# Assignment W3

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# 1 Q1.

(a) $\overline{D} \cup E = \{1, 2, 4, 5, 6, 7, 8, 9, 10\}$

(b) $C \cup (D \cap E) = \{2, 3, 5, 7, 8, 9, 10\}$

(c) $\overline{(C \cap E)} = \{1, 3, 4, 5, 6, 7, 8, 10\}$

(d) $(C \cup E) - D = \{1, 2, 7\}$

# 2 Q2.

Problem: For all intergers $x$, $y$, and $z$, if $7x^2 - 3y - z + 5xy + y^2z$ is odd, then $y$ is odd or $(x - z)$ is odd.

This will be a proof by contrapositive. We will prove that if $y$ is even and $(x - z)$ is even, then $7x^2 - 3y - z + 5xy + y^2z$ is even.

Notice that if $(x - z)$ is even, then either both $x$ and $z$ are even, or both are odd. Proof:
$2a - 2b = 2(a - b)$, even
$(2a + 1) - (2b + 1) = 2(a - b)$, even
$(2a + 1) - 2b = 2(a - b) + 1$, odd
$2a - 2b + 1 = 2(a - b) + 1$, odd
Note that $(a - b)$ is an integer because both $a$ and $b$ are integers.

Assumption: $2 \mid y$ and $2 \mid (x - z)$

Case 1: Given $x, y, z \in \mathbb{Z}$, and they are all even. Let $a, b, c \in \mathbb{Z} : x = 2a, y = 2b, z = 2c$

$$7(2a)^2 - 3(2b) - 2c + 5(2a)(2b) + (2b)^2(2c)$$
$$= 28a^2 - 6b - 2c + 20ab + 8b^2c$$
$$= 2(14a^2 - 3b - c + 10ab + 4b^2c)$$

$\because a, b, c \in \mathbb{Z}$, the expression inside the parenthesis is an integer,
$\therefore$ For $x, y, z$ being even, the expression is even.

Case 2: Given $x, y, z \in \mathbb{Z}$, and both $x$ and $z$ are odd, $y$ being even. Let $a, b, c \in \mathbb{Z} : x = 2a + 1, y = 2b, z = 2c + 1$

$$7(2a + 1)^2 - 3(2b) - (2c + 1) + 5(2a + 1)(2b) + (2b)^2(2c + 1)$$
$$= 7(4a^2 + 2a + 1) - 6b - 2c - 1 + (10a + 5)(2b) + (4b^2)(2c + 1)$$
$$= 28a^2 + 14a + 7 - 6b - 2c - 1 + 20ab + 10b + 8b^2c + 4b^2$$
$$= 28a^2 + 14a + 6 - 6b - 2c + 20ab + 10b + 8b^2c + 4b^2$$
$$= 2(14a^2 + 7a + 3 - 3b - c + 10ab + 5b + 4b^2c + 2b^2)$$

$\because a, b, c \in \mathbb{Z}$, the expression inside the parenthesis is an integer,
$\therefore$ For $y$ being even, $x, z$ being odd, the expression is even.

The expression is even for both cases, so we have proved the contrapositive. $\square$

3

# 3   Q3.

Problem: For all integers $r, s$, if $14|(r + s)$ and $21|r^3$ then $7|(3r^2(r - 5) + 15s^2)$

Assuming $14|(r + s)$ and $21|r^3$, and $r, s \in \mathbb{Z}$.

By Transitivity of Divisibility, $7|(r + s)$ and $7|r^3$, because $7|14$ and $7|21$.

By Divisibility of Integer Combinations, $7|3r^3 - 15(r - s)(r + s)$, because $3, (-15)(r - s) \in \mathbb{Z}$.

$$3r^3 - 15(r - s)(r + s) = 3r^3 - 15(r^2 - s^2)$$
$$= 3r^3 - 15r^2 + 15s^2$$
$$= 3r^2(r - 5) + 15s^2$$

Thus, $7|(3r^2(r - 5) + 15s^2)$, as required. $\qquad \square$

# 4   Q4.

Problem: $\forall x \in \mathbb{Z}, [(2x \neq 6) \implies (\forall z \in \mathbb{Z}, \exists y \in \mathbb{Z}, x - 7 \neq (z - y + 5)(z^2 + 1) - 4)]$

Proof: Assume that $2x \neq 6$. Notice that $2x \neq 6 \iff x \neq 3$.

Now, choose $y = z + 5$ for any $z \in \mathbb{Z}$. Clearly, $z + 5$ is an integer.

Substituting back:
$$(z - y + 5)(z^2 + 1) - 4 = (z - (z + 5) + 5)(z^2 + 1) - 4$$
$$= (0)(z^2 + 1) - 4 = -4$$

Therefore: $x - 7 \neq -4$, and
$$x - 7 \neq -4 \iff x \neq 3$$

Since we assumed $x \neq 3$ in our hypothesis, this always holds.

Thus, for any $x \neq 3$ and any $z \in \mathbb{Z}$, we can choose $y = z + 5$ to ensure that $x - 7 \neq (z - y + 5)(z^2 + 1) - 4$.
$\square$

# 5 Q5.

(a) This is false. Consider the number 2. It is not in $F$, because $4 \cdot \frac{1}{2} = 2$ and $\frac{1}{2} \notin \mathbb{Z}$, and also not in $G$, because $3 \nmid (2-1)$ Therefore $F \cup G$ does not contain 2, and so $\overline{F \cup G}$ contains 2, which means $\overline{F \cup G} \neq \emptyset$.

(b) This is true. $\forall x \in \mathbb{Z}$, if $x \in H$, then $24|(x+8)$, and by definition, $\exists k \in \mathbb{Z}, x + 8 = 24k$.

So $x + 8 = 24k$

$$x = 24k - 8$$

$$x = 4(6k - 2)$$

So there exists an integer, $(6k - 2)$, such that $x = 4(6k - 2)$, hence $4|x$, from which follows $x \in F$.

And also from $x + 8 = 24k$,

$$x - 1 = 24k - 9$$

$$x - 1 = 3(8k - 3)$$

So there exists an integer, $(8k - 3)$, such that $x - 1 = 3(8k - 3)$, hence $3|(x-1)$, from which follows $x \in G$.

Therefore, if $x \in H$, then $x \in F$ and $x \in G$, so $x \in F \cap G$.  $\square$

# 6 Q6.

(a) $S = K, P = P_3$
(b) $S = J, P = P_4$
(c) $S = L, P = P_1$
(d) $S = M, P = P_2$