 19 is applicable to all real numbers, so also to all rational numbers (a rational number is a real number)

Now choose a number $\frac{r+s}{2}$; then it is rational (by 18)

And r < $\frac{r+s}{2}$ < s (from 19), so there is a rational number between r and s (given by $\frac{r+s}{2}$)

1.4 Epp Ex 4.2 26 (3)

26. For any rational number s, $5s^3 + 8s^2 - 7$ is rational.

Let s be any arbitrary rational number. Then it can be written as $\frac{p}{q}$ with p and q integers, q$\neq$ 0

Then $5s^3 + 8s^2 - 7 = 5.s.s.s + 8.s.s - 7 = 5.\frac{p}{q}.\frac{p}{q}.\frac{p}{q} + 8\frac{p}{q}.\frac{p}{q} - 7$

$$= \frac{5p.p.p}{q.q.q} + \frac{8p.p}{q.q} - 7$$

$$= \frac{5p.p.p+8p.p.q-7q.q.q}{q.q.q}$$

$= \frac{k}{m}$ with k = $5p.p.p + 8p.p.q - 7q.q.q$; an integer (sum and product of integers) and m=q.q.q also an integer (product of integers).

Also q.q.q not zero (zero product rule applied twice)

Therefore $5s^3 + 8s^2 - 7$ is rational (definition rational)

1.5 Prove that any real decimal number in the form a.bc is rational, where a, b and c are integers (3)

Let a, b, and c be integers. Then a.bc has the value a + $\frac{b}{10} + \frac{c}{100}$

So a.bc can be written as $\frac{100a+10b+c}{100}$; with $100a + 10b + c$ an integer (sum and product of integers) and 100 also an integer, so $\frac{100a+10b+c}{100}$ therefore a rational number, per definition

Question 2 **(15)**

Epp Ex 4.3 p178 (206), Epp Ex 4.4 p 189 (217)

2.1 Epp Exercise 4.3 20 (3)

20. The sum of any three consecutive integers is divisible by 3. (Two integers are **consecutive** if, and only if, one is one more than the other.)

Let a be any arbitrary integer. Then a, a+1 and a+2 are three consecutive integers (definition consecutive)

Sum: a + a+1 + a+ 2 = 3a + 3

= 3 (a+1) distr law

= 3d (with d = a+1; substitution)

Therefore n = a + a+1 + a+ 2 is divisible by 3 (definition divisibility – there exists an integer d such that 3d = n)

2.2 Epp Exercise 4.3 29 (2)

Prove or disprove using counterexample

29. For all integers a and b, if $a \mid b$ then $a^2 \mid b^2$.

a| b means there exists integer k such that ak = b; then a = $\frac{b}{k}$ and a^2 = $\frac{b.b}{k.k}$ (substitution)

giving a^2 .k.k = b^2

with k.k an integer (product of integers)

so a^2 | b^2 definition divisibility

2.3 Epp Ex 4.4 21 (5)

21. Suppose b is an integer. If $b \; mod \; 12 = 5$, what is $8b \; mod \; 12$? In other words, if division of b by 12 gives a remainder of 5, what is the remainder when $8b$ is divided by 12?

b mod 12 = 5 if there exists integer d such that 12d + 5 = b

Then 8b = 8 (12d + 5) substitution

= 96d + 40 distr law

= 96d + 36 + 4 We are looking for mod 12; so look for multiples of 12

= 12(8d + 3) + 4 distr law

= 12m + 4 with m an integer, m= 8d + 3 (sum and product of integers)

So 8b mod 12 = 4 (definition mod

2.4 Epp Exercise 4.4 25 (5)

25. Prove that for all integers a and b, if $a \textit{ mod } 7 = 5$ and $b \textit{ mod } 7 = 6$ then $ab \textit{ mod } 7 = 2$.

a mod 7 = 5 if there exists integer k such that 7k+5 = a

b mod 7 = 6 if there exists integer p such that 7p+6 = b

Therefore ab = (7k+5) (7p+6) substitution

= 49kp +35p +42k +30 multiplication

=49kp +35p +42k +28 + 2 looking for multiples of 7 for mod 7…

= 7 (7kp + 5p + 6k + 4) + 2 distr law

= 7m + 2 with m = 7kp + 5p + 6k + 4 ; an integer (sum and prod of integers)

Therefore ab mod 7 = 2 (definition mod), **which was to be shown**

Question 3 **(10)**

Epp Ex 4.6 p205 (233), Epp Ex 4.7 p212 (240)

3.1 Epp Exercise 4.6 4 (5)

4. Use proof by contradiction to show that for all integers m, $7m + 4$ is not divisible by 7.

Suppose 7m + 4 is divisible by 7 (1)

Then there exists integer k such that 7k = 7m + 4 Definition divisibility

Then 7k-7m = 4 Cancellation law addition

7(k-m) = 4 distr law

k-m = $\frac{4}{7}$ cancellation law multiplication

but k and m are integers, so their difference must also be an integer (Integers closed under subtraction).

But $\frac{4}{7}$ not an integer, so leading to a **contradiction**

Therefore, supposition 1 is false, so 7m + 4 is not divisible by 7.

3.2 Epp Exercise 4.6 12 (5)

12. If a and b are rational numbers, $b \neq 0$, and r is an irrational number, then $a + br$ is irrational.

(Use contradiction to prove)

Let a and b be arbitrary rational numbers, b not=0 and r be an irrational number.

Then a = $\frac{p}{q}$ and b = $\frac{m}{n}$ with p,q, m and n integers, q and m $\neq$0. Definition rational numbers

Also m $\neq$0; since b $\neq$0

Suppose a + br is rational (1)

Then a+br can be written as $\frac{k}{s}$ with k and s integers, s $\neq$0.

Substituting a + br using definitions for a and b results in

a + br = $\frac{p}{q} + \frac{mr}{n}$ substitution

= $\frac{k}{s}$ Supposition

So $\frac{pn+rmq}{qn}$ = $\frac{k}{s}$; therefore s(pn + rmq) = kqn (qn $\neq$0 zero product rule, s $\neq$0)

Manipulation of this gives: spn + rsmq = kqn distr law + comm law

Giving r = $\frac{kqn-spn}{smq}$ but s and q $\neq$0 (def), and m $\neq$0

So r can be written as $\frac{t}{u}$ with t and u integers and u $\neq$0;

So r then a rational number (definition)

But r is irrational, **so contradiction with given statement**

So supposition 1 is false

So a + br with r an irrational number will always be irrational if b$\neq$ 0

Question 4 **(15)**

Epp Ex 5.2 p256 (284), Epp Ex 5.4 p277 (304)

4.1 Epp Exercise 5.2 7 (5)

7. For all integers $n \geq 1$,

$$1 + 6 + 11 + 16 + \cdots + (5n - 4) = \frac{n(5n-3)}{2}.$$

Mathematical induction:

Prove P(n) true for n = 1 (basis value is 1 in this case)

LHS: 1

RHS = $\frac{n(5n-3)}{2}$ = $\frac{1(2)}{2}$ = 1 = LHS, so P(1) is true

Assume P(k) is true, then

$\sum_1^k 5i - 4$ = $\frac{k(5k-3)}{2}$

LHS: P(k+1): $\sum_1^{k+1} 5i - 4$

=$\sum_1^k 5i - 4 + \sum_{k+1}^{k+1} 5i - 4$ Split sum into two parts

= $\frac{k(5k-3)}{2} + 5(k+1) - 4$ Inductive hypothesis

= $\frac{k(5k-3)+10k+10-8}{2}$ common denominator 2 and distr law

= $\frac{5k.k-3k+10k+10-8}{2}$ distr law

= $\frac{5k.k+7k+2}{2}$ adding terms

= $\frac{(k+1)(5k+2)}{2}$ factorization

RHS: P(k+1) = $\frac{(k+1)(5(k+1)-3)}{2}$

= $\frac{(k+1)(5k+2)}{2}$

= LHS ; so P(k+1) also true

So P(k) true implies that P(k+1) also true;

Also P(1) true, therefore, by induction, P(n) true for all n>=1

4.2 Epp Exercise 5.4 6 (5)

6. Suppose that $f_0, f_1, f_2, \ldots$ is a sequence defined as follows:

$$f_0 = 5, \ f_1 = 16$$
$$f_k = 7f_{k-1} - 10f_{k-2} \quad \text{for all integers } k \geq 2.$$

Prove that $f_n = 3 \cdot 2^n + 2 \cdot 5^n$ for all integers $n \geq 0$.

Since we see that f_k needs access to two previous elements f_{k-2}, we know that we need to have values for f_0, f_1

Prove that $f_n = 3.2^n + 2.5^n$ for all integers >=0

LHS: $f_0 = 5$ (initial values)

RHS: $f_0 = 3.2^0 + 2.5^0 = 3.1 + 2.1 = 5$ = LHS

LHS: $f_1 = 16$ (initial values)

RHS: $f_1 = 3.2^1 + 2.5^1 = 3.2 + 2.5 = 16$ = LHS

We have proven the statement for f_0 and f_1

Now assume statement is true for all i from 0 to k:

Then: $f_i = 7f_{i-1} - 10f_{i-2} = 3.2^i + 2.5^i$ Inductive hypothesis

$f_{k+1} = 7f_{(k+1)-1} – 10\, f_{(k+1)-2}$ for all k>=2; definition f_k

$= 7f_k - 10f_{k-1}$ simplifying subscripts

$= 7(3.2^k + 2.5^k) – 10(3.2^{k-1} + 2.5^{k-1})$ applying inductive hypothesis for f_k and f_{k-1}

$= 21.2^k + 14.5^k – 30.2^{k-1} - 20.5^{k-1}$

$= 21.2^k + 14.5^k – 30.2^k.2^{-1} - 20.5^k.5^{-1}$ law of exponents

$= 21.2^k + 14.5^k – 15.2^k - 4.5^k$ $\quad 5^{-1} = \frac{1}{5}$; $2^{-1} = \frac{1}{2}$ (trying to get similar exponents of 2 and 5)

$= (21-15)2^k + (14-4)5^k$ distr law

$= (6)2^k + (10)5^k$ adding terms

$= (3.2^1)2^k + (2.5^1)5^k$ factorising 6 and 10 to contain power of 2 and 5 respectively

$= 3.2^{k+1} + 2.5^{k+1}$ assoc law and law of exponents

RHS: $f^{k+1} = 3.2^{k+1} + 2.5^{k+1}$ = LHS

We have proven the statement for n=0, n=1, and that f_i true for i=0 to i=k implies that f_{k+1} is true

Therefore by strong induction the n-th term $f_n = 7f_{n-1} - 10f_{n-2} = 3.2^n + 2.5^n$ for all n >=0

4.3 Epp Exercise 5.4 7 (5)

7. Suppose that $g_1, g_2, g_3, \ldots$ is a sequence defined as follows:

$$g_1 = 3, \; g_2 = 5$$
$$g_k = 3g_{k-1} - 2g_{k-2} \quad \text{for all integers } k \geq 3.$$

Prove that $g_n = 2^n + 1$ for all integers $n \geq 1$.

Prove statement for k=1, 2

LHS: $g_1 = 3$ (initial value given)

RHS: $g_1 = 2^n + 1 = 2^1 + 1 = 2+1 = 3$ = LHS

LHS: $g_2 = 5$ (initial value given)

RHS: $g_2 = 2^2 + 1 = 4 + 1 = 5$ = LHS

Assume that $g_i = 3g_{i-1} - 2g_{i-2} = 2^i + 1$ for all i = 1 up to k (Inductive hypotheses)

Then $g_{k+1} = 3g_{k+1-1} - 2g_{k+1-2}$ (for all k>=3) definition g_n

$= 3g_k - 2g_{k-1}$	simplifying subscripts
$= 3(2^k + 1) - 2(2^{k-1} + 1)$	substituting inductive hypothesis for g_k and g_{k-1}
$= 3.2^k + 3 - 2.2^{k-1} - 2$	distr law
$= 3.2^k + 1 - 2.2^k.2^{-1}$	assoc law, adding terms, law of exponents
$= 3.2^k - 2^k + 1$	$2.2^{-1} = 1$
$= (3-1)2^k + 1$	distr law
$= 2.2^k + 1$	
$= 2^1.2^k + 1$	
$= 2^{k+1} + 1$	

RHS: $g_{k+1} = 2^{k+1} + 1$ = LHS

So we have proven g_n for n = 0 and n = 1, and also that assuming g_i true for all i = 1 up to k leads to g_{k+1} being true

So by strong induction g_n is true for all n>=1, meaning that $3g_{n-2} - 2g_{n-2} = 2^n+1$ for all n >=1